2(a) ← 2012-2013 $\boxed{10-11-6(a)}$ $\boxed{\text{page } 35 - \text{Kre}}$

1st step : Modeling

Time rate of change $= y' = \dfrac{dy}{dt} =$ inflow rate of salt $-$ outflow rate of salt

Rate of inflow $= 50(1+\cos t)$

Rate of outflow $= 0.05y$

$y' = 50(1+\cos t) - 0.05y$

1 gal contains $\dfrac{y(t)}{1000}$

$\therefore$ 50 " " $\dfrac{50y}{1000}$

$= 0.05y$

2nd Step :- Solution

$$y' + 0.05y = 50(1+\cos t)$$

$$p = 0.05$$

$$h = \int p\,dt = 0.05\int dt = 0.05t$$

(*) $$y(x) = e^{-h}\left[\int e^{h} r\,dx + c\right]$$

$$= e^{-0.05t}\left[\int e^{0.05t}\, 50(1+\cos t)\,dt + c\right] \quad \text{--- (1)}$$

Now $\int e^{0.05t}\, 50(1+\cos t)\,dt$

$$= 50\int e^{.05t}\,dt + 50\int e^{.05t}\cos t\,dt$$

$$= 50\,\frac{e^{.05t}}{0.05} + 50\int e^{.05t}\cos t\,dt \quad \text{--- (a)}$$

$$I = \int e^{.05t} \cos t \, dt$$

$$= \frac{e^{.05}}{(.05)^2 + (1)^2} (.05 \cos t + \sin t)$$

$$= e^{.05} \left\{ \frac{.05 \cos t}{(.05)^2 + 1} + \frac{1 \times \sin t}{(.05)^2 + 1} \right\}$$

$$= e^{.05} (.04987 \cos t + 1 \sin t)$$

ⓐ ⇒ $$= 50 \frac{e^{.05t}}{0.05} + 50 e^{.05} (.04987 \cos t + \sin t)$$

$$= 1000 e^{.05t} + 2.49 e^{.05} \cos t + 50 \sin t e^{.05}$$

① ⇒ $$y(x) = e^{-.05t} \left[e^{.05t} \{1000 + 2.49 \cos t + 50 \sin t\} + c \right]$$

$$= 1000 + 2.49 \cos t + 50 \sin t + c e^{-.05t}$$

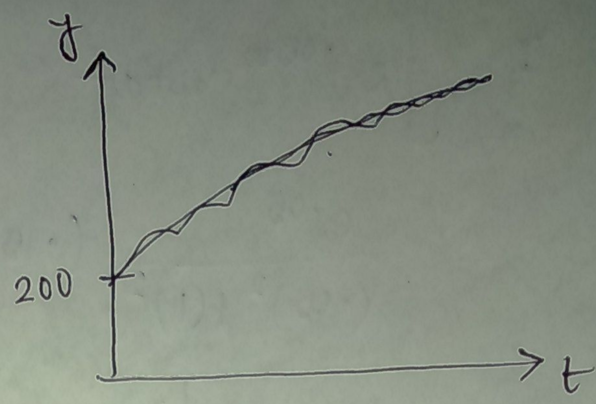
initial condition

$y(0) = 200$

$$\therefore \; y(0) = 200 = 1000 + 2.49 + c$$

$$\boxed{c = -802.5}$$

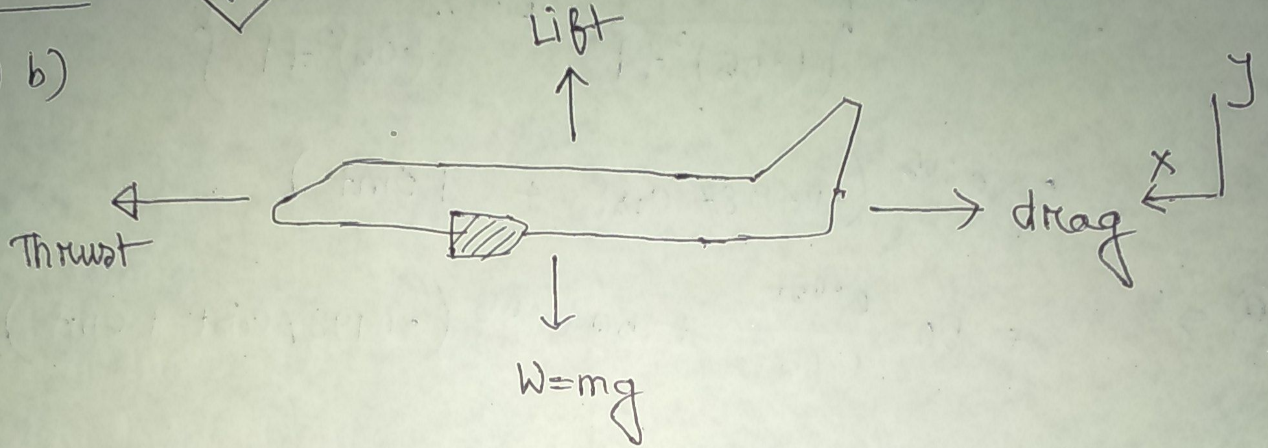
$$y(t) = 1000 + 2.49 \cos t + 50 \sin t - 802.5 e^{-.05t}$$



