CE 205: Numerical Methods

Partial Differential Equations

What is PDE?

An equation involving partial derivatives of an unknown function of two or more independent variables is called a partial differential equation (PDE)

$$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y$$

Linear

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x$$

$$\frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = x$$

Non-Linear

Because of their widespread application in engineering, we shall focus on linear, second-order equations

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, and C are functions of x and y and D is a function of x, y, u, $\partial u/\partial x$, and $\partial u/\partial y$.

Categories of Linear, 2nd order PDEs

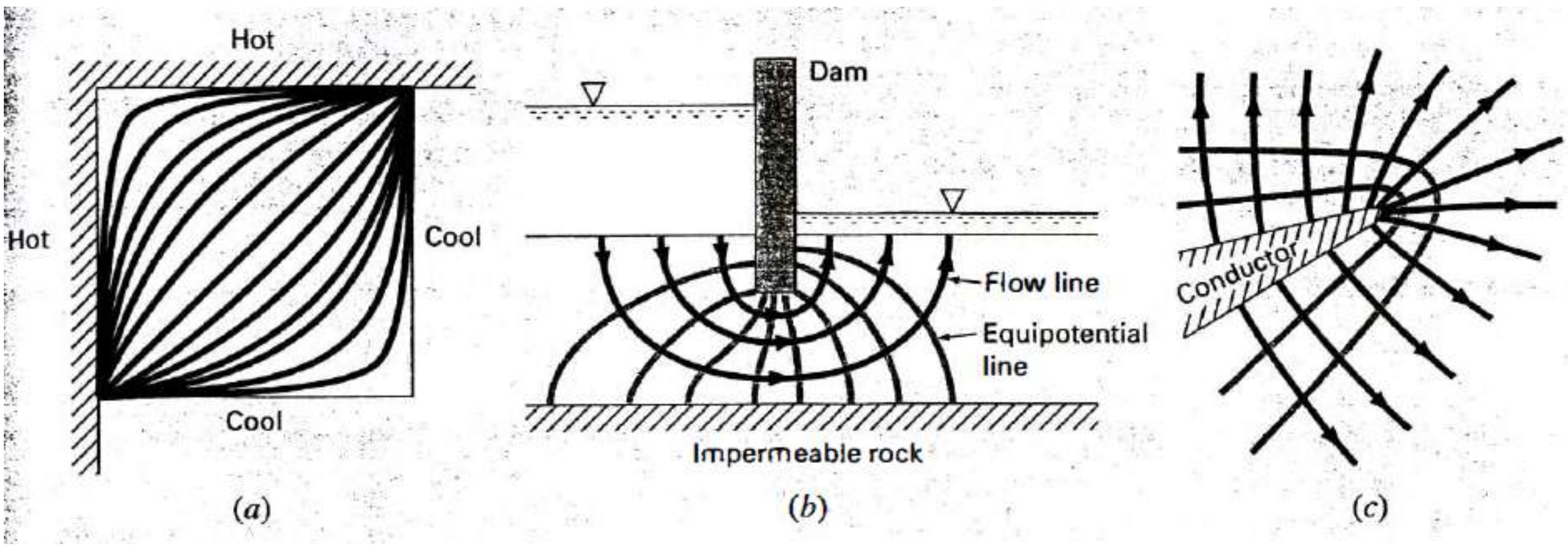
$B^2 - 4AC$	Category	Example
< 0	Elliptic	Laplace equation (steady state with two spatial dimensions) $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$
$= 0$	Parabolic	Heat conduction equation (time variable with one spatial dimension) $\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2}$
> 0	Hyperbolic	Wave equation (time variable with one spatial dimension) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$

Each category relates to specific and distinct engineering problem contexts that demand special solution techniques

A, B, and C depend on *x* and *y,* the equation may fall into a different Category depending on the solution domain.

