

CE 211: Mechanics of Solids | 3.00 credits, 3 hrs/week. Prereq. CE 101

Objective: This is a preparatory course for structural analysis and design. The concepts introduced in this course are essential for Level-3 and 4 courses of structural analysis and design and those of RCC.

Text book: Engineering Mechanics of Solids by Egor P. Popov, Second Edition, Prentice-Hall

Course Plan:

Course Content	Chapters	Weeks
Concepts of stress and strain, constitutive relationships	1, 2, 5	2
Deformations due to tension, compression and temperature change	3, 4	1
Beam statics: reactions, axial force, shear force and bending moments; axial force, shear force and bending moment diagrams using method of section and summation approach	7	3
Elastic analysis of circular shafts, solid non- circular and thin walled tubular members subjected to torsion	6	2
Flexural stresses in beams	8, 9	2
Shear stresses in beams, shear centre	10	1
Thin walled pressure vessels	-	1

Marks distribution:

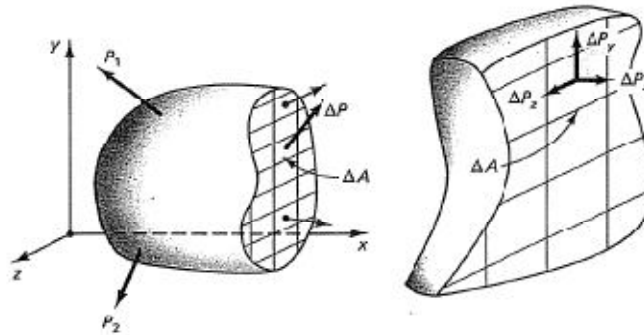
Item	Marks (%)
Class attendance	10
Class tests (best 3 out of 4)	20
Assignment: Only for practice	0
Term final exam	70

Tentative schedule of class tests:

Class test	Topic	Week
CT-1	Stress/Strain/Hook's law/Axial deformation	4th
CT-2	SFD, BMD	7th
CT-3	Torsion	9th
CT-4	Flexural stress	11th

Lecture 1: STRESS

1. Definition of Stress



Intensity of force per unit area is defined as stress.

Mathematical definition of stress is:

$$\tau_{xx} = \lim_{\Delta A \rightarrow 0} \frac{\Delta P_x}{\Delta A} \quad \tau_{xy} = \lim_{\Delta A \rightarrow 0} \frac{\Delta P_y}{\Delta A} \quad \text{and} \quad \tau_{xz} = \lim_{\Delta A \rightarrow 0} \frac{\Delta P_z}{\Delta A}$$

The intensity of the force perpendicular to or normal to the section is called the *normal stress* at a point. Normal stresses that cause traction or tension on the surface of a section are *tensile stresses*. On the other hand, those that are pushing against it are *compressive stresses*. Normal stresses are usually designated by the letter σ (sigma) .

The other components of the intensity of force act parallel to the plane of the elementary area. These components are called *shear or shearing stresses*. Shear stresses will be always designated by τ .

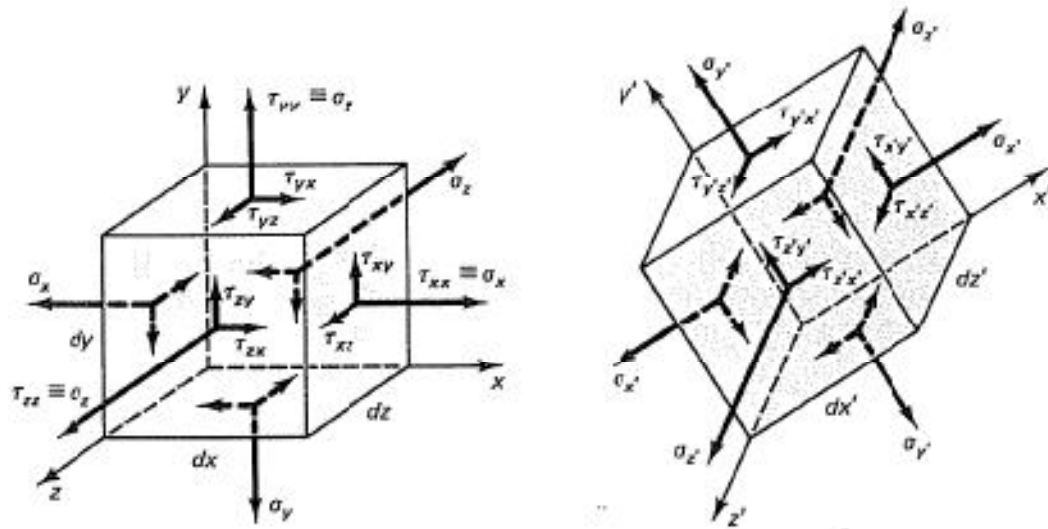
Unit:

US Customary - psi, ksi

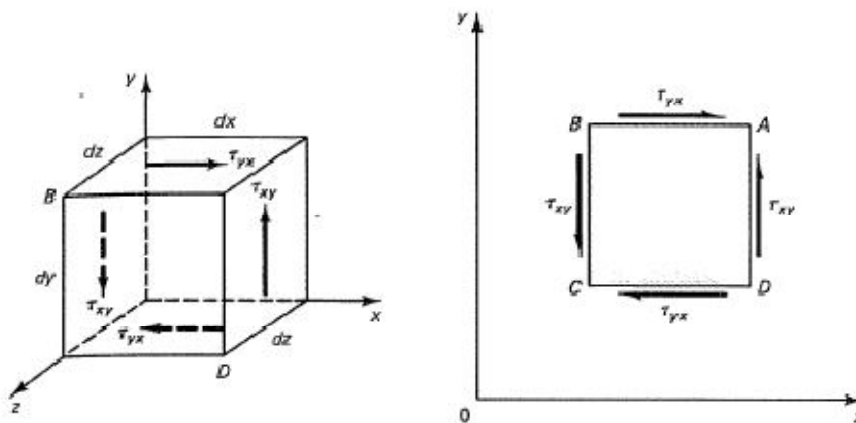
SI - N/m^2 (Pa), N/mm^2 (MPa)

Stresses multiplied by the respective areas on which they act give forces. At an imaginary section, a vector sum of these forces, called stress resultants, keeps a body in equilibrium.

2. Stress Tensor



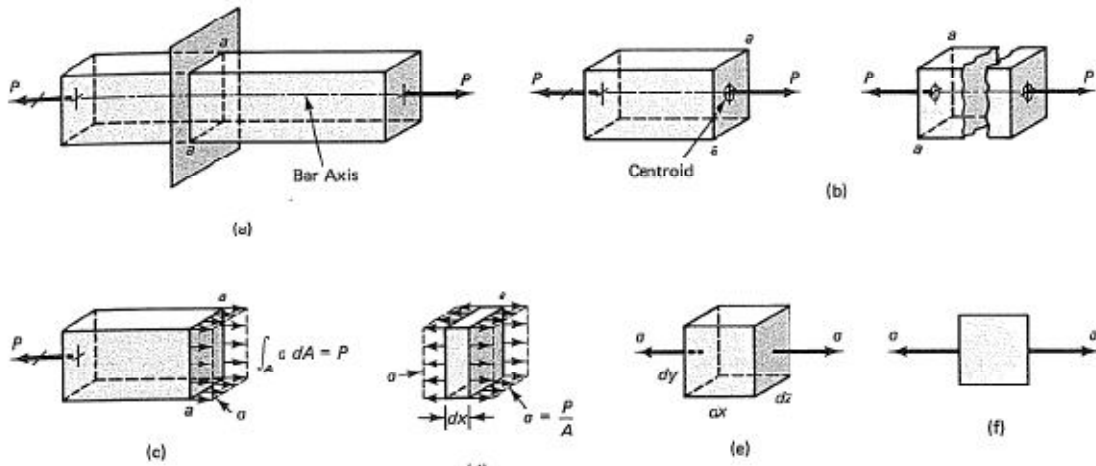
$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \equiv \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$



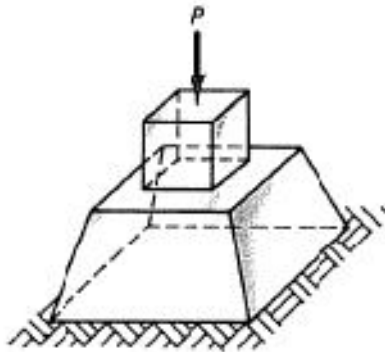
$$\tau_{yx} = \tau_{xy}$$

The stress tensor is symmetric.

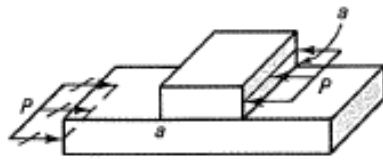
3. Normal Stress in Axially Loaded Bars



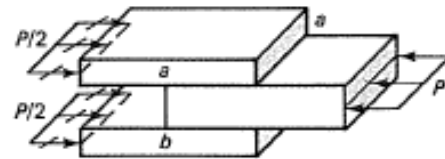
$$\sigma = \frac{\text{force}}{\text{area}} = \frac{P}{A} \quad \left[\frac{\text{N}}{\text{m}^2} \right] \text{ or } \left[\frac{\text{lb}}{\text{in}^2} \right]$$



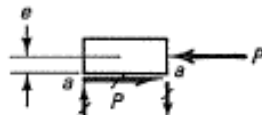
4. Shear Stresses



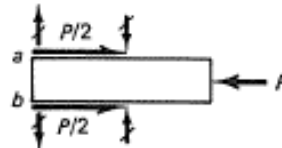
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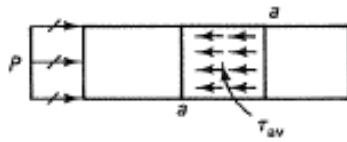
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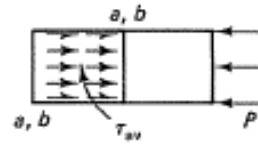
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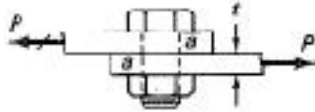


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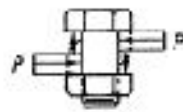


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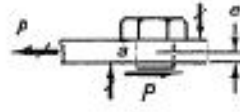
$$\tau_{av} = \frac{\text{force}}{\text{area}} = \frac{V}{A} \quad \left[\frac{\text{N}}{\text{m}^2} \right] \text{ or } \left[\frac{\text{lb}}{\text{in}^2} \right]$$



(a)



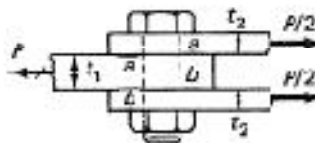
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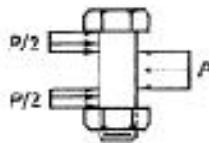
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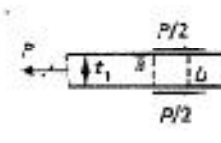
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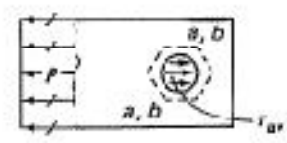
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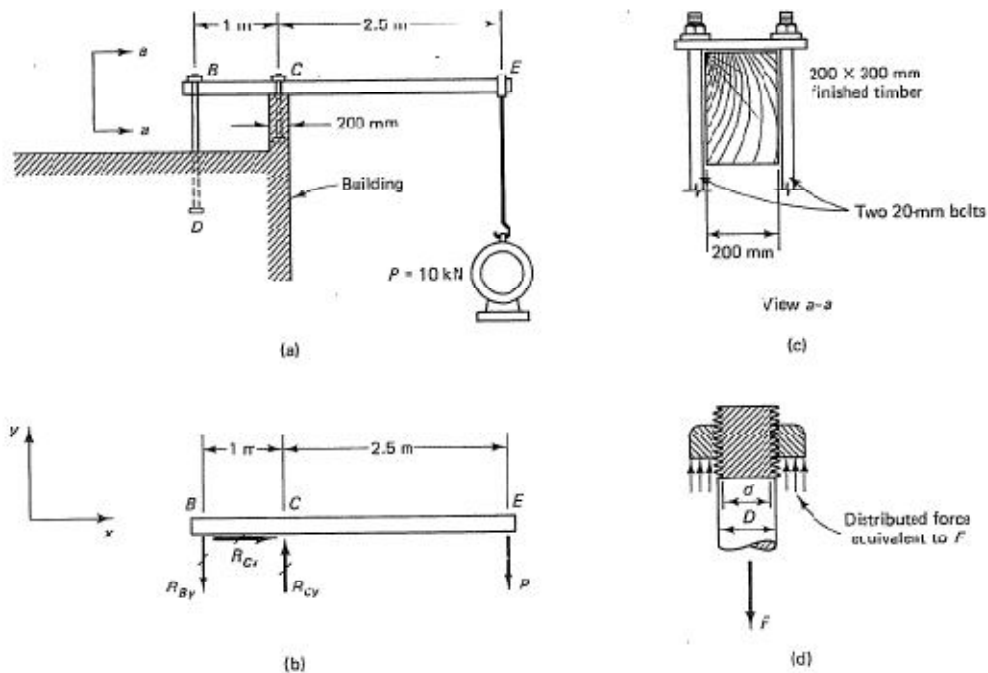
(g)



(h)

5. Analysis for Normal and Shear Stresses

Example 1-1



Determine the stress in bolts BD and the bearing stress at C. Note that the bolts are threaded with $d = 16$ mm at the root of the threads. Assume that the weight of the beam is negligible in comparison with the loads handled.

$$\begin{aligned} \sum F_x &= 0 & R_{Cx} &= 0 \\ \sum M_B &= 0 \curvearrowright + 10(2.5 + 1) - R_{Cy} \times 1 = 0 & R_{Cy} &= 35 \text{ kN } \uparrow \\ \sum M_C &= 0 \curvearrowright + 10 \times 2.5 - R_{By} \times 1 = 0 & R_{By} &= 25 \text{ kN } \downarrow \\ \text{Check: } \sum F_y &= 0 \uparrow + -25 + 35 - 10 = 0 \end{aligned}$$

The cross-sectional area of one 20-mm bolt at the root of the threads is

$$A_{\text{net}} = \pi 8^2 = 201 \text{ mm}^2$$

Maximum normal tensile stress in each of the two bolts BD:

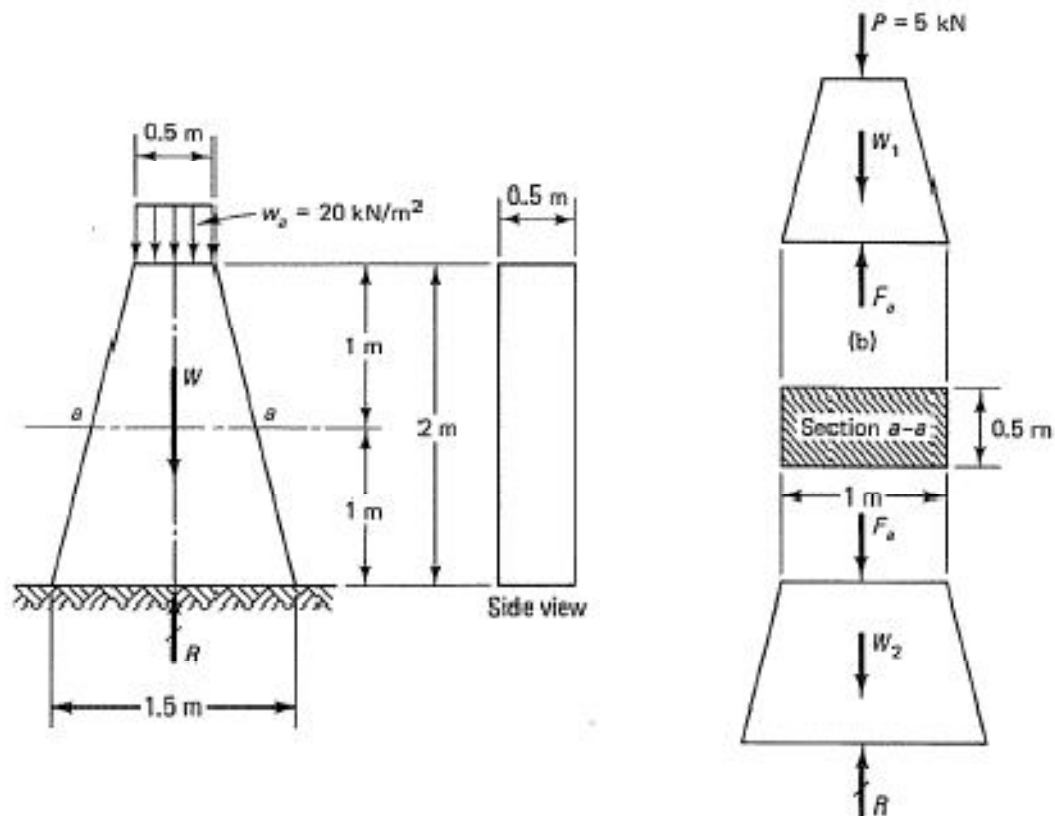
$$\sigma_{\text{tmax}} = \frac{R_{By}}{2A} = \frac{25 \times 10^3}{2 \times 201} = 62 \text{ N/mm}^2 = 62 \text{ MPa}$$

Contact area at C: $A = 200 \times 200 = 40 \times 10^3 \text{ mm}^2$

Bearing stress at C:

$$\sigma_b = \frac{R_{Cy}}{A} = \frac{35 \times 10^3}{40 \times 10^3} = 0.875 \text{ N/mm}^2 = 0.875 \text{ MPa}$$

Example 1-2



Investigate the state of stress at a level 1 m above the base. Concrete weighs approximately 25 kN/m^3 .

Weight of the whole pier:

$$W = [(0.5 + 1.5)/2] \times 0.5 \times 2 \times 25 = 25 \text{ kN}$$

Total applied force:

$$P = 20 \times 0.5 \times 0.5 = 5 \text{ kN}$$

From $\sum F_y = 0$, reaction at the base:

$$R = W + P = 30 \text{ kN}$$

Using the upper part of the pier as a free body, the weight of the pier above the section:

$$W_1 = (0.5 + 1) \times 0.5 \times 1 \times 25/2 = 9.4 \text{ kN}$$

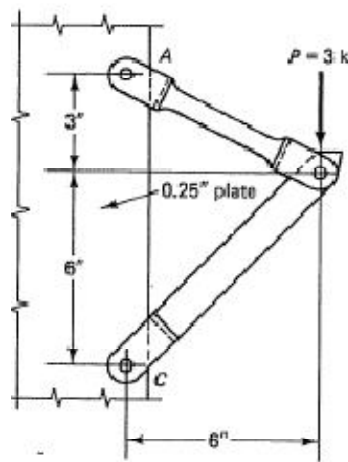
From $\sum F_y = 0$, the force at the section:

$$F_a = P + W_1 = 14.4 \text{ kN}$$

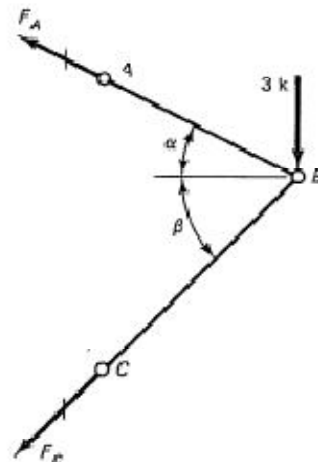
Hence, the normal stress at the level a-a is

$$\sigma_a = \frac{F_a}{A} = \frac{14.4}{0.5 \times 1} = 28.8 \text{ kN/m}^2$$

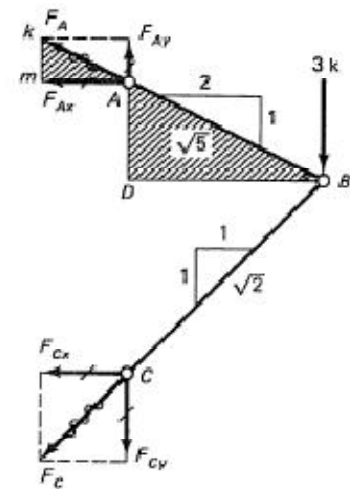
Example 1-3



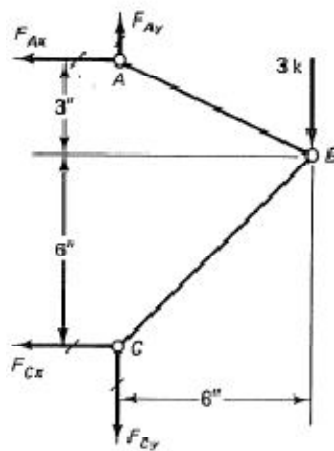
(a)



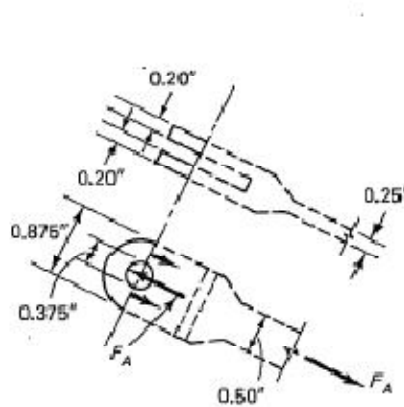
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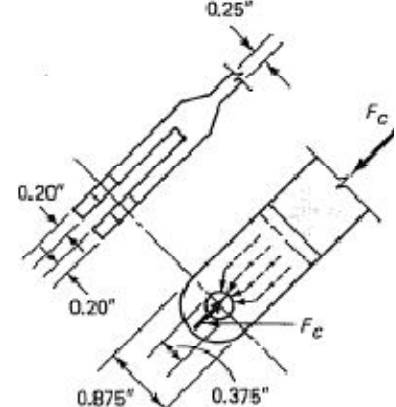
(c)



