Simpson's $\frac{1}{3}$rd Rule

If $f(x)$ is quadratic, then

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right)$$

where $h = b-a$ and the mid point is $m = \left(\frac{b-a}{2}\right)$

A general parabola looks like $Ax^2 + Bx + C1$

$$\int_a^b (Ax^2 + Bx + C1)\,dx = A\int_a^b x^2\,dx + B\int_a^b x\,dx + C\int_a^b 1\,dx$$

a) If $f(x) = 1$ then $\int_a^b f(x)\,dx = (b-a)$

and $\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{b-a}{3}\left(\frac{1}{2} + 2 + \frac{1}{2}\right)$

which is equal to $(b-a)$ so the formula works for $f(x) = 1$

b.) $f(x) = x$ $\quad \int_a^b x\,dx = \frac{b^2-a^2}{2}$

and $\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{b-a}{3}\left(\frac{a}{2} + 2\,\frac{a+b}{2} + \frac{b}{2}\right)$

$$= \frac{b-a}{3}\left(\frac{a}{2} + a + b + \frac{b}{2}\right)$$

$$= \frac{b-a}{3}\left(\frac{3}{2}a + \frac{3}{2}b\right)$$

$$= \frac{(b-a)(b+a)}{2}$$

$$= \frac{b^2-a^2}{2}$$

So the formula works for $f(x) = x$

c) For $f(x) = x^2$ $\quad \int_a^b f(x)\,dx = \frac{b^3 - a^3}{3}$

and $\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{b-a}{3}\left\{\frac{a^2}{2} + 2\left(\frac{a+b}{2}\right)^2 + \frac{b^2}{2}\right\}$

$$= \frac{b-a}{3}\left(\frac{a^2}{2} + \frac{a^2+2ab+b^2}{2} + \frac{b^2}{2}\right)$$

$$= \frac{b-a}{3}\left(\frac{2a^2 + 2ab + 2b^2}{2}\right)$$

$$= \frac{b-a}{3}\left(a^2 + ab + b^2\right)$$

$$= \frac{b^3 - a^3}{3}$$

For general quadratic $f(x) = Ax^2 + Bx + C$

$$\int_a^b Ax^2 + Bx + C\,1\,dx = A\int_a^b x^2\,dx + B\int_a^b x\,dx + C\int_a^b 1\,dx$$

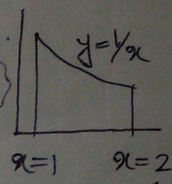
and using above results

$$\int_a^b f(x)\,dx = A\frac{h}{3}\left(\frac{a^2}{2} + 2m^2 + \frac{b^2}{2}\right) + B\frac{h}{3}\left(\frac{a}{2} + 2m + \frac{b}{2}\right) + C\frac{h}{3}(3)$$

$$= \frac{h}{3}\left(\frac{Aa^2 + Ba + C}{2} + 2(Am^2 + Bm + C) + \frac{Ab^2 + Bb + C}{2}\right)$$

$$= \frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right)$$

Problem $\int_1^2 \frac{1}{x}dx$ solve using trapezoidal and simpson's Rule for 1, 2, 4, 8, 16



Trapezoidal — $A = \frac{\Delta x}{2}(y_0 + y_n + 2(y_1 + y_2 + \cdots\cdots))$

simpsons — $A = \frac{\Delta x}{3}(y_0 + y_n + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots))$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{1} = 1$

