12-04-15

Why study differential equation

initial value problem → condition এ যেকোন একটা point দিবে
single value of x

Boundary Value Problem

→ Condition এ একের অধিক Point দিবে
একের অধিক constant

→ $y' = 0.85y \qquad y(0) = 19$

→ $y(0) = 0 \quad , \qquad y(t/2) = 2$

like simply supported beam

Euler's Condition

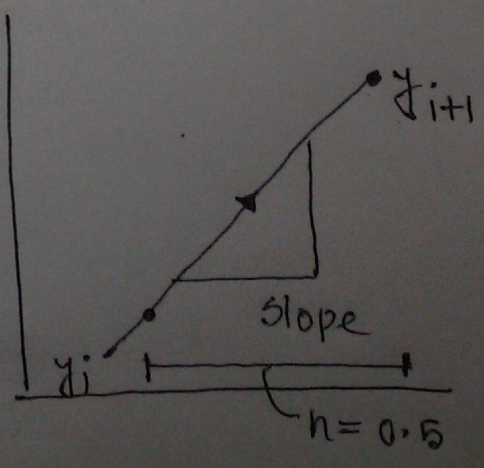
Problem $\dfrac{dy}{dx} = \boxed{-2x^3 + 12x^2 - 20x + 8.5}$ Slope

from $x = 0$ to $x = 4$ with a step size $h = 0.5$

using Euler's Method → "IVP"

Given $\underbrace{y(0) = 1}_{\text{initial condition}}$

$y_{i+1} = y_i + \underbrace{\text{slope}}_{dy/dx} \times h$



$y(0) = 1.0$ (given)

$y(0.5) = ?$

$y(1.0) = ?$

$y(1.5) = ?$

$y(2) =$

$$y(0.5) = \underset{\text{previous point}}{y(0)} + \underbrace{\left.\frac{dy}{dx}\right|}_{\text{Slope}} \times \underbrace{0.5}_{\text{step size}}$$

$\left(\frac{dy}{dx}\right) = 8.5$ at $x=0$

$= 1 + 8.5 \times 0.5$

$= 5.25$

$$y(1) = y(0.5) + \left(\frac{dy}{dx}\right) \times 0.5$$

$= 5.25 + (1.25 \times .5)$

$= 5.875$

$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$

at $x = .5$ $= 1.25$

$$y(1.5) = y(1) + \left(\frac{dy}{dx}\right) \times .5$$

$= 5.875 + (-1.5 \times .5)$

$= 5.125$

$y(2) = \ldots$

x	y Analytical	y Euler	Error
0	1	1	63.1%
.5	3.21	5.26	95.8%
1	3	5.87	131%
1.5	2.21	5.12	125%
2	2	4.5	74.7%
2.5	2.71	4.75	46.9%
3	4	5.87	51%
3.5	4.71	7.12	133.3%
4	3	7	

$$\frac{y_{analytical} - y_{Euler}}{y_{analytical}} \times 100\%$$

difference = 2.03
= LTE

Analytical Solution $y = -\frac{2x^4}{4} + 12\frac{x^3}{3} - 20\frac{x^2}{2} + 8.5x + c_1$

Round off errors:-

.333

.33

.666

.67

Error — Trancation Error — Local / propagated

Error — Round off

Global error = local + propagated

Taylor series

$$\underbrace{\boxed{y_{i+1} = y_i + y_i' h}}_{\text{Euler's Method}} + \underbrace{y_i'' \frac{h^2}{2!} + y_i''' \frac{h^3}{3!} + \cdots}_{\text{ignored} \rightarrow \text{truncation Error}}$$