11-12
5(b)

2012-13

1) b)



An airplane flies at a velocity v

The affecting factor's here are

(I) weight w with acts downward due to the gravitational acceration.

(II) An upward Force called lift acts opposite to g.

(III) Thrust in the direction of the airple flies. This causes ~~due to~~ by engine

(IV) Drag force in the opposite direction of the airplane velocity. due to air.

X direction

$$F_{thrust} - F_{drag} = m\frac{dv}{dt} \quad \text{--- (1)}$$

For calculating $F_{\cancel{drag}\ thrust} \Rightarrow$

$$dp = dm \times V \quad \text{--- (a)}$$

Again $F = \frac{dp}{dt}$

$$\Rightarrow dp = F\,dt$$

(a) $\Rightarrow$ $F\,dt = dm \times V$

$$F = \frac{dm}{dt} \times V$$

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① ⇒

$$F_{th} - F_{dr} = \frac{m\,dv}{dt}$$

$$\Rightarrow \boxed{\frac{dm}{dt} v - b v^2 = m \dot{v}} \quad \text{--- (1)}$$

This is the differential equation

And in y direction

$$F_{lift} = mg$$

in equation (1)

t = independent variable

m, v = dependent variable.

12-13

1) a) Independent variables are values that can be changed in a given equation or experiment. An independent variable represents information put into the equation.

Dependent variables are values that reasults directly from the independent variable. They depend on independent variables.

$y = 4x - 3$ x = independent variable
y = dependent ,,

* The values of independent value can be changed. dependent variables can be changed only by changing independent varable.

* indep. vari are manipulated
dep ,, ,, observed in the experiment

12-13

5(c) 11-12

1) c) Explict solution:- where dependent variable can be seperated.

$2y + 2x = 0$ is explict because if y dependent

Then $x = -2y$

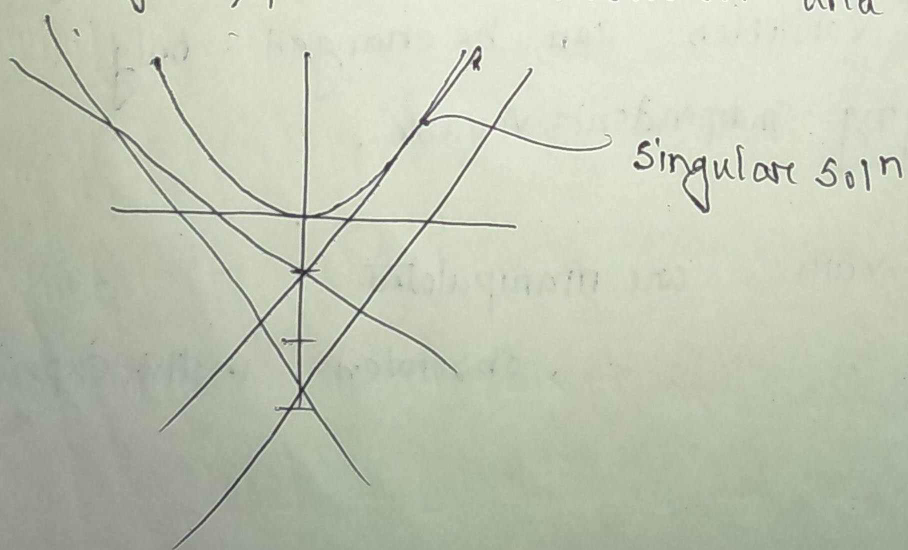
$\Rightarrow y = -x/2$ y has been seperated.

Singular Solution:- A differential equation may sometimes have an additional solution that can't be obtained from general soln. called sing. soln.

