CE 211: Mechanics of Solids I 3.00 credits, 3 hrs/week. Prereq. CE 101

Objective:
This is a preparatory course for structural analysis and design. The concepts introduced in this course are essential for Level-3 and 4 courses of structural analysis and design and those of RCC.

Text book:
Engineering Mechanics of Solids by Egor P. Popov, Second Edition, Prentice-Hall

Course Plan:

Course Content	Chapters	Weeks
Concepts of stress and strain, constitutive relationships	1, 2, 5	2
Deformations due to tension, compression and temperature change	3, 4	1
Beam statics: reactions, axial force, shear force and bending moments; axial force, shear force and bending moment diagrams using method of section and summation approach	7	3
Elastic analysis of circular shafts, solid non- circular and thin walled tubular members subjected to torsion	6	2
Flexural stresses in beams	8, 9	2
Shear stresses in beams, shear centre	10	1
Thin walled pressure vessels	-	1

Marks distribution:

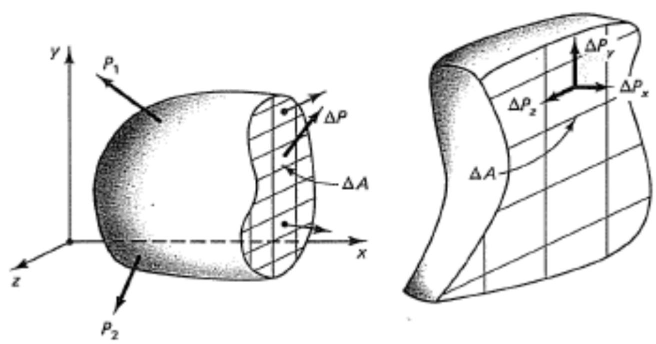
Item	Marks (%)
Class attendance	10
Class tests (best 3 out of 4)	20
Assignment: Only for practice	0
Term final exam	70

Tentative schedule of class tests:

Class test	Topic	Week
CT-1	Stress/Strain/Hook's law/Axial deformation	4th
CT-2	SFD, BMD	7th
CT-3	Torsion	9th
CT-4	Flexural stress	11th

Lecture 1: STRESS

1. Definition of Stress



Intensity of force per unit area is defined as stress.

Mathematical definition of stress is:

$$\tau_{xx} = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A} \qquad \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A} \qquad \text{and} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A}$$

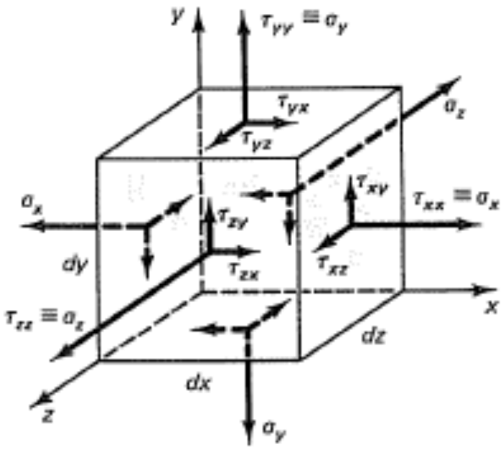
The intensity of the force perpendicular to or normal to the section is called the *normal stress* at a point. Normal stresses that cause traction or tension on the surface of a section are *tensile stresses.* On the other hand, those that are pushing against it are *compressive stresses.* Normal stresses are usually designated by the letter σ (sigma) .

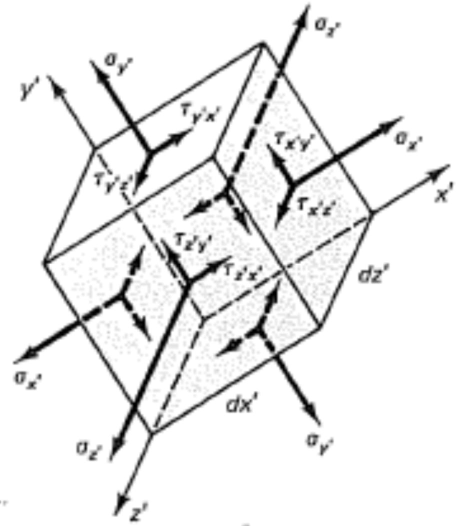
The other components of the intensity of force act parallel to the plane of the elementary area. These components are called *shear* or *shearing stresses.* Shear stresses will be always designated by τ.

Unit:
US Customary - psi, ksi
SI - N/m^2 (Pa), N/mm^2 (MPa)

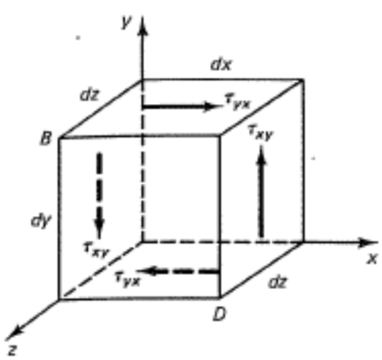
Stresses multiplied by the respective areas on which they act give forces. At an *imaginary* section, a vector sum of these forces, called *stress resultants, keeps a body in equilibrium.*

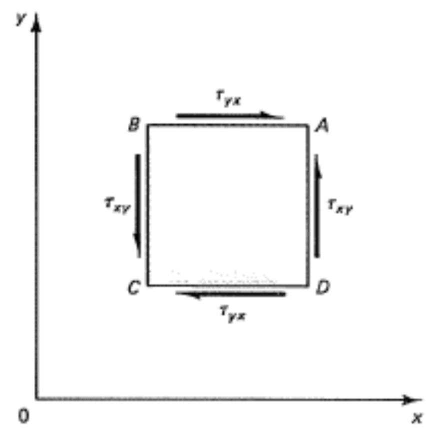
2. Stress Tensor





$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \equiv \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

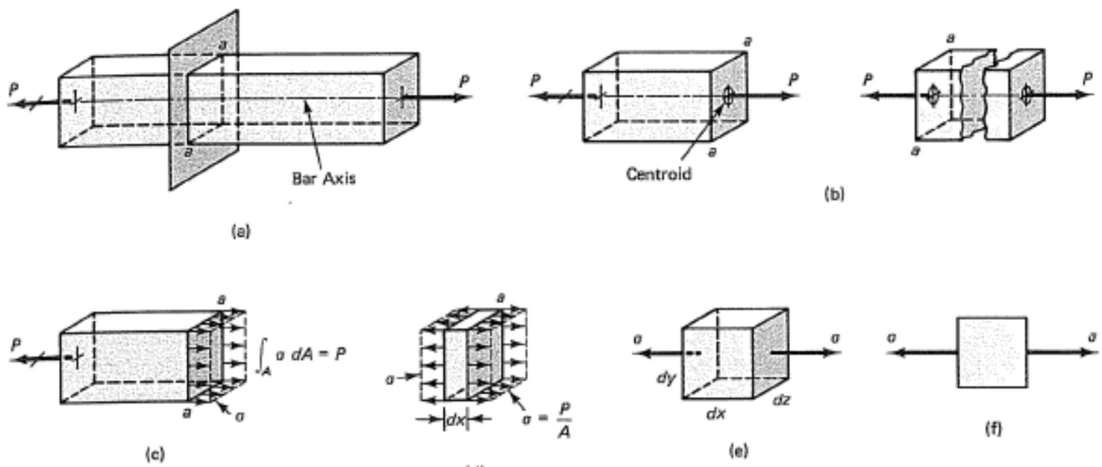




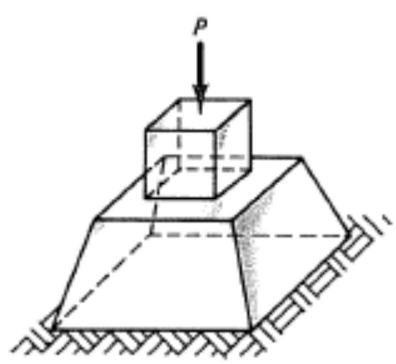
$$\tau_{yx} = \tau_{xy}$$

The stress tensor is symmetric.

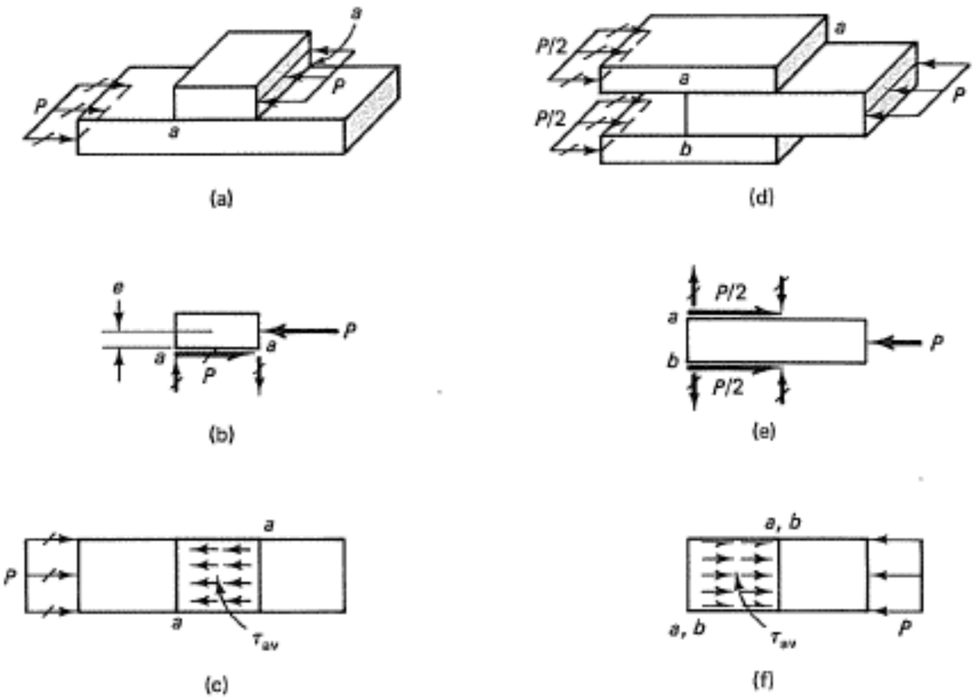
3. Normal Stress in Axially Loaded Bars



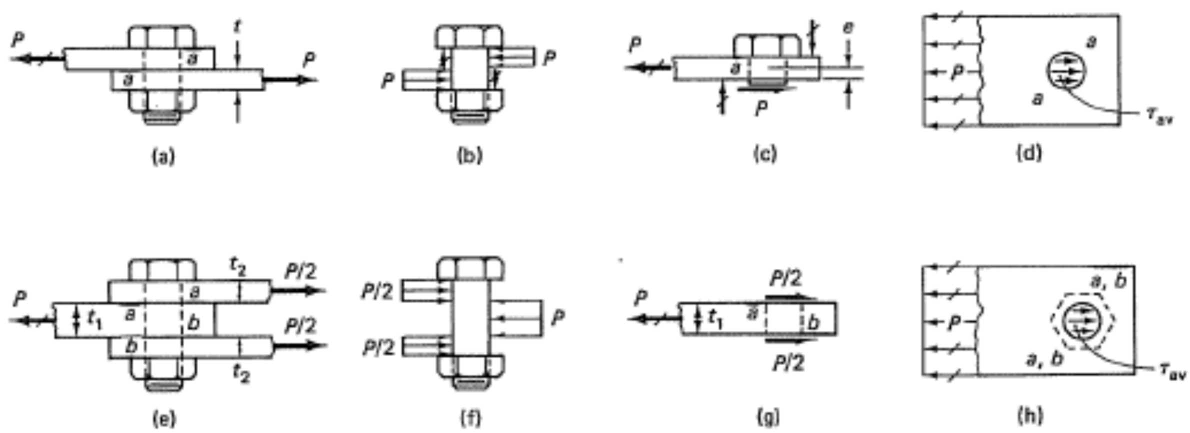
$$\sigma = \frac{\text{force}}{\text{area}} = \frac{P}{A} \quad \left[\frac{\text{N}}{\text{m}^2}\right] \text{ or } \left[\frac{\text{lb}}{\text{in}^2}\right]$$



4. Shear Stresses

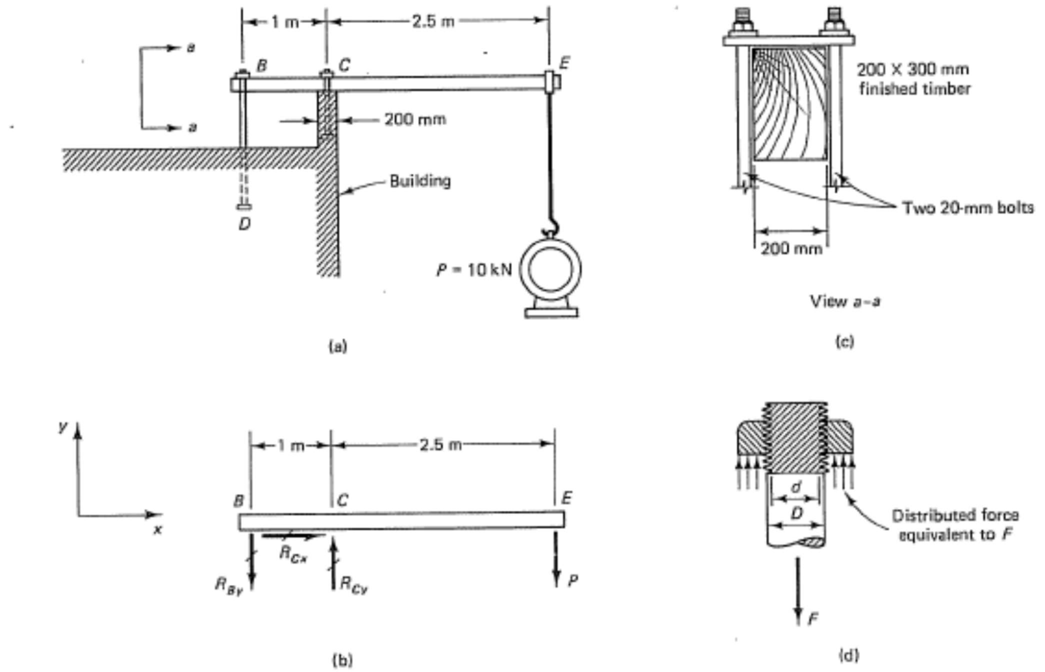


$$\tau_{av} = \frac{\text{force}}{\text{area}} = \frac{V}{A} \quad \left[\frac{\text{N}}{\text{m}^2}\right] \text{ or } \left[\frac{\text{lb}}{\text{in}^2}\right]$$



5. Analysis for Normal and Shear Stresses

Example 1-1



Determine the stress in bolts BD and the bearing stress at C. Note that the bolts are threaded with *d*= 16 mm at the root of the threads. Assume that the weight of the beam is negligible in comparison with the loads handled.

$$\sum F_x = 0 \qquad\qquad R_{Cx} = 0$$

$$\sum M_B = 0 \circlearrowleft + \qquad 10(2.5 + 1) - R_{Cy} \times 1 = 0 \qquad R_{Cy} = 35 \text{ kN} \uparrow$$

$$\sum M_C = 0 \circlearrowleft + \qquad 10 \times 2.5 - R_{By} \times 1 = 0 \qquad R_{By} = 25 \text{ kN} \downarrow$$

$$\textit{Check:} \quad \sum F_y = 0 \uparrow + \qquad -25 + 35 - 10 = 0$$

The cross-sectional area of one 20-mm bolt at the root of the threads is

$$A_{\text{net}} = \pi\, 8^2 = 201 \text{ mm}^2$$

Maximum normal tensile stress in each of the two bolts BD:

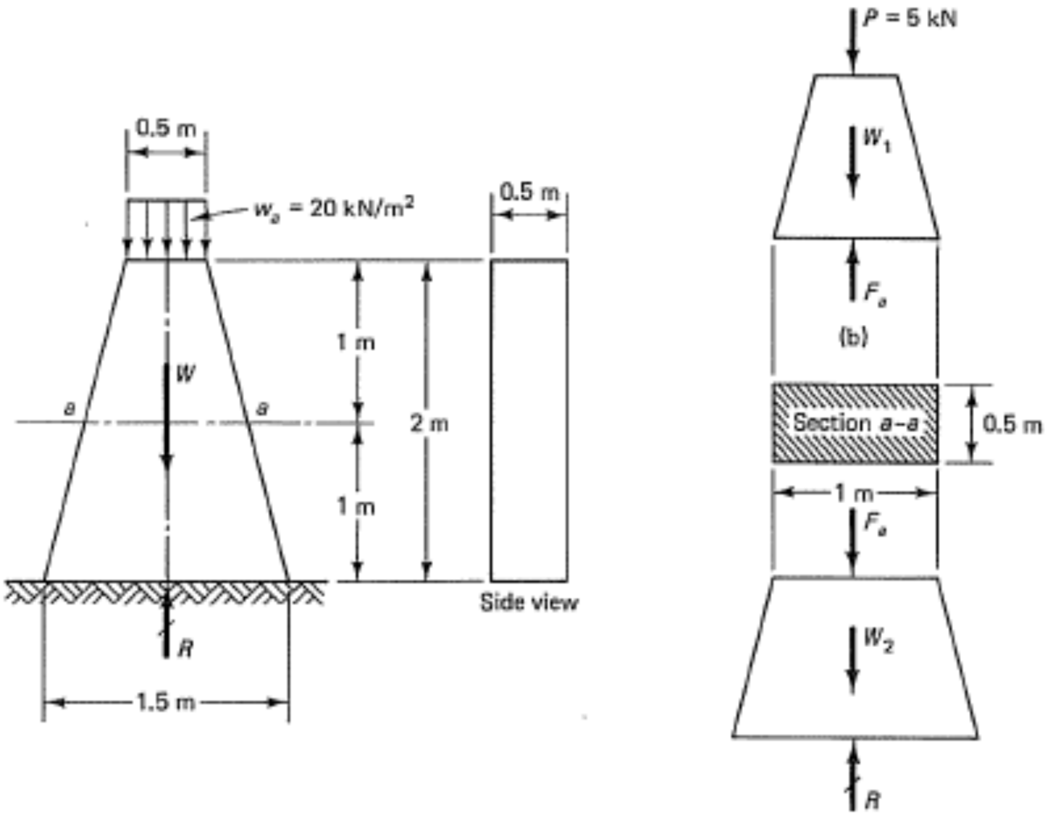
$$\sigma_{\max} = \frac{R_{By}}{2A} = \frac{25 \times 10^3}{2 \times 201} = 62 \text{ N/mm}^2 = 62 \text{ MPa}$$

Contact area at C: $A = 200 \text{ X } 200 = 40 \text{ X } 10^3 \text{ mm}^2$

Bearing stress at C:

$$\sigma_b = \frac{R_{Cy}}{A} = \frac{35 \times 10^3}{40 \times 10^3} = 0.875 \text{ N/mm}^2 = 0.875 \text{ MPa}$$

Example 1-2



Investigate the state of stress at a level 1 m above the base. Concrete weighs approximately 25 kN/m^3.

Weight of the whole pier:

$$W = [(0.5 + 1.5)/2] \text{ X } 0.5 \text{ X } 2 \text{ X } 25 = 25 \text{ kN}$$

Total applied force:

$$P = 20 \text{ X } 0.5 \text{ X } 0.5 = 5 \text{ kN}$$

From ∑Fy = 0, reaction at the base:

$$R = W + P = 30 \text{ kN}$$

Using the upper part of the pier as a free body, the weight of the pier above the section:

$$W_I = (0.5 + 1) \text{ X } 0.5 \text{ X } 1 \text{ X } 25/2 = 9.4 \text{ kN}$$

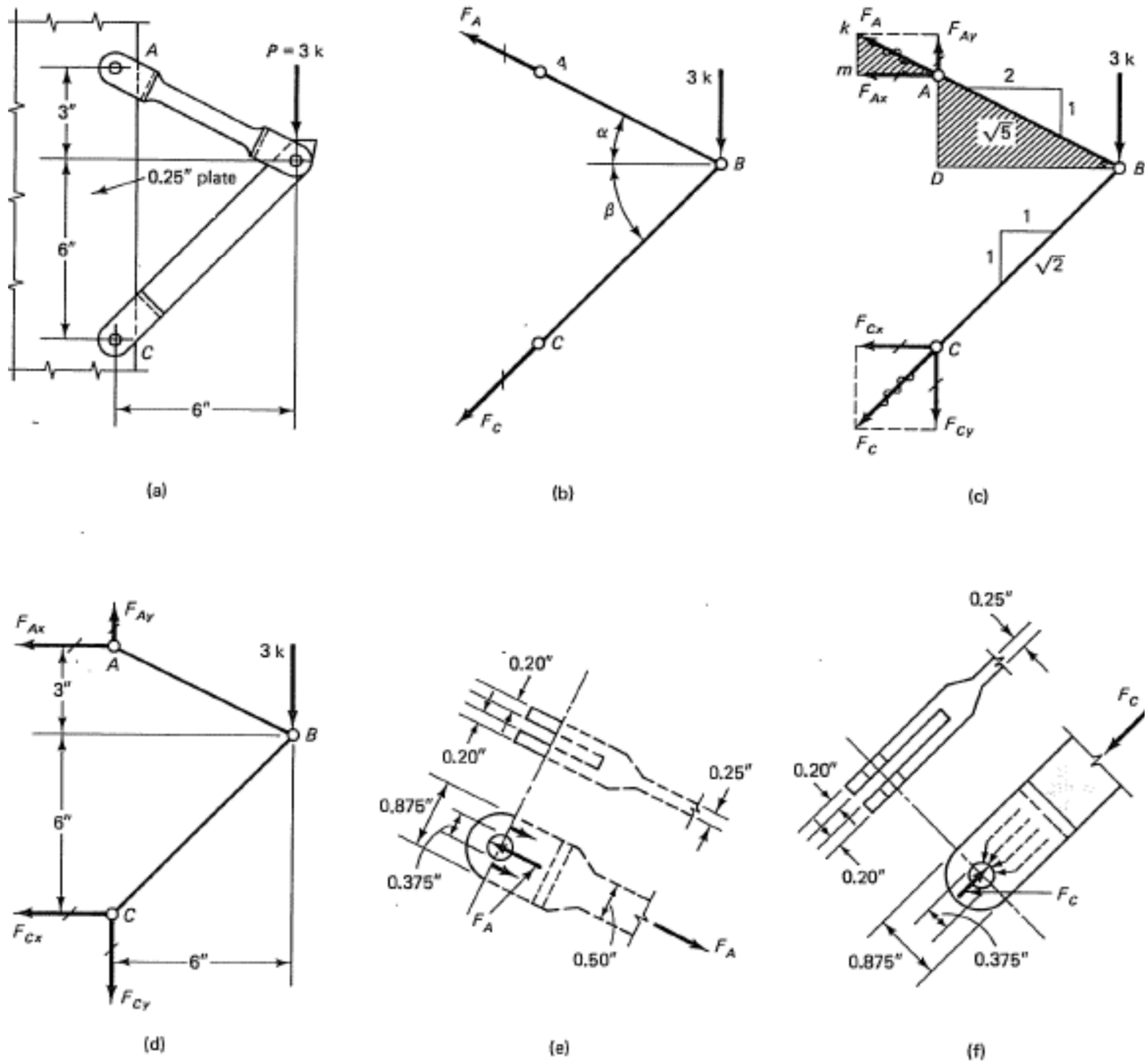
From ∑ Fy = 0, the force at the section:

$$Fa = P + WI = 14.4 \text{ kN}$$

Hence, the normal stress at the level a-a is

$$\sigma_a = \frac{F_a}{A} = \frac{14.4}{0.5 \times 1} = 28.8 \text{ kN/m}^2$$

Example 1-3



Find the axial stresses in members AB and BC and the bearing and shear stresses for pin C. All pins are 0.375 in in diameter.

$$F_A = (AB/DB)F_{Ax}$$

$$F_A = (\sqrt{5}/2)F_{Ax} \qquad \text{and} \qquad F_{Ay} = F_{Ax}/2$$

$$\sum M_C = 0 \circlearrowleft + \qquad + F_{Ax}(3 + 6) - 3(6) = 0 \qquad F_{Ax} = +2\text{ k}$$

$$F_{Ay} = F_{Ax}/2 = 2/2 = +1\text{ k}$$

$$F_A = 2(\sqrt{5}/2) = +2.23\text{ k}$$

$$\sum M_A = 0 \circlearrowright + \quad + 3(6) + F_{Cx}(9) = 0, \qquad F_{Cx} = -2 \text{ k}$$

$$F_{Cy} = F_{Cx} = -2 \text{ k}$$

$$F_C = \sqrt{2}(-2) = -2.83 \text{ k}$$

Tensile stress in main bar AB:

$$\sigma_{AB} = \frac{F_A}{A} = \frac{2.23}{0.25 \times 0.50} = 17.8 \text{ ksi}$$

Tensile stress in clevis of bar AB:

$$(\sigma_{AB})_{\text{clevis}} = \frac{F_A}{A_{\text{net}}} = \frac{2.23}{2 \times 0.20 \times (0.875 - 0.375)} = 11.2 \text{ ksi}$$

Compressive stress in main bar BC:

$$\sigma_{BC} = \frac{F_C}{A} = \frac{2.83}{0.875 \times 0.25} = 12.9 \text{ ksi}$$

Bearing between pin C and the clevis:

$$\sigma_b = \frac{F_C}{A_{\text{bearing}}} = \frac{2.83}{0.375 \times 0.20 \times 2} = 18.8 \text{ ksi}$$

Bearing between the pin C and the main plate:

$$\sigma_b = \frac{F_C}{A} = \frac{2.83}{0.375 \times 0.25} = 30.2 \text{ ksi}$$

Double shear in pin C:

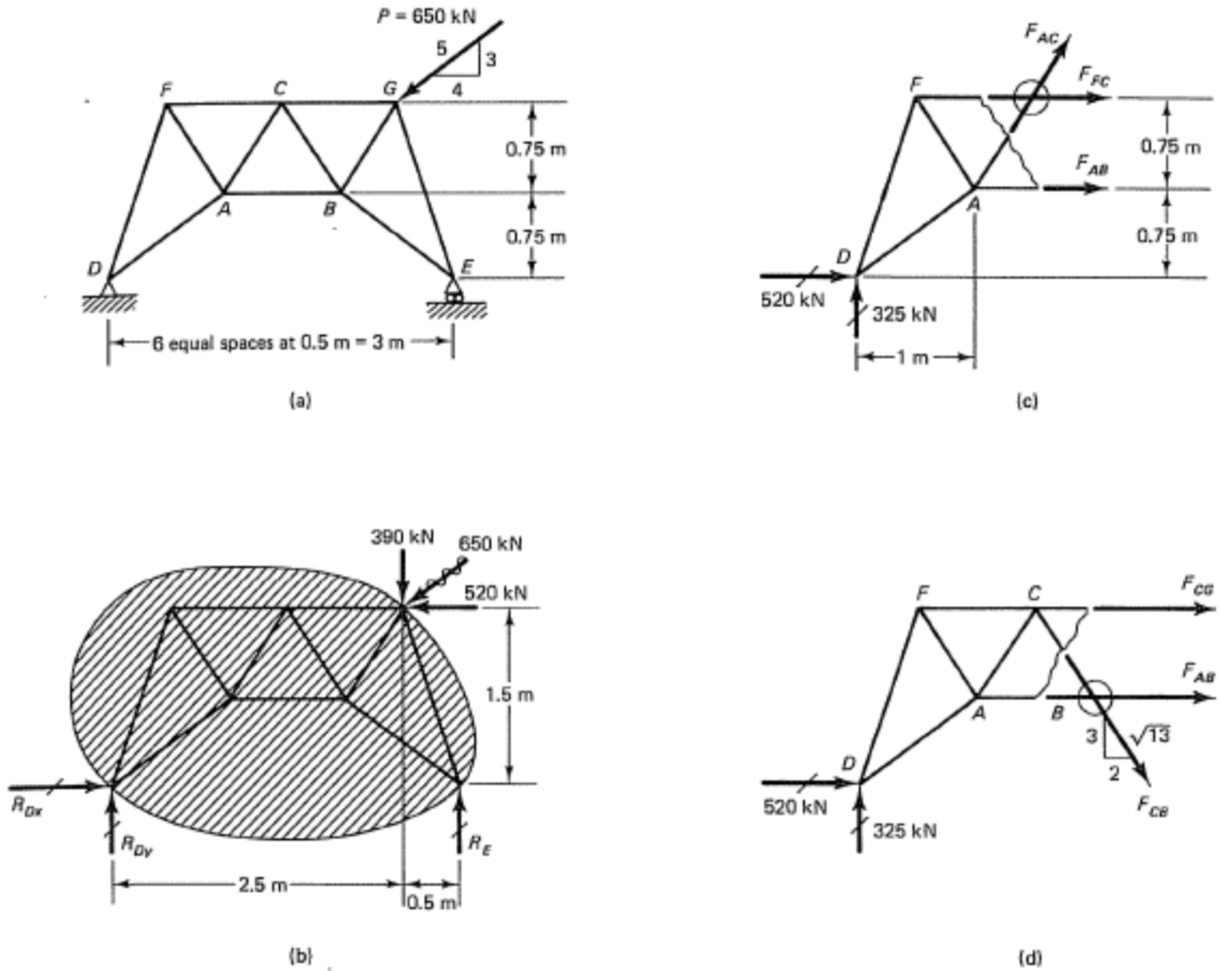
$$\tau = \frac{F_C}{A} = \frac{2.83}{2\pi(0.375/2)^2} = 12.9 \text{ ksi}$$

6. Deterministic Design of Members: Axially Loaded Bars

$$\text{F.S.} = \frac{\text{maximum useful material strength (stress)}}{\text{allowable stress}}$$

$$A = \frac{P}{\sigma_{\text{allow}}}$$

Example 1-4



Select members FC and CB. Set the allowable tensile stress at 140 MPa.

$$\sum F_x = 0 \qquad R_{Dx} - 520 = 0 \qquad R_{Dx} = 520 \text{ kN}$$

$$\sum M_E = 0 \circlearrowright + \qquad R_{Dy} \times 3 - 390 \times 0.5 - 520 \times 1.5 = 0$$

$$R_{Dy} = 325 \text{ kN}$$

$$\sum M_D = 0 \circlearrowleft + \qquad R_E \times 3 + 520 \times 1.5 - 390 \times 2.5 = 0$$

$$R_E = 65\text{kN}$$

$$\sum M_A = 0 \circlearrowright + \qquad F_{FC} \times 0.75 + 325 \times 1 - 520 \times 0.75 = 0$$

$$F_{FC} = +86.7 \text{ kN}$$

$$A_{FC} = F_{FC}/\sigma_{\text{allow}} = 86.7 \times 10^3/140 = 620 \text{ mm}^2$$