(d)



(e)



(f)

Find the axial stresses in members AB and BC and the bearing and shear stresses for pin C. All pins are 0.375 in in diameter.

$$F_A = (AB/DB)F_{Ax}$$

$$F_A = (\sqrt{5}/2)F_{Ax} \quad \text{and} \quad F_{Ay} = F_{Ax}/2$$

$$\begin{aligned} \sum M_C = 0 \quad \circlearrowleft + \quad + F_{Ax}(3 + 6) - 3(6) = 0 \quad F_{Ax} = +2 \text{ k} \\ F_{Ay} = F_{Ax}/2 = 2/2 = +1 \text{ k} \\ F_A = 2(\sqrt{5}/2) = +2.23 \text{ k} \end{aligned}$$

$$\sum M_A = 0 \text{ } \odot + \quad + 3(6) + F_{Cx}(9) = 0, \quad F_{Cx} = -2 \text{ k}$$

$$F_{Cy} = F_{Cx} = -2 \text{ k}$$

$$F_C = \sqrt{2}(-2) = -2.83 \text{ k}$$

Tensile stress in main bar AB:

$$\sigma_{AB} = \frac{F_A}{A} = \frac{2.23}{0.25 \times 0.50} = 17.8 \text{ ksi}$$

Tensile stress in clevis of bar AB:

$$(\sigma_{AB})_{\text{clevis}} = \frac{F_A}{A_{\text{net}}} = \frac{2.23}{2 \times 0.20 \times (0.875 - 0.375)} = 11.2 \text{ ksi}$$

Compressive stress in main bar BC:

$$\sigma_{BC} = \frac{F_C}{A} = \frac{2.83}{0.875 \times 0.25} = 12.9 \text{ ksi}$$

Bearing between pin C and the clevis:

$$\sigma_b = \frac{F_C}{A_{\text{bearing}}} = \frac{2.83}{0.375 \times 0.20 \times 2} = 18.8 \text{ ksi}$$

Bearing between the pin C and the main plate:

$$\sigma_b = \frac{F_C}{A} = \frac{2.83}{0.375 \times 0.25} = 30.2 \text{ ksi}$$

Double shear in pin C:

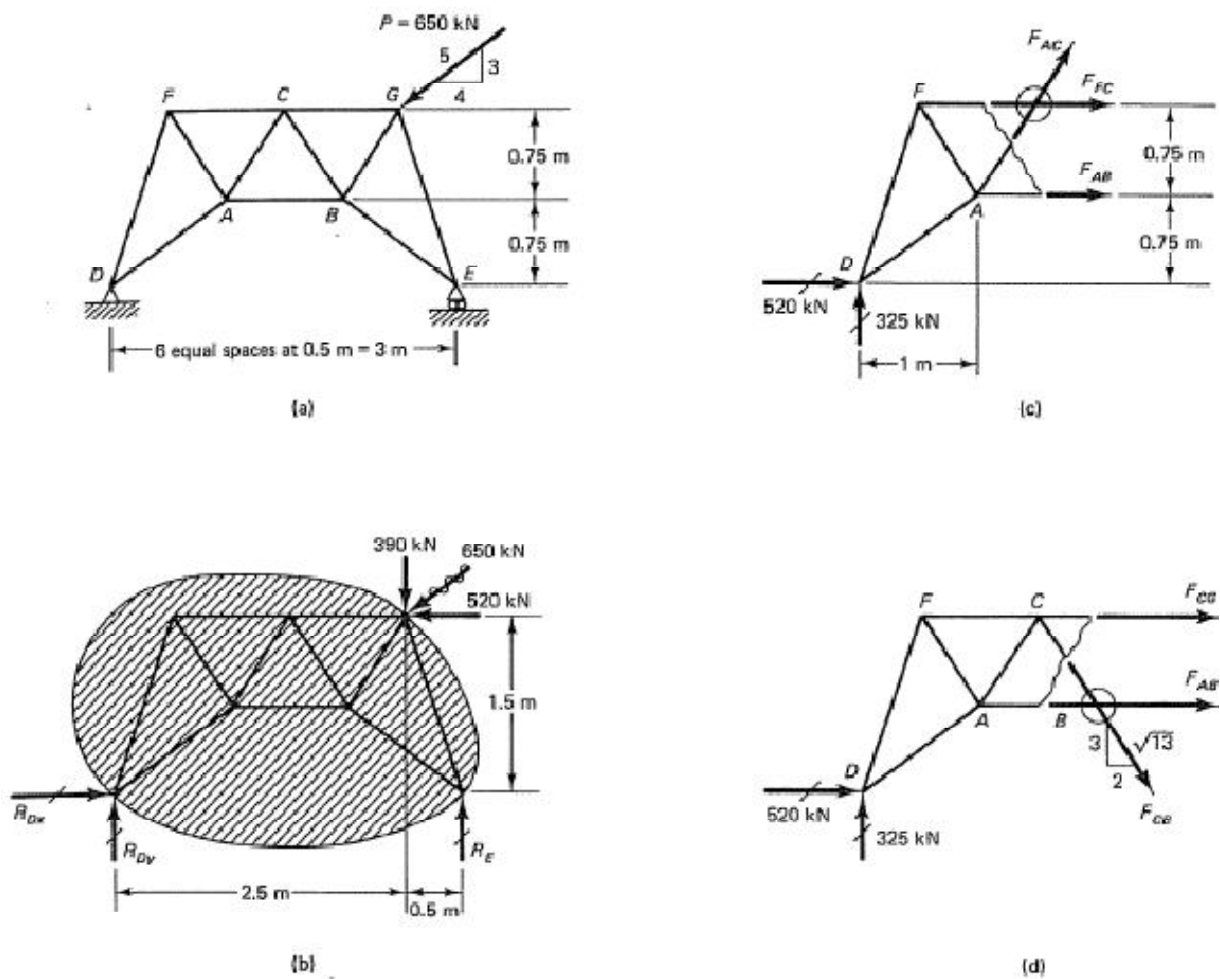
$$\tau = \frac{F_C}{A} = \frac{2.83}{2\pi(0.375/2)^2} = 12.9 \text{ ksi}$$

6. Deterministic Design of Members: Axially Loaded Bars

$$F.S. = \frac{\text{maximum useful material strength (stress)}}{\text{allowable stress}}$$

$$A = \frac{P}{\sigma_{\text{allow}}}$$

Example 1-4



Select members FC and CB. Set the allowable tensile stress at 140 MPa.

$$\sum F_x = 0 \quad R_{Dx} - 520 = 0 \quad R_{Dx} = 520 \text{ kN}$$

$$\sum M_E = 0 \text{ } \odot + \quad R_{Dy} \times 3 - 390 \times 0.5 - 520 \times 1.5 = 0$$

$$\sum M_D = 0 \text{ } \odot + \quad R_E \times 3 + 520 \times 1.5 - 390 \times 2.5 = 0$$

$$R_{Dy} = 325 \text{ kN}$$

$$R_E = 65 \text{ kN}$$

$$\sum M_A = 0 \text{ } \odot + \quad F_{FC} \times 0.75 + 325 \times 1 - 520 \times 0.75 = 0$$

$$F_{FC} = +86.7 \text{ kN}$$

$$A_{FC} = F_{FC} / \sigma_{\text{allow}} = 86.7 \times 10^3 / 140 = 620 \text{ mm}^2$$

(use 12.5 × 50-mm bar)

$$\sum F_y = 0 \quad -(F_{CB})_y + 325 = 0 \quad (F_{CB})_y = +325 \text{ kN}$$

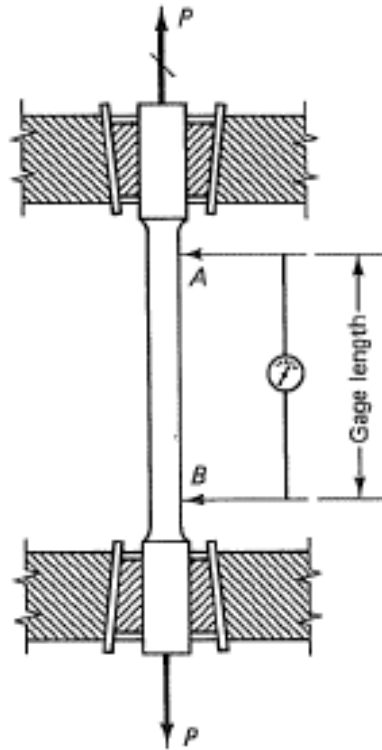
$$F_{CB} = \sqrt{13}(F_{CB})_y / 3 = +391 \text{ kN}$$

$$A_{CB} = F_{CB} / \sigma_{\text{allow}} = 391 \times 10^3 / 140 = 2790 \text{ mm}^2$$

(use two bars 30 × 50 mm)

Lecture 2: STRAIN

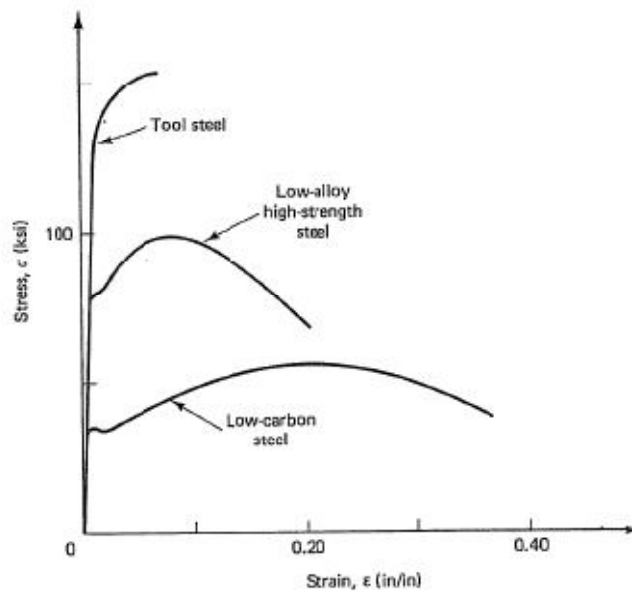
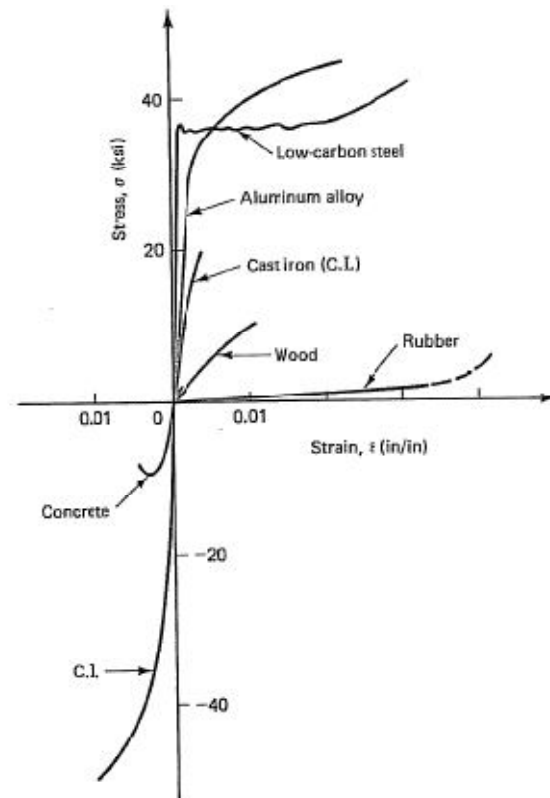
1. Normal Strain

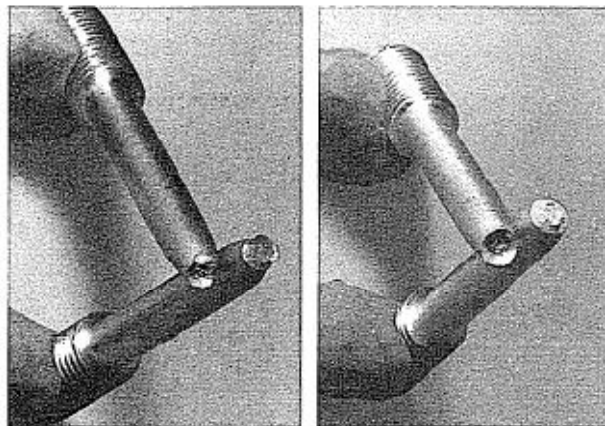
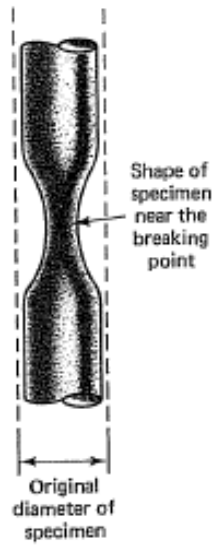
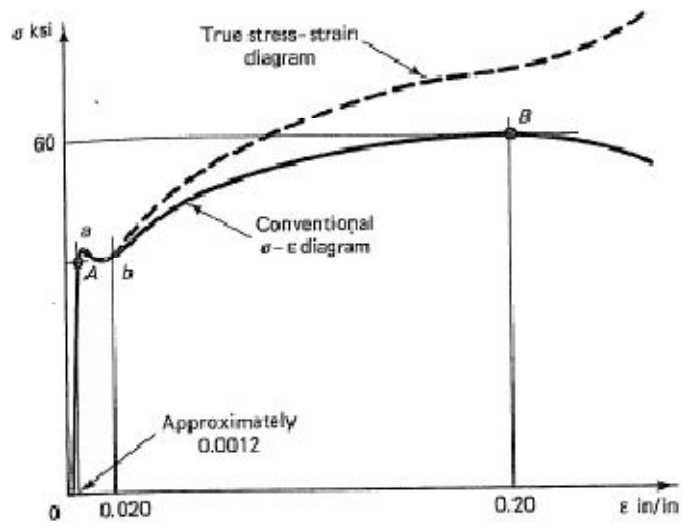


$$\varepsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0}$$

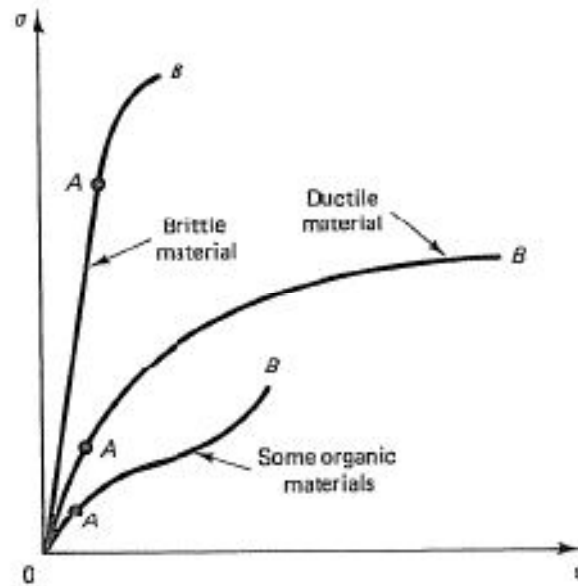
This expression defines the *extensional strain*. Since this strain is associated with the normal stress, it is usually called the *normal strain*. It is a dimensionless quantity, but it is customary to refer to it as having the dimensions of in/in, m/m, or $\mu\text{m}/\text{m}$ (microstrain). Sometimes it is given as a percentage.

2. Stress-Strain Relationship





Stresses are usually computed on the basis of the original area of a specimen; such stresses are often referred to as *conventional or engineering stresses*. On the other hand, it is known that some transverse contraction or expansion of a material always takes place. Dividing the applied force, at a given point in the test, by the corresponding actual area of a specimen at the same instant gives the so-called *true stress*. A plot of true stress vs. strain is called a *true stress-strain diagram*.



Materials capable of withstanding large strains without a significant increase in stress are referred to as *ductile materials*. The converse applies to *brittle materials*.

3. Hooke's Law

For a limited range from the origin, the experimental values of stress vs. strain lie essentially on a straight line. For all practical purposes, up to some point the relationship between stress and strain may be said to be linear for all materials. This sweeping idealization and generalization applicable to all materials is known as *Hooke's law*. Symbolically, this law can be expressed by the equation

$$\sigma = E\epsilon$$

which simply means that stress is directly proportional to strain, where the constant of proportionality is E . This constant E is called the *elastic modulus*, *modulus of elasticity*, or *Young's modulus*. As ϵ is dimensionless, E has the units of stress in this relation.

Graphically, E is interpreted as the slope of a straight line from the origin to the rather vague point A on a uniaxial stress-strain diagram. The stress corresponding to the latter point is termed the *proportional* or *elastic limit* of the material. Physically, the elastic modulus represents the stiffness of the material to an imposed load. The value of the elastic modulus is a definite property of a material. For all steels, E at room temperature is between 29 and 30 x .106 psi, or 200 and 207 GPa.

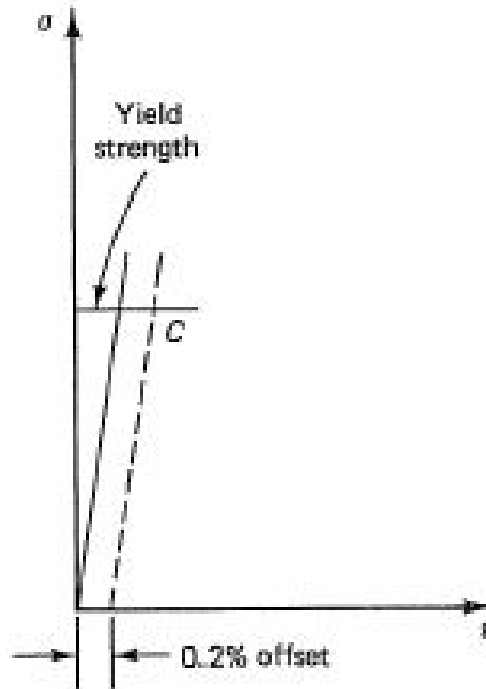
Hooke's law applies only up to the proportional limit of the material.

Some materials, notably single crystals and wood, possess different elastic moduli in different directions. Such materials, having different physical properties in different directions, are called *anisotropic*. The vast majority of engineering materials consist of a large number of randomly oriented crystals. Because of this random orientation, properties of materials become essentially alike in any direction. Such materials are called *isotropic*.

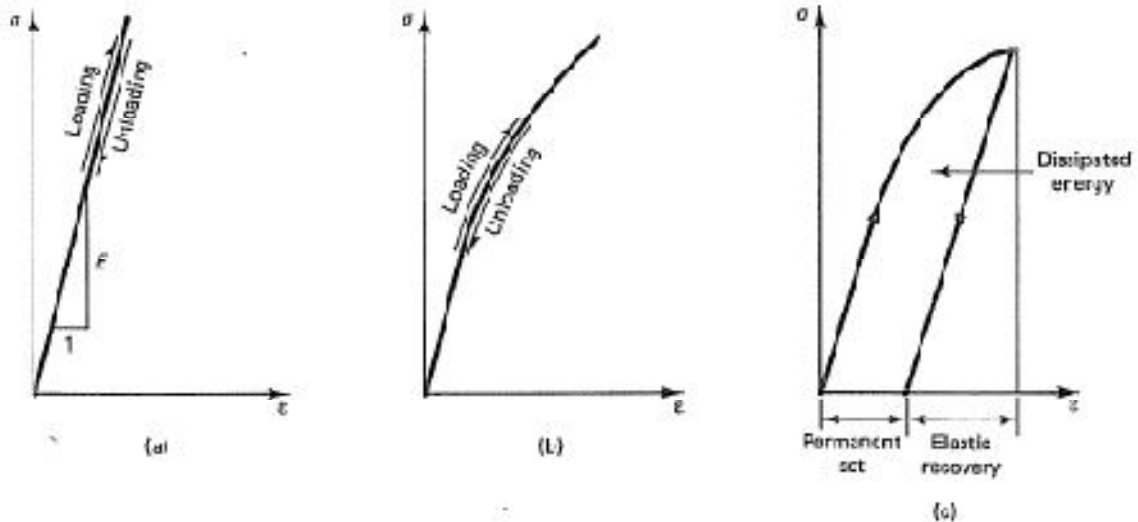
4. Further Remarks on Stress-Strain Relationships

In the stress-strain diagram, the highest points (B) correspond to the *ultimate strength* of a material. Stress associated with the long plateau is called the *yield strength* of a material. Note that at an essentially constant stress, strains 15 to 20 times those that take place up to the proportional limit occur during yielding. At the yield stress, a large amount of deformation takes place at a constant stress. The Yielding phenomenon is absent in most materials.

For materials that do not possess a well-defined yield strength, one is sometimes "invented" by the use of the so-called "offset method." In this method a line offset an arbitrary amount of 0.2 percent of strain is drawn parallel to the straight-line portion of the initial stress-strain diagram. Point C is then taken as the yield strength of the material at 0.2-percent offset.

