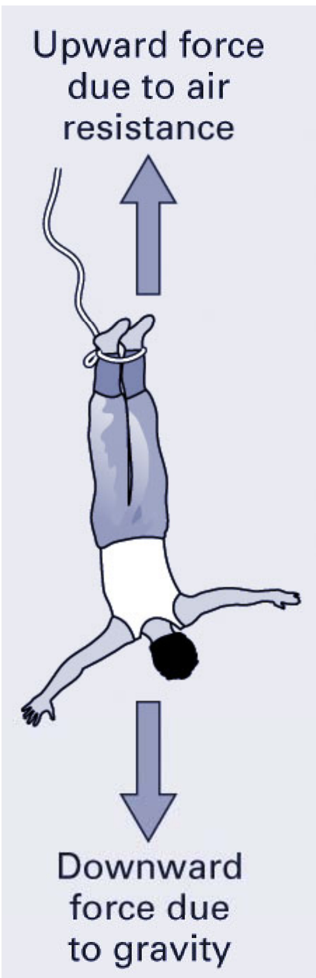
CE 205: Numerical Methods

Ordinary Differential Equations

Why study Differential Equations?

Many physical phenomena are best formulated mathematically in terms of their rate of change.

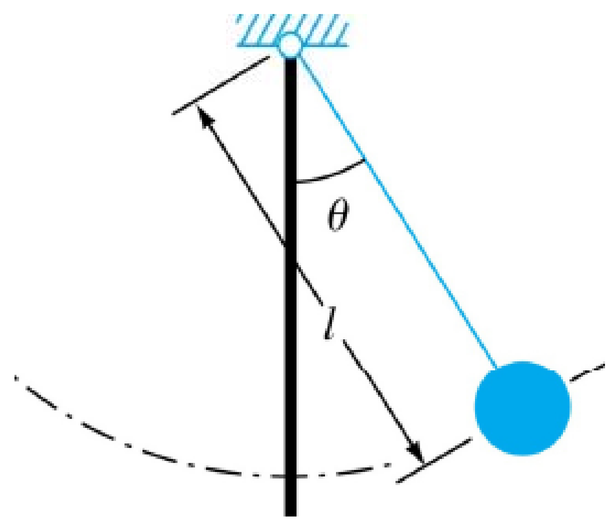


Bungee-jumper Equation

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

Motion of a swinging pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$



ODE and Engineering practice

$$F = ma$$

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

Analytical

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

Numerical

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m} v(t_i)^2\right](t_{i+1} - t_i)$$

Physical Laws

ODE

Solution

Independent variable: spatial or temporal

Initial value and boundary value problems

Boundary Value problems: Solve $y''(x) + y(x) = 0$

$$y(0) = 0,\ y(\pi/2) = 2.$$

General Solution $y(x) = A\sin(x) + B\cos(x).$

Applying B.C. $0 = A \cdot 0 + B \cdot 1 \quad 2 = A \cdot 1$

$$\boxed{y(x) = 2\sin(x).}$$

Initial Value problems: Solve $y' = 0.85y \qquad y(0) = 19$

General Solution $y = Ce^{0.85t}$

Applying I.C. $19 = Ce^{0.85*0} \qquad C = 19$

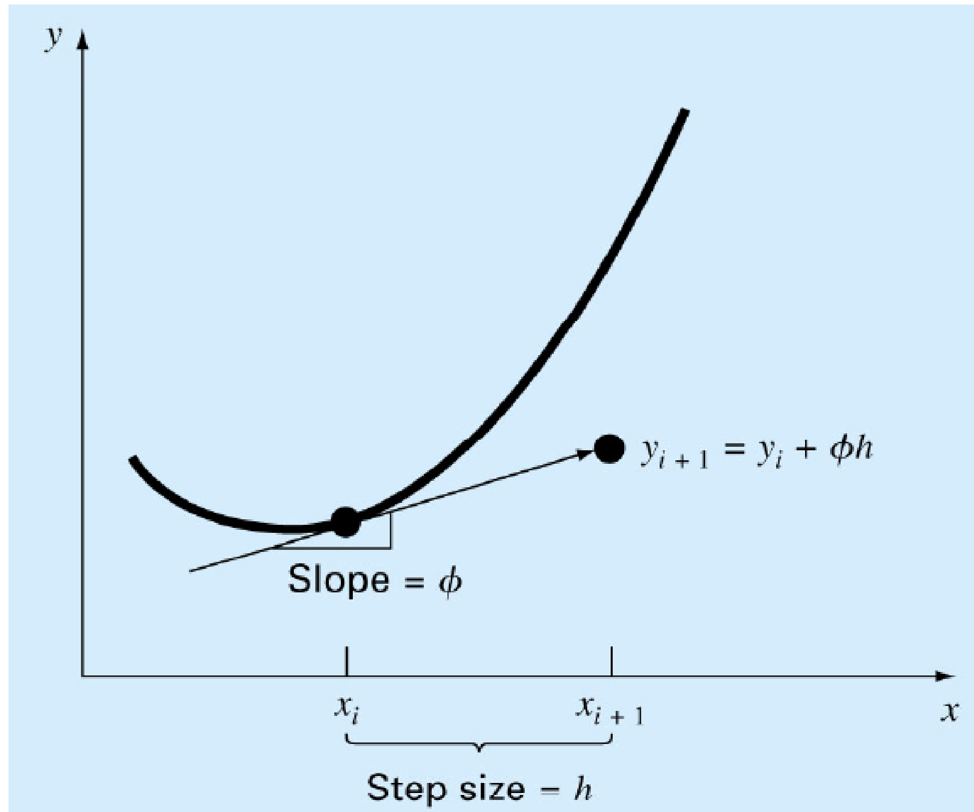
$$\boxed{y(t) = 19e^{0.85t}}$$

Solving initial value problems

ODE of the form $\frac{dy}{dt} = f(t,y)$ will be solved using the one-step methods having the general form:

$$y_{i+1} = y_i + \phi h$$

ϕ is called an *increment function*, and is used to extrapolate from an old value y_i to a new value y_{i+1}



Techniques:

