CHAPTER

1

ENGINEERING MECHANICS OF SOLIDS

Stress

Concept of Stress

The main objective of the study of mechanics of materials is to provide the future engineer with the means of analyzing and designing various machines and load bearing structures.

Both the analysis and design of a given structure involve the determination of stresses and deformations. This chapter is devoted to the concept of stress.

- **Stress: Stress is defined as the intensity of force per unit area at any point. To designate stress notations σ (normal stress) and τ (shear stress) are used.**

Free Body and Internal Linear Force

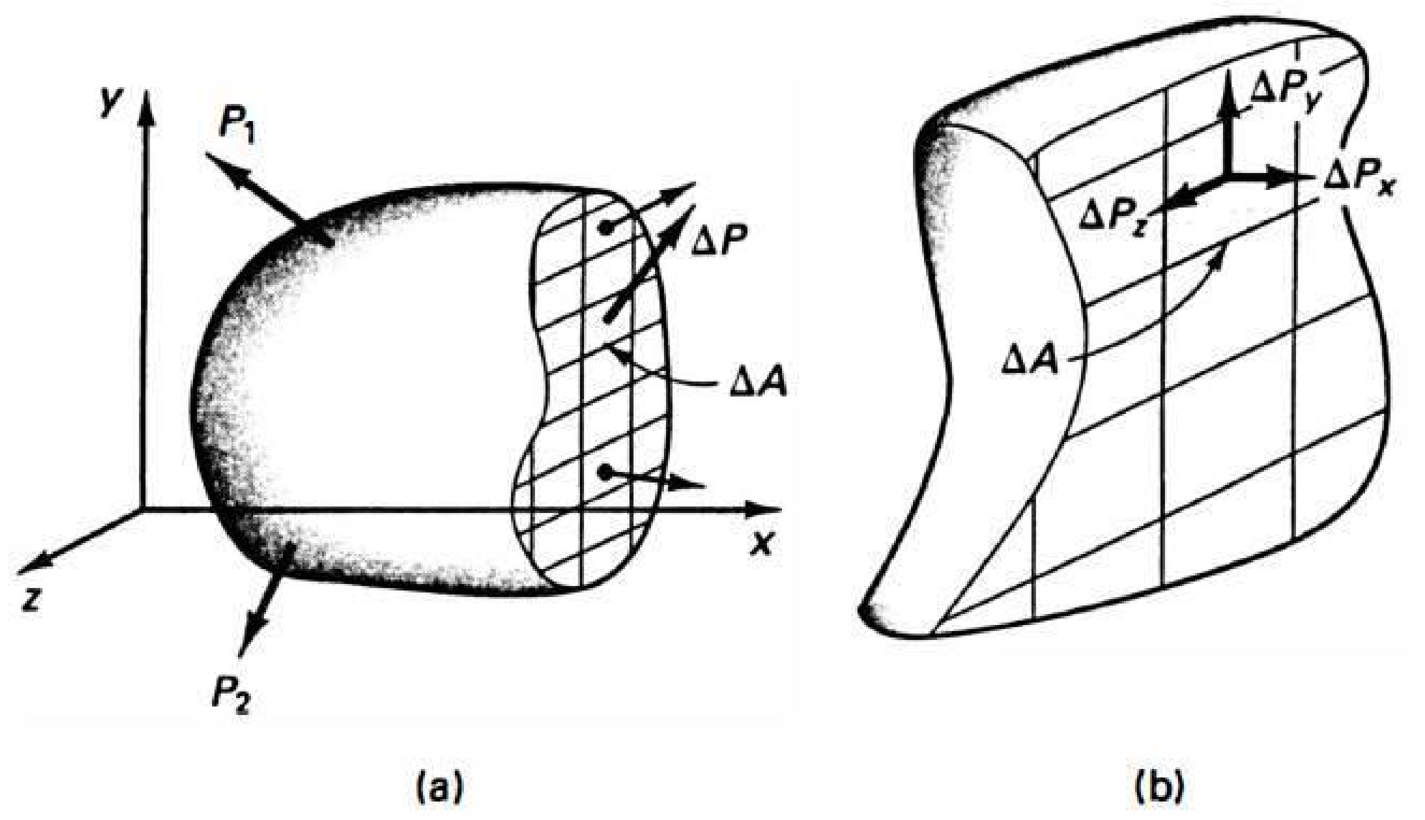


Fig. 1-2 Sectioned body: (a) free body with some internal forces, (b) enlarged view with components of **Δ*P*.**

Concept of Stress

- **Normal Stress**: Stress resulting from force that is normal to the surface is called normal stress. It can be either tensile stress or compressive stress. If the normal force is away from the surface, the resulting stress is **tensile normal stress** and if the force acts on an inward direction to the surface the resulting stress is **compressive normal stress**. Notations **σ** is used to designate normal stress.
- **Shear Stress**: Stress that results from forces that are along the surface is called shear stress. Notations **τ** is used to designate shear stress.