(use 12.5 × 50-mm bar)

$$\sum F_y = 0 \qquad -(F_{CB})_y + 325 = 0 \qquad (F_{CB})_y = +325 \text{ kN}$$

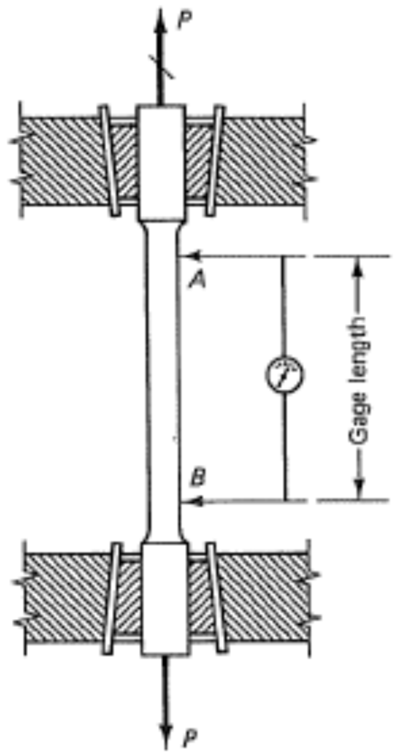
$$F_{CB} = \sqrt{13}(F_{CB})_y/3 = +391 \text{ kN}$$

$$A_{CB} = F_{CB}/\sigma_{\text{allow}} = 391 \times 10^3/140 = 2790 \text{ mm}^2$$

(use two bars 30 × 50 mm)

Lecture 2: STRAIN

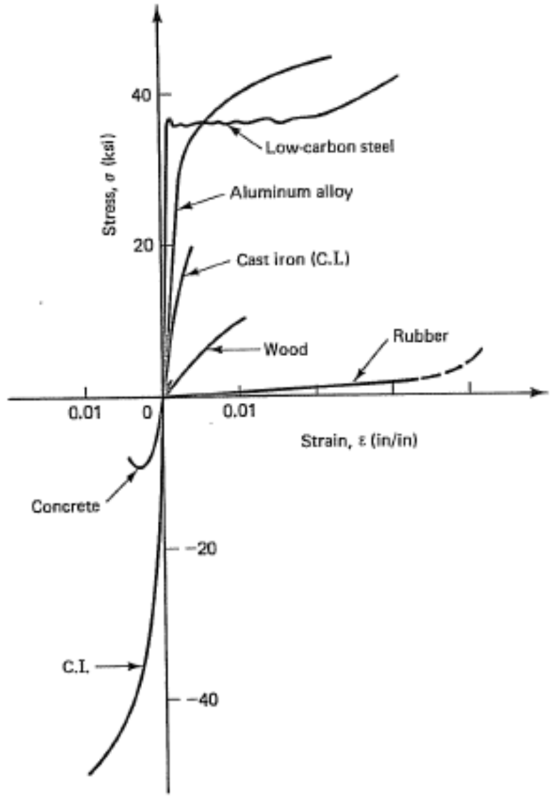
1. Normal Strain

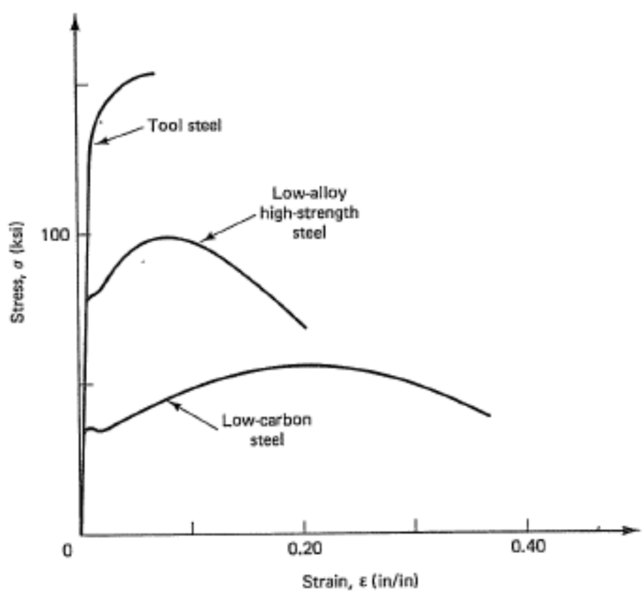


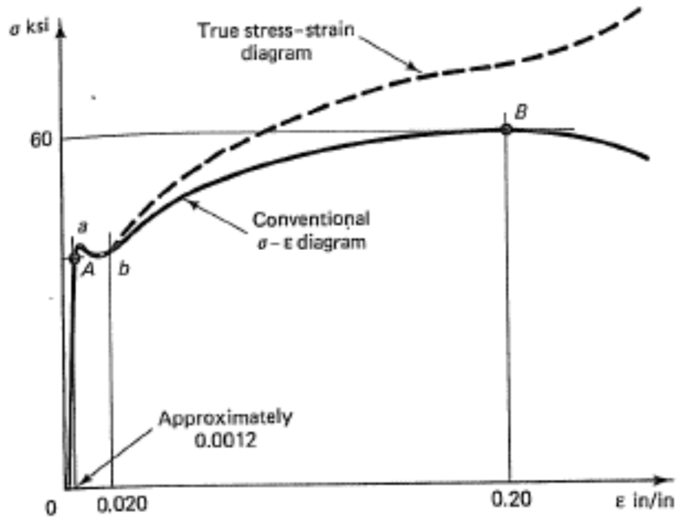
$$\varepsilon = \frac{L - L_o}{L_o} = \frac{\Delta L}{L_o}$$

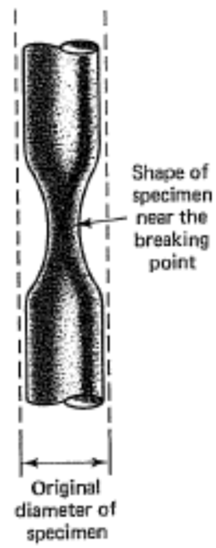
This expression defines the *extensional strain*. Since this strain is associated with the normal stress, it is usually called the *normal strain*: It is a dimensionless quantity, but it is customary to refer to it as having the dimensions of in/in, m/m, or μm/m (microstrain). Sometimes it is given as a percentage.

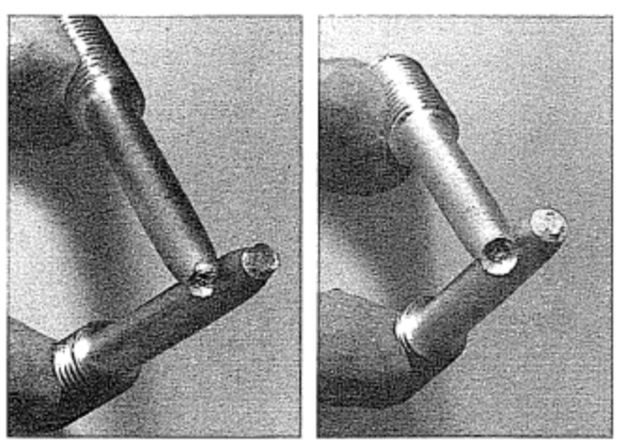
2. Stress-Strain Relationship



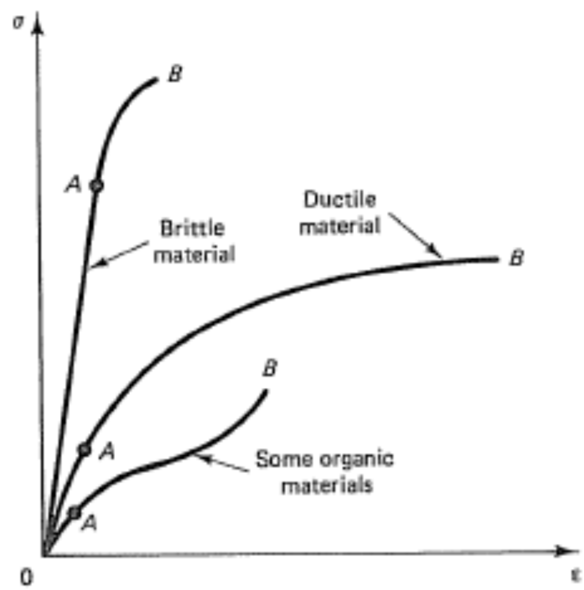



σ ksi
True stress–strain
diagram
B
60
a
A
b
Conventional
σ – ε diagram
Approximately
0.0012
0
0.020
0.20
ε in/in


Shape of
specimen
near the
breaking
point
Original
diameter of
specimen



Stresses are usually computed on the basis of the original area of a specimen; such stresses are often referred to as *conventional or engineering stresses*. On the other hand, it is known that some transverse contraction or expansion of a material always takes place. Dividing the applied force, at a given point in the test, by the corresponding actual area of a specimen at the same instant gives the so-called *true stress*. A plot of true stress vs. strain is called a *true stress-strain diagram*.



Materials capable of withstanding large strains without a significant increase in stress are referred to as ductile materials. The converse applies to brittle materials.

3. Hooke's Law

For a limited range from the origin, the experimental values of stress vs. strain lie essentially on a straight line. For all practical purposes, up to some point the relationship between stress and strain may be said to be linear for all materials. This sweeping idealization and generalization applicable to all materials is known as *Hooke's law*. Symbolically, this law can be expressed by the equation

$$\sigma = E\varepsilon$$

which simply means that stress is directly proportional to strain, where the constant of proportionality is *E*. This constant E is called the *elastic modulus*, *modulus of elasticity*, or *Young's modulus*. As ε is dimensionless, *E* has the units of stress in this relation.

Graphically, *E* is interpreted as the slope of a straight line from the origin to the rather vague point A on a uniaxial stress-strain diagram. The stress corresponding to the latter point is termed the *proportional* or *elastic limit* of the material. Physically, the elastic modulus represents the stiffness of the material to an imposed load. The value of the elastic modulus is a definite property of a material. For all steels, *E* at room temperature is between 29 and 30 x .106 psi, or 200 and 207 GPa.

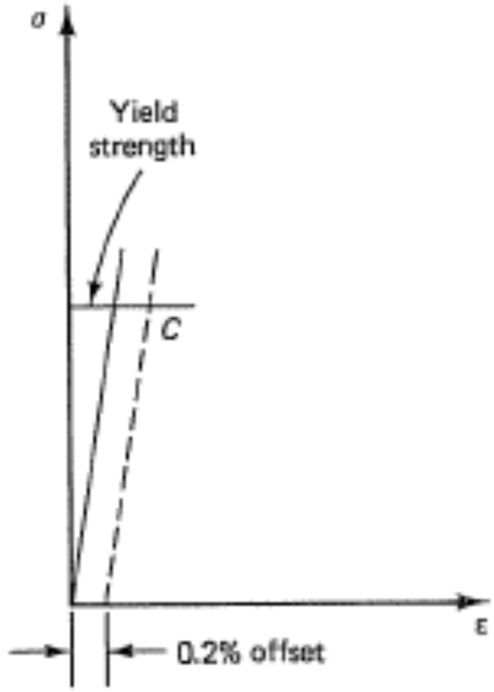
Hooke's law applies only up to the proportional limit of the material.

Some materials, notably single crystals and wood, possess different elastic moduli in different directions. Such materials, having different physical properties in different directions, are called *anisotropic*. The vast majority of engineering materials consist of a large number of randomly oriented crystals. Because of this random orientation, properties of materials become essentially alike in any direction. Such materials are called *isotropic*.

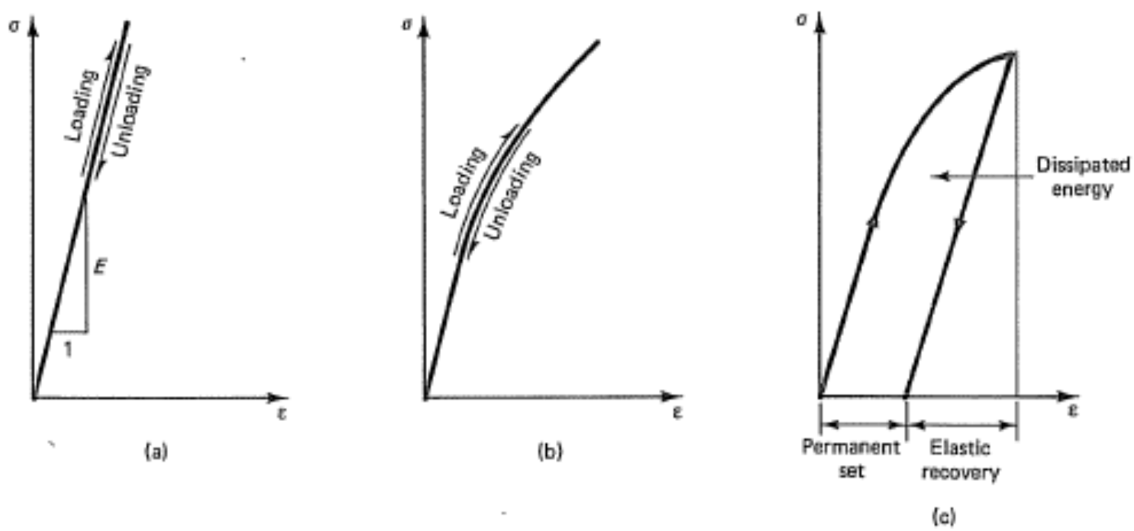
4. Further Remarks on Stress-Strain Relationships

In the stress-strain diagram, the highest points (B) correspond to the *ultimate strength* of a material. Stress associated with the long plateau is called the *yield strength* of a material. Note that at an essentially constant stress, strains 15 to 20 times those that take place up to the proportional limit occur during yielding. At the yield stress, a large amount of deformation takes place at a constant stress. The Yielding phenomenon is absent in most materials.

For materials that do not possess a well-defined yield strength, one is sometimes "invented" by the use of the so-called "offset method." In this method a line offset an arbitrary amount of 0.2 percent of strain is drawn parallel to the straight-line portion of the initial stress-strain diagram. Point C is then taken as the yield strength of the material at 0.2-percent offset.

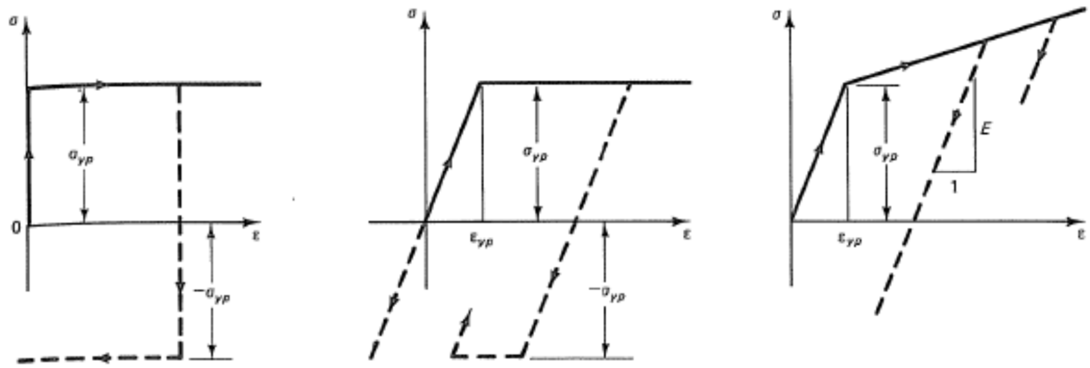


That a material is elastic usually implies that stress is directly proportional to strain, as in Hooke's law. Such materials are *linearly elastic* or *Hookean*. A material responding in a nonlinear manner and yet, when unloaded, returning back along the loading path to its initial stress-free state of deformation is also an elastic material. Such materials are called *nonlinearly elastic*.

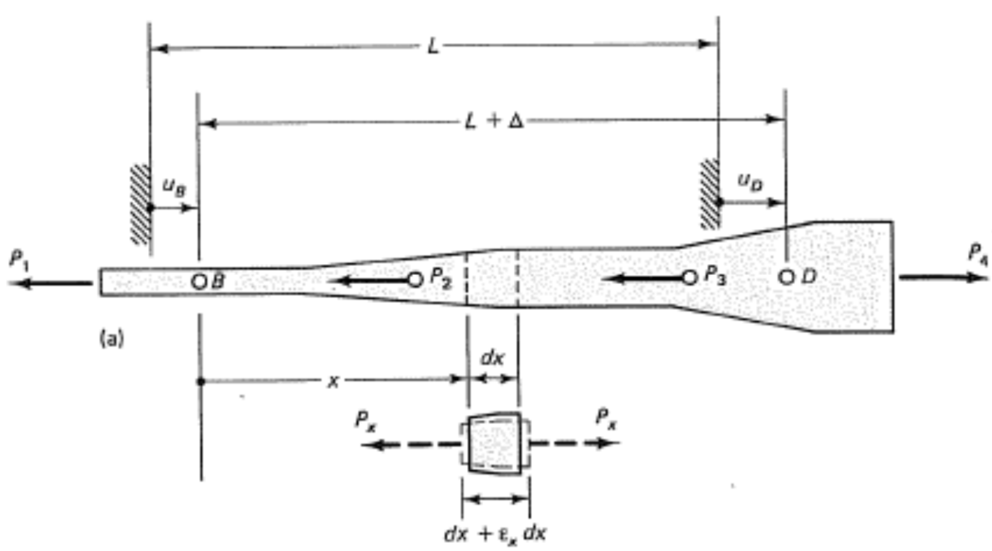


4. Other Idealizations of Constitutive Relations

In an increasingly larger number of technical problems, stress analyses based on the assumption of linearly elastic behavior are insufficient. For this reason, several additional stress-strain relations are now in general use. Such relations are frequently referred to as *constitutive relations* or *laws*.



5. Deformation of Axially Loaded Bars



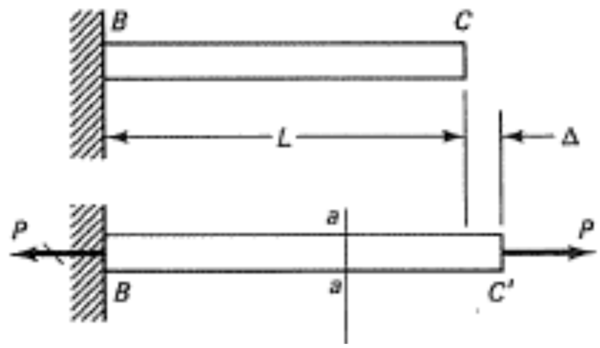
$$\varepsilon_x = \frac{du}{dx}$$

$$\int_0^L du = u(L) - u(0) = \int_0^L \varepsilon_x \, dx$$

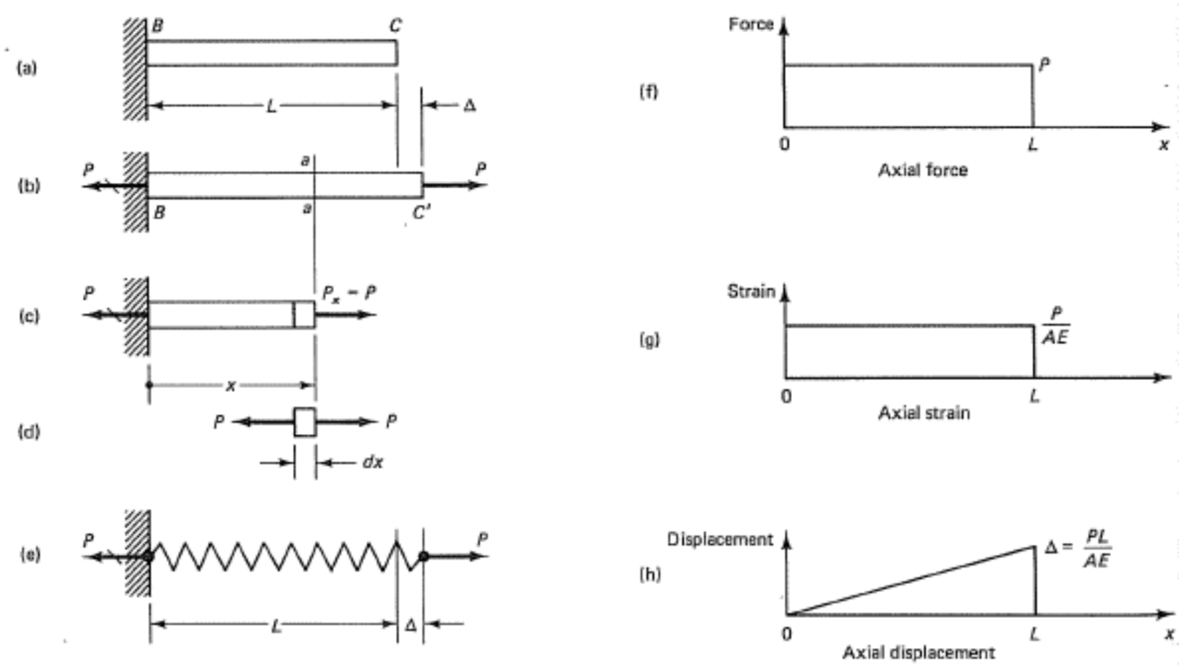
$$\Delta = \int_0^L \varepsilon_x \, dx$$

$$\Delta = \int_0^L \frac{P_x \, dx}{A_x E_x}$$

Example 2-1



Determine the deflection of the free end, caused by the application of a concentrated force *P*. The elastic modulus of the material is *E*.



$$\Delta = \int_A^B \frac{P_x\,dx}{A_x E} = \frac{P}{AE}\int_0^L dx = \frac{P}{AE}\,\Big|\, x \,\Big|_0^L = \frac{PL}{AE}$$

$$P = (AE/L)\Delta$$

This equation is related to the familiar definition for the *spring constant* or *stiffness* k reading

$$k = P/\Delta \qquad \text{[lb/in] or [N/m]}$$

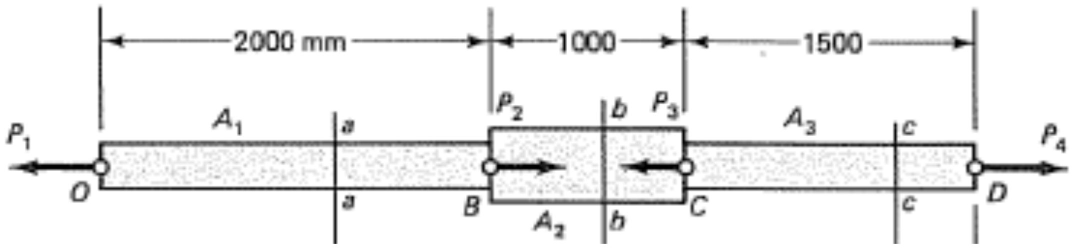
$$\boxed{k_i = \frac{A_i E_i}{L_i}}$$

The reciprocal of k defines the *flexibility* f, i.e.,

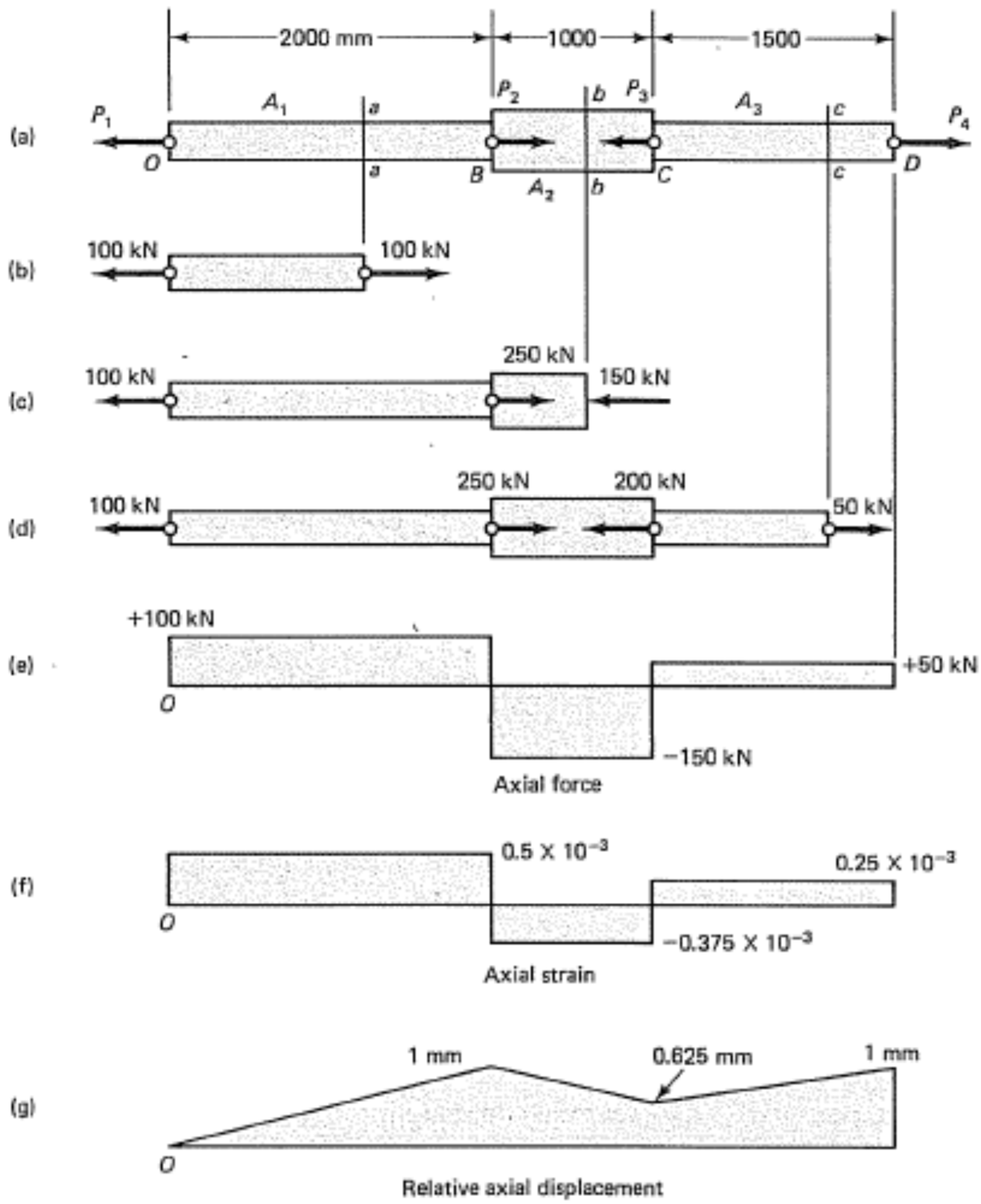
$$f = 1/k = \Delta/P \qquad \text{[in/lb] or [N/m]}$$

$$f_i = \frac{L_i}{A_i E_i}$$

Example 2-2



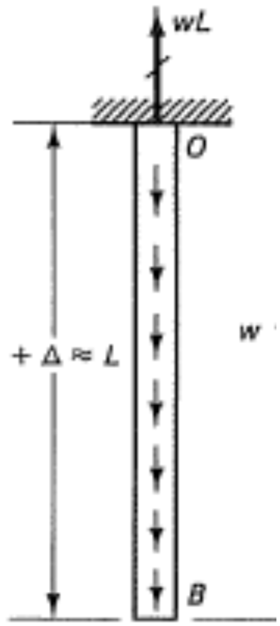
Determine the relative displacement of point *D* from *O*. The respective areas for bar segments *OB*, *BC*, and *CD* are 1000, 2000, and 1000 mm2. Let *E* = 200 GPa.



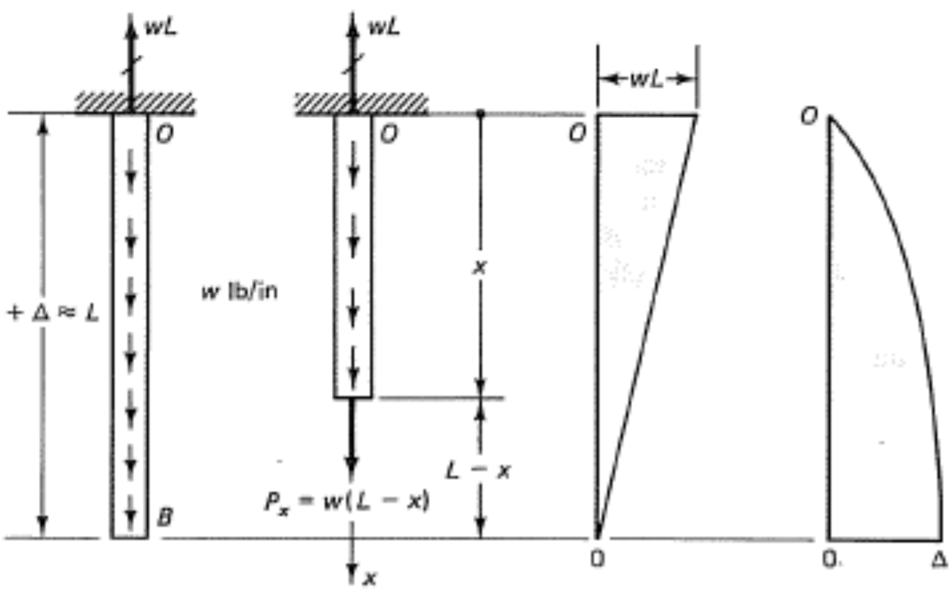
$$\Delta = \sum_i \frac{P_i L_i}{A_i E} = \frac{P_{OB} L_{OB}}{A_{OB} E} + \frac{P_{BC} L_{BC}}{A_{BC} E} + \frac{P_{CD} L_{CD}}{A_{C\dot{D}} EK}$$

$$\Delta = +\frac{100 \times 10^3 \times 2000}{1000 \times 200 \times 10^3} - \frac{150 \times 10^3 \times 1000}{2000 \times 200 \times 10^3} + \frac{50 \times 10^3 \times 1500}{1000 \times 200 \times 10^3}$$
$$= +1.000 - 0.375 + 0.375 = +1.000 \text{ mm}$$

Example 2-3



Determine the deflection of free end *B* of elastic bar *OB* caused by its own weight *w* lb/in. The constant cross-sectional area is *A*. Assume that *E* is given.