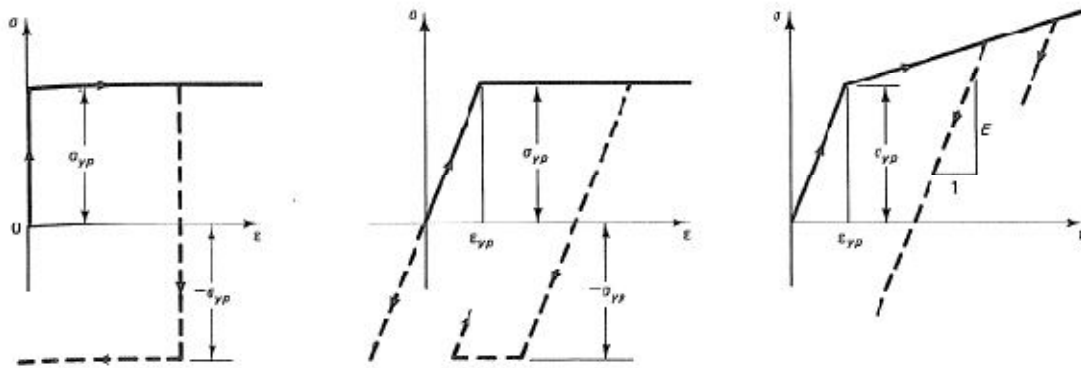


That a material is elastic usually implies that stress is directly proportional to strain, as in Hooke's law. Such materials are *linearly elastic* or *Hookean*. A material responding in a nonlinear manner and yet, when unloaded, returning back along the loading path to its initial stress-free state of deformation is also an elastic material. Such materials are called *nonlinearly elastic*.

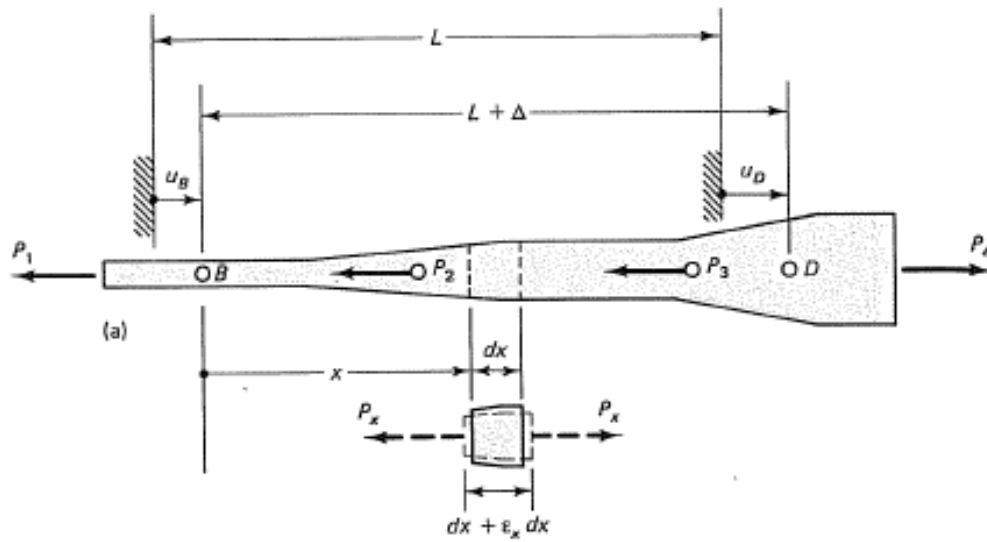


4. Other Idealizations of Constitutive Relations

In an increasingly larger number of technical problems, stress analyses based on the assumption of linearly elastic behavior are insufficient. For this reason, several additional stress-strain relations are now in general use. Such relations are frequently referred to as *constitutive relations* or *laws*.



5. Deformation of Axially Loaded Bars



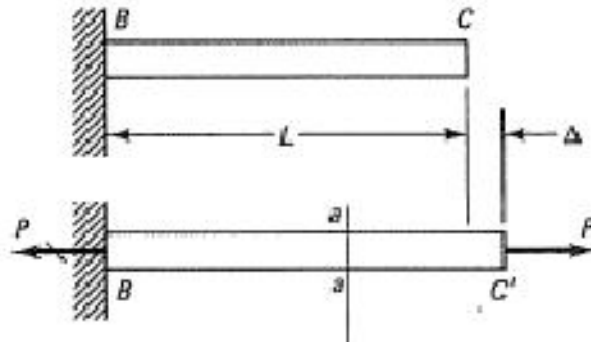
$$\epsilon_x = \frac{du}{dx}$$

$$\int_0^L du = u(L) - u(0) = \int_0^L \epsilon_x dx$$

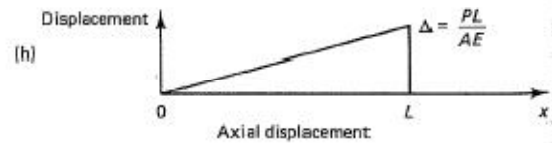
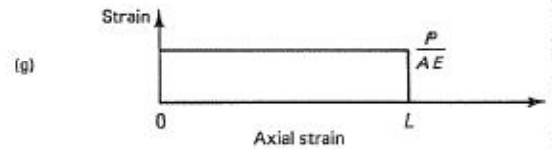
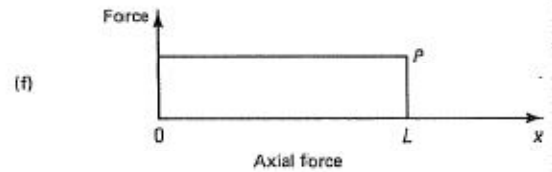
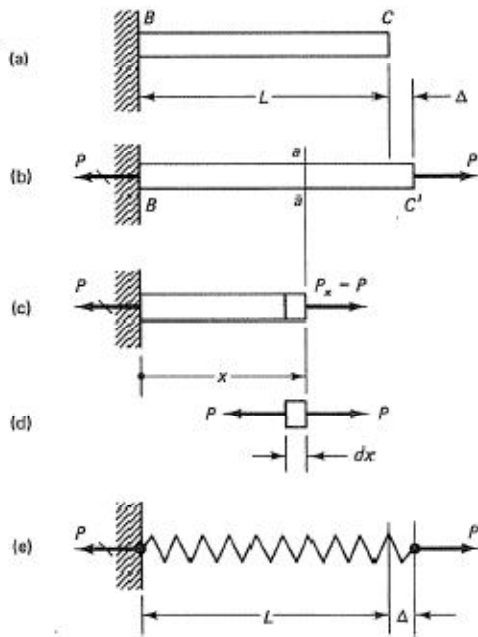
$$\Delta = \int_0^L \epsilon_x dx$$

$$\Delta = \int_0^L \frac{P_x dx}{A_x E_x}$$

Example 2-1



Determine the deflection of the free end, caused by the application of a concentrated force P . The elastic modulus of the material is E .



$$\Delta = \int_A^B \frac{P_x}{A_x E} dx = \frac{P}{AE} \int_0^L dx = \frac{P}{AE} \left| x \right|_0^L = \frac{PL}{AE}$$

$$P = (AE/L)\Delta$$

This equation is related to the familiar definition for the *spring constant* or *stiffness* k reading

$$k = P/\Delta \quad [\text{lb/in}] \text{ or } [\text{N/m}]$$

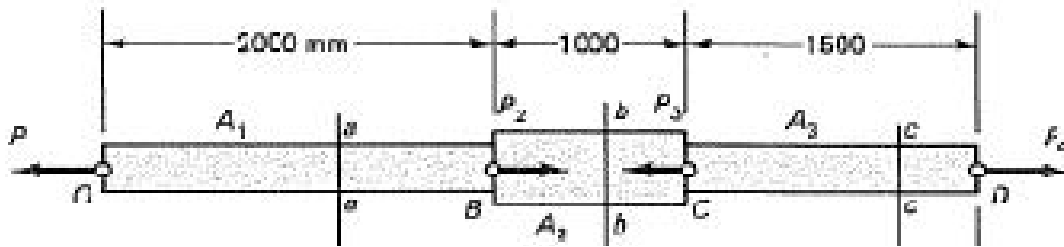
$$k_i = \frac{A_i E_i}{L_i}$$

The reciprocal of k defines the *flexibility* f , i.e.,

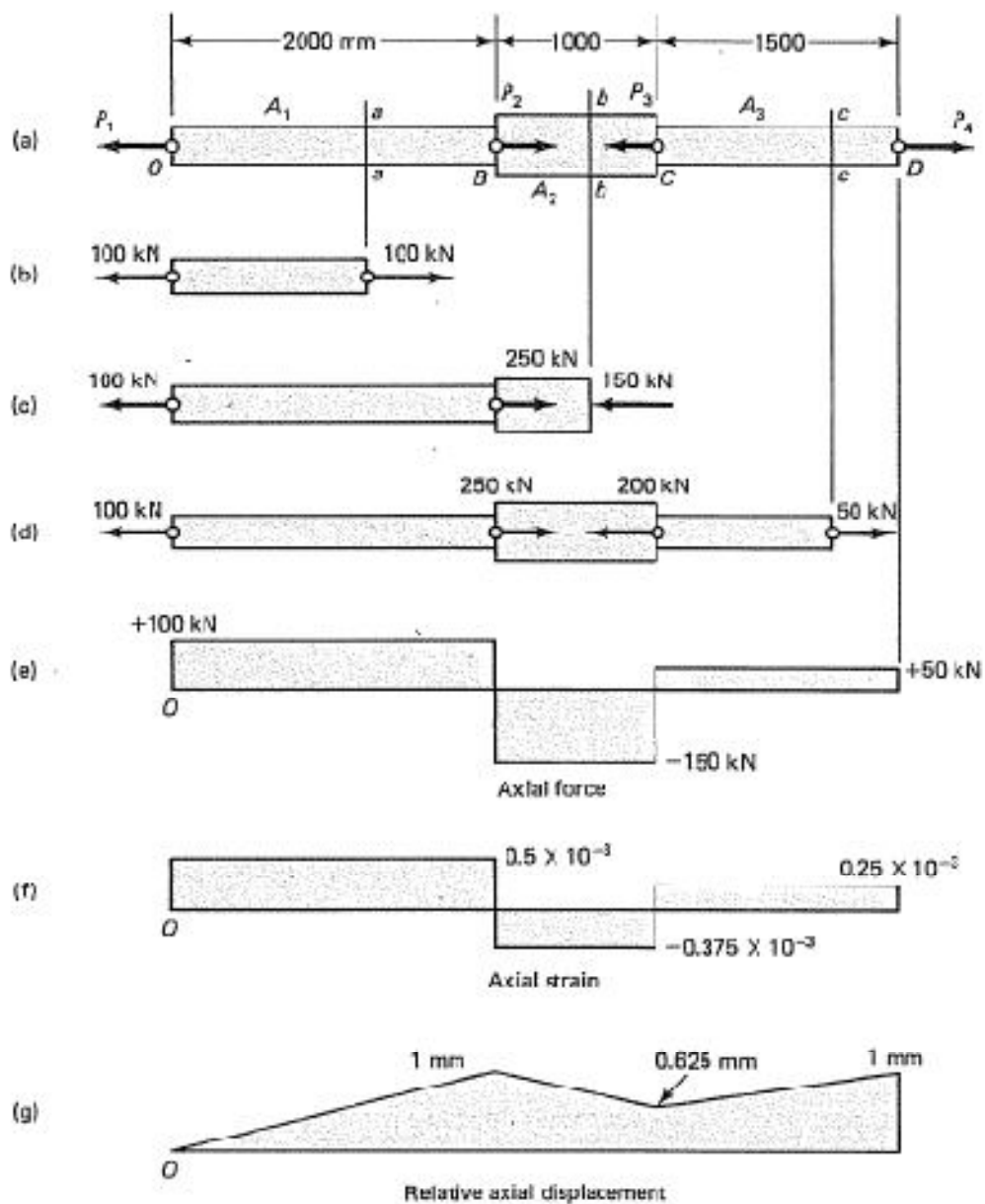
$$f = 1/k = \Delta/P \quad [\text{in/lb}] \text{ or } [\text{N/m}]$$

$$f_i = \frac{L_i}{A_i E_i}$$

Example 2-2



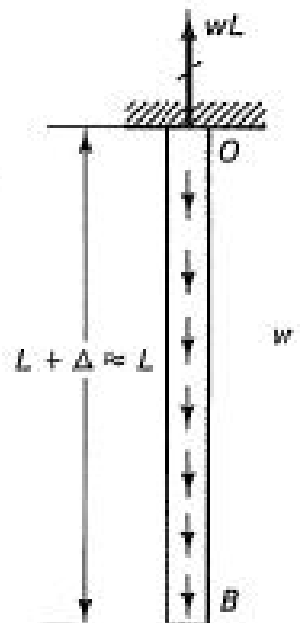
Determine the relative displacement of point D from O. The respective areas for bar segments OB , BC , and CD are 1000, 2000, and 1000 mm². Let $E = 200$ GPa.



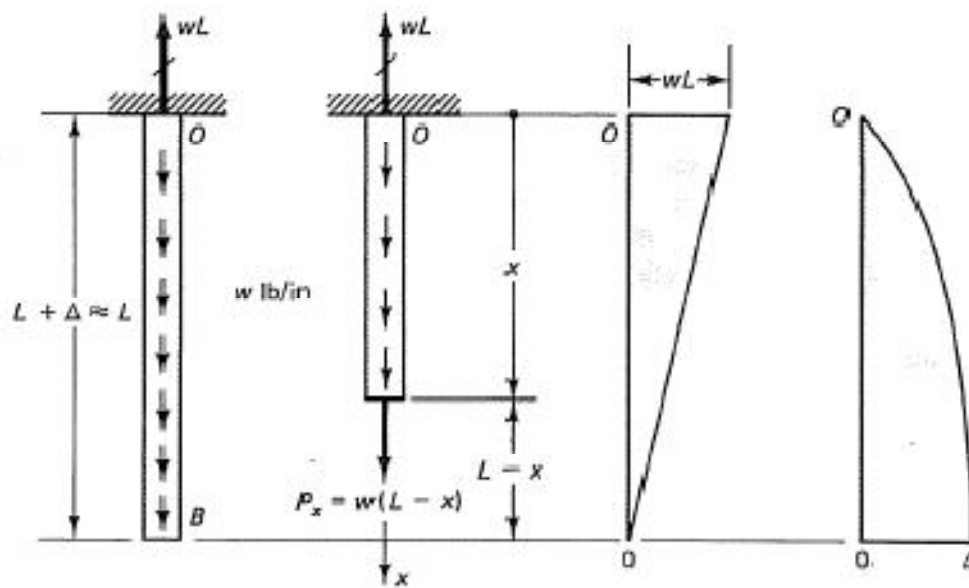
$$\Delta = \sum_i \frac{P_i L_i}{A_i E} = \frac{P_{OB} L_{OB}}{A_{OB} E} + \frac{P_{BC} L_{BC}}{A_{BC} E} + \frac{P_{CD} L_{CD}}{A_{CD} E K}$$

$$\begin{aligned} \Delta &= + \frac{100 \times 10^3 \times 2000}{1000 \times 200 \times 10^3} - \frac{150 \times 10^3 \times 1000}{2000 \times 200 \times 10^3} + \frac{50 \times 10^3 \times 1500}{1000 \times 200 \times 10^3} \\ &= +1.000 - 0.375 + 0.375 = +1.000 \text{ mm} \end{aligned}$$

Example 2-3



Determine the deflection of free end B of elastic bar OB caused by its own weight w lb/in. The constant cross-sectional area is A . Assume that E is given.



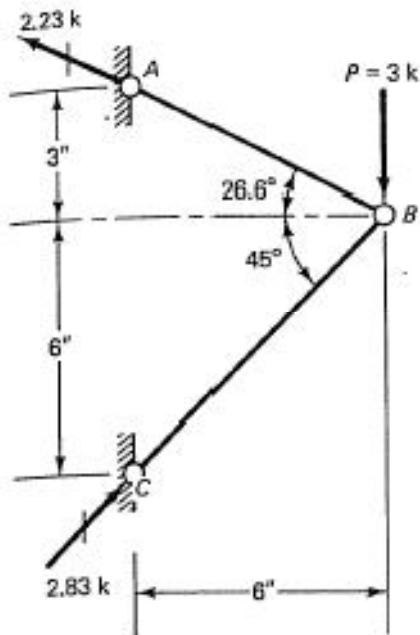
$$\Delta(x) = \int_0^x \frac{P_x dx}{A_x E} = \frac{1}{AE} \int_0^x w(L - x) dx = \frac{w}{AE} \left(Lx - \frac{x^2}{2} \right)$$

$$\Delta = \Delta(L) = \frac{w}{AE} \left(L^2 - \frac{L^2}{2} \right) = \frac{wL^2}{2AE} = \frac{WL}{2AE}$$

If a concentrated force P , in addition to the bar's own weight, were acting on bar OB at end B , the total deflection due to the two causes would be obtained by superposition as

$$\Delta = \frac{PL}{AE} + \frac{WL}{2AE} = \frac{[P + (W/2)]L}{AE}$$

Example 2-4



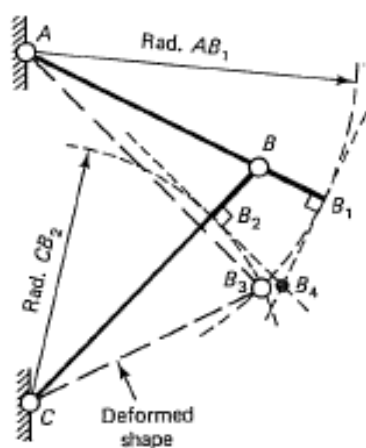
Determine the deflection of point B . Also determine the vertical stiffness of the bracket at B . Assume that the members are made of 2024-T4 aluminum alloy.

$$\Delta_{AB} = \left[\frac{PL}{AE} \right]_{AB} = \left[\sigma \frac{L}{E} \right]_{AB} = \frac{17.8 \times 6.71}{10.6 \times 10^3} = 11.3 \times 10^{-3} \text{ in}$$

(elongation)

$$\Delta_{BC} = - \frac{12.9 \times 8.29}{10.6 \times 10^3} = -10.3 \times 10^{-3} \text{ in}$$

(contraction)



$$\Delta_{BC} = \Delta \cos \theta_2 \quad \text{and} \quad \Delta_{AB} = \Delta \cos \theta_1$$

$$\frac{\cos \theta_2}{\cos \theta_1} = \frac{\Delta_{BC}}{\Delta_{AB}} = \frac{10.3 \times 10^{-3}}{11.3 \times 10^{-3}} = 0.912$$

$$\theta_2 = 180^\circ - 45^\circ - 26.6^\circ - \theta_1 = 108.4^\circ - \theta_1$$

$$\cos \theta_2 = \cos 108.4^\circ \cos \theta_1 + \sin 108.4^\circ \sin \theta_1$$

$$\frac{\cos \theta_2}{\cos \theta_1} = \cos 108.4^\circ + \sin 108.4^\circ \tan \theta_1 = 0.912$$

$$\tan \theta_1 = 1.29 \quad \text{and} \quad \theta_1 = 52.2^\circ$$

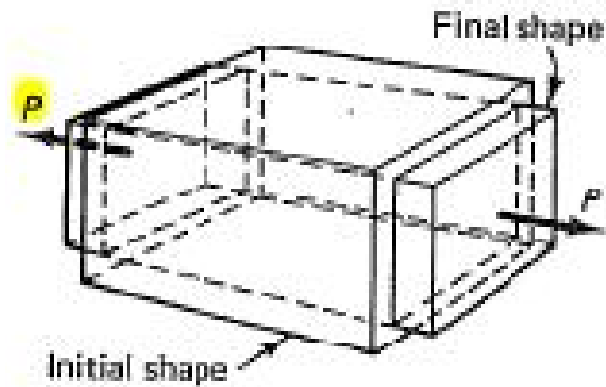
$$\Delta = \Delta_{AB} / \cos \theta_1 = 18.4 \times 10^{-3} \text{ in}$$

forming an angle of 11.2° with the vertical.

Since $\Delta_{\text{ver}} = \Delta \cos 11.2^\circ = 18.0 \times 10^{-3}$ in, the vertical stiffness of the bracket is given by the spring constant

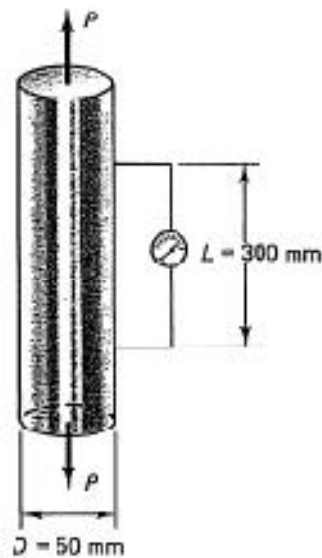
$$k = \frac{P}{\Delta_{\text{ver}}} = \frac{3}{18.0 \times 10^{-3}} = 167 \text{ kips/in}$$

6. Poisson's Ratio



$$\nu = - \frac{\text{lateral strain}}{\text{axial strain}}$$

Example 2-5



Consider a carefully conducted experiment where an aluminum bar of 50-mm diameter is stressed in a testing machine. At a certain instant the applied force P is 100 kN, while the measured elongation of the rod is 0.219 mm in a 300-mm gage length, and the diameter's dimension is decreased by 0.01215 mm. Calculate the two physical constants ν and E of the material.

