Eulers Condition

#Problem :- $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$; $y(0) = 1$

from $x=0$ to $x=4$ with a step size $h = 0.5$ using Eulers method solve it.

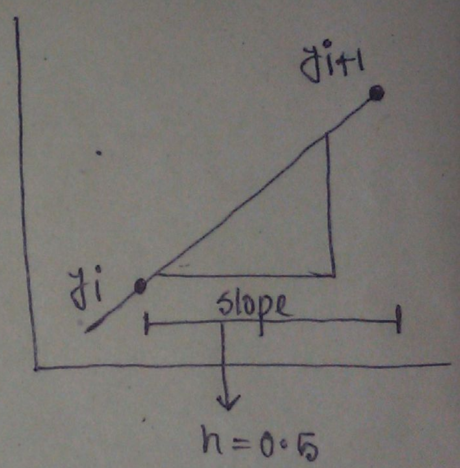
Solution Given $\underbrace{y(0)=1}_{\text{initial condition}}$

$$\boxed{y_{i+1} = y_i + \text{slope} \times h}$$

[h = step size]

slope = dy/dx

w.r.t. previous point



$\therefore\ y(0) = 1.0$ (given)

$y(0.5) = y(0) + \frac{dy}{dx} \times 0.5$

$= 1 + (8.5 \times 0.5)$

$= 5.25$

At $x=0$ $\frac{dy}{dx} = -2(0)^3 + 12(0)^2 - 0 + 8.5$

$= 8.5$

$y(1) = y(0.5) + \frac{dy}{dx} \times 0.5$

$= 5.25 + (1.25 \times 0.5)$

$= 5.875$

$x = 0.5$ $\frac{dy}{dx} = -2(.5)^3 + 12(.5)^2 - 20(.5) + 8.5$

$= 1.25$

$y(1.5) = y(1) + \frac{dy}{dx} \times 0.5$

$= 5.875 + (-1.5 \times 0.5)$

$= 5.125$

$x = 1$ $\frac{dy}{dx} = -1.5$

$y(2) = y(1.5) + \frac{dy}{dx} \times .5$

$= 5.125 + \cancel{(.5 \times .5)}\ (-1.25 \times .5)$

$= \cancel{5.375}\ 4.5$

$$y(2.5) = y(2) + \frac{dy}{dx} \times .5 \qquad \Big| \text{ at } x = 2 \ \frac{dy}{dx} = 0.5$$

$$= 4.5 + (.5 \times .5)$$

$$= 4.75$$

$$y(3) = y(2.5) + \frac{dy}{dx} \times .5 \qquad \Big| \text{ at } x = 2.5 \ \frac{dy}{dx} = 2.25$$

$$= 4.75 + \cancel{(2.5 \times 25)} + (2.25 \times .5)$$

$$= 5.875$$

$$y(3.5) = y(3) + \frac{dy}{dx} \times 0.5$$

$$= 5.875 + (2.5 \times .5) \qquad \Big| \ y' = 2.5$$

$$= 7.125$$

$$y(4) = y(3.5) + \frac{dy}{dx} \times .5$$

$$= 7.125 + (.25 \times .5) \qquad \Big| \ y' = 0.25$$

$$= 7$$

Analytical equation =

Given $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$

integrating $y = -\frac{2x^4}{4} + \frac{12x^3}{3} - \frac{20x^2}{2} + 8.5x + C_1$ [inte. consta

$$y = -\frac{x^4}{2} + 4x^3 - 10x^2 + 8.5x + C_1$$

when $x = 0$ $y = 1$

$$\therefore C_1 = 1$$

$$y = -\frac{x^4}{2} + 4x^3 - 10x^2 + 8.5x + 1$$

$y_{Analytical}$	y_{Euler}	$Error = \frac{y_{analy} - y_{Eu}}{y_{anal}} \times 100\%$
1	1	
3.21875	5.251	*63.55%
3	5.87	*95.6%
2.21875	5.125	131%
2	4.5	125%
2.71	4.75	75.27%
4	5.875	46.8%
4.71	7.125	51%
3	7	133.3%