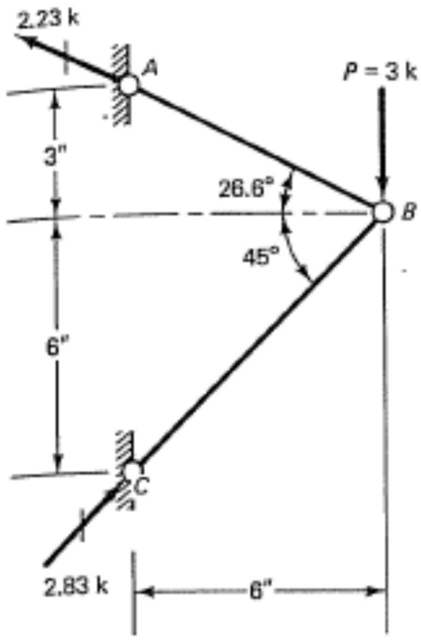
$$\Delta(x) = \int_0^x \frac{P_x\,dx}{A_x E} = \frac{1}{AE}\int w(L - x)\,dx = \frac{w}{AE}\left(Lx - \frac{x^2}{2}\right)$$

$$\Delta = \Delta(L) = \frac{w}{AE}\left(L^2 - \frac{L^2}{2}\right) = \frac{wL^2}{2AE} = \frac{WL}{2AE}$$

If a concentrated force *P*, in addition to the bar's own weight, were acting on bar *OB* at end *B*, the total deflection due to the two causes would be obtained by superposition as

$$\Delta = \frac{PL}{AE} + \frac{WL}{2AE} = \frac{[P + (W/2)]L}{AE}$$

Example 2-4

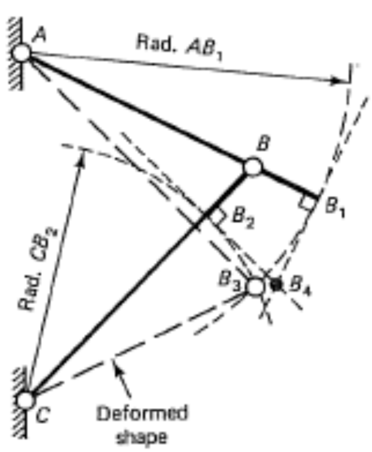


Determine the deflection of point *B*. Also determine the vertical stiffness of the bracket at *B*. Assume that the members are made of 2024-T4 aluminum alloy.

$$\Delta_{AB} = \left[\frac{PL}{AE}\right]_{AB} = \left[\sigma \frac{L}{E}\right]_{AB} = \frac{17.8 \times 6.71}{10.6 \times 10^3} = 11.3 \times 10^{-3} \text{ in}$$

(elongation)

$$\Delta_{BC} = -\frac{12.9 \times 8.29}{10.6 \times 10^3} = -10.3 \times 10^{-3} \text{ in} \qquad \text{(contraction)}$$



$$\Delta_{BC} = \Delta \cos \theta_2 \qquad \text{and} \qquad \Delta_{AB} = \Delta \cos \theta_1$$

$$\frac{\cos \theta_2}{\cos \theta_1} = \frac{\Delta_{BC}}{\Delta_{AB}} = \frac{10.3 \times 10^{-3}}{11.3 \times 10^{-3}} = 0.912$$

$$\theta_2 = 180° - 45° - 26.6° - \theta_1 = 108.4° - \theta_1$$

$$\cos \theta_2 = \cos 108.4° \cos \theta_1 + \sin 108.4° \sin \theta_1$$

$$\frac{\cos \theta_2}{\cos \theta_1} = \cos 108.4° + \sin 108.4° \tan \theta_1 = 0.912$$

$$\tan \theta_1 = 1.29 \qquad \text{and} \qquad \theta_1 = 52.2°$$

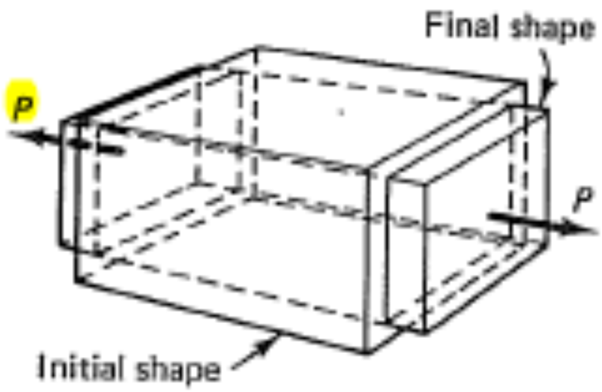
$$\Delta = \Delta_{AB}/\cos \theta_1 = 18.4 \times 10^{-3} \text{ in}$$

forming an angle of 11.2^0 with the vertical.

Since $\Delta_{ver} = \Delta \cos 11.2^0 = 18.0 \times 10^{-3}$ in, the vertical stiffness of the bracket is given by the spring constant

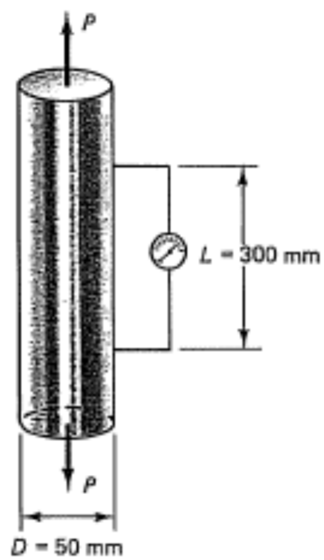
$$k = \frac{P}{\Delta_{vert}} = \frac{3}{18.0 \times 10^{-3}} = 167 \text{ kips/in}$$

6. Poinsson's Ratio



$$\vartheta = -\frac{\text{lateral strain}}{axial\ strain}$$

Example 2-5



Consider a carefully conducted experiment where an aluminum bar of 50-mm diameter is stressed in a testing machine. At a certain instant the applied force P is 100 kN, while the measured elongation of the rod is 0.219 mm in a 300-mm gage length, and the diameter's dimension is decreased by 0.01215 mm. Calculate the two physical constants v and E of the material.

Transverse or lateral strain:

$$\varepsilon_t = \frac{\Delta_t}{D} = -\frac{0.01215}{50} = -0.000243 \text{ mm/mm}$$

Axial strain:

$$\varepsilon_a = \frac{\Delta}{L} = +\frac{0.219}{300} = 0.00073 \text{ mm/mm}$$

Poinsson's Ratio:

$$\nu = -\frac{\varepsilon_t}{\varepsilon_a} = -\frac{(-0.000243)}{0.00073} = 0.333$$

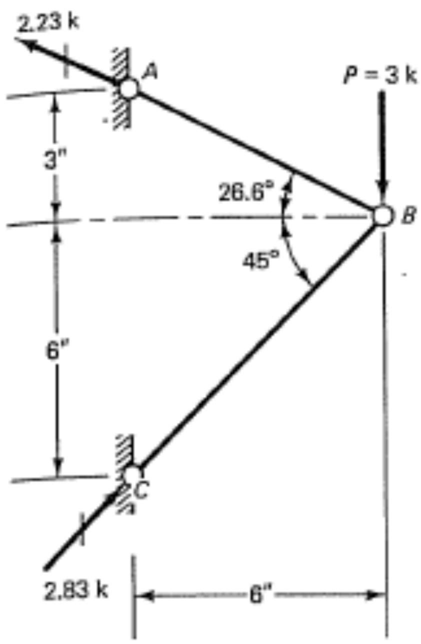
$$E = \frac{PL}{A\Delta} = \frac{100 \times 10^3 \times 300}{1960 \times 0.219} = 70 \times 10^3 \text{ N/mm}^2 = 70 \text{ GPa}$$

7. Thermal Strain and Deformation

$$\varepsilon_T = \alpha(T - T_o)$$

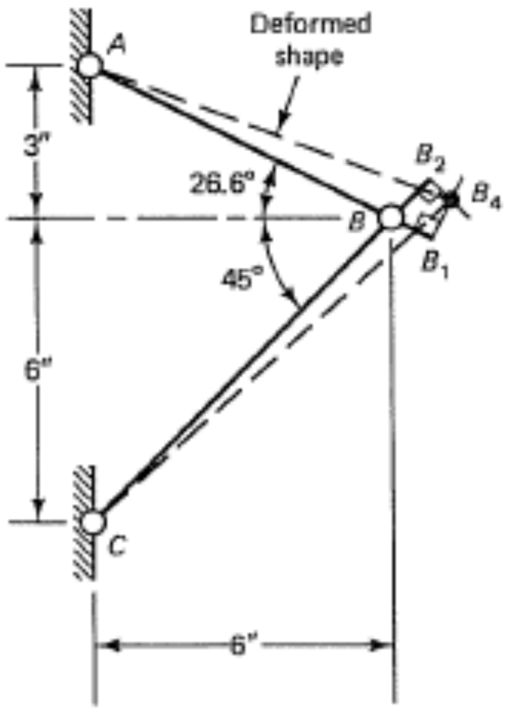
$$\Delta_T = \alpha(\delta T)L$$

Example 2-7

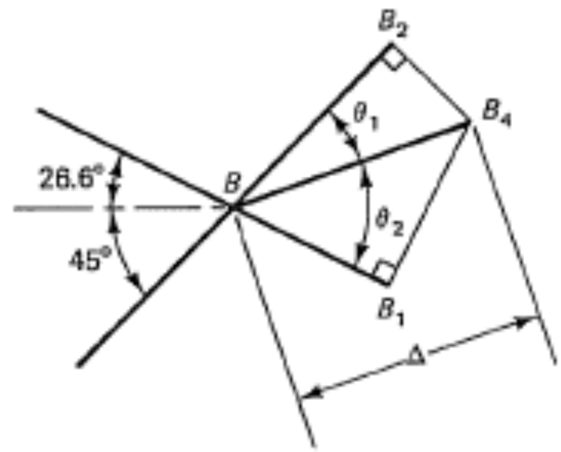


Determine the displacement of point B caused by an increase in temperature of 100°F. Given, coefficient of expansion is 12.9 x $10^{-6}/^{0}$ F.

$$\Delta_{AB} = 12.9 \times 10^{-6} \times 100 \times 6.71 = 8.656 \times 10^{-3} \text{ in}$$
$$\Delta_{BC} = 12.9 \times 10^{-6} \times 100 \times 8.49 = 10.95 \times 10^{-3} \text{ in}$$



(a)



(b)

$$\Delta_T \cos \theta_2 = \Delta_{AB} \quad \text{and} \quad \Delta_T \cos \theta_1 = \Delta_{BC}$$

$$\frac{\cos \theta_2}{\cos \theta_1} = \frac{\Delta_{AB}}{\Delta_{BC}} = \frac{8.656 \times 10^{-3}}{10.95 \times 10^{-3}} = 0.7905$$

Here, however, $\theta_2 = 45^0 + 26.6^0 - \theta_1 = 71.6^0 - \theta_1$; therefore,

$$\cos \theta_2 = \cos 71.6° \cos \theta_1 + \sin 71.6° \sin \theta_1$$

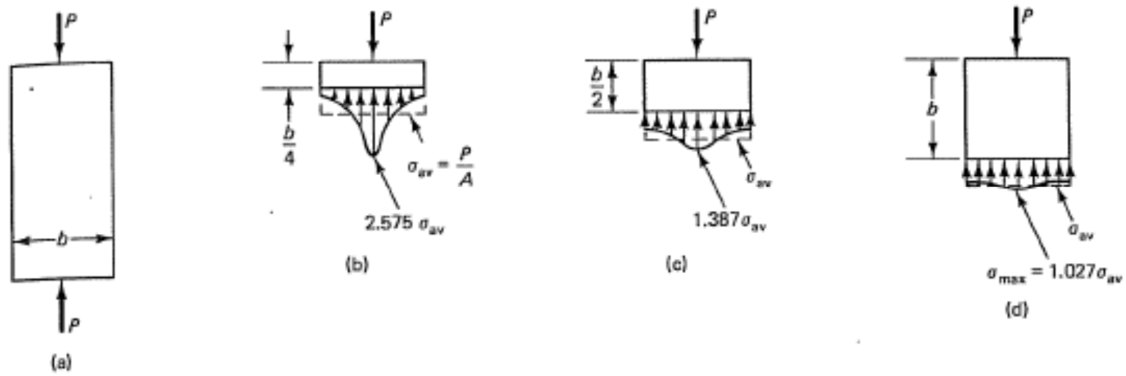
$$\frac{\cos \theta_2}{\cos \theta_1} = \cos 71.6° + \sin 71.6° \tan \theta_1 = 0.7905$$

$$\tan \theta_1 = 0.500$$

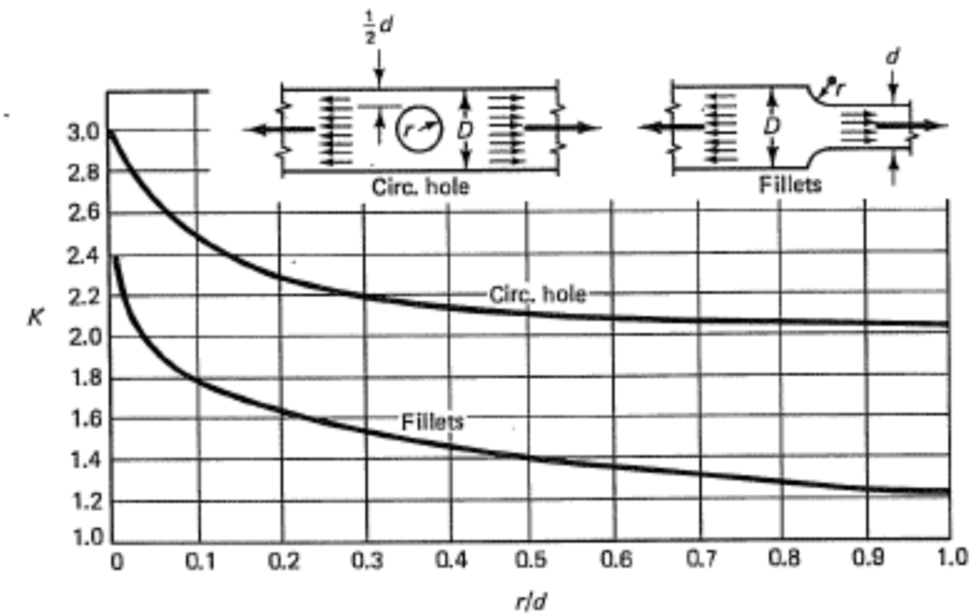
$$\theta_1 = 26.6°$$

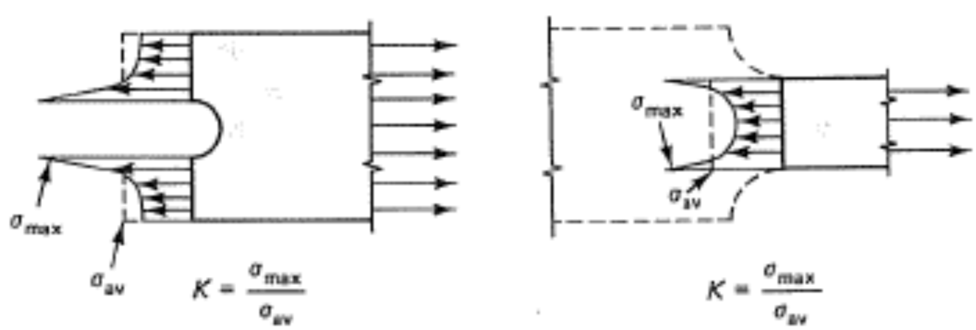
$$\Delta_T = \Delta_{BC}/\cos \theta_1 = 12.2 \times 10^{-3} \text{ in}$$

8. Saint-Venant's Principle and Stress Concentrations



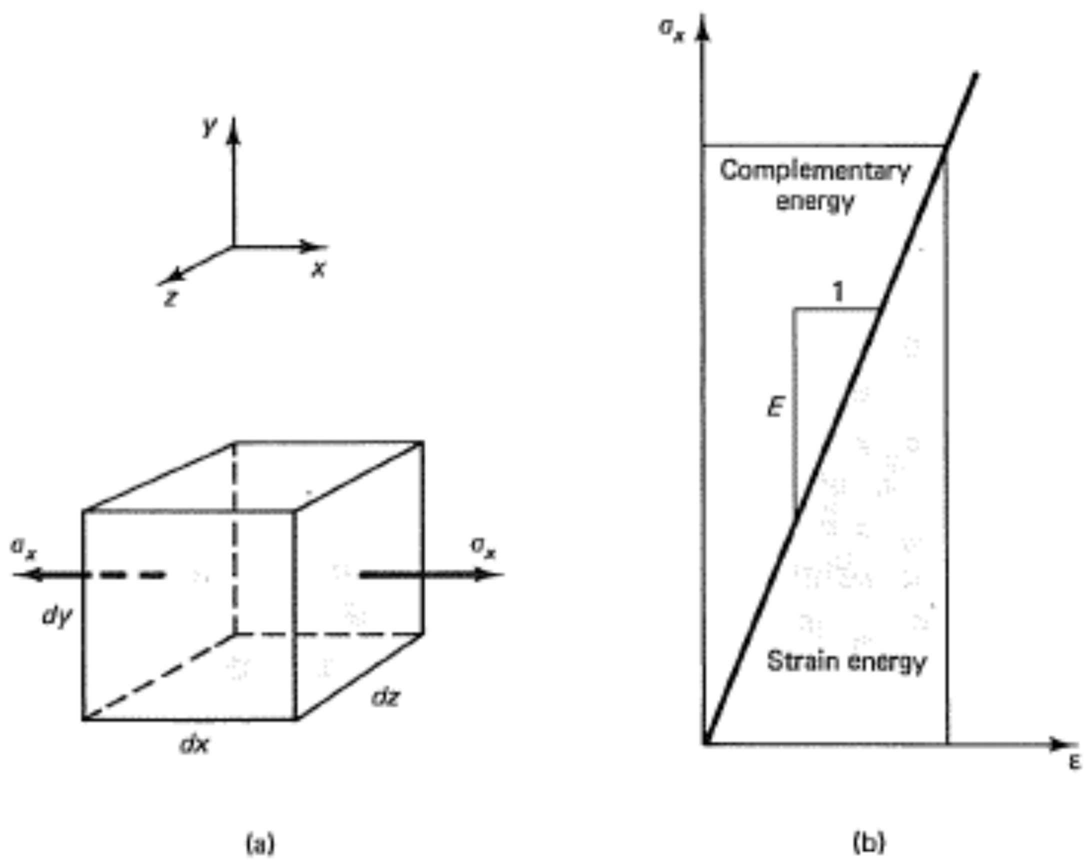
The manner of force application on stresses is important only in the vicinity of the region where the force is applied. This also holds true for the disturbances caused by changes in cross section.





9. Elastic Strain Energy for Uniaxial Stress

In mechanics, energy is defined as the capacity to do work, and work is the product of a force times the distance in the direction that the force moves. In solid deformable bodies, stresses multiplied by their respective areas are forces, and deformations are distances. The product of these two quantities is the *internal work* done in a body by externally applied forces. This internal work is stored in an elastic body as the *internal elastic energy* of deformation, or the *elastic strain energy*.



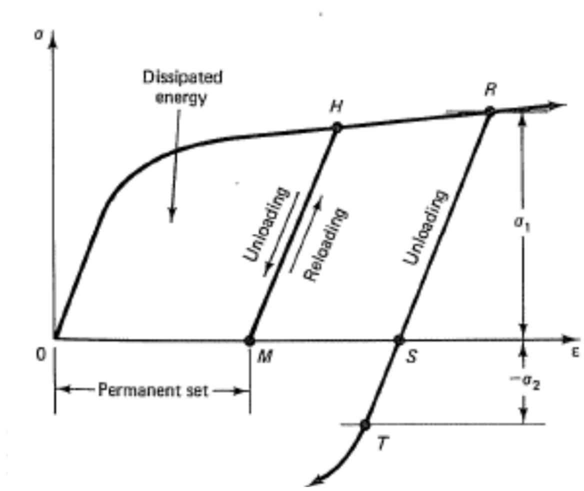
$$dU = \underbrace{\underbrace{\tfrac{1}{2}\sigma_x\,dy\,dz}_{\text{average force}} \times \underbrace{\varepsilon_x\,dx}_{\text{distance}}}_{\text{work}} = \tfrac{1}{2}\sigma_x\varepsilon_x\,dx\,dy\,dz = \tfrac{1}{2}\sigma_x\varepsilon_x\,dV$$

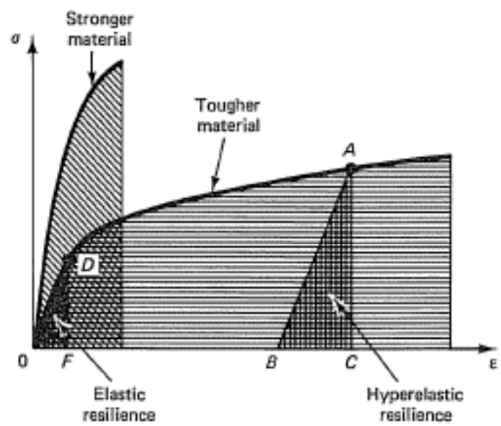
One obtains the strain energy stored in an elastic body per unit volume of the material, or its *strain-energy density* U_o .

$$U_o = \frac{dU}{dV} = \frac{\sigma_x \varepsilon_x}{2}$$

$$U_o = \frac{dU}{dV} = \frac{E\varepsilon_x^2}{2} = \frac{\sigma_x^2}{2E}$$

$$U = \int_{vol} \frac{\sigma_x^2}{2E} dV$$

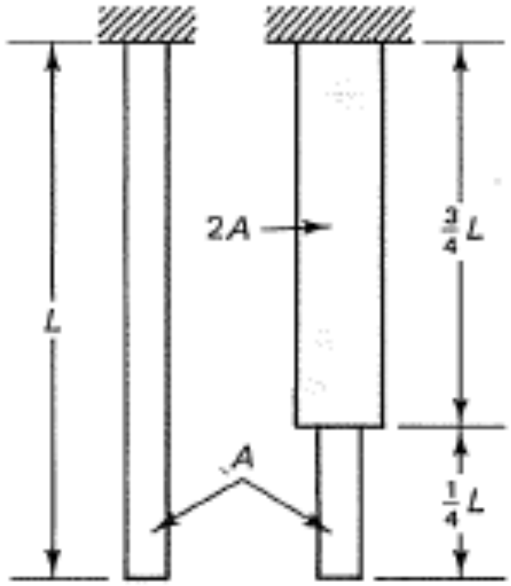




For a particular material, substitution into Eq. 2-22 of the value of the stress at the proportional limit gives an index of the material's ability to store or absorb energy without permanent deformation. The quantity so found is called the *modulus of resilience* and is used to differentiate materials for applications where energy must be absorbed by members.

The area under a complete stress-strain diagram gives a measure of a material's ability to absorb energy up to fracture and is called its *toughness*. The larger the total area under the stress-strain diagram, the tougher the material.

Example 2-9



Two elastic bars, whose proportions are shown in the figure above, are to absorb the same amount of energy delivered by axial forces. Neglecting stress concentrations, compare the stresses in the two bars. The cross-sectional area of the left bar is *A*, and that of the right bar is *A* and *2A* as shown.

$$U_1 = \int_V \frac{\sigma_1^2}{2E}\,dV = \frac{\sigma_1^2}{2E}\int_V dV = \frac{\sigma_1^2}{2E}(AL)$$

$$U_2 = \int_V \frac{\sigma^2}{2E}\,dV = \frac{\sigma_2^2}{2E}\int_{\text{lower part}} dV + \frac{(\sigma_2/2)^2}{2E}\int_{\text{upper part}} dV$$

$$= \frac{\sigma_2^2}{2E}\left(\frac{AL}{4}\right) + \frac{(\sigma_2/2)^2}{2E}\left(2A\,\frac{3L}{4}\right) = \frac{\sigma_2^2}{2E}\left(\frac{5}{8}AL\right)$$

$$\frac{\sigma_1^2}{2E}(AL) = \frac{\sigma_2^2}{2E}\left(\frac{5}{8}AL\right) \qquad \text{or} \qquad \sigma_2 = 1.265\sigma_1$$

10. Deflections by the Energy Method

$$U = W_e$$

Example 2-10

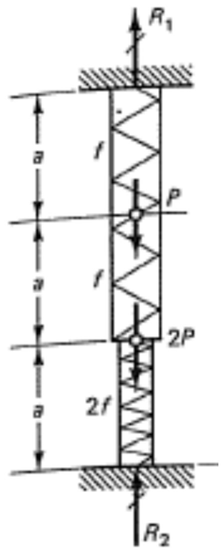
Find the deflection of the free end of an elastic rod of constant cross-sectional area *A* and length *L* due to axial force *P* applied at the free end.

$$U = \frac{\sigma_1^2}{2E} AL = \frac{P^2 L}{2AE}$$

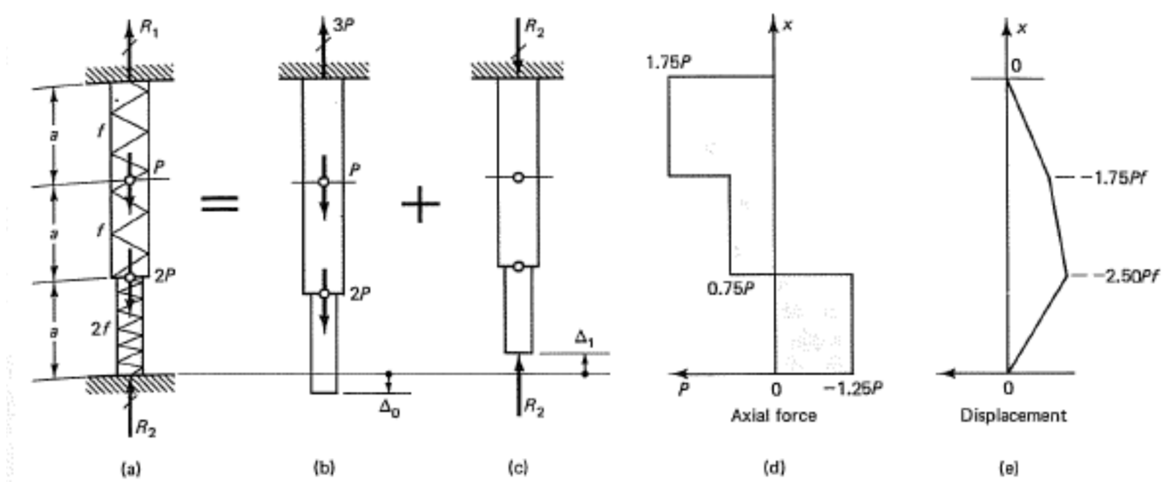
$$\frac{P\Delta}{2} = \frac{P^2 L}{2AE} \quad \text{and} \quad \Delta = \frac{PL}{AE}$$

11. Statically Indeterminate Systems: Force Method of Analysis

Example 2-12



Determine the reactions and plot the axial force and the axial displacement diagrams for the bar.

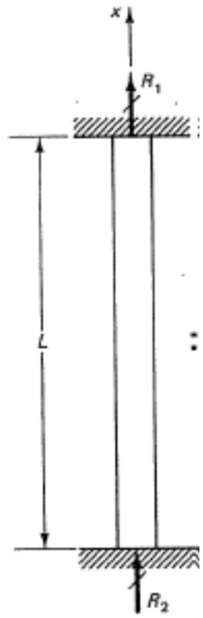


$$\Delta_0 = \sum_i f_i P_i = -2fP - f(2P + P) = -5fP$$

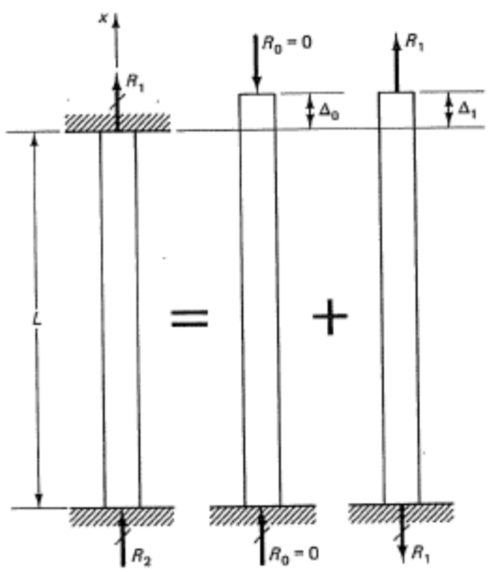
$$\Delta_1 = (2f + f + f)R_2 = 4fR_2$$

$$\Delta_1 + \Delta_2 = 0, \qquad R_2 = 1.25P$$

Example 2-13



If the bar temperature increases by 8T, what axial force develops in the bar? AE for the bar is constant.



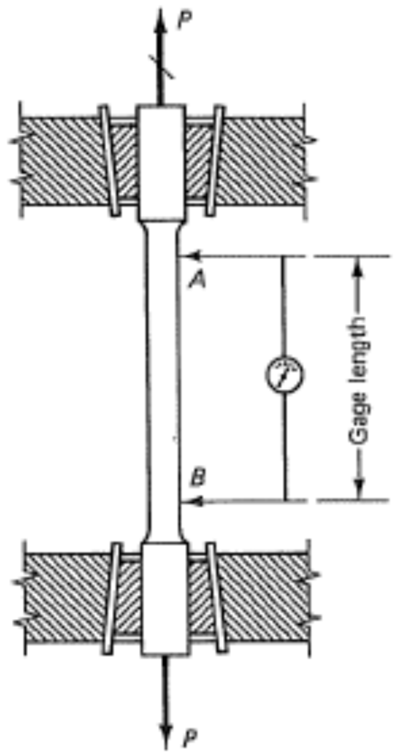
$$\Delta_0 = \alpha(\delta T)L$$

$$\Delta_1 = R_1 f = \frac{R_1 L}{AE}$$

$$\Delta_0 + \Delta_1 = 0, \qquad R_1 = -\alpha(\delta T)AE$$

Lecture 3: Generalized Hooke's Law

1. Normal Strain



$$\varepsilon = \frac{L - L_o}{L_o} = \frac{\Delta L}{L_o}$$

This expression defines the *extensional strain*. Since this strain is associated with the normal stress, it is usually called the *normal strain*: It is a dimensionless quantity, but it is customary to refer to it as having the dimensions of in/in, m/m, or µm/m (microstrain). Sometimes it is given as a percentage.