Transverse or lateral strain:

$$\epsilon_t = \frac{\Delta_t}{D} = -\frac{0.01215}{50} = -0.000243 \text{ mm/mm}$$

Axial strain:

$$\epsilon_a = \frac{\Delta}{L} = +\frac{0.219}{300} = 0.00073 \text{ mm/mm}$$

Poisson's Ratio:

$$\nu = -\frac{\epsilon_t}{\epsilon_a} = -\frac{(-0.000243)}{0.00073} = 0.333$$

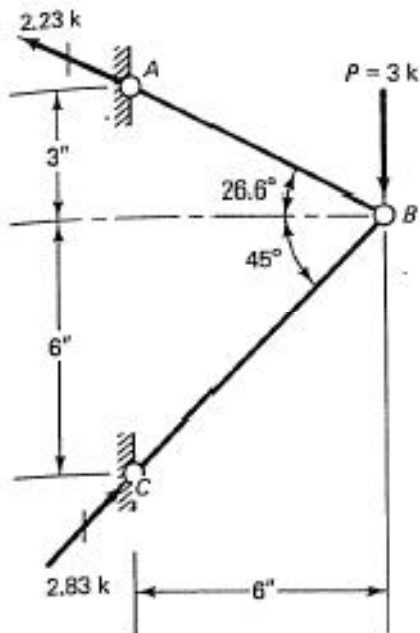
$$E = \frac{PL}{A\Delta} = \frac{100 \times 10^3 \times 300}{1960 \times 0.219} = 70 \times 10^3 \text{ N/mm}^2 = 70 \text{ GPa}$$

7. Thermal Strain and Deformation

$$\epsilon_T = \alpha(T - T_0)$$

$$\Delta_T = \alpha(\delta T)L$$

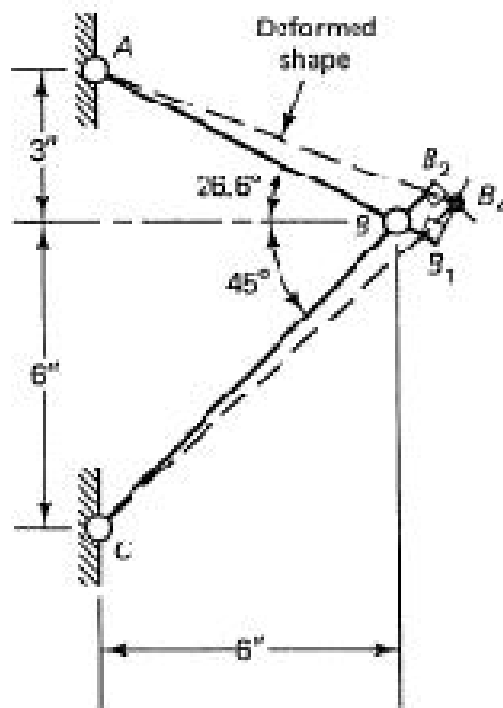
Example 2-7



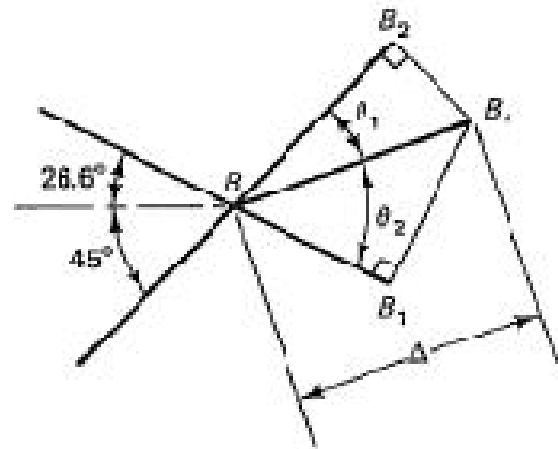
Determine the displacement of point B caused by an increase in temperature of 100°F. Given, coefficient of expansion is $12.9 \times 10^{-6}/^\circ\text{F}$.

$$\Delta_{AB} = 12.9 \times 10^{-6} \times 100 \times 6.71 = 8.656 \times 10^{-3} \text{ in}$$

$$\Delta_{BC} = 12.9 \times 10^{-6} \times 100 \times 8.49 = 10.95 \times 10^{-3} \text{ in}$$



(a)



(b)

$$\Delta_T \cos \theta_2 = \Delta_{AB} \quad \text{and} \quad \Delta_T \cos \theta_1 = \Delta_{BC}$$

$$\frac{\cos \theta_2}{\cos \theta_1} = \frac{\Delta_{AB}}{\Delta_{BC}} = \frac{8.656 \times 10^{-3}}{10.95 \times 10^{-3}} = 0.7905$$

Here, however, $\theta_2 = 45^\circ + 26.6^\circ - \theta_1 = 71.6^\circ - \theta_1$; therefore,

$$\cos \theta_2 = \cos 71.6^\circ \cos \theta_1 + \sin 71.6^\circ \sin \theta_1$$

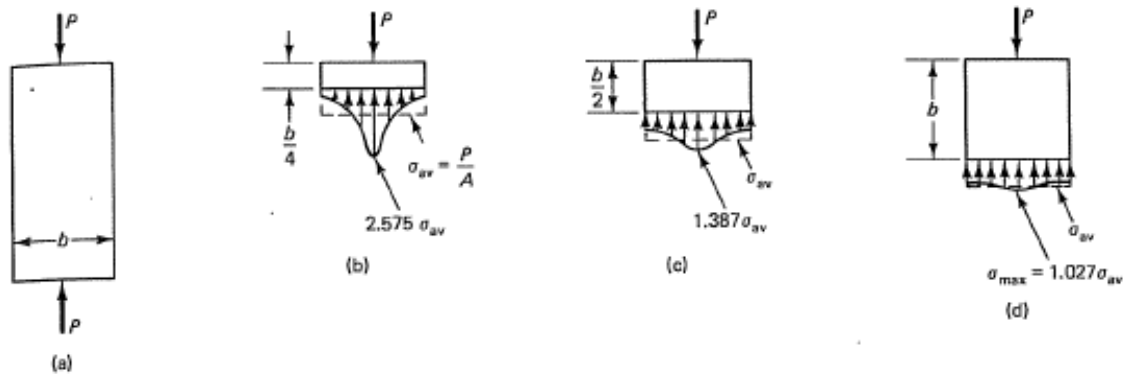
$$\frac{\cos \theta_2}{\cos \theta_1} = \cos 71.6^\circ + \sin 71.6^\circ \tan \theta_1 = 0.7905$$

$$\tan \theta_1 = 0.500$$

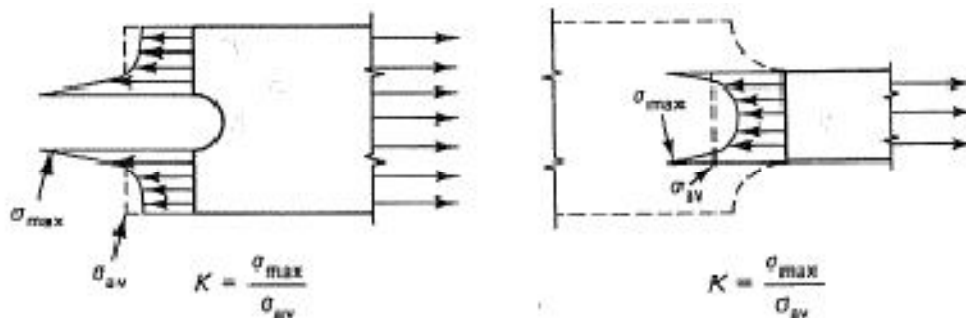
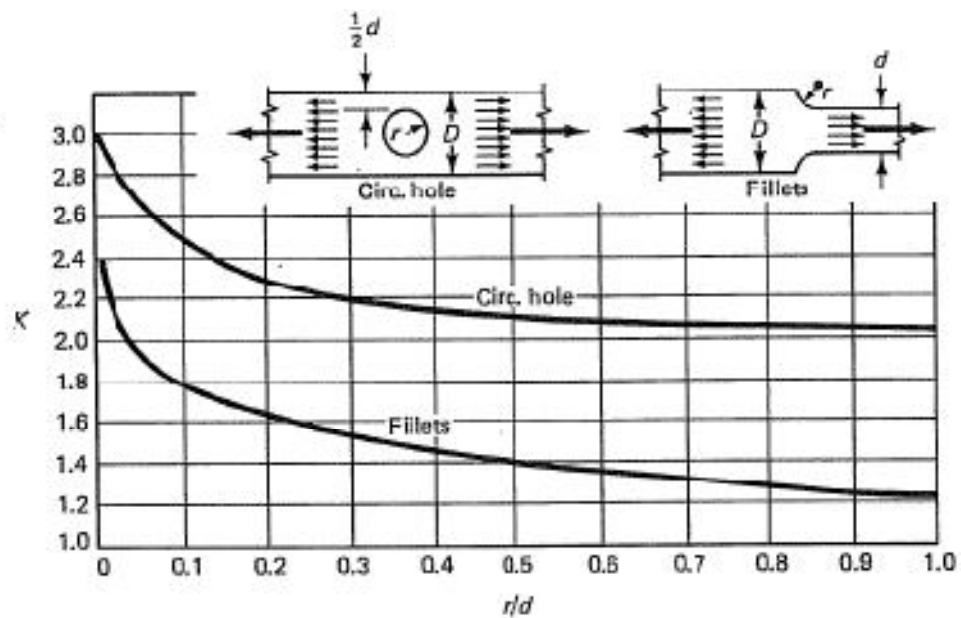
$$\theta_1 = 26.6^\circ$$

$$\Delta_T = \Delta_{BC} / \cos \theta_1 = 12.2 \times 10^{-3} \text{ in}$$

8. Saint-Venant's Principle and Stress Concentrations

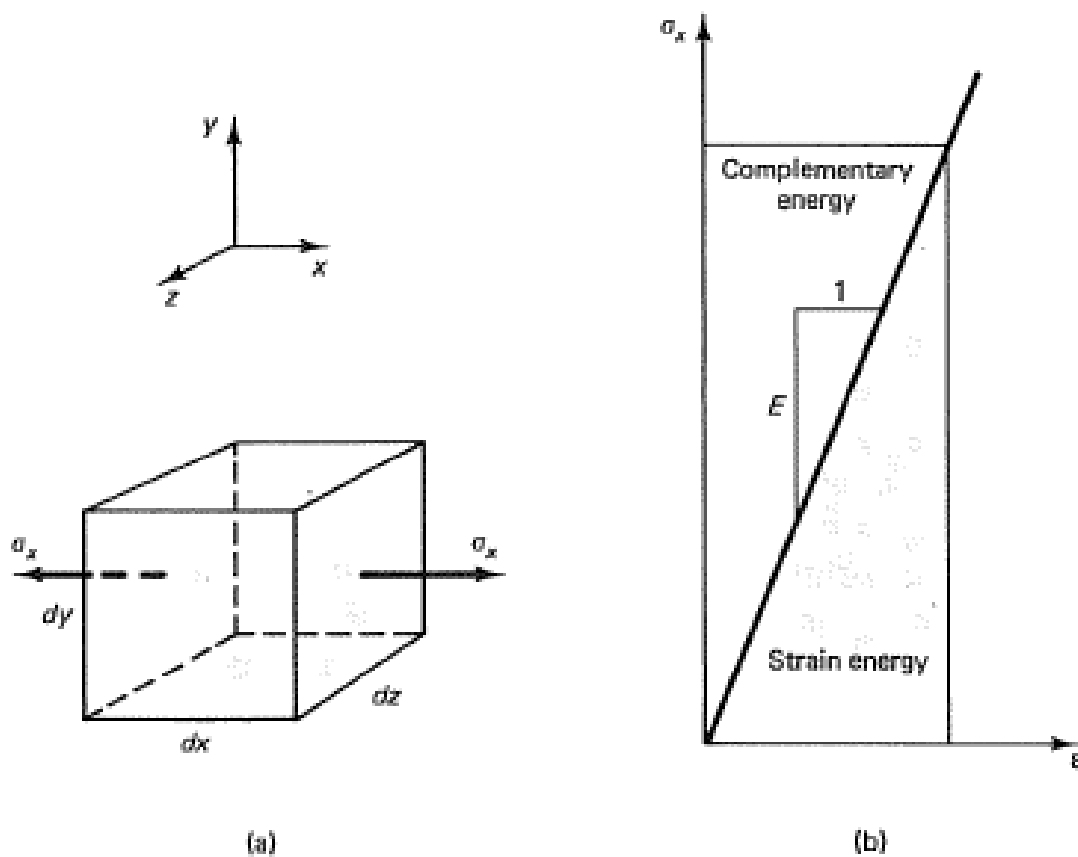


The manner of force application on stresses is important only in the vicinity of the region where the force is applied. This also holds true for the disturbances caused by changes in cross section.



9. Elastic Strain Energy for Uniaxial Stress

In mechanics, energy is defined as the capacity to do work, and work is the product of a force times the distance in the direction that the force moves. In solid deformable bodies, stresses multiplied by their respective areas are forces, and deformations are distances. The product of these two quantities is the *internal work* done in a body by externally applied forces. This internal work is stored in an elastic body as the *internal elastic energy* of deformation, or the *elastic strain energy*.



$$dU = \underbrace{\frac{1}{2}\sigma_x dy dz}_{\text{average force}} \times \underbrace{\epsilon_x dx}_{\text{distance}} = \frac{1}{2}\sigma_x \epsilon_x dx dy dz = \frac{1}{2}\sigma_x \epsilon_x dV$$

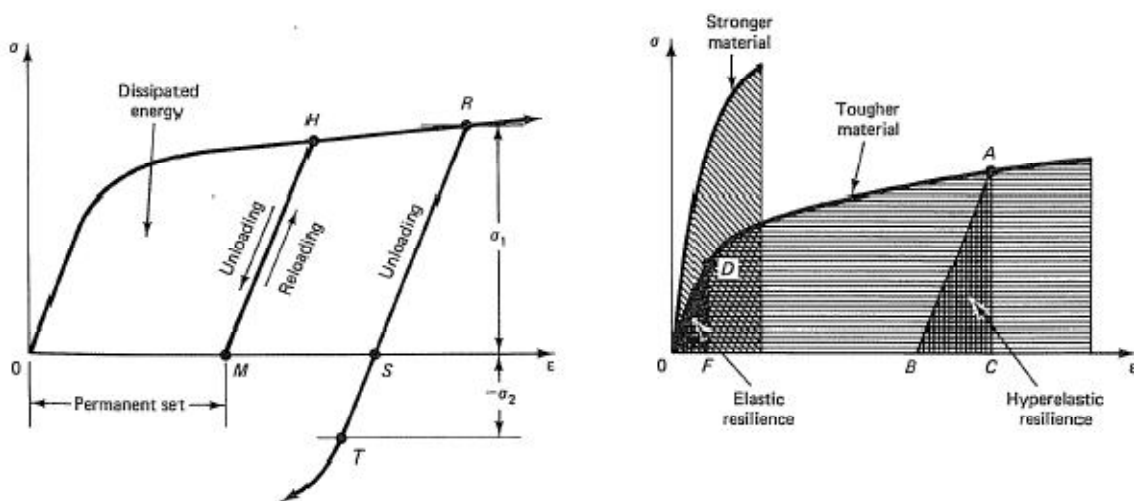
work

One obtains the strain energy stored in an elastic body per unit volume of the material, or its *strain-energy density* U_0 .

$$U_o = \frac{dU}{dV} = \frac{\sigma_x \epsilon_x}{2}$$

$$U_o = \frac{dU}{dV} = \frac{E \epsilon_x^2}{2} = \frac{\sigma_x^2}{2E}$$

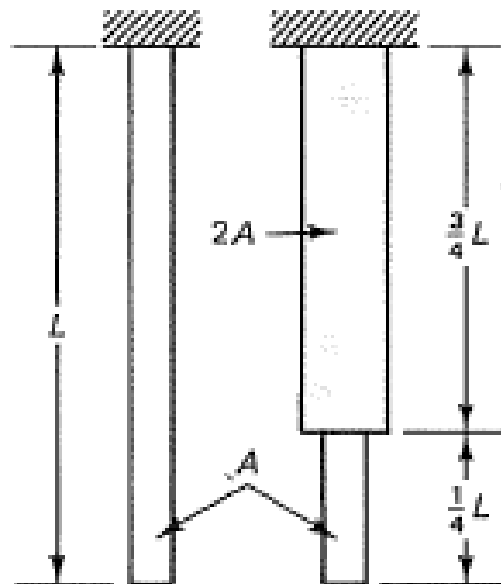
$$U = \int_{vol} \frac{\sigma_x^2}{2E} dV$$



For a particular material, substitution into Eq. 2-22 of the value of the stress at the proportional limit gives an index of the material's ability to store or absorb energy without permanent deformation. The quantity so found is called the *modulus of resilience* and is used to differentiate materials for applications where energy must be absorbed by members.

The area under a complete stress-strain diagram gives a measure of a material's ability to absorb energy up to fracture and is called its *toughness*. The larger the total area under the stress-strain diagram, the tougher the material.

Example 2-9



Two elastic bars, whose proportions are shown in the figure above, are to absorb the same amount of energy delivered by axial forces. Neglecting stress concentrations, compare the stresses in the two bars. The cross-sectional area of the left bar is A , and that of the right bar is A and $2A$ as shown.

$$U_1 = \int_V \frac{\sigma_1^2}{2E} dV = \frac{\sigma_1^2}{2E} \int_V dV = \frac{\sigma_1^2}{2E} (AL)$$

$$\begin{aligned} U_2 &= \int_V \frac{\sigma^2}{2E} dV = \frac{\sigma_2^2}{2E} \int_{\text{lower part}} dV + \frac{(\sigma_2/2)^2}{2E} \int_{\text{upper part}} dV \\ &= \frac{\sigma_2^2}{2E} \left(\frac{AL}{4} \right) + \frac{(\sigma_2/2)^2}{2E} \left(2A \frac{3L}{4} \right) = \frac{\sigma_2^2}{2E} \left(\frac{5}{8} AL \right) \end{aligned}$$

$$\frac{\sigma_1^2}{2E} (AL) = \frac{\sigma_2^2}{2E} \left(\frac{5}{8} AL \right) \quad \text{or} \quad \sigma_2 = 1.265\sigma_1$$

10. Deflections by the Energy Method

$$U = W_e$$

Example 2-10

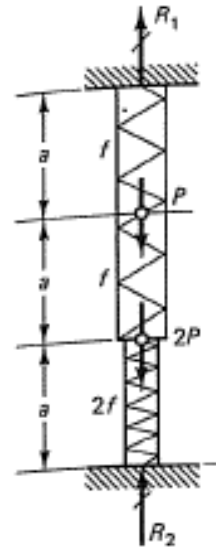
Find the deflection of the free end of an elastic rod of constant cross-sectional area A and length L due to axial force P applied at the free end.

$$U = \frac{\sigma_1^2}{2E} AL = \frac{P^2 L}{2AE}$$

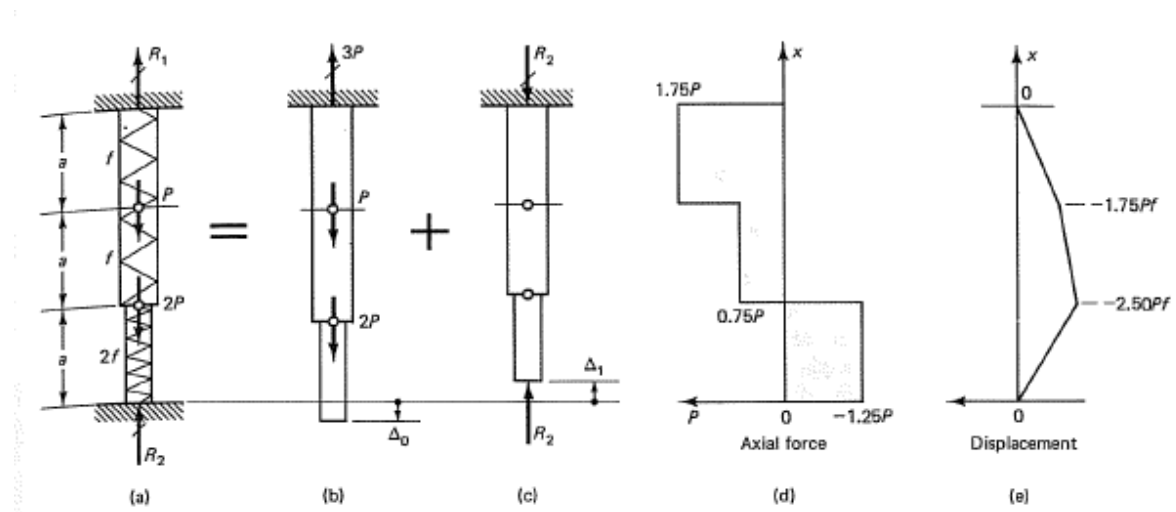
$$\frac{P\Delta}{2} = \frac{P^2 L}{2AE} \quad \text{and} \quad \Delta = \frac{PL}{AE}$$

11. Statically Indeterminate Systems: Force Method of Analysis

Example 2-12



Determine the reactions and plot the axial force and the axial displacement diagrams for the bar.

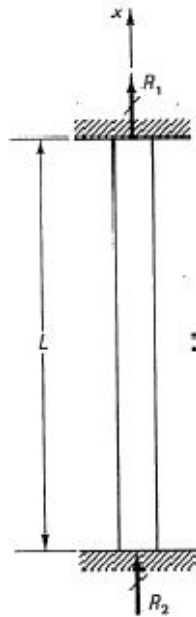


$$\Delta_0 = \sum_i f_i P_i = -2fP - f(2P + P) = -5fP$$

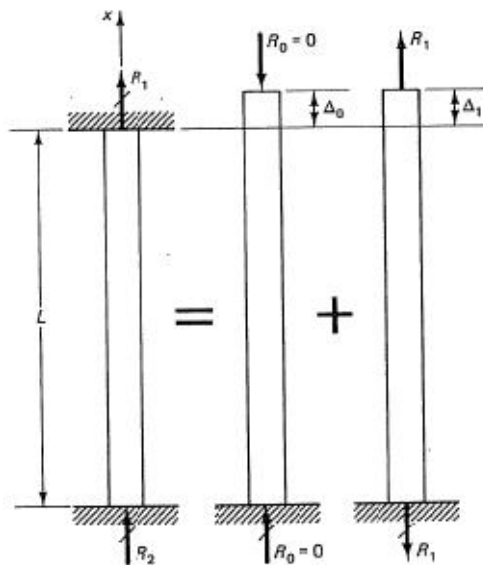
$$\Delta_1 = (2f + f + f)R_2 = 4fR_2$$

$$\Delta_1 + \Delta_2 = 0, \quad R_2 = 1.25P$$

Example 2-13



If the bar temperature increases by δT , what axial force develops in the bar? AE for the bar is constant.