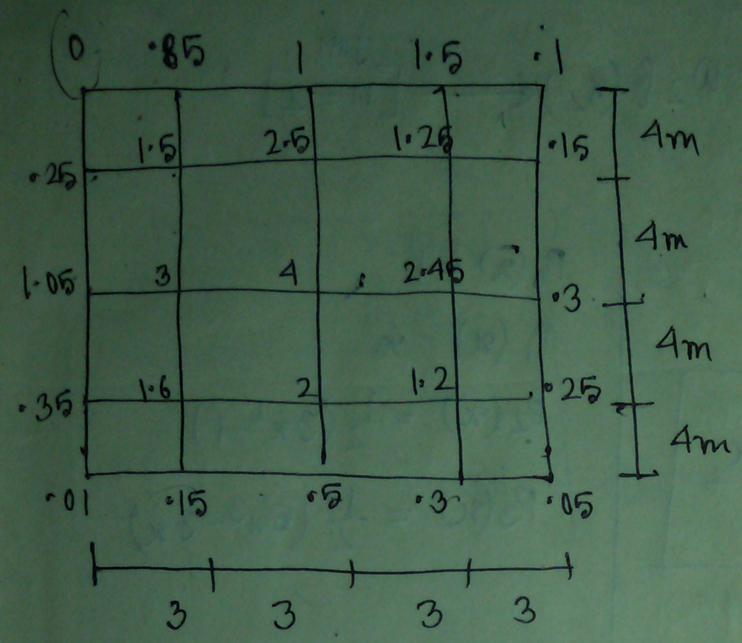
n	Δx	A Trapezoidal	A simpson's
n=1	$\frac{2-1}{1}=1$	$\frac{1}{2}[\frac{1}{1}+\frac{1}{2}] = .75$	
n = 2	$\frac{2-1}{2}=.5$	$\frac{.5}{2}[1+2^{-1}+2(\frac{1}{1.5})]$ = .6970238	$\frac{.5}{3}[1+2^{-1}+4(\frac{1}{1.5})]$
n = 4	$\frac{2-1}{4}=.25$	$\frac{.25}{2}[1+.5+2(\frac{1}{1.25}+\frac{1}{1.5}+\frac{1}{1.75})$ =0.69702	$=\frac{.25}{3}[1+.5+4(\frac{1}{1.25}+\frac{1}{1.75})+2(\frac{1}{1.5})] = .6933$
n=8	$\frac{2-1}{8}=.125$	$\frac{.125}{2}[1+.5+2(\frac{1}{1.125}+\frac{1}{1.25}+\frac{1}{1.375}+\frac{1}{1.5}+\frac{1}{1.625}+\frac{1}{1.75}+\frac{1}{1.875})]$ = .6973	$=\frac{.125}{3}[1+0.5+4(\frac{1}{1.125}+\frac{1}{1.375}+\frac{1}{1.625}+\frac{1}{1.875})+2(\frac{1}{1.25}+\frac{1}{1.5}+\frac{1}{1.75})]$ = .6931
n=16	$\frac{2-1}{16}=$		

Rombergs Table $\quad I = \int_1^2 \left(\frac{1}{x}\right) dx \quad E = 0.0001$

	Trapezoidal	Simpsons	C	D	E
1	T_0 (1 division) $= 0.75$				
		$S_0 = \frac{4T_1 - T_0}{4-1} = \frac{4(.7083) - .75}{3} = .6944$			
2	$T_1 = 0.7083$ (2 division)		$C_0 = \frac{4^2 S_1 - S_0}{4^2 - 1} = .6931$		
		$S_1' = \frac{4T_2 - T_1}{4-1} = \frac{4(.6970) - .7083}{3} = .69323$		$D_0 = \frac{4^3 C_1 - C_0}{4^3 - 1} = \frac{4^3(.6931) - .6231}{4^3 - 1} = .6931$	
4	$T_2 = .697024$ (4 division)		$C_1 = \frac{4^2(.6338) - .6231}{15} = .629$		$E_0 = \frac{4^4 D_1 - D_0}{4^4 - 1} = .7130$
		$S_2 = \frac{4T_3 - T_2}{4-1} = \frac{4(.6496) - .6970}{3} = .6338$		$D_1 = \frac{4^3 C_2 - C_0}{4^3 - 1} = \frac{4^3(.712) - .629}{4^3 - 1} = .713$	
8	$T_3 = -.649677$ (8 division)		$C_2 = \frac{4^2(.7079) - .6338}{15} = .712$		
		$S_3 = \frac{4T_4 - T_3}{3} = .7079$			
16	$T_4 = .693391$ (16 division)				

