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1 (a) Determine Fourier Transform of $\frac{2x}{x^2+4x+8}$.

Sol^n:

Let $f(x) = \frac{1}{x^2+4x+8}$

$= \frac{1}{(x+2)^2+2^2}$

Now, $F\left\{\frac{1}{x^2+2^2}\right\} = \frac{\pi}{2} e^{-2|\omega|}$ $\left[\text{if } f(x) = \frac{1}{x^2+a^2} \cdot (a>0) \text{ then } F\{f(x)\} = \frac{\pi}{a} e^{-a|\omega|}\right]$

and so. $F\left\{\frac{1}{(x+2)^2+2^2}\right\}$

$= \frac{\pi}{2} e^{2i\omega-2|\omega|}$ $\left[\text{if } f(ax+b),\ a>0 \text{ then } \hat{f}(\omega) = \frac{1}{a} e^{ib\omega/a} \hat{f}\left(\frac{\omega}{a}\right)\right]$

Since $x^n f(x)$ $(n=1,2,3,\ldots)$

$\hat{f}(\omega) = i^n \frac{d^n}{d\omega^n} \hat{f}(\omega)$

$\therefore F\left\{2x \cdot \frac{1}{(x+2)^2+2^2}\right\} = 2i \frac{d}{d\omega} \frac{\pi}{2} e^{2i\omega-2|\omega|}$

$= \pi i\, e^{2(i\omega-|\omega|)}\left(2i - \frac{\omega}{|\omega|}\right)$ $\left[\frac{d}{dx}|x| = \frac{x}{|x|},\ x \neq 0\right]$

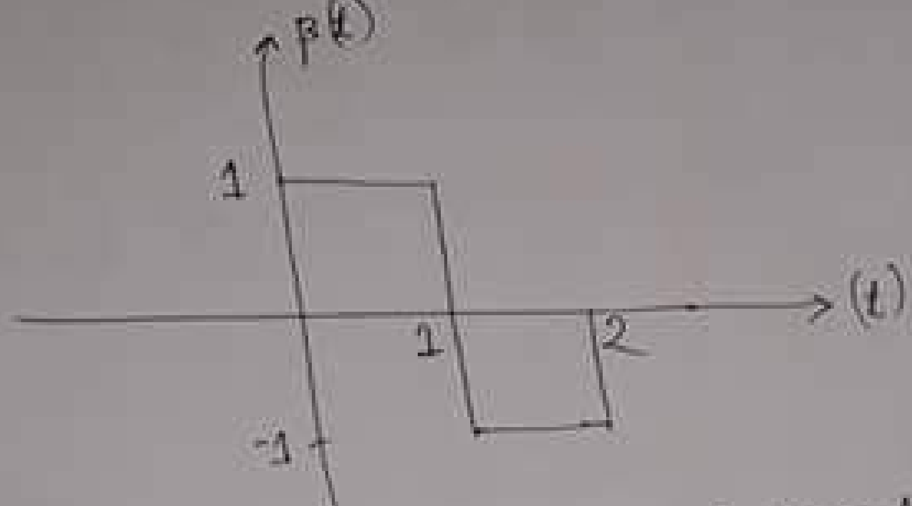
$= \pi\left(2i^2 - \frac{\omega}{|\omega|} i\right) e^{2(i\omega-|\omega|)}$

$= -\pi\left(2+\frac{\omega}{|\omega|} i\right) e^{2(i\omega-|\omega|)}, \quad \omega \neq 0$

(Ans.)

4 (b) An Oscillatory system is subjected to a dynamic force $p(t) = H(t) - 2H(t-1) + H(t-2)$; where H is the Heaviside Step Function. Equation of motion of the oscillatory system is given by $4\ddot{u} + 16u = p(t)$. Determine the response $u(t)$.