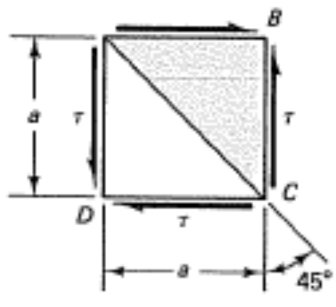
Example 3-2

A 50 mm cube of steel is subjected to a uniform pressure of 200 MPa acting on all faces. Determine the change in dimension between two parallel faces of the cube. Let E = 200 GPa and ν= 0.25.

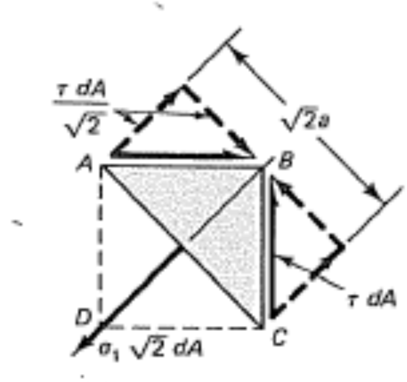
$$\varepsilon_x = \frac{(-200)}{200 \times 10^3} - \left(\frac{1}{4}\right)\frac{(-200)}{200 \times 10^3} - \left(\frac{1}{4}\right)\frac{(-200)}{200 \times 10^3}$$
$$= -5 \times 10^{-4} \text{ mm/mm}$$
$$\Delta_x = \varepsilon_x L_x = -5 \times 10^{-4} \times 50 = -0.025 \text{ mm (contraction)}$$

In this case $\Delta_x = \Delta_y = \Delta_z$.

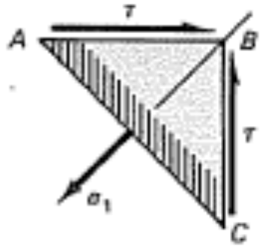
2. E, G and ν Relationship



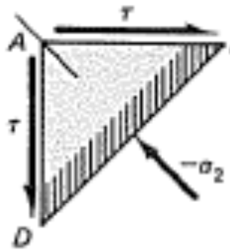
(a)



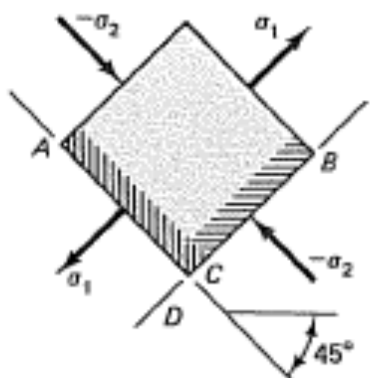
(b) Force diagram



(c)

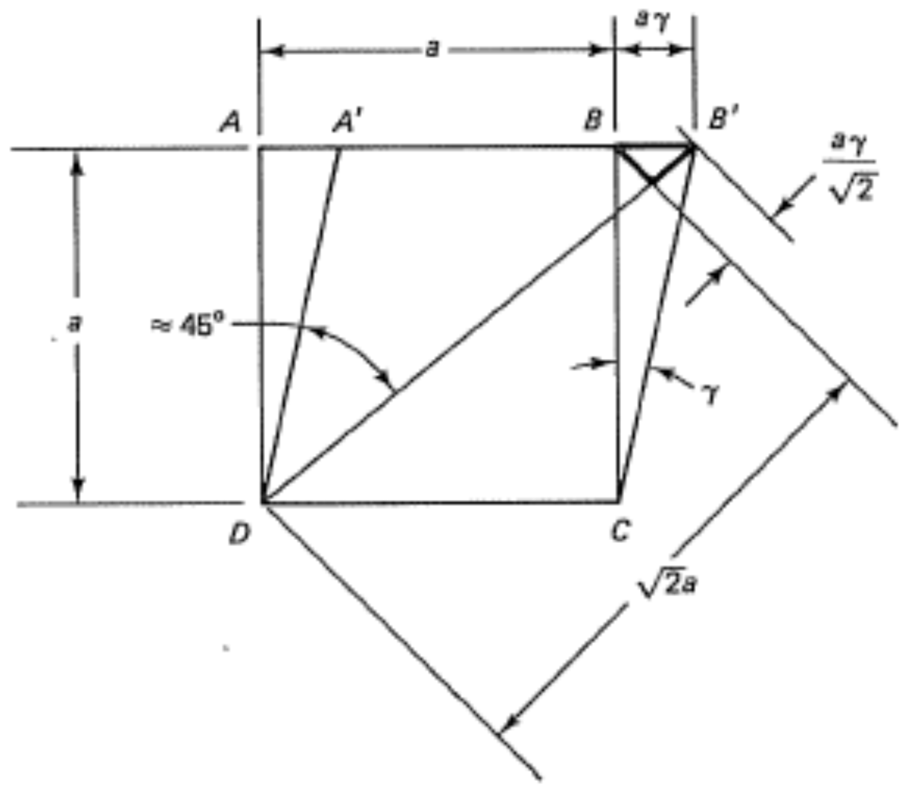


(d)



(e)

$$\sigma_1 = -\sigma_2 = \tau$$



$$\varepsilon_{45^\circ} = \frac{\tau}{2G}$$

$$\varepsilon_{45^\circ} = \frac{\sigma_1}{E} - \nu\,\frac{\sigma_2}{E} = \frac{\tau}{E}\,(1 + \nu)$$

$$\boxed{G = \frac{E}{2(1 + \nu)}}$$

3. Dilation and Bulk Modulus

After subtracting the initial volume from the volume of the strained element, the change in volume is determined. This is

$$(1 + \varepsilon_x)\,dx\,(1 + \varepsilon_y)\,dy\,(1 + \varepsilon_z)\,dz - dx\,dy\,dz \approx (\varepsilon_x + \varepsilon_y + \varepsilon_z)\,dx\,dy\,dz$$

e, the change in volume per unit volume, often referred to as *dilatation*, is defined as

$$\boxed{e = \varepsilon_x + \varepsilon_y + \varepsilon_z}$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

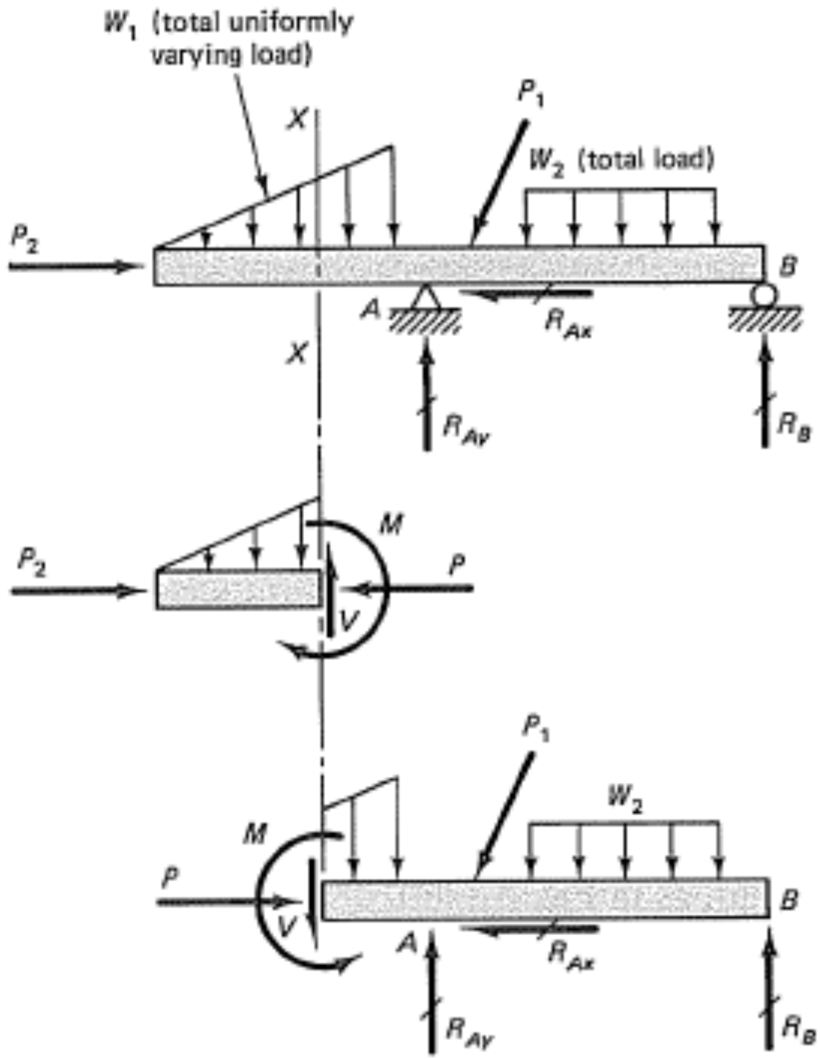
If an elastic body is subjected to hydrostatic pressure of uniform intensity *p*, so that $\sigma_x = \sigma_y = \sigma_z = -p$,

$$e = -\frac{3(1 - 2\nu)}{E}p \qquad \text{or} \qquad \boxed{\frac{-p}{e} = k = \frac{E}{3(1 - 2\nu)}}$$

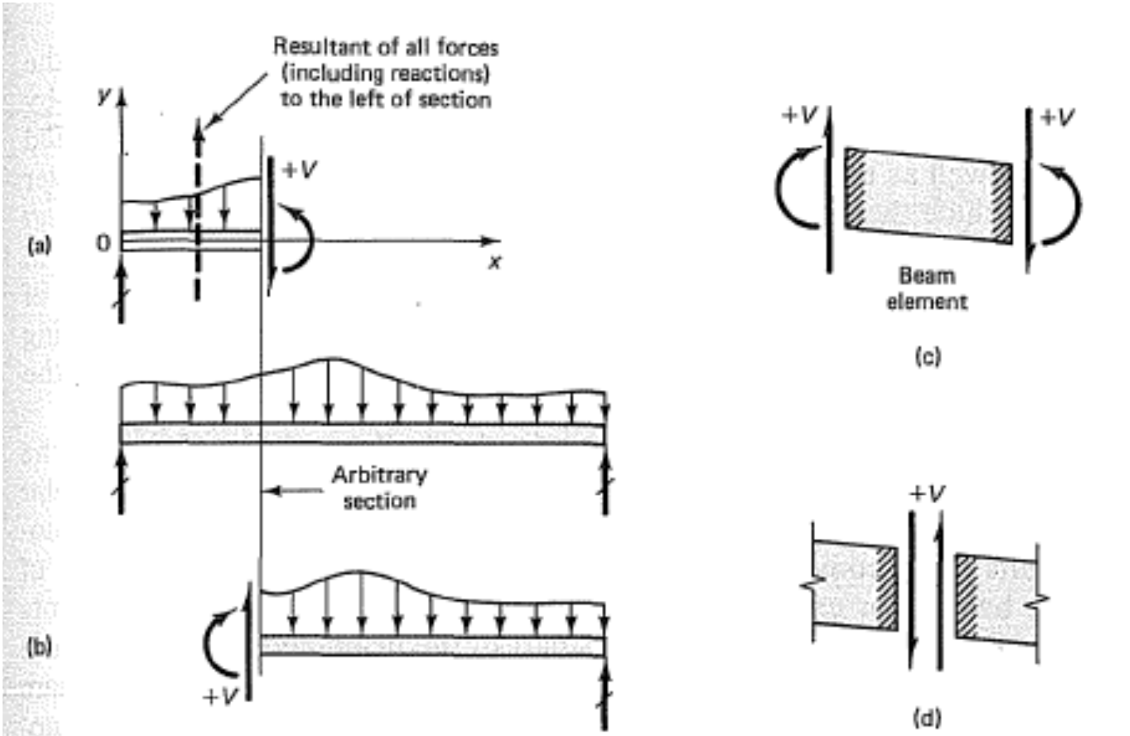
The quantity *k* represents the ratio of the hydrostatic compressive stress to the decrease in volume and is called the *modulus of compression* or *bulk modulus*.

Lecture 4: Axial Force, Shear and Bending Moment

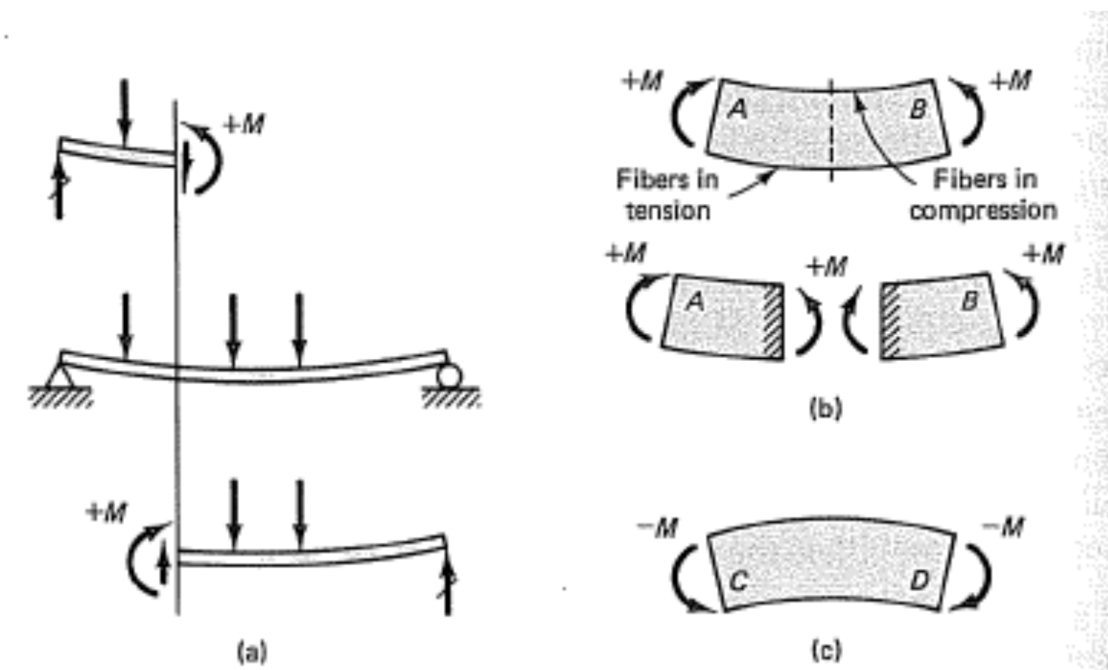
1. Axial Force in Beams



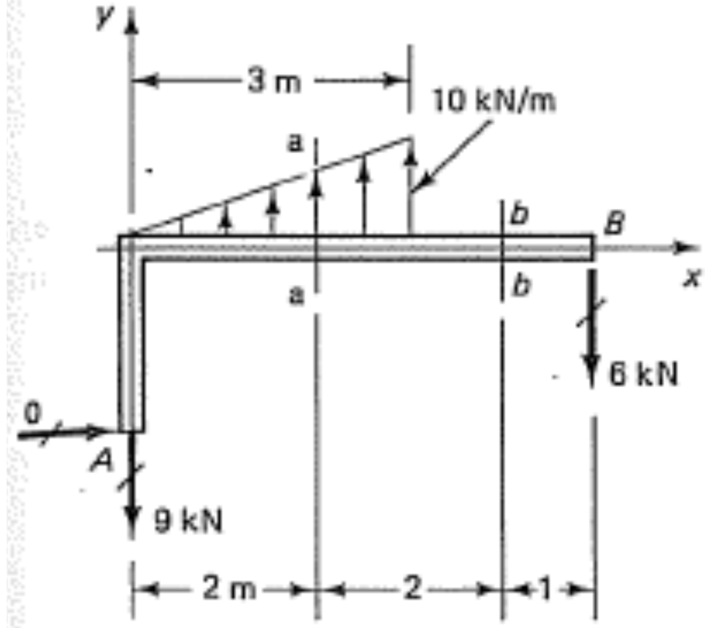
2. Shear in Beams



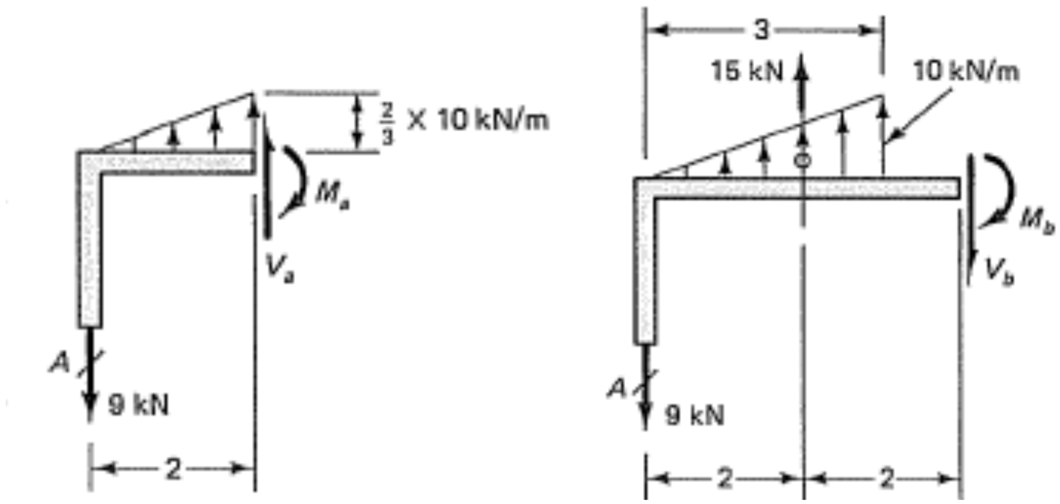
3. Bending Moment in Beams



Example 5-4



Determine the internal system of forces at sections a-a and b-b.

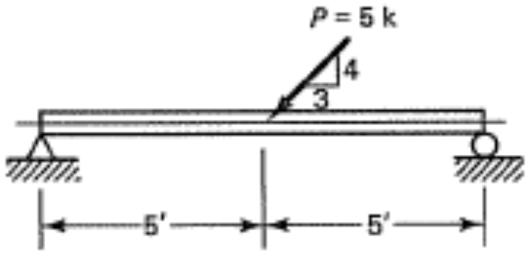


$$V_a = -9 + \frac{1}{2} \times 2 \times \frac{2}{3} \times 10 = -2.33 \text{ kN}$$

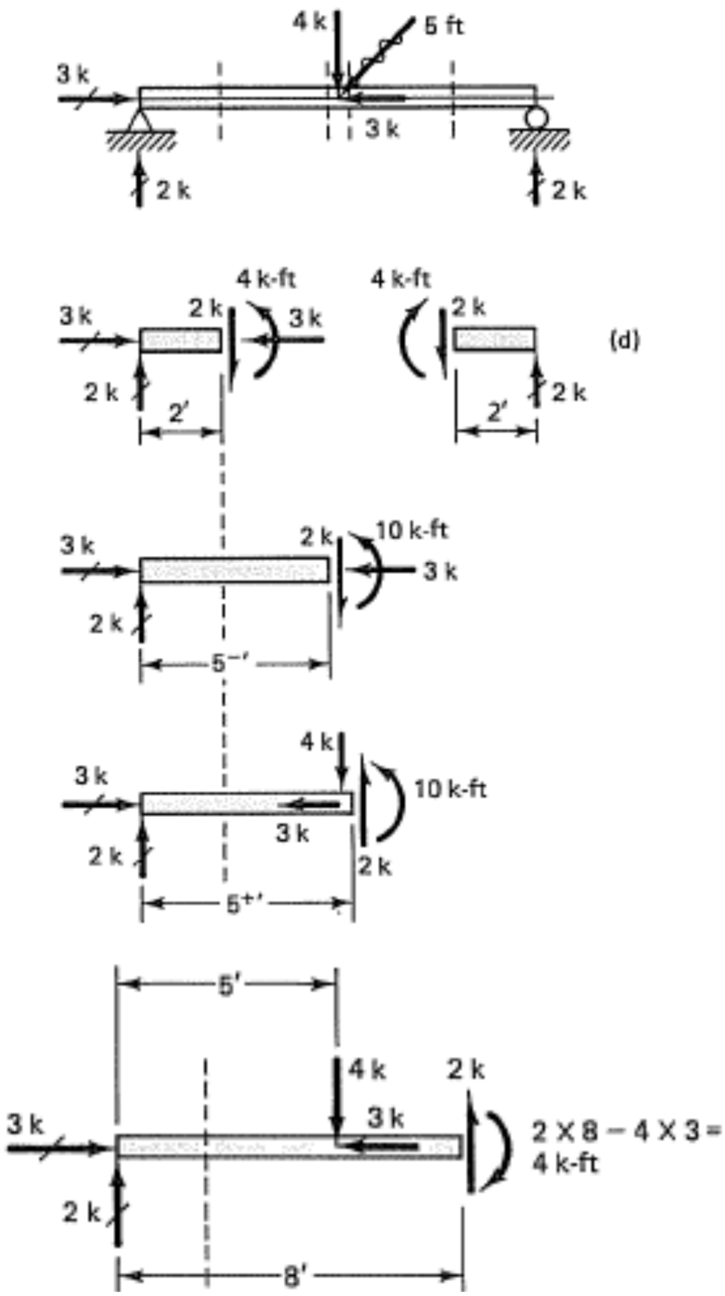
$$M_a = -9 \times 2 + \frac{1}{2} \times 2 \times \frac{2}{3} \times 10 \times \frac{1}{3} \times 2 = -13.6 \text{ kN·m}$$

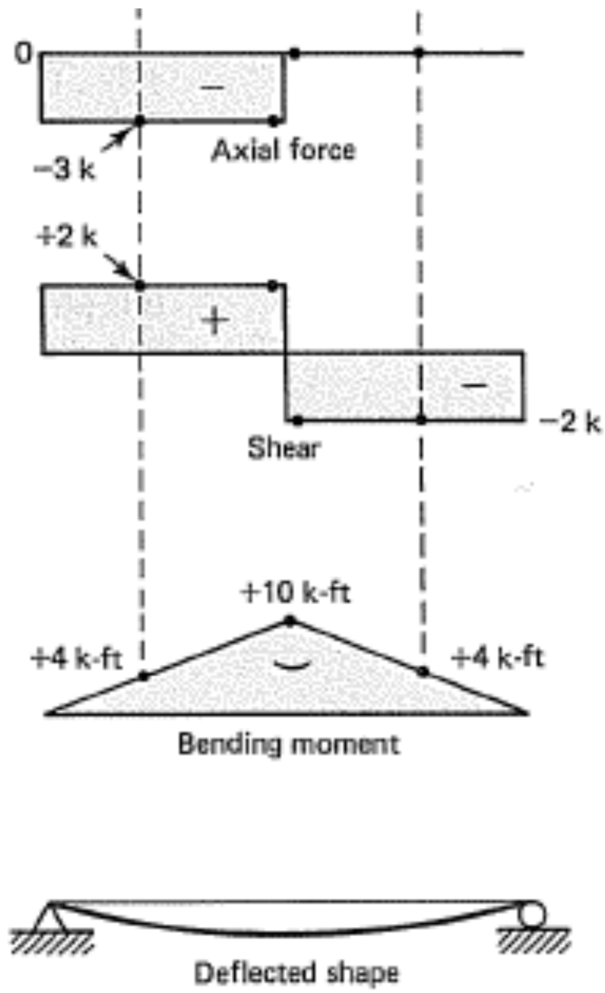
$$V_b = +6 \text{ kN} \quad M_b = -6 \times 1 = -6 \text{ kN·m}$$

Example 5-5

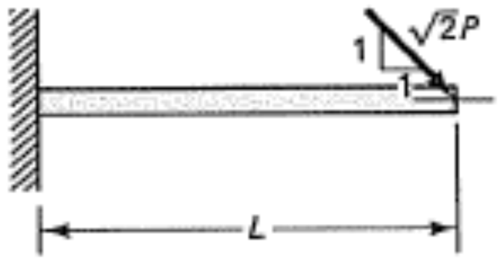


Construct axial-force, shear, and bending-moment diagrams for the beam.

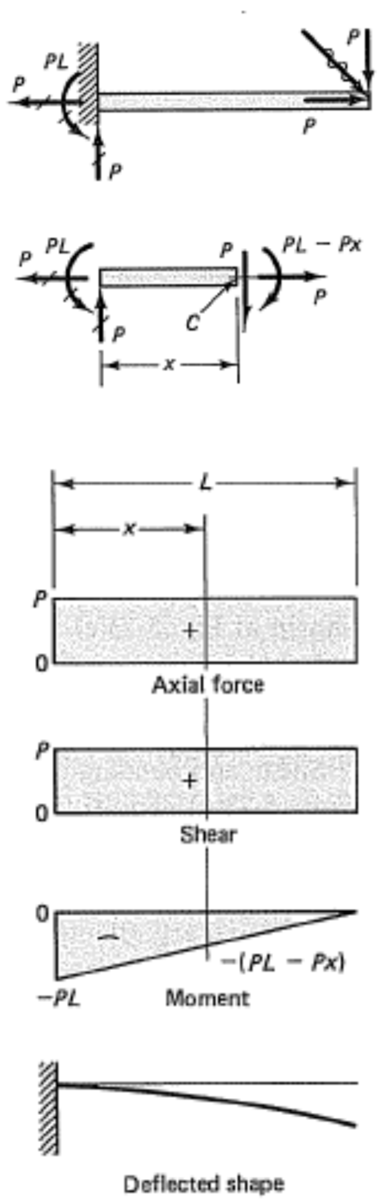



0
−
−3 k
Axial force
+2 k
+
−
−2 k
Shear
+10 k-ft
+4 k-ft
+4 k-ft
Bending moment
Deflected shape

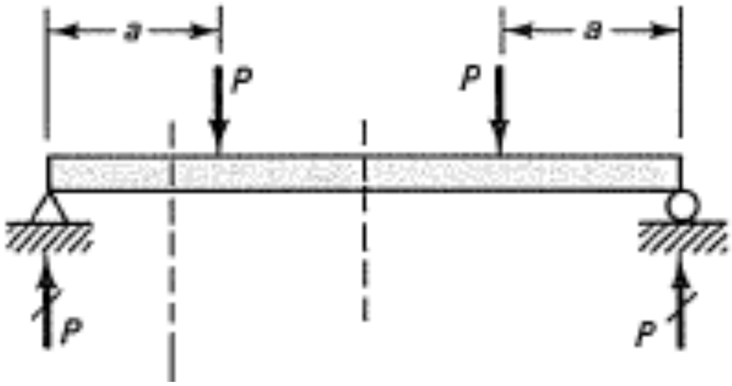
Example 5-6



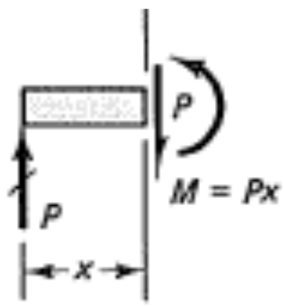
Determine axial-force, shear, and bending-moment diagrams.

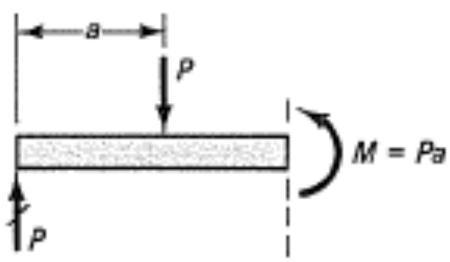


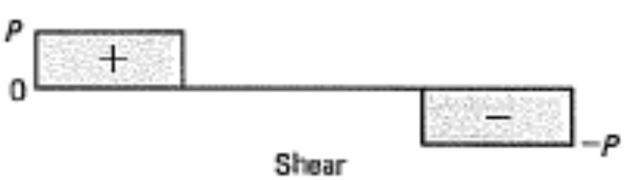
Example 5-7



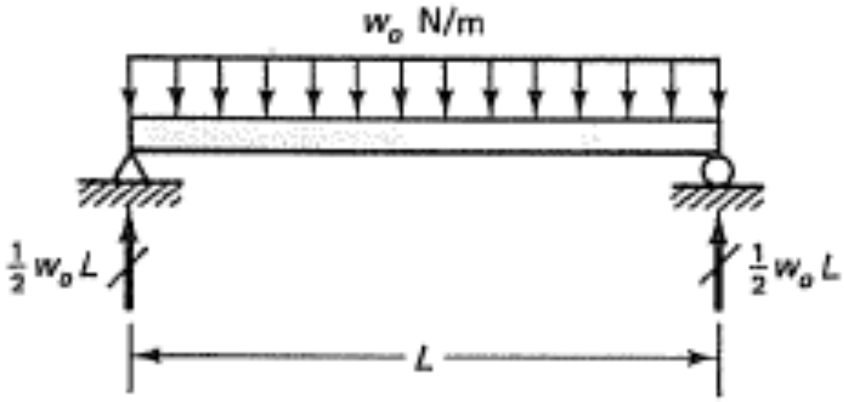
Construct shear and bending-moment diagrams.



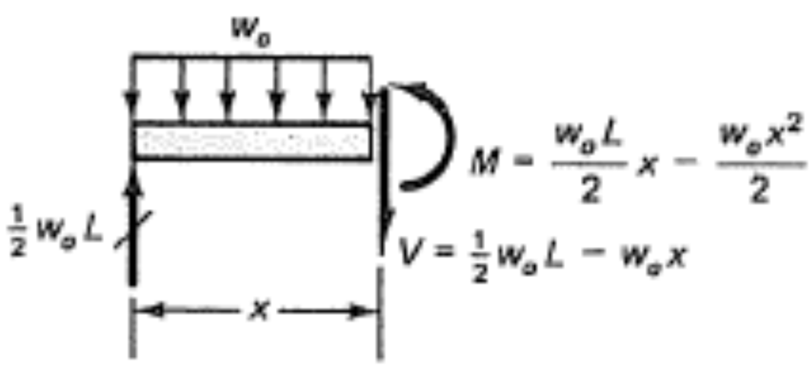


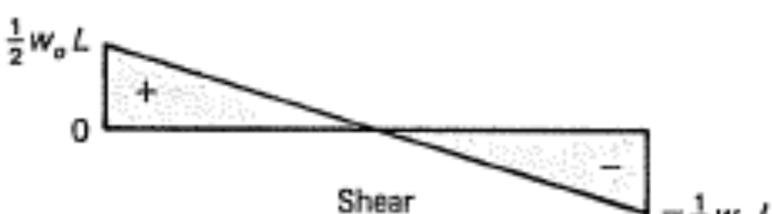


Example 5-8

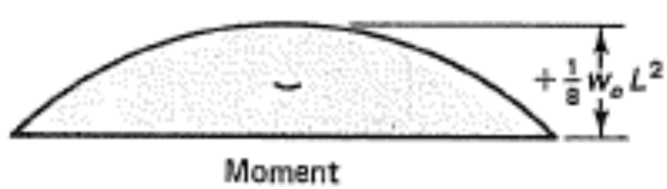


Plot shear and a bending-moment diagrams.