$$\left| \frac{S_0}{T_0} - 1 \right| = |-.07411| = .0741$$

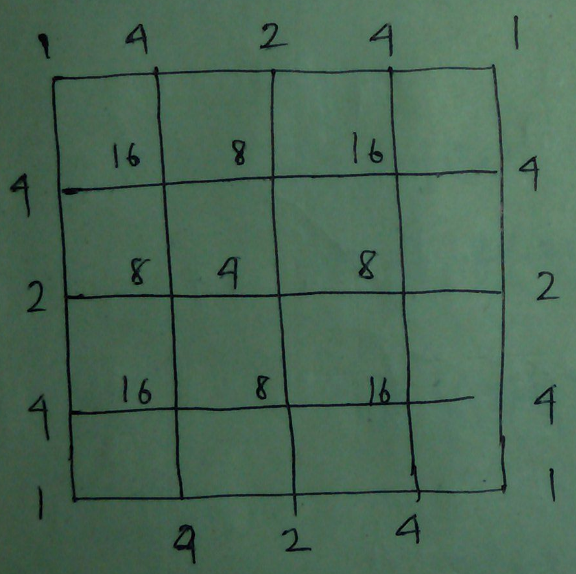
Spot height



$$\text{Volume} = \frac{4 \times 3}{4} \cdot \Big[1 \times (0 + \cdot 1 + \cdot 05 + \cdot 01)$$

$$+ 2(\cdot 5 + 1 + 1\cdot 5 + \cdot 15 + \ldots$$

$$+ 4(1\cdot 5 + 2\cdot 5 + 1\cdot 25 + 3 + 4 + \ldots$$



Gauss Quadrature

$$w_K = \frac{1}{P'_{n+1}(x_K)} \int_{-1}^{1} \frac{P_{n+1}(x)}{x - x_0}\, dx$$

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n} (x^2 - 1)^n$$

For $n = 2$

[* কিছু বলা না থাকলে $n = 2$]

$$P_{n+1} = P_{2+1} = P_3 = \frac{1}{2^3 \lfloor 3} \frac{d^3}{dx^3} (x^2 - 1)^3$$

$$= \frac{1}{8 \times 6} \frac{d^3}{dx^3} (x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48} \frac{d^2}{dx^2} (6x^5 - 12x^3 + 6x)$$

$$= \frac{1}{48} \frac{d}{dx} (30x^4 - 36x^2 + 6)$$

$$= \frac{1}{48} (120x^3 - 72x)$$

$$= \frac{24}{48} (5x^3 - 3x) = \frac{1}{2}(5x^3 - 3x)$$

Now $P'_3(x) = P'_{2+1}(x) = \frac{1}{2}(5x^3 - 3x) = \frac{1}{2}(5x^2 - 3)x$

Let $P_3(x) = 0 = P_{n+1}(x) = \frac{1}{2}(5x^3 - 3x) = 0$

$$5x^3 - 3x = 0 \Rightarrow \boxed{x = 0, \sqrt{3/5}, -\sqrt{3/5}}$$

For $x_k = 0$ $\quad P'_{n+1}(x_k) = \frac{1}{2}(15 \cdot 0^2 - 3) = -3/2$

$$w_K = \frac{1}{(-3/2)} \int_{-1}^{1} \frac{\frac{1}{2}(15x^2 - 3)}{(x-0)} dx$$

$$= -\frac{2}{3} \int_{-1}^{1} \left(\frac{15}{2}x - \frac{3}{2}\frac{1}{x}\right) dx$$

$$= -\frac{2}{3}\left[\frac{15}{2}\frac{x^2}{2} - \frac{3}{2}\ln x\right]_{-1}^{1} = 5/9$$