Elliptic Equations

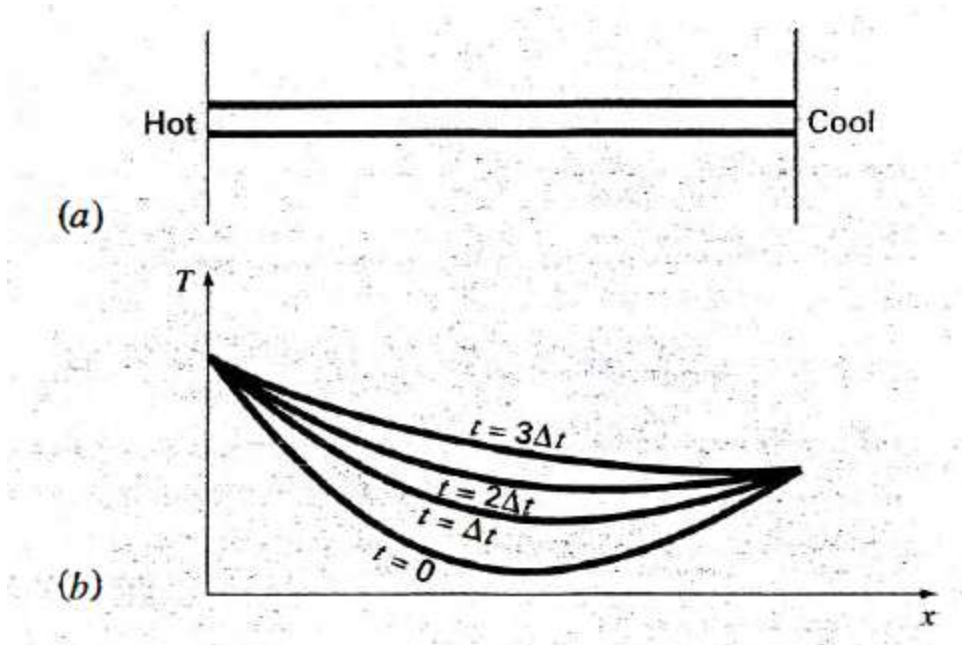
Elliptic equations are typically used to characterize steady-state systems. (the absence of a time derivative)
Typically employed to determine the steady-state distribution of an unknown in two spatial dimensions.



(a) Temperature distribution in a heated plate, (b) seepage of water under a dam, and (c) Electric field near the point of a conductor

Parabolic Equations

Parabolic equations determine how an unknown varies in both space and time. (the presence of both spatial and temporal derivatives)



A long, thin rod that is insulated everywhere but at its end. The dynamics of the onedimensional distribution of temperature along the rod's length can be described by a parabolic PDE.

Hyperbolic Equations

The hyperbolic category also deals with propagation problems.

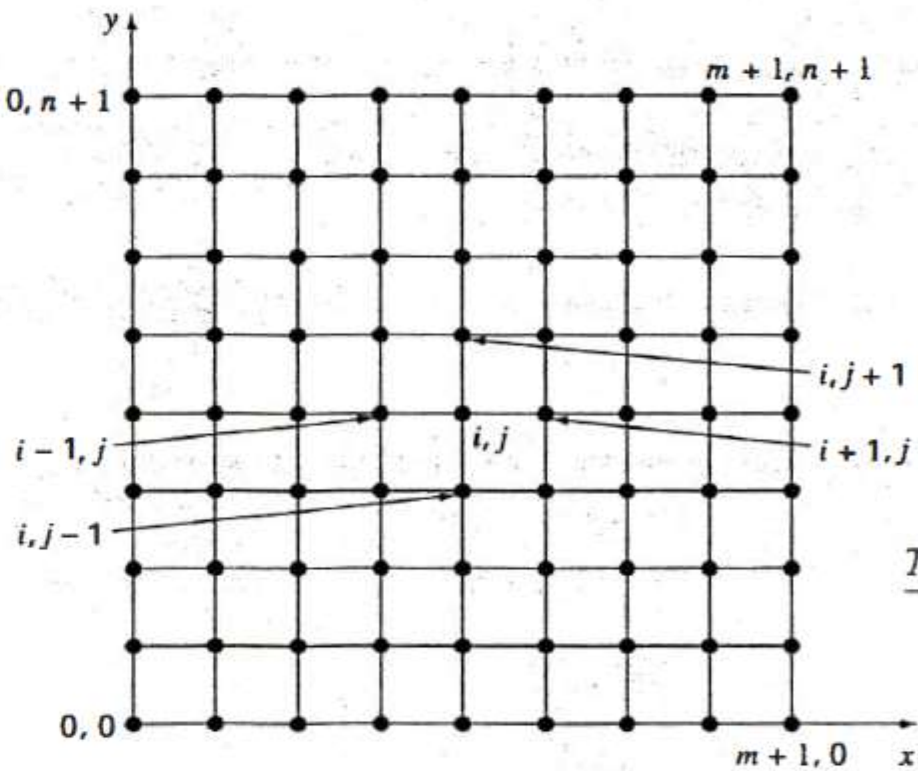
An important distinction manifested by the wave equation is that the unknown is characterized by a second derivative with repsect to time. As a consequence, the solution oscillates.