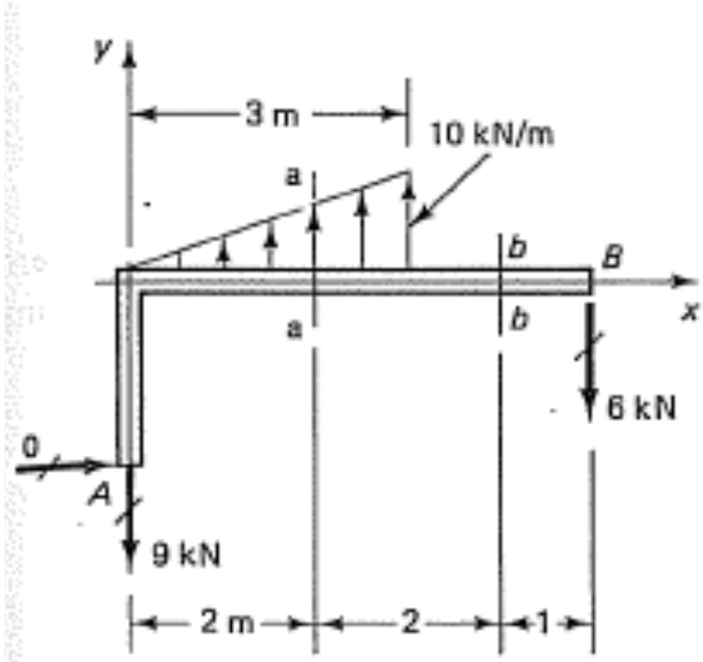


(c)

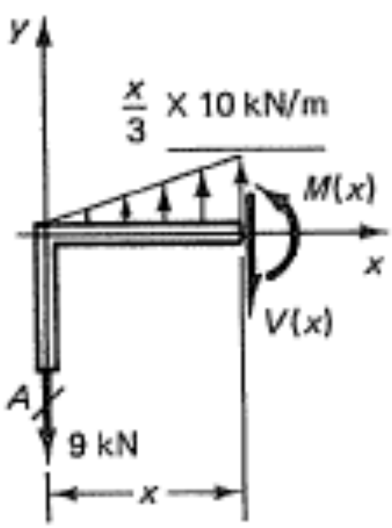


(d)

Example 5-9



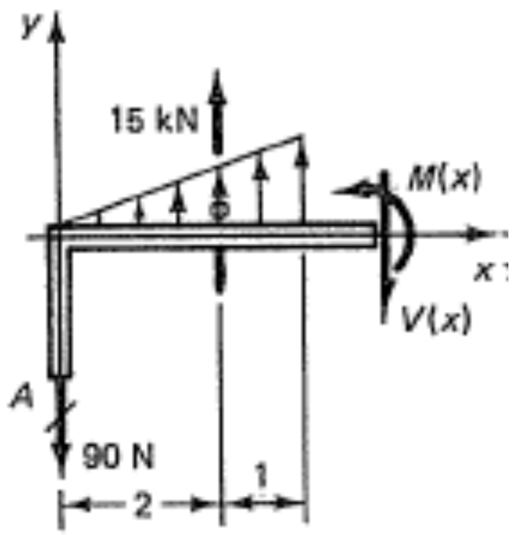
Express the shear V and the bending moment M as a function of x along the horizontal member.



For 0 < x < 3,

$$V(x) = -9 + \frac{1}{2}x\left(\frac{x}{3} \times 10\right) = -9 + \frac{5}{3}x^2 \text{ kN}$$

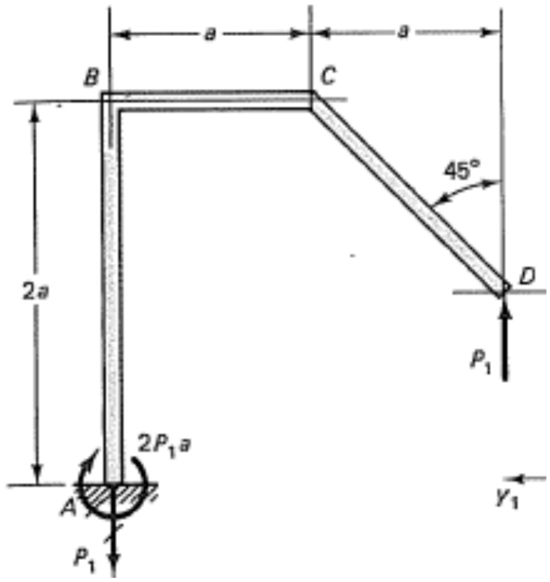
$$M(x) = -9x + \frac{1}{2}x\left(\frac{x}{3} \times 10\right)\left(\frac{x}{3}\right) = -9x + \frac{5}{9}x^3 \text{ kN·m}$$



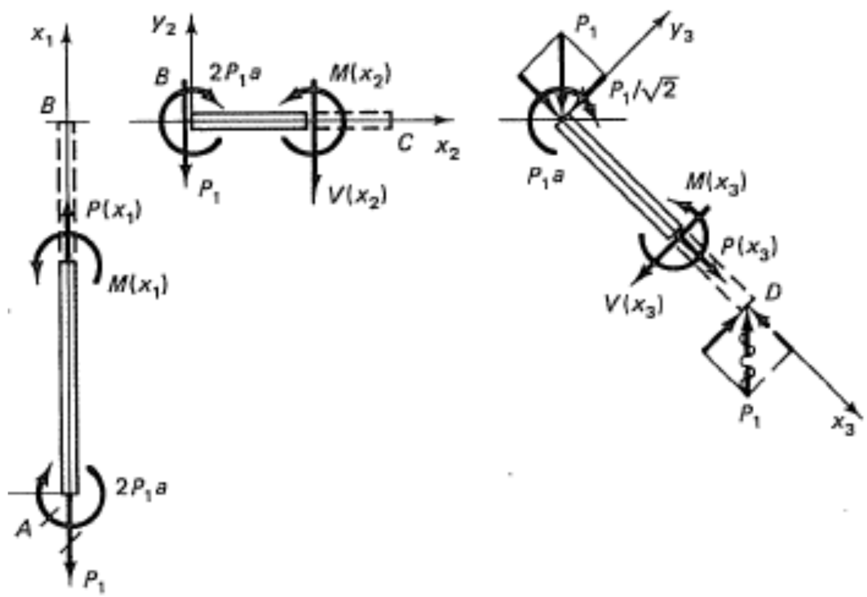
For 3 < x < 5,

$$V(x) = -9 + 15 = +6 \text{ kN}$$
$$M(x) = -9x + 15(x - 2) = 6x - 30 \text{ kN·m}$$

Example 5-11



At arbitrary sections, determine the internal forces P, V, and M in the members caused by the application of a vertical force PI at D.

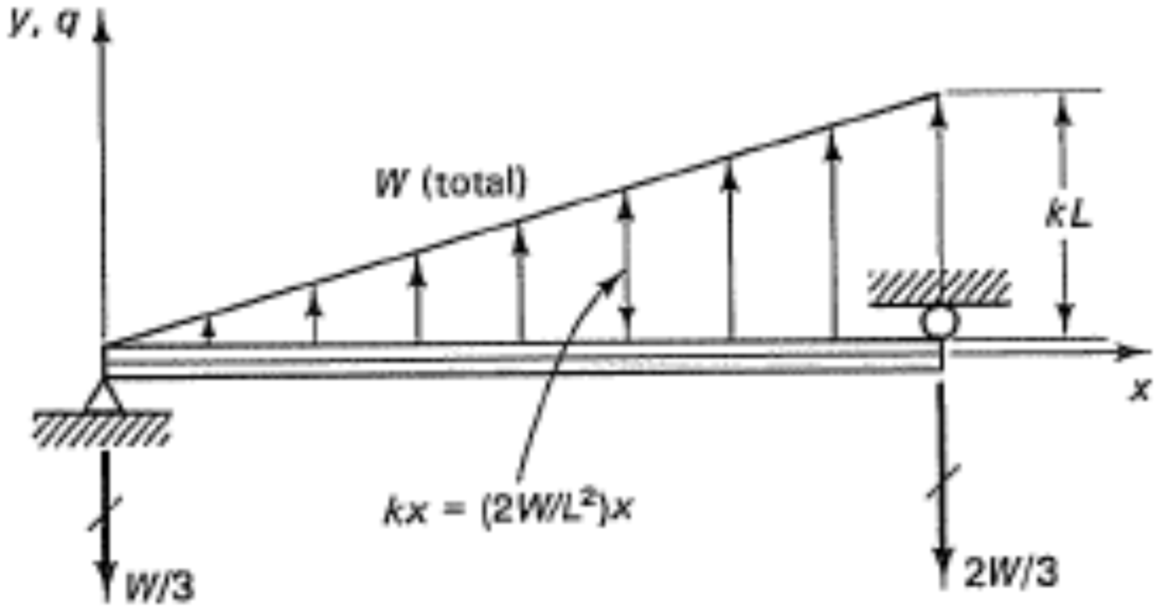


$$P(x_1) = +P_1,\ V(x_1) = 0 \quad \text{and} \quad M(x_1) = +2P_1a$$

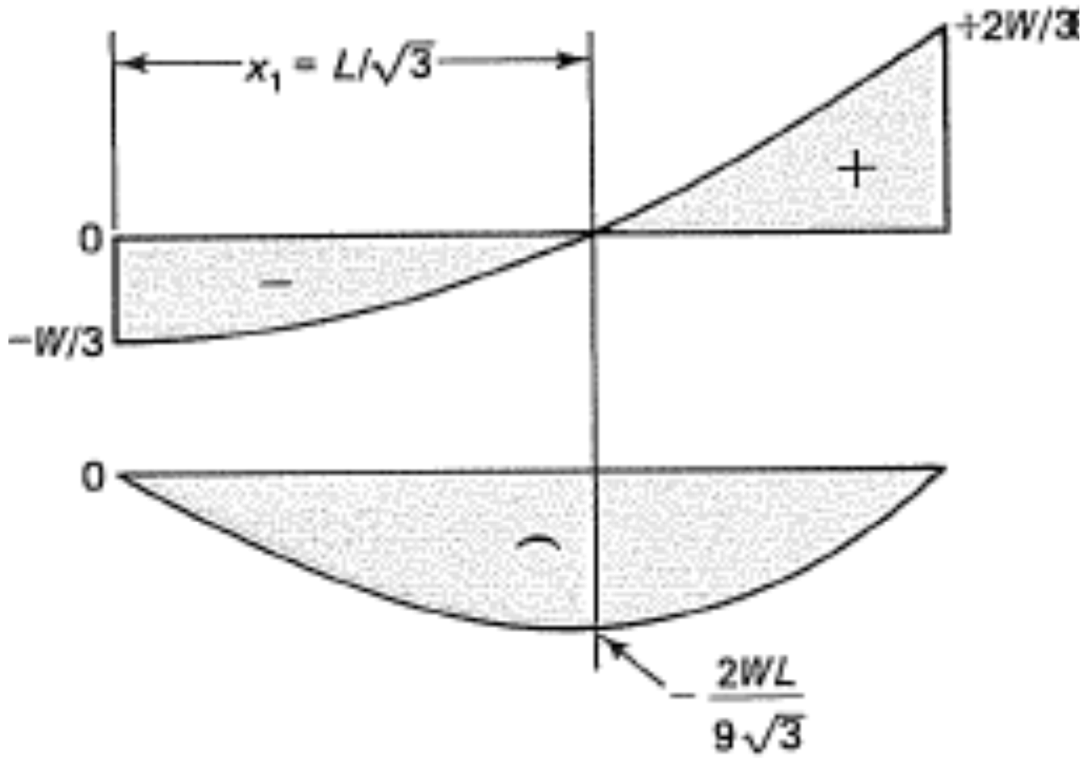
$$P(x_2) = 0,\ V(x_2) = -P_1 \quad \text{and} \quad M(x_2) = +2P_1a - P_1x_2$$

$$P(x_3) = -P_1/\sqrt{2}, V(x_3) = -P_1/\sqrt{2} \quad \text{and} \quad M(x_3) = +P_1a - P_1x_3/\sqrt{2}$$

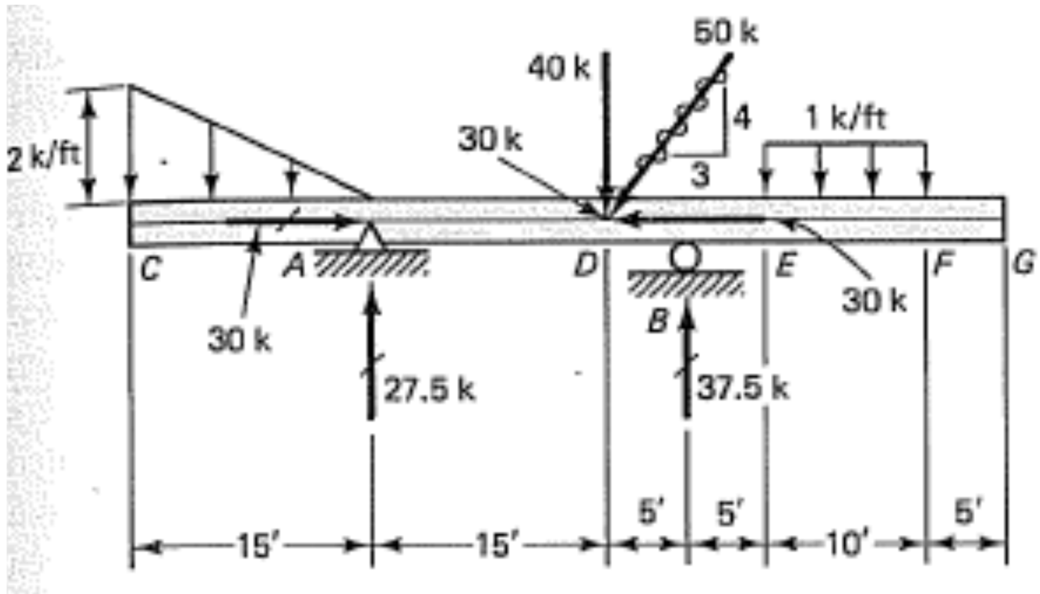
Example 5-14



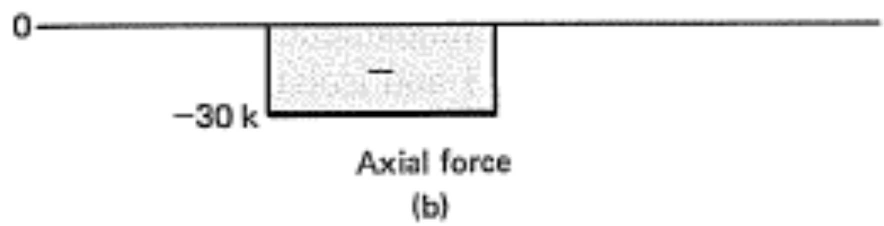
Construct shear and moment diagrams.

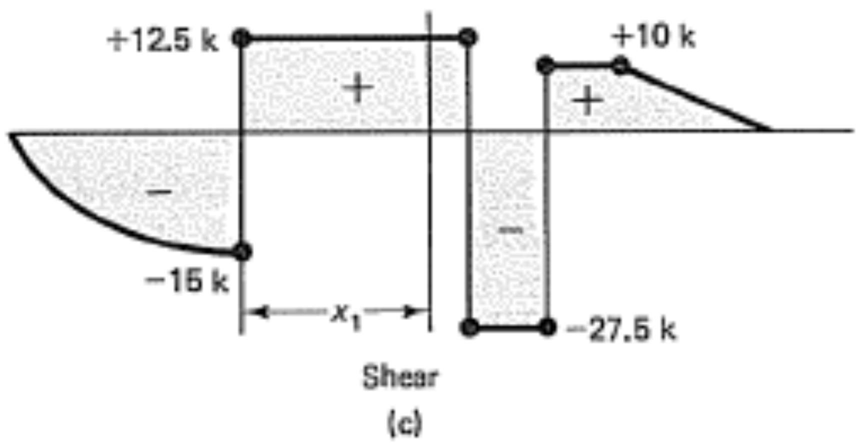


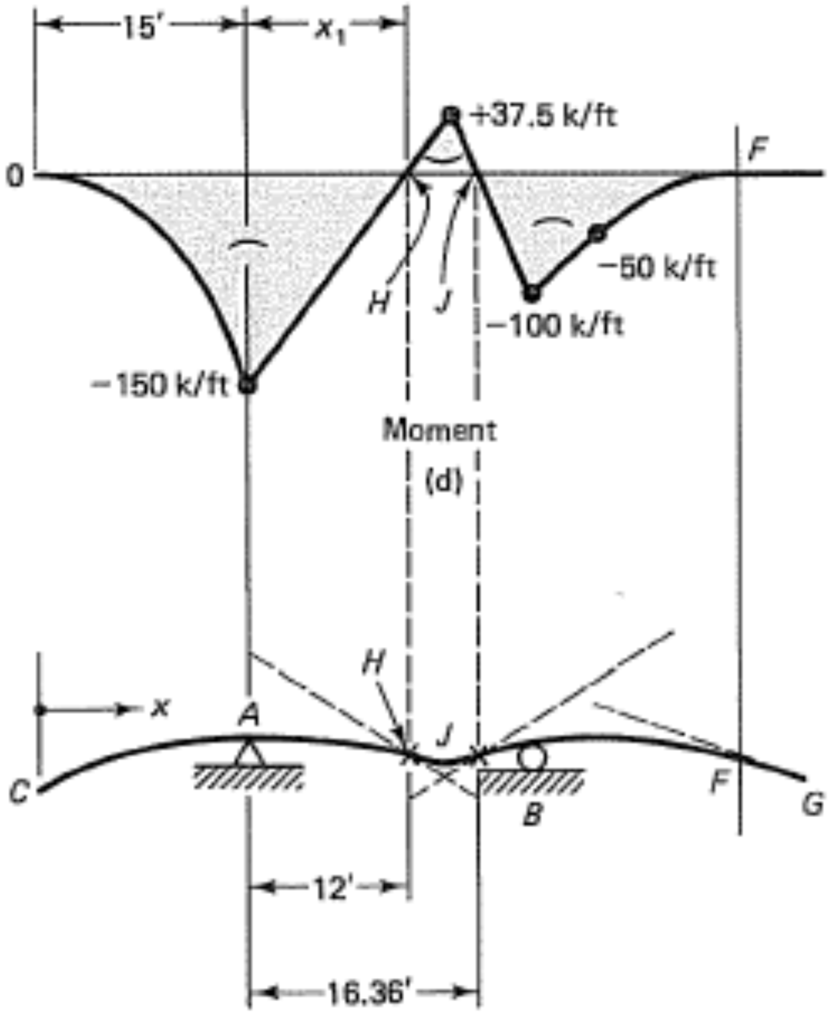
Example 5-15



Construct shear and moment diagrams.







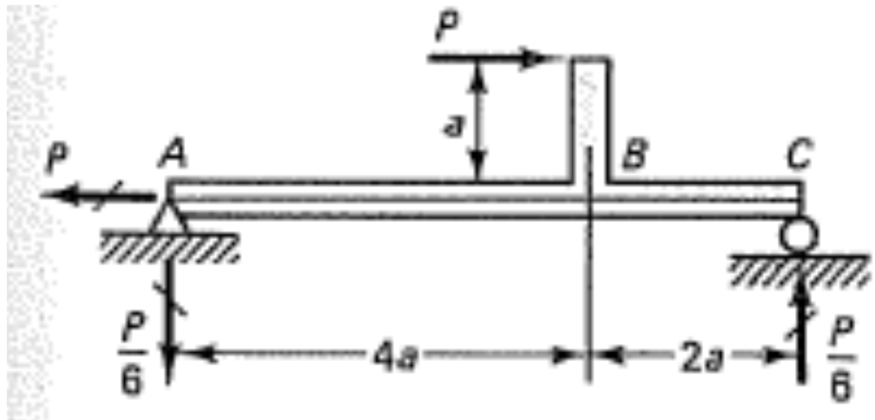
Moment

(d)

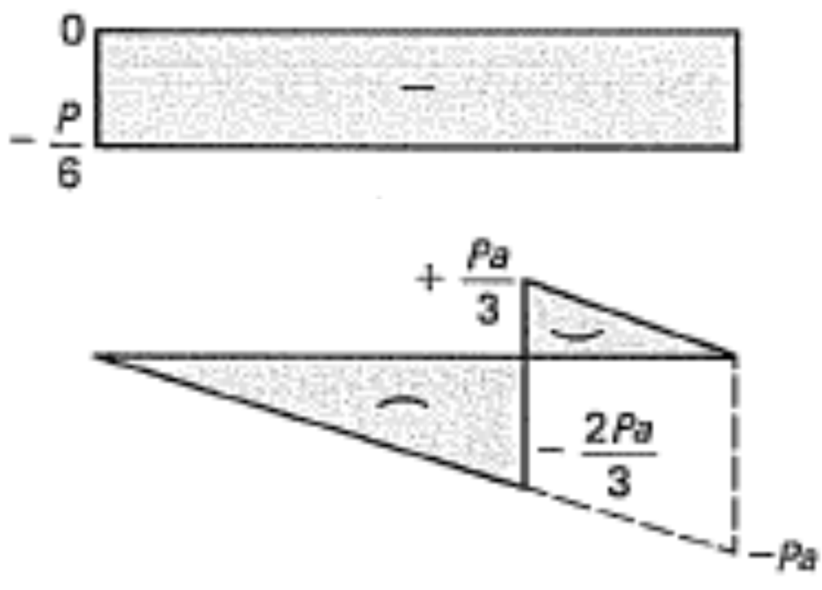
Deflected shape

(e)

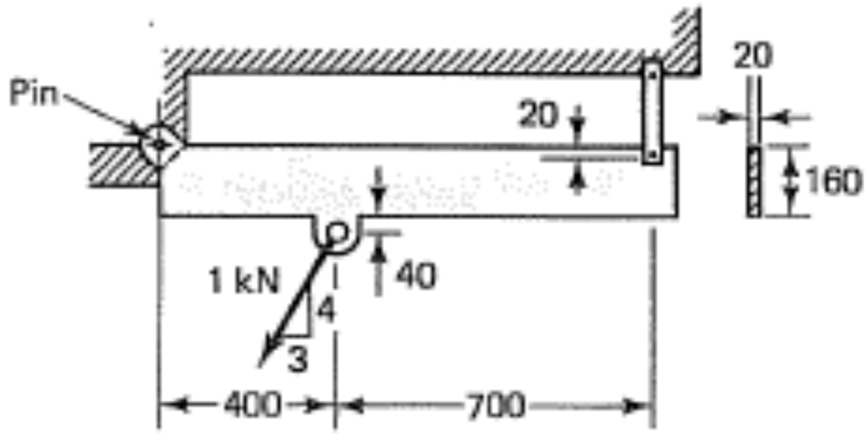
Example 5-16



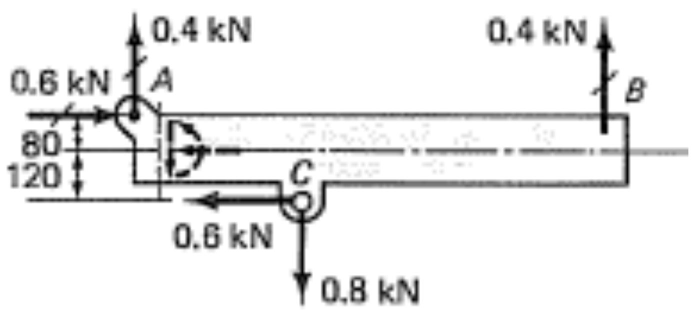
Construct the bending-moment diagram for the horizontal beam.

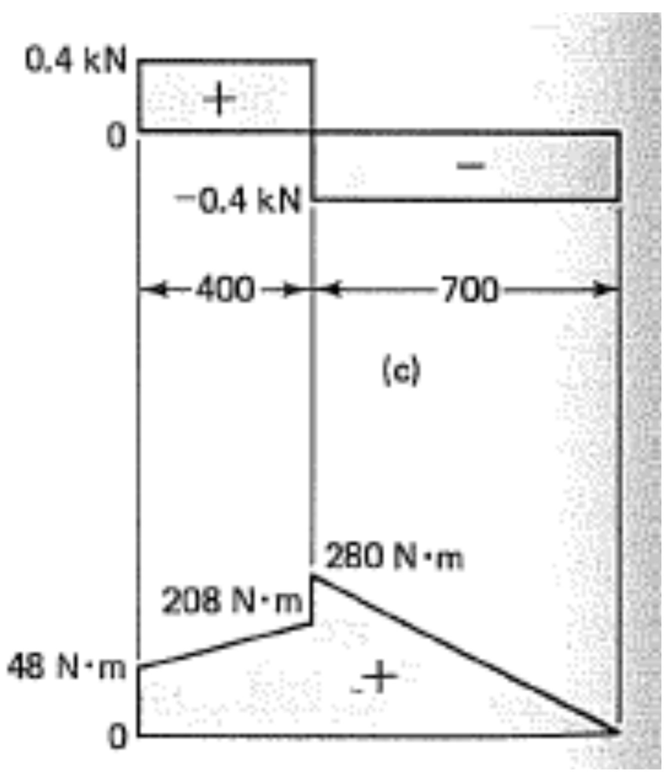


Example 5-17



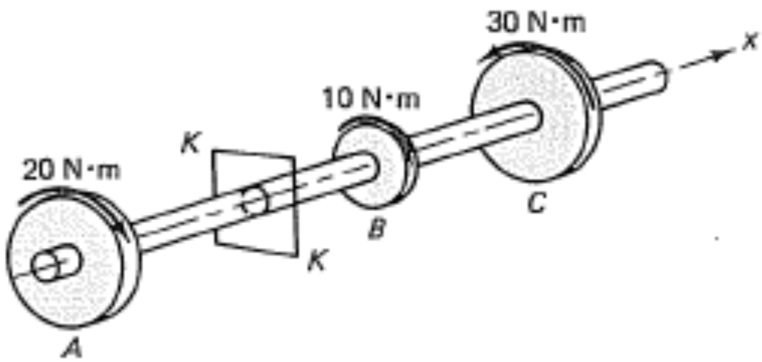
Construct shear arid moment diagrams for the member.





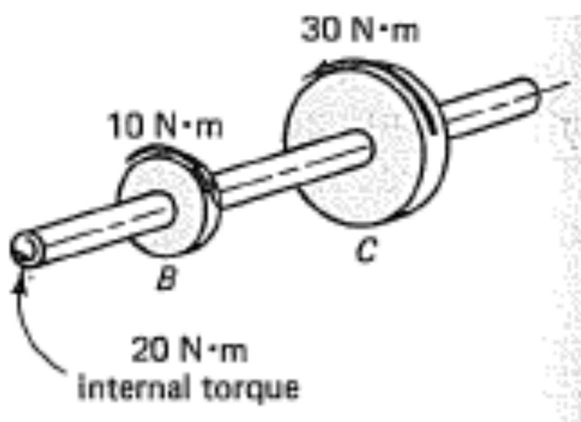
Lecture 5: Torsion

Example 4-1



Find the internal torque at section K-K.

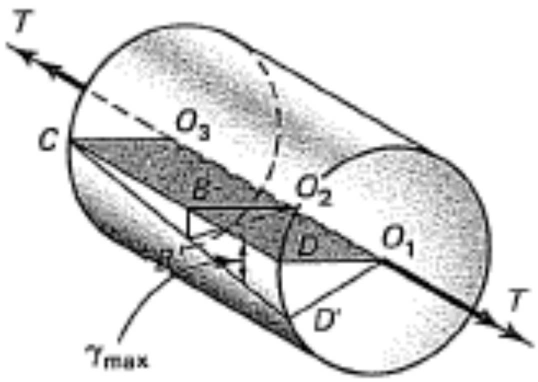
externally applied torque = internal torque



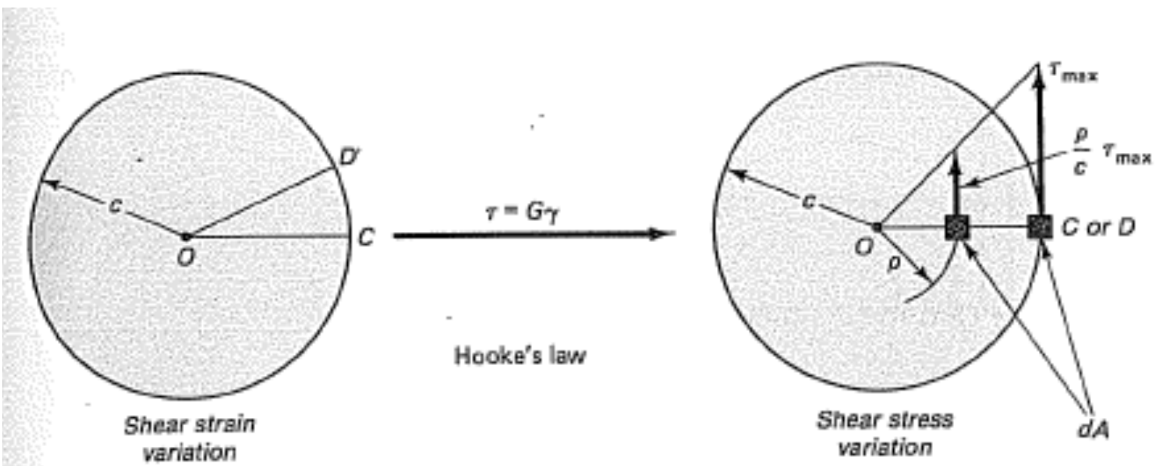
1. Torsion of Circular Elastic Bars

Basic Assumptions for Circular Members:

1. A plane section of material perpendicular to the axis of a circular member remains plane after the torques are applied, i.e., no warpage or distortion of parallel planes normal to the axis of a member takes place.
2. In a circular member subjected to torque, shear strains vary linearly from the central axis reaching maximum at the periphery. It must be emphasized that these assumptions hold only for circular solid and tubular members.



