Chapter XIII

PLANE MOTION

168. Introduction. This chapter and the next one deal briefly with the subdivision of mechanics which is called *kinematics*. As defined in § 1, kinematics is the study of the motion of particles (whose dimensions are infinitesimal) and bodies without regard to the forces which affect the motion. Some study of this phase of mechanics is necessary at this time, because *unbalanced* force systems, which is the next subject for consideration, always produce motion or a change of motion of the bodies on which they act.

The laws of motion were investigated scientifically first by Galileo Galilei (1564–1642), who was born at Pisa to an impoverished noble family. Although he was educated for medicine, his interest shifted to the physical sciences. Most of his experiments on motion were conducted in later life, although the famous one of two weights released from the Tower of Pisa was performed when he was 26.

Aristotle had reasoned that heavy bodies fall faster than light bodies, a logical and plausible conclusion, and nearly every one of Galileo's day believed that this was true. No one thought to question Aristotle's idea until, some 2000 years later, Galileo decided to drop simultaneously a half-pound weight and a 100-lb. cannonball from the Leaning Tower of Pisa. When the spectators observed that both weights struck the ground at the same time, they roundly hissed Galileo, thinking that perhaps he was practicing witchcraft. People always "know" so many things which are not so, and such a result was contrary to what the people "knew".

In passing, it will be interesting to suggest further how authoritative the writings of Aristotle became. When Galileo discovered sun spots, a monk heard of his discovery and, having access to a telescope which had just then been invented, he verified Galileo's observation. The monk then reported the spots to his bored and dubious superior. The superior searched his Aristotle in vain. Finding no mention of sun spots, he told the monk, "Be assured therefore that it is a deception of your senses or of your telescope." Today, we know that the very essence of science is experimental verification. We may theorize first and then verify by experiment, or we may evolve a earth, and therefore would be useless, and therefore do not exist. Besides the Jews and other ancient nations, as well as modern Europeans, have adopted the division of the week into seven days, and have named them after the seven planets. Now if we increase the number of planets, this whole and beautiful system falls to the ground." Such are the pitfalls of logic.

The best that Aristotle could do was to classify motion as natural (such as the motion of the stars and planets) or unnatural (such as the thrown stone, which would naturally soon come to rest—with respect to the earth). We shall study general (Newton's) laws of motion later, which apply to earthly bodies and roughly to heavenly bodies, and in the meantime learn the names of a few kinds of motion of which Aristotle never heard.

169. Displacement. Displacement is a directed distance, and as such it is a vector quantity. In common with force vectors, it has the properties of magnitude, sense, and location (line of action). Thus, suppose a person starts at some point designated as the origin O, Fig. 443, and walks for 2 miles in a straight line directed N 30° E (read this, north 30° east). The corresponding displacement is represented by the vector OA laid out in magnitude and pointing in the direction of the displacement. Then if this person walks one mile due east, the vector representing this displacement is AB. It is supposed, of course, that the terrain is perfectly level, so that these displacements are in the same plane.

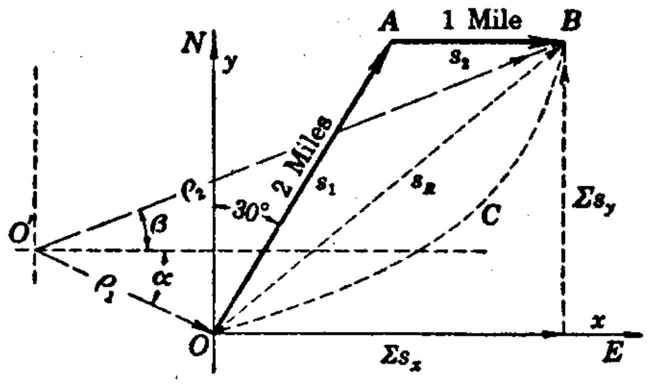

Fig. 443. Displacement.

At the point B in space, the pedestrian is not 3 miles from the starting point (the origin O), but some shorter distance OB. The pedestrian's *resultant* displacement from O is evidently the resultant (vector *sum*) of the vectors OA and AB. This resultant vector OB may be obtained:

1. By graphical solution,
2. From the law of cosines (§ 6), or
3. By summing horizontal and vertical components of the displacements

$$(\Sigma s_x \qquad \text{and} \qquad \Sigma s_y)$$

and then finding the resultant

$$s_R = [(\Sigma s_x)^2 + (\Sigma s_y)^2]^{1/2}.$$

We recognize that each one of these methods of solution is an exact counterpart of a method that we might have used if these vectors had been force vectors.

Suppose that the point of reference is some point O', Fig. 443, other than the point O from which the displacement occurred. The simplest way of expressing the resultant displacement is in terms of radius vectors and angles (polar coordinates). Relative to O', the point O is defined by the radius vector ρ_1 and the angle α, which is, of course, a negative angle as measured in Fig. 443. The point B is defined by ρ_2 and β. Since the vector sum of ρ_1 and OB is equal to ρ_2, it is evident that the vector *difference* is

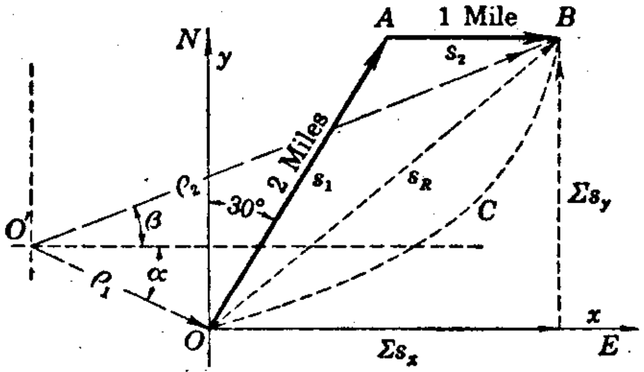

Fig. 443. **Repeated.**

$$\rho_2 \rightarrow \rho_1 = OB = s_R,$$

that is, the resultant displacement. This method of finding a resultant displacement tells nothing of the path that may have been followed between O and B, as the path may have been *any* one, such as OCB. Observe in the foregoing illustration that the pedestrian walked 3 miles, which is the *distance* that he traversed, but his displacement from the starting point is something less than 3 miles.

Displacement problems, such as the preceding, will not occupy much of our time here; but the significant point, that displacement is a vector quantity, should be fixed firmly in mind. Soon we shall be dealing simultaneously with velocities, accelerations, and forces, as well as displacements, all of which are represented by vectors which may or may not point in the same direction. In general, we shall follow the usual convention; component displacements in the upward direction and component displacements toward the right are positive. In many problems, it may be convenient to change this convention, but consistency in the use of algebraic signs must be maintained in any particular problem.

170. Speed and Velocity. The ***speed*** of a particle at any instant is the time rate with which it is traversing distance. Technically, although the words *speed* and *velocity* are often used as though their meanings were identical, ***velocity*** is a vector quantity, and therefore possesses both properties of magnitude and *sense*; whereas speed is a scalar quantity. In other words, the speed is only the magnitude of the velocity. This distinction is important because changes of velocity are important; and a change in the *sense* (*direction*) of a velocity is as truly a *change of velocity* as is a change in the *magnitude* of a velocity. Consider an airplane, whose engine is warming up at a constant angular speed, in revolutions per minute, before the plane takes off. A point on the tip of the propeller is moving at constant *speed* in the path of a circle, but its velocity is continuously varying, since the direction of the velocity changes through 360° with every revolution of the propeller.

Let a point traverse a distance Δs during some time interval Δt. Then the *average* speed of the point is $v = \Delta s/\Delta t$. As the distance and the time interval become smaller and smaller, we may write as the limit

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{33}$$

Thus we see that when s is expressed as a continuous function of the time t, the derivative of s with respect to t gives the instantaneous speed at any time t. If a constant speed is maintained, the instantaneous speed and the average speed are of course the same. Speed may be expressed in any convenient units of distance and time. In engineering work, the most common units are feet per second v_s (fps), feet per minute v_m (fpm), and miles per hour (mph).

A method of notation which is becoming popular because of its simplicity is to indicate differentials *with respect to time* by a dot over the symbol. For example,

$$\begin{aligned}
\dot{x} &\text{ stands for } dx/dt = v_x,\\
\dot{y} &\text{ stands for } dy/dt = v_y,\\
\dot{z} &\text{ stands for } dz/dt = v_z,\\
\dot{s} &\text{ stands for } ds/dt = v \text{ or } v_s,
\end{aligned}$$

where v_x, v_y, and v_z may be considered as components in the x, y, and z directions respectively of the resultant velocity v. Similarly, two dots over the symbol represent the second differentiation with respect to time; thus

$$\begin{aligned}
\ddot{x} &\text{ stands for } d^2x/dt^2 = dv_x/dt = a_x,\\
\ddot{y} &\text{ stands for } d^2y/dt^2 = dv_y/dt = a_y,\\
\ddot{z} &\text{ stands for } d^2z/dt^2 = dv_z/dt = a_z,\\
\ddot{s} &\text{ stands for } d^2s/dt^2 = dv_s/dt = a_s,
\end{aligned}$$

where a_x, a_y, and a_z may be components of some acceleration vector a. Acceleration is discussed in § 172. We shall use both the differential and dotted forms, and the reader may adopt the one he prefers. It follows from the defined meaning of the dotted symbol that

$$\dot{v} = \frac{dv}{dt} = a.$$

171. Examples. (a) If the motion of a parachutist (before the parachute opens) is expressed by the relation $s = Ct^2$ ft., where C is a constant, what is his speed 2 sec. after he leaves the plane, if $C = 15$?

Solution. Since $v = ds/dt$, we first differentiate the given expression with respect to t, and find

$$\dot{s} = v_s = \frac{ds}{dt} = 2Ct.$$

Thus, after 2 sec., $v_s = (2)(15)(2) = 60$ fps, which is equal to $(60)(3600)/5280 = 41$ mph, approximately.

(b) A car is moving in such a manner that $s = 88t$, where s is in feet and t in seconds. What is the car's speed after 10 sec.?

SOLUTION. By differentiation, we get

$$\dot{s} = v_s = \frac{ds}{dt} = 88, \text{ a constant.}$$

Therefore, after 10 sec., the car is moving with a speed of 88 fps (= 60 mph) and it continues to move with this speed as long as the car's movement conforms to the given rule that distance is proportional to the time.

172. Acceleration. ***Acceleration a*** is the time rate of change of velocity. The velocity may change in either magnitude or direction (or both); hence the acceleration of a point (or particle) may be due to a change in the magnitude or in the direction, or in both the magnitude and direction, of the velocity. Moreover, the acceleration too is a vector quantity, since it has both magnitude and sense. The acceleration a may be constant or variable.

Suppose a point is moving in a straight line (rectilinear motion), so that there is no change in the sense of the velocity. Then the acceleration is due to a change in the magnitude of the velocity only. Thus let Δv be the change in speed during a time interval Δt. The average acceleration during rectilinear motion is $a = \Delta v/\Delta t$. Letting the increments Δv and Δt decrease until they become infinitesimal, we get the instantaneous acceleration as

$$\text{(a)} \qquad a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt},$$

which gives the total acceleration only for the case of rectilinear motion. Observe that since

$$\dot{s} = v = \frac{ds}{dt},$$

we find

$$\dot{v} = \ddot{s} = \frac{dv}{dt} = \frac{d^2s}{dt^2};$$

thus, for rectilinear motion,

$$\text{(34)} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Another useful differential equation for the acceleration of a point is obtained by multiplying the numerator and denominator of the right-hand side of (a) by ds; that is, for rectilinear motion,

$$\text{(35)} \qquad a = \frac{dv}{dt}\frac{ds}{ds} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds}, \qquad \text{or} \qquad v\,dv = a\,ds.$$

If the displacement were represented by some other symbol, for instance, x instead of s, (35) would take the form $v\,dv = a\,dx$. In rectilinear motion, the resultant acceleration is in the direction of the straight-line path. However, the acceleration may be either in the sense in which the point is moving or in the opposite sense. When the sense of the acceleration is the *same* as that of the velocity, the velocity is *increasing.* When the sense of the acceleration is *opposite* to that of the velocity, the velocity is *decreasing.* When a moving point is slowing down, its acceleration is often termed ***deceleration*** or ***retardation.***

Keep in mind that the sign of the acceleration designates the sense of the acceleration *vector.* Since we often *choose the direction of motion as the positive direction,* a negative acceleration in *this* event means a slowing down or retardation.

The most common unit of acceleration in engineering is feet per second-second (fps^2). Other units sometimes used include feet per minute-minute (fpm^2), miles per hour-second.

173. Example. What is the acceleration of the parachutist of § 171(a) 2 sec. after he leaves the plane? Assume that he moves in a vertical path.

SOLUTION. Using the equation given, $s = Ct^2$, we obtain

$$\dot{s} = v = \frac{ds}{dt} = 2Ct,$$

$$\ddot{s} = \dot{v} = a = \frac{d^2s}{dt^2} = 2C = (2)(15) = 30 \text{ fps}^2,$$

when $C = 15$, as given in § 171(a). Note that this is a constant acceleration, and is the same for any time t as long as the rule $s = Ct^2$ holds true.

174. Example. A particle is moving with rectilinear motion such that the acceleration $a = 4t$ fps^2. When we start counting time, the speed is 10 fps (initial speed). After 4 sec., what are the speed and the distance traversed?

SOLUTION. Using equation (34), $a = dv/dt$, we may write

$$\int_{10}^{v_2} dv = 4\int_0^4 t\,dt,$$

where the limits of the integrals are set in accordance with the statement of the problem. Integration gives

$$v_2 - 10 = 2t^2\Big]_0^4 = 32,$$

or

$$v_2 = 42 \text{ fps}.$$

To find the distance, first integrate $dv = 4t\,dt$ as an indefinite integral and find

$$v = \dot{s} = 2t^2 + C,$$

where C is the constant of integration. The value of C may be found from the simultaneous values $v_o = 10$ when $t_o = 0$; or $C = 10$. The subscript o for v_o and t_o indicate values at the origin. Using this value of $C = 10$ and using $v = ds/dt$ in the preceding equation, we get

$$\int_0^{s_2} ds = 2\int_0^4 t^2 dt + 10\int_0^4 dt,$$

$$s_2 = \frac{2t^3}{3} + 10t\Big]_0^4 = 82.7 \text{ ft}.$$

175. Example. A particle is moving in a straight line so that $\ddot{s} = a = 2s$. If it starts from rest, what is its speed after it has moved 10 ft.? What time has elapsed?

SOLUTION. Using equation (35), $v\,dv = a\,ds$, and $a = 2s$ we get

$$\int v\,dv = 2\int s\,ds,$$

$$\frac{v^2}{2} = s^2 + C,$$

where C is the constant of integration. Since the problem states that the particle starts from rest, $v_o = 0$ when $s_o = 0$. These values in the foregoing equation give $C = 0$. The speed after any distance s is traversed is then

$$v^2 = 2s^2.$$

After $s = 10$ ft., the speed is

$$v = s\sqrt{2} = 10\sqrt{2} = 14.1 \text{ fps.}$$

Using $v = ds/dt$ in the previous equation, we find a relation between distance and time; thus

$$v = \frac{ds}{dt} = \sqrt{2}s,$$

or

$$\int_0^{10} \frac{ds}{s} = \sqrt{2}\int_0^t dt,$$

from which

$$\log_e s \Big]_0^{10} = \sqrt{2}t,$$

$$t = \frac{\log_e 10}{\sqrt{2}} = 1.63 \text{ sec.},$$

the time for the particle to move 10 ft. with the motion as defined by $a = 2s$.