- A member subjected to a general combination of loads is cut into two segments by a plane passing through Q
- The distribution of internal stress components may be defined as,

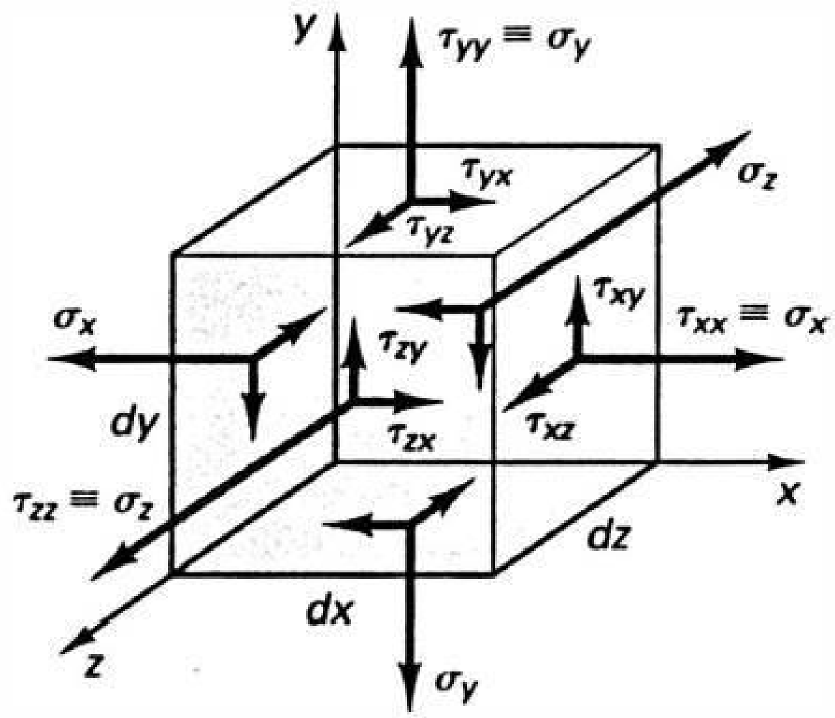
$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$

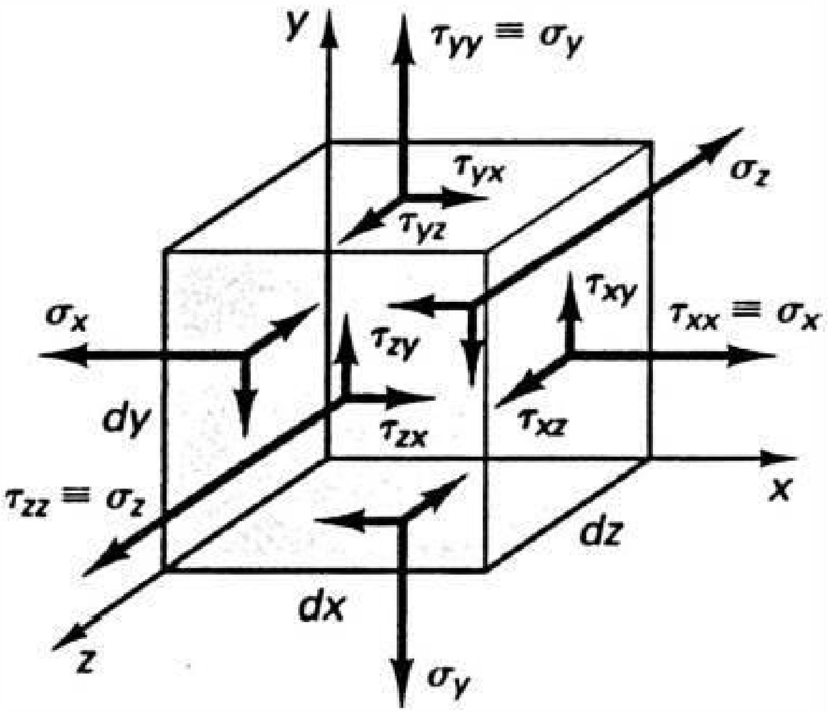
$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_{xy}}{\Delta A} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_{xz}}{\Delta A}$$

- For equilibrium, an equal and opposite internal force and stress distribution must be exerted on the other segment of the member.

Concept of Stress

- **Direction of Stress**: Direction of stress is the direction of force from which the stress occurs. Two subscripts are required to completely indicate the direction of stress: the first subscript indicates the direction of normal to the plane on which the stress acts, the second subscript indicates the direction of the force from which the stress occurs.





Thus $\boldsymbol{\sigma}_{xx}$ is complete notations of normal stress that occurs on a plane for which direction of normal is the direction of x axis and direction of force is the direction of x axis, and $\boldsymbol{\tau}_{xy}$ is complete notations of shear stress that occurs on a plane for which direction of normal is the direction of x axis and direction of force is the direction of y axis. For the plane shown direction of normal is the x axis, thus first subscript of any stress acting on this plane is x and the second subscript is the direction of the force from which the stress occurs.