- Euler
- Heun
- Midpoint
- Fourth-Order Runge-Kutta

Euler's Method

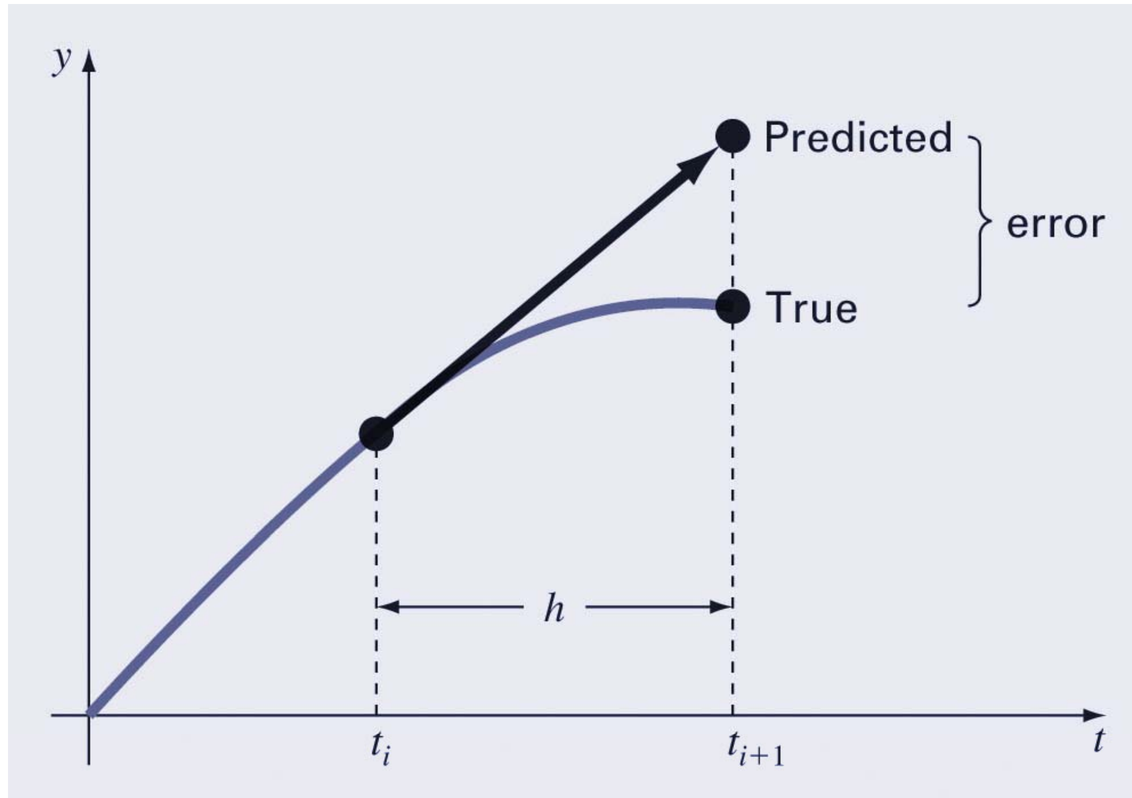
The first derivative provides a direct estimate of the slope at t_i:

$$\left.\frac{dy}{dt}\right|_{t_i} = f(t_i, y_i)$$

Euler method uses that estimate as the increment function:

$$\phi = f(t_i, y_i)$$

$$y_{i+1} = y_i + f(t_i, y_i)h$$



Errors in ODE

The numerical solution of ODEs involves two types of error:

- *Truncation errors*, caused by the nature of the techniques employed
- *Roundoff errors*, caused by the limited numbers of significant digits that can be retained

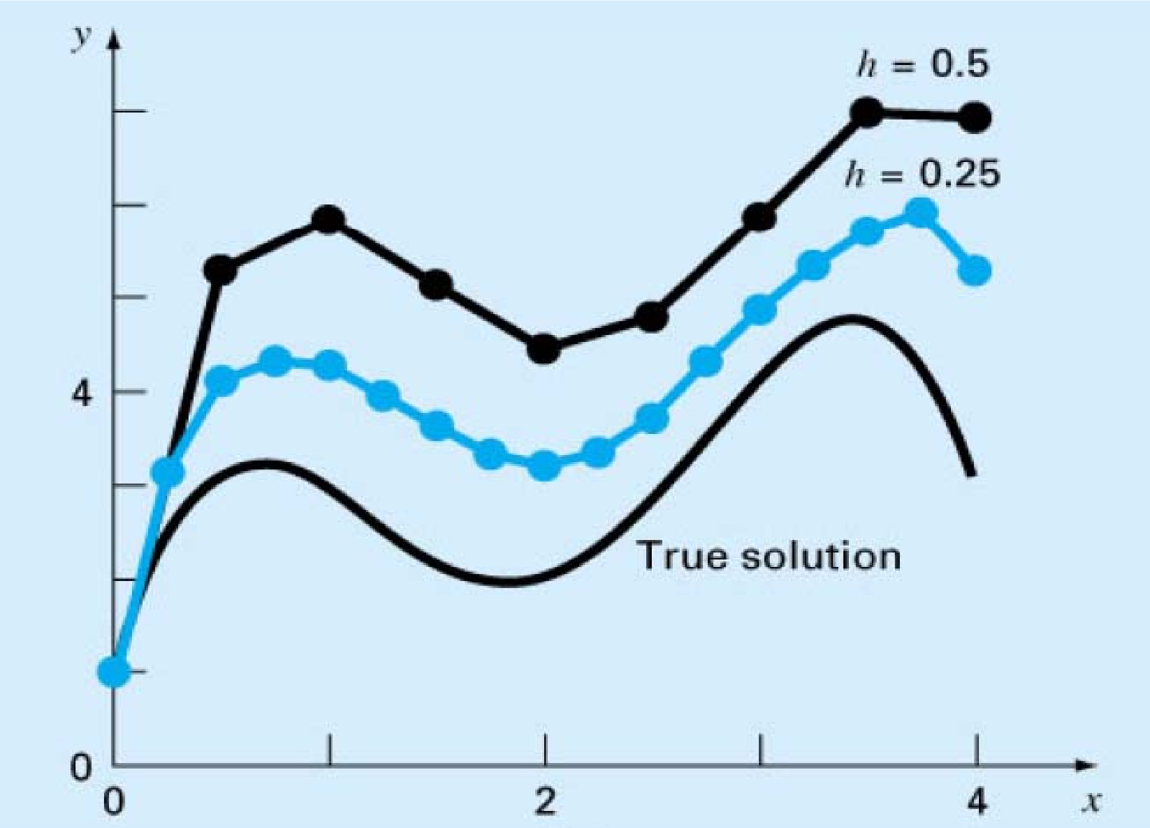
The total, or *global* truncation error can be further split into:

- *local truncation error* that results from an application method in question over a single step, and
- *propagated truncation error* that results from the approximations produced during previous steps.