176. Constant Acceleration—Rectilinear Motion. When we know the law that governs the acceleration of a particle which moves on a straight line, we may use equations (34) and (35) to derive the detailed relations between a, v, s, and t, as suggested by the examples of §§ 174 and 175. A special case of importance is that in which the acceleration a is *constant* or is assumed to be constant. Thus from (34), we have

(b) $$\dot{v} = \frac{dv}{dt} = a \qquad \text{or} \qquad dv = a\,dt.$$

Integrating **(b)**, we find

(c) $$\dot{s} = v = at + C_1,$$

where C_1 is a constant of integration. Let $v = v_o$ when $t = 0$ and substitute these values in **(c)**: This substitution gives $C_1 = v_o$; hence equation **(c)** becomes

(d) $$v = v_o + at \qquad \text{or} \qquad a = \frac{\Delta v}{t},$$

where Δv is the change in speed (second speed minus the first speed, $v - v_o$) during the time t. Since $v = ds/dt$, we have, from (d),

$$\int ds = v_o \int dt + a \int t\, dt,$$

$$\text{(e)} \qquad s = v_o t + \frac{at^2}{2} + C_2.$$

If we start measuring space and counting time at the same instant, that is, if $s = 0$ when $t = 0$, we find $C_2 = 0$; in which case, (e) becomes

$$\text{(f)} \qquad s = v_o t + \frac{at^2}{2}.$$

Now, if the acceleration a is constant, we find from (35) a useful relation;

$$\int v\, dv = a \int ds,$$

whence

$$\frac{v^2}{2} = as + C_3.$$

If the particle is moving with a velocity v_o when we start measuring space, then $v = v_o$ when $s = 0$, from which $C_3 = v_o^2/2$. Using this value of C_3 in the preceding equation, we get

$$\text{(g)} \qquad \frac{v^2}{2} = as + \frac{v_o^2}{2} \qquad \text{or} \qquad v^2 = v_o^2 + 2as.$$

It should be kept in mind that these relations hold only if the acceleration a is *constant*. If the body starts from rest, $v_o = 0$, and equation (g) becomes

$$v^2 = \dot{s}^2 = 2as.$$

You are probably familiar with the special case of this equation, $v^2 = 2gh$, that for a falling body, where g is the acceleration of gravity and h is the height through which the body has fallen from rest (without air or other resistance) when the speed is v.

In the equations (d) and (f), v is the speed of a point at any instant t, s is the distance the point has moved at the instant t, v_o is the initial speed (the speed at the origin of time), and a is the *constant* acceleration of the point. In equation (g), v is the speed of a point with constant acceleration a after it has moved a distance s from an origin where the speed of the point is v_o. If the acceleration is opposite to the velocity and to the displacement, all of these equations hold good, but the value of a is *negative* when the sense of the velocity is taken as positive.

If the student uses these equations in any problem, he must be sure that the problem falls within the limitations of this article, that is, that the acceleration is constant,

$$a = C = \text{a constant}.$$

177. Uniform Motion. The term *uniform motion* is applied to a particle or body which moves with a constant speed, $v = C$. If v is constant,

$$a = \frac{dv}{dt} = 0, \qquad \text{or} \qquad a = 0,$$

the acceleration is zero. This is recognized as the special case of statics, as far as forces are concerned. When $a = 0$, we have $v = s/t$.

178. Example. The velocity of a freight train increases at a constant rate from 15 mph to 45 mph in half an hour. What is its acceleration (fps²)?

SOLUTION. Convert mph to fps.

$$45 \text{ mph} = \frac{(45)(5280)}{3600} \text{ fps} = 66 \text{ fps}.$$

$$15 \text{ mph} = \frac{(15)(5280)}{3600} \text{ fps} = 22 \text{ fps}.$$

Therefore, by (d), § 176, the acceleration is

$$a = \frac{\Delta v}{t} = \frac{66 - 22}{1800} = 0.0244 \text{ fps}^2,$$

where $t = 0.5$ hr. $= 1800$ sec.

179. Example. A body A is projected vertically downward from a 500-ft. cliff with an initial velocity of 10 fps. One second later, a body B is projected vertically upward from the bottom of the cliff with an initial velocity of 70 fps. (a) When do these bodies pass one another? Time is to be measured from the beginning of the motion of A. (b) How far above the bottom of the cliff are the bodies when they pass? (c) What is the direction of motion and speed of B when they pass? Neglect air resistance.

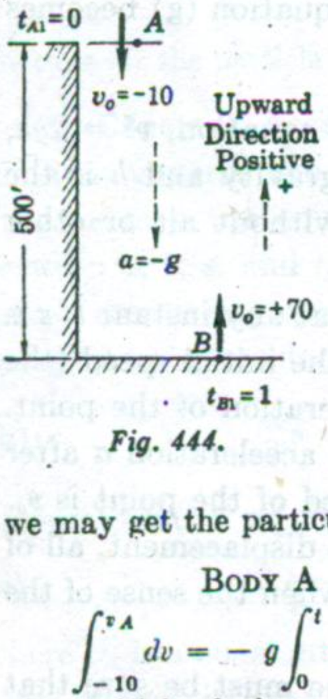

Fig. 444.

SOLUTION. (a) The first step is to get a clear mental picture of what is taking place. To do this, make a sketch to show the pertinent information (Fig. 444). We may choose either sense as positive. In this solution, we shall let the upward sense be positive. We notice that the bodies, considered as particles, are acted upon by gravity, so that the acceleration of the bodies is the acceleration of gravity $g = 32.2$ fps². However, since the acceleration of gravity is *downward*, $a = -g$ in equations (34) and (35), if the upward sense is positive. Also, let the origin of space be at the level of B, which places A at an initial position of $+500$ ft. By (34), the equation of motion for either body A or B is $\dot{v} = dv/dt = -g$, or $dv = -g\,dt$. By integration, we may get the particular relations of s, v, t, and a for each body as follows:

BODY A	BODY B
$\int_{-10}^{v_A} dv = -g\int_0^t dt$	$\int_{70}^{v_B} dv = -g\int_1^t dt$
(Initial velocity $= -10$)	(Initial $v_o = +70$, initial $t = 1$)
$v_A + 10 = -gt$	$v_B - 70 = -gt + g$
$v_A = \dfrac{ds}{dt} = -10 - gt$	$v_B = \dfrac{ds}{dt} = 70 - gt + g$

$$\int_{500}^{s_A} ds = -10\int_0^t dt - g\int_0^t t\,dt \qquad \int_0^{s_B} ds = 70\int_1^t dt - g\int_1^t t\,dt + g\int_1^t dt$$

(Initial $s_o = +500$ ft.) (Initial $s_o = 0$)

$$s_A = 500 - 10t - \frac{gt^2}{2} \qquad s_B = 70t - 70 - \frac{gt^2}{2} + \frac{g}{2} + gt - g$$

$$= 102.2\,t - \frac{gt^2}{2} - 86.1$$

When the bodies pass, they will be at that instant at the same elevation; that is, $s_A = s_B$. Equating the preceding values of s_A and s_B, we get

$$500 - 10t - \frac{gt^2}{2} = 102.2\,t - \frac{gt^2}{2} - 86.1,$$

from which $t = 5.22$ sec., the elapsed time after A is projected downward.

(b) The distance of the bodies from the level of B may be found from the expression for either s_A or s_B by substituting $t = 5.22$ sec. Thus we find

$$s_A = 500 - (10)(5.22) - \frac{(32.2)(5.22)^2}{2} = 9.2 \text{ ft.},$$

which is also equal to s_B at this instant. Note that, since the answer for s_A is positive, the distance 9.2 ft. is measured upward from B. (Both A and B have the same origin of *distance*. In this instance, *distance* and *displacement* have the same meaning because the motion is all in the vertical direction.)

(c) From the preceding expression for v_B, the velocity of the body B is

$$v_B = 70 - gt + g = 70 - (32.2)(5.22) + 32.2 = -65.9 \text{ fps}.$$

The negative answer in this case shows that B is moving *downward*; B is returning from its highest point. For another verification that B is moving down, we may find the time it takes for B to reach its highest point by using our knowledge that at the peak of its travel its speed is momentarily zero. Setting $v_B = 0$, we have

$$v_B = 0 = 70 - gt + g,$$

or $t = 3.18$ sec. Since this time is less than $t = 5.22$ sec. obtained in part (a), body B must be moving down when it and A are at the same height.

180. Variable Acceleration. In the event of variable acceleration, it is in general necessary to express a as a function of s or t, as some of the preceding examples have shown. Such a relation would be of the form

$$a = f(s) \qquad \text{or} \qquad a = f(t).$$

In the first form, we may use the fundamental expression $v\,dv = a\,ds$, equation (35), and we obtain a differential equation with two variables, as

$$v\,dv = f(s)\,ds.$$

If $a = f(t)$, the fundamental expression $dv = a\,dt$, equation (34), will take the form

$$dv = f(t)\,dt.$$

Even with a simple law expressing the variation of acceleration, the solution of the resulting differential equation may be difficult.

181. Example. A particle whose acceleration is $a = 3t - 12$ fps², is moving at a certain instant in a straight line with an initial velocity of 15 fps in the same sense as the initial acceleration. (a) At the end of 10 sec., what is the velocity of the particle? (b) At the end of 14 sec., what is its displacement from the origin?

SOLUTION. (a) At $t = 0$, $a = -12$ fps². Therefore, the given initial velocity is $v_o = -15$ fps, that is, it is negative if it is to have the same sense as the acceleration. Using $a = \dot{v} = dv/dt$, we get

$$\int_{-15}^{v} dv = \int_{0}^{10} (3t - 12)dt,$$

$$v + 15 = \left[\frac{3t^2}{2} - 12\,t\right]_0^{10} = \frac{300}{2} - 120,$$

$$v = +15 \text{ fps.}$$

(b) To find the displacement in a second integration, make the first integration indefinite. An indefinite integration gives

$$\frac{dx}{dt} = v = \frac{3t^2}{2} - 12t + C,$$

where the constant of integration C is determined by the conditions that $v = v_o = -15$ when $t = 0$. Using these simultaneous values, we find $C = -15$. Substituting this value of C in the preceding equation and integrating, we find

$$\int_0^x dx = \int_0^{14} (1.5t^2 - 12t - 15)dt$$

$$x = \left[\frac{1.5t^3}{3} - \frac{12t^2}{2} - 15\,t\right]_0^{14} = -14 \text{ ft.}$$

Assuming that the positive sense is toward the right, the negative sign for x shows that the displacement is to the left from the origin.

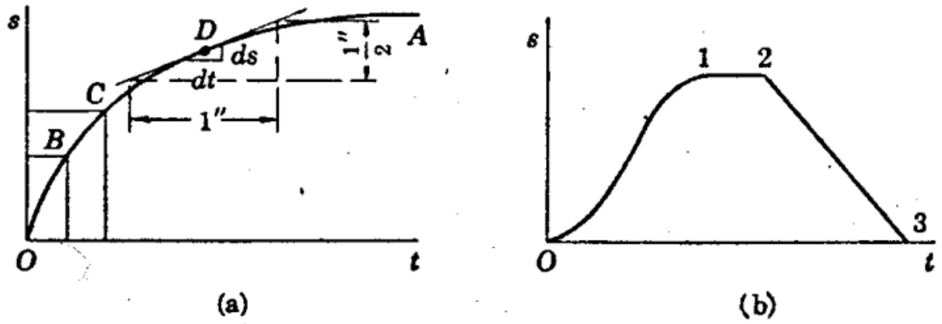

Fig. 445. s-t Diagrams. **The slope represents velocity.**

182. Graphical Representations. The variables which appear in the motion equations, s, t, v, and a, may be combined to plot curves representing the relation between any two of them. However, certain pairs of these variables produce curves of particular usefulness. We shall now consider these advantageous groupings briefly.

(*a*) ***Displacement Diagram*** (*s–t* curve). A curve with the coordinates *s–t*, Fig. 445, is sometimes called a *displacement diagram*, inasmuch as the ordinates *s* represent displacements. In some circumstances, it may be convenient to measure the distances a body has moved after various times t. If so, a series of points such as B and C, Fig. 445 (a), may be plotted for simultaneous values of s and t and a smooth curve drawn through these points to obtain some curve OA. The slope of the curve at any point D is ds/dt, which is recognized as the instantaneous speed, $v = \dot{s} = ds/dt$. Hence, the steeper the curve, the greater is the speed. This curve near A is horizontal (zero slope), indicating that the velocity is zero (no change in s).

Suppose that the tangent drawn at D, Fig. 445(a), has a slope of

$$\text{Slope} = \frac{\tfrac{1}{2}\text{ in. of distance}}{1\text{ in. of time}}.$$

The actual velocity is determined from the scales used for s and t in drawing the curve. Suppose the scales were

$$1\text{ in.} = 4\text{ sec. (for } t),$$
$$1\text{ in.} = 20\text{ ft. (for } s).$$

Then the slope expressed in feet per second (velocity) units is

$$v = \frac{(\tfrac{1}{2})(20)}{(1)(4)} = 2.5\text{ fps.}$$

Keeping in mind the significant characteristic of this diagram, namely, $v = ds/dt$ = slope of the curve, we may easily interpret such a curve as O-1-2-3, Fig. 445(b). We see that the speed (slope) increases to a maximum at a point about midway between O and 1, that the speed reaches zero at the point 1, that no motion occurs from 1 to 2, and that the speed (slope) is constant from 2 to 3. The negative slope from 2 to 3 indicates that the velocity vector is pointing in the negative sense.

On occasion, we may be able to plot an s-t curve, or one of the other curves, when a mathematical relationship is impossible or very difficult to obtain. In cases of this nature, the graphical procedure readily yields important information on the motion of a point.

(*b*) ***The Velocity-Time Curve.*** The curve OB, Fig. 446, defines a relation between v and t, $v = f(t)$. At any point D, we observe that the slope of the curve is dv/dt, which is recognized as the acceleration $a = \dot{v} = dv/dt$. Knowing the scales used for v and t, we may convert the actual slope to acceleration. Suppose the slope at D is 0.8 and the scales used in plotting the curve are

$$1\text{ in.} = 20\text{ fps (for } v),$$
$$1\text{ in.} = 5\text{ sec. (for } t);$$

then

$$a = \text{Slope}\left(\frac{\text{Speed scale}}{\text{Time scale}}\right) = \text{Slope}\left(\frac{\dfrac{20\text{ ft.}}{\text{sec-in.}}}{\dfrac{5\text{ sec.}}{\text{in.}}}\right) = \frac{(0.8)(20)}{5} = 3.2\text{ fps}^2.$$

When the slope is negative, the speed is decreasing and the body or particle is decelerating.

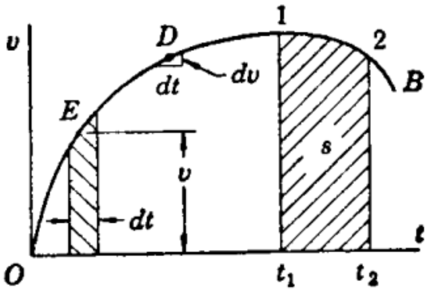

Fig. 446. v-t Diagram. The slope represents acceleration. The area represents displacement.

Another significant characteristic of the v–t diagram is that the area "under" the curve represents to scale the distance traversed by the point in question. Observe that the area of the differential element at E, Fig. 446, is $v\,dt$; but since $v = ds/dt$, we have $v\,dt = ds$. Thus

$$\int ds = s = \int v\,dt,$$

which is the area "under" the curve when the correct limits of the integral are used. Suppose we wish to know the distance traversed between 1 and 2, Fig. 446, then the integration is made from t_1 to t_2. If the curve OB is obtained from experimental data and $v\,dt$ is difficult to integrate, the area under the curve between 1 and 2 may be found by a planimeter or other means. This area in square inches, say, may be converted to distance when the scales are known. Thus, if the area under 1–2 is 0.6 sq. in. and the scales are as before, we have

$$\text{(Area in sq. in.)(Speed scale)(Time scale)} = \text{Distance}$$
$$(0.6 \text{ in.}^2) \times \left(\frac{20 \text{ ft.}}{\text{sec-in.}}\right) \times \left(\frac{5 \text{ sec.}}{\text{in.}}\right) = 60 \text{ ft.}$$

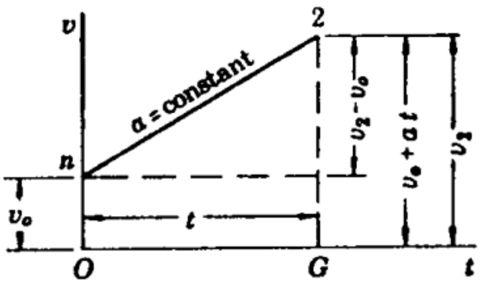

Fig. 447. v-t Diagram, Acceleration Constant.