Concept of Stress

Sign convention for stress components: If the direction of the positive normal stress is the direction of the positive axis on any plane, the direction of positive shear stress is the direction of respective positive axis and if the direction of the positive normal stress is the direction of the negative axis on any plane, the direction of positive shear stress is the direction of respective negative axis.

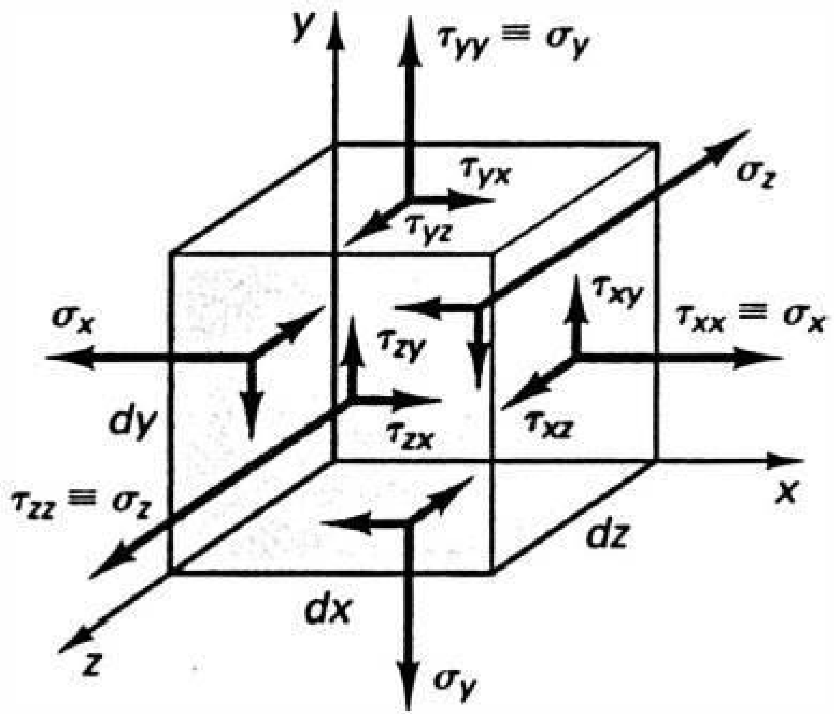
Stress Tensor: A force P can be resolved into three components (in the direction of the axis system in Cartesian coordinate system) , load can be written in vector form.

$$P = \begin{pmatrix} Px \\ Py \\ Pz \end{pmatrix}$$

Analogously, the stress components can be assembled as follows:

$$\equiv \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

Concept of Stress



At first glance it appears that a total of nine stress components (three on each face x three face) are required to fully define the state of stress at a point.

Stress components are defined for the planes cut parallel to the x, y and z axes. For equilibrium, equal and opposite stresses are exerted on the hidden planes.

The combination of forces generated by the stresses must satisfy the conditions for equilibrium:

$$\sum F_x = \sum F_y = \sum F_z = 0$$

$$\sum M_x = \sum M_y = \sum M_z = 0$$

State of Stress

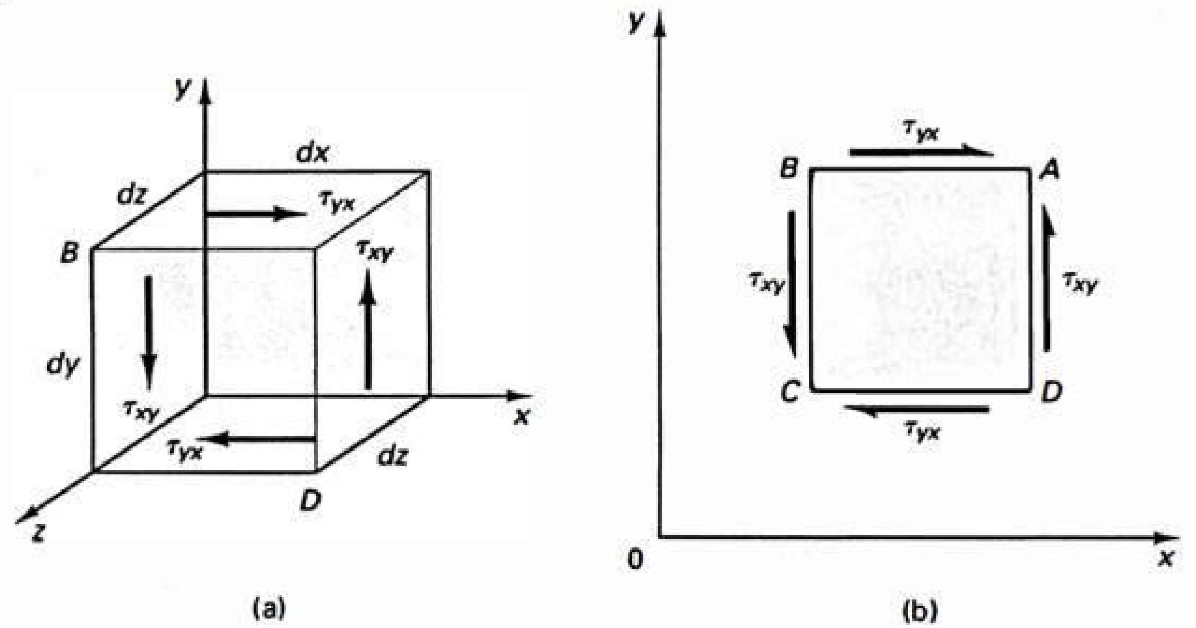


Fig. 1-4 Elements in pure shear.