A taut string vibrating at a low amplitude is a simple physical system that con be characterized by a hyperbolic POE.

Other problems: vibrations of rods and beams, motion of fluid waves, and transmission of sound and electrical signals.

Solution of the Laplace Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



Central finite-difference scheme:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2}$$

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$$

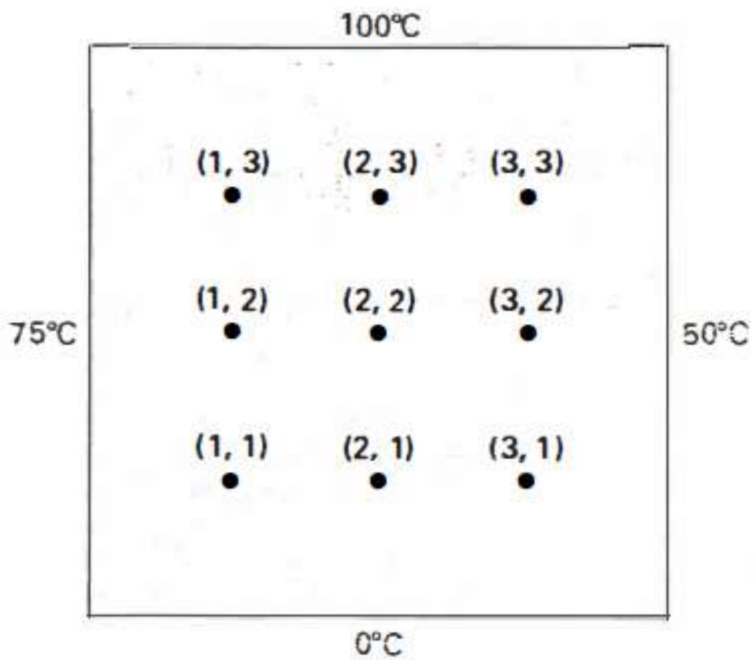
For square grids (i.e. Δx = Δy)

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Problem: Drichlet Boundary Condition

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

A heated plate where boundary temperatures are held constant (Drichlet boundary condition)



For node (1,1)

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

Since T_{01} = 75 and T_{10} = 0

$$-4T_{11} + T_{12} + T_{21} = -75$$

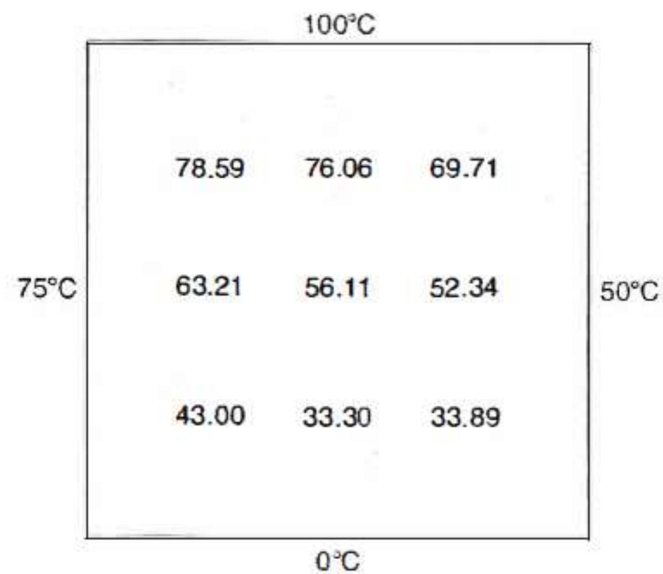
Similar equations can be developed for the other interior points

Solution with Drichlet Boundary Condition

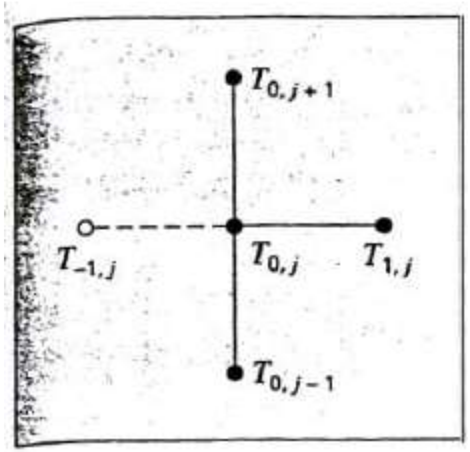
The result is the following set of nine simultaneous equations with nine unknowns

$$
\begin{array}{rrrrrrrrrcl}
4T_{11} & -T_{21} & & -T_{12} & & & & & & = & 75 \\
-T_{11} & +4T_{21} & -T_{31} & & -T_{22} & & & & & = & 0 \\
 & -T_{21} & +4T_{31} & & & -T_{32} & & & & = & 50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & = & 75 \\
 & -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & = & 0 \\
 & & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & = & 50 \\
 & & & -T_{12} & & & +4T_{13} & -T_{23} & & = & 175 \\
 & & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & = & 100 \\
 & & & & & -T_{32} & & -T_{23} & +4T_{33} & = & 150
\end{array}
$$