— How to estimate local function error for Euler's Method

For previous Problem

$$y'' = -6x^2 + 24x - 20$$

$$y''' = -12x + 24$$

$$y^{iv} = -12$$

$$y^{v} = 0 = y^{vi} = y^{vii} = \cdots$$

$$LTE = y'' \frac{h^2}{2!} + y''' \frac{h^3}{3!} + y^{iv} \frac{h^4}{4!} + 0$$

$$= (-6x^2 + 24x - 20) \cdot \frac{h^2}{2!} + (-12x + 24) \frac{h^3}{3!} + (-12) \frac{h^4}{4!}$$

for $x = 0$ $y = 1$ — starting point

$$L.T.E = -2.5 + .5 - .0315$$

$$= -2.03$$

$$E_t = \left| \frac{2.03}{3.21} \right| \times 100 = 63.1\%$$

At least 1st point can be explained by LTE

Other points can be of other errors. LTE is the major part of the error

$h \to 0$ এর মান যত কম হবে error will reduced

Effect of LTE on global errors

change in sign of LTE reduces global error

if all the LTE are the same sign

└ Numerical Soln diverges further

↓

Becomes unstable

Why we need to understand / determine LTE

27.04.15

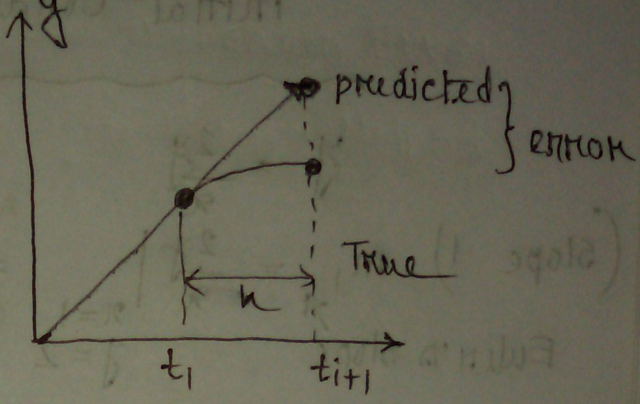
Initial value and boundary value problem

Wall's Method- Simplest method of Numerical solution

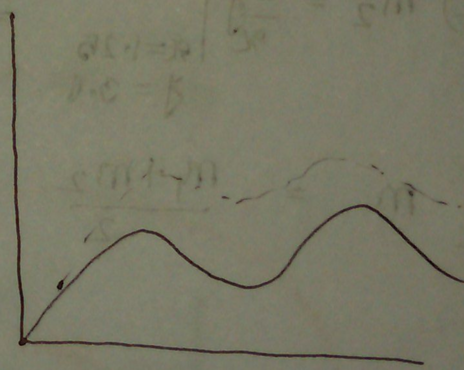
Euler's Method

$$\frac{dy}{dt} = f(t_1, y_1)$$

true solution= analytic solution

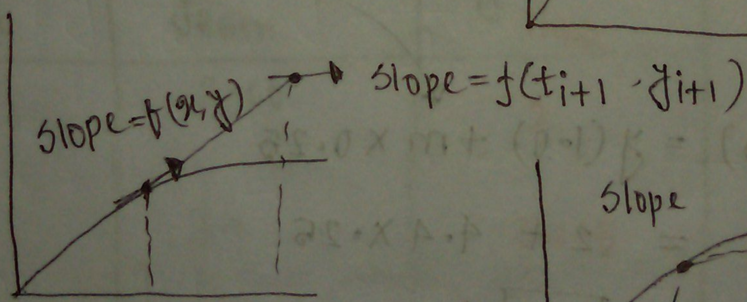


Errors in Euler's Method

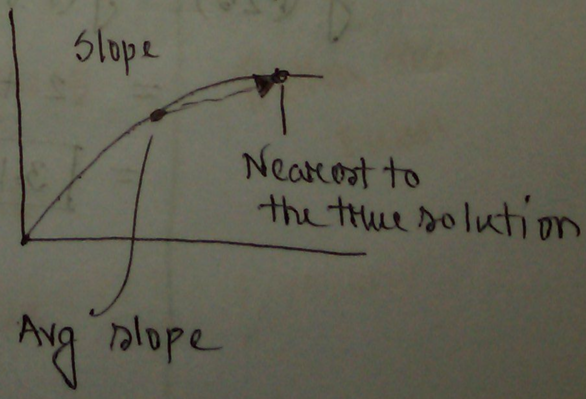


Improvement of Euler's Method

Hen's Method



দুটি থেকে
↳ Average slope



Real slope is not 1st or second its midway of two slope

Solve the differential eqn $\frac{dy}{dx} = \frac{2y}{x}$ with $y(1) = 2$, from $x=1$ to 2 with $\underbrace{h = 0.25}_{\text{Step size}}$, using Heun's Method.