For $x = \sqrt{3/5}$ $\quad P'_{n+1}(x_k) = \frac{1}{2}\left(15 \times \frac{3}{5} - 3\right) = 3$

$$w_K = \frac{1}{3} \int_{-1}^{1} \frac{1}{2} \frac{(15x^2 - 3)}{x - \sqrt{3/5}} dx = 8/9$$

again $\quad x = -\sqrt{3/5}$ $\quad P'_{n+1}(x_K) = \frac{1}{2}\left(15 \frac{3}{5} - 3\right) = 3$

$$w_K = \frac{1}{3} \int_{-1}^{1} \frac{\frac{1}{2}(15x^2 - 3)}{x + \sqrt{3/5}} dx = 8/9$$

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For n = 3

$$P_4(x) = P_{3+1}(x) = P_{n+1}(x) = \frac{1}{2^4 \lfloor 4} \frac{d^4}{dx^4} (x^2-1)^4$$

$$= \frac{1}{16 \times \lfloor 4} \frac{d^4}{dx^4} \left(x^8 - 4x^6 + 6x^4 - 4x^2 + 1\right)$$

$$\left[(x^2-1)^2 = (x^2)^2 - 2(x^2)(1) + 1 \quad = (x^4 - 2x^2 + 1)\right.$$

$$(a+b+c)^2 = a^2+b^2+c^2 + 2ab + 2bc + 2ca$$

$$(x^4 - 2x^2 + 1)^2$$

$$= (x^4)^2 + (-2x^2)^2 + 1^2 + 2(x^4)(-2x^2) + 2(-2x^2)(1) + 2(1)(x^4)$$

$$= x^8 + 4x^4 + 1 - 4x^6 - 4x^2 + 2x^4$$

$$= (x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$$

$$= \frac{1}{16 \times \lfloor 4} \frac{d^3}{dx^3} \left(8x^7 - 24x^5 + 24x^3 - 8x\right)$$

$$= \frac{1}{16 \times \lfloor 4} \frac{d^2}{dx^2} \left(56x^6 - 120x^4 + 72x^2 - 8\right)$$

$$= \frac{1}{16 \times \lfloor 4} \frac{d}{dx} \left(336x^5 - 480x^3 + 144x\right)$$

$$= \frac{1}{16 \times \lfloor 4} \left(1680x^4 - 1440x^2 + 144\right)$$

Now $\frac{1}{16 \times \lfloor 4} \left(1680x^4 - 1440x^2 + 144\right) = 0$

$$1680x^4 - 1440x^2 + 144 = 0$$

$$\Rightarrow x = \pm 0.34, \pm 0.8641$$

$$w_k = 0.6521, .3479$$

Evaluate $\int_{-1}^{1} x^2 \cos x \, dx$

Soln:- Let $I = \int_{-1}^{1} x^2 \cos x \, dx$

Here $n = 2$ so

x_0	w_k
$-\sqrt{3/5}$	$5/9$
0	$8/9$
$+\sqrt{3/5}$	$5/9$

We know $\int_{-1}^{1} f(x)\,dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$

Here $f(x) = x^2 \cos x$

(*) All angles are calculated in radian

$$f(-\sqrt{3/5}) = \left(-\sqrt{3/5}\right)^2 \cos\left(\sqrt{3/5}\right) = \frac{3}{5} \times 0.7147 = 0.42882$$

$$f(0) = (0)^2 \cos(0) = 0$$

$$f(\sqrt{3/5}) = \left(\sqrt{3/5}\right)^2 \cos\left(\sqrt{3/5}\right) = \frac{3}{5} \times 0.7147 = 0.42882$$

$w_0 = 5/9$, $w_1 = 8/9$ $w_2 = 5/9$

$$\int_{-1}^{1} x^2 \cos dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

$$= \frac{5}{9} \times .42882 + 0 + \frac{5}{9} \times .42882$$

$$= .4764$$

Evaluate b) $\int_{2}^{3} x^2 \cos x \, dx$ ⊗ Not sure

$$x = \frac{(b-a)t + (b+a)}{2} \quad \text{— (I)}$$

$$\frac{dx}{dt} = \frac{(b-a)}{2}$$

$$\Rightarrow dx = \frac{(b-a)}{2} dt \quad \text{— (II)} \qquad \text{from (II)} \Rightarrow \quad dx = \frac{3-2}{2} dt = \frac{1}{2} dt$$