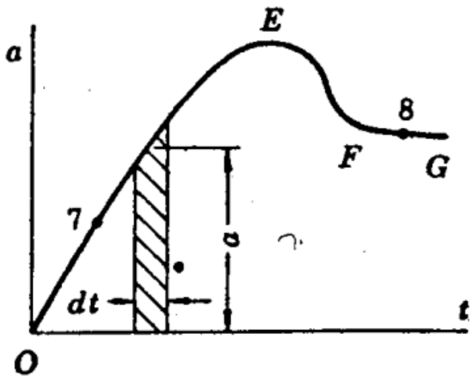

Fig. 448. a-t Diagram. **The area represents speed.**

In Fig. 447, the curve $n2$ represents a motion with constant acceleration (constant slope). When $t = 0$, the ordinate is some finite value v_o, which is the initial speed. At the point 2, the ordinate is v_2, the speed of the point after the elapsed time t. The slope of the line $n2$, which is equal to the acceleration, is

$$\frac{v_2 - v_o}{t} = a \qquad \text{or} \qquad v_2 = v_o + at.$$

Compare this expression with equation (d), § 176, and observe how the equations of § 176 may be obtained from graphical representations. To get equation (f), § 176 (the expression for s the distance moved), find the area under $n2$, Fig. 447. We see that this area is equal to a rectangular area $v_o t$ *plus* a triangular area $(\frac{1}{2})(v_2 - v_o)t$. The whole area and the distance is given by

$$s = v_o t + \frac{(v_2 - v_o)t}{2} = v_o t + \frac{(v_2 - v_o)t^2}{2t},$$

$$s = v_o t + \frac{at^2}{2},$$

where we have used $a = (v_2 - v_o)/t$. Compare with equation (f).

(*c*) ***The Acceleration-Time Curve.*** The significant characteristic of the a-t curve, Fig. 448, is that areas "under" the curve represent speed. To prove this, set up the shaded differential element, whose area is $a\,dt$. Since $a = dv/dt$, we have $a\,dt = dv$. Hence, $\int a\,dt$, which represents the area, is equal to $\int dv$, which is the change of speed. For example, the area under the curve from point 7 to point 8, Fig. 448, represents the change of speed between the times t_7 and t_8. The *scale* of the area is the product of the acceleration scale and the time scale.

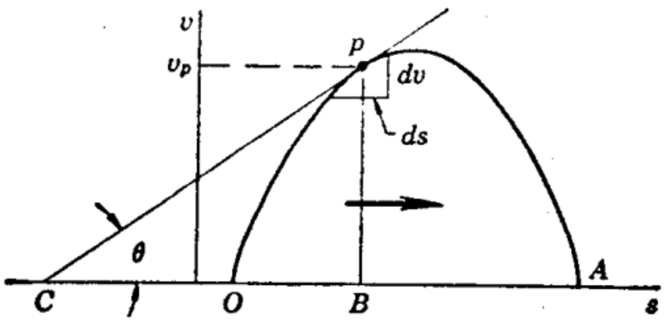

Fig. 449. v-s (Velocity) Diagram.

(*d*) ***Velocity Diagrams.*** In studying the motion of machine parts, we often construct a so-called velocity diagram, which is a v-s curve. For example, the curve OpA of Fig. 449 represents the relation between the velocity and displacement of a shaper cutting tool on the return stroke. The distance OA is laid out to scale to represent the length of the stroke of the tool. The velocities at various points of the stroke (that is, the curve) are found easiest by graphical means (as explained in books on kinematics and mechanism). This diagram is used principally to study the variation of velocity, but the acceleration at any point p may be found from it by using the known relation $v\,dv = a\,ds$.

Tangent and vertical lines drawn at p form the triangle pBC. Since this triangle is similar to the one whose legs are dv/ds, Fig. 449, we may find dv/ds from pB/BC. Knowing this ratio, we may multiply it by the actual velocity at p to get the acceleration at p. Thus suppose $v_p = 100$ fpm and

$BC = 9$ in. (These are actual values for the machine.) Then (9 in. $= 0.75$ ft. and $dv/ds = 100/0.75$)

$$a_p = v_p \frac{dv}{ds} = (100)\left(\frac{100}{0.75}\right) = 13{,}333 \text{ fpm}^2,$$

or converting the time unit to seconds, $a_p = 3.7$ fps². Observe that at the highest point of the curve, the slope (acceleration, too) is zero. To the right of the highest point, the slope (acceleration) is negative and the velocity is decreasing.

In connection with the foregoing diagrams, it is worth while noting that, if the displacement *s-t* diagram of the rectilinear motion of a point is known, the other diagrams, for *v-t* and *a-t*, may be drawn. The procedure is to measure carefully the slope (speed) of the *s-t* curve at a series of points, and plot this slope against the corresponding time. A smooth curve through the points obtained gives the *v-t* curve. Then plotting slopes (accelerations) of the *v-t* curve against time will give the *a-t* diagram.

183. Angular Velocity. If a body is rotating about either a fixed or a moving axis, it is said to have angular velocity. The angular velocity is measured by the time rate of change of angular displacement of a line in the body. Suppose the body B (*not* a point or a particle), Fig. 450, turns about an axis O so that the line OA takes the position OA'. The resulting angular displacement of the body is the angle $\Delta\theta$ between the two positions of the line, OA and OA'.

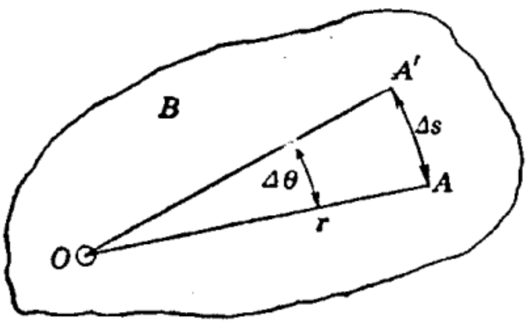

Fig. 450.

Now suppose that this rotation through the angle $\Delta\theta$ occurred during a time interval of Δt. Then the average angular velocity ω of the body B is $\dot{\theta} = \omega = \Delta\theta/\Delta t$. Letting the increments $\Delta\theta$ and Δt become progressively smaller, we find

$$(36) \qquad \dot{\theta} = \omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt},$$

the instantaneous angular velocity of the body. The usual units of angular velocity are: radians per second, radians per minute, revolutions per minute (rpm), and revolutions per second (rps).

184. Relations between Angular and Linear Speeds. Observe in Fig. 450 that the distance moved by point A along the arc AA' is $s = r\,\Delta\theta$, where $\Delta\theta$ is measured in *radians*. In terms of differentials, we have $ds = r\,d\theta$. Using this value of ds in equation (33), we get

$$(37\text{A}) \qquad v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \text{ fps} \qquad \text{or} \qquad v = r\omega,$$

where r is in feet and ω is in radians per second in order to give $v = v_s$ fps. Since there are 2π radians in one revolution, $\omega = 2\pi n_s$, where n_s is in revolutions per second; or $\omega = 2\pi n_m$ radians per minute, where n_m is in revolutions per minute.

(37B) $v_s = 2\pi r n_s$ fps and $v_m = 2\pi r n_m$ fpm,

(37C) $v_s = \pi D n_s$ fps and $v_m = \pi D n_m$ fpm,

where r and D are expressed in feet. Equations (37) are important relations between the linear and angular speeds. As we see from these equations, the linear speeds of any two points in a rotating rigid body are directly proportional to their distances from the center of rotation of the body, that is

$$\frac{v_A}{v_B} = \frac{r_A}{r_B},$$

where v_A is the speed of point A, v_B is the speed of point B, r_A is the radius of A, and r_B is the radius of B.

185. Angular Acceleration. Using a definition analogous to that given for linear acceleration,* we may say that angular acceleration α is the time rate of change of angular velocity. During a time interval Δt, let the change of angular velocity be $\Delta\omega$. Then, as before, the average angular acceleration is $\Delta\omega/\Delta t$; and the instantaneous angular acceleration is ($\alpha = \dot{\omega} = \ddot{\theta}$)

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

Since $\omega = d\theta/dt$, we have $d\omega/dt = d^2\theta/dt^2$, and therefore

$$\ddot{\theta} = \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{38}$$

By multiplying the numerator and denominator of the middle term of (38) by $d\theta$, we get,

$$\alpha = \frac{d\omega}{dt}\frac{d\theta}{d\theta} = \frac{d\theta}{dt}\frac{d\omega}{d\theta} = \omega\frac{d\omega}{d\theta} \qquad \text{or} \qquad \alpha\, d\theta = \omega\, d\omega. \tag{39}$$

The equations (38) and (39) are basic for analyzing the angular motion of a body which turns about an axis, whether the angular acceleration is constant or variable. The positive sense of an angular velocity or an angular acceleration may be chosen arbitrarily to be either clockwise or counterclockwise. When the angular acceleration α is in the same sense as the angular velocity ω, the angular velocity is increasing. When the sense of α is opposite to that of ω, then ω is decreasing. Angular acceleration is usually expressed in radians per second-second (rad. per sec.^{2}).

*When the words *acceleration* or *velocity* appear alone, *linear acceleration* or *linear velocity* is understood. When angular motion is referred to, the adjective *angular* is used.

186. Examples. (a) A body starting from rest rotates counterclockwise according to the law $\theta = 0.1t^3 - 0.3t^2 + 0.8t$. After 6 sec., what is (a) the angular displacement, (b) the angular velocity, and (c) the angular acceleration?

SOLUTION. (a) The angular displacement is obtained directly from the given law by substituting $t = 6$:

$$\theta = (0.1)(6)^3 - (0.3)(6)^2 + (0.8)(6) = 15.6 \text{ rad.} = \frac{15.6}{2\pi} \text{ rev.} = 2.48 \text{ rev.}$$

(b) Differentiating θ with respect to t [equation (36)], we get

$$\omega = \dot{\theta} = \frac{d\theta}{dt} = \frac{d(0.1t^3 - 0.3t^2 + 0.8t)}{dt} = 0.3t^2 - 0.6t + 0.8.$$

After 6 sec., the instantaneous angular velocity is

$$\omega = (0.3)(6)^2 - (0.6)(6) + 0.8 = 8 \text{ rad. per sec.,}$$

which is equivalent to $(8)(60)/(2\pi) = 76.4$ rpm.

(c) Differentiating ω with respect to t [equation (38)], we get

$$\alpha = \frac{d\omega}{dt} = \frac{d(0.3t^2 - 0.6t + 0.8)}{dt} = 0.6t - 0.6.$$

Observe that the acceleration is not constant. After 6 sec., the instantaneous angular acceleration is

$$\alpha = \dot{\omega} = (0.6)(6) - 0.6 = 3 \text{ rad. per sec.}^2$$

The reader should note the characteristics of this motion as revealed by the equations. For example, the origin of displacement is the same as the origin of time ($\theta = 0$ when $t = 0$). Moreover, the value of ω never becomes zero, yet the angular acceleration has a negative value at first, is zero at $t = 1$ sec., and is positive for all values of t greater than 1.

(b) A body is rotating about a fixed axis so that its angular acceleration is $\alpha = \ddot{\theta} = 4t^2 - t + 4$ rad. per sec.² If the initial angular velocity is 10 rad. per sec., what are the angular velocity and angular displacement after 2 sec.?

SOLUTION. Using (38), $\alpha = d\omega/dt$, we get

$$\int d\omega = \int (4t^2 - t + 4)dt$$

$$\omega = \frac{4t^3}{3} - \frac{t^2}{2} + 4t + C.$$

The constant of integration C is found from the simultaneous values $\omega = \omega_o = 10$ when $t = 0$; which gives $C = 10$. Then after 2 sec.,

$$\omega_2 = \frac{(4)(8)}{3} - \frac{4}{2} + (4)(2) + 10 = 26.7 \text{ rad. per sec.}$$

Letting $\omega = d\theta/dt$, we may integrate for θ as follows,

$$\int_0^\theta d\theta = \int_0^2 \left(\frac{4t^3}{3} - \frac{t^2}{2} + 4t + 10\right)dt,$$

$$\theta = \frac{4t^4}{(4)(3)} - \frac{t^3}{6} + \frac{4t^2}{2} + 10t \Big]_0^2 = 32 \text{ rad.,}$$

which is equivalent to $32/(2\pi) = 5.1$ rev.

187. Constant Angular Acceleration. When the law of variation of the angular acceleration can be expressed as a function of t or θ, equations (38) or (39) may be used to derive detailed relations between α, ω, θ, and t, as suggested by § 186. In the special case of *constant* angular acceleration, we find forms of the equations analogous to those obtained for constant linear acceleration. In order to benefit by this analogy, the reader should review § 176 while studying the following derivations and observe the similarities. With α constant, we get, from (38)

$$\int d\omega = \alpha \int dt, \qquad \text{whence} \qquad \omega = \alpha t + C_1,$$

where C_1 is a constant of integration. If $\omega = \omega_o$ when $t = 0$, we find $C_1 = \omega_o$, and the preceding equation becomes ($\omega = \dot{\theta}$)

(h) $$\omega = \omega_o + \alpha t. \qquad \text{(Compare with } v = v_o + at\text{)}$$

Since $\omega = d\theta/dt$, we find, from (h),

$$\int d\theta = \omega_o \int dt + \alpha \int t\, dt, \qquad \text{whence} \qquad \theta = \omega_o t + \frac{\alpha t^2}{2} + C_2.$$

If $\theta = 0$ when $t = 0$, then $C_2 = 0$, and this equation becomes

(i) $$\theta = \omega_o t + \frac{\alpha t^2}{2} \qquad \left(\text{Compare with } s = v_o t + \frac{at^2}{2}\right)$$

If α is a constant in (39), we have

$$\int \omega\, d\omega = \alpha \int d\theta, \qquad \text{whence} \qquad \frac{\omega^2}{2} = \alpha\theta + C_3.$$

If the body is moving with an angular velocity ω_o when $\theta = 0$, we get $C_3 = \omega_o^2/2$. This value of C_3 in the previous equation gives

(j) $$\omega^2 = \omega_o^2 + 2\,\alpha\theta. \quad \text{(Compare with } v^2 = v_o^2 + 2as\text{)}$$

In this equation, the origin of time and angular displacement occur simultaneously, that is, when $t = 0$, $\theta = 0$. At this same instant, however, the angular velocity is ω_o. If $\omega_o = 0$, this term simply drops from the equations. If the sense of α is opposite to that of ω, its value is negative if the sense of ω is taken as positive. A quick and easy procedure is to set up the differential equation, (38) or (39), to fit a particular problem and integrate for the desired results.

188. Example. A wheel which is rotating 300 rpm is slowing down at the rate of 2 rad. per sec.² (a) What time will elapse before the wheel stops? (b) At what rate in rpm is the wheel revolving after 10 sec.? (c) Through how many revolutions has it turned during the first 10 sec.? (d) What is the total angular displacement? (e) Compute the number of revolutions from the time $t = 10$ sec. until the wheel stops.

SOLUTION. (a) As sketched in Fig. 451, the sense of α is opposite to that of ω; therefore, if the direction of motion of the wheel is taken as positive, $\alpha = -2$ rad. per sec.² Moreover, $\omega_o = (300)(2\pi)/60 = 10\pi$ rad. per sec. Using equation (38), we get

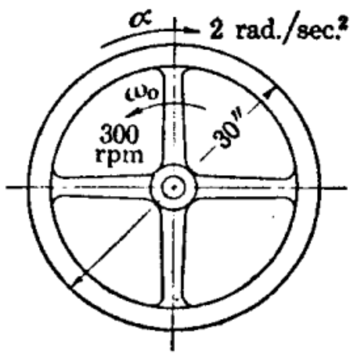

Fig. 451.

$$\int_{10\pi}^{0} d\omega = -2\int_0^t dt,$$

whence

$$0 - 10\pi = -2t \qquad \text{or} \qquad t = 15.7 \text{ sec.},$$

the elapsed time when the wheel has stopped.

(b) In making an integration, the limits are always from the first condition to the second condition. When we begin counting time in this example (when $t = 0$), $\omega_1 = 10\pi$ rad. per sec. The second limits are then $\omega = \omega_2$ and $t_2 = 10$ sec. Thus, we find ω_2 from (38) as follows:

$$\int_{10\pi}^{\omega_2} d\omega = -2\int_0^{10} dt,$$

$$\omega_2 - 10\pi = (-2)(10) = -20,$$

$$\omega_2 = 31.4 - 20 = 11.4 \text{ rad. per sec.},$$

which is equivalent to $(11.4)(60)/(2\pi) = 109$ rpm, approximately, the speed after 10 sec.