Sol^n:



Transforming $p(t)$ into Fourier integral

$$p(t) = \int_0^\infty \{a(\omega)\cos\omega t + b(\omega)\sin\omega t\} \, d\omega$$

where, $a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} p(t)\cos\omega t \, dt$

$$= \frac{1}{\pi}\left[\int_0^1 \cos\omega t \, dt - \int_1^2 \cos\omega t \, dt\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega t}{\omega}\Big|_0^1 - \frac{\sin\omega t}{\omega}\Big|_1^2\right]$$

$$= \frac{1}{\pi\omega}(2\sin\omega - \sin 2\omega)$$

and $b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} p(t)\sin\omega t\, dt$

$$= \frac{1}{\pi}\left[\int_0^1 \sin\omega t\, dt - \int_1^2 \sin\omega t\, dt\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos\omega t}{\omega}\Big|_0^1 + \frac{\cos\omega t}{\omega}\Big|_1^2\right]$$

$$= \frac{1}{\pi\omega}\left(1 + \cos 2\omega - 2\cos\omega\right)$$

$$\therefore p(t) = \int_0^{\infty} \frac{1}{\pi\omega}\Big\{\left(2\sin\omega - \sin 2\omega\right)\cos\omega t$$
$$+ \left(1 + \cos 2\omega - 2\cos\omega\right)\sin\omega t\Big\}\, d\omega$$

The differential eq^n of oscillatory system is

$$4\frac{d^2u}{dt^2} + 16u = p(t) \, ;$$

let $u(t) = \int_0^{\infty}\left\{c(\omega)\cos\omega t + d(\omega)\sin\omega t\right\} d\omega$

$$\therefore \dot{u} = \int_0^{\infty}\left\{- c(\omega)\sin\omega t\cdot\omega + d(\omega)\cos\omega t\cdot\omega\right\} d\omega$$

and $\ddot{u} = \int_0^{\infty}\left\{- \omega^2 c(\omega)\cos\omega t - \omega^2 d(\omega)\sin\omega t\right\} d\omega$

putting these values into the eq^n

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we get,

$$\int_0^\infty \left\{-4w^2 c(w) \cos wt - 4w^2 d(w) \sin wt\right\} dw$$

$$+ \int_0^\infty \left\{c(w) \cos wt + d(w) \sin wt\right\} dw$$

$$= \int_0^\infty \frac{1}{\pi w}\left\{(2\sin w - \sin 2w) \cos wt + (1+\cos 2w - 2\cos w) \sin wt\right\}$$

equating coefficients of $\cos wt$ and $\sin wt$

$$\Rightarrow c(w)(1-4w^2) = \frac{1}{\pi w}(2\sin w - \sin 2w)$$

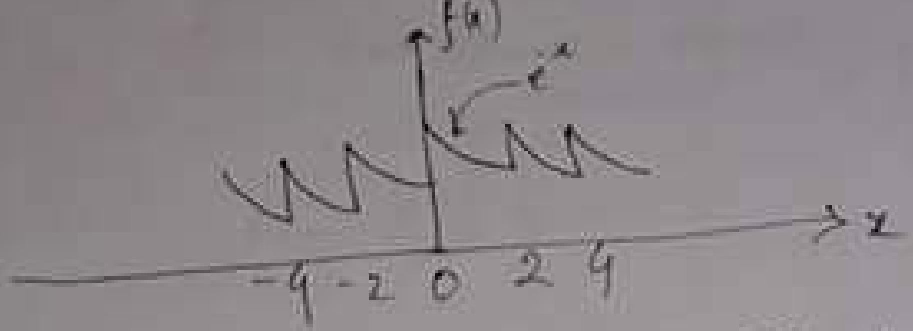
$$\therefore c(w) = \frac{1}{\pi w(1-4w^2)}(2\sin w - \sin 2w)$$

$$\text{and } d(w) = \frac{1}{\pi w(1-4w^2)}(1+\cos 2w - 2\cos w)$$

$$\therefore u(t) = \int_0^\infty \left[\frac{1}{\pi w(1-4w^2)}\left\{(2\sin w - \sin 2w) \cos wt + (1+\cos 2w - 2\cos w) \sin wt\right\}\right] dw$$

(Ans)

8] (a) Determine Fourier series of the periodic function shown -



Sol^n: $f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right]$

period, $2l = 2$

$\Rightarrow l = 1$.