Error Analysis in Euler's Method

The local truncation error for Euler's method is $O(h^2)$ and proportional to the derivative of $f(t,y)$ while the global truncation error is $O(h)$.

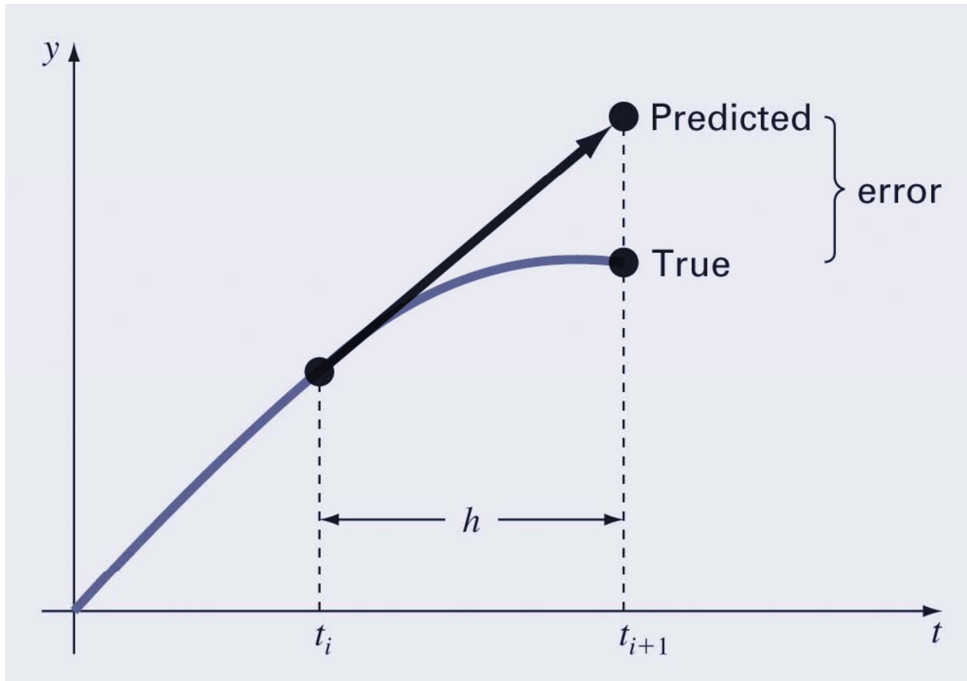
$$E_a = \frac{f'(x_i, y_i)}{2!} h^2$$



Global error can be reduced by decreasing the step size

Improvement of Euler's Method

Main limitation: the derivative at the beginning of the interval is assumed to apply across the entire interval



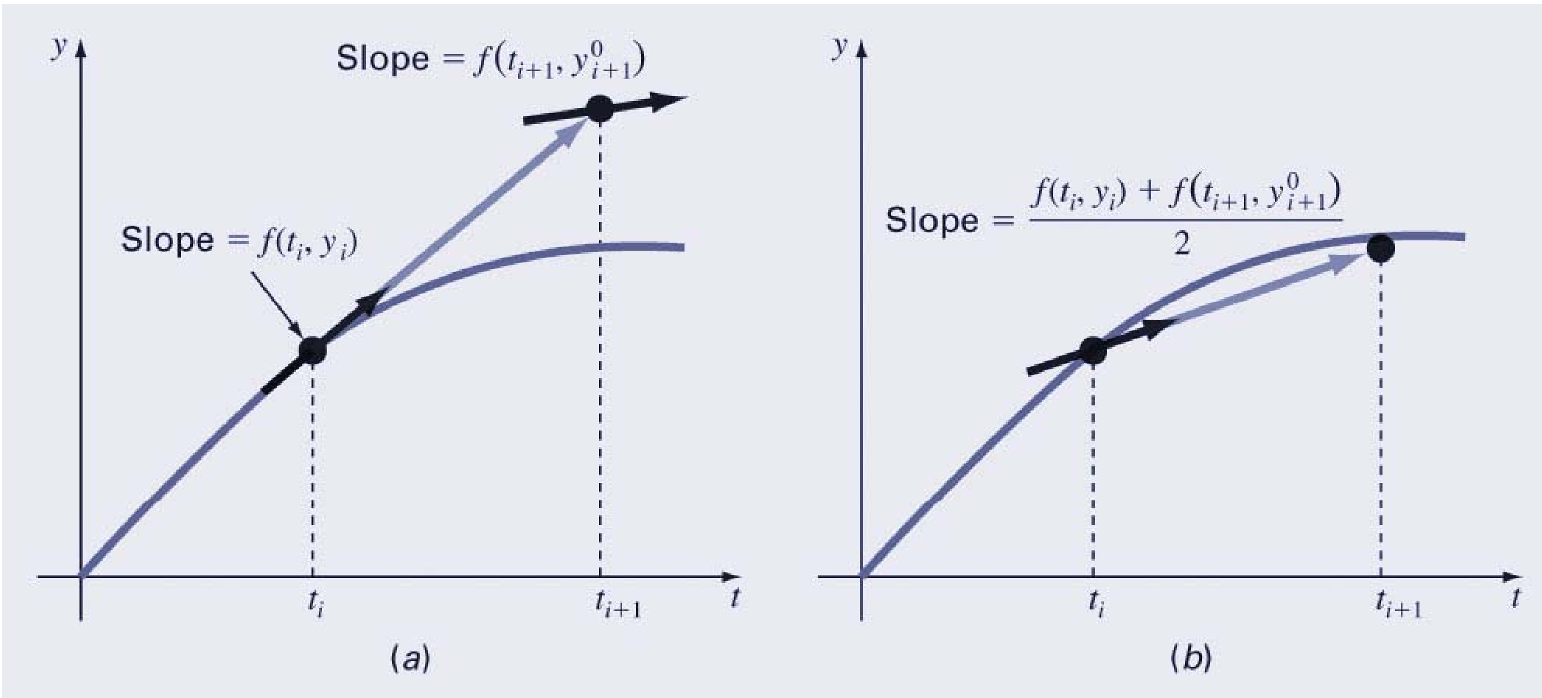
Two modifications can be made:

(1) Heun's method

(2) Mid-point method

Heun's Method (Predictor-corrector approach)

Determine derivatives at the beginning and predicted ending of the interval and average them to obtain an improved estimate of the slope