<u>Error</u> The numerical solution of ODE involves types of error

Extra

Use Euler's Method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t=0$ to 4 with a step size of 1. The initial condition at $t=0$ is $y=2$. Note that the exact solution can be determined analytically as

$$y = \frac{4}{1.3}\left(e^{0.8t} - e^{-0.5t}\right) + 2e^{-0.5t}$$

Solution :- $y(0) = 2$ – initial condition

$$\boxed{y_{i+1} = y_i + \text{slope} \times h}$$

$$y(1) = y(0) + \frac{dy}{dx}(1)$$
$$= 2 + 3\times 1$$
$$= 5$$

When $t=0$

$$\frac{dy}{dx} = 4e^{0.8\times 0} - .5(2)$$
$$= 3$$

$$y(2) = y(1) + \frac{dy}{dx}\times 1$$
$$= 5 + (6.4021\times 1)$$
$$= 11.4021$$

$$\frac{dy}{dx} = 4e^{0.8\times 1} - .5(5)$$
$$= 6.4021$$

$$y(3) = y(2) + \frac{dy}{dx}\times 1$$
$$= 11.402 + (14.1111\times 1)$$
$$= 25.5131$$

$$\frac{dy}{dx} = 14.1111$$

$$y(4) = y(3) + \frac{dy}{dx}\times 1$$
$$= 25.5131 + 31.336\times 1$$
$$= 56.8492$$

$$\frac{dy}{dx}$$

True solution of this equation when $t = 0, 1, 2, 3, 4$

2, 6.19463, 14.84392, 33.6771, 75.33896

t	y_{true}	y_{Euler}	$\varepsilon_E \times 100\%$
0	2	2	
1	6.19463	5	19.28
2	14.843	11.40216	23.19
3	33.6771	25.51	24.25
4	75.33	56.84	24.54

Hen's Method

\# Solve the differential equation $\frac{dy}{dx} = \frac{2y}{x}$ with $y(1)=2$ from $x=1$ to 2 with $h=0.25$ using Hen's Method.

Solution :- $y' = \frac{2y}{x}$

1st step :- $m_1 = \text{slope } 1 = \left.\frac{2y}{x}\right|_{\substack{x=1 \\ y=2}} = \frac{2\times 2}{1} = 4$

Euler's slope

$m_2 = \text{slope } 2 = \left.\frac{2y}{x}\right|_{\substack{x=1.25 \\ y=3}} = \frac{2\times 3}{1.25} = 4.8$

$$y_e(1.25) = y(1) + m_1 \times .25$$
$$= 2 + \left.\left(\frac{2y}{x}\right)\right|_{\substack{x=1 \\ y=2}} \times .25$$
$$= 2 + \left(\frac{2\times 2}{1}\right) \times .25$$
$$= 2+1$$
$$= 3$$

Avg slope $= \frac{m_1+m_2}{2} = \frac{4+4.8}{2} = 4.4 = m$

$$y(1.25) = y(1) + m \times 0.25$$
$$= 2 + (4.4 \times .25)$$
$$= \boxed{3.1} \text{ Ans}$$

2nd step :- $y(1.5) = ?$ using $y(1.25) = 3.1$

$m_1 = \frac{2y}{x} = \frac{2\times 3.1}{1.25} = 4.96$

$m_2 = \left.\frac{2y}{x}\right|_{\substack{x=1.5 \\ y=4.34}} = \frac{2\times 4.34}{1.5} = 5.789$

$$y_e(1.5) = y(1.25) + \left.\frac{dy}{dx}\right|_{\substack{x=1.25 \\ y=3.1}} \times .25$$
$$= 3.1 + \frac{2\times 3.1}{1.25} \times .25$$
$$= 3.1 + \frac{\cancel{4.96}}{1.24} = 4.34$$