initial condition

$$y' = \frac{2y}{x}$$

$y(1.25) = ?$
$y(1.5) = ?$
$y(1.75) = ?$
$y(2.0) = ?$
Soln

(Slope 1) $m_1 = \left.\frac{2y}{x}\right|_{x=1, y=2} = \frac{2\times 2}{1} = 4$

Euler's slope

(*) below

(Slope 2) $m_2 = \left.\frac{2y}{x}\right|_{x=1.25, y=3.0} = \frac{2\times 3}{1.25} = 4.8$

avg slope $m = \frac{m_1+m_2}{2} = \frac{4+4.8}{2} = 4.4$

$$y(1.25) = y(1.0) + m \times 0.25$$
$$= 2 + 4.4 \times .25$$
$$= \boxed{3.1} \text{ Ans}$$

(*) $y_e(1.25) = y(1.0) + m_1 \times .25$ — Euler's Method
$= 2 + 4 \times .25 = 3.0$

$x = 1.25$ then $y = 3$

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Now $y = 1.5 = ?$ using $y(1.25) = 3.1$

Often mistake to calculate. (3.1 must be used) Most updated Reasult.

$m_1 = \frac{2y}{x} = \frac{2 \times 3.1}{1.25} = 4.96$

$m_2 = ?$

(*) $y_e(1.5) = 3.1 + 4.96 \times .25 = 4.34$

$m_2 = \frac{2y}{x} = \frac{2 \times 4.34}{1.5} = 5.79$

$m = \frac{m_1 + m_2}{2} = \frac{4.96 + 5.79}{2} =$

$y(1.5) = 3.1 + m \times 0.25 = 4.44$ Ans

Comparision :-

Fundamental Method

x	y_{Euler}	y_{Heun}	y_{True}
1.0	2.00	2.00	2
1.25	3	3.1	3.125
1.5	4.2	4.44	4.5
1.75	5.6	6.03	6.125
2.0	7.2	7.86	8

Better than Euler

Midpoint Method

Slove the same problem using Mid Point Method

$y(1) = 2$ initial condition

$h = 0.25$ step size

$y' = \frac{2y}{x} =$

$$y\Big(1 + \underbrace{\frac{0.25}{2}}_{h/2}\Big) = y(1) + \text{Slope}\Big|_{\substack{x=1 \\ y=2}} \times \underbrace{\frac{0.25}{2}}_{h/2}$$

$(2y/x)$

$$\Rightarrow y(1.125) = 2 + \left(\frac{2\times 2}{1}\right) \times .125 = 2.5$$

slope at midpoint

$$y'(1.125) = \frac{2y}{x}\Big|_{\substack{x=1.125 \\ y=2.5}} = \frac{2\times 2.5}{1.125} = 4.44$$

$$y(1.25) = y(1) + \text{slope} \times 0.25$$

$$= 2 + 4.44 \times 0.25 = 3.11 \text{ Ans}$$

similarly $y(1.5) = \cdots \quad \cdots = 4.47$ Ans

and so on

$\frac{dy}{dx}$ = this is easy. using this eq^n

$(y+1)^2 = dy/dx\,(1+x^2) + 2x$ solve something like this

not so easy. Rearrange

$$\frac{dy}{dx} = \frac{(1+y)^2 - 2x}{(1+x)^2} = \underline{\underline{y'}}$$

Next class — CT 2 —— 1pm

(curve fitting)

$$\frac{dy}{dx} = \frac{(1+y)^2 - 2x}{(1+x)^2} = y'$$

Next class — CT 2 — 1 pm

(curve fitting)