Predictor: $$y^0_{i+1} = y_i + f(t_i, y_i)h$$

Corrector: $$y_{i+1} = y_i + \frac{f(t_i, y_i) + f(t_{i+1}, y^0_{i+1})}{2}h$$

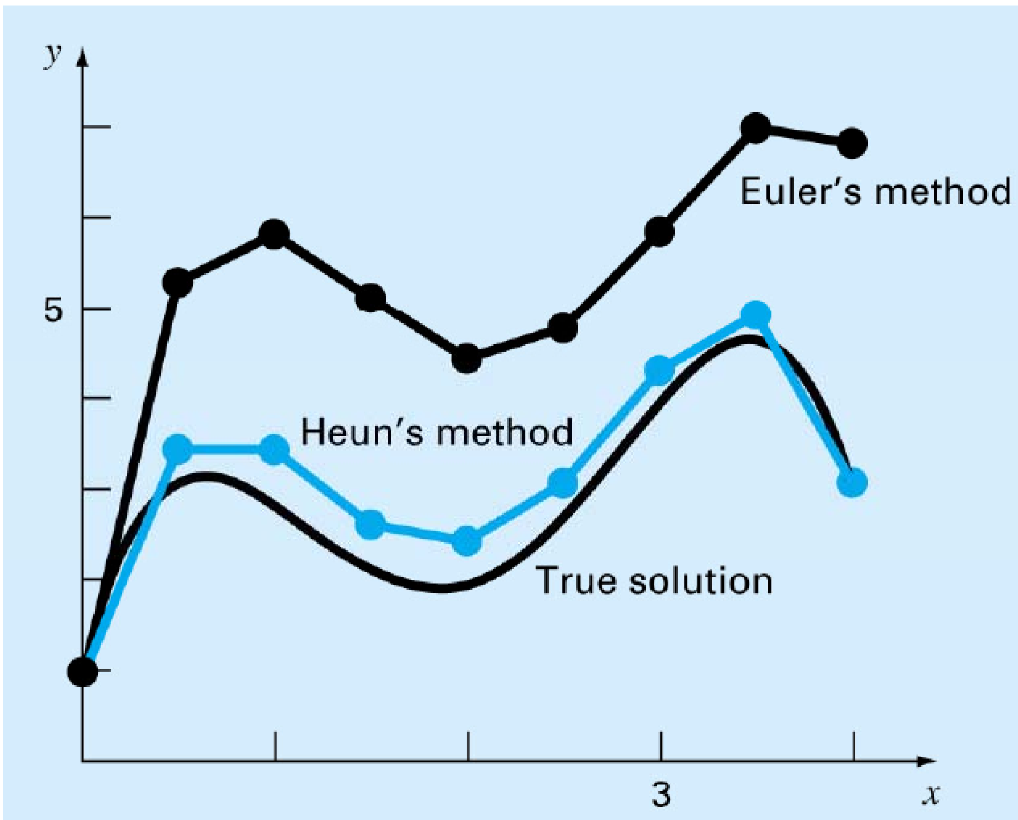
$$E_a = \frac{f''(\xi)}{12}h^3$$

$$E_a = O(h^3)$$

Heun's Method (Predictor-corrector approach)

predictor-corrector approach can be iterated to convergence:

$$y_{i+1}^{j} \longleftarrow y_i^m + \frac{f(t_i, y_i^m) + f(t_{i+1}, y_{i+1}^{j-1})}{2} h$$

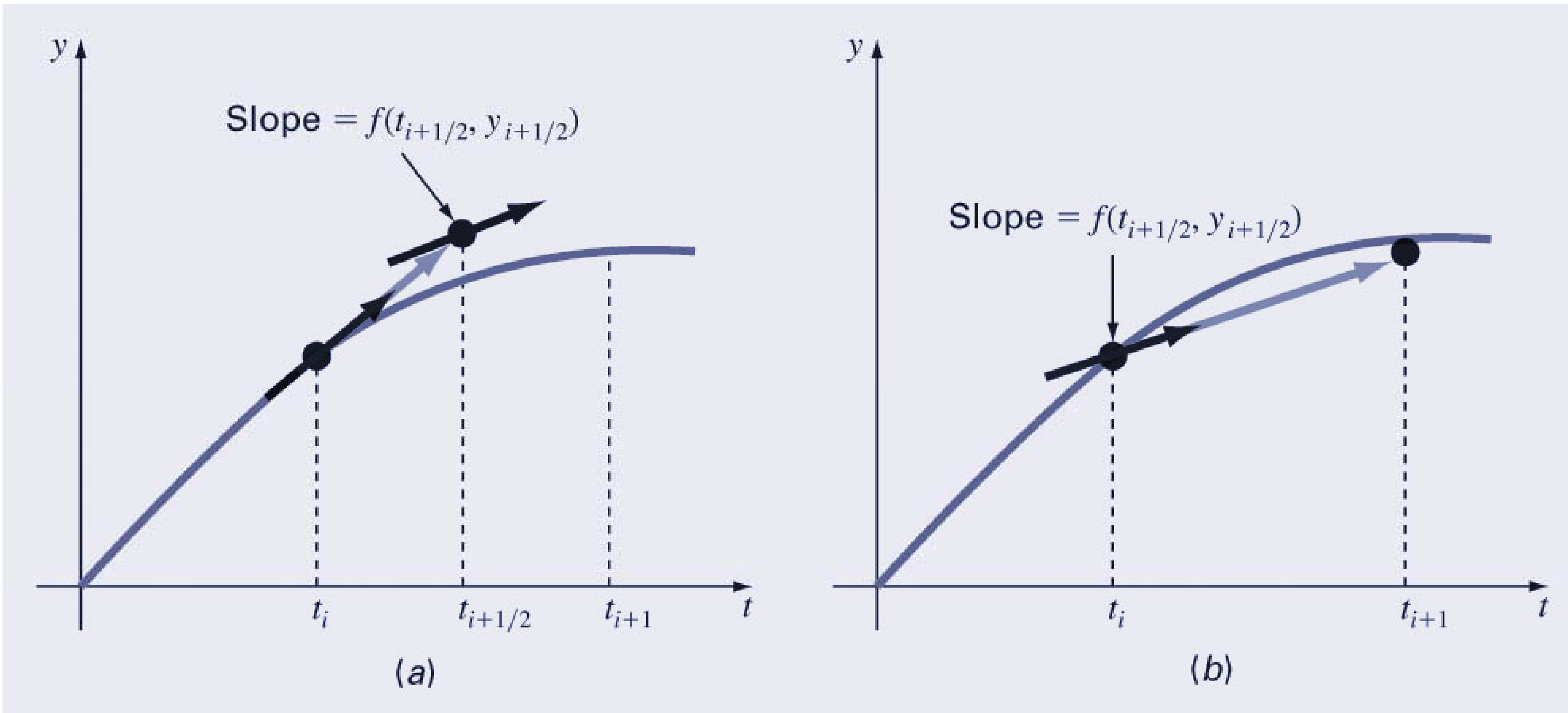


$$E_a = \frac{f''(\xi)}{12} h^3$$

$$E_a = O(h^3)$$

Midpoint Method

Similar to Heun's method, but predicts the slope at the midpoint of an interval rather than at the end