$$m = \frac{m_1+m_2}{2} = \frac{4.96 + 5.789}{2} = 5.37$$

$$y(1.5) = 3.1 + 5.37 \times .25 = \boxed{4.44} \text{ Ans}$$

3rd Step :- $y(1.75) = ?$

$$y_e(1.75) = 4.4 + 5.92 \times .25$$
$$= 5.88$$

$$m_1 = \frac{2y}{x} = \frac{2 \times 4.44}{1.5} = 5.92$$

$$m_2 = \left.\frac{2y}{x}\right|_{\substack{x=1.75 \\ y=5.88}} = \frac{2 \times 5.88}{1.75} = 6.72$$

$$m = \frac{m_1+m_2}{2} = \cancel{\frac{5.92}{6.72}} \frac{5.92 + 6.72}{2} = 6.32$$

$$y(1.75) = 4.44 + 6.32 \times .25 = 6.02$$

4th Step $y(2) = ?$

$$m_1 = \frac{2y}{x} = \frac{2 \times 6.02}{1.75} = 6.88$$

$$y_e(2) = 6.02 + 6.88 \times .25$$
$$= 7.74$$

$$m_2 = \left.\frac{2y}{x}\right|_{\substack{x=2 \\ y=7.74}} = \frac{2 \times 7.74}{2} = 7.74$$

$$m = m_1 + m_2 = 7.31$$

$$y(2) = 6.02 + (7.31 \times .25)$$
$$= 7.8475 = 7.85$$

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Comparison

x	y_{Euler}	y_{Heun}	y_{true}
1	2.00	2.00	2
1.25	3	3.1	3.125
1.5	4.2	4.44	4.5
1.75	5.6	6.03	6.125
2.0	7.2	7.86	8

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Forth order Runge-Kutta Method

$$y_{i+1} = y_i + \boxed{\frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)}\, h$$ Slope

where
$$K_1 = f(x_i, y_i)$$
$$K_2 = f\left(x_i + \tfrac{1}{2}h;\ y_i + \tfrac{1}{2}K_1 h\right)$$
$$K_3 = f\left(x_i + \tfrac{1}{2}h;\ y_i + \tfrac{1}{2}K_2 h\right)$$
$$K_4 = f(x_i + h,\ y_i + K_3 h)$$

Solve the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \qquad y(0) = 0$$

Estimate $y(0.4)$ with $h = 0.2$ using 4th order R.K.

$$K_1 = f(0,0) = 0^2 + 0^2 = 0$$
$$K_2 = f\left(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}K_1 h\right)$$
$$= f\left(0 + \tfrac{1}{2}\times .2,\ 0 + \tfrac{1}{2}0\times .2\right)$$
$$= f(.1, 0) = (.1)^2 + 0^2 \quad [f = x^2 + y^2]$$
$$= .01$$
$$K_3 = f\left(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}K_2 h\right)$$
$$= f\left(0 + \tfrac{.2}{2},\ 0 + \tfrac{1}{2}\times .01 \times .2\right)$$
$$= f(.1, .001)$$
$$= (.1)^2 + (.001)^2$$
$$= .010001$$

$$K_4 = f(x_i + h, y_i + h)$$
$$= f(0 + .2,\ 0 + .2 \times .010001)$$
$$= f(.2, .0020002)$$
$$= (.2)^2 + (.0020002)^2$$
$$= .0400040008$$

$$y(0.2) = y(0) + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6} \times .2$$

$$= 0 + \frac{0 + (.01 \times 2) + (2 \times .01) + .04}{6} \times .2$$

$$= 0.0026667$$

Now $y(0.4) = ?$

$\cancel{K(.2, .2}$

$K_1 = f(.2, .2) = (.2)^2 + (.2)^2 =$

$K_1 = f(.2, 0) = (.2)^2 + 0^2 = .04$

$K_2 = f\left(.2 + \frac{.2}{2},\ 0 + \frac{.04 \times .2}{2}\right)$

$= f(0.3, 0.004)$

$\cancel{= (.3)^2 + (0.004)^2}$

$= 0.09$

$K_3 = f\left(.3,\ 0 + \frac{1}{2} \times .09 \times .2\right)$ ..