18.5.2015

4th order Runge Kutta Method

The most popularly used Method

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h$$

Slope

where $K_1 = f(t_i, y_i)$

$K_2 = f(t_i + \frac{1}{2}h, y_i + \ldots$

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Solve the differential equation

$$\frac{dy}{dx} = x^2+y^2 \qquad y(0)=0$$

Estimate $y(0.4)$ with $h=0.2$ using 4th order R.K

$k_1 = f(0,0) = 0$

$k_2 = f\left(0+\frac{h}{2}, 0+\frac{k_1 h}{2}\right) = f(.1, 0) = .01$

$k_3 = f\left(0+\frac{h}{2}, 0+\frac{k_2 h}{2}\right) = f\left(\frac{0.2}{2}, \frac{.01\times .2}{2}\right) = .01$

$k_4 = f(0+h, 0+k_3 h) = \cdots = 0.04$

$$y(0.2) = y(0) + \frac{k_1+2k_2+2k_3+k_4}{6} \times .2$$

$$= 0 + \frac{0+.01\times 2+.01\times 2+.04}{6} \times .2$$

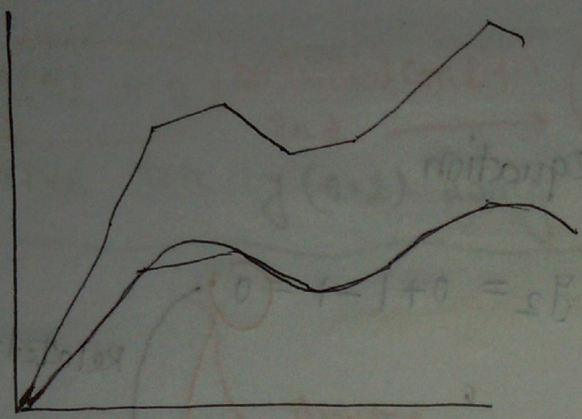
$$= .002667$$

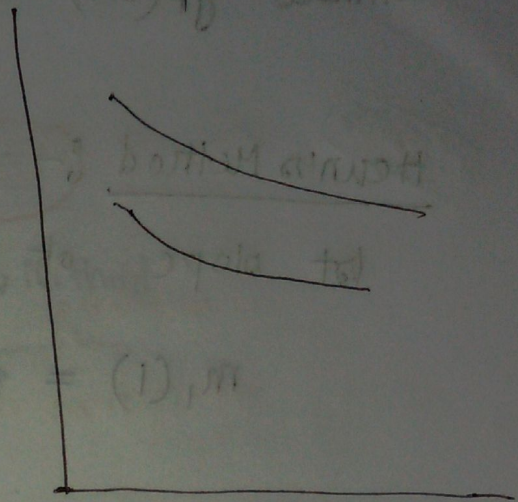
similarly

$$y(.4) = y(0.2) + \cdots$$

$$= \boxed{0.02136} \text{ Ans}$$

performance of different Method





- 4th R.K need less effort
- 4th better than other

A system of ODE :-

→ y, z both functn of x

- ideal practice to solve them seperately
- In order to use 1st one some extent in case of 2nd

Similarly " " 2nd " " " 1st

System of ODE (two variable)

$$\frac{dy_1}{dx} = x + y_1 + y_2 \quad \text{(i)} \quad (1)$$

$y_1(0) = 1$ belong to (i) not (2)

(in order to solve one we have to depend on other)

$$\frac{dy_2}{dx} = 1 + y_1 + y_2 \quad \text{(ii)} \quad (2)$$

$y_2(0) = -1$

So both of them should be solve simultaneously.