Consider the moments about the *z* axis:

$$\sum M_z = 0 = \left(\tau_{yx} dx.dz\right).dy - \left(\tau_{xy}.dy.dz\right).dx$$

$$\tau_{xy} = \tau_{yx}$$

similarly, $\tau_{yz} = \tau_{zy}$ and $\tau_{yz} = \tau_{zy}$

It follows that only 6 components of stress are required to define the complete state of stress

1-5. Differential Equations of Equilibrium

An infinitesimal element of a body must be in equilibrium. For the two-dimensional case, the system of stresses acting on an infinitesimal element $(dx)(dy)(1)$ is shown in Fig. 1-6. In this derivation, the element is of unit thick-

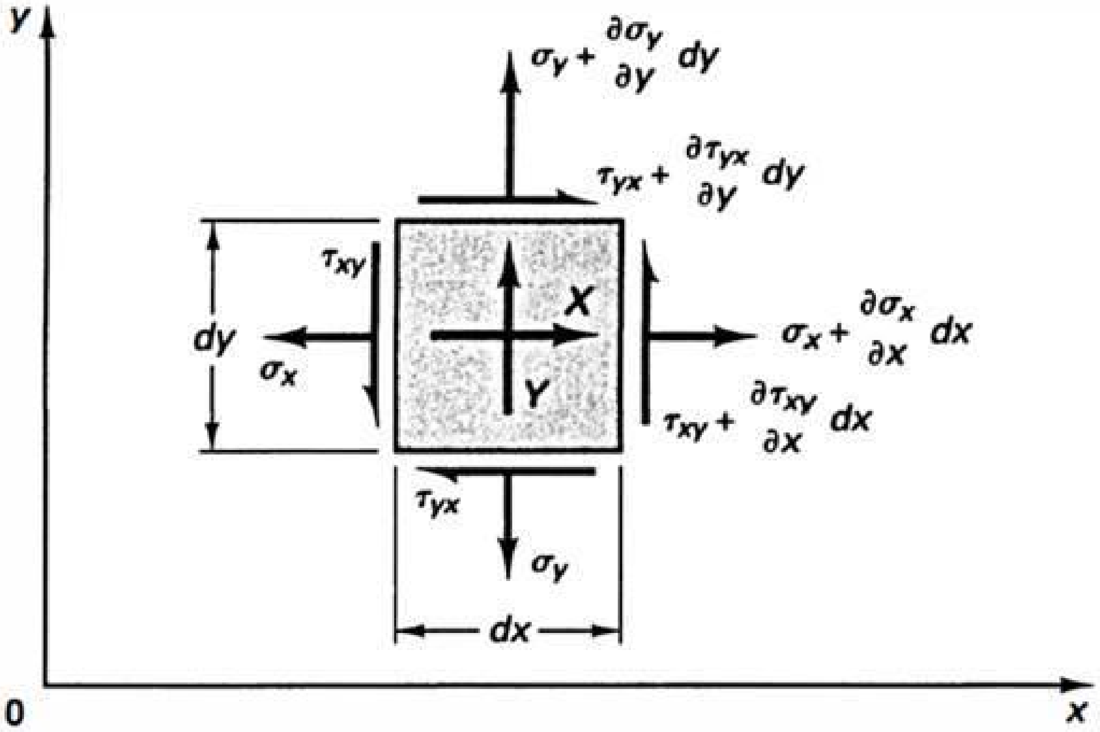


Fig. 1-6 Infinitesimal element with stresses and body forces.

ness in the direction perpendicular to the plane of the paper. Note that the possibility of an increment in stresses from one face of the element to another is accounted for. For example, since the rate of change of σ_x in the x direction is $\partial\sigma_x/\partial x$ and a step of dx is made, the increment is $(\partial\sigma_x/\partial x)\,dx$. The partial derivative notation has to be used to differentiate between the directions.

The inertial or body forces, such as those caused by the weight or the magnetic effect, are designated X and Y and are associated with the unit volume of the material. With these notations,

$$\sum F_x = 0 \rightarrow +, \qquad \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\,dx\right)(dy \times 1) - \sigma_x(dy \times 1)$$

$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\,dy\right)(dx \times 1) - \tau_{yx}(dx \times 1) + X(dx\,dy \times 1) = 0$$

Simplifying and recalling that $\tau_{xy} = \tau_{yx}$ holds true, one obtains the basic equilibrium equation for the x direction. This equation, together with an analogous one for the y direction, reads

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \end{aligned} \qquad (1\text{-}5)$$

The moment equilibrium of the element requiring $\sum M_z = 0$ is assured by having $\tau_{xy} = \tau_{yx}$.