(c) To get the angular displacement after 10 sec., we may integrate

$$\int_{10\pi}^{11.4} \omega\, d\omega = -2\int_0^{\theta} d\theta,$$

where the second limit for ω ($= 11.4$) is the angular velocity after 10 sec. Another method of obtaining θ is to integrate

$$\omega = \frac{d\theta}{dt} = \omega_0 + \alpha t,$$

equation (h), § 187. Using the latter method, we have

$$\int_0^{\theta} d\theta = 10\pi\int_0^{10} dt - 2\int_0^{10} t\, dt,$$

$$= (10\pi)(10) - (2)\left(\frac{100}{2}\right) = 214 \text{ rad.}$$

In order to convert to revolutions, divide by 2π rad. per rev., or $\theta = 214/(2\pi) = 34.06$ rev.

(d) The total angular displacement may be obtained either from equation (h) or from $\omega\, d\omega = \alpha\, d\theta$. Since we found in part (a) that it took 15.7 sec. for the wheel to come to rest, equation (h) integrated from 0 to 15.7 sec. with respect to time will give the desired θ; thus

$$\int_0^{\theta} d\theta = \omega_0\int_0^{15.7} dt - 2\int_0^{15.7} t\, dt.$$

The other procedure, to use equation (39), is somewhat simpler. The limits of ω are from 10π, the original angular velocity, to 0; the corresponding limits for θ are 0 to θ; hence

$$\int_{10\pi}^{0} \omega\, d\omega = -2\int_{0}^{\theta} d\theta,$$

$$0 - \frac{100\pi^2}{2} = -2\theta \qquad \text{or} \qquad \theta = 247 \text{ rad.}$$

(e) The angular displacement from $t = 10$ sec. until the wheel stops is found as $247 - 214 = 33$ rad., or $(33)/(2\pi) = 5.25$ rev. during 5.7 sec. This result may be obtained also, within slide-rule error, by the integration of the equation

$$\int_{11.4}^{0} \omega\, d\omega = -2\int_{0}^{\theta} d\theta$$

for the value of θ, where $11.4 = \omega$ at the instant when $t = 10$ sec.

189. Curvilinear Motion. A point moving in the path of a plane curve which is not a straight line is said to have curvilinear motion. Since the velocity of the point at any instant is in the direction of the displacement at that instant and since the direction of the displacement at a point on a curve is the direction of the tangent at that point, the velocity of a point in a curved path is in the direction of a tangent to the path at the instantaneous position of the point. In short, the velocity vector is always tangent to the path of the point.

A point may move in a curved path with a constant *speed*, but inevitably its velocity varies because the direction in which the point travels varies. This variation of the *direction* of the velocity results in an acceleration, even with constant speed—an acceleration which we term the ***normal acceleration.***

190. Tangential and Normal Accelerations. Let a point move in the curved path shown in Fig. 452. Let the two positions A and B, defined by radius vectors ρ_A and ρ_B and the angles θ and $\theta + \Delta\theta$, be a distance Δs apart. Let the speed of the point at A be v, and at B, $v + \Delta v$. And let the time taken by the point to move from A to B be Δt. Remembering that the *change* of velocity is that vector which when added to the first velocity gives the second velocity, we may construct the vector diagram in Fig. 452(b). Thus, with a change of velocity equal to the vector EF, the acceleration is $EF/\Delta t$, since this change EF occurs in the time Δt. Hence, the instantaneous acceleration is

$$a = \lim_{\Delta t \to 0} \frac{EF}{\Delta t}.$$

However, to evaluate this limit, we find it convenient to use the components EG and GF of the vector EF. Substituting $EG \nrightarrow GF$ for EF above, we get

$$\text{(k)} \qquad a = \lim_{\Delta t \to 0} \frac{EG \nrightarrow GF}{\Delta t} = \lim_{\Delta t \to 0} \frac{EG}{\Delta t} \nrightarrow \lim_{\Delta t \to 0} \frac{GF}{\Delta t}.$$

In Fig. 452(b), we observe that, in the limit with $\Delta\phi = 0$, the component EG is $(v + \Delta v) - v = \Delta v$, which is the change in the speed. Therefore, the first term of equation (**k**) becomes

$$a_t = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt},$$

which is recognized as the time rate of change of the speed, an acceleration which is directed tangent to the curved path. This acceleration due to a change of the magnitude of the velocity is called the ***tangential acceleration*** a_t, and it is a component of the resultant or absolute acceleration.

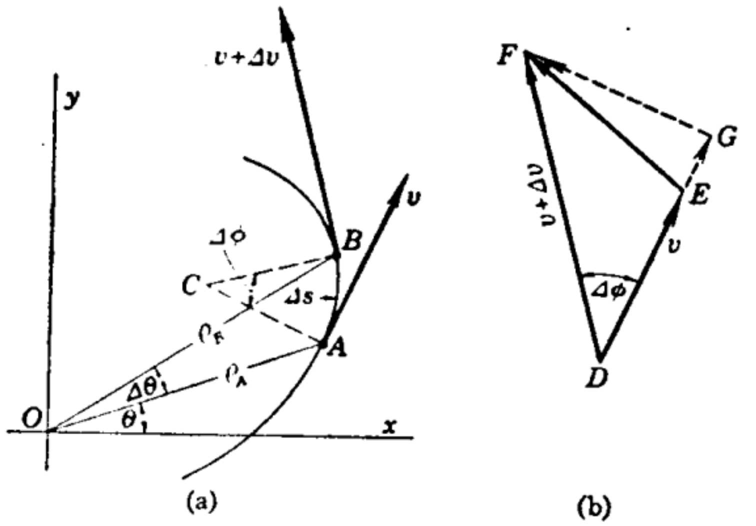

Fig. 452. Normal Acceleration. **Point C is the center of curvature of an arc Δs. In the limit as Δt approaches 0, point C is the center of curvature at point A.**

The other rectangular component, the second term in equation (**k**), may be transformed as follows. We find

$$GF = (v + \Delta v)\sin \Delta\phi$$

from the triangle DGF. Thus, the average acceleration due to the component GF of the velocity change EF is

$$\text{(l)} \qquad \frac{(v + \Delta v)\sin \Delta\phi}{\Delta t}.$$

The following conditions apply as Δt approaches zero:

The point B approaches point A.
The direction of GF is normal to the curve at A (giving rise to the name ***normal acceleration*** a_n for this component).
Δv approaches zero, so that $v + \Delta v$ approaches v.
The $\sin \Delta\phi$ approaches $\Delta\phi$.

With these conditions in mind, multiply and divide expression (*l*) by $r = AC$, the radius of curvature at the *point* A (Fig. 452) and then transform the expression as follows:

$$a_n = \lim_{\Delta t \to 0} \frac{r(v + \Delta v)\sin \Delta\phi}{r\,\Delta t} = \lim_{\Delta t \to 0} \frac{vr\,\Delta\phi}{r\,\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{v\,\Delta s}{r\,\Delta t} = \frac{v}{r}\frac{ds}{dt} = \frac{v^2}{r},$$

since $ds/dt = v$, which is the speed of the point at the position and instant in question. If either the speed v of the point or the radius of curvature r of the path of the point varies, this value of the normal acceleration a_n is an instantaneous value. Since, by (37), $v = r\omega$, we may write a_n in the form

$$a_n = \frac{v^2}{r} = r\omega^2 = v\omega, \tag{40}$$

in which ω is the instantaneous value of the angular velocity of a line connecting the position of the moving point with the center of *curvature* at that position. For the case of Fig. 452, ω is the instantaneous angular velocity of the line CA. As previously mentioned, the vector for a_n is directed *normal to the curve* toward the *center of curvature.*

For a point in curvilinear motion, then, the ***magnitude of the total acceleration*** (resultant acceleration) may be found from its two rectangular components, in the form (curved path is stationary)

$$a = (a_t^2 + a_n^2)^{1/2}, \tag{m}$$

in which a_t is zero if the *speed* of the point is constant and a_n is zero only for rectilinear motion or as the velocity passes through a zero value. When the radius of curvative r is constant (circle), we get, from $a_t = dv/dt$ and $v = r\omega$,

$$a_t = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad \text{or} \qquad a_t = r\alpha, \tag{41}$$

where α is the instantaneous angular acceleration of a radius to the point whose tangential acceleration is a_t. The resultant or total acceleration may also be defined by the vector sum

$$a = a_t \nrightarrow a_n,$$

where a is the resultant of the vectors for a_t and a_n.

191. Example. In the wheel of § 188, a point A is located on the horizontal centerline (position A_1, Fig. 453) at the instant the wheel begins to slow down. What is the total acceleration of this point after 10 sec.?

Solution. The angular displacement after 10 sec. as obtained from § 188 is $\theta = 214$ rad. $= 34.06$ rev. Therefore, after 10 sec., the wheel has made 34 complete revolutions plus 0.06 of a revolution. This puts the second position of A at A_2, Fig. 453, which is $(0.06)(360°) = 21.6°$ above position A_1. Since the radius of the wheel is 15 in. = 1.25 ft., we have

$$a_t = r\alpha = (1.25)(2) = 2.5 \text{ fps}^2,$$

directed oppositely to the rotation of the wheel, because the wheel is slowing down and we are considering the direction of motion as positive. At this instant, $\omega = 11.4$ rad. per sec. (§ 188); hence,

$$a_n = r\omega^2 = (1.25)(11.4)^2 = 162.5 \text{ fps}^2,$$

directed toward the center of the wheel. The total acceleration is

$$a = [(2.5)^2 + (162.5)^2]^{1/2} = 162.52 \text{ fps}^2,$$

a value which is of course very nearly equal to a_n, since a_n is very large compared to a_t in this instance. The angle that the resultant vector makes with the vector a_n is

$$\theta = \tan^{-1}\frac{a_t}{a_n} = \tan^{-1}\frac{2.5}{162.5} = 0.88°.$$

The total acceleration is therefore directed downward toward the left at an angle of $\theta = 21.6° + 0.88° = 22.48°$ with the horizontal, as shown exaggerated in Fig. 453.

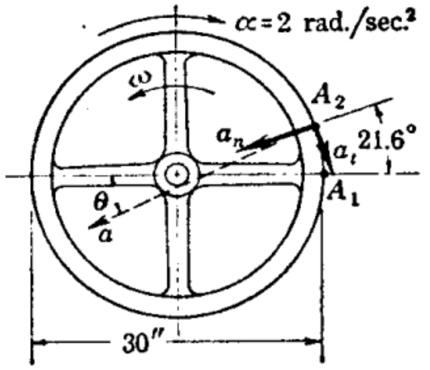

Fig. 453.

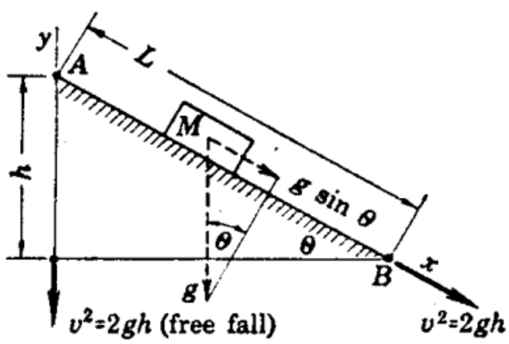

Fig. 454.

192. Component Motion. Since displacement, velocity, and acceleration are vector quantities, they may be resolved into any convenient components. The components a_t and a_n are examples of such convenient components. Frequently, rectangular components in certain x and y directions are desired. Thus, let a_x represent the component of the acceleration, v_x the component of the velocity, and let x represent the component of the displacement, each in the x direction. Since the x axis is a straight line, this component of the motion is necessarily rectilinear, no matter what is happening to the actual point. Hence, the equations for rectilinear motion apply. We have

$$v_x = \dot{x} = \frac{dx}{dt}, \qquad a_x = \dot{v}_x = \frac{dv_x}{dt} = \frac{d_2x}{dt^2}, \qquad v_x\,dv_x = a_x\,dx.$$

Similarly, the equations for the motion in the y-direction are

$$v_y = \dot{y} = \frac{dy}{dt}, \qquad a_y = \dot{v}_y = \frac{dv_y}{dt} = \frac{d^2y}{dt_2}, \qquad v_y\,dv_y = a_y\,dy.$$

The magnitude of the resultant velocity at a particular instant is obtained from the relation

$$v = (v_x^2 + v_y^2)^{1/2},$$

where v_x and v_y are simultaneous values. The angle between the resultant vector v and the vector v_x is

$$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{\dot{y}}{\dot{x}},$$

after the manner of getting the sense of a resultant force vector, equation (c), p. 19. The magnitude of the resultant acceleration is

$$a = (a_x^2 + a_y^2)^{1/2},$$

where a_x and a_y are simultaneous values. The direction of the vector for a is found from

$$\theta = \tan^{-1}\frac{a_y}{a_x}.$$

The total acceleration is also defined by the vector sum

$$a = a_x \nrightarrow a_y.$$

193. Example. A body M, considered as a particle, moves from rest down a *smooth* inclined plane (Fig. 454). If it moves solely under the influence of gravity and the normal reaction on the plane, what is its speed after it has moved from A to B?

Solution. If the body M fell freely under the action of gravity alone, its acceleration would be $g = 32.2$ fps² vertically downward. Therefore, if it is constrained to move along a smooth plane, its acceleration would be that component of the acceleration of gravity in the direction of the plane; in this case $a_x = g \sin\theta$, where the x axis is parallel to the plane. Use the equation $v_x dv_x = a_x\, dx$ and integrate:

$$\int_0^{v_x} v_x\, dv_x = g \sin\theta \int_0^L dx,$$

$$\frac{v_x^2}{2} = g(\sin\theta)L;$$

or, since $\sin\theta = h/L$,

(n) $$v_x^2 = 2gh,$$

from which the speed at B may be determined. If this body should fall freely through a vertical distance h from point A, the speed would be obtained from the equation*

$$\int_0^{v_y} v_y\, dv_y = g \int_0^h dy,$$

whence

(o) $$v_y^2 = 2gh.$$

Since the values of v_x and v_y in equations (n) and (o) are identical, we conclude that the *speed* attained by a body sliding a distance L down an inclined plane without friction of any kind is the same as the speed attained by the body in falling without friction (and

*Note that in this example the x and y axes are not perpendicular to each other.

without air resistance) through the vertical component h of that distance. It is also true that the speed attained by a body moving downward *along any other smooth path is the same* as that which would be attained in a free fall through the vertical distance defined by the path. However, the *velocity* is *not* necessarily the same in each case. In a free fall, the velocity is directed vertically downward; the velocity of a particle as it leaves a sloping or curved path is in the direction of the tangent to the path at the point B of departure (Fig. 454).

194. Example—Trajectory. Investigate the motion of a projectile (considered as a particle) in a vacuum. (This analysis neglects the effects of the rifling of the gun barrel, wind movement, and the wind resistance.) The initial velocity of the projectile is v_o, directed upward at an angle θ with the horizontal (Fig. 455).