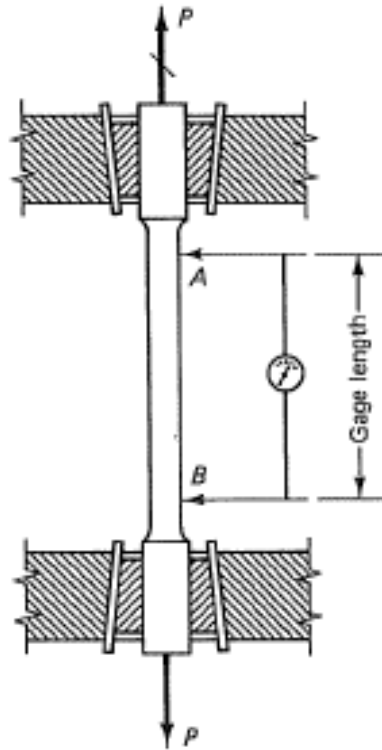
$$\Delta_0 = \alpha(\delta T)L$$

$$\Delta_1 = R_1 j = \frac{R_1 L}{AE}$$

$$\Delta_0 + \Delta_1 = 0, \quad R_1 = -\alpha(\delta T)AE$$

Lecture 3: Generalized Hooke's Law

1. Normal Strain



$$\epsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0}$$

This expression defines the *extensional strain*. Since this strain is associated with the normal stress, it is usually called the *normal strain*. It is a dimensionless quantity, but it is customary to refer to it as having the dimensions of in/in, m/m, or $\mu\text{m}/\text{m}$ (microstrain). Sometimes it is given as a percentage.

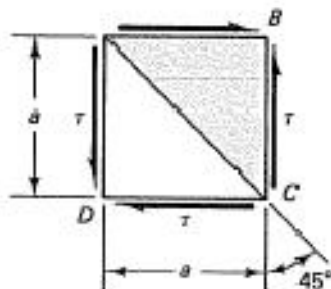
Example 3-2

A 50 mm cube of steel is subjected to a uniform pressure of 200 MPa acting on all faces. Determine the change in dimension between two parallel faces of the cube. Let $E = 200$ GPa and $\nu = 0.25$.

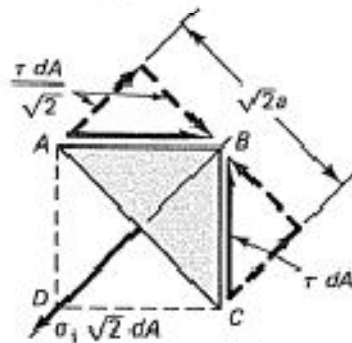
$$\begin{aligned}\epsilon_x &= \frac{(-200)}{200 \times 10^3} - \left(\frac{1}{4}\right) \frac{(-200)}{200 \times 10^3} - \left(\frac{1}{4}\right) \frac{(-200)}{200 \times 10^3} \\ &= -5 \times 10^{-4} \text{ mm/mm} \\ \Delta_x &= \epsilon_x L_x = -5 \times 10^{-4} \times 50 = -0.025 \text{ mm (contraction)}\end{aligned}$$

In this case $\Delta_x = \Delta_y = \Delta_z$.

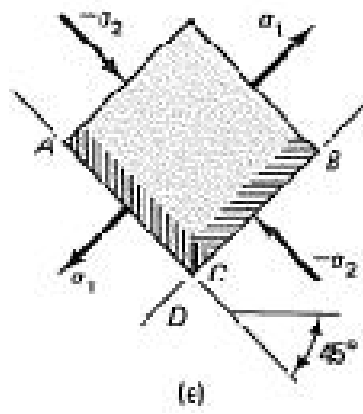
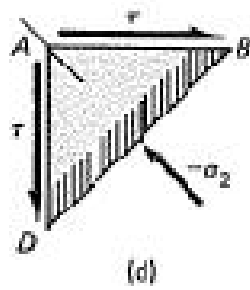
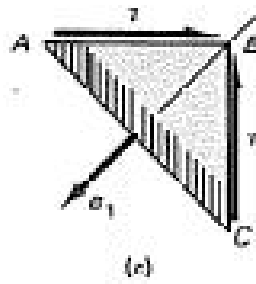
2. E, G and ν Relationship



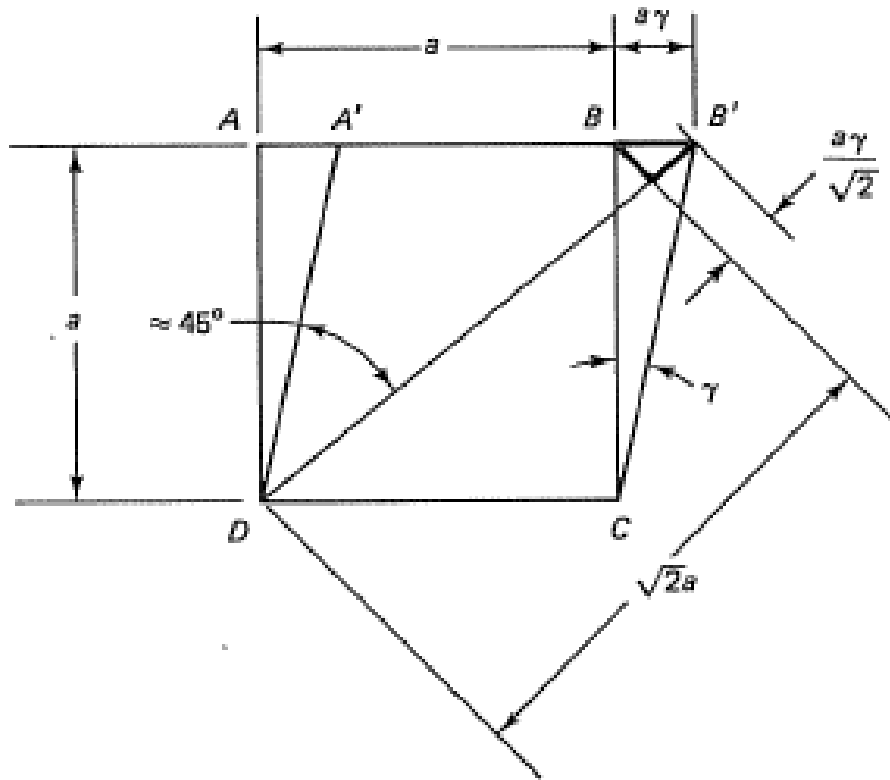
(a)



(b) Force diagram



$$\sigma_1 = -\sigma_2 = \tau$$



$$\epsilon_{45^\circ} = \frac{\tau}{2G}$$

$$\epsilon_{45^\circ} = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E} = \frac{\tau}{E} (1 + \nu)$$

$$G = \frac{E}{2(1 + \nu)}$$

3. Dilation and Bulk Modulus

After subtracting the initial volume from the volume of the strained element, the change in volume is determined. This is

$$(1 + \epsilon_x) dx (1 + \epsilon_y) dy (1 + \epsilon_z) dz - dx dy dz = (\epsilon_x + \epsilon_y + \epsilon_z) dx dy dz$$

e , the change in volume per unit volume, often referred to as *dilatation*, is defined as

$$e = \epsilon_x + \epsilon_y + \epsilon_z$$

$$e = \epsilon_x + \epsilon_y + \epsilon_z = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y + \sigma_z)$$

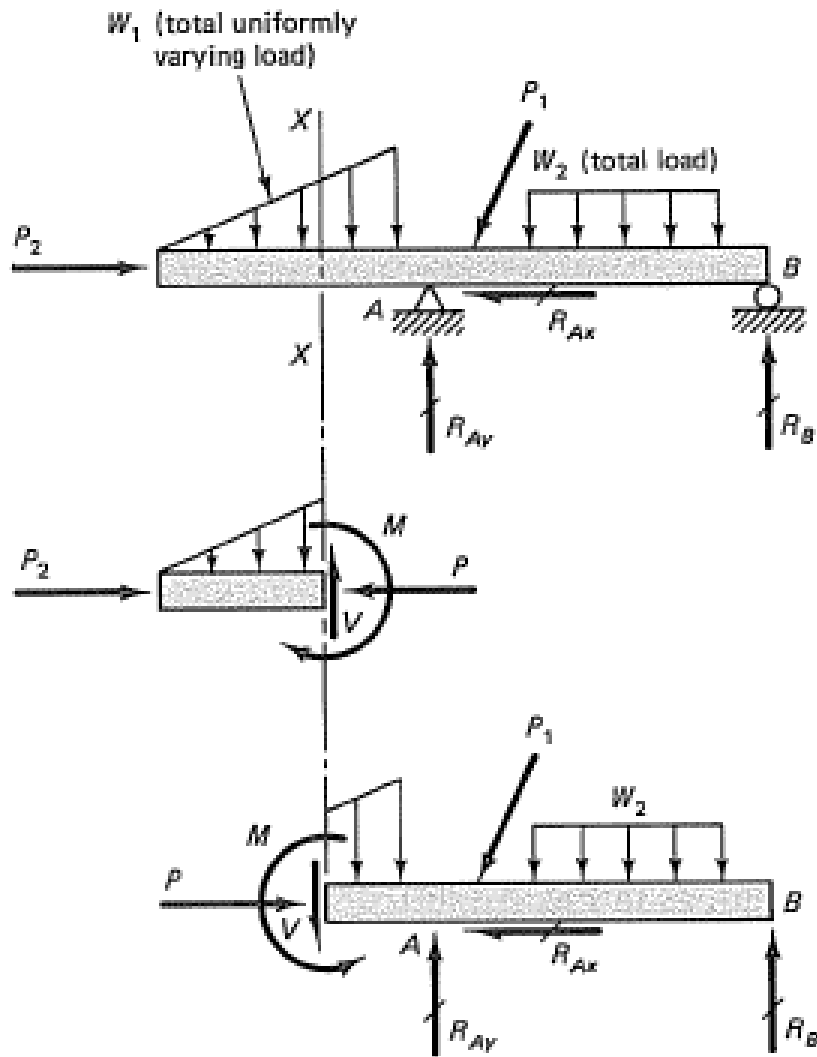
If an elastic body is subjected to hydrostatic pressure of uniform intensity p , so that $\sigma_x = \sigma_y = \sigma_z = -p$,

$$e = -\frac{3(1 - 2\nu)}{E} p \quad \text{or} \quad \frac{-p}{e} = k = \frac{E}{3(1 - 2\nu)}$$

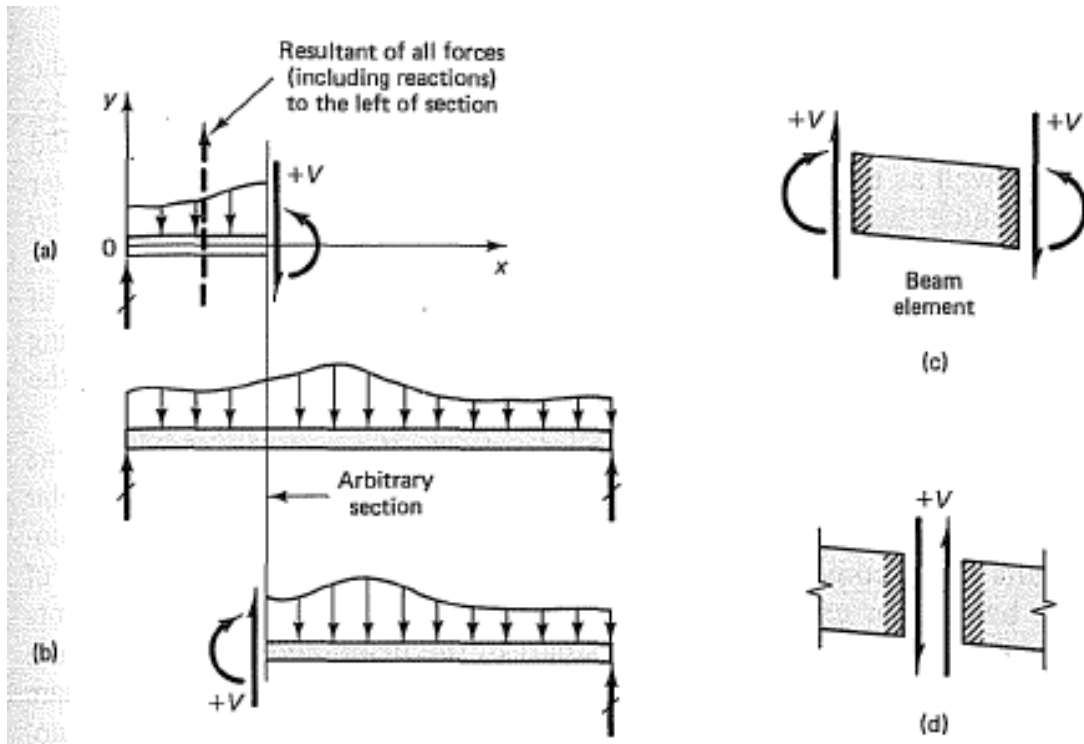
The quantity k represents the ratio of the hydrostatic compressive stress to the decrease in volume and is called the *modulus of compression* or *bulk modulus*.

Lecture 4: Axial Force, Shear and Bending Moment

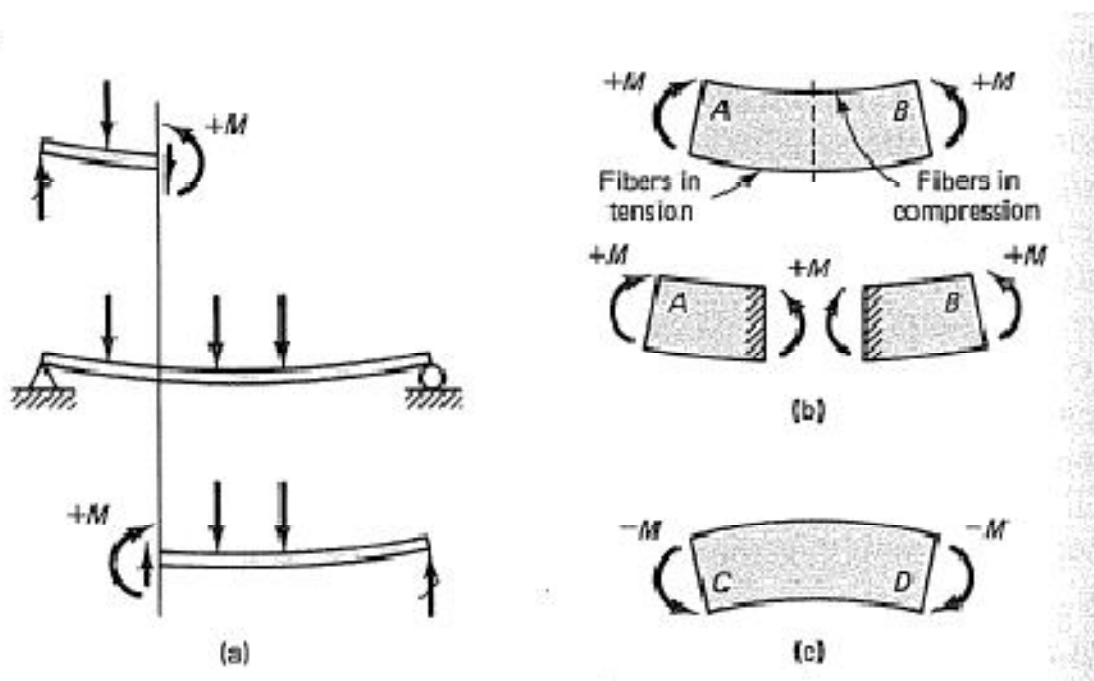
1. Axial Force in Beams



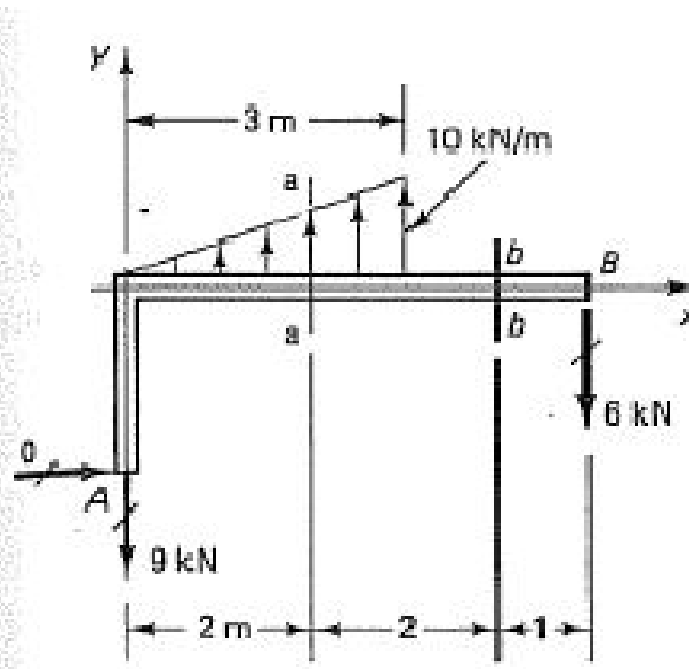
2. Shear in Beams



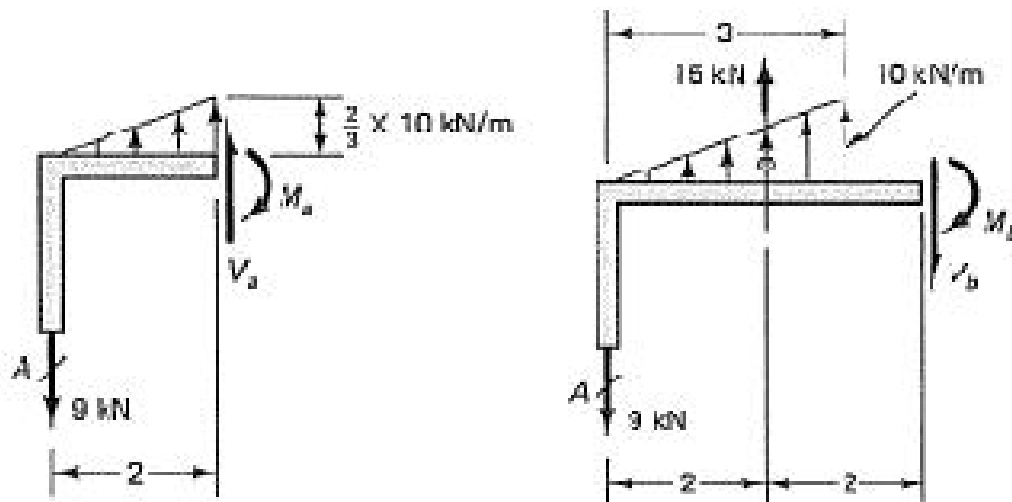
3. Bending Moment in Beams



Example 5-4



Determine the internal system of forces at sections a-a and b-b.