3. If attention is confined to the linearly elastic material, Hooke's law applies, and, it follows that shear stress is proportional to shear strain.



$$\int_A \underbrace{\underbrace{\underbrace{\frac{\rho}{c}\tau_{max}}_{\text{stress}}\ \underbrace{dA}_{\text{area}}}_{\text{force}}\ \underbrace{\rho}_{\text{arm}}}_{\text{torque}} = T$$

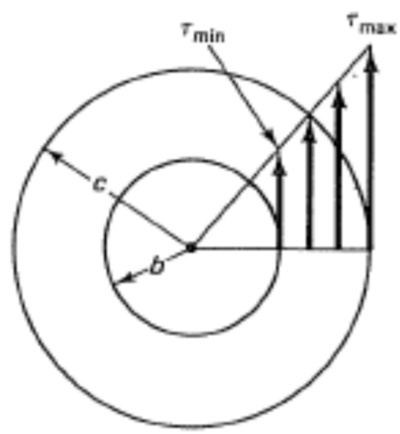
$$\frac{\tau_{max}}{c}\int_A \rho^2\, dA = T$$

$$J = \int_A \rho^2 \, dA = \int_0^c 2\pi\rho^3 \, d\rho = 2\pi \left| \frac{\rho^4}{4} \right|_0^c = \frac{\pi c^4}{2} = \frac{\pi d^4}{32}$$

$$\boxed{\tau_{max} = \frac{Tc}{J}}$$

$$\tau = \frac{\rho}{c}\tau_{max} = \frac{T\rho}{J}$$

For a circular tube,

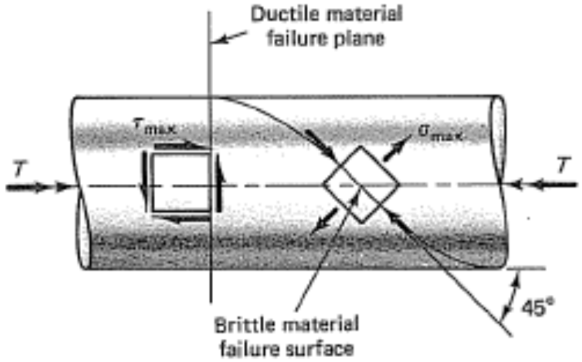


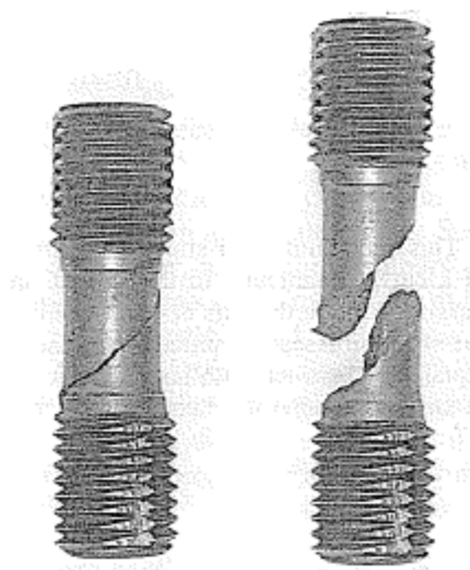
$$J = \int_A \rho^2 \, dA = \int_b^c 2\pi\rho^3 \, d\rho = \frac{\pi c^4}{2} - \frac{\pi b^4}{2}$$

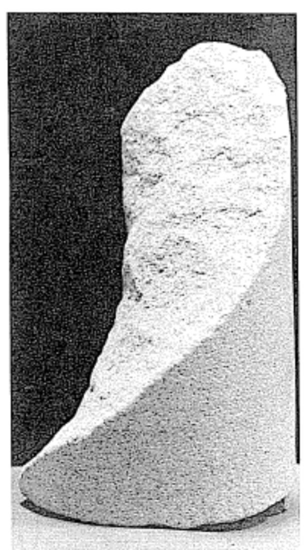
For very thin tubes,

$$J \approx 2\pi R_{av}^3 t$$

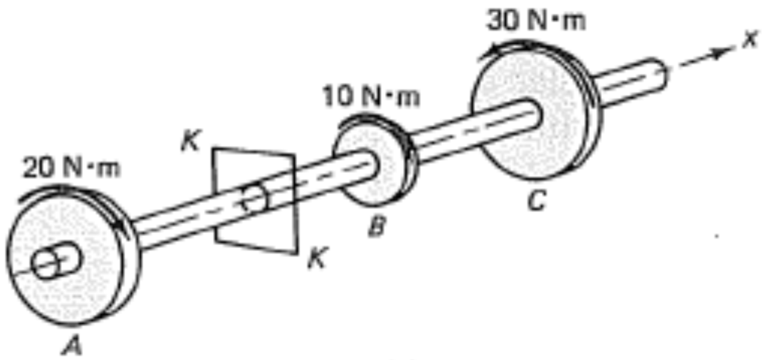
2. Fracture in Torsion







Example 4-2

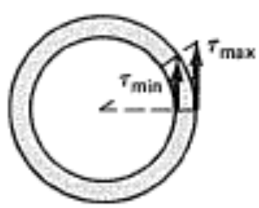


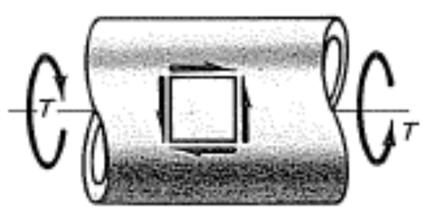
Find the maximum torsional shear stress in shaft AC. Assume the shaft from A to C is 10 mm in diameter.

$$J = \frac{\pi d^4}{32} = \frac{\pi \times 10^4}{32} = 982 \text{ mm}^4$$

$$\tau_{max} = \frac{Tc}{J} = \frac{30 \times 10^3 \times 5}{982} = 153 \text{ MPa}$$

Example 4-3





Consider a long tube of 20 mm outside diameter and of 16 mm inside diameter, twisted about its longitudinal axis with a torque T of 40 N·m. Determine shear stresses at the outside and the inside of the tube

$$J = \frac{\pi(c^4 - b^4)}{2} = \frac{\pi(d_o^4 - d_i^4)}{32} = \frac{\pi(20^4 - 16^4)}{32} = 9270 \text{ mm}^4$$

$$\tau_{max} = \frac{Tc}{J} = \frac{40 \times 10^3 \times 10}{9270} = 43.1 \text{ MPa}$$

$$\tau_{min} = \frac{T\rho}{J} = \frac{40 \times 10^3 \times 8}{9270} = 34.5 \text{ MPa}$$

3. Design of Circular Members in Torsion

$$\text{hp} \times 745.7 = 2\pi f T \ [\text{N·m/s}]$$

$$T = \frac{119 \times \text{hp}}{f} \quad [\text{N·m}]$$

$$T = \frac{159 \times \text{kW}}{f} \ [\text{N·m}]$$

$$\boxed{T = \frac{63{,}000 \times \text{hp}}{N} \ [\text{in-lb}]}$$

Example 4-4

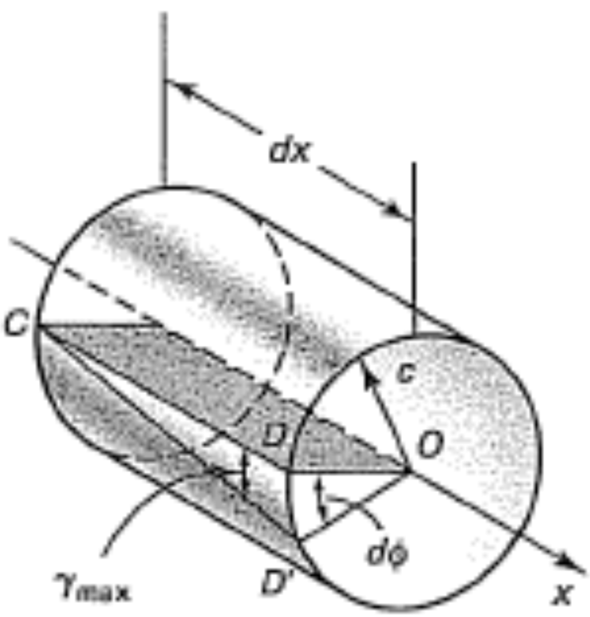
Select a solid shaft for a 10-hp motor operating at 30 Hz. The maximum shear stress is limited to 55 MPa.

$$T = \frac{119 \times \text{hp}}{f} = \frac{119 \times 10}{30} = 39.7\ \text{N·m}$$

$$\frac{J}{c} = \frac{T}{\tau_{max}} = \frac{39.7 \times 10^3}{55} = 722\ \text{mm}^3$$

$$\frac{J}{c} = \frac{\pi c^3}{2} \qquad \text{or} \qquad c^3 = \frac{2}{\pi}\frac{J}{c} = \frac{2 \times 722}{\pi} = 460\ \text{mm}^3$$

4. Angle of Twist of Circular Members



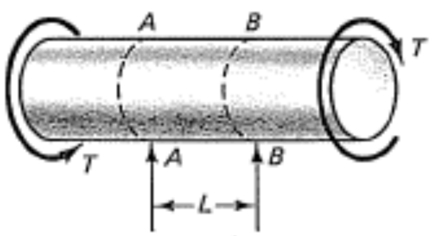
$$\text{arc } DD' = \gamma_{max}\, dx \qquad \text{or} \qquad \text{arc } DD' = d\phi\, c$$

$$\gamma_{max}\, dx = d\phi\, c$$

$$\frac{d\phi}{dx} = \frac{T}{JG} \qquad \text{or} \qquad d\phi = \frac{T\,dx}{JG}$$

$$\boxed{\phi = \phi_B - \phi_A = \int_A^B d\phi = \int_A^B \frac{T_x\,dx}{J_x G}}$$

Example 4-6



Find the relative rotation of section B-B with respect to section A-A.

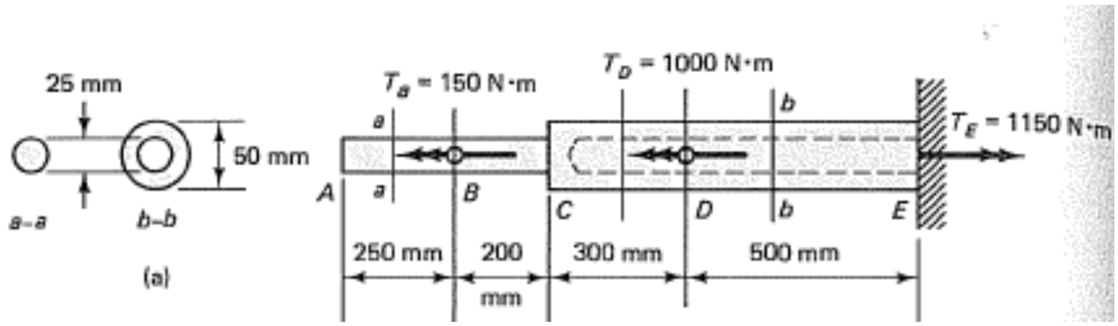
$$\phi = \int_A^B \frac{T_x\,dx}{J_x G} = \int_0^L \frac{T\,dx}{JG} = \frac{T}{JG}\int_0^L dx = \frac{TL}{JG}$$

$$\boxed{\phi = \frac{TL}{JG}}$$

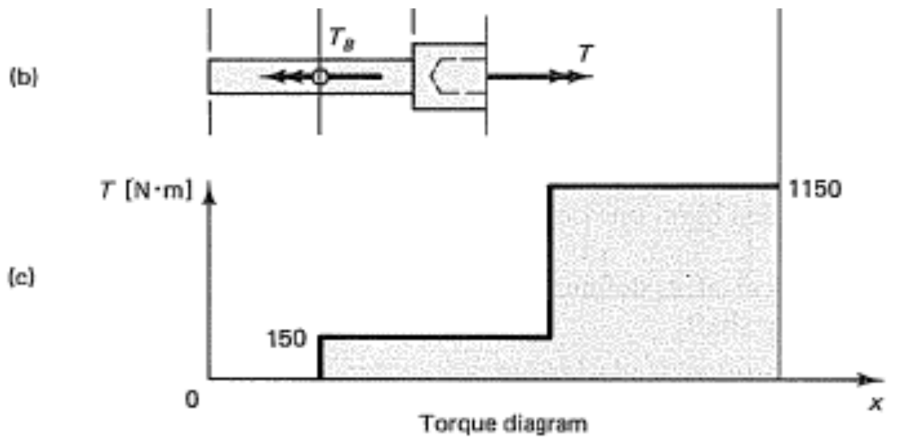
$$\boxed{k_t = \frac{T}{\phi} = \frac{JG}{L}}$$

$$f_t = \frac{1}{k_t} = \frac{L}{JG}$$

Example 4-7



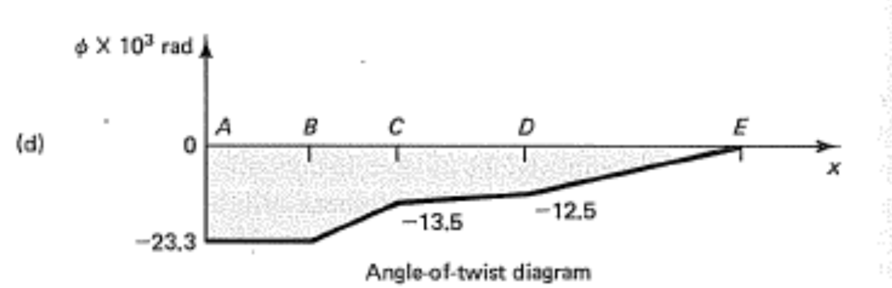
Determine the angle-of-twist of the end A. Assume the shear modulus G to be 80 GPa, a typical value for steels.



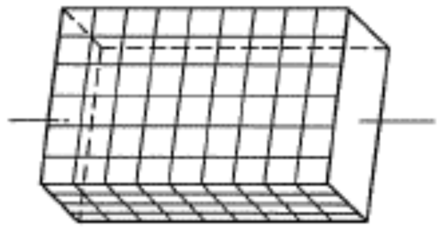
$$\phi = \sum_i \frac{T_i L_i}{J_i G_i} = \frac{T_{AB} L_{AB}}{J_{AB} G} + \frac{T_{BC} L_{BC}}{J_{BC} G} + \frac{T_{CD} L_{CD}}{J_{CD} G} + \frac{T_{DE} L_{DE}}{J_{DE} G}$$

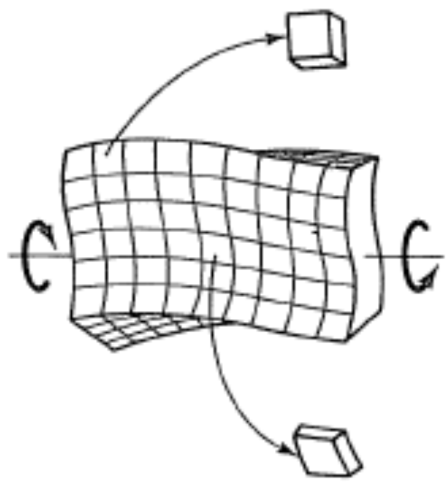
$$= 0 + \frac{150 \times 10^3 \times 200}{38.3 \times 10^3 \times 80 \times 10^3} + \frac{150 \times 10^3 \times 300}{575 + 10^3 \times 80 \times 10^3} + \frac{1150 \times 10^3 \times 500}{575 \times 10^3 \times 80 \times 10^3}$$

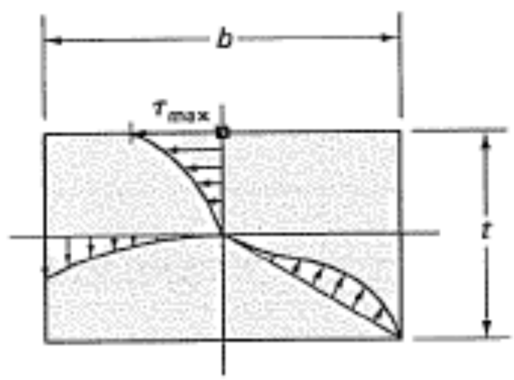
$$= 0 + 9.8 \times 10^{-3} + 1.0 \times 10^{-3} + 12.5 \times 10^{-3} = 23.3 \times 10^{-3} \text{ rad}$$

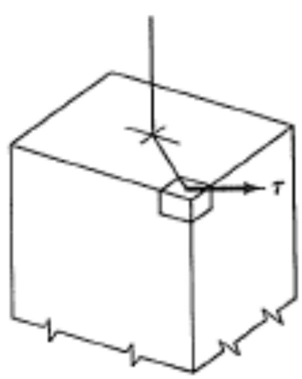


5. Torsion of Solid Non-Circular Members







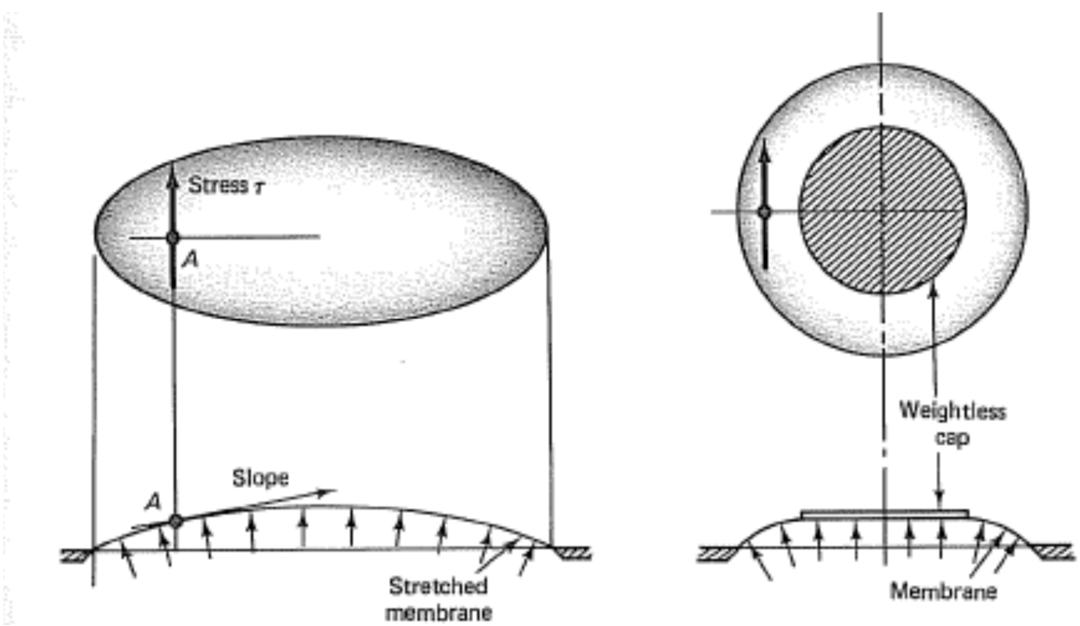


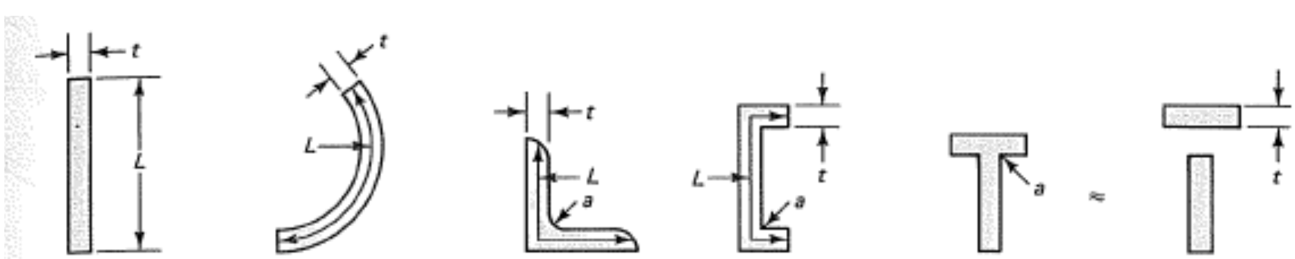
$$\tau_{max} = \frac{T}{\alpha b t^2} \quad \text{and} \quad \phi = \frac{TL}{\beta b t^3 G}$$

Table of Coefficients for Rectangular Bars[17]

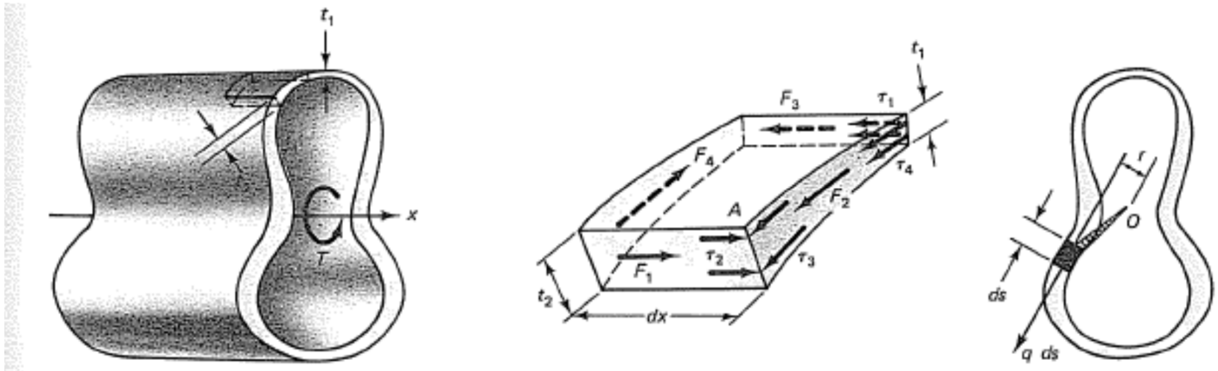
b/t	1.00	1.50	2.00	3.00	6.00	10.0	∞
α	0.208	0.231	0.246	0.267	0.299	0.312	0.333
β	0.141	0.196	0.229	0.263	0.299	0.312	0.333

It happens that the solution of the partial differential equation that must be solved in the elastic torsion problem is mathematically identical to that for a thin membrane, such as a soap film, lightly stretched over a hole.





6. Torsion of Thin-Walled Tubular Members



$$T = \oint rq\, ds$$

$$T = q \oint r\, ds$$

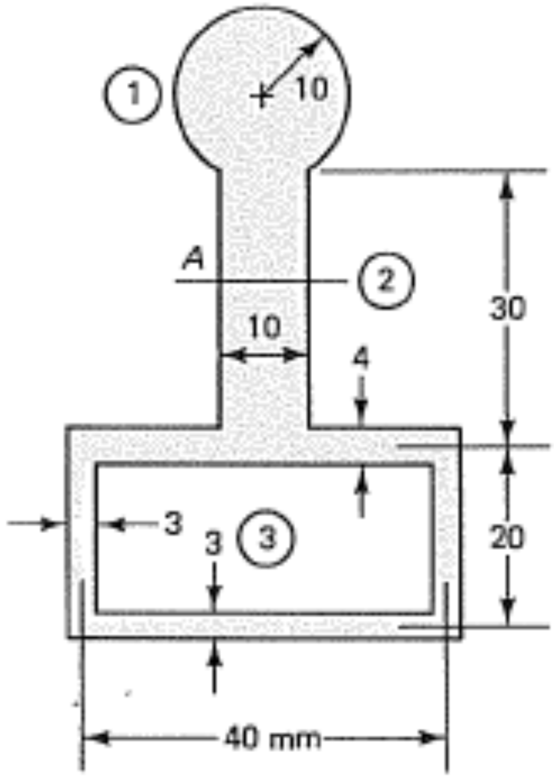
$$T = 2\textcircled{A}q \qquad \text{or} \qquad q = \frac{T}{2\textcircled{A}}$$

$$\boxed{\tau = \frac{q}{t}}$$

$$\theta = \frac{d\phi}{dx} = \frac{T}{4\textcircled{A}^2 G} \oint \frac{ds}{t}$$

$$\boxed{k_t = \frac{T}{\phi} = \frac{4\textcircled{A}^2}{\oint ds/t} \frac{G}{L}}$$

Example 4-17



If torque T = 300 N·m is applied, (a) determine the maximum shear stresses that would develop in the three different parts of the member, and (b) find the torsional stiffness of the member.

$$(k_t)_1 = J\frac{G}{L} = \frac{\pi \times 10^4}{2}\frac{G}{L} = 1.57 \times 10^4 \frac{G}{L}$$

$$(k_t)_2 = \beta b t^3 \frac{G}{L} = 0.263 \times 30 \times 10^3 \frac{G}{L} = 0.789 \times 10^4 \frac{G}{L}$$

$$(k_t)_3 = \frac{4\textcircled{A}^2}{\oint ds/t}\frac{G}{L} = \frac{4 \times (40 \times 20)^2}{(40 + 2 \times 20)/3 + 40/4}\frac{G}{L} = 6.98 \times 10^4 \frac{G}{L}$$

The applied torque is distributed among the three parts in a ratio of *Individual Component Stiffness/Total Stiffness*. On this basis, the torques are 300 x (1.57 x 10^4 G/L)/(9.34 x 10^4G/L) = 50.4 N·m for the knob, 25.3 N·m for the bar, and 224 N·m for the box.

$$\tau_{1\text{-max}} = \frac{Tc}{J} = \frac{50.4 \times 10^3 \times 10}{\pi \times 10^4/2} = 32.1 \quad \text{MPa}$$

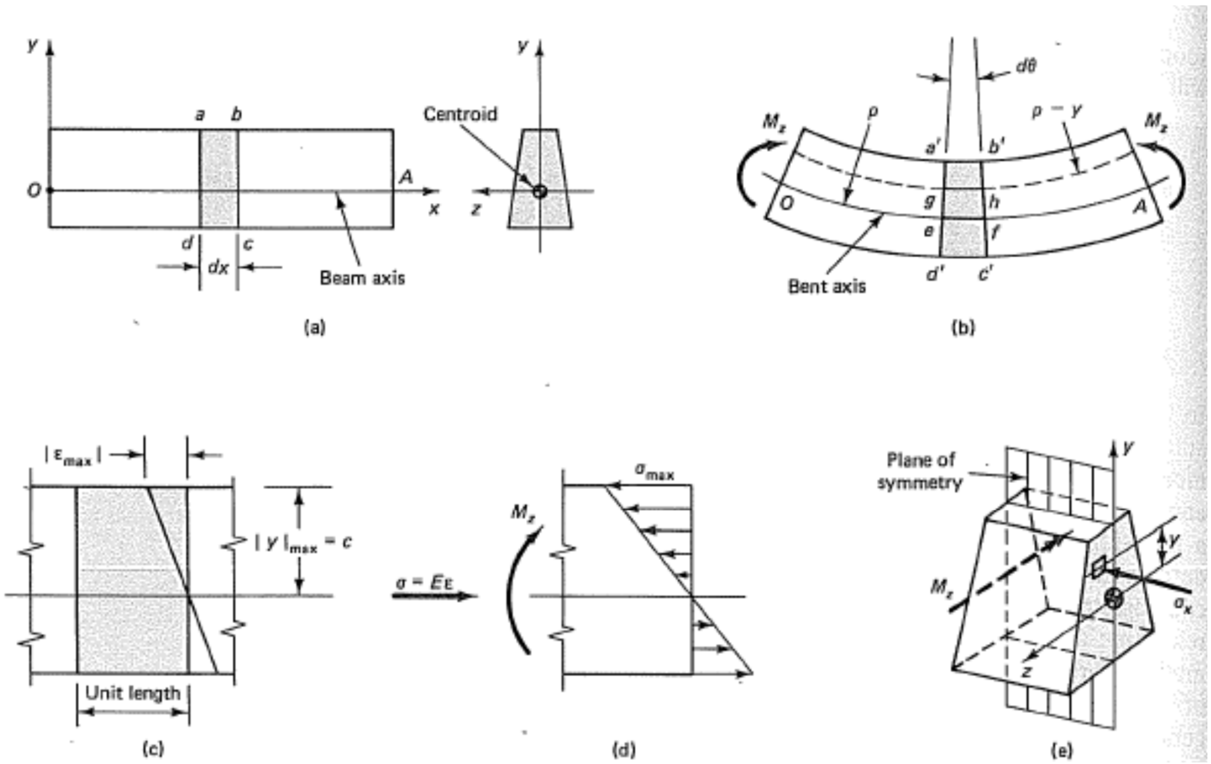
$$\tau_{2\text{-max}} = \frac{T}{\alpha b t^2} = \frac{25.3 \times 10^3}{0.267 \times 30 \times 10^2} = 31.6 \quad \text{MPa}$$

$$\tau_{3\text{-max}} = \frac{T}{2\textcircled{A}t} = \frac{224 \times 10^3}{2 \times 40 \times 20 \times 3} = 46.7 \quad \text{MPa}$$

Lecture 6: Flexural Stresses in Beams

1. Bending of Beams with Symmetric Cross-Sections

Basic Kinematic Assumption:



Plane sections through a beam taken normal to its axis remain plane after the beam is subjected to bending.

$$\frac{d\theta}{ds} = \frac{1}{\rho} = \kappa$$

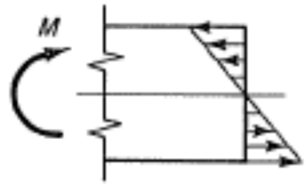
$$d\hat{u} = (\rho - y)\, d\theta - \rho\, d\theta = -y\, d\theta$$

$$\varepsilon_x = -\kappa\, y$$

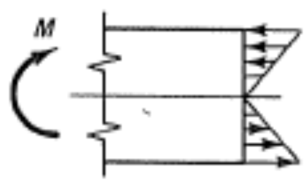
$$\boxed{\sigma_x = E\,\varepsilon_x = -E\,\kappa\,y}$$

$$\sum F_x = 0 \qquad \int_A \sigma_x\, dA = 0$$

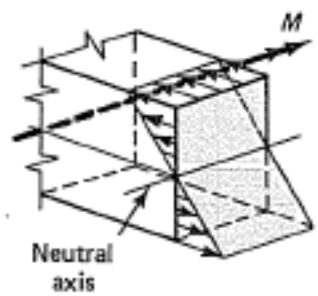
$$\int_A -E\,\kappa y\, dA = -E\,\kappa \int_A y\, dA = 0$$