The solution



Neumann Boundary Condition Problem



A boundary node (0, j) on the left edge of a heated plate. (To approximate the derivative normal to the edge). An imaginary node (-1, j) is located at a distance Δx beyond the edge

Finite difference approximation for derivative at node (0, j):

$$\frac{\partial T}{\partial x} \cong \frac{T_{1,j} - T_{-1,j}}{2\,\Delta x}$$

Re-arranging

$$T_{-1,j} = T_{1,j} - 2\,\Delta x \frac{\partial T}{\partial x}$$

Equation for node (0,j)

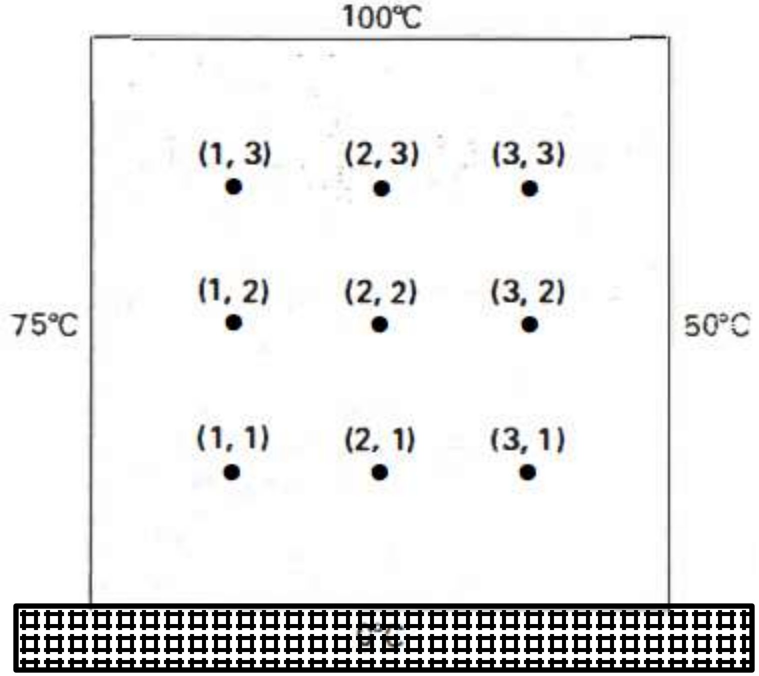
$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

Replacing $T_{-i,j}$ in the equation gives:

$$2T_{1,j} - 2\,\Delta x \frac{\partial T}{\partial x} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

Solution with Neumann Boundary Condition

Problem: Repeat the following problem with an insulated edge at the bottom



The general equation to characterize a derivative at the lower edge (i.e. *j* = 0)

$$T_{i+1,0} + T_{i-1,0} + 2T_{i,1} - 2\,\Delta y \frac{\partial T}{\partial y} - 4T_{i,0} = 0$$