$$\therefore a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} f(x)\, dx$$

$$= \frac{1}{2} \left[\int_{-1}^{0} e^{-x}\, dx + \int_{0}^{1} e^{-x}\, dx \right]$$

$$= \left(1 - \frac{1}{e}\right)$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \int_{-1}^{1} f(x) \cos n\pi x\, dx$$

$$= 2\int_0^1 e^{-x}\cos n\pi x\,dx$$

$$= 2\left[\frac{e^{-x}}{1+n^2\pi^2}\left(n\pi\sin n\pi x - \cos n\pi x\right)\right]_0^1$$

$$= \frac{2}{1+n^2\pi^2}\left\{1-\frac{(-1)^n}{e}\right\}$$

and $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \int_{-1}^{1} e^{-x}\sin n\pi x\,dx$$

$$= -\frac{2}{1+n^2\pi^2}\left[e^{-x}\left(\sin n\pi x - n\pi\cos n\pi x\right)\right]_0^1$$

$$= \frac{2}{1+n^2\pi^2}\left[n\pi\{(-1)^n-1\}-\frac{1}{e}\right]$$

$$\therefore f(x) = \left(1-\frac{1}{e}\right) + \sum_{n=1}^{\infty}\left[\frac{2}{1+n^2\pi^2}\left[\left\{1-\frac{(-1)^n}{e}\right\}\cos n\pi x + \left[\{n\pi(-1)^n-1\}-\frac{1}{e}\right]\sin n\pi x\right]\right]$$

(Ans)

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let, $f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right]$

Where, $a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$

$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |\sin x|\, dx$

$= \frac{2}{\pi} \int_{0}^{\pi/2} |\sin x|\, dx$

$= \frac{2}{\pi} \int_{0}^{\pi/2} \sin x\, dx$ $\left[\text{since } \sin x \text{ is +ve in } 0 \le x \le \pi/2 \right]$

$= \frac{2}{\pi} \left[-\cos x \right]_{0}^{\pi/2}$

$= \frac{2}{\pi}$

and,
$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$

$= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} |\sin x| \cos 2nx\, dx$

$= \frac{2}{\pi} \int_{0}^{\pi/2} 2 \sin x \cos 2nx\, dx$

$= \frac{2}{\pi} \int_{0}^{\pi/2} \left\{ \sin(x + 2nx) + \sin(x - 2nx) \right\} dx$

$= \frac{2}{\pi} \left[\frac{1}{1+2n} - \frac{1}{1-2n} \right] = \frac{8}{\pi} \frac{n}{4n^2 - 1}$

$b_n = 0$ [Since is an even function.]

$$\therefore P(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left[\frac{8}{\pi} \frac{n}{4n^2-1} \cos 2nx \right]$$

Now,

$$T \frac{d^2y}{dx^2} - ky = p(x) \quad \cdots \cdots (1)$$

$$\therefore T \frac{d^2y_1}{dx^2} - ky_1 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left[\frac{8}{\pi} \frac{n}{4n^2-1} \cos 2nx \right]$$

$$\therefore y_1 = -\frac{2}{\pi K} \quad \text{[since } y_1 \text{ is constant]}$$

let, $y_2 = a \cos 2nx + b \sin 2nx$

$$\therefore y_2'' = -4n^2 a \cos 2nx - 4n^2 b \sin 2nx$$

putting these value in eq^n (1)

$$\Rightarrow -4n^2 a T \cos 2nx - 4n^2 b T \sin 2nx - ka \cos 2nx - kb \sin 2nx = \cos 2nx$$

$$\Rightarrow \cos 2nx (-4n^2 a T - ka) + \sin 2nx (-4n^2 b T - kb) = \cos 2nx$$

equating coefficients

$$\Rightarrow a(-4n^2 T - K) = 1 \Rightarrow a = -\frac{1}{4n^2 T + K}$$

and $b = 0$.

$\therefore$ Deformation

$$y(x) = -\frac{2}{\pi K} - \sum_{n=1}^{\infty}\left[\frac{8}{\pi}\,\frac{n}{4n^2-1}\,\frac{1}{4n^2T+K}\cos 2n\pi x\right]$$

(Ans)