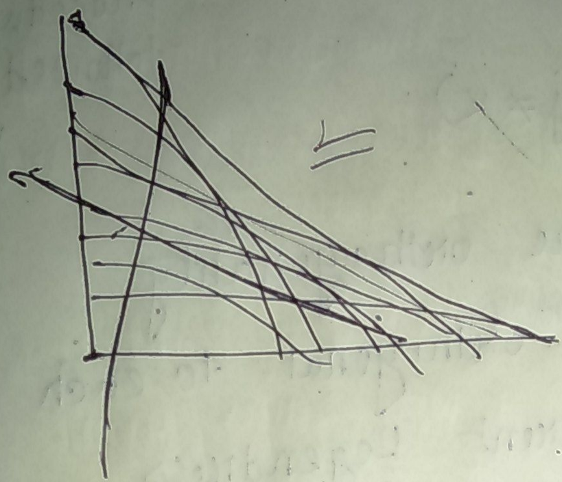
$y'^2 - xy' + y = 0$

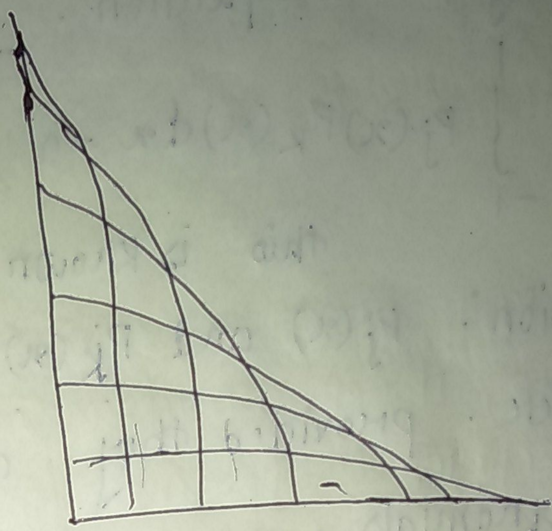
general soln $y = cx - c^2$

$y = x^2/4$ is also a solution and it is singular soln.



Example of singular Solution   Jatiyo Smriti Soudho





12-13 2) c) $\underline{11-12 \ . \ 8(d)}$ $\underline{10-11 \ 8(b)}$ 1st

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

#Pg - 236 Greenberg

where λ = constant. This eqn known as Legendre's equation.

Another procedure is followed here

$$\int_{-1}^{1} P_j(x) P_K(x)\, dx = 0 \qquad (j \neq K)$$

this is known as the orthogonality relation. $P_j(x)$ and $P_K(x)$ are orthogonal to each other. provided they are different Legendre's polynomials.

Proof

The legendre equation is

$$\frac{d}{dx}\left[(1-x^2) P_\ell(x)'\right] + \ell(\ell+1)\, P_\ell(x) = 0 \quad -(1)$$

multiply by P_m

$$P_m \frac{d}{dx}\left[(1-x^2) P_\ell(x)'\right] + P_m (\ell+1)\ell\, P_\ell(x) = 0$$

$$\Rightarrow P_m\left[(1-x^2) P_\ell(x)'\right]' + P_m P_\ell\, \ell(\ell+1) = 0$$

Now write similar eqn $\ell \leftrightarrow m$ and subtract it

$$\Rightarrow P_m\left[(1-x^2)P_\ell'\right]' - P_\ell\left[(1-x^2)P_m'\right]' + P_m P_\ell (\ell+1)\ell - P_m P_\ell (m+1)m = 0$$

2) $\Rightarrow \quad \dfrac{d}{dx}\left[(1-x^2)(P_m P_l' - P_l P_m')\right] + \left[l(l+1) - m(m+1)\right] P_l P_m = 0$

$$\Rightarrow \int_{-1}^{1} \left\{ \underbrace{\frac{d}{dx}\left[(1-x^2)(P_m P_l' - P_l P_m')\right]} + \left[l(l+1) - m(m+1)\right] P_l P_m \right\} dx = 0$$

$1 - (\pm 1)^2 = 0$ So this vanishes

$$\Rightarrow \int_{-1}^{1} \underbrace{\left[l(l+1) - m(m+1)\right]} P_l P_m \, dx = 0$$

not zero if $m \neq l \Rightarrow \int_{-1}^{1} P_l P_m \, dx = 0$

So legendre's polynomial are orthogonal.