$$y_{i+1/2} = y_i + f(t_i, y_i)h/2$$

$$y_{i+1} = y_i + f(t_{i+1/2}, y_{i+1/2})h$$

local truncation error $O(h^3)$
global error $O(h^2)$

Runge-Kutta Methods: general form

Can achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives.

The increment function ϕ can be generally written as

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

$$\begin{aligned}
k_1 &= f(t_i, y_i) \\
k_2 &= f(t_i + p_1 h, y_i + q_{11} k_1 h) \\
k_3 &= f(t_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \\
&\vdots \\
k_n &= f(t_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)
\end{aligned}$$

a's, p's and q's are constants

4^{th} order Runge-Kutta method

The most popularly used method

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

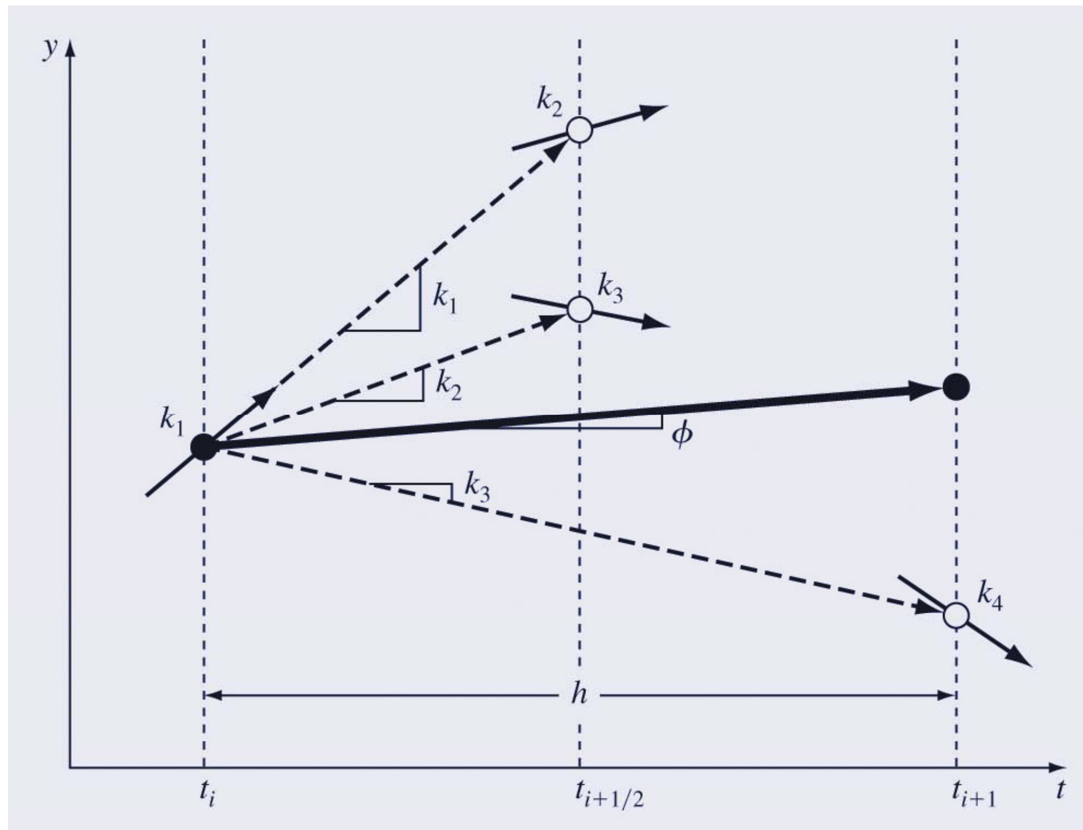
where

$$k_1 = f\left(t_i, y_i\right)$$

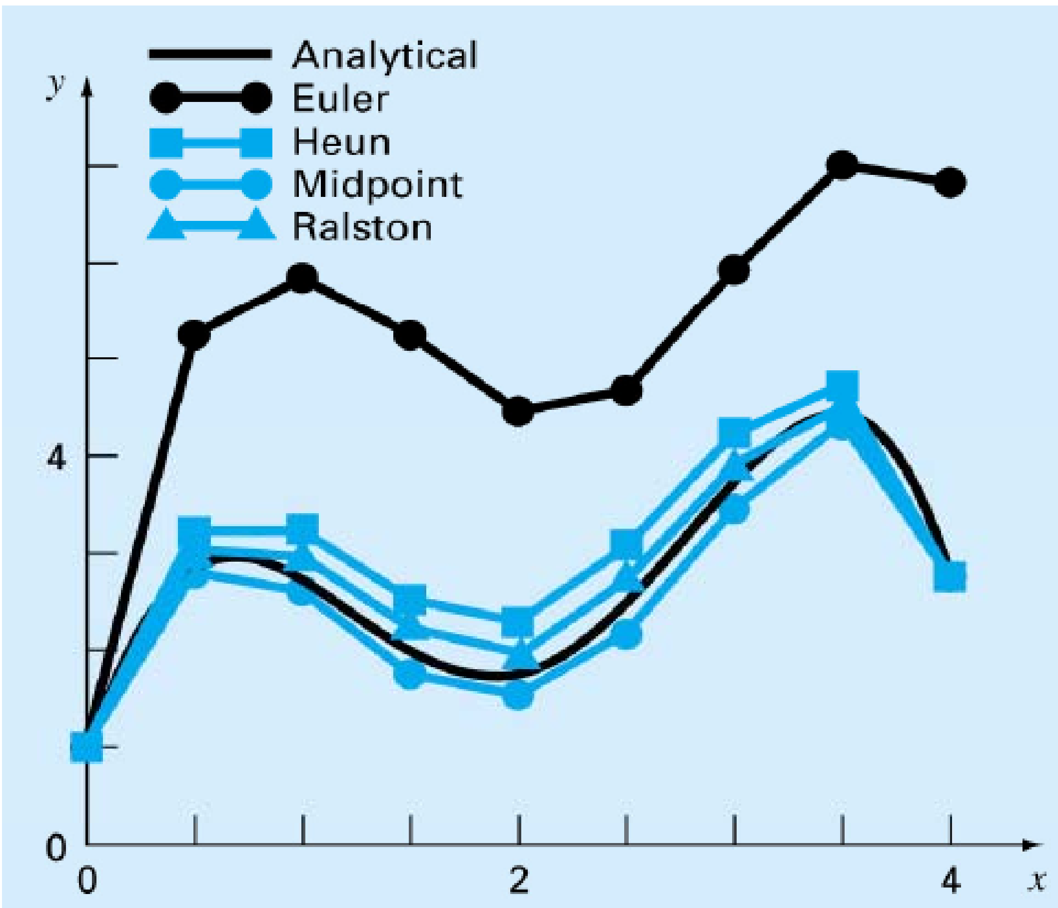
$$k_2 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