It can be shown that for the three-dimensional case, a typical equation from a set of three is

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

Note that in deriving the previous equations, mechanical properties of the material have not been used. This means that these equations are applicable whether a material is elastic, plastic, or viscoelastic. Also it is very important to note that there are not enough equations of equilibrium to solve for the unknown stresses. In the two-dimensional case, given by Eq. 1-5, there are three unknown stresses, and only two equations. For the three-dimensional case, there are six stresses, but only three equations.

Stress Analysis of Axially Loaded Bars

1-6. Maximum Normal Stress in Axially Loaded Bars

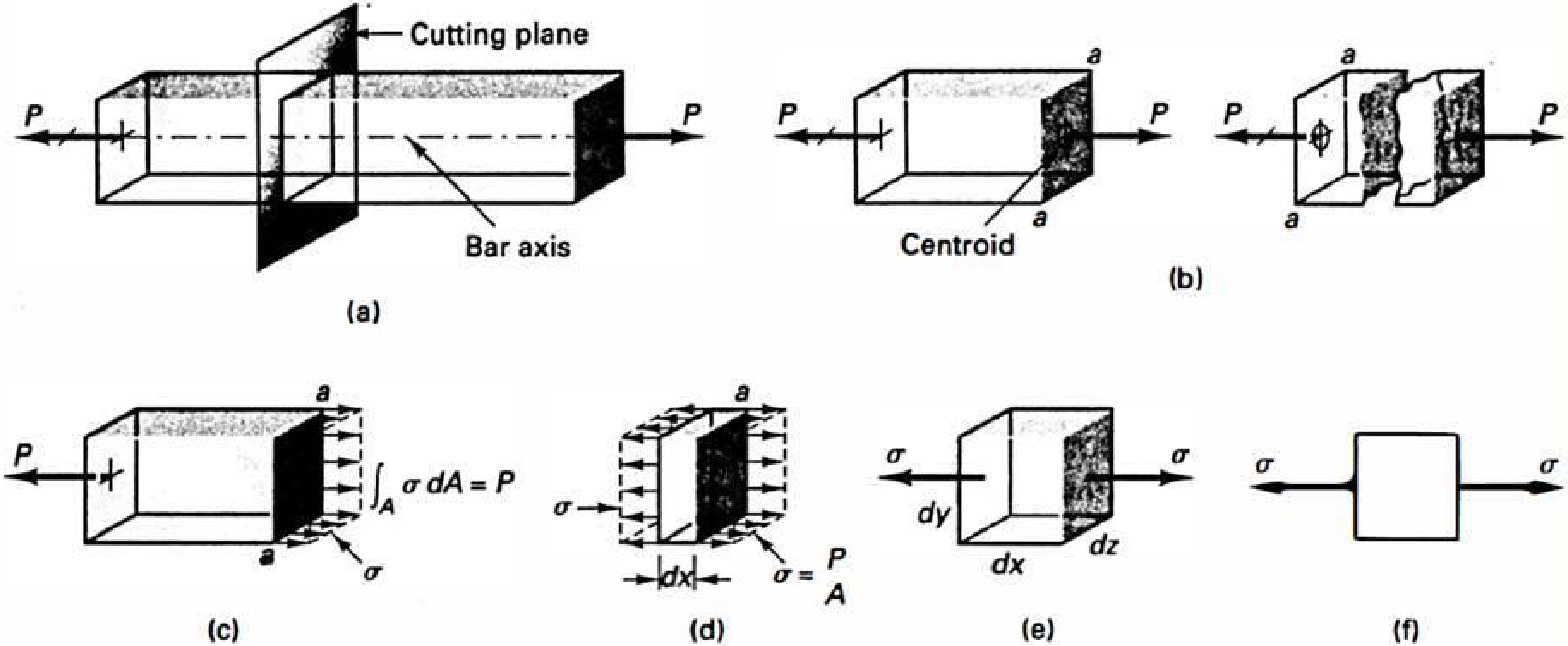


Fig. 1-7 Successive steps in determining the largest normal stress in an axially loaded bar.

Bearing Stress

Sometimes compressive stresses arise where one body is supported by another. If the resultant of the applied forces coincides with the centroid of the contact area between the two bodies, the intensity of force, or stress, between the two bodies can again be determined from Eq. 1-6. It is customary to refer to this normal stress as a **bearing stress**. Figure 1-9, where a short block bears on a concrete pier and the latter bears on the soil, illustrates such a stress.

These bearing stresses can be approximated by dividing the applied force P by the corresponding contact area giving a useful nominal bearing stress.