$= 0.09$

$K_4 = f(.2 + .2,\ 0 + .09 \times .2)$

$= f(.4, .018)$

$= (.4)^2 + (0.018)^2$

$= 0.160324$

$y(0.2) = 0.002667$

when $x = 0.2$ $y = 0.002667$

$K_1 = f(0.2, .002667)$

$= (.2)^2 + (.002667)^2$

$= 0.04$

$K_2 = \left(.2 + \frac{.2}{2},\ .002667 + \frac{.04 \times .2}{2}\right)$

$= (0.03, 0.0000106)$

$= \cancel{.0004}$ 0.09

$K_3 = \left(0.03,\ .002667 + \frac{.09 \times .2}{2}\right)$

$= (0.03, 0.011667)$

$= 0.0901361$

$K = (.2 + .2,\ .002667 + .09 \times .2)$

$= (.4, .020667)$

$= 0.160$

$$y(0.4) = y(0.2) + \frac{(K_1 + 2K_2 + 2K_3 + K_4)}{6} h$$

$$= 0.002667 + \frac{(.04 + 2 \times .09 + 2 \times .09 + .160\cancel{324})}{6} \times .2$$

$= \boxed{0.02133366}$ Ans

System of ODE (two variable)

$$\frac{dy_1}{dx} = x + y_1 + y_2 \quad \text{--- (i)} \qquad y_1(0) = 1 \rightarrow \text{belongs to equation } 1$$

$$\frac{dy_2}{dx} = 1 + y_1 + y_2 \quad \text{--- (ii)} \qquad y_2(0) = -1 \rightarrow \text{" " " } 2$$

Estimate $y_1(0.1)$ and $y_2(0.1)$ using Heun's Method.

$h = .1$

* in order to solve one we have to depend on other. So both of them should be solved simultaneously.

$m_1(1)$ 1st slope of 1st equation $= x + y_1 + y_2 = 0 + 1 - 1 = 0$

$m_1(2) =$ " " 2nd " $= 1 + y_1 + y_2 = 1 + 1 - 1 = 1$

$$m_2(1) = x + y_1 + y_2 = .1 + [y_1|_{x=0} + .1 \times \text{slope}] + [y_2|_{x=0} + .1 \times \text{slope}]$$

$$= .1 + [1 + .1 \times 0] + [-1 + .1 \times 1] =$$

$$= 0.2$$

$$m_2(2) = 1 + y_1 + y_2 = 1 + [1 + .1 \times 0] + [-1 + .1 \times 1]$$

$$= 1 + 1 - 1 + (.1 \times 1)$$

$$= 1 + .1$$

$$= 1.1$$

$$m(1) = \frac{m_1(1) + m_2(1)}{2} = \frac{0 + 0.2}{2} = 0.1$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2} = \frac{1 + 1.1}{2} = \frac{2.1}{2} = 1.05$$

$$y_1(0.1) = y_1(0) + m(1) \times .1 = 1 + (.1 \times .1) = 1.01$$

$$y_2(0.1) = y_2(0) + m(2) \times .1 = -1 + (1.05 \times .1) = -0.895$$

Higher order ODE :-

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$$

$y(0) = 0$ $\qquad y'(0) = 1$ $\qquad h = \cdot 2$

Solve for $y(0\cdot2)$ using Hen's Method.