Estimate $y_1(0.1)$ and $y_2(0.1)$ using Hen's Method

Heun's Method :-

1st slope of 1st equation

$m_1(1) = x + y_1 + y_2 = 0 + 1 - 1 = 0$

Related to equation(1)

2nd "

$m_1(2) = 1 + y_1 + y_2 = 1 + 1 - 1 = 1$

0 থেকে শুরু
Step size .5 হলে
পরের Point .5

$m_2(1) = x + y_1 + y_2 = .1 + [y_1|_{x=0} + .1 \times 0] + [y_2|_{x=0} + 1 \times .1]$

$= 0.2$ same

$m_2(2) = 1 + y_1 + y_2 = = 1.1$

$$\boxed{m(1)} = \frac{m_1(1) + m_2(2)}{2} = \frac{0 + 0.2}{2} = \boxed{0.1}$$

$$\boxed{m(2)} = \frac{m_1(2) + m_2(2)}{2} = \frac{1 + 1.1}{2} = \boxed{1.05}$$

$y_1(0.1) = y_1(0) + m(1) \times .1 = 1 + 0.1 \times .1 = 1.01$

$y_2(0.1) = y_2(0) + m(2) \times 0.1 = -1 \quad 1.05 \times 0.1$

$= -.895$ Ans

Higher Order ODE

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$$

$y(0) = 0$ ← 1st initial conditn

2nd → $y'(0) = 1$

Solve for $y(0.2)$ using Heun's Method

$y = y_1$

$\frac{dy}{dx} = y_2$

Assume

$$\frac{dy_1}{dx} = y_2 \quad \text{---- (1)} \qquad y_1(0) = 0$$

$$\frac{dy_2}{dx} = 6x + 3y_1 - 2y_2 \quad \text{(2)} \qquad y_2(0) = 1$$

A system of ODE

initial condition

Solve it the same way as for a system of ODE (Previous Example)

Ans :- $y_1(0.2) = 0.16 = y(0.2)$

$y_2(0.2) = 0.86 = \frac{dy}{dx}(0.2)$ – good idea to solve this

2nd Degree equation most common

— Boundary Value Problem Next class

19/5/15

Boundary Value Problem

i) shooting Method

ii) Method of finite differences.

Prob solve the diff eqn $\frac{d^2y}{dx^2} = 6x$ $\boxed{y(1) = 2}$

in the interval (1,2) use $h = 0.5$ $y(2) = 9$

— shooting Methods strategy same as last class's higer order

Try to employ same method as solving higher order ODE

Assume $\frac{dy}{dx} = z$ --- (i)

$\frac{dz}{dx} = 6x$ - (ii)

$y(1) = 2$

$z(1) = ?$

$\underbrace{z(1) = 2}_{\text{assume}}$

Iteration 1 :- use Heun's Method to solve $h = .5$

$m_1(1) = 2$

$m_1(2) = 6$

$m_2(1) = 2 + 6 \times .5 = 5$

$m_2(2) = 6(1 + 0.5) = 9$

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$m(1) = \frac{2+5}{2} = 3.5$