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12-13 10-11 8(a)

(2b) An equation has power series solution if it can be represented in the form

(i) $y'' + p(x)y' + q(x)y = r(x)$ [p, q co-efficient and function of r]

(ii) $\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = \tilde{r}(x)$ [$\tilde{h}(x_0) \neq 0$, x_0 centre of series]

* A real function $f(x)$ is called analytic at a point $x = x_0$ if it can be represented by a power series in powers of $(x - x_0)$ with radius of convergence $R > 0$.

* If p, q, R in eqn (i) are ~~any~~ analytic at a point $x = x_0$ ~~if it can be represented~~ then every solution of (i) is analytic at $x = x_0$ and thus can be represented by a power series in powers of $(x - x_0)$ with $R > 0$ [R = Radius of convergence]

11-12

6(a) Radio active substances ~~den~~ decompose at a rate proportional to the amount present.

Consider the amount of substance present at time t is $y(t)$

$$\therefore \quad \frac{dy}{dt} = y' = Ky \qquad K = \text{constant}$$

$$\Rightarrow \int \frac{dy}{y} = \int K\,dt$$

$$\Rightarrow \quad y = ce^{Kt}$$

At $t=0$ the $y = y_0$ [$\because y_0$ denotes initial amount]

$$\therefore y = y_0 e^{Kt}$$

Here, K is unknown.

* Half life is the time after which the amount of radio active substance has decreased half to its original amount. for ${}_{6}C^{14}$ half life is 5730

$\therefore$ when $t = 5730$ & $y = y_0/2$

$$\Rightarrow \frac{y_0}{2} = y_0 e^{Kt}$$

$$\Rightarrow \left(\frac{1}{2}\right) = e^{Kt}$$

$$\Rightarrow \frac{\ln(1/2)}{5730} = \boxed{K = -0.00012}$$

$\therefore\ y = y_0 e^{-.00012t}$

given $y_0 = 2$

$\therefore\ \boxed{y = 2e^{-.00012t}}$

10-11

7(c) $\sum_{m=0}^{\infty} a_m (x - x_n)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$

Here x_0 = center

a_0, a_1 -- coefficient.

The nth partial sum is

$S_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots a_n(x - x_0)^n$

$n = 0, 1 \ldots$

The Remaining expression

$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \ldots$

Expl :- $1 + x + x^2 + \ldots$

$S_0 = 1 \qquad R_0 = x + x^2 + \ldots$

$S_1 = 1 + x \qquad R_1 = x^2 + \ldots$

12-13 | 11-12 8(a) ~~2010-11 8(a)~~ 7(a) 2010-11

~~3(b)~~ 3(a) ~~3(b)~~ There is no easy procedure for solving linear differential equations with variable co-efficient, at least not routinely and infinitely many steps. With the expectation of special types such as occasional equation can be solved by inspection, linear equation with variable co-efficients generally requires power series method.

Example. $(x-1)^2 y'' + (x-1)y' - 4y = 0$

A equation is non lineare when

1) Power of dependent variable is greater than 1 like $y^2, y^3 \dots$

ii) If it varies like $\frac{dy}{dx}, y\frac{dy}{dx}, \dots$

iii) If there are e^y, $\sin y$ -- constant term

Any equation which is other than these that means liner with variable co-efficient. P. S can be applied.

Example $x(1-x)y'' + 2(1-2x)y' - 2y = 0$

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12-13 11-12 8(e)

3(c) Indicial equation

obtained during Application of Frobenius method of solving a second order ordinary differential equation.

Pg - 212-Kre

Derivation Frobenius equation

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$$\Rightarrow x^2 y'' + x b(x) y' + c(x) y = 0 \quad \text{—— (1)}$$ [multiplying by x^2]

Now $b(x) = b_0 + b_1 x + b_2 x^2 + \ldots\ldots$

$c(x) = c_0 + c_1 x + c_2 x^2 + \ldots\ldots$

$$y(x) = x^p \sum_{m=0}^{\infty} a_m x^m = x^p (a_0 + a_1 x + a_2 x^2 + \ldots\ldots)$$

$$y'(x) = \sum_{m=0}^{\infty} (m+p) a_m x^{m+p-1} = x^{p-1} [p a_0 + \ldots\ldots]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+p)(m+p-1) a_m x^{m+p-2}$$

$$= x^{p-2} [p(p-1) a_0 + (p+1) p a_1 x + \ldots]$$

(1) $\Rightarrow$ $x^p [p(p-1) a_0 + \ldots\ldots] + (b_0 + b_1 x + \ldots) x^p [p a_0 + \ldots\ldots$

$+ (c_0 + c_1 x + \ldots) x^p (a_0 + a_1 x + \ldots) = 0$

equating the co-efficient

for x^p we get

$$\left[p(p-1) + b_0 p + c_0\right] a_0 = 0$$

$$p(p-1) + b_0 p + c_0 = 0$$

indicial equation.