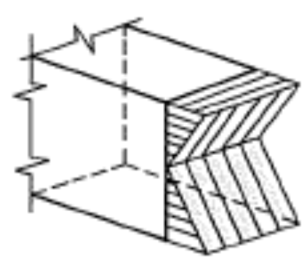
(a)



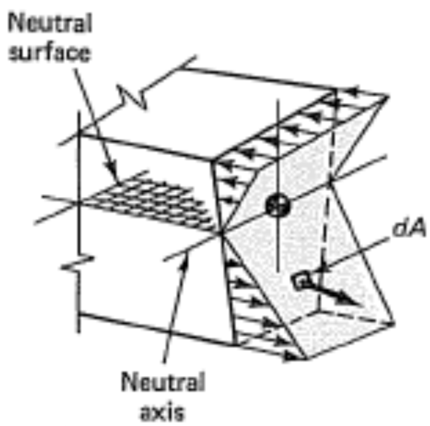
(b)



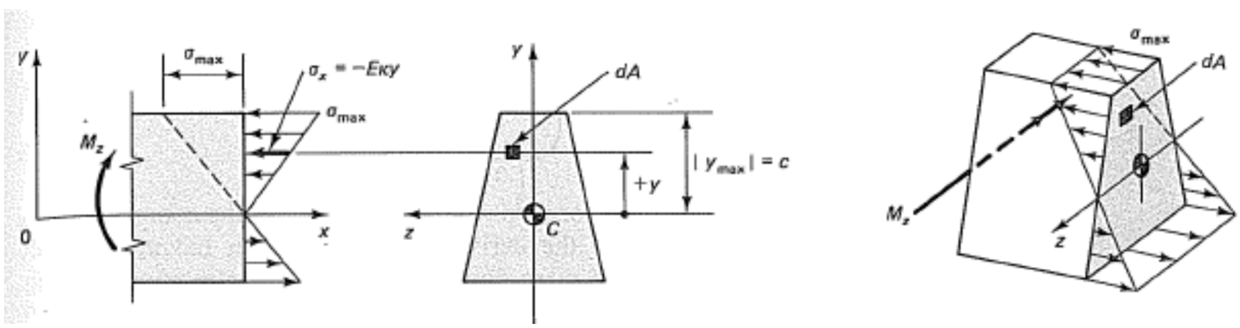
(c)



(d)



$$\sum M_O = 0 \circlearrowright + \qquad M_z - \int_A \underbrace{\underbrace{E\kappa y}_{\text{stress}} \quad \underbrace{dA}_{\text{area}}}_{\text{force}} \qquad \underbrace{y}_{\text{arm}} = 0$$

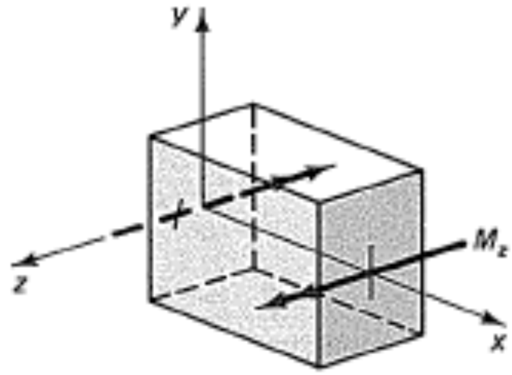


$$M_z = E\kappa \int_A y^2 \, dA$$

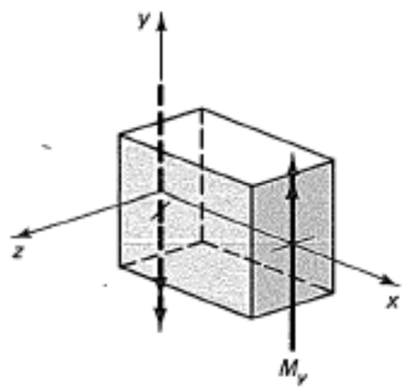
$$\boxed{I_z = \int_A y^2 \, dA}$$

$$\boxed{\kappa = \frac{M_z}{E\, I_z}}$$

$$\boxed{\sigma_x = -\frac{M_z}{I_z} y}$$



$$\sigma_x = +\frac{M_y}{I_y} z$$

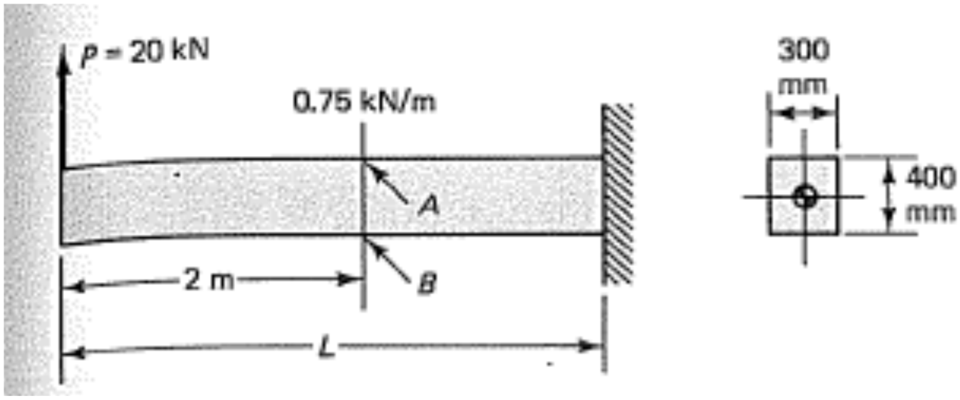


$$\sigma_{\max} = \frac{Mc}{I}$$

$$\sigma_{\max} = \frac{Mc}{I} = \frac{M}{I/c} = \frac{M}{S}$$

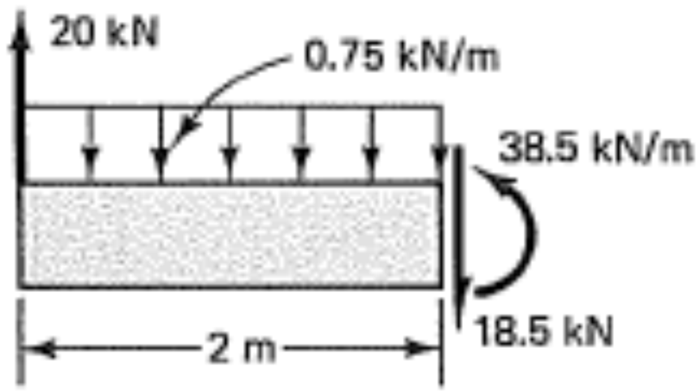
$$\text{maximum bending stress} = \frac{\text{bending moment}}{\text{elastic section modulus}}$$

Example 6-4

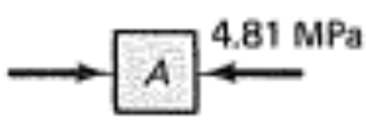


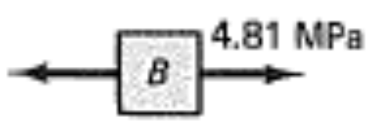
Determine the maximum bending stresses at a section 2 m from the free end.

$$I_z = \frac{bh^3}{12} = \frac{300 \times 400^3}{12} = 16 \times 10^8 \text{ mm}$$

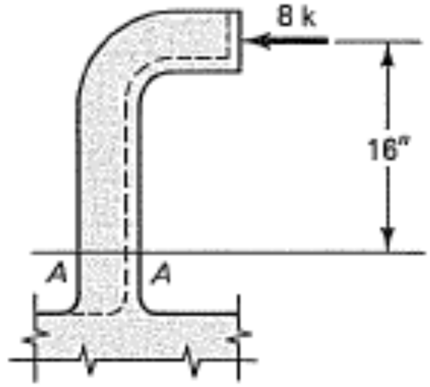


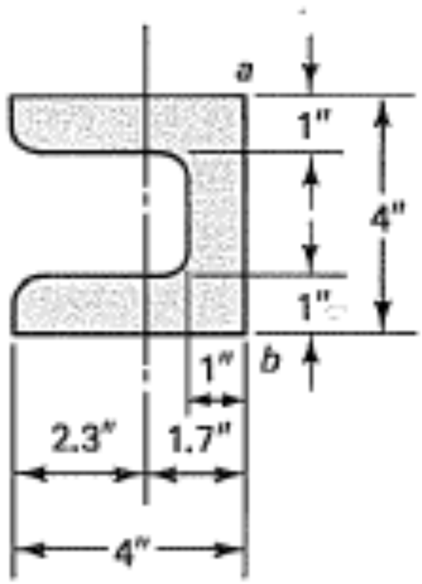
$$\sigma_{\max} = \frac{Mc}{I} = \frac{38.5 \times 10^6 \times 200}{16 \times 10^8} = \pm 4.81 \text{ MPa}$$





Example 6-5



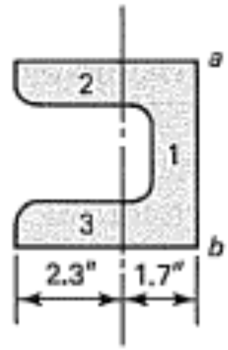


Section A–A

Find the maximum tensile and compressive stresses acting normal to section A-A.

Area Number	A [in^2]	y [in] (from ab)	Ay
1	4.0	0.5	2.0
2	3.0	2.5	7.5
3	3.0	2.5	7.5
	$\sum A = 10.0$ in^2		$\sum Ay = 17.0$ in^3

$$\bar{y} = \frac{\sum Ay}{\sum A} = \frac{17.0}{10.0} = 1.70 \text{ in} \qquad \text{from line } ab$$

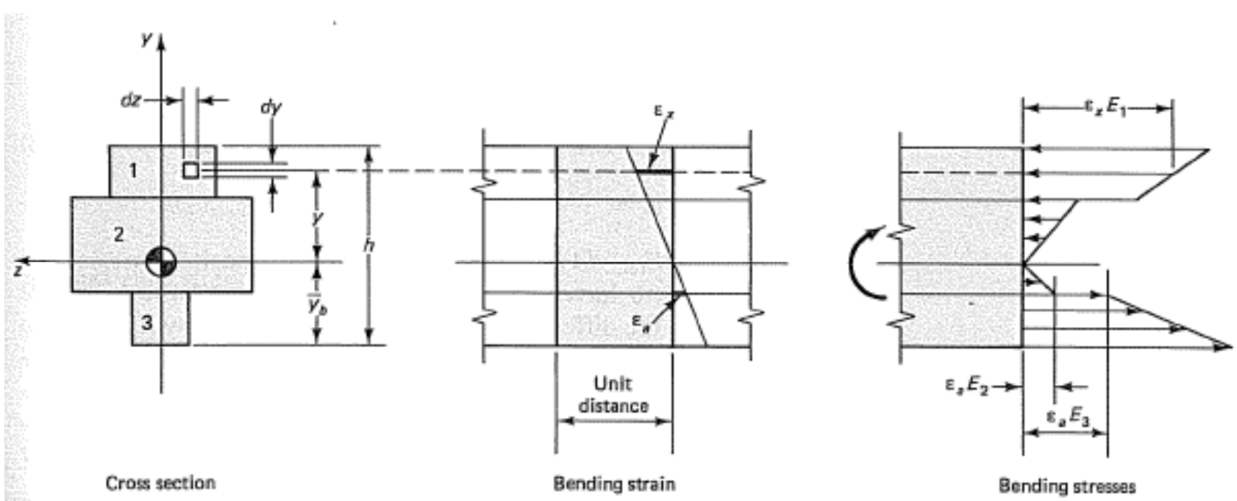


$$I = \sum (I_o + Ad^2) = \frac{4 \times 1^3}{12} + 4 \times 1.2^2 + \frac{2 \times 1 \times 3^3}{12} + 2 \times 3 \times 0.8^2 = 14.43 \text{ in}^4$$

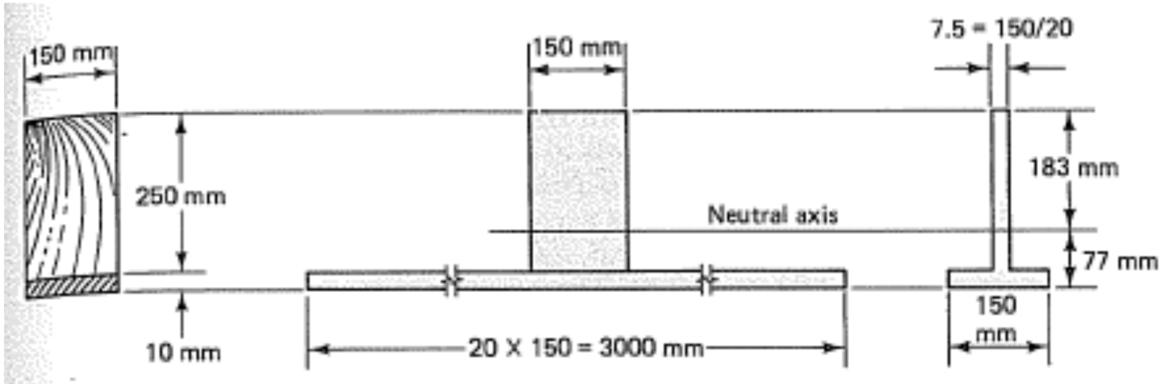
$$\sigma_{max} = \frac{Mc}{I} = \frac{8 \times 16 \times 2.3}{14.43} = 20.4 \text{ ksi} \qquad \text{(compression)}$$

$$\sigma_{max} = \frac{Mc}{I} = \frac{8 \times 16 \times 1.7}{14.43} = 15.1 \text{ ksi} \qquad \text{(tension)}$$

2. Beams of Composite Cross-Section



Example 6-8



The upper 150 by 250 mm part is wood, Ew = 10 GPa; the lower 10 by 150 mm strap is steel, Es 200 GPa. If this beam is subjected to a bending moment of 30 kN'm around a horizontal axis, what are the maximum stresses in the steel and wood?

$$\bar{y} = \frac{150 \times 250 \times 125 + 10 \times 3000 \times 255}{150 \times 250 + 10 \times 3000} = 183 \text{ mm} \qquad \text{(from the top)}$$

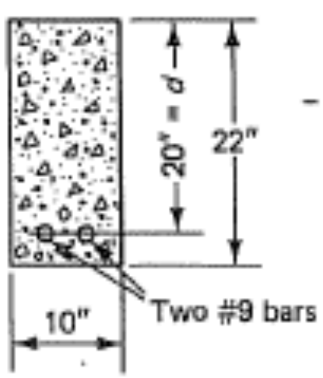
$$I_z = \frac{150 \times 250^3}{12} + 150 \times 250 \times 58^2 + \frac{3000 \times 10^3}{12} + 10 \times 3000 \times 72^2$$

$$= 478 \times 10^6 \text{ mm}^4$$

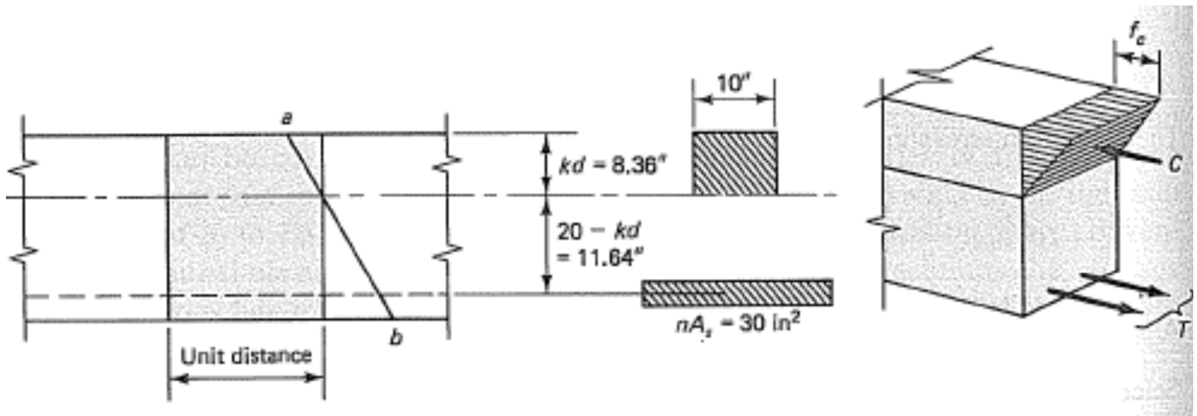
$$(\sigma_w)_{max} = \frac{Mc}{I} = \frac{0.03 \times 10^9 \times 183}{478 \times 10^6} = 11.5 \text{ MPa}$$

$$(\sigma_s)_{max} = n\sigma_w = 20 \times \frac{0.03 \times 10^9 \times 77}{478 \times 10^6} = 96.7 \text{ MPa}$$

Example 6-9



Determine the maximum stress in the concrete and the steel for a reinforced concrete beam with the section shown in the figure, if it is subjected to a bending moment of 50,000 ft-Ib. The reinforcement consists of two #9 steel bars. (These bars are 1.5 in in diameter and have a cross-sectional area of 1 in^2). Assume the ratio of *E* for steel to that of concrete to be 15, i.e., $n = 15$.



$$\underbrace{10(kd)}_{\text{concrete area}}\ \underbrace{(kd/2)}_{\text{arm}} = \underbrace{30}_{\text{transformed steel area}}\ \underbrace{(20 - kd)}_{\text{arm}}$$

$$5(kd)^2 = 600 - 30(kd)$$

$$(kd)^2 + 6(kd) - 120 = 0$$

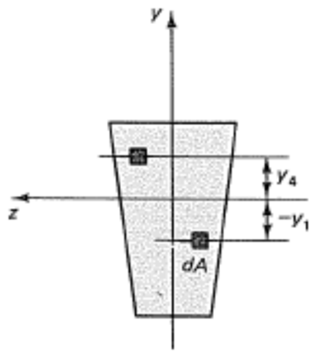
$$kd = 8.36 \text{ in} \qquad \text{and} \qquad 20 - kd = 11.64 \text{ in}$$

$$I = \frac{10(8.36)^3}{12} + 10(8.36)\left(\frac{8.36}{2}\right)^2 + 0 + 30(11.64)^2 = 6020 \text{ in}^4$$

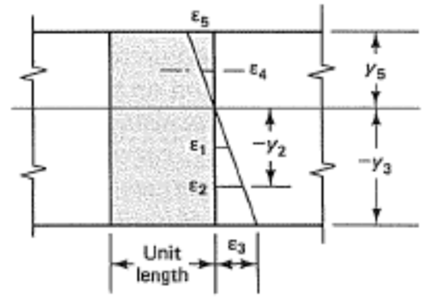
$$(\sigma_c)_{max} = \frac{Mc}{I} = \frac{50{,}000 \times 12 \times 8.36}{6020} = 833 \text{ psi}$$

$$\sigma_s = n\frac{Mc}{I} = \frac{15 \times 50{,}000 \times 12 \times 11.64}{6020} = 17{,}400 \text{ psi}$$

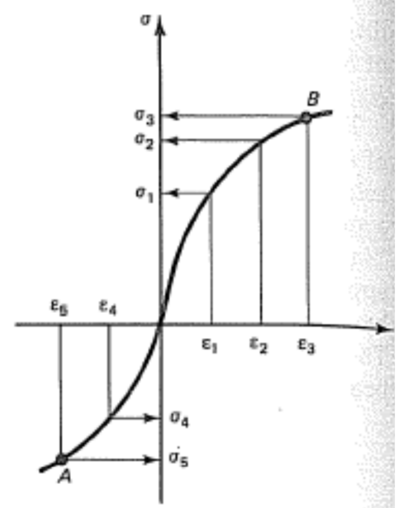
3. Inelastic Bending of Beams



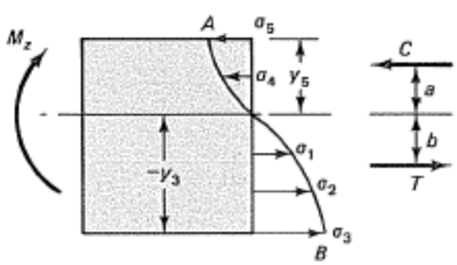
(a) Beam section



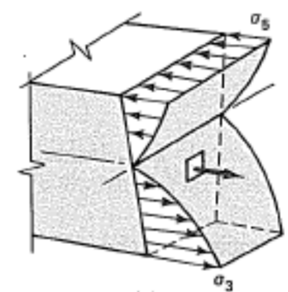
(b) Bending strain



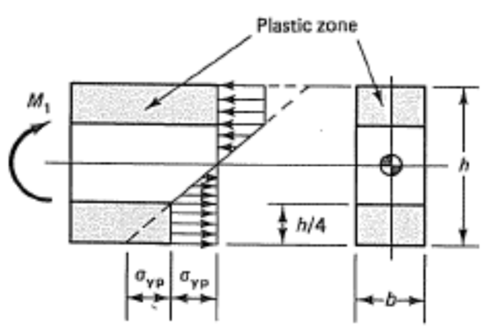
(c) Stress-strain diagram

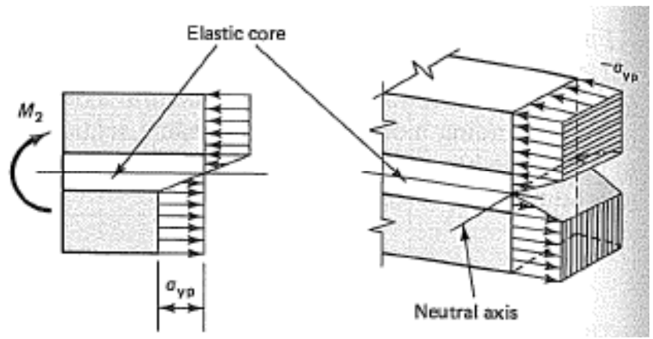


(d) Bending stresses



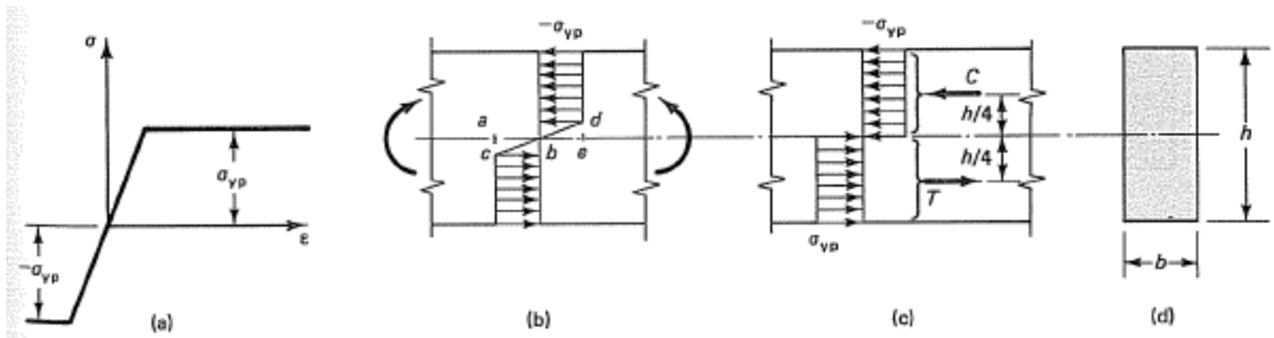
(e)





Example 6-11

Determine the ultimate plastic capacity in flexure of a mild steel beam of rectangular cross section. Consider the material to be ideally elastic-plastic.



$$C = T = \sigma_{yp}(bh/2) \qquad \text{i.e., stress} \times \text{area}$$

$$M_p = M_{\text{ult}} = C\left(\frac{h}{4} + \frac{h}{4}\right) = \sigma_{yp}\frac{bh^2}{4}$$

$$M_p \equiv M_{\text{ult}} = -2\int_0^{h/2} (-\sigma_{yp})yb\,dy = \sigma_{yp}bh^2/4$$

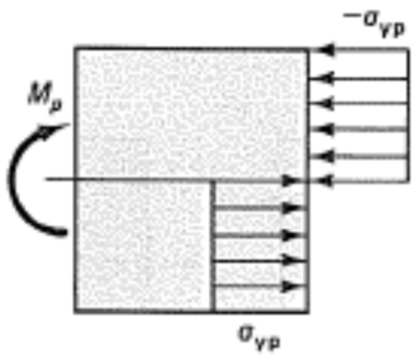
$$M_{yp} = \sigma_{yp}I/c = \sigma_{yp}(bh^2/6) \qquad \text{therefore, } M_p/M_{yp} = 1.50$$

The ratio M_p/M_{yp} depends only on the cross-sectional properties of a member and is called the *shape factor*.

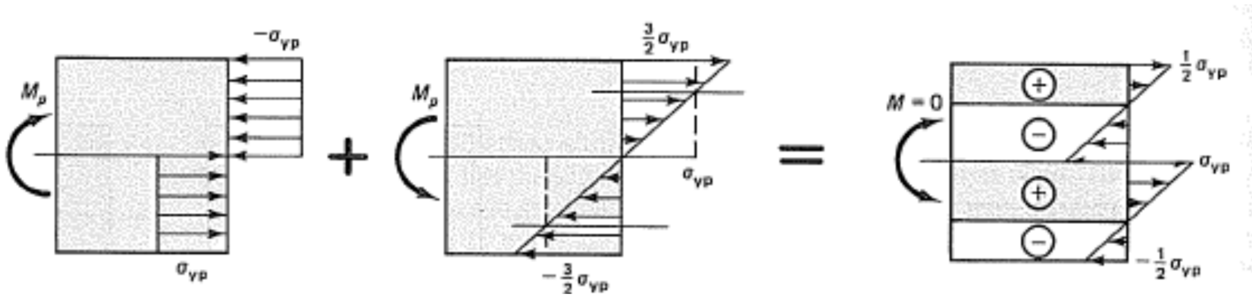
Ultimate capacities can be approximately determined using plastic moments. The procedures based on such concepts are referred to as the *plastic method of analysis or design*. For such work, *plastic section modulus Z* is defined as follows:

$$M_p = \sigma_{yp}Z$$

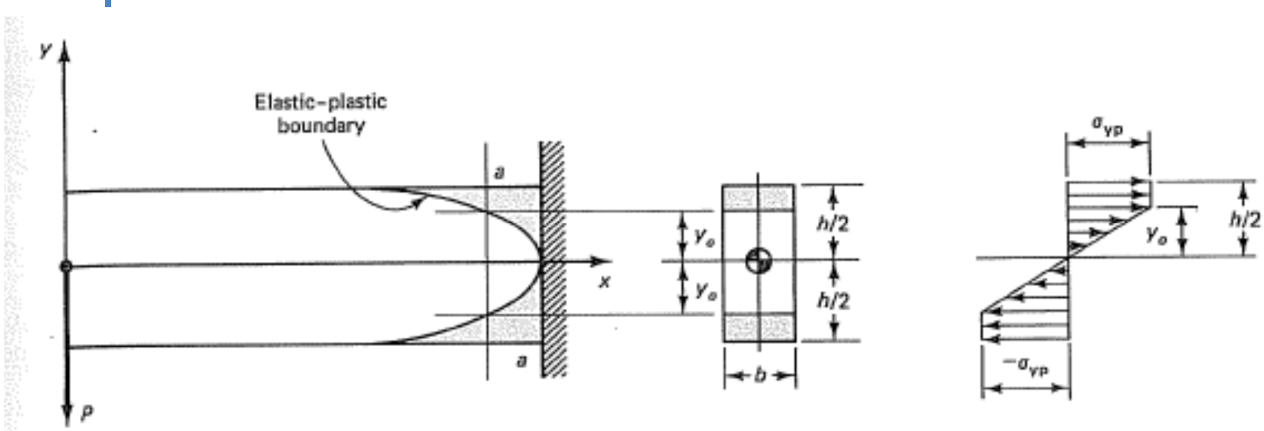
Example 6-12



Find the residual stresses in a rectangular beam upon removal of the ultimate plastic bending moment.



Example 6-13

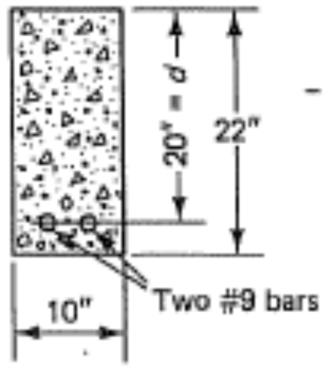


Determine the moment resisting capacity of an elastic-plastic rectangular beam.

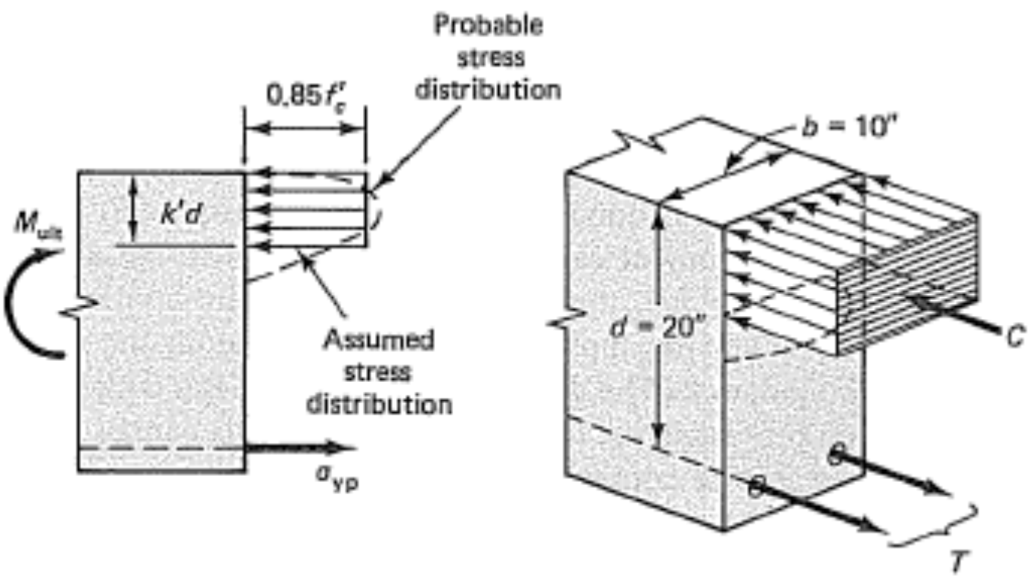
$$M = -2\int_0^{y_o}\left(-\frac{y}{y_o}\sigma_{yp}\right)(b\,dy)y - 2\int_{y_o}^{h/2}(-\sigma_{yp})(b\,dy)y$$

$$= \sigma_{yp}\frac{bh^2}{4} - \sigma_{yp}\frac{by_o^2}{3} = M_p - \sigma_{yp}\frac{by_o^2}{3}$$

Example 6-14



Determine the plastic moment strength for the reinforced concrete beam. Assume that the steel reinforcement yields at 40,000 psi and that ultimate strength of concrete $f_c^{/}$= 2500 psi.