Here $a = 2$

$b = 3$

t = transformation function

$$\text{(I)} \Rightarrow \quad x = \frac{(3-2)t + (3+2)}{2}$$

$$x = \frac{t+5}{2}$$

Now $f(x) = x^2 \cos x$

$$f\left(\frac{t+5}{2}\right) = \left(\frac{t+5}{2}\right)^2 \cos\left(\frac{t+5}{2}\right) \times \frac{dt}{2}$$

$=$

$$f\left(\frac{t+5}{2}\right) = \frac{1}{2}\left(\frac{t+5}{2}\right)^2 \cos\left(\frac{t+5}{2}\right) dt$$

$$= \frac{1}{2\times 4} (t+5)^2 \cos\left(\frac{t+5}{2}\right) dt$$

$$= \frac{1}{8} (t+5)^2 \cos\left(\frac{t+5}{2}\right) dt$$

$$= \frac{1}{8} \int_{-1}^{1} (t+5)^2 \cos\left(\frac{t+5}{2}\right) dt$$

$$\boxed{f(t) = \frac{1}{8}\left[(t+5)^2 \cos\left(\frac{t+5}{2}\right)\right]}$$

Now $t_0 = -\sqrt{3/5}$

$t_1 = 0$

$t_2 = +\sqrt{3/5}$

$$f(t_0) = \frac{1}{8}\left(-\sqrt{\tfrac{3}{5}} + 5\right)^2 \cos\left(\frac{-\sqrt{3/5} + 5}{2}\right) = -1.151$$

$$f(t_1) = -2.50$$

$$f(t_2) = \frac{1}{8}\left(\sqrt{\tfrac{3}{5}} + 5\right)^2 \cos\left(\frac{\sqrt{3/5} + 5}{2}\right) =$$

$$= -4.03$$

$$f(x) = -1.15 \times \frac{8}{9} - 2.50 \times \frac{5}{9} \left(4.03 \times \frac{5}{9}\right)$$

$$f(x) = -4.65$$

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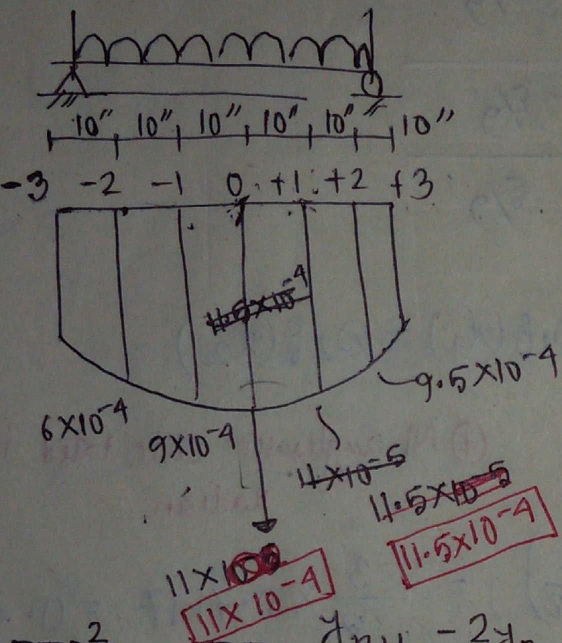
Problem Experimentally observed values of deflections of a beam are shown below. Calculate the bending moment at point −1, 0, +1

Given $E = 30\times10^6$ psi
$I = 1000\,in^4$

Also estimate slope and shear force at those points.



$$M = EID^2y = -EI\,\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

$$= 30\times10^6\times1000\times\cancel{\frac{(11.5\times10^{-4})\ \cancel{11\times10^{-5}} - 2\times(11\times10^{-4}) + (9\times10^{-4})}{(10)^2}}$$

$$\to EI\times\frac{11.5\times10^{-4} - (2\times11\times10^{-4}) + (9\times10^{-4})}{(10)^2}$$

$$= 45 \text{ kip-in}$$

$$\text{Shear} = \frac{dM}{dx} = EId^3y$$

$$= EI\,\frac{-y_{n-2} + 2y_{n-1} - 2y_{n+1} + y_{n+2}}{2h^3}$$

$$= 30\times10^6\times1000\times\frac{-6\times10^{-4} + (2\times9\times10^{-4}) - (2\times11.5\times10^{-4}) + (9.5\times10^{-4})}{2\times(10)^3}$$