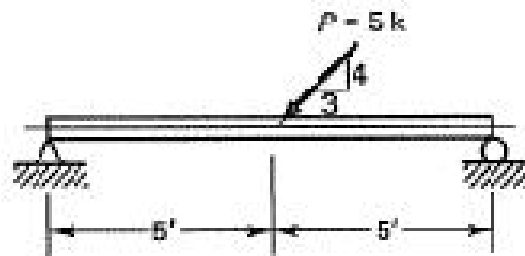


$$V_a = -9 + \frac{1}{2} \times 2 \times \frac{2}{3} \times 10 = -2.33 \text{ kN}$$

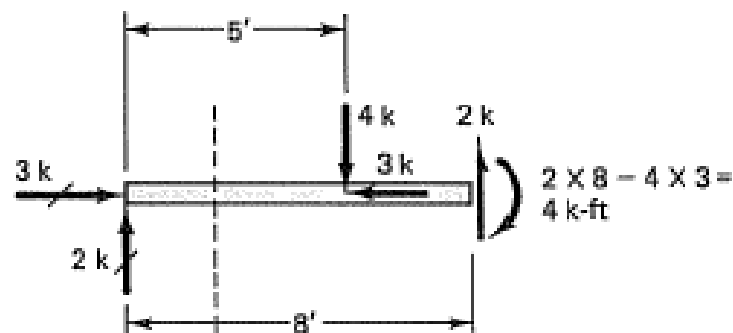
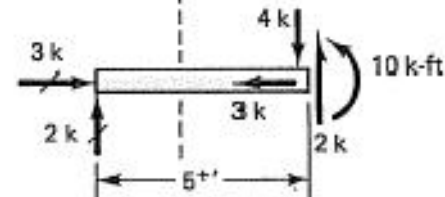
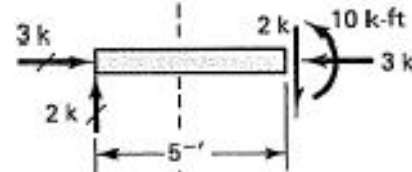
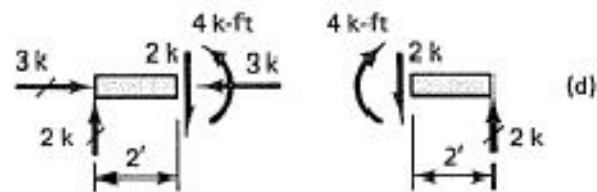
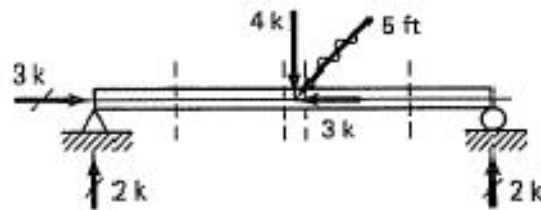
$$M_a = -9 \times 2 + \frac{1}{2} \times 2 \times \frac{2}{3} \times 10 \times \frac{1}{3} \times 2 = -13.6 \text{ kN}\cdot\text{m}$$

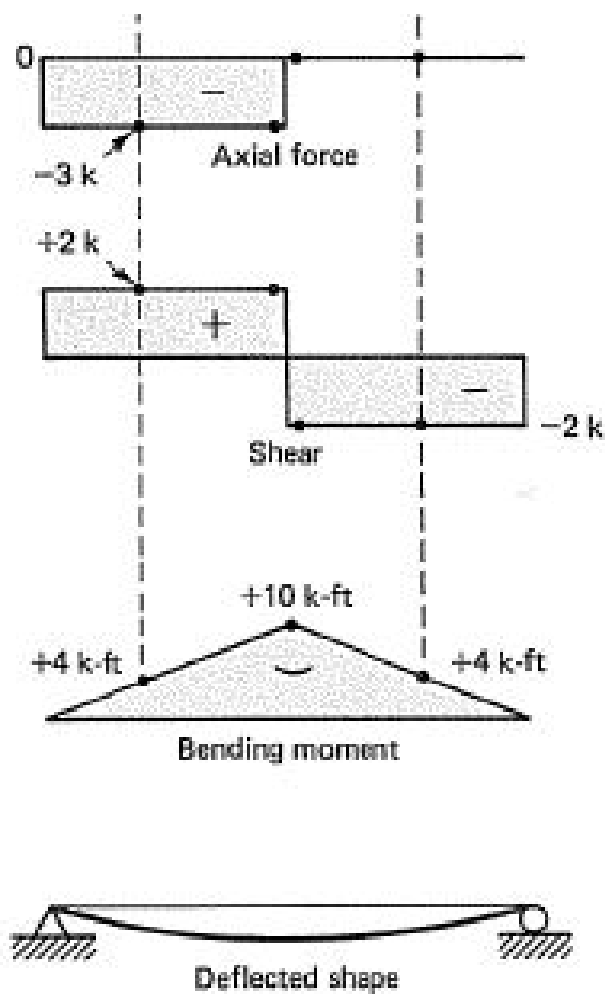
$$V_b = +6 \text{ kN} \quad M_b = -6 \times 1 = -6 \text{ kN}\cdot\text{m}$$

Example 5-5



Construct axial-force, shear, and bending-moment diagrams for the beam.

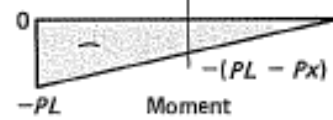
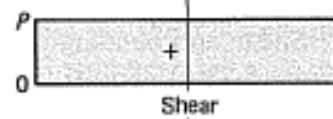
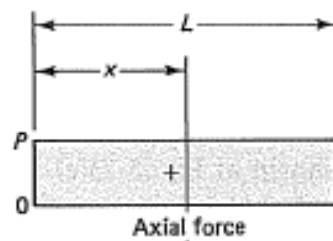
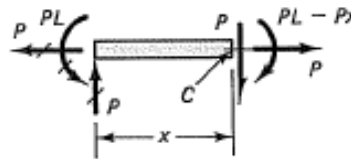
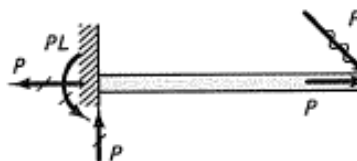




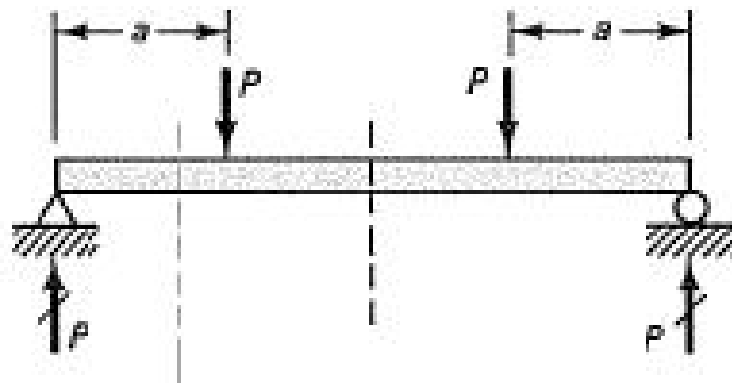
Example 5-6



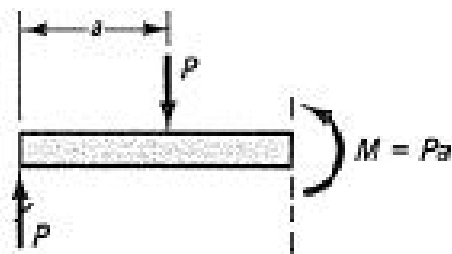
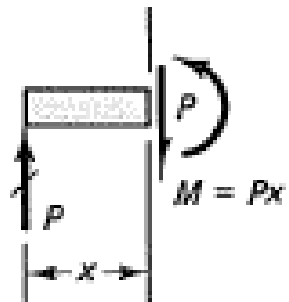
Determine axial-force, shear, and bending-moment diagrams.



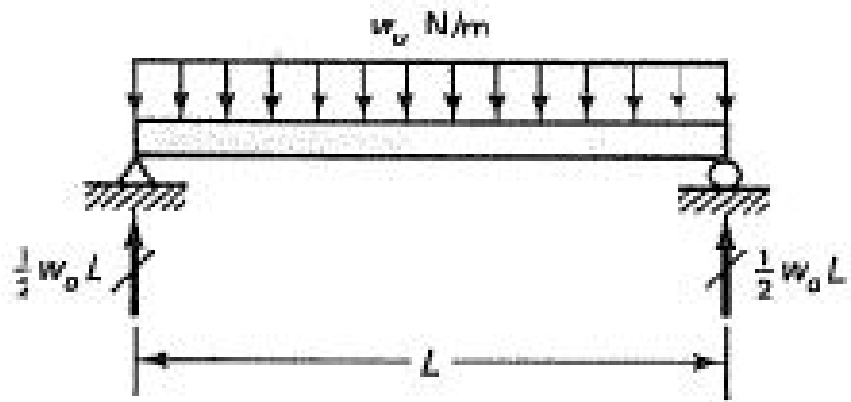
Example 5-7



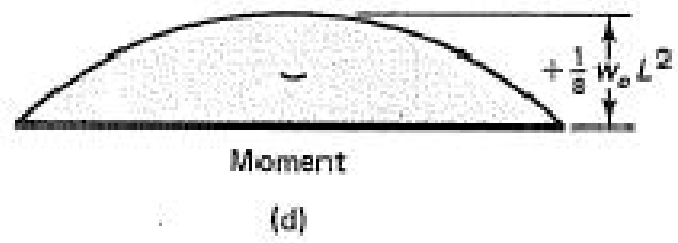
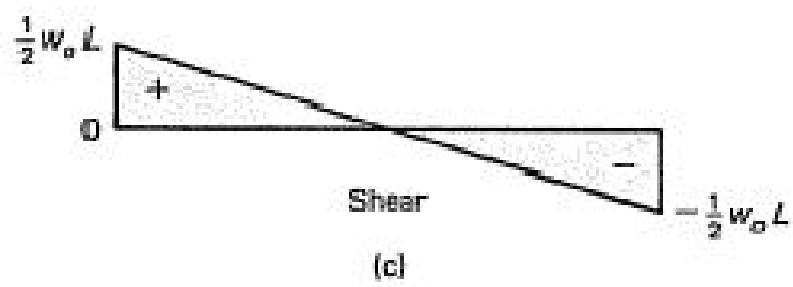
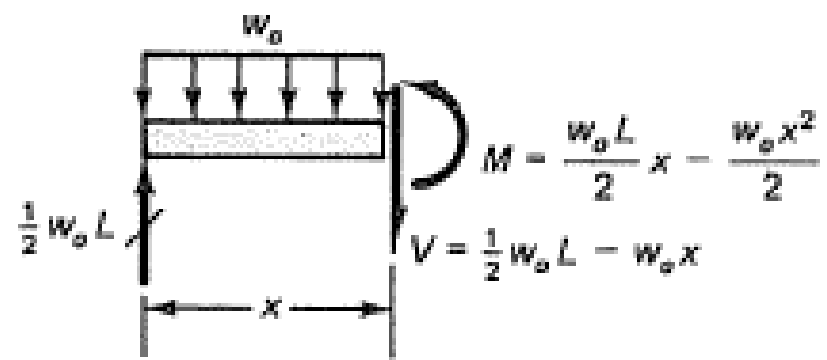
Construct shear and bending-moment diagrams.



Example 5-8

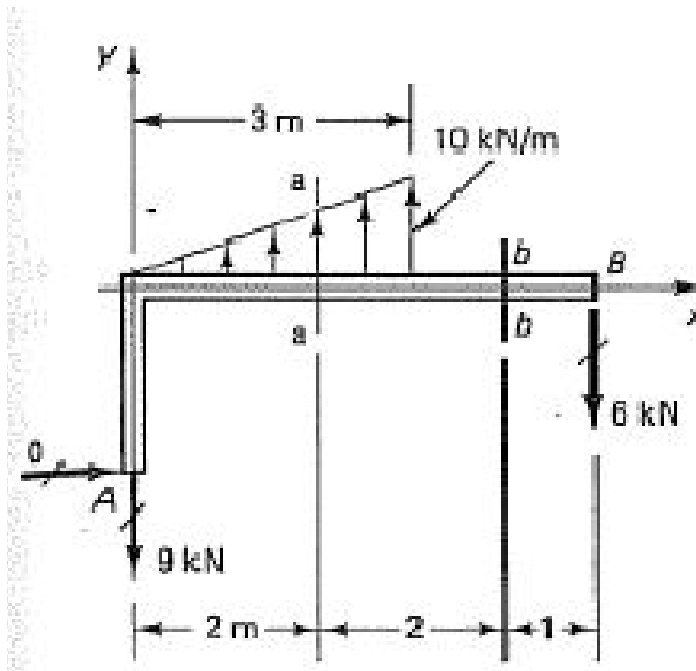


Plot shear and a bending-moment diagrams.

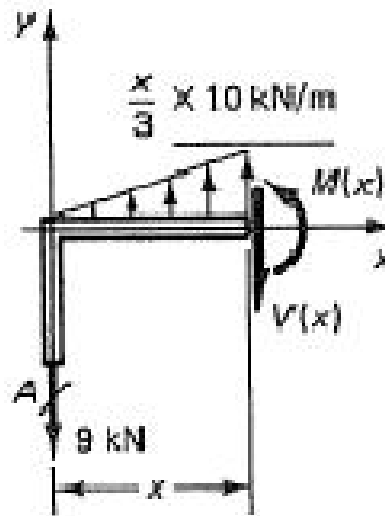


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Example 5-9



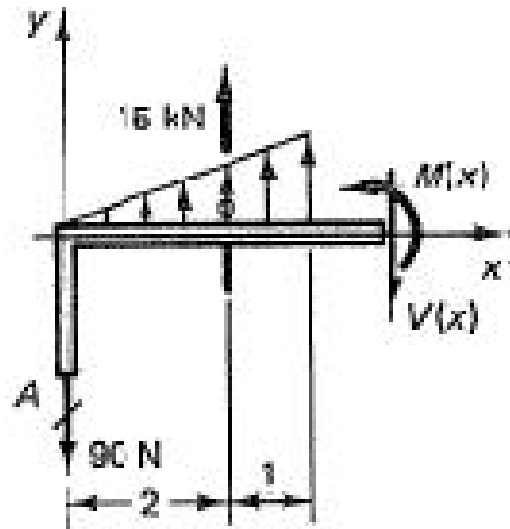
Express the shear V and the bending moment M as a function of x along the horizontal member.



For $0 < x < 3$,

$$V(x) = -9 + \frac{1}{2}x\left(\frac{x}{3} \times 10\right) = -9 + \frac{5}{3}x^2 \text{ kN}$$

$$M(x) = -9x + \frac{1}{2}x\left(\frac{x}{3} \times 10\right)\left(\frac{x}{3}\right) = -9x + \frac{5}{9}x^3 \text{ kN}\cdot\text{m}$$

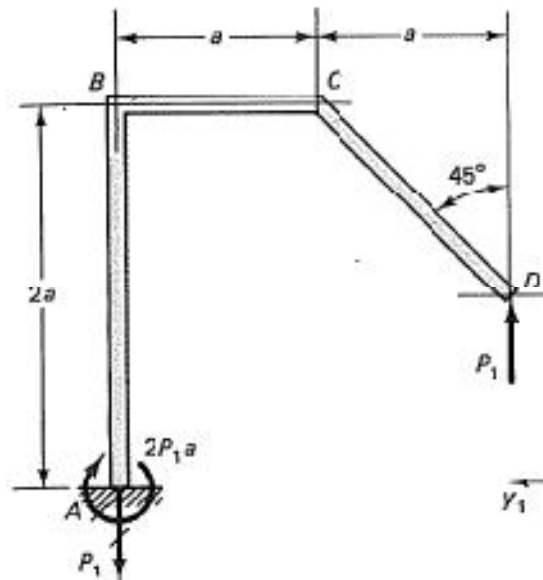


For $3 < x < 5$,

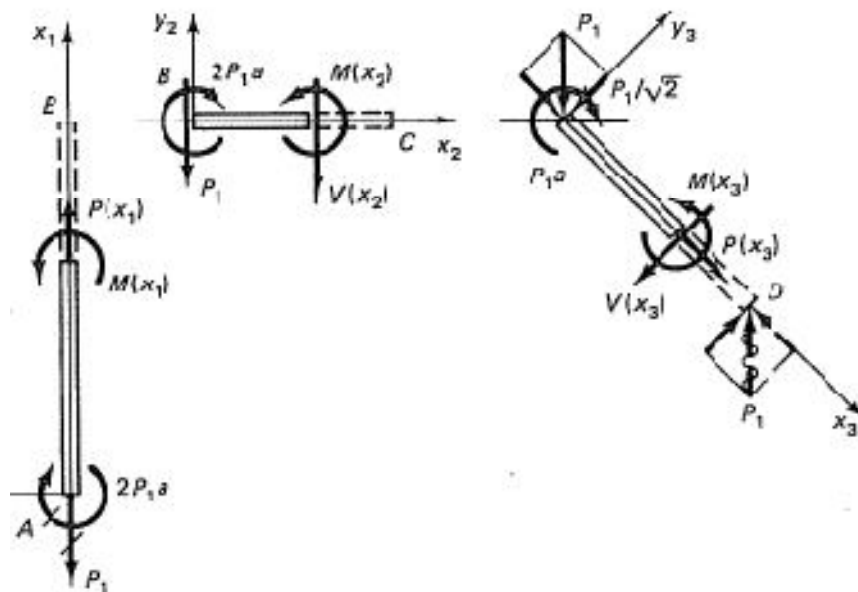
$$V(x) = -9 + 15 = +6 \text{ kN}$$

$$M(x) = -9x + 15(x - 2) = 6x - 30 \text{ kN}\cdot\text{m}$$

Example 5-11



At arbitrary sections, determine the internal forces P , V , and M in the members caused by the application of a vertical force P_1 at D .

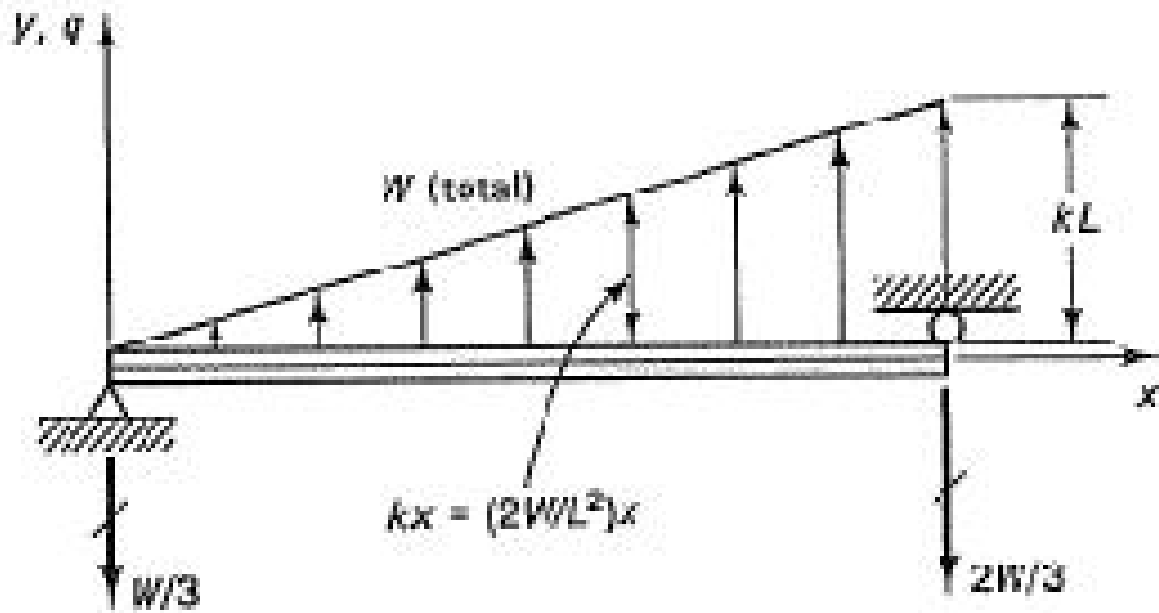


$$P(x_1) = +P_1, V(x_1) = 0 \quad \text{and} \quad M(x_1) = +2P_1a$$

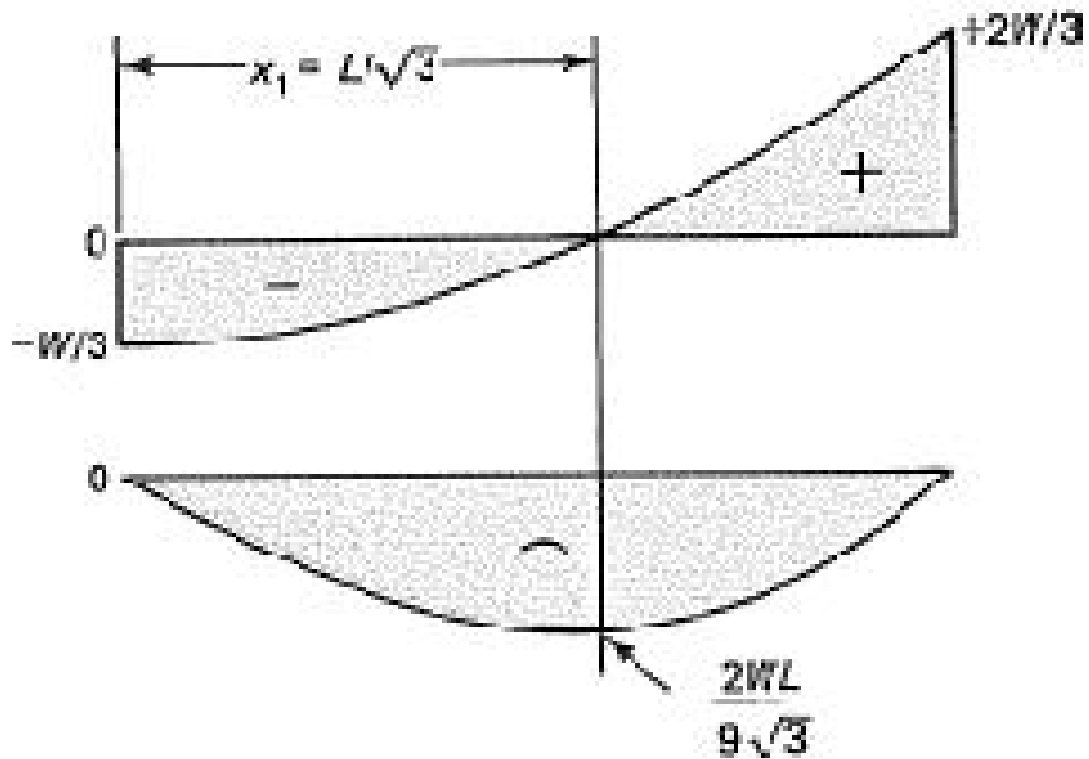
$$P(x_2) = 0, V(x_2) = -P_1 \quad \text{and} \quad M(x_2) = +2P_1a - P_1x_2$$

$$P(x_3) = -P_1/\sqrt{2}, V(x_3) = -P_1/\sqrt{2} \quad \text{and} \quad M(x_3) = +P_1a - P_1x_3/\sqrt{2}$$