$m(2) = \frac{6+9}{2} = 7.5$

$y(1.5) = y(1.0) + 3.5 \times 0.5 = 3.75$

$z(1.5) = z(1.0) + 7.5 \times 0.5 = 5.75$

iteration 2 :- same procedure

$y(2.0) = \ldots \ldots = 7.75 < 9.0$ (given

so it is incorrect assumpti

Let's take other assumption

$z(1) = 4 \rightarrow$ solve again

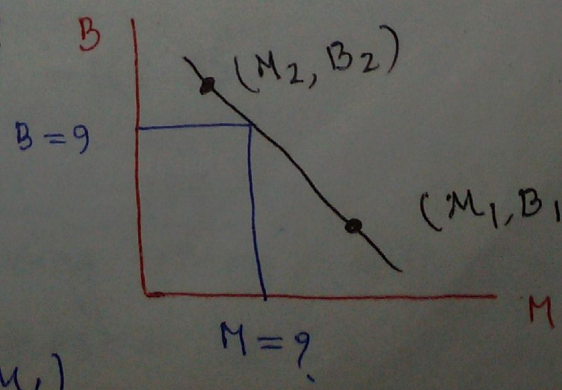
$y(2.0) = 9.75 > 9.0$

→ So this procedure doesn't seem very nice

$z(1) = 2.0 = M_1$ $\qquad y(2.0) = 7.75 = B_1$

$z(1) = 4 = M_2$ $\qquad y(2.0) = 9.75 = B_2$

M, B linearly related



As linearly Related

$$\frac{M-M_2}{B-B_2} = \frac{M_2-M_1}{B_2-B_1}$$

$$M = M_2 - \frac{B_2-B}{B_2-B_1}(M_2-M_1)$$

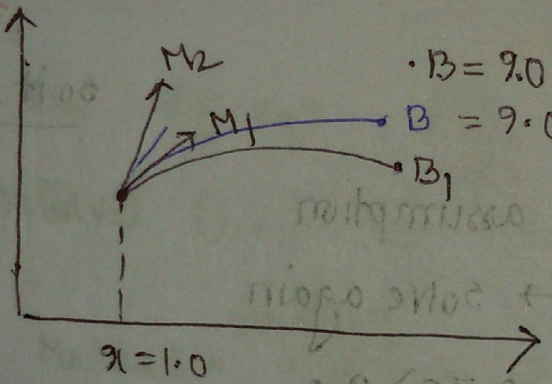
$$= 4 - \frac{9.75-9}{9.75-7.75}(4-2) = \underline{3.25}$$

The correct value of $z(1)$

Perform the same calculation again to get

$$\left.\begin{array}{l} y(1.0) = 2 \\ \boxed{y(1.5) = 4.375} \\ y(2.0) = 4 \end{array}\right\} \text{Solution}$$

shooting Method



Part of question will be given in exam

\# Assume to be slope

diff equation

initial condition 2

have to solve the problem

Sequence of Procedure

1. Convert the problem to an ~~VIP~~ IVP

2. Initialize the variables at the initial slope

3. Solve the equation with these guesses.

4. Interpolate from the reasults to find an improved value of the slopes obtained.

5. Repeat the process until a specified accuracy in the final functional value is obtained.

Prob:- Solve $\frac{d^2T}{dx^2} + 0.01(20-T) = 0$

$T(0) = 40$ $\quad T(10) = 200$ use shooting method

Assume $\frac{dT}{dx} = Z$ $\quad T(0) = 40$ — (1)

$\frac{dZ}{dx} = -0.01(20-T)$ $\quad Z(0) = ?$ — (2)

?

$Z(0) = 10 \longrightarrow T(10) = 168.37$

if $Z(0) = 20 \longrightarrow \underbrace{T(10) = 285.89}$

using linear interpolation

for $T(10) = 200 \longrightarrow Z(0) = 12.69$

Ans

- in case the relation is parabolic, 3 initial guess are required
- relation might not be linear for all cases.

- Round off error
- Trancation "
- How to quantify Trancetion error problem 1 and 2
- Tranction error in numerical diff

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Boundary Value Problem

Method of Finite difference. (F-D)

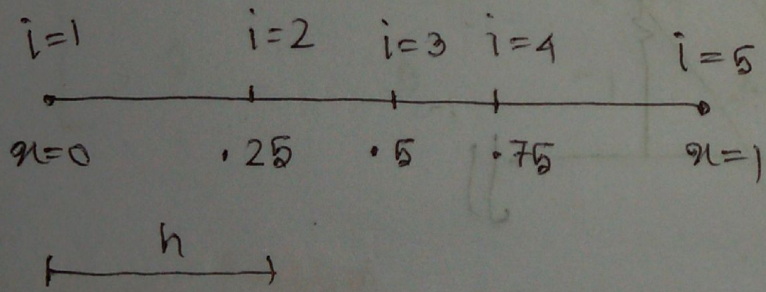
Given $\frac{d^2y}{dx^2} = e^{x^2}$ with $y(0) = 0$ $y(1) = 0$

estimate the values of $y(x)$ at $x = 0.25$, 0.5 and $.75$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