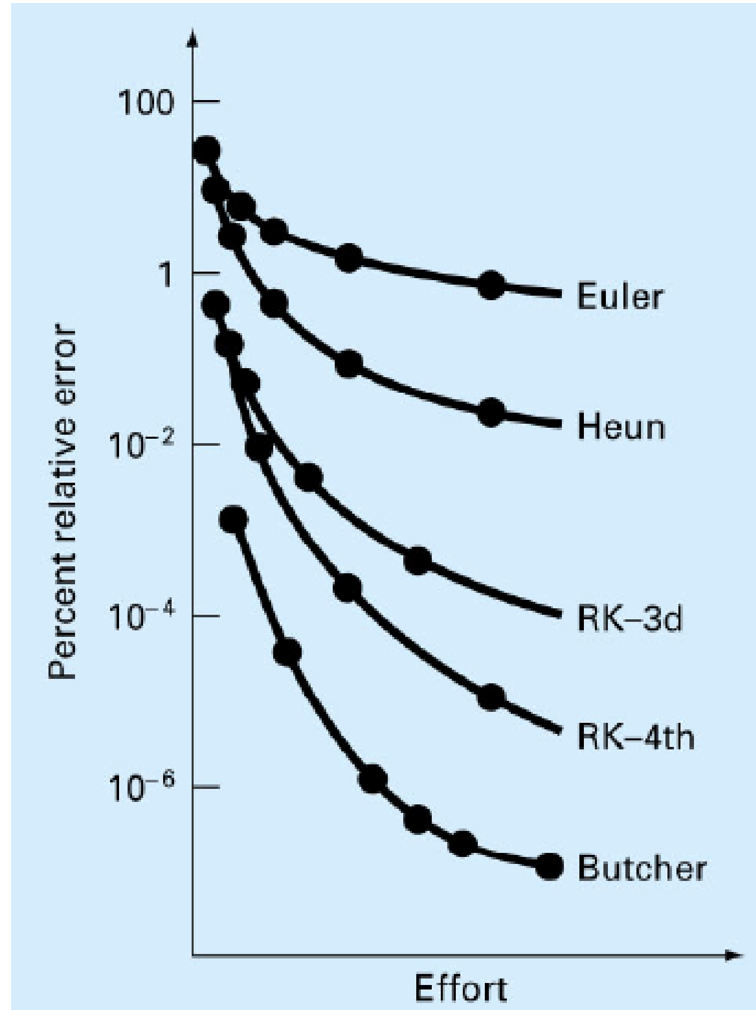
$$k_3 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(t_i + h, y_i + k_3 h\right)$$



Performance of different methods





A system of ODE

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2, \cdots, y_n)$$
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2, \cdots, y_n)$$
$$\vdots$$
$$\frac{dy_n}{dt} = f_n(t, y_1, y_2, \cdots, y_n)$$

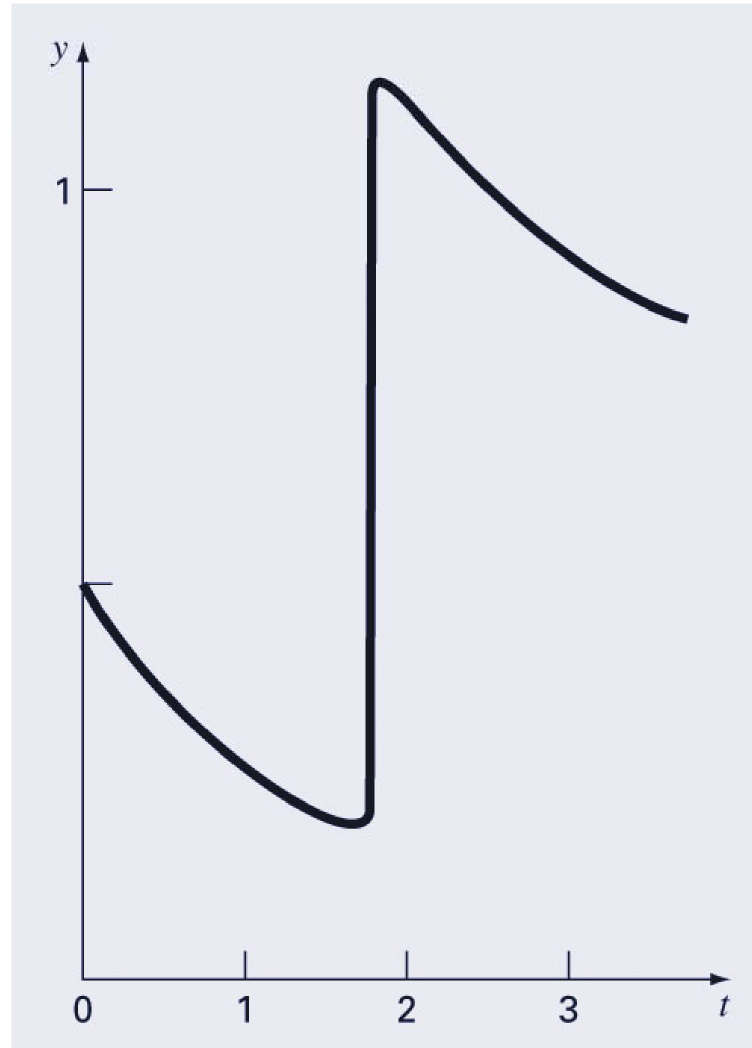
n initial conditions be known at the starting value of *t*.