For an insulated edge, the derivative is zero and the equation becomes

$$T_{i+1,0} + T_{i-1,0} + 2T_{i,1} - 4T_{i,0} = 0$$

Solution with Neumann Boundary Condition

The result is the following set of 12 simultaneous equations with 12 unknowns

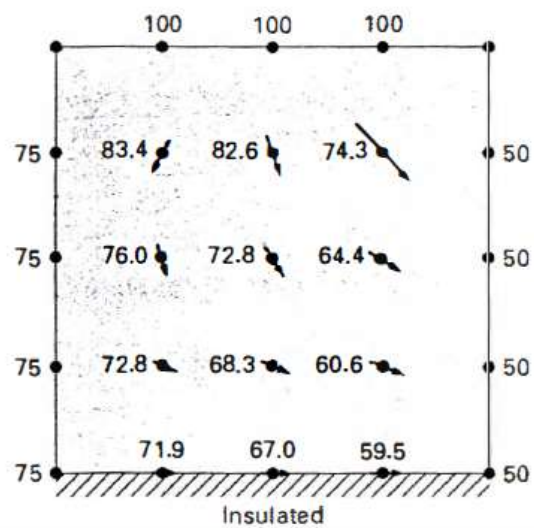
$$
\begin{bmatrix}
4 & -1 & & -2 & & & & & & & & \\
-1 & 4 & -1 & & -2 & & & & & & & \\
& -1 & 4 & & & -2 & & & & & & \\
-1 & & & 4 & -1 & & -1 & & & & & \\
& -1 & & -1 & 4 & -1 & & -1 & & & & \\
& & -1 & & -1 & 4 & & & -1 & & & \\
& & & -1 & & & 4 & -1 & & -1 & & \\
& & & & -1 & & -1 & 4 & -1 & & -1 & \\
& & & & & -1 & & -1 & 4 & & & -1 \\
& & & & & & -1 & & & 4 & -1 & \\
& & & & & & & -1 & & -1 & 4 & -1 \\
& & & & & & & & -1 & & -1 & 4
\end{bmatrix}
\begin{Bmatrix} T_{10} \\ T_{20} \\ T_{30} \\ T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{Bmatrix}
=
\begin{bmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{bmatrix}
$$

$T_{10} = 71.91 \qquad T_{20} = 67.01 \qquad T_{30} = 59.54$

$T_{11} = 72.81 \qquad T_{21} = 68.31 \qquad T_{31} = 60.57$

$T_{12} = 76.01 \qquad T_{22} = 72.84 \qquad T_{32} = 64.42$

$T_{13} = 83.41 \qquad T_{23} = 82.63 \qquad T_{33} = 74.26$

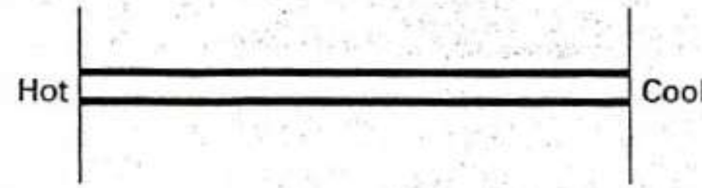


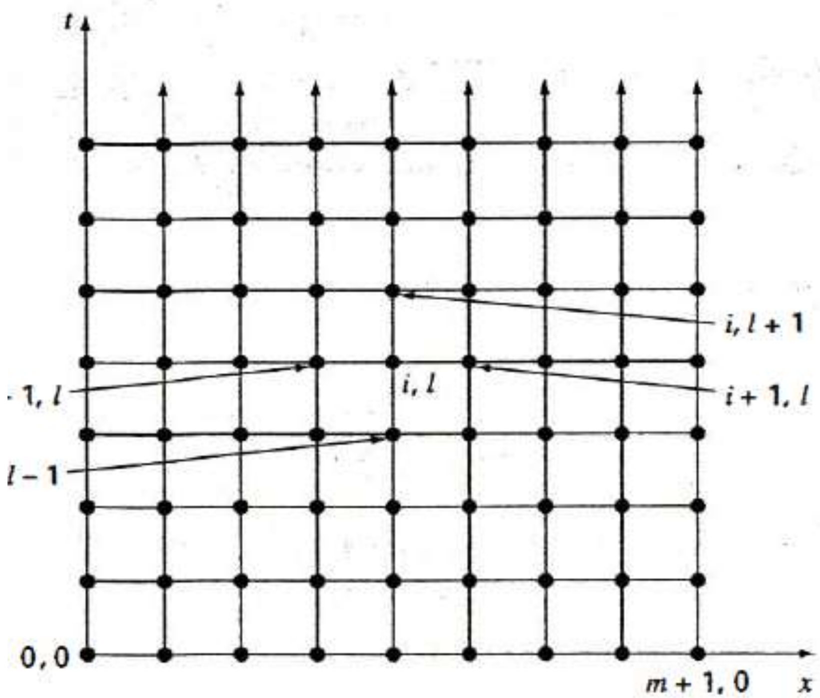
Parabolic PDEs

The *heat conduction* equation:

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Example: A thin rod, insulated at all points except at its end



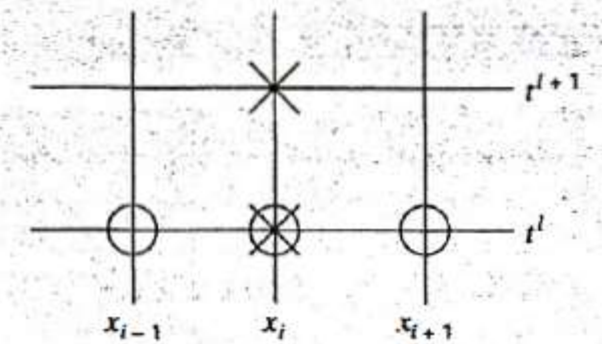


Whereas elliptic equations were bounded in all relevant dimensions, **parabolic PDEs are temporally open-ended**