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$$\theta_0 = \frac{y_1 - y_{-1}}{2h} \times EI = \frac{(11.5 - 9) \times 10^{-4}}{2 \times 10} =$$

$$M_1 = -EI \quad \frac{y_0 - 2y_1 + y_2}{(10)^2}$$

$$= 30 \times 10^6 \times 1000 \times \frac{(11 \times 10^{-5}) - (2 \times 11.5 \times 10^{-5}) + (9.5 \times 10^{-4})}{10^2}$$

$$= + \frac{7500000}{1000} \text{ lb-in}$$

$$M_{-1} = EI \quad \frac{y_{-2} - 2y_{-1} + y_0}{h^2}$$

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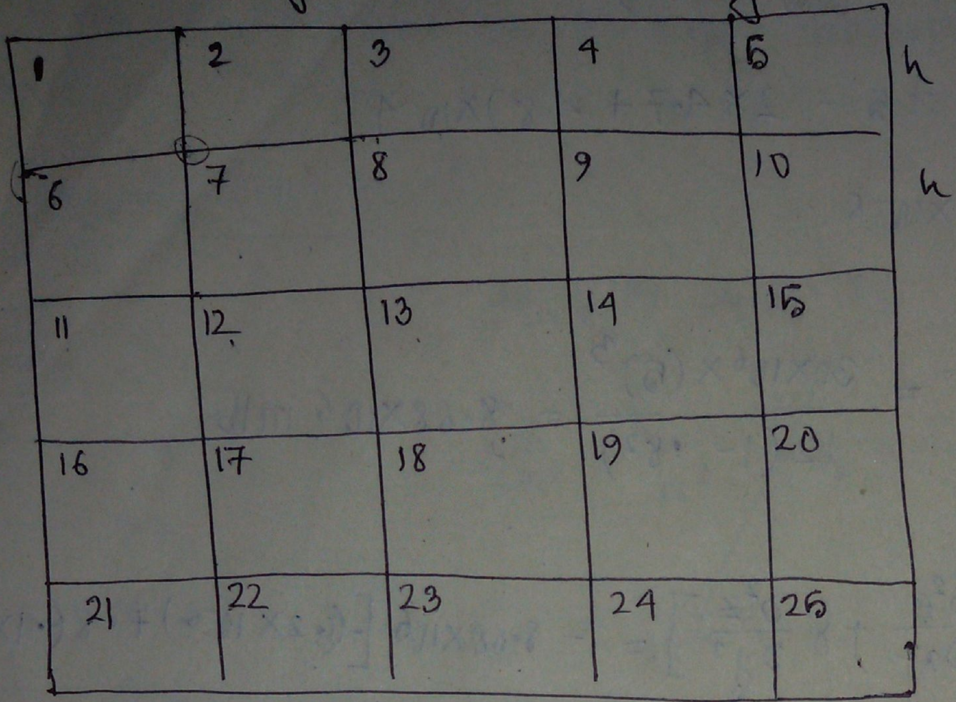
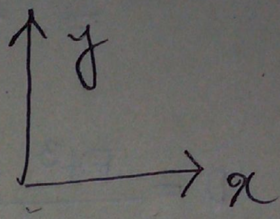
The deflection at various points on a normally loaded plate are shown in the fig. Estimate the bending moment at point 6, 7 and 11, 5.





??