One-step method is applied for every equation at each step before proceeding to the next step

-Euler's method
-4th order R-K

Adaptive solution methods

The solutions to some ODE problems exhibit multiple time scales - for some parts of the solution the variable changes slowly, while for others there are abrupt changes.



Adaptive algorithms can change step-size depending on the region

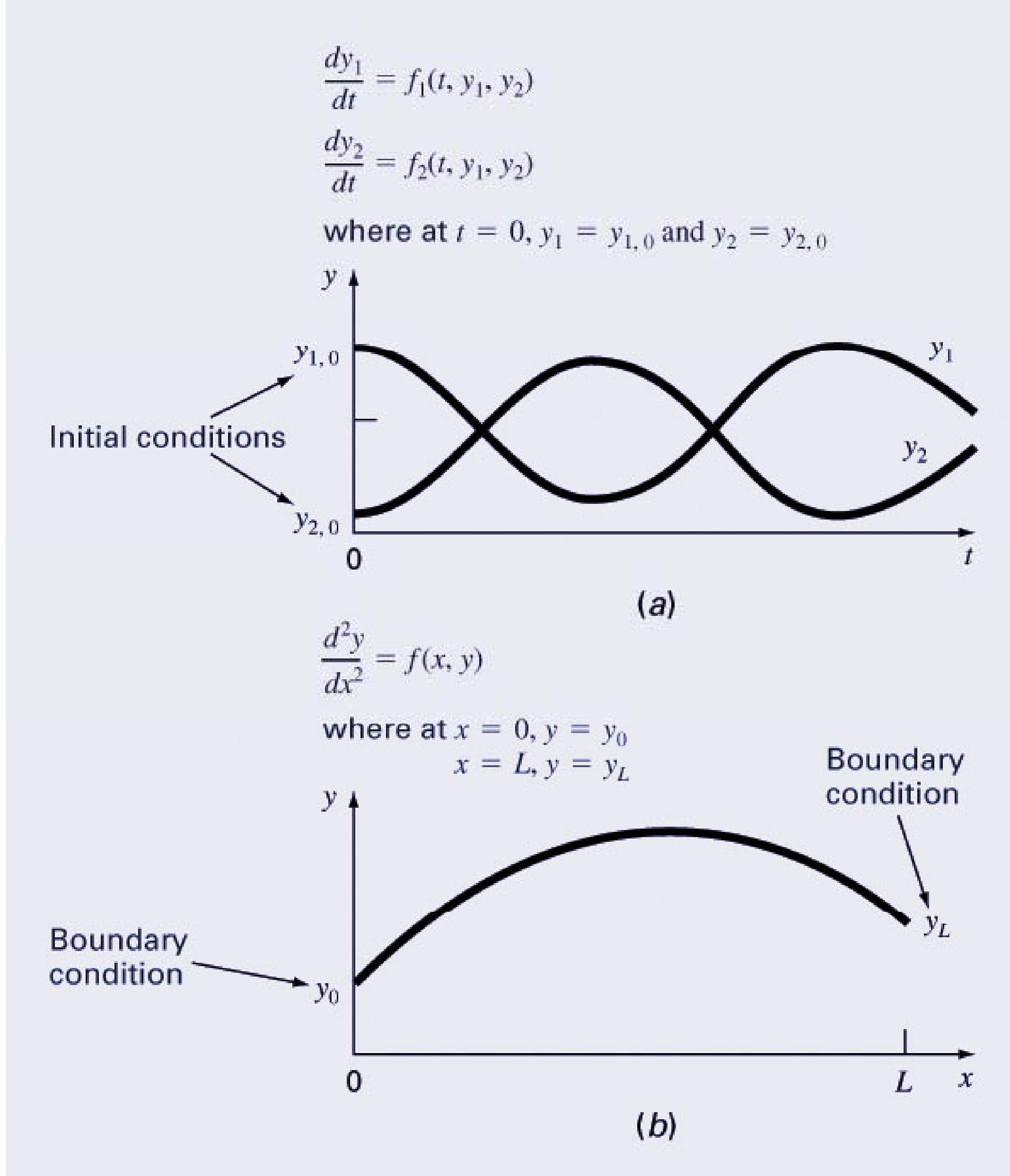
-Step-halving

-Performing two RK predictions of different order

Boundary value problems

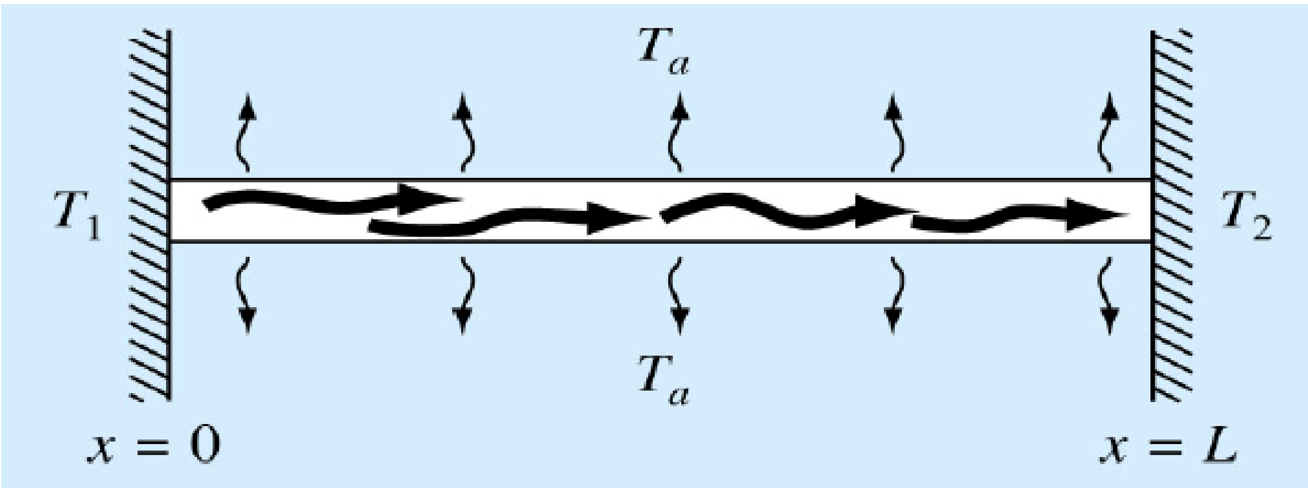
Conditions are not known at a single point but rather are given at different values of the independent variable