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{\Delta x^2}$$

$$\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

Parabolic PDEs: Explicit Method of Solution



$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{\Delta x^2}$$

$$\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

$$k\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

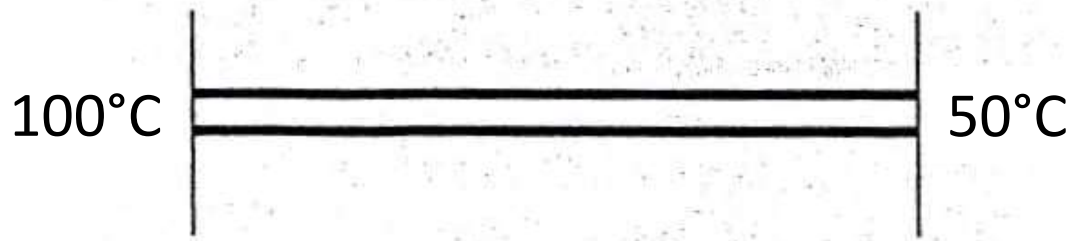
which can be solved for

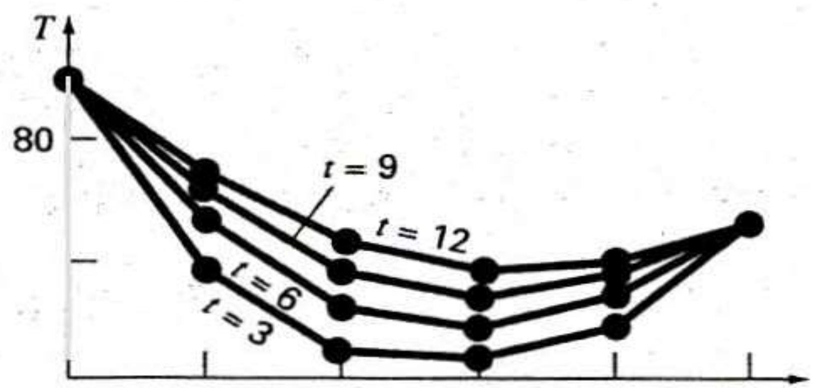
$$T_i^{l+1} = T_i^l + \lambda(T_{i+1}^l - 2T_i^l + T_{i-1}^l)$$

where $\lambda = k\,\Delta t/(\Delta x)^2$.

Problem: Explicit Method

Use the explicit method to solve for the temperature distribution of a long, thin rod with a length of 10 cm and the following values: k = 0.835 cm^2/sec, Δx = 2 cm and Δt = 0.1s. At t = 0, the temperature of the rod is zero and the boundary conditions are fixed for all times at T(0) = 100°C and T(10) = 50°C.





Convergence and Stability

Convergence means that as Δx and Δt approach zero, the results of the finite-difference technique approach the true solution.

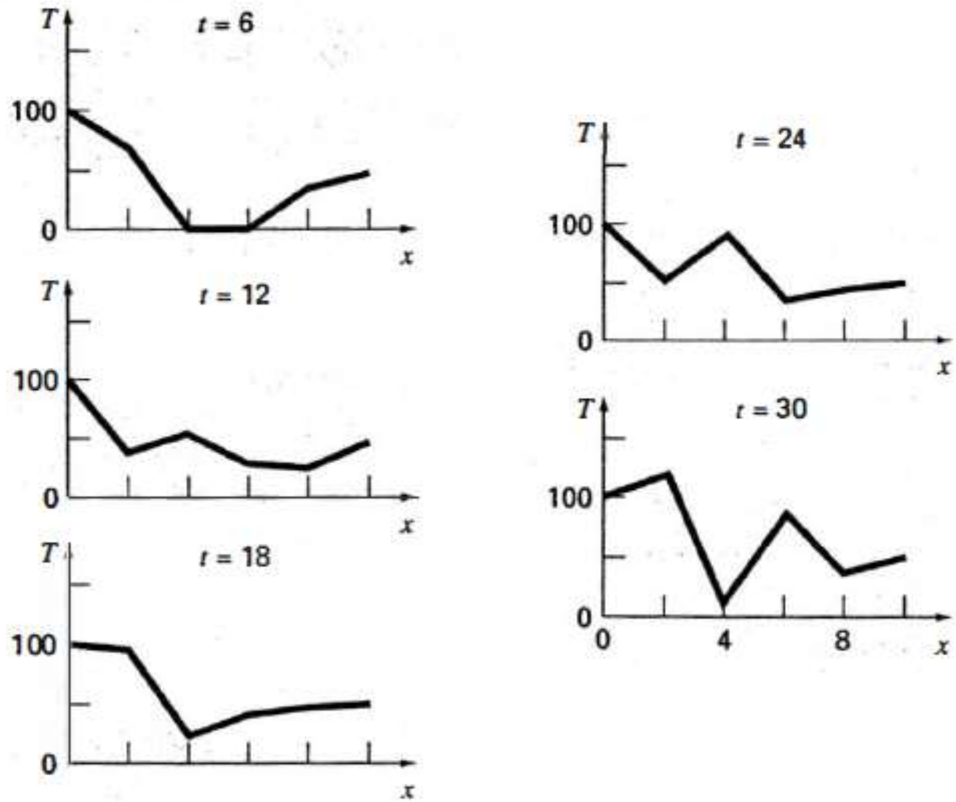
Stability means that errors at any stage of the computation are not amplified but are attenuated as the computation progresses.