SOLUTION. (a) The origin of coordinates is taken at the point where the projectile leaves the gun, at which point the time t is zero. Since the projectile is subjected only to the acceleration of gravity, the acceleration in the horizontal direction is $a_x = \ddot{x} = 0$,

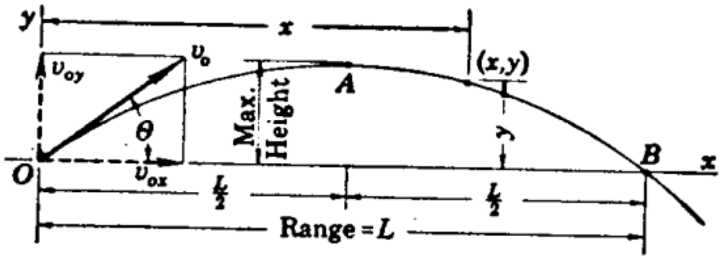

Fig. 455. Trajectory.

and that in the vertical direction is $a_y = \ddot{y} = -g$, the negative sign indicating that the acceleration is downward (the upward sense being chosen as positive). Investigating the horizontal motion first, we have, using equation (34),

$$a_x = \frac{dv_x}{dt} = 0.$$

Since the differential of v_x is zero, v_x is a constant which is equal to the horizontal component of the initial velocity v_o. See Fig. 456. Thus

(p) $$v_x = v_{ox} = v_o \cos\theta,$$

where θ is a constant for a particular firing of the gun. Since $v_x = dx/dt$, we have

$$\frac{dx}{dt} = v_o \cos\theta,$$

$$\int dx = v_o \cos\theta \int dt,$$

$$x = (v_o \cos\theta)t + C_1,$$

where C_1 is the constant of integration. Since $x = 0$ when $t = 0$, then $C_1 = 0$; thus

(q) $$x = (v_o \cos\theta)t.$$

This equation shows that the horizontal displacement is directly proportional to the time t. For the motion in the vertical direction, we have

$$a_y = \frac{dv_y}{dt} = -g,$$

$$\int dv_y = -g \int dt,$$

$$v_y = -gt + C_2,$$

where C_2 is the constant of integration. When $t = 0$, $v_y = v_{oy} = v_o \sin\theta$, the vertical component of the initial velocity; so $C_2 = v_o \sin\theta$. Thus

(r) $$v_y = \frac{dy}{dt} = -gt + v_o \sin\theta,$$

$$\int dy = -g\int t\,dt + v_o \sin\theta \int dt,$$

$$y = \frac{-gt^2}{2} + (v_o \sin\theta)t + C_3.$$

Since $y = 0$ when $t = 0$, the constant of integration $C_3 = 0$; therefore,

(s) $$y = -\frac{gt^2}{2} + (v_o \sin\theta)t.$$

Equation (r) gives the vertical component of the velocity of the projectile at any time t after firing. At first, $v_o \sin\theta$ is greater than gt. During this period the value of v_y is positive, showing that the vector points upward. Later, gt becomes greater than $v_o \sin\theta$, in which event, the value of v_y is negative, showing that the vector points downward. Evidently, then, the path of the projectile (the trajectory) moves upward from the origin (when $0 < \theta < 90°$) and then turns downward (Fig. 456). In equation (s), when $gt^2/2$ becomes greater than $(v_o \sin\theta)t$, the value of y is negative and the projectile has passed the point B.

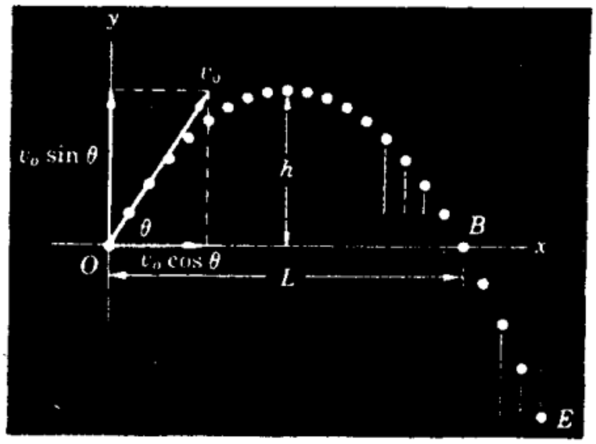

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Fig. 456. Multiflash Photograph of Projectile. The white dots are photographs of a projectile (a small ball) taken by high-speed flash photography. The time interval between flashes is constant. Notice that the horizontal distances between adjacent positions of the projectile are all the same. The vertical distances between successive positions decreases as the projectile moves upward, increases as it moves downward—the acceleration is constant downward. Even though the target may be at E, where y has a negative value, the _range_ is defined by L. The maximum value of y is $y_{max} = h$.

(b) The equation (t) of the trajectory OAB, Fig. 455, in terms of x and y is obtained by eliminating t from equations (q) and (s); this gives

(t) $$y = x\tan\theta - \frac{gx^2}{2\,v_o^2\cos^2\theta}.$$

This is an equation for a parabola.

(c) The ***time of flight*** t_f is by definition the time for the projectile to return to the level of the origin. The condition here (at B, Fig. 456) is that $y = 0$. Letting y in equation (s) equal to zero, we get

$$y = -\frac{gt^2}{2} + (v_o \sin\theta)t = 0,$$

whence the time of flight is

(u) $$t = t_f = \frac{2v_o \sin\theta}{g}.$$

(d) The ***range*** L is by definition the horizontal distance OB, Figs. 455 and 456, although the projectile might actually strike a target ahead of or beyond point B.

The range ($x = L$) may be determined by letting $y = 0$ in equation (t); or by substituting the value of t_f from (u) in (q). The latter procedure gives

$$x = L = (v_o \cos\theta)t_f = \frac{2v_o^2 \sin\theta \cos\theta}{g} = \frac{v_o^2 \sin 2\theta}{g}.$$

For a particular initial velocity v_o, the maximum range L_{max} is obtained when the sin 2θ is a maximum. The corresponding value of θ is 45°.

(e) The maximum height is the maximum value of y. This value of y may be obtained by letting $x = L/2$ in equation (t), or by letting $t = t_f/2 = v_o \sin\theta/g$ in equation (s). Using the latter plan, we find

$$y_{max} = -\frac{g(v_o \sin\theta)^2}{2g^2} + \frac{(v_o \sin\theta)^2}{g} = \frac{v_o^2 \sin^2\theta}{2g}.$$

See also Fig. 457.

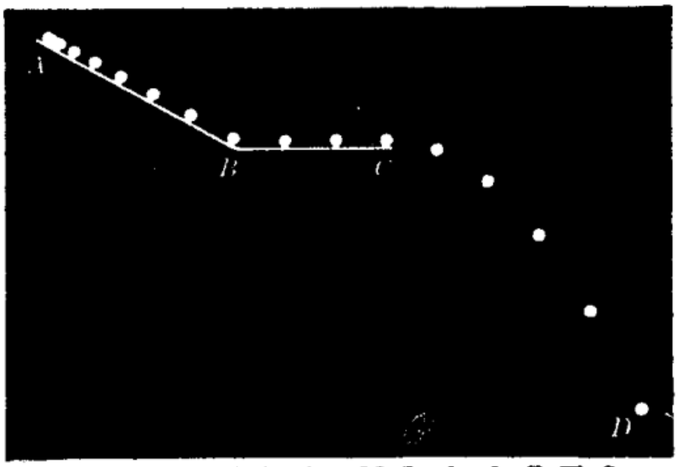

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Fig. 457. Multiflash Photograph of a Ball. **The ball starts from rest at *A* and flash pictures are taken at constant time intervals. The ball accelerates as it rolls down the incline *AB*, moves with constant speed on the horizontal from *B* to *C* (notice that the distance between positions remains constant), then it moves as a projectile from *C* to *D*. The path from *C* to *D* is a parabola for a projectile given an initial velocity in a horizontal direction ($\theta = 0$, Fig. 456), and the equations for a projectile apply to this part of the movement.**

195. Example. A moving point follows the path of the hyperbola $xy = 16$. The y component of the velocity is constant at the value $v_y = dy/dt = 6$ fps. At the point (8, 2) on this curve, determine (a) the tangential (absolute) speed, and (b) the x component of the acceleration a_x.

SOLUTION. (a) Differentiate $xy = 16$ with respect to t.

$$y\frac{dx}{dt} + x\frac{dy}{dt} = 0.$$

Since $\dot{y} = dy/dt = 6$, this equation becomes

$$\text{(v)} \qquad y\frac{dx}{dt} + 6x = 0.$$

When $x = 8$ and $y = 2$, we get

$$v_x = \frac{dx}{dt} = -\frac{6\,x}{y} = -24 \text{ fps.}$$

$$v = (v_x^2 + v_y^2)^{1/2} = [(-24)^2 + (6)^2]^{1/2} = 24.75 \text{ fps},$$

the speed at the point (8, 2). With $v_x = -24$ and $v_y = 6$, the vector v evidently points upward toward the left at an angle $\theta = \tan^{-1}(v_y/v_x)$ with the horizontal.

(b) Differentiating equation (v) with respect to t, we find

$$\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right) + y\frac{d^2x}{dt^2} + 6\frac{dx}{dt} = 0$$

$$6\frac{dx}{dt} + y\frac{d^2x}{dt^2} + 6\frac{dx}{dt} = 0$$

$$a_x = \frac{d^2x}{dt^2} = -\left(\frac{12}{y}\right)\left(\frac{dx}{dt}\right).$$

Since $dx/dt = -6\,x/y$, we get

$$a_x = \frac{d^2x}{dt^2} = \frac{72\,x}{y^2} = \frac{(72)(8)}{4} = 144\text{ fps}^2,$$

directed toward the right, since the sign is positive. Since $\dot{y} = 6$ fps is a constant, $\ddot{y} = a_y = 0$.

196. Simple Harmonic Motion. One of the most important examples of variable acceleration in engineering problems is simple harmonic motion, because this motion, which is vibratory or periodic, is the starting point in the study of vibrations (Chapter XX). Since the modern tendency is to run machines at higher and higher speeds, the study of vibrations was never more important than it is now. A point describes ***simple harmonic motion*** when its acceleration varies directly as the displacement of the point from an origin, but is in the sense opposite to that of the displacement. That is, when the displacement is positive, the acceleration is negative, and vice versa. In mathematical form, the definition is

$$(42) \qquad a = \ddot{x} = \frac{d^2x}{dt^2} = -Cx \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -Cx.$$

The physical significance of this equation is shown in Fig. 458. As the point P moves along Ox away from the origin O toward the right, it is slowing down, because the acceleration is toward the left. It will come to rest at some position A and then begin to move toward the left, gaining speed until it passes the origin O. On the *left* side of O, the acceleration is toward the *right*; so that the particle again slows down until it stops at some point B. Since the acceleration is still toward the right, the point P now moves toward the right with increasing speed until the origin O is reached, after which it is again retarded. Thus, the motion occurs entirely between some definite limits A and B. Equations for harmonic motion may be found by integrating the equation $v\,dv = -Cx\,dx$, followed by another integration of the expression for $v = dx/dt$. However, we can find by simpler means a solution to the differential equation (42).

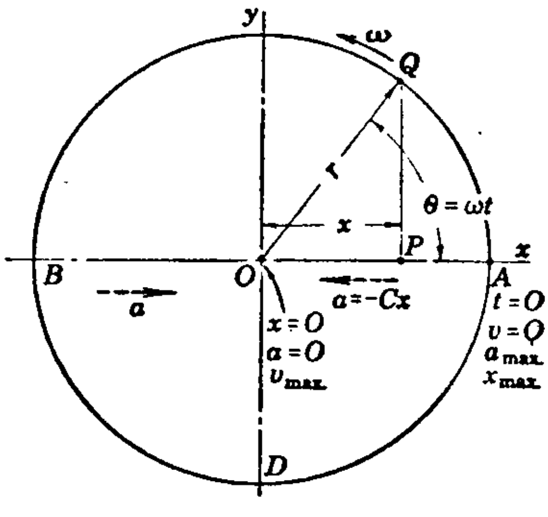

Fig. 458. Harmonic Motion.

In Fig. 458, imagine a point Q moving at constant speed on the circumference of the circle whose center is at O and whose radius r is OA. We shall find that the motion of the projection of this point Q on a diameter of the circle is simple harmonic motion. Let $t = 0$ when the point P (and Q) is at A. The point A then is the origin of *time*, a condition not to be overlooked. When the radius to the point Q makes an angle θ with Ox, the displacement of the point P is $x = r \cos \theta$. If ω is the constant angular velocity of the radial line to the point Q, and if $t = 0$ at A, then $\theta = \omega t$. Thus we have

(w) $x = r \cos \omega t$

(x) $v = \dot{x} = \dfrac{dx}{dt} = -\omega r \sin \omega t$

$$a = \ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 r \cos \omega t,$$

or

(y) $\dfrac{d^2x}{dt^2} = -\omega^2 x,$

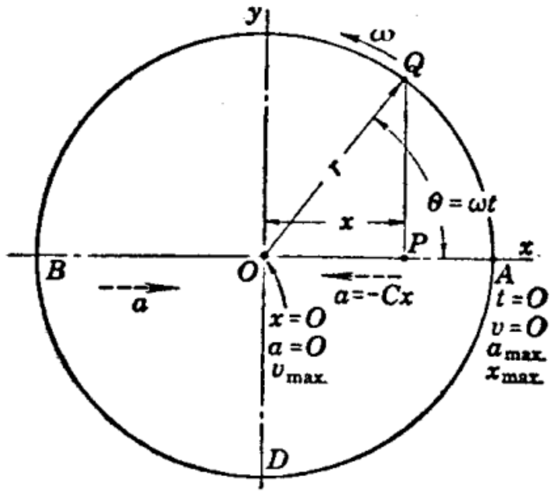

Fig. 458. **Repeated.**

where v and a are, respectively, the velocity and acceleration of the point P. Since, by successive differentiations of $x = r \cos \omega t$, we arrived at the differential equation $d^2x/dt^2 = -\omega^2 x$, it follows that $x = r \cos \omega t$ is a solution of this equation. Moreover, comparing equations (42) and (y), we observe that they are the same in form. Hence, if $C = \omega^2$, equation (42) defines the motion of the projection P of the point Q, so that this projected motion *must* be harmonic.

The maximum distance reached by the oscillating point P from the origin O is called the ***amplitude*** (distance $r = OA$, Fig. 458). The ***period*** T of harmonic motion is the time taken by the point P to complete a cycle or oscillation; for example, the time to make the two strokes AB and BA, Fig. 458. If ω is the angular velocity of the radius vector OQ in radians per second, the period is

(z) $T = \dfrac{\text{rad. per rev. (cycle)}}{\text{rad. per sec.}}$ or $T = \dfrac{2\pi}{\omega}$ sec. per cycle.

Since $\omega = C^{1/2}$, the period is also $T = 2\pi/C^{1/2}$, where C is defined by equation (42).

The ***frequency*** ϕ is the number of cycles per second (or other unit of time) and is therefore seen to be the reciprocal of the period; or

$$\phi = \frac{\omega}{2\pi} = \frac{C^{1/2}}{2\pi}.$$

Observe that the acceleration is a maximum when the velocity is zero; that is, although the velocity is zero at points A and B, Fig. 458, the time rate of change of velocity at these points is a maximum. Chapter XX gives a more extended discussion of harmonic motion.

197. Example. A point moves according to the law $a = -16\,x$ with an amplitude of 3 in. (a) Find the period and frequency. (b) Determine the displacement, velocity, and acceleration after 10 sec.

SOLUTION. (a) From the law of motion, $C = \omega^2 = 16$, whence $\omega = 4$ rad. per sec. Thus the period and frequency are

$$T = \frac{2\pi}{4} = 1.57 \text{ sec.}, \qquad \phi = \frac{1}{1.57} = 0.637 \text{ cycle per sec.}$$

(b) After 10 sec., the number of cycles is $(10)(0.637) = 6.37$ cycles. Hence, the point has completed 0.37 of a cycle from point A, the origin of time. The corresponding time is $(0.37)(1.57) = 0.581$ sec. In order that the angle ωt be less than 360°, we thus use $t = 0.581$ sec., instead of 10 sec., in the equations of harmonic motion. The angle ωt is then $(4)(0.581) = 2.32$ rad. $= 133°$. The displacement is

$$x = r \cos \omega t = 3 \cos 133° = -2.045 \text{ in.},$$

where the negative sign shows that the point is on the left side of the origin of displacement. The velocity is

$$v = -r\omega \sin \omega t = -(3)(4)\sin 133° = -8.78 \text{ in. per sec.},$$

where the negative sign shows that the point is moving toward the left. The acceleration is

$$a = -r\omega^2 \cos \omega t = -(3)(16)\cos 133° = +32.75 \text{ in. per sec.}^2,$$

where the positive sign shows that the acceleration is directed toward the right.

The student should practice the derivation of equations (**x**) and (**y**) in solving this example.