Let $y = y_1$

$$\frac{dy}{dx} = y_2$$

$\Rightarrow \frac{dy_1}{dx} = y_2$ ------ (i) $\qquad y_1(0) = 0$

$\Rightarrow \frac{dy_2}{dx} = 6x + 3y_1 - 2y_2$ -- (ii) $\qquad y_2(0) = 1$

$m_1(1) = y_2\big|_{x=0} = 1$

$m_1(2) = 6x + 3y_1\big|_{x=0} - 2y_2\big|_{x=0}$

$= 6\times0 + (3\times0) - 2(1)$

$= -2$

$m_2(1) = y_2\big|_{x=0} + m_1(2)\times h$

$= 1 + (-2\times\cdot2)$

$= 0\cdot6$

$m_2(2) = 6x + 3y_1 - 2y_2$ -

$= 6\times\cdot2 + 3(0 + \cdot2\times1) - 2(1 - 2\times\cdot2)$

$= 1\cdot2 + \cdot6 \qquad - 1\cdot2$

$= \cdot6$

$m(1) = \frac{m_1(1) + m_2(1)}{2}$

$= \frac{1 + 0\cdot6}{2}$

$= 0\cdot8$

$m(2) = \frac{m_1(2) + m_2(2)}{2}$

$= \frac{-2 + 0\cdot6}{2}$

$= -0\cdot7$

$$y_1(0\cdot2) = y_1(0) + m(1) \times \cdot 2$$

$$= 0 + \cdot 8 \times \cdot 2$$

$$= 0\cdot16$$

$$y_2(0\cdot2) = y_2(0) + m(2) \times \cdot 2$$

$$= 1 + (-\cdot 7 \times \cdot 2)$$

$$= 1 - \cdot 14$$

$$= \cdot 86$$

Boundary value Problem

Method of finite difference (F.D)

Given $\frac{d^2y}{dx^2} = e^{x^2}$ with $y(0)=0$
$y(1)=0$.

Estimate the values of $y(x)$ at $x = 0.25, 0.5, .75$

Solution Here step size $h = .25$

$$\boxed{y_i' = \frac{y_{i+1} - y_{i-1}}{2h}}$$

$$\boxed{y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}}$$

$x=0$	$x=.25$	$x=.5$	$x=.75$	$x=1$
y_0	y_1	y_2	y_3	y_4

For this point

$$\frac{d^2y}{dx^2} = e^{x^2}$$

$$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = e^{x^2}$$

$$\Rightarrow \frac{y_2 - 2y_1 + y_0}{h^2} = e^{(.25)^2} \quad \left[\text{At that point } x = .25\right]$$

$$\Rightarrow \frac{y_2 - 2y_1 + 0}{(.25)^2} = 1.064494 \quad \left[y_0 = y(0) = 0 \text{ given}\right]$$

$$\Rightarrow y_2 - 2y_1 = 0.06653$$

$$\Rightarrow \boxed{2y_1 + y_2 = 0.066530} \quad --(1)$$

At this point

Similarly $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = e^{x^2}$

$$\Rightarrow \frac{y_3 - 2y_2 + y_1}{(.25)^2} = e^{.5^2} \quad \left[\text{At } y_2 \; x = .5\right]$$

$$\Rightarrow \boxed{y_1 - 2y_2 + y_3 = 0.080251} \quad --(11)$$

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At this point y_3 then the value of $x = .75$

similarly $$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = e^{x^2}$$

$$\Rightarrow \frac{y_4 - 2y_3 + y_2}{(.25)^2} = e^{(.75)^2}$$

$$\Rightarrow y_4 - 2y_3 + y_2 = 0.1097$$

$$\boxed{y_2 - 2y_3 = 0.1097} \quad [y_4 = y(1) = 0] \quad \text{--- (iii)}$$

Solving these (i), (ii) & (iii) equation

$$\boxed{\begin{aligned} y_1 &= y(0.25) = -0.117448 \\ y_2 &= y(0.5) = -0.168366 \\ y_3 &= y(.75) = -0.139033 \end{aligned}}$$

Ans

Solve $\frac{d^2T}{dx^2} - .01(T-20) = 0$

$T(0) = 40$ and $\boxed{\Delta x = 2}$

$T(10) = 200$

Solution:-

$$\frac{d^2T}{dx^2} = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} - 0.01(T-20) = 0$$

$$= \frac{T_{i+1} - 2T_i + T_{i-1}}{2^2} - 0.01(T_i - 20) = 0$$

Given $\Delta x = 2$

Means stepsize

x_1 x_2 x_3

$\Delta x = h$ $\Delta x = h$

$$= T_{i+1} - 2T_i + T_{i-1} = 4 \times 0.01 (T_i - 20)$$

$$\Rightarrow T_{i+1} - 2T_i + T_{i-1} = 4 \times .01 \times T_i - 4 \times 0.01 \times 20$$