F-D approx at derivation
└ central diff schemes



for $i = 0$ $y(0) = 0$
$x = 0$

for $i = 1$ $\frac{y_2 - 2y_1 + y_0}{(0.25)^2} = e^{(0.25)^2}$ $y(0) = 0$
$x = .25$

F.D approx wpr $\frac{d^2y}{dx^2} = e^{x^2}$

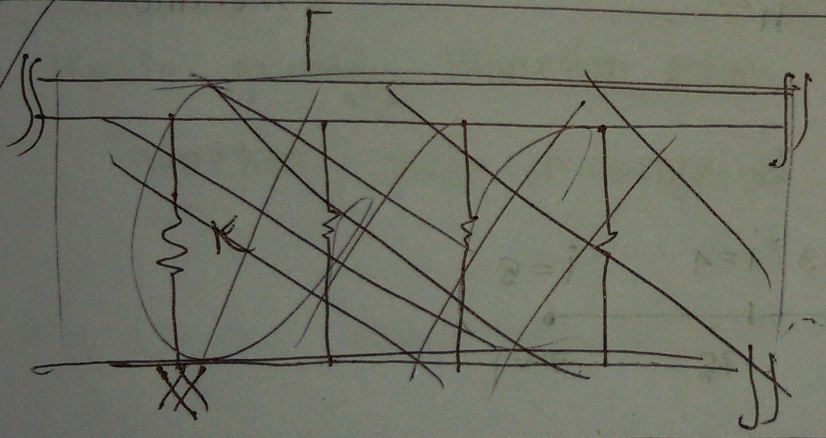
$\Rightarrow$ $y_2 - 2y_1 = 0.0665$ Eq (1) for $i = 2$ $\quad x = 0.5$ $\quad \dfrac{y_3 - 2y_2 - y_1}{(0.25)^2} = e^{(0.5)^2}$

$$\Rightarrow y_3 - 2y_2 + y_1 = 0.0803 \quad -(2)$$

$i = 3$ $\quad x = 0.75$

$$\cancelto{0}{y_4} - 2y_3 + y_2 = 0.1097 \quad - (3)$$

$$y(1) = 0$$



solve eqn (1), (2), (3)

$$\left.\begin{aligned} y_1 &= y(0.25) = -0.1175 \\ y_2 &= y(0.5) = -.1684 \\ y_3 &= y(0.75) = -.1391 \end{aligned}\right\} \text{Ans}$$

Solve $\frac{d^2T}{dx^2} - .1(T-20) = 0$

$T(0) = 40$ $\Delta x = 2$

$T(10) = 200$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{2^2} - 0.01(T_i - 20) = 0$$

$$\boxed{-T_{i-1} + (2 + 0.01 \times 2^2)T_i - T_{i+1} = 0.01 \times 2^2 \times 20}$$

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{bmatrix}$$

Same Problem

Different B.C $\rightarrow$ $\left\{ \left.\frac{dT}{dx}\right|_{x=0} = 0 \quad T(0) = 400 \right\}$ Newman BC

$\frac{T_1 - T_{-1}}{2 \times 2} = 0 \Rightarrow$

$T_1 = T_{-1}$

$- T_{i-1} + 2.04T_i + T_{i+1} = 0.8$

for

for

$\underline{\underline{i=0}}$ $\quad -T_{-1} + 2.04T_0 - T_1 = 0.8$

Additional eqn

Additional unknown

$$\begin{bmatrix} \boxed{2.04 \quad -2 \quad 0 \quad 0 \quad 0} \\ -1 \quad 2.04 \quad -1 \quad 0 \quad 0 \\ 0 \quad -1 \quad 2.04 \quad -1 \quad 0 \\ 0 \quad 0 \quad -1 \quad 2.04 \quad -1 \\ 0 \quad 0 \quad 0 \quad -1 \quad 2.04 \end{bmatrix} \begin{bmatrix} \boxed{T_0} \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} .8 \\ .8 \\ .8 \\ .8 \\ 400.8 \end{bmatrix}$$