198. Instantaneous Center or Centro. If a body moves so that each particle of the body remains in a particular plane, the body is said to have ***plane motion.*** No matter how complicated the plane motion of a body may be, the body is rotating at any particular *instant* about some axis. Imagine that a thin plate M, Fig. 459, is tossed vertically into the air. The plate will have a vertical translation as a whole, but it is likely to be spinning about some axis perpendicular to the plate at the same time. Thus, a particular point in the plate will have an undefinable curvilinear motion that in general is not the same as the curvilinear motion of some other point. Suppose that Fig. 459 is the position of the plate at a particular instant. Suppose that by some means we had learned that a point A was moving in the direction indicated by the arrow at A; and that the point B was moving in the direction shown by the arrow at B. Recalling that the velocity vector is tangent to the curve described by a point and that a normal to the curve passes through the center of curvature and is also normal to a tangent line (see Fig. 452), we draw the lines AO and BO of indefinite extent and perpendicular, respectively, to the vectors v_A and v_B. The intersection O of these lines is the ***instantaneous center*** of rotation or the ***centro;*** because, since the point B moves about a center of curvature along the line BO and

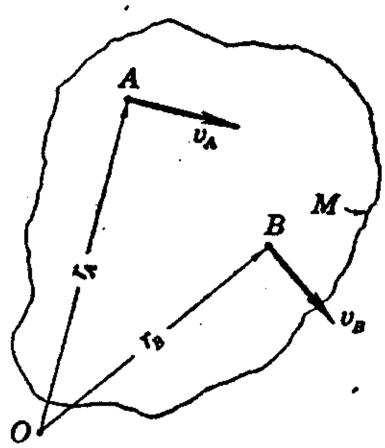

Fig. 459. Centro.

since the point A moves about a center of curvature along the line AO, the intersection of these lines must represent the center of rotation of the whole body, provided, of course, that M is a rigid body, so that there is no *relative* motion of the points A and B, for example. The point O is momentarily stationary when the velocities used in this construction are absolute velocities. By *absolute velocity*, we mean the velocity of a particle relative to the earth.

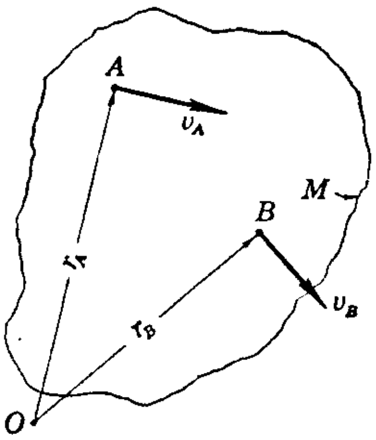

Fig. 459. **Repeated.**

Thus this point O is a point of *zero velocity* (as is any center of a rotating body); that is, it is a point without velocity when the earth is the reference body. Although the body M is rotating about O at this particular *instant*, it will probably be rotating about some other point in the next instant, and still another in the next, etc. If we say the center of rotation is moving, the reader tries to imagine a *moving* point that has *no velocity* and is naturally confused. The fact is that this body M rotates about a series of fixed points and about each one for an instant. A smooth curve drawn through these various fixed centers of rotation is the locus of the centros for the body, a curve called a ***centrode.***

An illustration should help. Consider a wheel. If the wheel is slipping, as in the case of a sharply braked, screaming automobile wheel, there is rubbing of the bottom of the wheel on the road. If the wheel is *rolling*, i.e., if it is not skidding on the road, the point on the bottom of the wheel in contact with the road is *momentarily stationary*. This point of contact O, Fig. 460, is the point of zero velocity and is therefore the instantaneous center or the centro of the rolling wheel. As the wheel rolls along, the centro is continuously moving ahead at the speed with which the wheel advances. See Fig. 461 for evidence that the point of contact between the ground and the wheel is stationary. If the wheel *is* slipping, there is a point of zero velocity relative to the ground (a centro), but it is not the point of contact with the ground.

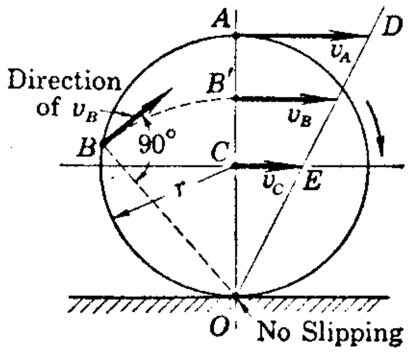

Fig. 460. **There are applications of this principle in § 209 and elsewhere in this book.**

Suppose the wheel, Fig. 460, is moving with a velocity of v_C. This means that the axis (and the car) is moving at a velocity of v_C. The speed of any point in a rotating body is proportional to its distance from the center of rotation. Thus, if the radius of the wheel is r, we have $v_A/v_C = 2r/r$, or $v_A = 2v_C$; that is, the top point of the wheel moves at twice the speed of the axis. This same result is obtained by graphical construction. Let the vector v_C, Fig. 460, be laid out to scale. Draw the line OD through the end of vector v_C. Draw a line AD perpendicular to OCA. The intercept v_A represents the speed of the point A, as we see from the similar triangles OCE and OAD. Since the speed of any point B is proportional to the instantaneous radius OB, v_B may be found from similar triangles by swinging an arc of radius OB to locate the point B' and then erecting a perpendicular to OCA at the point B'. It is seen from Fig. 460 that $v_B/v_C = OB/OC$, and that the vector v_B therefore represents the speed of point B.

Courtesy H. E. Edgerton, author of Flash.

Fig. 461. Rolling Automobile Wheel. **This photograph provides visual evidence that the center of rotation or centro of a rolling wheel is at the point of contact with the ground. Observe how the curved streaks on the wheel bend about the contact point in near-circular arcs. Notice too that the markings on the tire are clear cut at the bottom, but blurred at other points where the velocities were greater. This car was traveling at 25 mph when the picture was taken.**

It should be said in passing that while the point O on the wheel is the point of zero velocity, it is *not* a point of zero acceleration. Consequently, this method of handling velocity vectors cannot be applied to finding absolute accelerations. Acceleration problems of this type are discussed in the next chapter. Considerably more detail on instantaneous centers is found in books on kinematics and mechanism.

199. Velocity Ratio. The ***velocity ratio*** VR between two members of a machine, both of which are undergoing plane curvilinear motion, is the angular velocity of the *driving* member divided by the angular velocity of the *driven* member. This is illustrated by two rolling wheels or two meshing gears, Fig. 462. Suppose that the wheel A drives the wheel B. Then the velocity ratio is

$$VR = \frac{\omega_A}{\omega_B} = \frac{n_A}{n_B},$$

where the angular velocities may be expressed in any convenient units, say radians per second or rpm, but the units must evidently be the same for both parts in a particular ratio. Now if these wheels *roll* on each other at the point of contact D, that is, if there is *no slipping* at D, the tangential velocities of any points P and Q on the surfaces of the wheels are the same, that is, $v_P = v_Q$. Since

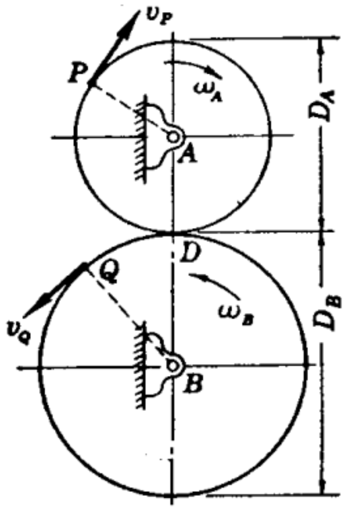

Fig. 462.

$$v_P = r\omega = \frac{D_A\omega_A}{2}, \qquad \text{and} \qquad v_Q = \frac{D_B\omega_B}{2},$$

we find

$$\frac{D_A\omega_A}{2} = \frac{D_B\omega_B}{2}, \qquad \text{or} \qquad \frac{\omega_A}{\omega_B} = \frac{D_B}{D_A},$$

which is the correct ratio when there is *no slipping* at the point of contact. For this case, then,

$$VR = \frac{\omega_A}{\omega_B} = \frac{D_B}{D_A},$$

or the velocity ratio is the diameter of the *driven* wheel divided by the diameter of the *driving* wheel. The definition of velocity ratio may be applied to any two members in a machine whose angular motions are related to one another; for example, the statement that the velocity ratio between the engine and rear wheels of an automobile is 3, means that the engine turns through three times as many revolutions as the wheels in a unit of time.

200. Closure. The principal objectives of this chapter are to acquaint you with the conception of acceleration, especially normal acceleration, with which you may not be too familiar, and with the usefulness and some applications of the basic kinematic equations of motion:

$$v = \dot{s} = ds/dt, \qquad \omega = \dot{\theta} = d\theta/dt,$$
$$a = \dot{v} = dv/dt, \qquad \alpha = \dot{\omega} = d\omega/dt,$$
$$v\,dv = a\,ds, \qquad \omega\,d\omega = \alpha\,d\theta.$$

Of course, there are some other related details that you need to know, but observe that these equations are as basic to kinematics as $\Sigma F = 0$ and

$\Sigma M = 0$ are to statics. You will make frequent use of the relations $v = r\omega$ and $a = r\alpha$.

In passing, it may be interesting and instructive to note that a point may have zero velocity but not zero acceleration, or zero acceleration but not zero velocity.

Problems

SUGGESTION TO STUDENT. *In solving problems involving constant acceleration, start in each case with the basic equations (for example, $a = dv/dt$ and $v\,dv = a\,ds$) and proceed through the integration (or differentiation) in order to build up your confidence in your ability to use the basic equations. This practice will tax your memory the least and give you a good acquaintance with some excellent tools. You gain so little when all you do is to substitute numbers in an equation.*

RECTILINEAR MOTION

891. A passenger on a train counts the clicks of the wheels as they pass over the joints of the rails, noting 62 clicks in 20 sec. (a) If the rails are 29 ft. long, what is the approximate speed of the train in mph? (b) If the passenger counts the clicks for only one minute, what round number when divided into the total clicks per minute would give the approximate speed in mph?
Ans. (a) 61.3 mph; (b) 3.

892. A train moving at 45 mph is decelerated to 15 mph in 22 sec. What is the average acceleration in fps²? in fpm²? In miles per hr-sec.? The direction of motion is taken as positive.

893. A fighter plane pilot uses his wing flaps to decelerate the plane at an average rate of $g/3$, or 10.7 fps². (a) How long must he apply the flap controls to decrease the speed from 400 to 250 mph? (b) What is the speed of the plane 4 sec. after the flap controls are applied in fps, in fpm, and in mph?
Ans. (a) 20.6 sec.; (b) 543.3, 32,600, 371.

894. An automobile is retarded uniformly at the rate of 15 fps² from a speed of 60 mph to 9 yd. per sec. What is the elapsed time? How far does the car travel in this time?

895. How far does a car travel in changing its speed from 10 fps to 60 fps when $a = 8$ fps²?

896. A car made a test stop from 60 mph in 300 ft. on a certain type road. Should the driver see a roadblock 250 ft. ahead and should it take him 0.6 sec. to apply the brakes, find the speed (in fps, fpm, and mph) with which he would hit the roadblock, assuming uniform deceleration at the test rate.
Ans. 51.5 fps, 3090 fpm, 35.1 mph.

897. A car made a test stop from 60 mph in 300 ft. If the deceleration was uniform, find the speed in mph (a) 1 sec. after the brakes were applied, (b) after the car has traveled 100 ft. (convert to fps and to fpm), and (c) find the time required for the stop.

898. A ball is thrown vertically upward and observed to go level with the top of a 161-ft. tree. Find (a) the initial velocity, (b) the distance traveled during the first second of flight, (c) the distance traveled during the third second of flight (from $t = 2$ sec. to $t = 3$ sec.), and (d) the time required for the ball to return to the starting point.
Ans. (a) 101.9 fps, (b) 85.8 ft., (c) 21.4 ft., (d) 6.32 sec.

899. A stone is thrown downward from a 100-ft.-high tower with an initial velocity of 20 fps. (a) With what speed did it hit the ground? (b) What is the velocity when $t = 2$ sec.? (c) What was the time of flight? (d) What initial speed would reduce the time of flight to 50% of that found in part (c)? (Errors are less likely in this problem if one assumes the downward direction as positive.)
Ans. (a) 82.75 fps, (b) zero, (c) 1.95 sec., (d) 86.8 fps.

900. The same as 899 except that the initial velocity is 10 fps *upward*.

901. As a balloon is rising at a speed of 10 fps, a sandbag is pushed overboard. Four seconds later the sandbag hit the ground. Find the maximum altitude reached by the sandbag and the altitude when it was pushed overboard. *Ans.* 219 ft., 217.5 ft.

902. A balloon is ascending vertically with an acceleration of 2 fps². At an altitude of 200 ft., while the balloon is moving at 20 fps, a rock is released. With what velocity does the rock strike the ground? What is the elapsed time? What total distance did the rock move? Neglect air resistance.

903. Two elevators in adjoining shafts approach one another after starting simultaneously from rest when they are 500 ft. apart. The top elevator A travels down with a constant acceleration of 1 fps². The other elevator B has a uniform acceleration of 2 fps² upward. After what time are they opposite each other? How far has each traveled at this instant?
Ans. 18.25 sec., $s_A = 166.7$ ft., $s_B = 333.3$ ft.

904. The same as **903** except that the lower car has a uniform acceleration of 0.5 fps² upward.

905. The same as **903** except that the top elevator starts 2 sec. ahead of the bottom elevator.

906. Car A, traveling at a constant speed of 60 mph on a straight road, passed a parked police car P. Car P passed A 51 sec. later. If P delayed 1 sec. in getting started and its continuous acceleration is considered constant, find a_p and v_p at the instant P passes A. *Ans.* 3.59 fps², 179.5 fps.

907. Two automobiles, A and B, are traveling in line on a straight highway at the same speed of 100 fps. At time $t = 0$, A's brakes are applied to give a constant acceleration of $a = -15$ fps². One second later, B's brakes are applied to decelerate it at 20 fps². If the distance between the cars is reduced to zero without collision, what is the least distance at which B could have been following A? Find the time t and velocities v_A and v_B when this bumper-to-bumper condition occurs. The initial position of A is suggested for the $s = 0$ position.
Ans. 30 ft., 4 sec., $v_A = v_B = 40$ fps.

908. Two automobiles are traveling in line along a straight highway at 100 fps. The front driver A applies his brakes and decelerates his car uniformly at 15 fps². One second later, the rear driver B applies his brakes. At what minimum uniform rate must B decelerate in order to avoid a collision if the initial distance between cars is 160 ft.? (Note: At the crucial moment, the cars have the same velocity and are at the same point, pratically.)

909. A stone is dropped into a vertical mine shaft. It is heard to strike the bottom of the shaft after 6 sec. Estimate the depth of shaft. Use the speed of sound as 1130 fps and neglect the effect of air resistance.
Ans. 497.2 ft.

910. The same as **909** except that the stone is given an initial downward velocity of 30 fps.

911. Two mortar shells are fired vertically into a vacuum. The muzzle velocity of the first shell A fired is 600 fps, of the second shell B, 900 fps. If these two shells are to be together when the first shell is at its highest point, what would be the time interval between the firings? *Ans.* 11.5 sec.

912. A rock is dropped from a bridge across a canyon 1000 ft. deep. At the same instant another rock is projected vertically upward from the bottom of the canyon. If they pass at a point 300 ft. from the bottom, what is the initial velocity of the rock that is projected upward?

RECTILINEAR MOTION = VARIABLE ACCELERATION

913. (a) When $a = 2s$, derive a general expression for the velocity v. Assume the initial conditions to be v_o and s_o when $t = 0$. The motion is in a straight line. (b) If $v_o = 5$ fps and $s_o = 10$ ft., find the velocity when $t = 2$ sec.
Ans. (a) $v = \pm[v_o^2 + 2(s^2 - s_o^2)]^{1/2}$

914. The linear acceleration of a particle is defined by the equation $a = (t^3 + 3t^2)$ fps². If it starts from rest, find the displacement, velocity, and acceleration when $t = 4$ sec. Could such motion long continue?
Ans. 115.2 ft., 128 fps, 112 fps².

915. The motion of a point moving along a horizontal straight line varies according to the equation $a = 12\sqrt{s}$. When $t = 2$ sec., the point is 16 ft. to the right of the origin, has a velocity of 32 fps to the right, and has an acceleration of 48 fps² to the right. Determine the velocity and acceleration when the $t = 3$ sec.

916. A particle moves according to the equation $a = 5t^{1/2}$. If the initial conditions are $t = 0$, $v_o = 6$ fps, and $s_o = 0$, find the displacement during the interval between $t = 4$ sec. and $t = 10$ sec. *Ans.* 340 ft.

917. The same as **916** except that $a = 28t^{1/3}$.

918. The acceleration of a particle moving with rectilinear motion is $a = Cx$, where C is a constant. At a certain instant, its speed is 15 fps and its displacement from an origin is 30 ft. Somewhat later, its speed is 60 fps and its displacement is 130 ft. What is the value of C? What are the units of C? *Ans.* 0.211 rad. per sec.²

919. The acceleration of a particle moving with rectilinear motion is $a = 5t^{1/2}$. If the initial velocity is 6 fps, what space is traversed in 16 sec.? What is the space traversed during the last 4 sec. of this 16 sec.? *Ans.* 1461 ft., 724 ft.