It can be shown that the explicit method is both convergent and stable if $\lambda \le 0.5$ or

$$\Delta t \le \frac{1}{2}\frac{\Delta x^2}{k}$$

setting $\lambda \le 1/2$ could result in a solution in which errors do not grow, but oscillate. Setting $\lambda \le 1/4$ ensures that the solution will not oscillate.

Convergence and Stability

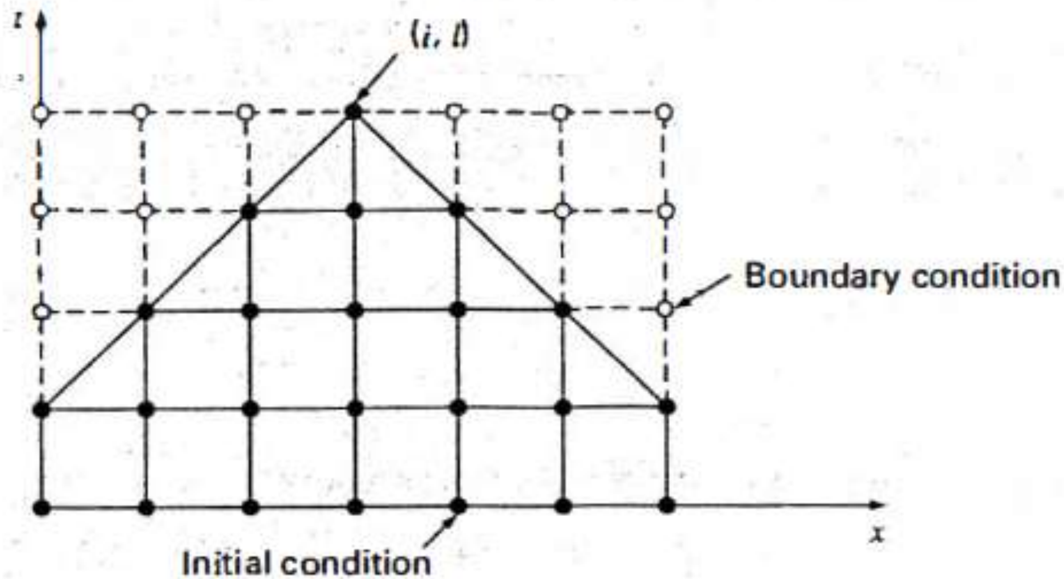


If Δx is halved to improve the approximation of the spatial second derivative, the time step must be quartered to maintain convergence and stability.
for the one-dimensional case, halving Δx results in an eightfold increase in the number of calculations.