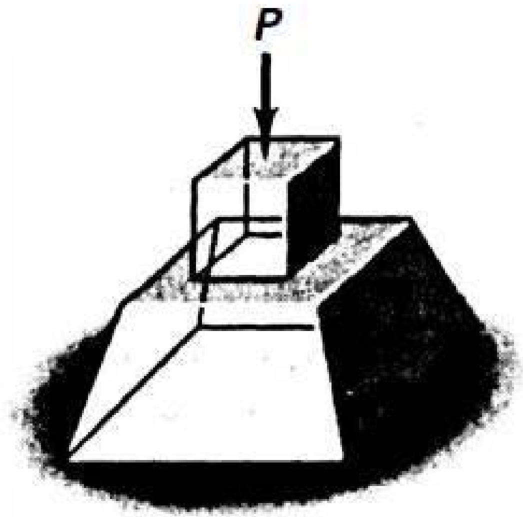


Fig. 1-9 Bearing stresses occur between the block and pier, as well as between the pier and soil.

Shear Stresses

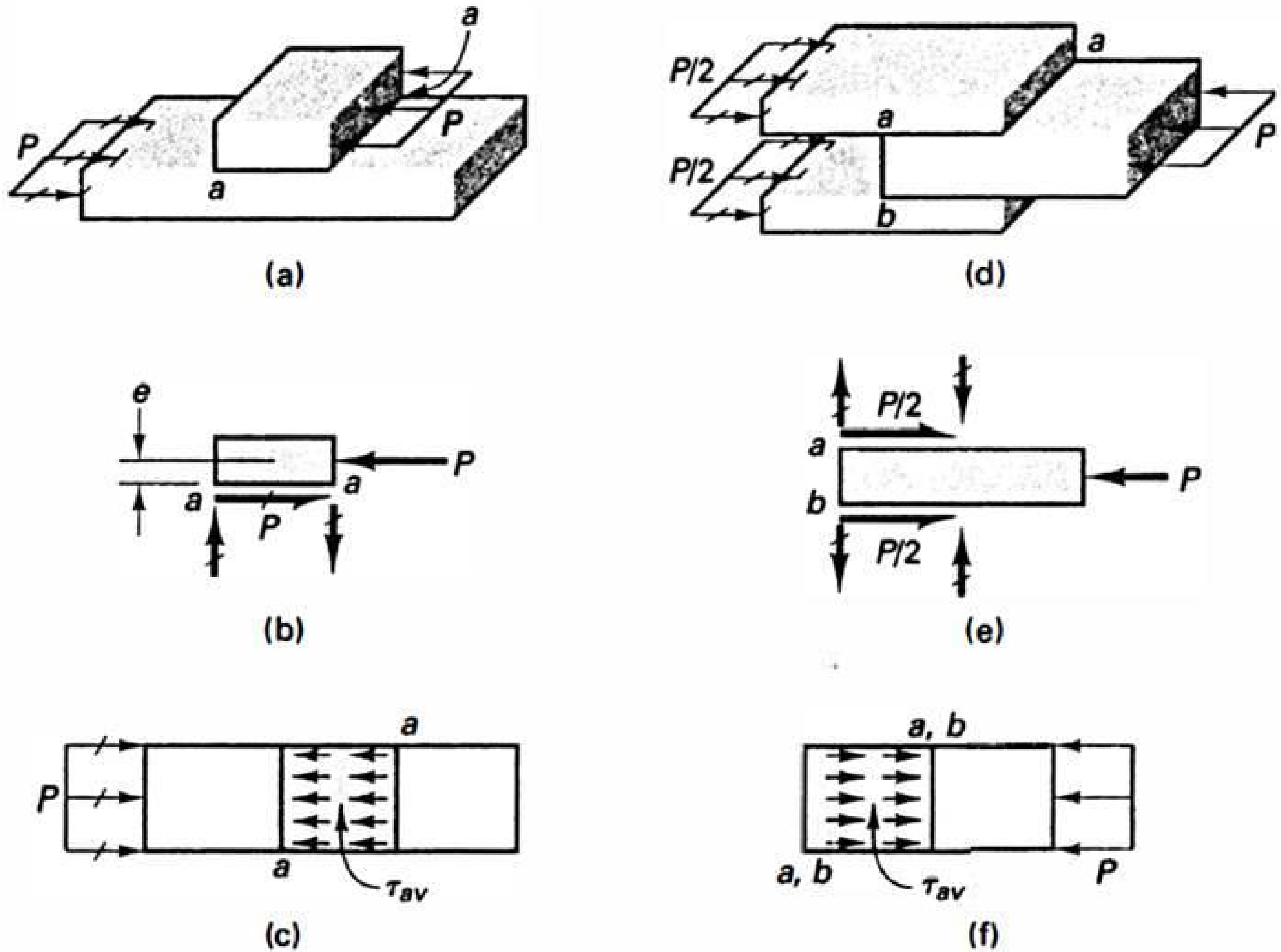


Fig. 1-13 Loading conditions causing shear stresses between interfaces of glued blocks.