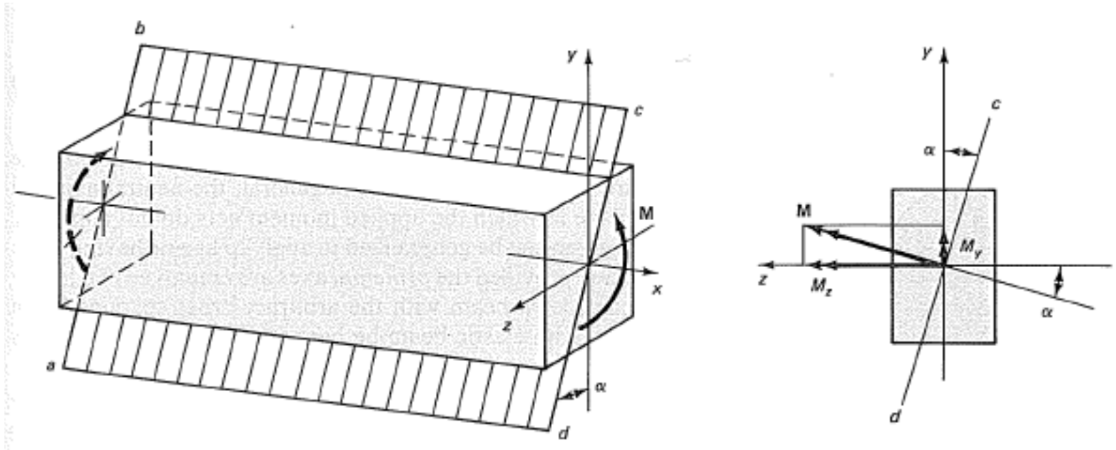


$$T_{ult} = \sigma_{yp} A_s = 40{,}000 \times 2 = 80{,}000 \text{ lb} = C_{ult}$$

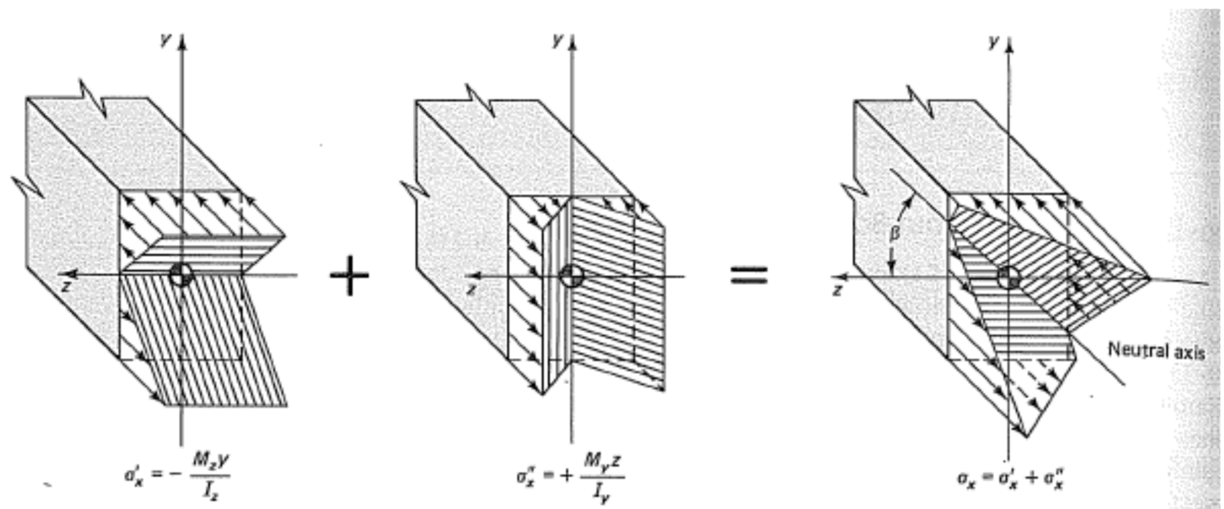
$$k'd = \frac{C_{ult}}{0.85 f'_c b} = \frac{80{,}000}{0.85 \times 2{,}500 \times 10} = 3.77 \text{ in}$$

$$M_{ult} = T_{ult}(d - k'd/2) = 80{,}000(20 - 3.77/2)/12 = 121{,}000 \text{ ft-lb}$$

4. Bending about both Principal Axes



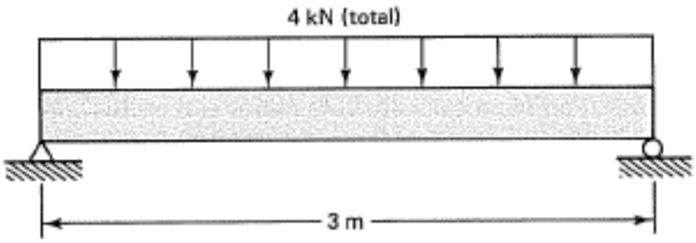
$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

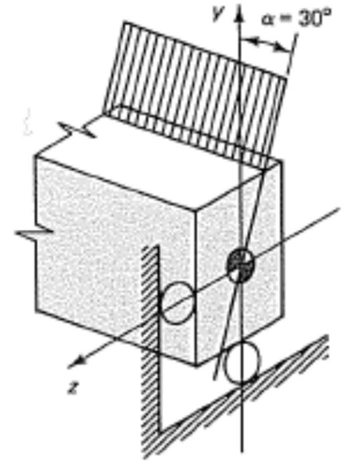


$$-\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0 \qquad \text{or} \qquad \tan\beta = \frac{y}{z} = \frac{M_y I_z}{M_z I_y}$$

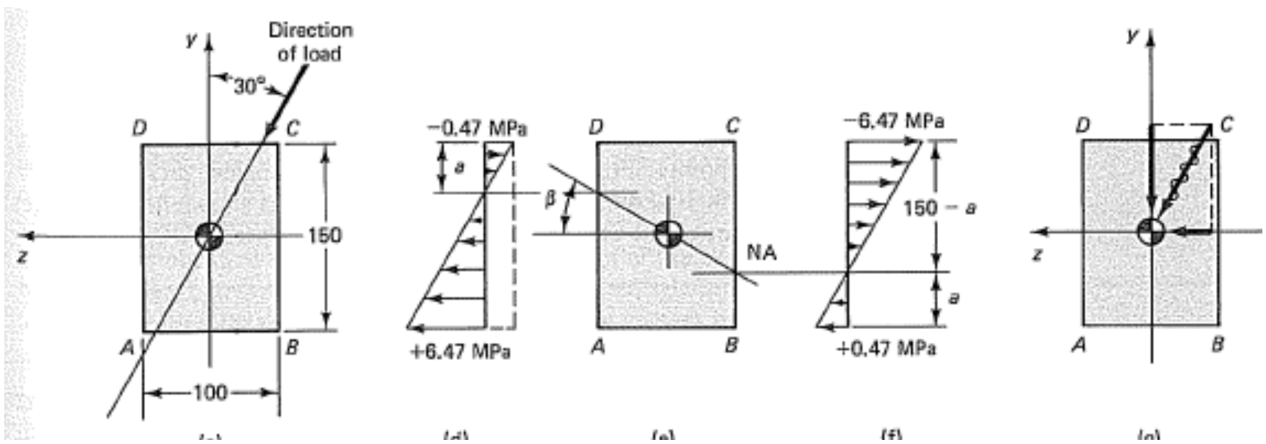
$$\tan\beta = \frac{I_z}{I_y}\tan\alpha$$

Example 6-15





The 100 by 150 mm wooden beam is used to support a uniformly distributed load of 4 kN (total) on a simple span of 3 m. The applied load acts in a plane making an angle of 30° with the vertical. Calculate the maximum bending stress at midspan, and, for the same section, locate the neutral axis. Neglect the weight of the beam.



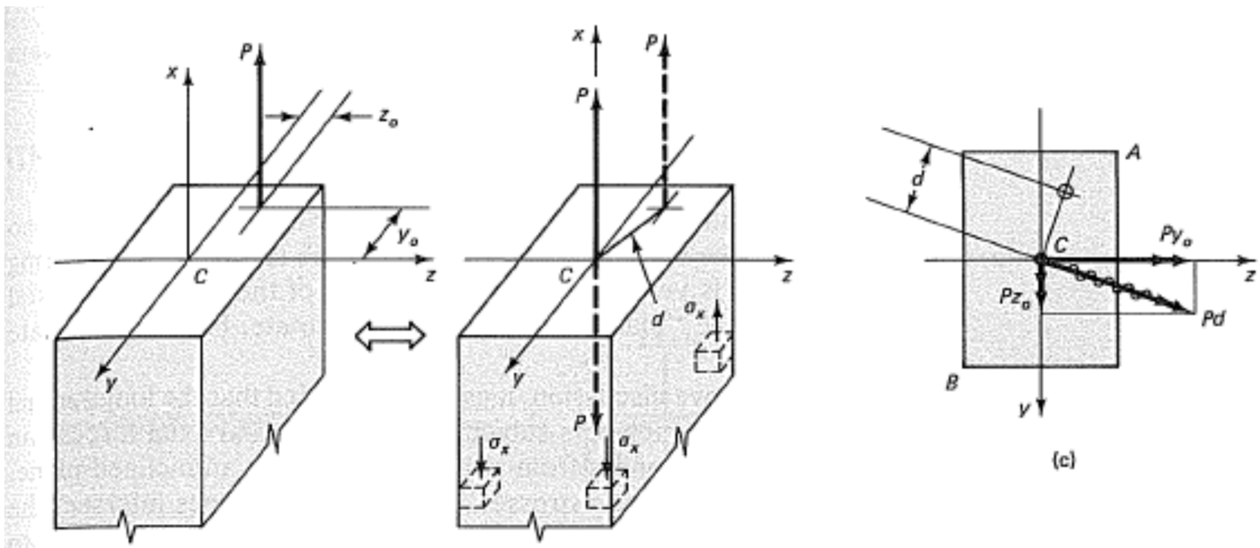
$$M = \frac{WL}{8} = \frac{4 \times 3}{8} = 1.5 \text{ kN·m}$$

$$M_z = M \cos\alpha = 1.5 \times \sqrt{3}/2 = 1.3 \text{ kN·m}$$
$$M_y = M \sin\alpha = 1.5 \times 0.5 = 0.75 \text{ kN·m}$$
$$I_z = 100 \times 150^3/12 = 28.1 \times 10^6 \text{ mm}^4$$
$$I_y = 150 \times 100^3/12 = 12.5 \times 10^6 \text{ mm}^4$$

$$\sigma_A = -\frac{M_z(-c_1)}{I_z} + \frac{M_y c_2}{I_y} = \frac{1.3 \times 10^6 \times 75}{28.1 \times 10^6} + \frac{0.75 \times 10^6 \times 50}{12.5 \times 10^6}$$
$$= +3.47 + 3.00 = +6.47 \text{ MPa}$$
$$\sigma_B = +3.47 - 3.00 = +0.47 \text{ MPa}$$
$$\sigma_C = -3.47 - 3.00 = -6.47 \text{ MPa}$$
$$\sigma_D = -3.47 + 3.00 = -0.47 \text{ MPa}$$

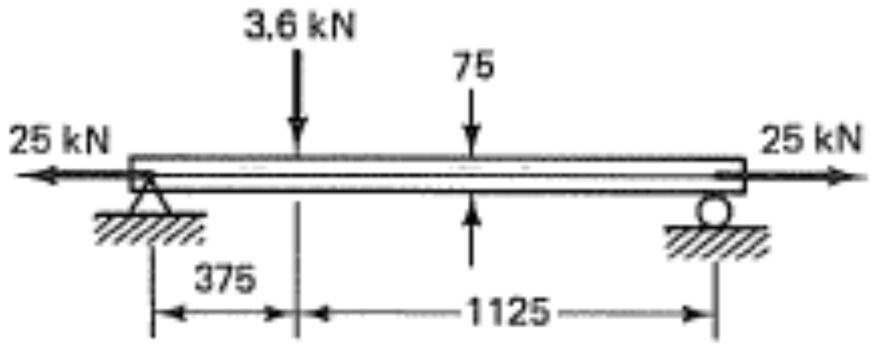
$$\tan\beta = \frac{28.1 \times 10^6}{12.5 \times 10^6} \tan 30° = 1.30 \qquad \text{or} \qquad \beta = 52.4°$$

5. Elastic Bending with Axial Loads

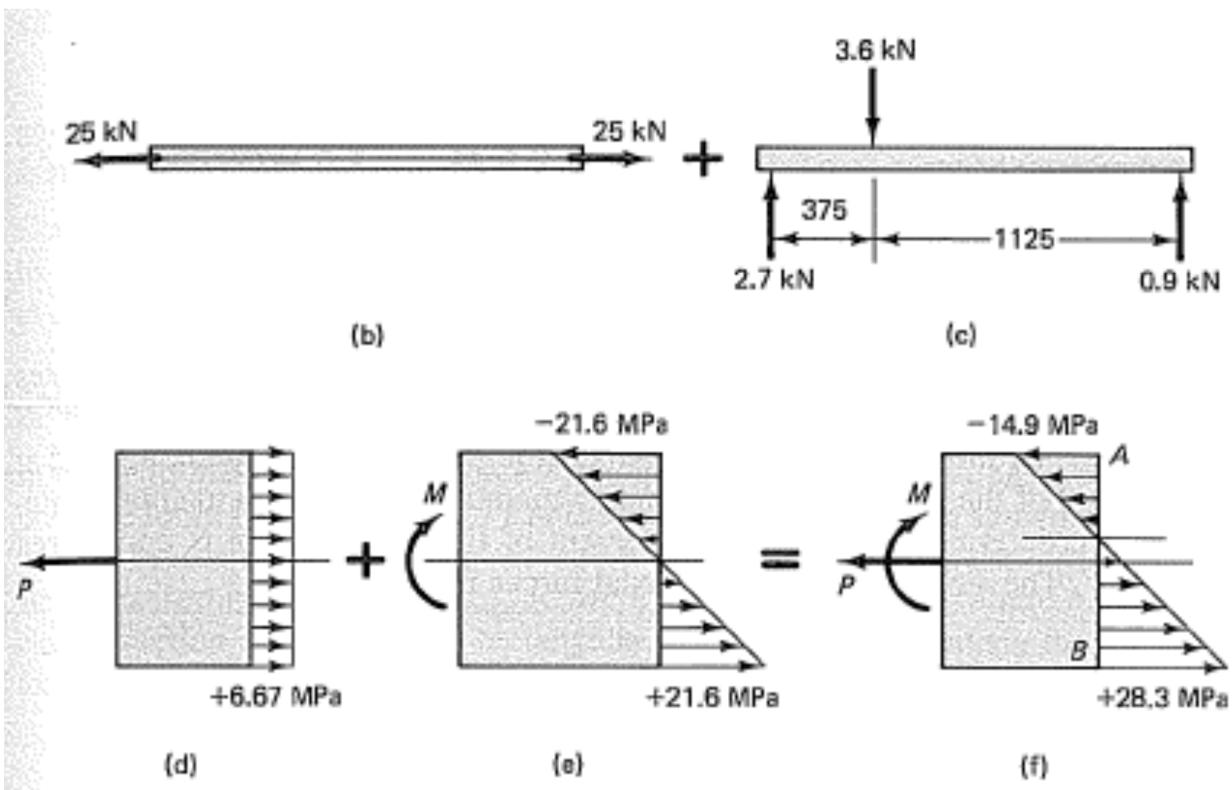


$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

Example 6-17



A 50 by 75 mm, 1.5 m long elastic bar of negligible weight is loaded. Determine the maximum tensile and compressive stresses acting normal to the section through the beam.

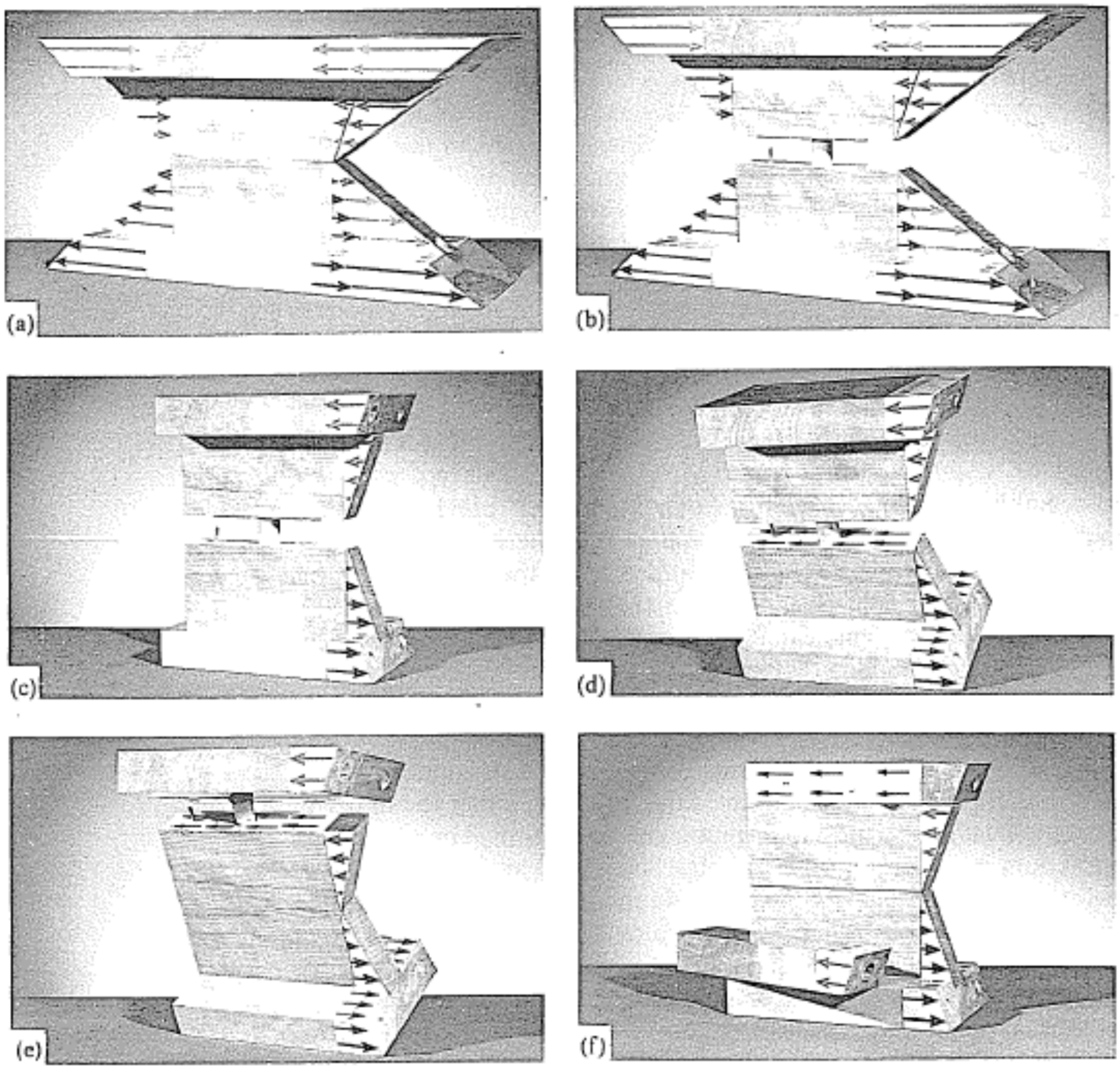


$$\sigma = \frac{Mc}{I} = \frac{6M}{bh^2} = \frac{6 \times 1.013 \times 10^6}{50 \times 75^2} = \pm 21.6 \text{ MPa}$$

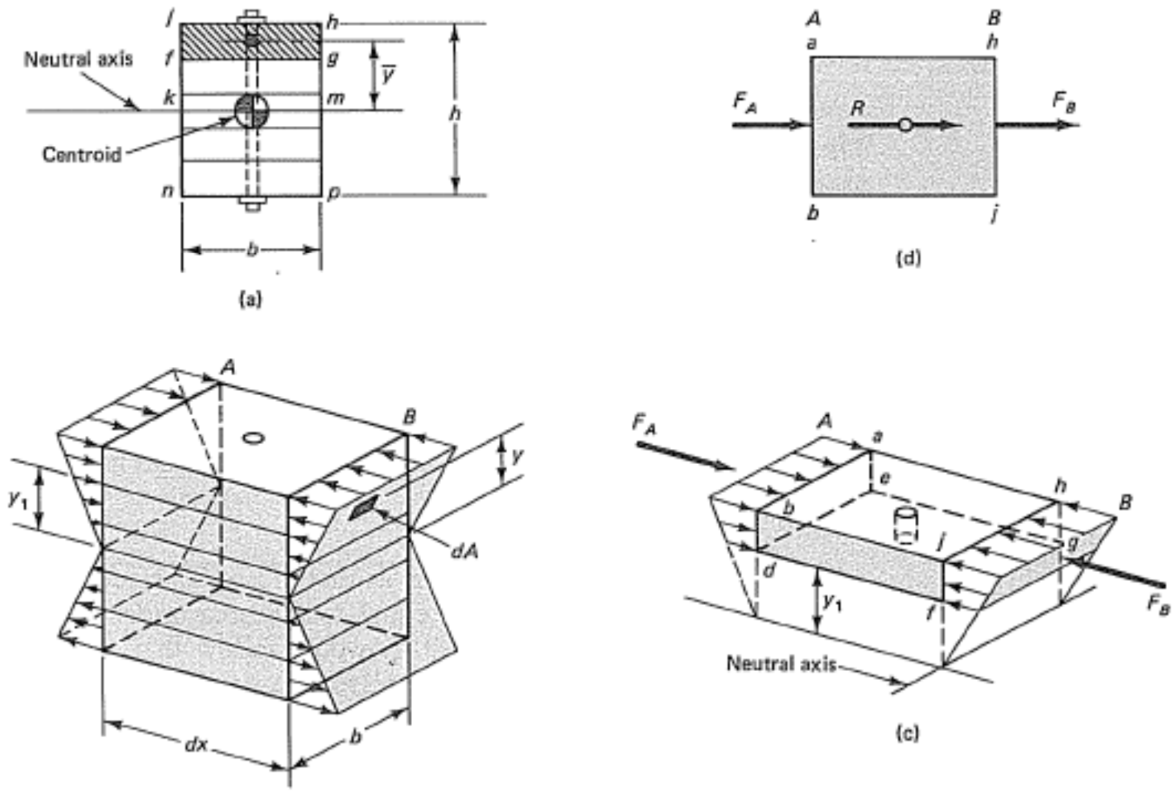
Lecture 7: Shear Stresses in Beams

1. Preliminary Remarks

$$dM = V\,dx \qquad \text{or} \qquad \frac{dM}{dx} = V$$



2 Shear Flow



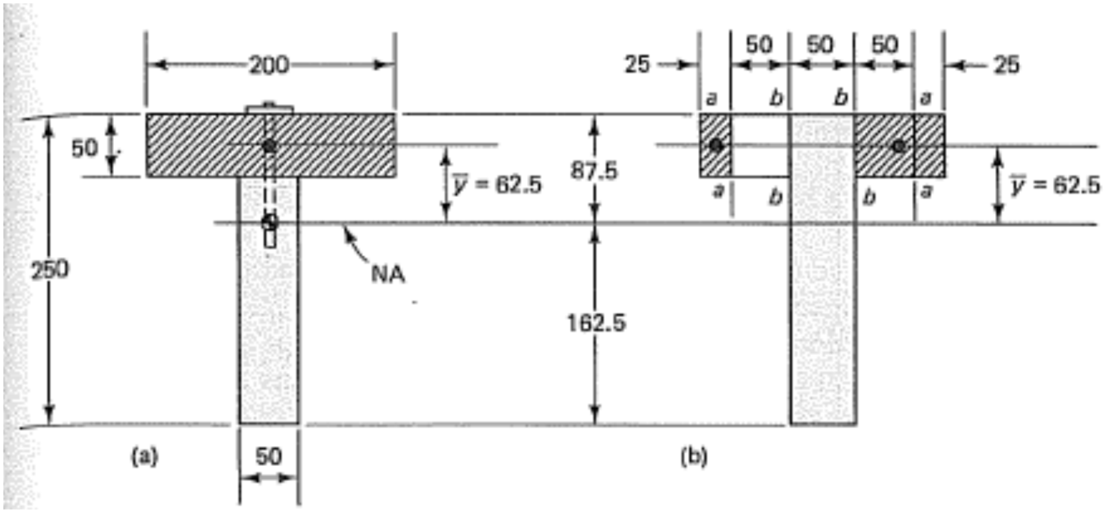
$$F_B = \int_{\substack{\text{area}\\ fghj}} -\frac{M_B y}{I}\, dA = -\frac{M_B}{I}\int_{\substack{\text{area}\\ fghj}} y\, dA = -\frac{M_B Q}{I}$$

$$F_A = -\frac{M_A}{I}\int_{\substack{\text{area}\\ abde}} y\, dA = -\frac{M_A Q}{I}$$

$$dF = |F_B| - |F_A| = \left(\frac{M_A + dM}{I}\right)Q - \left(\frac{M_A}{I}\right)Q = \frac{dM}{I}Q$$

$$q = \frac{dF}{dx} = \frac{dM}{dx}\frac{1}{I}\int_{\substack{\text{area}\\ fghj}} y\, dA = \frac{VA_{fghj}\bar{y}}{I} = \frac{VQ}{I}$$

Example 7-1



If this beam transmits a constant vertical shear of 3000 N, find the necessary spacing of the nails between the two planks to make the beam act as a unit. Assume that the allowable shear force per nail is 700 N.

$$y_c = \frac{50 \times 200 \times 25 + 50 \times 200 \times 150}{50 \times 200 + 50 \times 200} = 87.5 \text{ mm}$$

$$I = \frac{200 \times 50^3}{12} + 50 \times 200 \times 62.5^2 + \frac{50 \times 200^3}{12} + 50 \times 200 \times 62.5^2$$

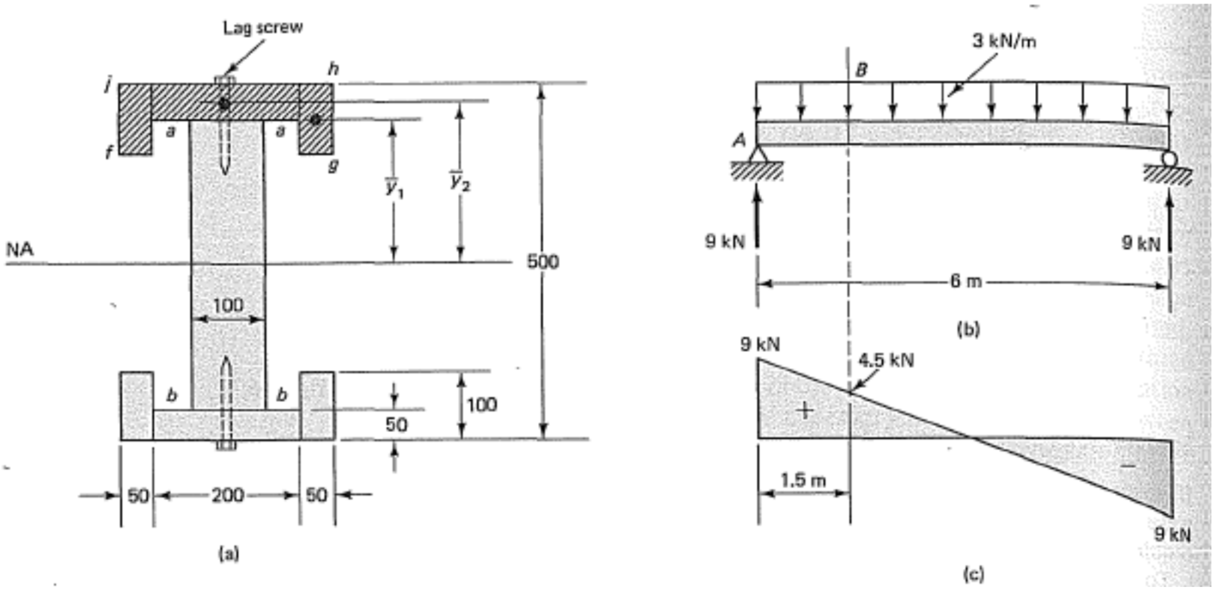
$$= 113.54 \times 10^6 \text{ mm}^4$$

$$Q = A_{fghj}\bar{y} = 50 \times 200 \times (87.5 - 25) = 625 \times 10^3 \text{ mm}^3$$

$$q = \frac{VQ}{I} = \frac{3000 \times 625 \times 10^3}{113.54 \times 10^6} = 16.5 \text{ N/mm}$$

One nail is adequate for transmitting shear 700/16.5 = 42 mm of the beam length.

Example 7-2



A simple beam on a 6-m span carries a load of 3 kN/m including its own weight. The beam's cross section is to be made from several wooden pieces, as is shown in mm. Specify the spacing of the 10 mm lag screws shown that is necessary to fasten this beam together. Assume that one 10 mm lag screw, as determined by laboratory tests, is good for 2 kN when transmitting lateral load parallel to the grain of the wood. For the entire section, I is equal to 2.36 x 10 mm^2.

$$Q = A_{fghj}\bar{y} = 2A_1\bar{y}_1 + A_2\bar{y}_2$$
$$= 2 \times 50 \times 100 \times 200 + 50 \times 200 \times 225 = 4.25 \times 10^6 \text{ mm}^3$$
$$q = \frac{VQ}{I} = \frac{9 \times 4.25 \times 10^9}{2.36 \times 10^9} = 16.2 \text{ N/mm}$$

At the supports, the spacing of the lag screws must be 2 x 10^3/16.2 = 123 rom apart. This spacing of the lag screws applies only at a section where shear V is equal to 9 kN. Similar calculations for a section where V = 4.5 kN gives q = 8.1 N/mm; and the spacing of the lag screws becomes 2 x 103/8.1 = 246mm.