There are three cases

(*) Case 1 :- two roots are equal only one solution is obtained

(*) (2) If two roots differ by a non integer, two soln can be obtained.

(*) Case 3 :- two roots differ by a integer the larger will yield a solution. The smaller may or maynot.

~~5(b)~~ $\frac{11-12}{8\,(b)}$

operations in Power Series

Termwise differenciation

$$y(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m$$

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x-x_0)^{m-1}$$

Termwise addition

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m \qquad \sum_{m=0}^{\infty} b_m (x-x_0)^m$$

adding

$$\sum_{m=0}^{\infty} (a_m+b_m)(x-x_0)^m$$

Termwise Multiplication

$$\sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} \cdots)(x-x_0)^m$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots$$

Vanishing of All co-efficient

~~shuff~~ shifting summation indices :

$\boxed{11-12}$

8 (c)

Legendre Equation :-

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Legendre polynomial :-

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2}$$

$$a_n = \frac{\lfloor 2n}{2^n (\lfloor n)^2}$$

$$a_{n-2} = \frac{\lfloor 2n-2}{2^n \lfloor n-1 \lfloor n-2}$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$= \frac{\lfloor (2n-4)}{2^n \lfloor 2 \lfloor n-2 \lfloor n-4}$$

$$a_{n-2m} = (-1)^m \frac{\lfloor (2n-2m)}{2^m \lfloor m \lfloor (n-m) \lfloor n-2m}$$

12-13

3(b)

Bessel Equation

$$x^2 y'' + xy' + (x^2 - v^2) y = 0 \quad -- (I)$$

Conditions it can be solved using Frobenius Method

Case 1 :- Frobenius equation $y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \quad -- (II)$

comparing two eq^n ⇒ $\frac{c(x)}{x} = (x^2 - v^2)$

$$\frac{b(x)}{x} = x$$

$$x^2 = 1$$

when these are equal then (II) converts to (I)

Case 2 :- In Fobenius method

$$\tilde{h}(x) y'' + \tilde{p}(x) y'(x) + \tilde{q}(x) y = 0$$

$$\tilde{h}(x_0) = 0$$

Now from eq^n (I) we see x^2 can't be zero.

Case 3

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$$\Rightarrow x^2 y'' + x b(x) y' + c(x) y = 0$$

Bsq

$$p(p-1) + b_0 p + c_0 = 0 \quad \text{--- (IV)}$$

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \quad \text{--- (V)}$$

$$x^2 y'' + b(x) x y' + c(x) y = 0 \quad \text{--- (VII)}$$

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \quad \text{--- (VI)}$$

comparing $\quad b(x) = 1 \quad c(x) = x^2 - \nu^2$

$$b_0 + b_1 x + b_2 x^2 = 1$$

$$b_0 = 1$$

$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots = x^2 - \nu^2$$

$$\Rightarrow c_0 = -\nu^2$$

$\therefore \quad p(p-1) + b_0 p + c_0 = 0$

$\Rightarrow p^2 - p + p - \nu^2 = 0$

$\Rightarrow$ So this is true

7(b) 2010-11

$(1+x)y' = y$ can be solved by Power series Method

$y = a_0 + a_1 x + a_2 x^2$
$y' = \quad a_1 + 2a_2 x + \sim$
$y_2 = \quad - \quad 2a_2 + \cdots$

$\Rightarrow (1+x)(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \cdots\cdots$

$= (a_0 + a_1 x + a_2 x + \cdots \quad \cdots)$

$a_1 + (a_1 + 2a_2)x +$

$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 +)$

$+ (\quad a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 \cdots)$

$= (a_0 + a_1 x + a_2 x^2 + \cdots \quad \cdots$

Equating co-efficient

$\Rightarrow$ constant $\Rightarrow$
$a_1 = a_0$

$x \Rightarrow$
$2a_2 + a_1 = a_1$
$\Rightarrow a_2 = 0$

$x^2 \Rightarrow$
$3a_3 + 2a_2 = a_2$
$\Rightarrow 3a_3 = -a_2$
$\Rightarrow a_3 = 0$

$\therefore \quad y = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$

$= a_0 + a_0 x + 0 + 0 \cdots$

$\boxed{y = a_0(1+x)}$ Ans

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