Shear Stresses

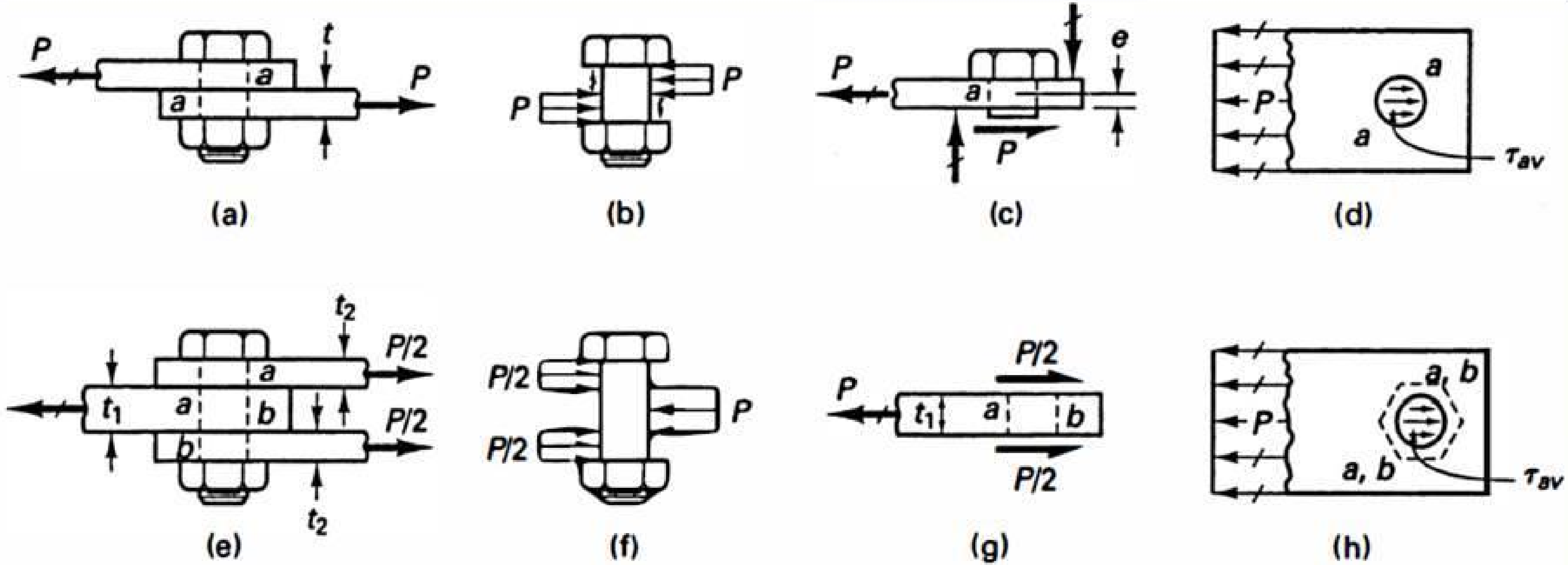


Fig. 1-14 Loading conditions causing shear and bearing stress in bolts.

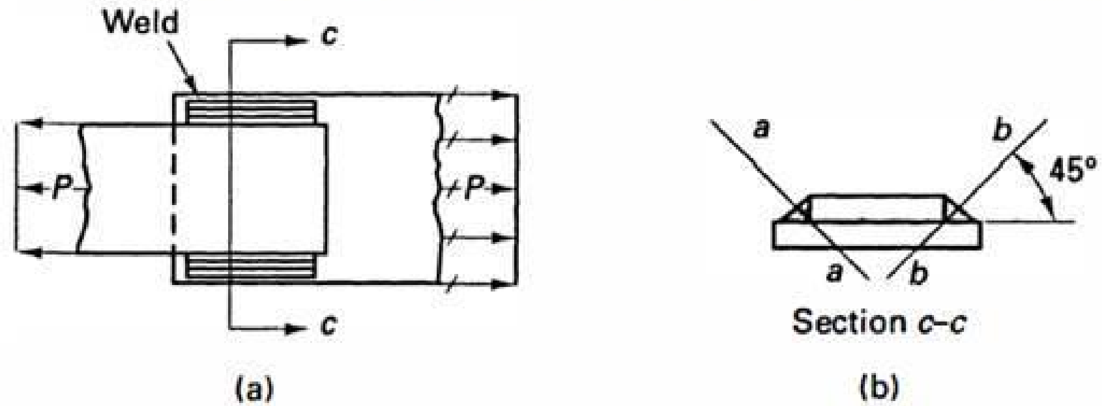
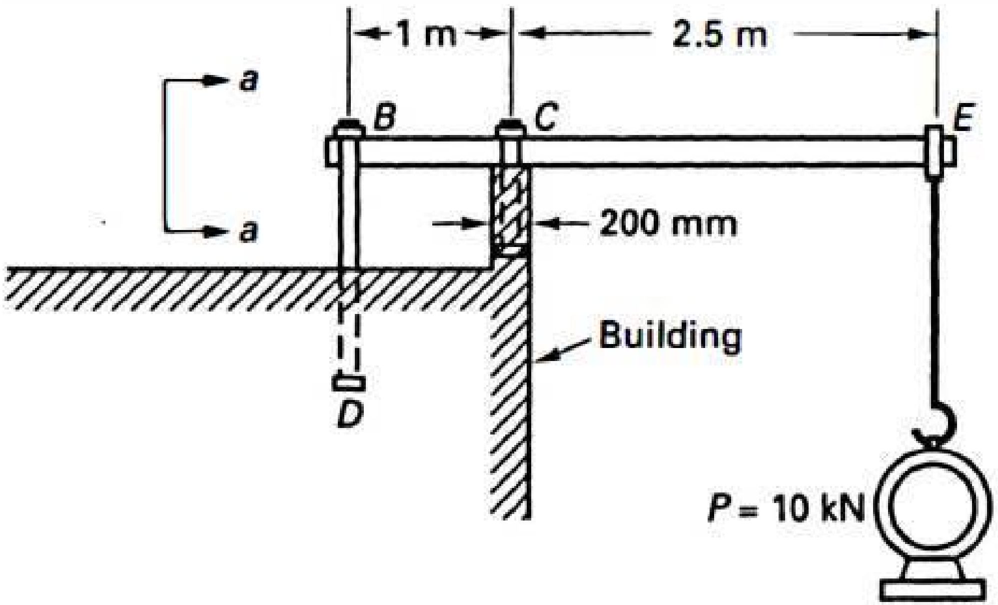