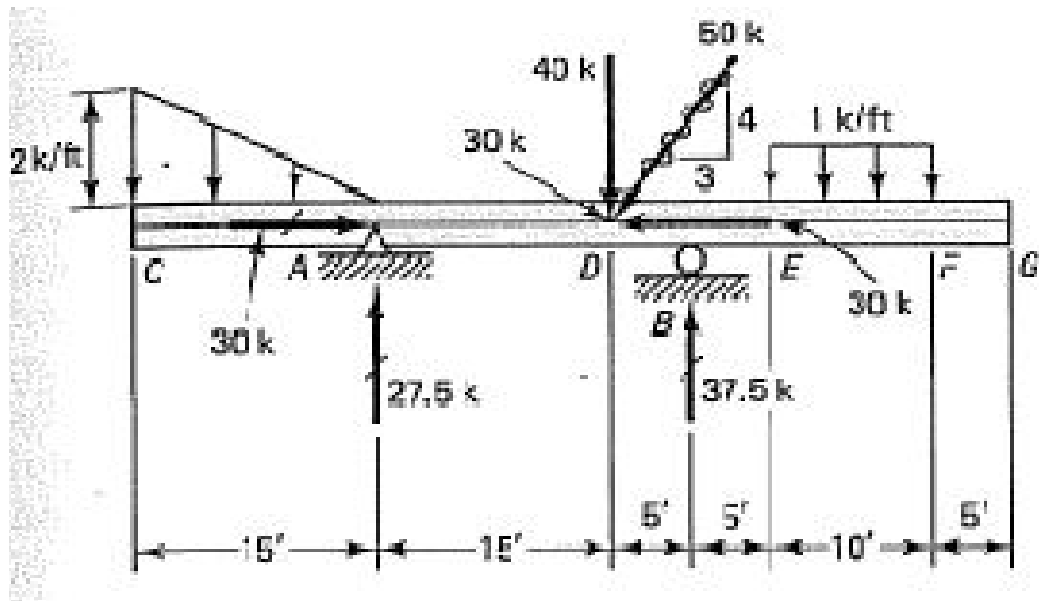
Example 5-14



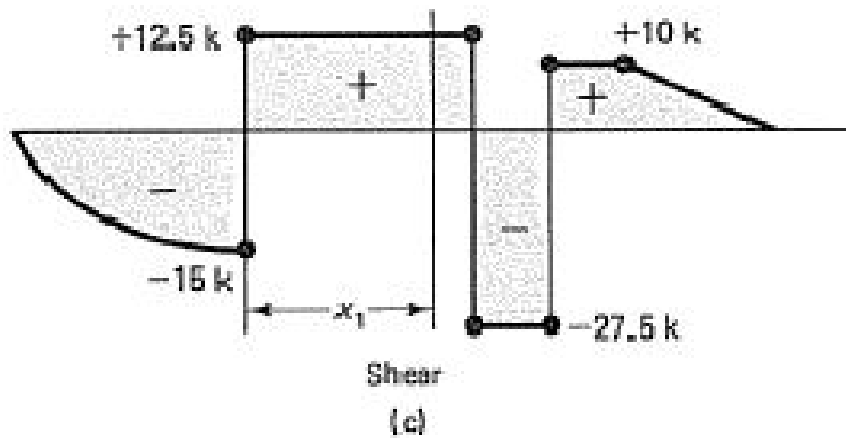
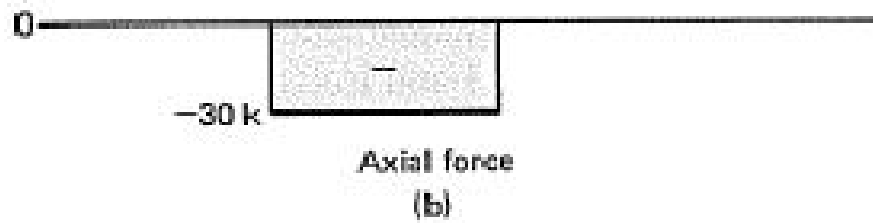
Construct shear and moment diagrams.

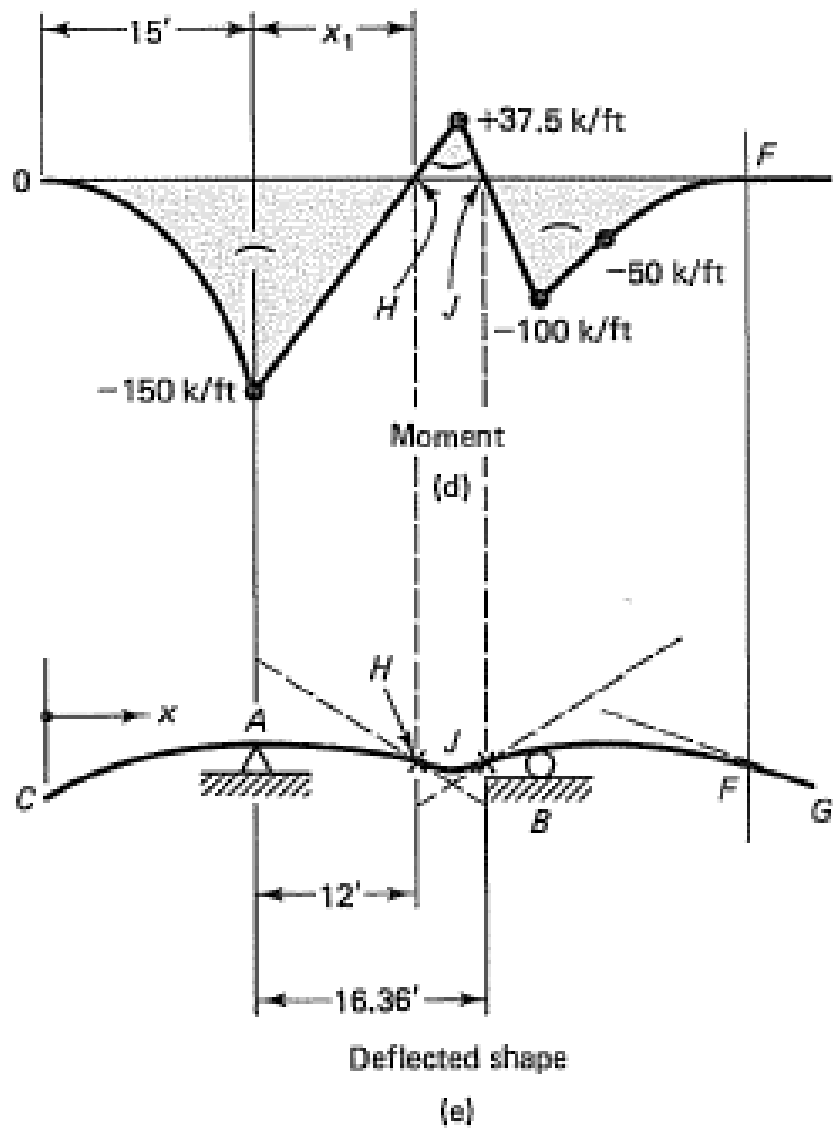


Example 5-15

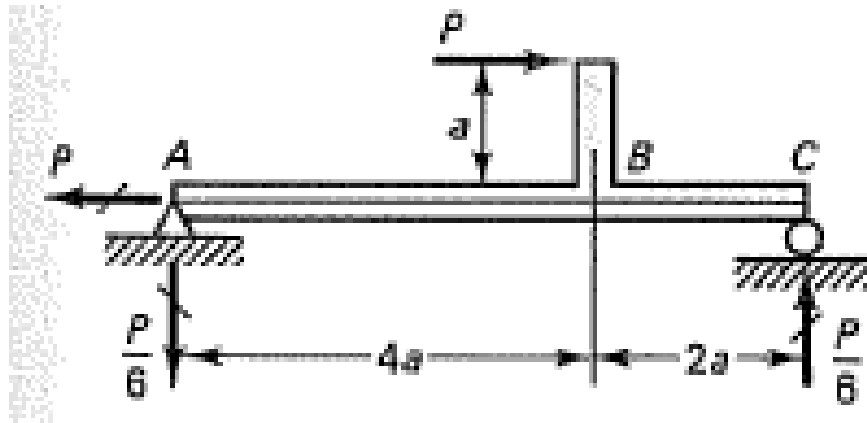


Construct shear and moment diagrams.

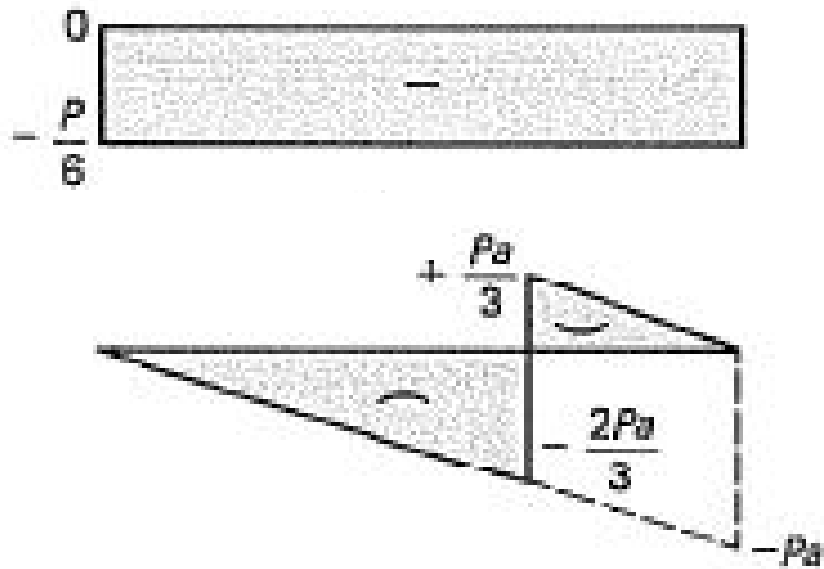




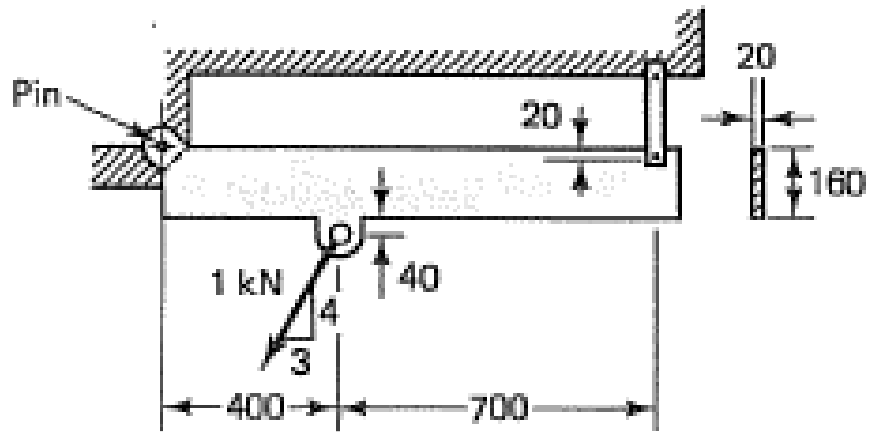
Example 5-16



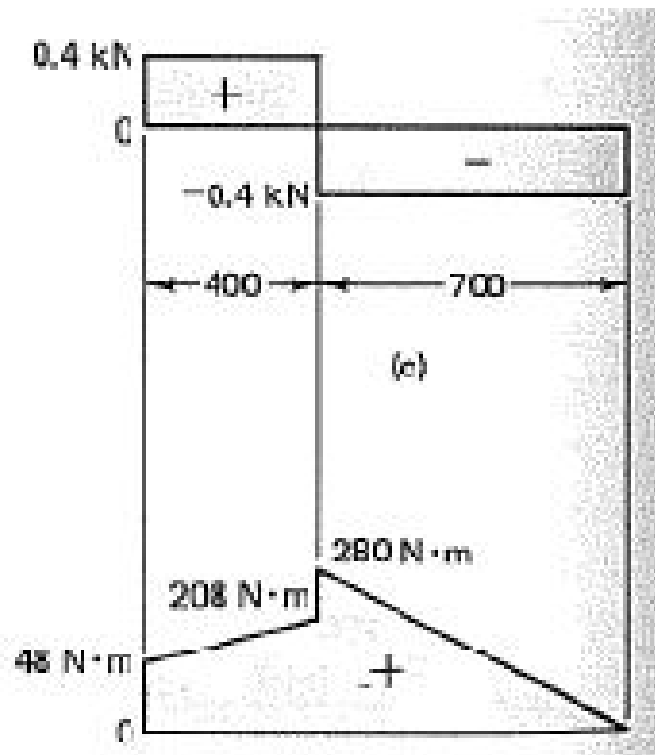
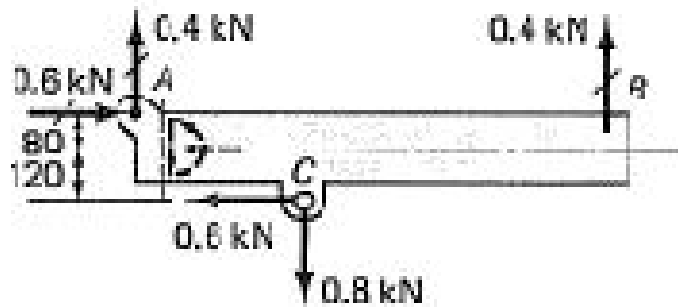
Construct the bending-moment diagram for the horizontal beam.



Example 5-17

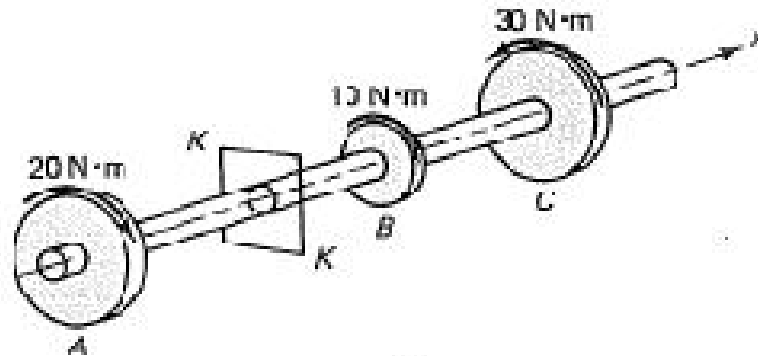


Construct shear and moment diagrams for the member.



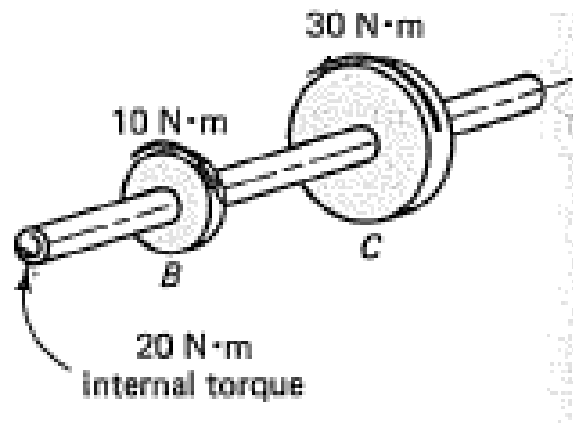
Lecture 5: Torsion

Example 4-1



Find the internal torque at section K-K.

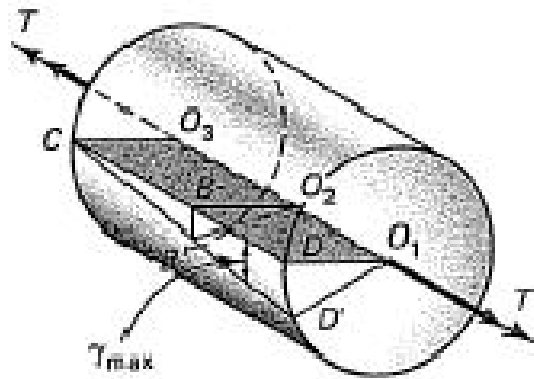
externally applied torque = internal torque



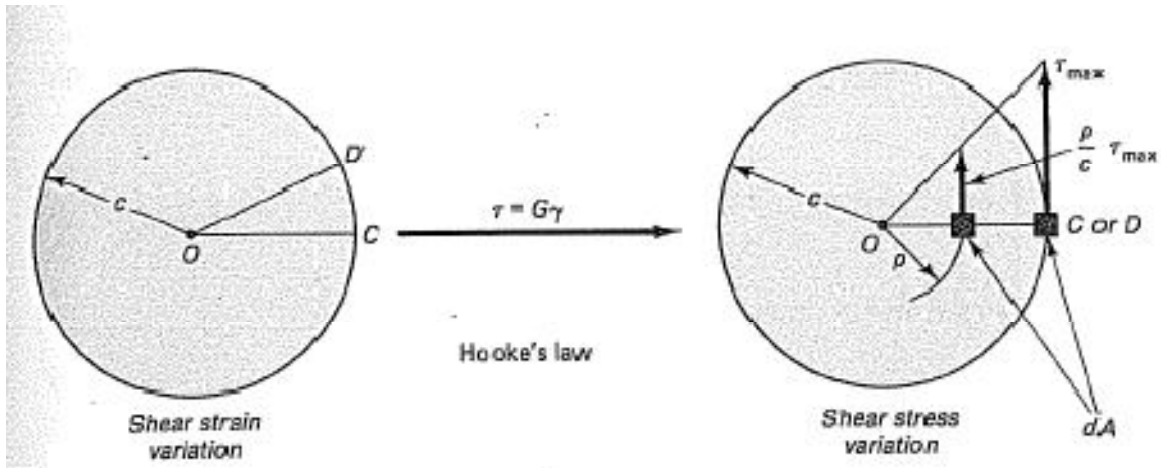
1. Torsion of Circular Elastic Bars

Basic Assumptions for Circular Members:

1. A plane section of material perpendicular to the axis of a circular member remains plane after the torques are applied, i.e., no warpage or distortion of parallel planes normal to the axis of a member takes place.
2. In a circular member subjected to torque, shear strains vary linearly from the central axis reaching maximum at the periphery. It must be emphasized that these assumptions hold only for circular solid and tubular members.



3. If attention is confined to the linearly elastic material, Hooke's law applies, and, it follows that shear stress is proportional to shear strain.



$$\int_A \underbrace{\frac{\rho}{c} \tau_{\max}}_{\text{stress}} \underbrace{dA}_{\text{area}} \quad \rho = T$$

$$\underbrace{\text{force}} \quad \underbrace{\text{arm}}_{\text{torque}}$$

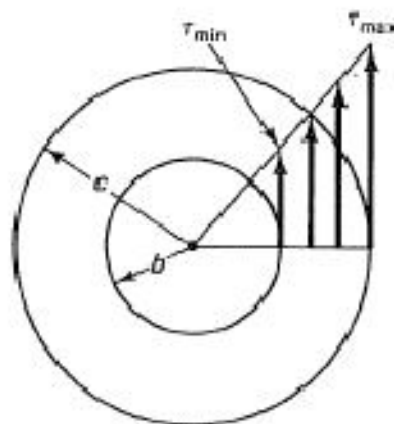
$$\frac{\tau_{\max}}{c} \int_A \rho^2 dA = T$$

$$J = \int_A \rho^2 dA = \int_0^c 2\pi\rho^3 d\rho = 2\pi \left| \frac{\rho^4}{4} \right|_0^c = \frac{\pi c^4}{2} = \frac{\pi d^4}{32}$$

$$\tau_{\max} = \frac{Tc}{J}$$

$$\tau = \frac{\rho}{c} \tau_{\max} = \frac{T\rho}{J}$$

For a circular tube,

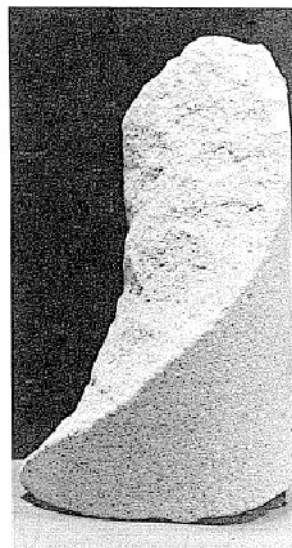
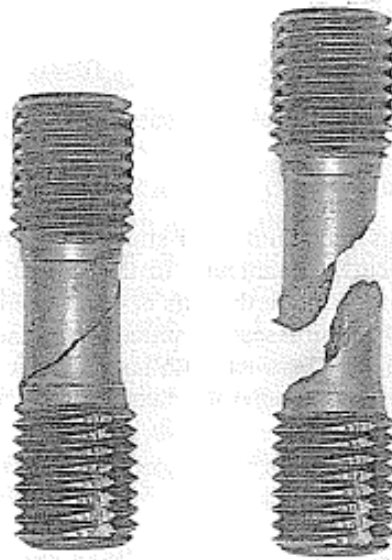
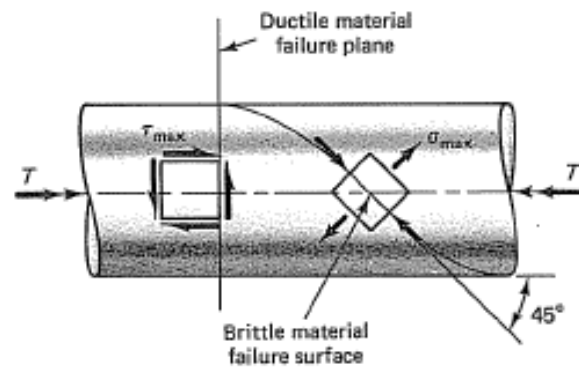


$$J = \int_A \rho^2 dA = \int_b^c 2\pi\rho^3 d\rho = \frac{\pi c^4}{2} - \frac{\pi b^4}{2}$$