Boundary Conditions:
-values of variables
-values of derivatives of variables



Example: Boundary value problem

Non-insulated rod positioned between bodies of constant but different temperature



$$\frac{d^2T}{dx^2} + h'(T_\infty - T) = 0$$

Boundary conditions: $T(0) = T_1 = 40, \quad T(L) = T_2 = 200$

$$L = 10m \quad T_a = 20 \quad h' = 0.01m^{-2}$$

Analytical solution:

$$T = 73.4523e^{0.1x} - 53.4523e^{-0.1x} + 20$$

Numerical solution: the shooting method

the boundary-value problem is converted into an equivalent initial-value problem.

$$\frac{d^2T}{dx^2} + h'(T_\infty - T) = 0 \Rightarrow \begin{cases} \dfrac{dT}{dx} = z \\ \dfrac{dz}{dx} = -h'(T_\infty - T) \end{cases}$$

Generally, the equivalent system will not have sufficient initial conditions

- A guess is made for any undefined values.
- The guesses are changed until the final solution satisfies all the B.C.

For linear ODEs, only two "shots" are required - the proper initial condition can be obtained as a linear interpolation of the two guesses.

Numerical solution: the shooting method

Solution using RK method:

$$\left.\begin{aligned}\frac{dT}{dx} &= z\\ \frac{dz}{dx} &= h'(T-T_a)\end{aligned}\right\} \Rightarrow \begin{aligned} T_{i+1} &= T_i + (a_1 k_{T,1} + a_2 k_{T,2})h \\ z_{i+1} &= z_i + (a_1 k_{z,1} + a_2 k_{z,2})h \end{aligned}$$

$$T_{i+1} = T_i + \left[\frac{1}{3}k_{T,1} + \frac{2}{3}k_{T,2}\right]h \quad ; \quad k_{T,1} = f_T(x_i, z_i) \quad ; \quad k_{T,2} = f_T(x_i + \frac{3}{4}h, z_i + \frac{3}{4}k_{T,1}h)$$

or

$$k_{T,1} = z_i \quad ; \quad k_{T,2} = z_i + \frac{3}{4}k_{T,1}h = z_i + \frac{3}{4}z_i h$$

$$z_{i+1} = z_i + \left[\frac{1}{3}k_{z,1} + \frac{2}{3}k_{z,2}\right]h \quad ; \quad k_{z,1} = f_z(x_i, T_i) \quad ; \quad k_{z,2} = f_z(x_i + \frac{3}{4}h, T_i + \frac{3}{4}k_{T,1}h)$$

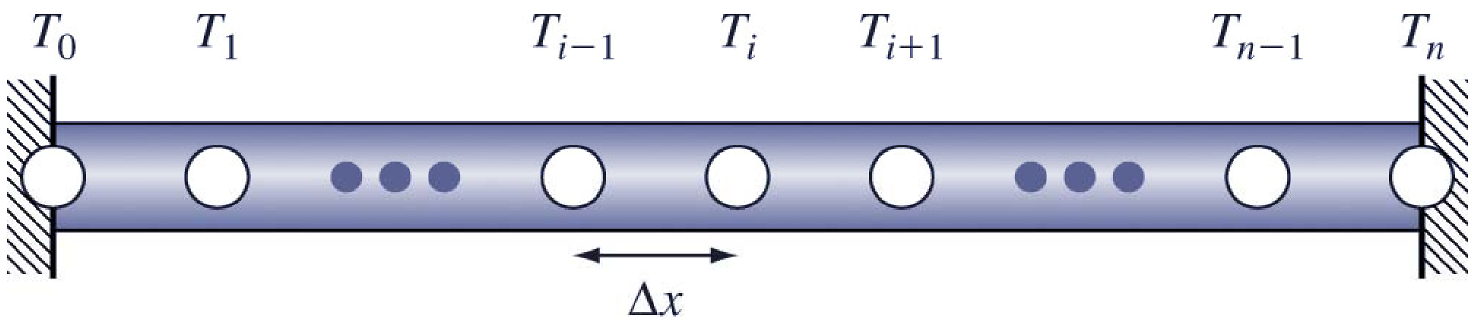
or

$$k_{z,1} = h'(T_a - T_i) \quad ; \quad k_{z,2} = h'[T_a - (T_i + \frac{3}{4}k_{z,1}h)]$$

Numerical solution: finite difference

finite differences are substituted for the derivatives in the original equation

$$\frac{d^2T}{dx^2} + h'(T_\infty - T) = 0$$



$$\frac{d^2T}{dx^2} = \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2}$$

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} + h'(T_\infty - T_i) = 0$$

$$-T_{i-1} + (2 + h'\Delta x^2)T_i - T_{i+1} = h'\Delta x^2 T_\infty$$

Numerical solution: finite difference

The linear differential equation is transformed into a set of simultaneous algebraic equations.

Since T_0 and T_n are known (Drichlet boundary conditions), they will be on the right-hand-side of the linear algebra system

$$\begin{bmatrix} 2+h'\Delta x^2 & -1 & & \\ -1 & 2+h'\Delta x^2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2+h'\Delta x^2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{n-1} \end{Bmatrix} = \begin{Bmatrix} h'\Delta x^2 T_\infty + T_0 \\ h'\Delta x^2 T_\infty \\ \vdots \\ h'\Delta x^2 T_\infty + T_n \end{Bmatrix}$$

Numerical solution: finite difference

If there is a derivative boundary condition (Neumann B.C.), the centered difference equation is solved at the point and rewriting the system equation accordingly.

For a Neumann condition at T_0 point:

$$\left.\frac{dT}{dx}\right|_0 = \frac{T_1 - T_{-1}}{2\Delta x} \Rightarrow T_{-1} = T_1 - 2\Delta x\left(\left.\frac{dT}{dx}\right|_0\right)$$

$$-T_{-1} + \left(2 + h'\Delta x^2\right)T_0 - T_1 = h'\Delta x^2 T_\infty$$

$$-\left[T_1 - 2\Delta x\left(\left.\frac{dT}{dx}\right|_0\right)\right] + \left(2 + h'\Delta x^2\right)T_0 - T_1 = h'\Delta x^2 T_\infty$$

$$\left(2 + h'\Delta x^2\right)T_0 - 2T_1 = h'\Delta x^2 T_\infty - 2\Delta x\left(\left.\frac{dT}{dx}\right|_0\right)$$