920. A particle starting from rest and moving in a straight line has a displacement $s = 3t^3 - 8t^2 + 5$. What is the acceleration after 5 sec.? What is the *change* of acceleration during the 10th sec.?

921. A particle moves with rectilinear motion so that its displacement is given by $s = (t^2 - 2t + 9)^{1/2}$ ft., where t is in seconds. Compute its velocity and its acceleration at the end of 3 sec. What is its arithmetic average acceleration during the 3d sec.? *Ans.* 0.578 fps, 0.193 fps², 0.245 fps².

922. A particle whose acceleration $a = 3t - 12$ fps² is moving at a certain instant in a straight line with an initial velocity of 15 fps in the same sense as the initial acceleration. At the end of $t = 3$ sec., what are the velocity and displacement of the particle? *Ans.* −37.5 fps, −85.5 ft

923. The same as **922** except that $t = 30$ sec.

924. Newton's law for the motion of bodies falling toward the earth from great distances (millions of miles) is $a = -C/s^2$, where the constant of proportionality $C = gr^2$. Neglect all resistances. (a) Let a body start from rest at a distance h from the surface of the earth (radius r), choose the center of the earth as the origin, and show that $v^2 = 2ghr/(h + r)$, where v is the velocity after falling to the earth's surface. (b) Let $r = 4000$ miles. If h is small, say, a few thousand feet, compared to r, to what form does this equation reduce?

925. The general expression obtained in accordance with Newton's law (problem **924**) for the velocity after a fall from a distance s_o to some distance s from the center of the earth is $v^2 = 2C(1/s - 1/s_o)$. Using $v = ds/dt$, show that

$$t = \left(\frac{s_o}{2gr^2}\right)^{1/2}\left[s_o \cos^{-1}\left(\frac{s}{s_o}\right)^{1/2} + (ss_o - s^2)^{1/2}\right].$$

The integration may be made by substituting $s = s_o \cos^2 \theta$.

926. A body falling in air actually undergoes a diminishing acceleration, and a point is reached at which the acceleration is zero. At the instant the body starts falling, the acceleration is g. For such bodies as rocks, bullets, and bombs, the acceleration $a = g - gv^2/v'^2 = (g/v'^2)(v'^2 - v^2)$, where v' is the limiting speed. Show that

$$t = \frac{v'}{2g} \log_e \left(\frac{v' + v}{v' - v}\right).$$

GRAPHICAL SOLUTIONS

927. The v-t diagram of a point is shown in Fig. 463. Let the area of this diagram be 0.58 in.², the time scale 1 in. = 0.12 sec., and the velocity scale 1 in. = 30 fps. Find the acceleration of the point at (a) position B, (b) position A, (c) position C. (d) Determine the displacement corresponding to OD.

Ans. (a) 150 fps², (b) 0, (c) −267 fps², (d) 2.09 ft.

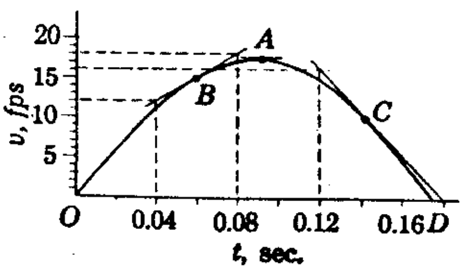

Fig. 463. Problem 927.

928. The v-t diagram for the rectilinear motion of an automobile between two points shown in Fig. 464. (a) What is the acceleration during the first 10 sec.? during the

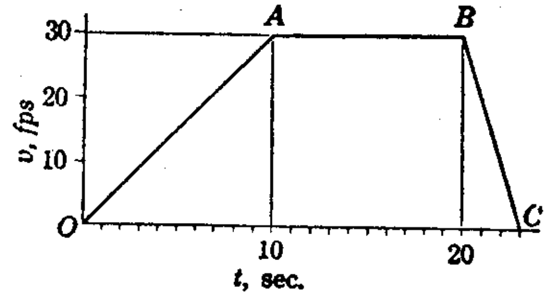

Fig. 464. Problem 928.

next 10 sec.? during the last 3 sec.? (b) What distance is traveled by the car?

929. (a) A body falls freely for a time t. Sketch the s-t, v-t, and a-t diagrams. What does the area under each curve represent? What does the slope of each curve represent? (b) A body is projected upward with an initial velocity v and permitted to fall back to the initial point. Neglecting air resistance, sketch the s-t, v-t, and a-t diagrams.

930. Sketch the s-t, v-t, and a-t diagrams for the stone in **900**. Numerical values are not required.

931. A ball is dropped from a height h above a floor. If the ball is perfectly elastic and there are no losses, it will distort elastically as it comes to rest after initially touching the floor. It then rebounds and theoretically returns to the initial point (see Fig. 691). Assume the accelerations during the distortion phase (while it is in contact with the floor) to be constant and sketch the a-t, v-t, and s-t diagrams for two complete bounces of the ball.

932. A reciprocating steam-engine mechanism, with a 1-ft. crank and a 4-ft. connecting rod, is represented in Fig. 465. The crank turns at a uniform speed of 50 rpm. Thus, each of the divisions 0-1, 1-2, etc., on the crank circle represents a certain time unit. Let 1 in. = 0.2 sec. of time and 1 in. = 1 ft. of displacement. (a) Construct a displacement s-t diagram for the piston, using the positions of the crank pin numbered in Fig. 465. (b) From the s-t diagram construct the v-t diagram. (c) From the v-t diagram, construct the a-t diagram. (d) From these diagrams, determine the maximum velocity and the maximum acceleration.
Ans. (d) Approximately -5.23 fps, -27.4 fps².

933. High-speed photography is now used as a means of measuring the displacement of moving objects. Thereby Δs is measured by comparing successive pictures taken Δt time apart (see Fig. 457 and several other flash photographs in this text). Then by graphical analysis the accelerations can be quite accurately found. The exposure rate is 120 frames per second. Let $\Delta s_1 = 1$ in., $\Delta s_2 = 1.5$ in., $\Delta s_3 = 2.25$ in., $\Delta s_4 = 3.25$ in., $\Delta s_5 = 4$ in., $\Delta s_6 = 4$ in., $\Delta s_7 = 3$ in., $\Delta s_8 = 2$ in., $\Delta s_9 = 1$ in., and $\Delta s_{10} = 0.5$ in. Plot the s-t curve. Then plot the v-t curve and the a-t curve. What are v_{max} and a_{max} during this 10-frame exposure?

Ans. $v_{max} = 1$ fps, $a_{max} = -1.05$ fps².

934. The displacement of a particle is given by the equation $s = t - 0.2t^2$ ft., where t is in seconds. (a) Plot an s-t diagram for an interval of 5 sec. What is the displacement after 5 sec.? (b) Derive a v-t diagram from the s-t curve, and check by differentiation the velocities obtained graphically for $t = 1$ sec. and $t = 3$ sec. (c) Derive an a-t diagram from the v-t curve, and check the acceleration by differentiation.

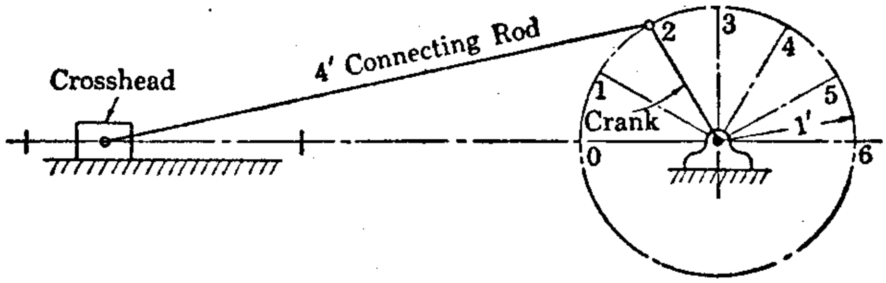

Fig. 465. Problem 932.

ANGULAR MOTION

935. A 12-ft. flywheel attains a speed of 120 rpm from rest in 60 sec. (a) What is the average angular acceleration? (b) What is the final velocity in fps of a point on the rim?
Ans. (a) 0.2095 rad. per sec.², (b) 75.4 fps.

936. A flywheel attains a speed of 120 rpm from rest after turning through 360 revolutions. What is the average angular acceleration?

937. (a) Through how many revolutions will a drum turn in 40 sec. when the initial angular velocity is 5 rad. per sec. and the acceleration is constant at 2 rad. per sec.²? (b) After this 40 sec., what angular acceleration will bring the drum to rest in 3 min.?
Ans. (a) 287 rev., (b) 0.472 rad. per sec.²

938. The speed of an electric motor is changed from 10 rpm to 1800 rpm in 4 sec. For this interval, find (a) the arithmetic average angular acceleration and (b) the angular displacement in revolutions.

939. (a) A turbine turning at 1200 rpm is brought to rest at the uniform rate of 100

rpm². What are the number of revolutions turned and the elapsed time? (b) If this turbine slows to 300 rpm from 1200 rpm during 7500 rev., what is the corresponding uniform acceleration in rad. per sec.²?
Ans. (a) 7200 rev., 12 min.; (b) 0.157 rad. per sec.²

940. From a speed of 2000 rpm, a flywheel will come to rest with constant acceleration in 3 min. after the power is cut off. (a) When $t = 1$ min., find ω and θ. (b) When $\theta = 1200$ rev., find ω and t. Time is measured from the instant that deceleration begins.

941. A flywheel may be coupled to a driving shaft by a jaw clutch. The flywheel initially has a speed of 200 rad. per sec. and is decelerating at a constant rate of 2 rad. per sec.². The shaft is started from rest in the same sense at the same time and is accelerated at a uniform rate of 10 rad. per sec.². The operator desires to make a "flying" engagement of the clutch, which is possible when their velocities are the same. At what time t should he engage the clutch?

942. An automobile engine turning at 2600 rpm is driving the car through high gear. The wheels are 30 in. in diameter. If the velocity ratio to the rear wheels is 3.42 in high gear and 6.5 in second gear, what is the speed in mph when the car is (a) in high gear and (b) in second gear?
Ans. (a) 67.8 mph, (b) 35.7 mph.

943. The angular displacement of a rotating body follows the law $\theta = t - 0.1t^2$ rad., where t is in seconds. Determine the angular displacement, velocity, and acceleration (a) after 3 sec., (b) after 10 sec. (c) What is the maximum displacement during the first 10 sec.?
Ans. (a) 2.1 rad., 0.4 rad. per sec., −0.2 rad. per sec.²; (b) 0 rad., −1 rad. per sec., −0.2 rad. per sec.²; (c) 2.5 rad.

944. The same as **943** except that $\theta = 1 - 0.1t^2$.

945. A rotating body follows the law $\theta = 2t^3 + 4t^2 + 10$ rad., when t is in seconds. Determine the angular displacement, velocity, and acceleration after 4 sec. Is the acceleration constant?
Ans. 202 rad., 128 rad. per sec., 56 rad. per sec.²

946. The same as **945** except that $\theta = 3t^2 - 2t + 4$.

947. If $\alpha = 2t$, derive general expressions for ω and θ when the initial angular velocity is ω_o and when the initial angular displacement is θ_o.

948. If $\alpha = 2\theta$, what is the general expression for ω when the initial angular velocity is ω_o and when $\theta_o = 0$?
Ans. $\omega = \pm(\omega_o^2 + 2\theta^2)^{1/2}$.

949. The same as **948.** except that $\alpha = \theta + 3\theta^2$.

950. Figure 466 represents the reciprocating mechanism in an airplane engine, with a crank OA, a connecting rod AB, and a piston B. The displacement of the piston as measured from the center of the crank shaft O is seen to be

$$s = r\cos\theta + (L^2 - r^2\sin^2\theta)^{1/2}.$$

Show that the expressions for the velocity and the acceleration of the piston B are

(a) $$v = -r\omega\left(\sin\theta + \frac{r}{2L}\sin 2\theta\right),$$

(b) $$a = -r\omega^2\left(\cos\theta + \frac{r}{L}\cos 2\theta\right).$$

HINT: Expand the second term of the equation for s to two terms, and obtain

$$s = r\cos\theta + L - \frac{r^2}{2L}\sin^2\theta + \ldots,$$

then differentiate. If the second term in the equations (a) and (b) were zero, what

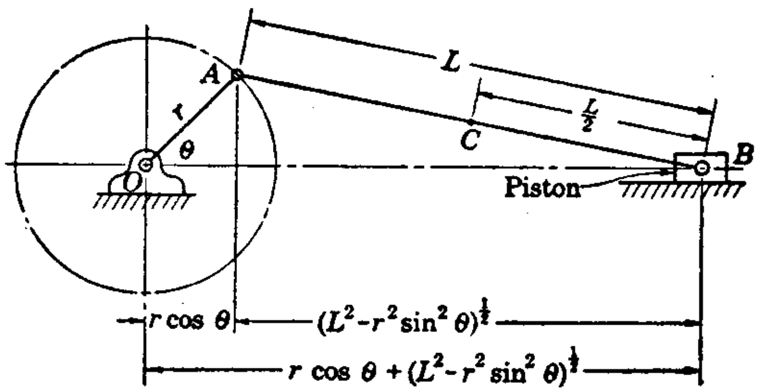

Fig. 466. Problems 950-952, 1001.

kind of motion would the piston describe? See § 196.

951. If $r = 2$ in., $L = 10$ in., and if the engine of **950** turns 2000 rpm, determine the velocity and acceleration of the piston when $\theta = 45°$.

952. In the airplane engine mechanism, Fig. 466, the point C is the center of mass of the connecting rod. With the origin at the center of the crankshaft O, the coordinates of C are

$$x = r \cos \theta + \frac{1}{2} (L^2 - r^2 \sin^2\theta)^{1/2},$$

$$y = \frac{r}{2} \sin \theta.$$

Determine the horizontal and the vertical components of the velocity and of the acceleration of the center of mass C, when $\theta = 60°$, $L = 6$ in., $r = 1.5$ in., and $n = 1500$ rpm. See the hint in **950**.

Ans. $v_x = -18.1$ fps, $a_x = -17.15$ fps², $v_y = 4.8$ fps, $a_y = -8.5$ fps².

NORMAL AND TANGENTIAL ACCELERATION

953. A motorcycle goes around a circular track whose radius is 200 ft. at 80 mph. What is its normal acceleration?

Ans. 68.8 fps².

954 A train moves at constant speed around a railroad curve of 2000-ft. radius. If the normal acceleration is 3.872 fps², what is the speed of the train in mph?

955. A point on a rotating body changes its speed uniformly from 10 fps to 20 fps while it moves 120 ft. If the radius of the point is 6 ft., what is its absolute acceleration at the instant its speed is 20 fps?

Ans. 66.7 fps².

956. A point moves in the circumference of a 20-ft. circle through an arc distance of 320 ft. in 10 sec. Its initial speed is 10 fps and the tangential acceleration is constant. At the instant that it has moved 200 ft., what are the values of a_n and a_t?

957. A point at a radius of 2 ft. in a rotating body has an initial speed of 160 fps. During a period of 6 sec., the angular deceleration of the body is 5 rpm each 1.5 sec. Compute the tangential and normal components of the acceleration (a) after 3 sec.; (b) after 6 sec.

Ans. (a) 0.698 fps², 12,450 fps²; (b) 0.698 fps², 12,120 fps².

958. A body A, Fig. 467, is suspended from a cable wound around a 5-ft. drum and is moving down with a constant velocity of 10 fps. When $t = 3$ sec., (a) determine the angular and linear velocities of point P which is on the flywheel that turns with the drum and (b) the normal and tangential accelerations of point P.

959. The same as **958** except that $a_A = 2$ fps². Let $v_o = 10$ fps.

Ans. (a) 6.4 rad. per sec., 25.6 fps, (b) 164 fps², 3.2 fps².

960. The same as **958** except that $a_A = g/2 - t/12$ and $v_o = 10$ fps.

961. The angular displacement of a rotating body follows the law $\theta = t - 0.1t^2$ rad. when t is in seconds. What are the tangential and normal accelerations of a point whose radius is 2 ft. after (a) 3 sec.; (b) 5 sec.?