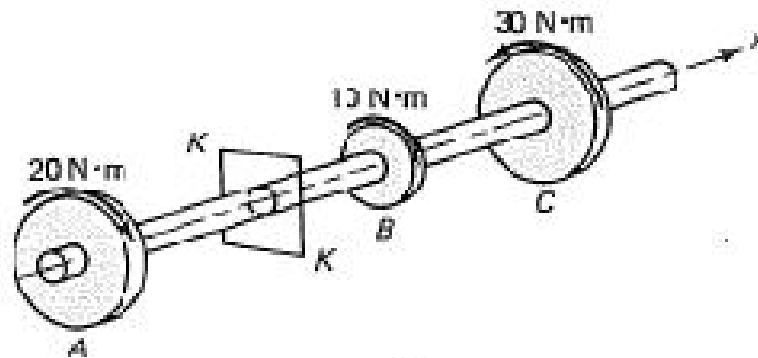
For very thin tubes,

$$J \approx 2\pi R_{av}^3 t$$

2. Fracture in Torsion



Example 4-2

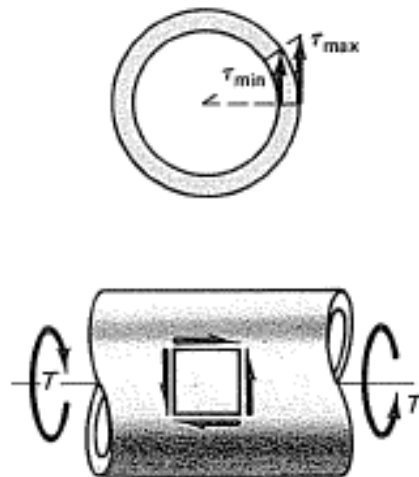


Find the maximum torsional shear stress in shaft AC. Assume the shaft from A to C is 10 mm in diameter.

$$J = \frac{\pi d^4}{32} = \frac{\pi \times 10^4}{32} = 982 \text{ mm}^4$$

$$\tau_{\max} = \frac{Tc}{J} = \frac{30 \times 10^3 \times 5}{982} = 153 \text{ MPa}$$

Example 4-3



Consider a long tube of 20 mm outside diameter and of 16 mm inside diameter, twisted about its longitudinal axis with a torque T of 40 N·m. Determine shear stresses at the outside and the inside of the tube

$$J = \frac{\pi(c^4 - b^4)}{2} = \frac{\pi(d_o^4 - d_i^4)}{32} = \frac{\pi(20^4 - 16^4)}{32} = 9270 \text{ mm}^4$$

$$\tau_{\max} = \frac{Tc}{J} = \frac{40 \times 10^3 \times 10}{9270} = 43.1 \text{ MPa}$$

$$\tau_{\min} = \frac{Tp}{J} = \frac{40 \times 10^3 \times 8}{9270} = 34.5 \text{ MPa}$$

3. Design of Circular Members in Torsion

$$\text{hp} \times 745.7 = 2\pi fT \text{ [N}\cdot\text{m/s]}$$

$$T = \frac{119 \times \text{hp}}{f} \text{ [N}\cdot\text{m]}$$

$$T = \frac{159 \times \text{kW}}{f} \text{ [N}\cdot\text{m]}$$

$$T = \frac{63,000 \times \text{hp}}{N} \text{ [in}\cdot\text{lb]}$$

Example 4-4

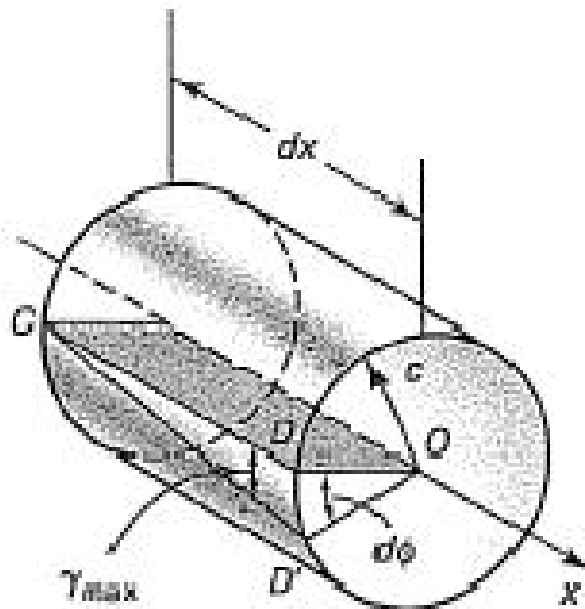
Select a solid shaft for a 10-hp motor operating at 30 Hz. The maximum shear stress is limited to 55 MPa.

$$T = \frac{119 \times \text{hp}}{f} = \frac{119 \times 10}{30} = 39.7 \text{ N}\cdot\text{m}$$

$$\frac{J}{c} = \frac{T}{\tau_{\max}} = \frac{39.7 \times 10^3}{55} = 722 \text{ mm}^3$$

$$\frac{J}{c} = \frac{\pi c^3}{2} \quad \text{or} \quad c^3 = \frac{2J}{\pi c} = \frac{2 \times 722}{\pi} = 460 \text{ mm}^3$$

4. Angle of Twist of Circular Members



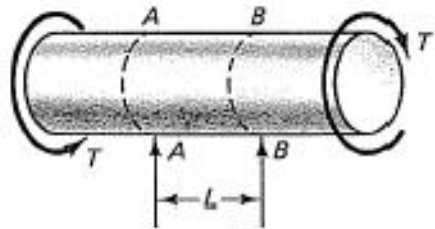
$$\text{arc } DD' = \gamma_{\max} dx \quad \text{or} \quad \text{arc } DD' = d\phi c$$

$$\gamma_{\max} dx = d\phi c$$

$$\frac{d\phi}{dx} = \frac{T}{JG} \quad \text{or} \quad d\phi = \frac{T dx}{JG}$$

$$\phi = \phi_B - \phi_A = \int_A^B d\phi = \int_A^B \frac{T_x dx}{J_x G}$$

Example 4-6



Find the relative rotation of section B-B with respect to section A-A.

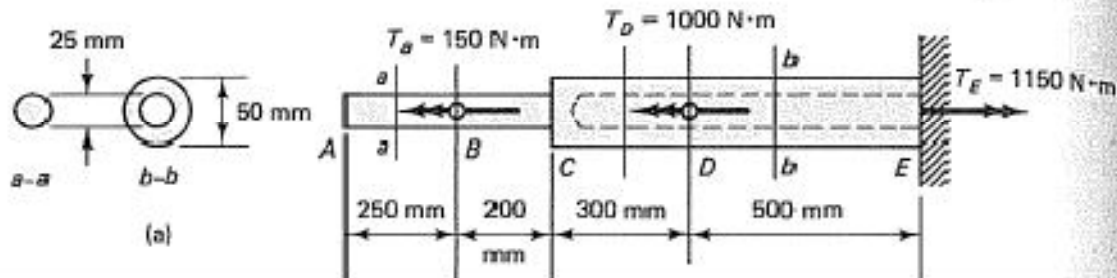
$$\phi = \int_A^B \frac{T_x dx}{J_x G} = \int_0^L \frac{T dx}{JG} = \frac{T}{JG} \int_0^L dx = \frac{TL}{JG}$$

$$\phi = \frac{TL}{JG}$$

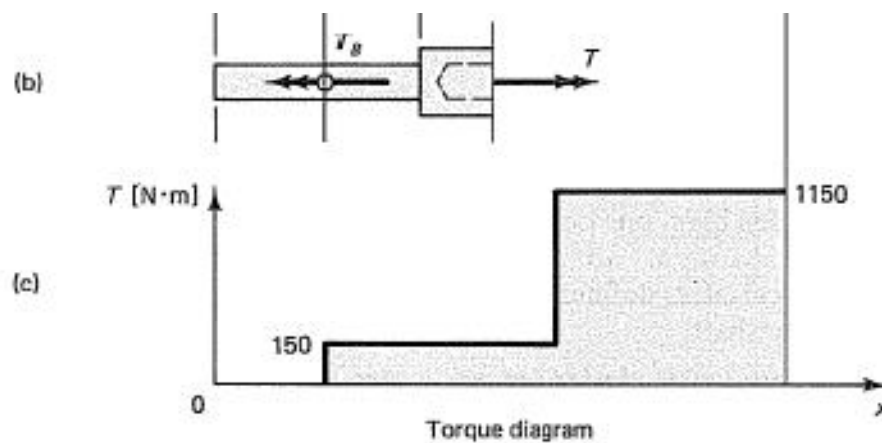
$$k_t = \frac{T}{\phi} = \frac{JG}{L}$$

$$j_t = \frac{1}{k_t} = \frac{L}{JG}$$

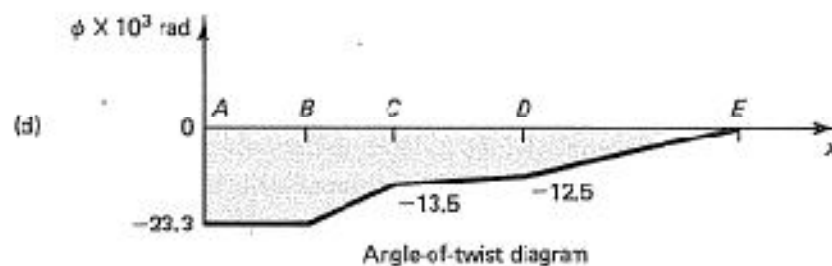
Example 4-7



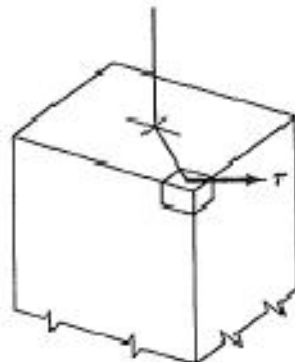
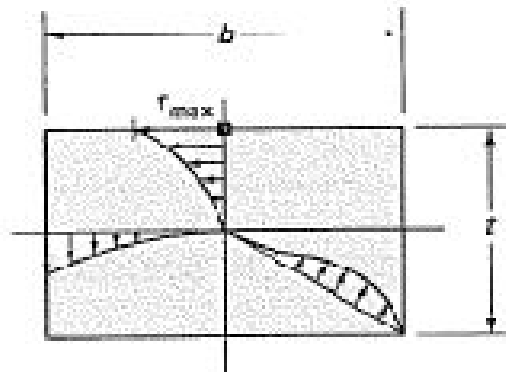
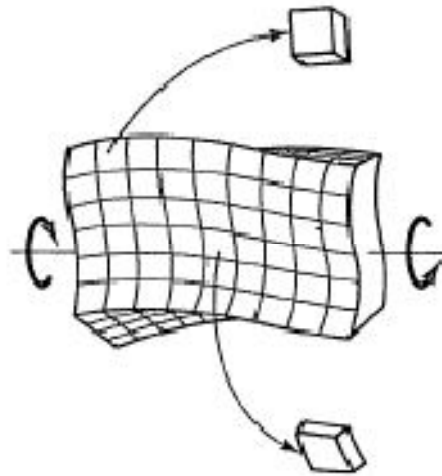
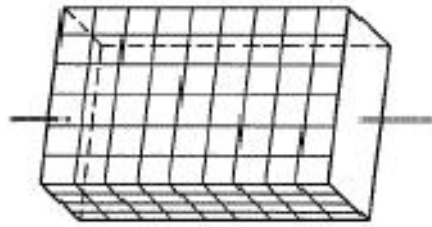
Determine the angle-of-twist of the end A. Assume the shear modulus G to be 80 GPa, a typical value for steels.



$$\begin{aligned} \phi &= \sum_i \frac{T_i L_i}{J_i G_i} = \frac{T_{AB} L_{AB}}{J_{AB} G} + \frac{T_{BC} L_{BC}}{J_{BC} G} + \frac{T_{CD} L_{CD}}{J_{CD} G} + \frac{T_{DE} L_{DE}}{J_{DE} G} \\ &= 0 + \frac{150 \times 10^3 \times 200}{38.3 \times 10^3 \times 80 \times 10^3} + \frac{150 \times 10^3 \times 300}{575 \times 10^3 \times 80 \times 10^3} \\ &\quad + \frac{1150 \times 10^3 \times 500}{575 \times 10^3 \times 80 \times 10^3} \\ &= 0 + 9.8 \times 10^{-3} + 1.0 \times 10^{-3} + 12.5 \times 10^{-3} = 23.3 \times 10^{-3} \text{ rad} \end{aligned}$$



5. Torsion of Solid Non-Circular Members



$$\tau_{\max} = \frac{T}{\alpha b t^2}$$

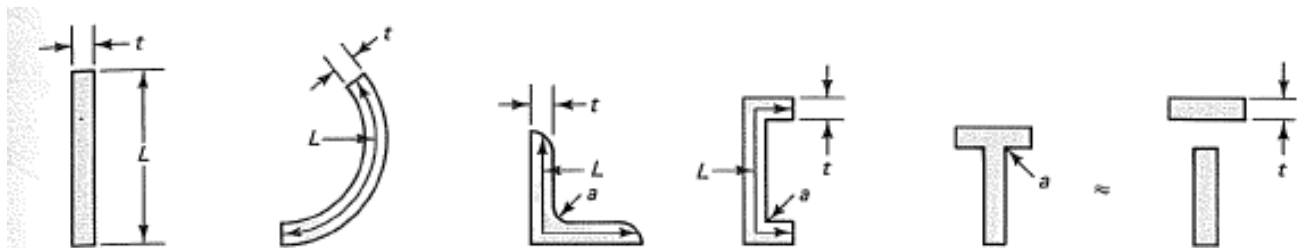
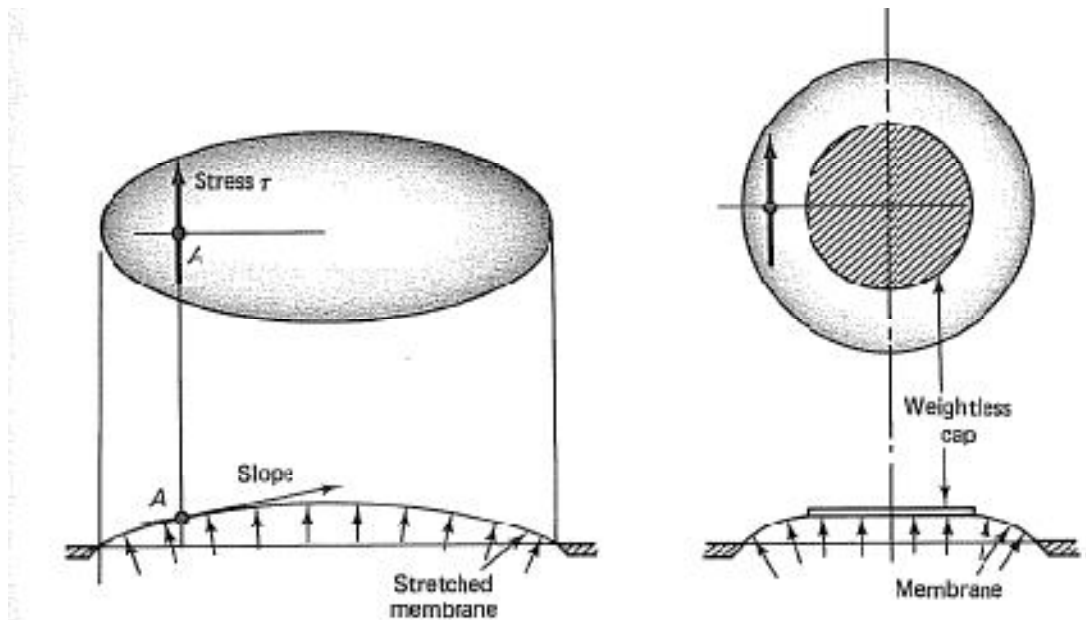
and

$$\phi = \frac{TL}{\beta b t^3 G}$$

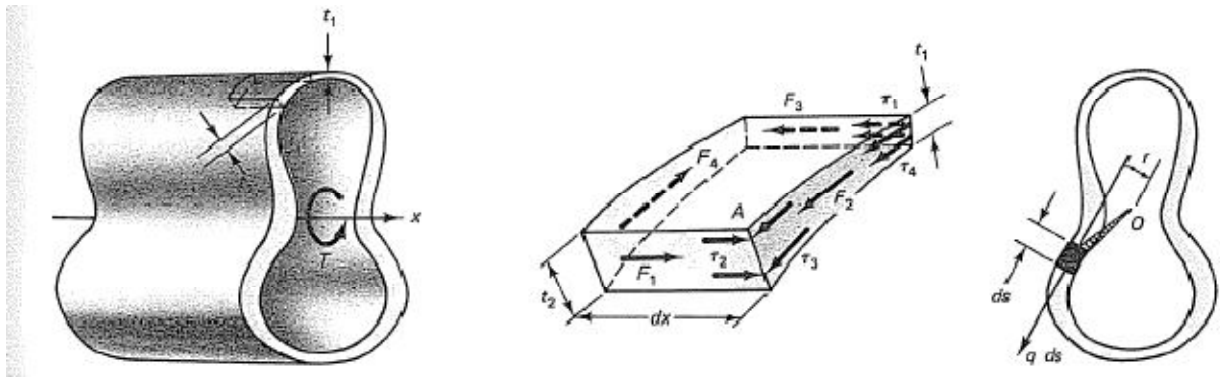
Table of Coefficients for Rectangular Bars¹⁷

b/t	1.00	1.50	2.00	3.00	6.00	10.0	∞
α	0.208	0.231	0.246	0.267	0.299	0.312	0.333
β	0.141	0.196	0.229	0.263	0.299	0.312	0.333

It happens that the solution of the partial differential equation that must be solved in the elastic torsion problem is mathematically identical to that for a thin membrane, such as a soap film, lightly stretched over a hole.



6. Torsion of Thin-Walled Tubular Members



$$T = \oint r q ds$$

$$T = q \oint r ds$$

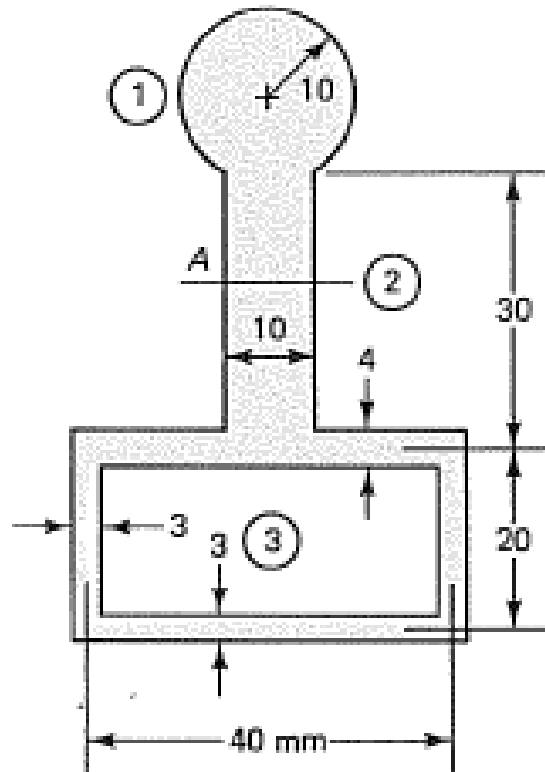
$$T = 2(A)q \quad \text{or} \quad q = \frac{T}{2(A)}$$

$$\tau = \frac{q}{t}$$

$$\theta = \frac{d\phi}{dx} = \frac{T}{4(A)^2 G} \oint \frac{ds}{t}$$

$$k_t = \frac{T}{\phi} = \frac{4(A)^2 G}{\oint ds/t} L$$

Example 4-17



If torque $T = 300 \text{ N}\cdot\text{m}$ is applied, (a) determine the maximum shear stresses that would develop in the three different parts of the member, and (b) find the torsional stiffness of the member.

$$(k_r)_1 = J \frac{G}{L} = \frac{\pi \times 10^4}{2} \frac{G}{L} = 1.57 \times 10^4 \frac{G}{L}$$

$$(k_r)_2 = \beta b r^3 \frac{G}{L} = 0.263 \times 30 \times 10^3 \frac{G}{L} = 0.789 \times 10^4 \frac{G}{L}$$

$$(k_r)_3 = \frac{4(A)^2}{\oint ds/t} \frac{G}{L} = \frac{4 \times (40 \times 20)^2}{(40 + 2 \times 20)/3 + 40/4} \frac{G}{L} = 6.98 \times 10^4 \frac{G}{L}$$

The applied torque is distributed among the three parts in a ratio of *Individual Component Stiffness/Total Stiffness*. On this basis, the torques are $300 \times (1.57 \times 10^4 \text{ G/L}) / (9.34 \times 10^4 \text{ G/L}) = 50.4 \text{ N}\cdot\text{m}$ for the knob, $25.3 \text{ N}\cdot\text{m}$ for the bar, and $224 \text{ N}\cdot\text{m}$ for the box.

$$\tau_{1-\max} = \frac{Tc}{J} = \frac{50.4 \times 10^3 \times 10}{\pi \times 10^4 / 2} = 32.1 \text{ MPa}$$

$$\tau_{2-\max} = \frac{T}{\alpha bt^2} = \frac{25.3 \times 10^3}{0.267 \times 30 \times 10^2} = 31.6 \text{ MPa}$$

$$\tau_{3-\max} = \frac{T}{2 \textcircled{A} t} = \frac{224 \times 10^3}{2 \times 40 \times 20 \times 3} = 46.7 \text{ MPa}$$