Point	Deflection (10^{-4})
1	6
2	5
3	4
4	7
5	9
6	11
7	14
8	6
9	5
10	2
11	11
12	13
13	20
14	11
15	1
16	5
17	7
18	9
19	4

Point	Deflection (10^{-4})
20	13
21	5
22	3
23	2
24	1
25	4

$E = 30\times10^{6}$ psi

$\gamma = .2$

$t = \cancel{15}\ 5''$

$h = 10''$

$$\boxed{D = \frac{Et^3}{12(1-\gamma^2)}} = \frac{30\times10^6\times(5)^3}{12(1-.2^2)}$$

$$= 3.25\times10^{8}\ \text{lb-in}$$

$$M_x = -D\left[\frac{\partial^2 z}{\partial x^2} + \gamma \frac{\partial^2 z}{\partial y^2}\right] \quad \text{— (I)}$$

$$M_y = -D\left[\frac{\partial^2 z}{\partial y^2} + \gamma \frac{\partial^2 z}{\partial x^2}\right] \quad \text{— (II)}$$

At point 7

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{h^2}\left(y_{n+1} - 2y_n + y_{n-1}\right) \quad [\because \text{Central Difference}]$$

$$= \frac{1}{10^2}\left(y_8 - 2\times y_7 + y_6\right)$$

$$= \frac{1}{10^2}\left(6 - (2\times 14) + 11\right)\times 10^{-4}$$

$$= -0.11\times 10^{-4} = -1.1\times 10^{-5}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{h^2}\left(y_{n+1} - 2y_n + y_{n-1}\right)$$

$$\cancel{= \frac{1}{10^2}\left(y_2 - 2y_7\right.}$$

$$= \frac{1}{10^2}\left(y_2 - 2y_7 + y_{12}\right)$$

$$= \frac{1}{10^2}\left(5 - 2\times 14 + 13\right)\times 10^{-4}$$

$$= -10\times 10^{-6}$$

$$\text{(I)} \Rightarrow M_x = -3.25\times 10^{8}\left[-.11\times 10^{-4} + .2(-10\times 10^{-6})\right]$$

$$= 4225 \text{ lb}$$

$$\text{(II)} \Rightarrow M_y = -3.25\times 10^{8}\left[-10\times 10^{-6} + .2(-.11\times 10^{-4})\right]$$

$$= 3965 \text{ lb}$$

At point 6

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{h^2}\left(y_n + y_{n+2} - 2y_{n+1}\right) \quad \text{[(forward difference)}$$

$$= \frac{1}{h^2}\left(y_6 + y_8 - 2y_7\right)$$

$$= \frac{1}{10^2}\left(11 + 6 - (2\times 14)\right)\times 10^{-4}$$

$$= -1.1\times 10^{-5}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{h^2}\left(y_{n-1} - 2y_n + y_{n+1}\right) \quad \text{[central differen}$$

$$= \frac{1}{h^2}\left(y_1 - 2y_6 + y_{11}\right)$$

$$= \frac{1}{10^2}\left(6 - (2\times 11) + 11\right)\times 10^{-4}$$

$$= -5\times 10^{-6}$$

$$M_x = -D\left[\frac{\partial^2 z}{\partial x^2} + \gamma\frac{\partial^2 z}{\partial y^2}\right]$$

$$= -3.25\times 10^8\left[-1.1\times 10^{-5} - \times(.2\times 5\times 10^{-6})\right]$$

$$= 3900\text{ lb}$$

$$M_y = -D\left[\frac{\partial^2 z}{\partial y^2} + \gamma\frac{\partial^2 z}{\partial x^2}\right]$$

$$= +3.25\left[5\times 10^{-6} + .2\times 1.1\times 10^{-5}\right]\times 10^8$$

$$= 3412.5\text{ lb}$$

For point 5

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{h^2}\left[y_n - 2y_{n-1} + y_{n-2} \right]$$ (Backward difference)

$$= \frac{1}{h^2}\left[y_5 - 2y_4 + y_3 \right]$$

$$= \frac{1}{10^2}\left[\qquad \right]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{h^2}\left[y_n - 2y_{n-1} + y_{n-2} \right]$$ (Backward difference)

$$= \frac{1}{h^2}\left[y_5 - 2y_{10} + y_{15} \right]$$

$$=$$