$$\Rightarrow T_{i+1} - T_i(2 + 4 \times 0.01) + T_{i-1} = -0.8$$

$$\Rightarrow -T_{i-1} + 2.04 T_i - T_{i+1} = 0.8$$

*if $-T_{i-1} + (2 + h'\Delta x^2) T_i - T_{i+1} = h'\Delta x^2 T_\infty$

Then

$$\begin{bmatrix} 2+h'\Delta x^2 & -1 & & \\ -1 & 2+h'\Delta x^2 & -1 & \\ & & \ddots & \\ & & -1 & 2+h'\Delta x^2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{n-1} \end{Bmatrix} = \begin{Bmatrix} h'\Delta x^2 T_\infty + T_0 \\ h'\Delta x^2 T_\infty \\ \vdots \\ h'\Delta x^2 T_\infty + T_n \end{Bmatrix}$$

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{bmatrix} .8 + 40 = \boxed{40.8} \\ .8 \\ .8 \\ .8 + 200 = \boxed{200.8} \end{bmatrix}$$

if one boundary condition is given

Then $T(0) = 400$

$$-T_{i-1} + 2.04T_i - T_{i+1} = 0.8$$

Additional eqn — addi ~~constant~~/unknown

$$\begin{bmatrix} 2.04 & -2 & 0 & 0 & 0 \\ -1 & 2.04 & -1 & 0 & 0 \\ 0 & -1 & 2.04 & -1 & 0 \\ 0 & 0 & -1 & -2.04 & -1 \\ 0 & 0 & 0 & -1 & -2.04 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} .8 \\ .8 \\ .8 \\ .8 \\ 400.8 \end{bmatrix}$$

* $.8 + 400$

Midpoint Method $\quad \frac{dy}{dx} = \frac{2y}{x} \quad y(1) = 2 \quad$ step size $h = .25$

Solve:

$$y' = \frac{2y}{x} = \text{slope}$$

$$y\Big(1 + \underbrace{\frac{0.25}{2}}_{h/2}\Big) = y(1) + \text{slope}\Big|_{\substack{x=1 \\ y=2}} \times \underbrace{\Big(\frac{.25}{2}\Big)}_{h/2}$$

$$y(1.125) = 2 + \Big(\frac{2\times 2}{1}\Big) \times \Big(\frac{.125}{2}\Big) =$$

$$= 2 + 2\times .25 = 2.5$$

$$\text{Slope} = y'(1.125) = \frac{2y}{x}\Big|_{\substack{x=1.125 \\ y=2.5}} = \frac{2\times 2.5}{1.125} = 4.444$$

$$y(1.125) = y(1) + \text{slope}\times .25$$

$$= 2 + 4.444\times .25 = 3.11$$

$$y\Big(1.125 + \frac{.25}{2}\Big) = y(1.125) + \text{slope}\Big|_{\substack{x=1.125 \\ y=3.11}} \times \frac{.25}{2}$$

$$= \cancel{3.11 + 2}$$

$$= 3.11 + \Big(\frac{2\times 3.11}{1.125}\Big)\times .25/2 = \cancel{3.80}\ 3.732$$

$$y'(1.25) = \frac{2\times 3.732}{1.25} = 5.97 \quad 6.63$$

$$\text{Slope } y(1.5) = 3.11 + 5.97\times .25$$

$y(1.25) = 3.11$

$$\frac{2 \times 3.11}{1.25}$$

$$= 4.976$$

<u>2nd Step:-</u> $y(1.5) = ?$

$$y\left(1.25 + \frac{.25}{2}\right) = y(1.25) + \text{slope}\Big|_{\substack{x=1.25 \\ y=3.11}} \times \frac{.25}{2}$$

$$y(1.375) \quad \cancel{= 3.11}$$

$$y(1.375) = 3.11 + \left(4.976 \times \frac{.25}{2}\right)$$

$$y(1.375) = 3.732$$

$$\text{Slope} = y'(1.375) = \frac{2 \times 3.732}{1.375} = 5.42$$

$$y(1.5) = y(1.25) + \text{slope} \times .25$$

$$= 3.11 + 5.42 \times .25$$

$$= 4.4670$$