Limitations of Explicit Method

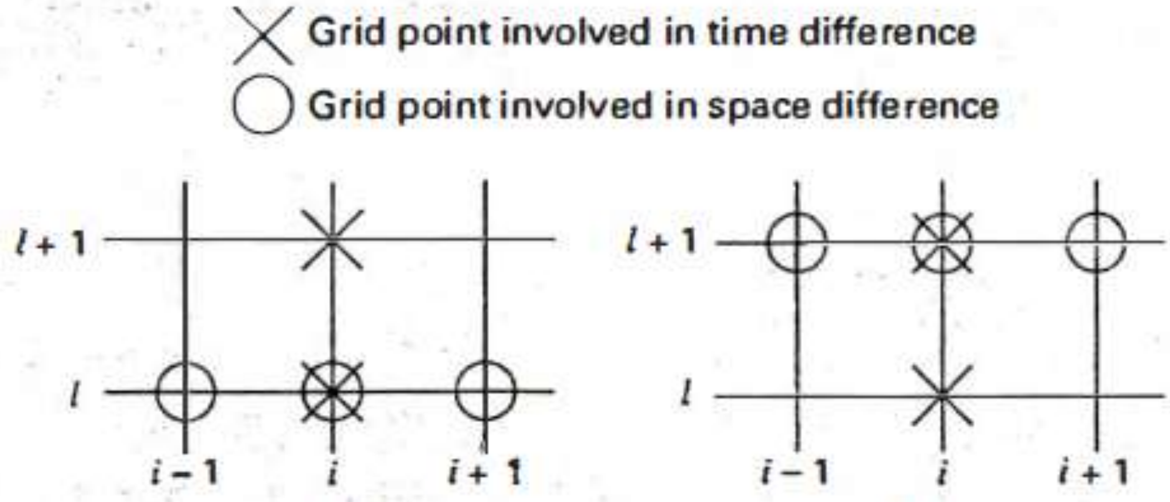
Explicit finite-difference formulations have **problems related to stability**.

Explicit methods **exclude information** that has a bearing on the solution.



Implicit methods overcome both these difficulties at the expense of somewhat more complicated algorithms.

Parabolic PDEs: Implicit Method of Solution



(a) Explicit (b) Implicit

In implicit methods, the spatial derivative is approximated at an advanced time level

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}$$

Parabolic PDEs: Implicit Method of Solution

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad \Rightarrow \quad k\frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

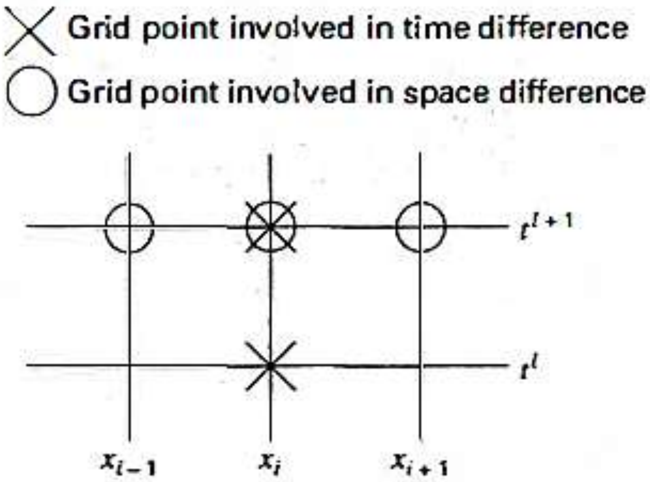
Which can be expressed as: $$-\lambda T_{i-1}^{l+1} + (1 + 2\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^l$$

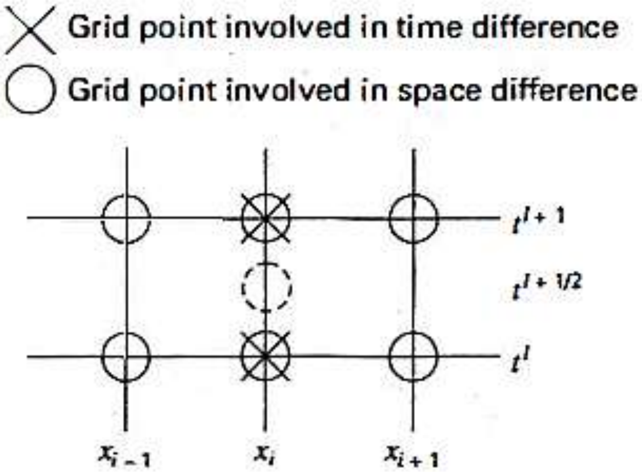
where $\lambda = k\,\Delta t/(\Delta x)^2$.

The resulting difference equations contains several unknowns which cannot be solved explicitly; **the entire system of equations must be solved simultaneously**

Implicit methods are unconditionally stable; but there is an accuracy issue for the use of large time steps

Crank-Nicolson Method





General Implicit Method | **Crank-Nicolson Method**

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{1}{2}\left[\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right]$$

$$\frac{\partial T}{\partial t} \cong \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^{l} + 2(1-\lambda)T_i^{l} + \lambda T_{i+1}^{l}$$

where $\lambda = k\,\Delta t/(\Delta x)^2$.