Ans. (a) -0.4 fps², 0.32 fps²; (b) -0.4 fps², 0.

962. A rotating body whose motion follows the law $\alpha = 0.1t$ has an initial angular velocity of 5 rad. per sec. After 10 sec., what are the tangential and normal accelerations of a point whose radius is 6 in.?

963. A rotating body whose motion follows the law $\alpha = -4\theta^{1/2}$ has an initial angular velocity of 50 rad. per sec. After a rotation of 7 revolutions, what are the tangential and normal accelerations of a point whose radius is 18 in.?

Ans. -39.8 fps², 1410 fps².

964. At a particular instant, a point is on the horizontal centerline of a wheel whose angular velocity is 300 rpm counterclockwise and whose acceleration is 2 rad. per sec.² clockwise. When $t = 2$ sec., find

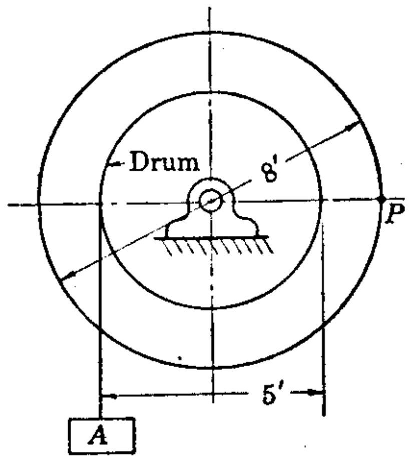

Fig. 467. Problems 958-960.

the position of the point and the magnitude of its absolute acceleration. The radius to the point is 6 in.

965. An automobile starts from rest and moves around a circular path whose radius is 600 ft. Its tangential acceleration is $a_t = (s + 6)^{1/2}$. Determine the tangential and normal accelerations after the car has gone 100 ft.

966. A body has an angular motion such that $\alpha = C \sin \theta$, where C is a constant. Determine the expressions for the normal and tangential accelerations for a point on this body at a radius r.

Ans. $a_t = Cr \sin \theta$, $a_n = 2\ Cr(1 - \cos \theta)$.

967. A crank moves with an acceleration of $\alpha = (1 - \sin \theta)^{1/2}/3$. Its initial speed is 20 rpm. After 10 revolutions, find (a) the angular velocity in rpm, (b) the magnitudes of the normal and tangential accelerations of a point whose radius is 3 ft.

968. The same as **967** except that $\alpha = (1 - \cos \theta)^{1/2}/3$.

Ans. (a) 20 rpm; (b) 13.17 fps², 0.

COMPONENT MOTION

969. If an automobile travels with the uniform speed of 40 mph up a 6% grade, what are the vertical and horizontal component velocities? *Ans.* 2.4 mph, 39.9 mph.

970. An automobile starts down a 7% grade at a speed of 10 mph. After 20 sec., its speed is 50 mph. At the moment its speed is 50 mph, what are the horizontal and vertical components of the velocity v and acceleration a, if a is constant? How far has the car traveled?

971. A body slides down a smooth 15° incline with an initial velocity of 6 fps. How far does it slide in 7 sec.? How far does it slide during the seventh second?

Ans. 246 ft., 60 ft.

972. A body slides down a smooth 60° incline which is 10 ft. long. If its initial velocity is 5 fps, what is its velocity at the bottom of the incline?

973. A body moves freely up a smooth 30° incline. After traveling 70 ft., its velocity is 10 fps. What was the initial velocity?

Ans. 48.5 fps.

974. A body moving freely up a smooth incline slows down from 60 fps to 20 fps in a distance of 100 ft. What is the inclination of the plane?

975. A good golfer imparts an initial velocity of 136.5 mph to a good golf ball. If the direction of this velocity is 30° with the horizontal, find (a) the horizontal distance traveled in flight, (b) the time of flight, (c) the maximum height reached by the ball. Assume that the ground is level, and neglect air resistance.

Ans. (a) 1077 ft.; (b) 6.22 sec.; (c) 155.6 ft.

976. The same as **975** except that the direction of the velocity is 45° above the horizontal.

977. A mortar projectile has a muzzle velocity of 900 fps. What must be the angle of elevation if the shell is to hit a target 2000 ft. away on a level with the gun? Neglect air resistance.

Ans. 2.29°, 87.71°.

978. A shell has just sufficient muzzle velocity to clear a hill, the crest of which is 2 miles away horizontally and 1500 ft. above the cannon. With the elevation of the gun set for maximum range, what is this muzzle velocity? Neglect air resistance.

Ans. 630 fps.

979. A bombing plane in level flight at an altitude of 40,000 ft. is moving 400 mph. How far ahead of the target, as measured horizontally, must a bomb be released in order to strike the target? What is the time of flight? Neglect air resistance.

Ans. 29,300 ft., 49.8 sec.

980. Assume that sound travels Q fps, and that t sec. elapse between the time of discharge of a field-artillery gun and the time at which the gunner hears the sound of the shell burst. Neglecting air resistance to the shell, show that the range is given by $Q(gt - 2v_1 \sin \theta)/g$, where v_1 is the initial velocity of the shell and θ is the elevation of the gun.

981. A body A slides 10 ft. down a smooth plane that is inclined at 45° with the horizontal. The end of the plane is 40 ft. above the ground. At the instant that the body A leaves the inclined plane, a body B is projected vertically upwards from the ground with an initial velocity of 30 fps. (a) With what velocity does A leave the plane? (b) Where does the body A strike the ground, as measured horizontally from the end of the incline? (c) As measured from the instant the body A leaves the incline, what time has elapsed when the bodies A and B are the same distance from the ground? (d) How far above the ground are the bodies when they are at the same

level? (e) What is the magnitude and sense of the velocity of body B?
Ans. (a) 21.35 fps; (b) 17.72 ft.; (c) 0.88 sec.; (d) 13.9 ft.; (e) 1.4 fps up.

982. The same as **981** except that the inclination of the plane is 30°.

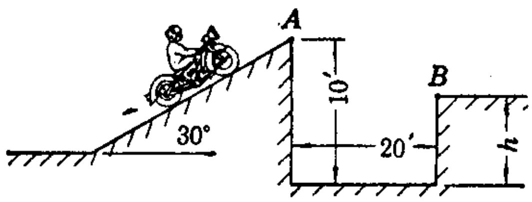

Fig. 468. Problems 983, 984.

983. A motorcycle stunt rider passes point A, Fig. 468, at a speed of 75 mph. What is the maximum value of h if the motorcycle (considered as a particle) is to jump the 20-ft ditch? Neglect the air resistance and somersaults. *Ans.* 20.84 ft.

984. The same as **983** except the rider passes A with a velocity of 80 fps.

985. A particle A slides down a parabolic chute whose equation is $x^2 = 4y$. It starts at a value of $y = h$. Show that the speed at point (0,0) is $v = \sqrt{2gh}$, friction neglected.

986. A point P moves in the path of the hyperbola $x^2/36 - y^2/16 = 1$. The x component of the velocity is constant at $v_x = 9$ fps. At the instant that P is at the position $(12, 4\sqrt{3})$, what is the acceleration a_y in the y direction and what is the tangential velocity? *Ans.* -1.73 fps², 11.36 fps.

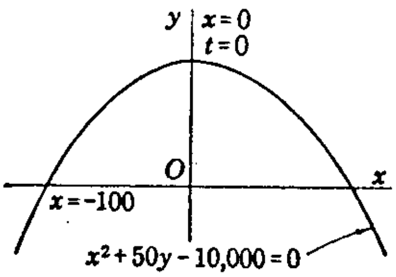

Fig. 469. Problem 987.

987. A point P moves in the path of the parabola $x^2 + 50y = 10{,}000$ (Fig. 469), so that v_x = a constant and $a_y = -g$. Determine the magnitude and direction of its velocity when the point is at the position $(-100, 0)$.

988. A point P moves in the path of a curve defined by the equation $y = e^x$. Its tangential velocity is constant and is equal to 12 fps. At a position defined by $y = 10$ ft., what are the components v_x and v_y of the velocity? *Ans.* 1.195 fps, 11.95 fps.

989. A point moves along the parabola $x^2 = 36y$. At the position where $x = 40$, the tangential velocity is $v = 12$ fps. At this position, what are the components v_x and v_y of the velocity.

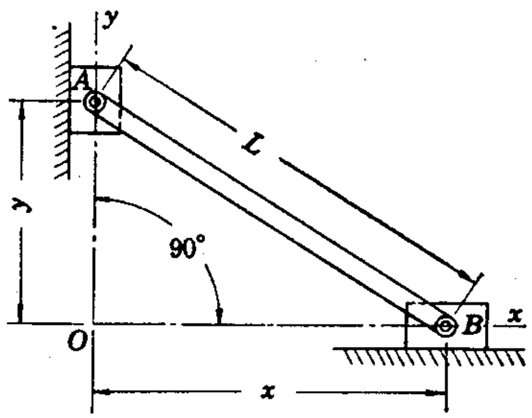

Fig. 470. Problems 990, 1002.

990. The sliding members A and B, Fig. 470, are constrained to move at all times in the y and x directions, respectively. They are connected by the rod whose length is $L = 10$ ft. At the instant when $x = 8$ ft., $v_B = 20$ fps toward the right and $a_B = -15$ fps² toward the left. Determine the velocity and acceleration of A at this instant. *Ans.* -26.67 fps, -165.1 fps².

991. A point moves so that

$$x = 5 \cos Ct \text{ and } y = 5 \sin Ct,$$

when t is in seconds. Determine the equations of the path of the point, its velocity, and its acceleration. In general, how is the resultant acceleration directed?
Ans. $x^2 + y^2 = 25$ ft.²; $+5C$ fps; $+5C^2$ at $\tan^{-1}(180° + Ct)$.

HARMONIC MOTION

992. A point moves according to the law $a = -16x$ with an amplitude of 3 in. (a) Find the period and frequency. (b) Determine the displacement, velocity, and acceleration when $t = 3$ sec.

993. A particle moves with a harmonic motion whose amplitude is 4 ft. The maximum acceleration is 20 fps². Determine the maximum velocity of the particle and the period of the motion.
Ans. 8.96 fps, 2.81 sec.

994. In Fig. 458, p. 269, show that the acceleration of P at the position B is equal to the normal acceleration of Q when it is at B. The point P is moving with harmonic motion. Also show that at any position of P, its acceleration is the horizontal component of the normal acceleration of Q.

995. In Fig. 458, p. 269, the point Q moves in the circle with a constant tangential speed of 10 fps. The radius of the circle is 4 ft. Five seconds after Q, going counterclockwise, passes the point D, what is the velocity of P, the projection of Q on the diameter? *Ans.* 9.98 fps, 0.279 ft.

996. The radius of the circle in Fig. 458, p. 269, is 20 in. The period of P is 3 sec. Determine for P (a) its maximum velocity, (b) its maximum acceleration, (c) its acceleration when $\theta = 30°$.

997. The cam shown in Fig. 471 raises and lowers the follower a distance $s = 3$ in. with harmonic motion. If the cam makes 60 rpm, determine the maximum velocity and maximum acceleration of the follower. The follower makes a stroke in each half revolution of the cam.

Ans. 9.42 in. per sec., 59.2 in. per sec.[2]

998. The cam shown in Fig. 471 raises and lowers the follower a distance $s = 5$ in. with harmonic motion. If the period of oscillation is 2 sec., find the maximum velocity and acceleration of the follower. The follower makes one oscillation for each revolution of the cam.

999. A body oscillates twice a second with an amplitude of 3 in. Use the relation $a = v\,dv/ds = -C^2 s$ and integrate to find (a) the maximum velocity, (b) the displacement when $v = 10$ in. per sec., and (c) the velocity when $s = 2$ in.
Ans. (a) 37.7 in. per sec.; (b) 2.89 in.; (c) 28.1 in. per sec.

1000. The same as **999** except that the frequency is such that $T = 1$ sec.

The complete movement of a machine part esp. a reciprocating part in one direction

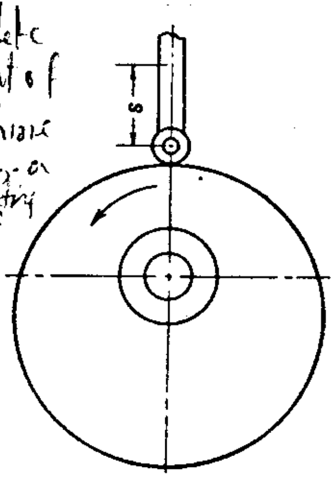

Fig. 471. Problems 997, 998

CENTROS

1001. Note that the direction of motion of each end of the connecting rod in Fig. 466 is known and locate its centro for $\theta = 45°$, $r = 2$ in., $L = 10$ in. Using this centro, determine the instantaneous velocity of the piston B when the engine turns at 2000 rpm. (See problem **951**.)

1002. Locate the centro of the rod AB in Fig. 470 for the position defined in Problem **990**. Using this centro, check the velocity of A as found in **990**.

1003. A ladder 25 ft. long is in a vertical position against a vertical wall. If the bottom of the ladder is dragged outward on a horizontal surface at the constant speed of 4 fps, what is the velocity of the top of the ladder when the bottom point is 15 ft. from the wall? *Ans.* −3 fps.

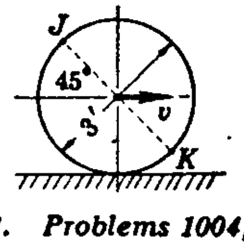

Fig. 472. Problems 1004, 1005.

1004. A 3-ft. wheel, Fig. 472, rolls toward the right. If the velocity of the center of the wheel is $v = 30$ fps, what are the absolute velocities of the points J and K?
Ans. 55.5 fps, 22.95 fps.

1005. The same as **1004** except that $v = 10$ fps.

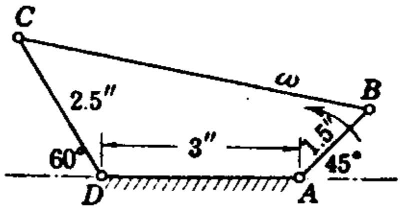

Fig. 473. Problems 1006, 1007.

1006. (a) The pin B, Fig. 473, has an absolute velocity of 10 fps. What is the absolute velocity of the pin C? (b) What is the velocity ratio between the links AB and DC for the instant defined in (a)? The link AB is the driving link.
Ans. (a) 11.2 fps; (b) 1.485.

1007. The same as **1006** except that the link AB is 2 in. long.

1008. The wheel in Fig. 474 is rolling so that the absolute velocity of the pin A is 10 fps when $\theta = 60°$. What is the absolute velocity of the slider B? The pin at B is level with the bottom of the wheel.

Ans. 7.4 fps.

1009. The same as **1008** except that $\theta = 150°$. Point A remains 2 in. from the instant center.

1010. Figure 475 shows a sliding block B that is driven by a crankpin A through link AB. The block B is constrained to move on a circular arc whose center is at C.

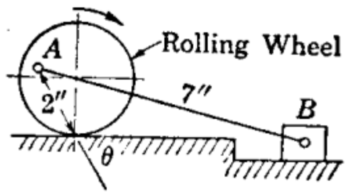

Fig. 474. Problems 1008, 1009.

For the position shown, let $\omega_A = 40$ rpm counterclockwise and find the speed of B and the angular velocity of link AB. Solve graphically for $r = 5$ ft. and $\theta = 30°$.

Ans. $\omega_{AB} = 2.23$ rad. per sec.

1011-1020. These numbers may be used for other problems.

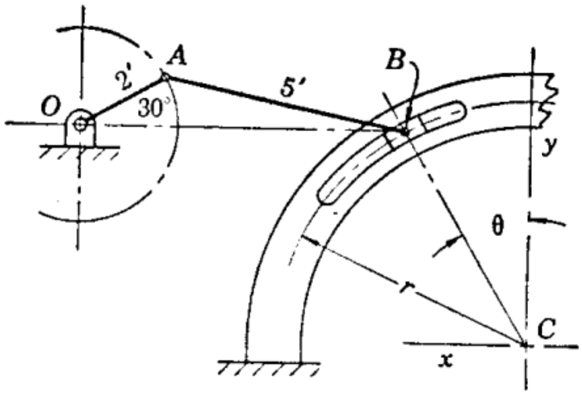

Fig. 475. Problem 1010.