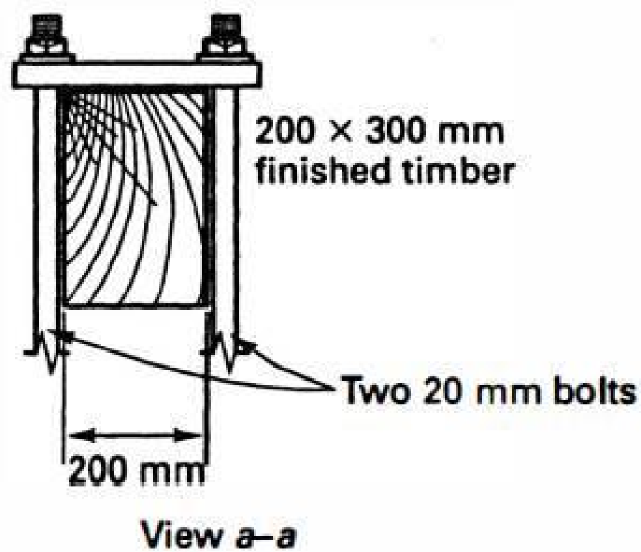
Fig. 1-16 Loading condition causing critical shear in two planes of fillet welds.

Example 1-1

The beam BE in Fig. 1-18(a) is used for hoisting machinery. It is anchored by two bolts at B, and at C it rests on a parapet wall. The essential details are given in the figure. Note that the bolts are threaded, as shown in Fig. 1-18(d), with d = 16 mm at the root of the threads. If this hoist can be subjected to a force of 10 kN, determine the stress in bolts BD and the bearing stress at C. Assume that the weight of the beam is negligible in comparison with the loads handled.



(a)



(c)

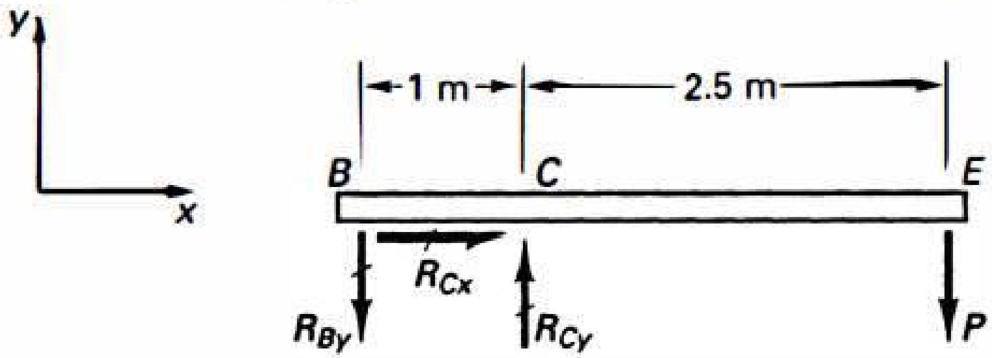
To solve this problem, the actual situation is idealized and a free-body diagram is made on which all known and unknown forces are indicated. This is shown in Fig. 1-18(b). The vertical reactions of B and C are unknown. They are indicated, respectively, as R_{By} and R_{Cy}, where the first subscript identifies the location, and the second the line of action of the unknown force. As the long bolts BD are not effective in resisting the horizontal force, only an unknown horizontal reaction at C is assumed and marked as R_{Cx}. The applied known force P is shown in its proper location. After a free-body diagram is prepared, the equations of statics are applied and solved for the unknown forces.

$$\sum F_x = 0 \qquad\qquad R_{Cx} = 0$$

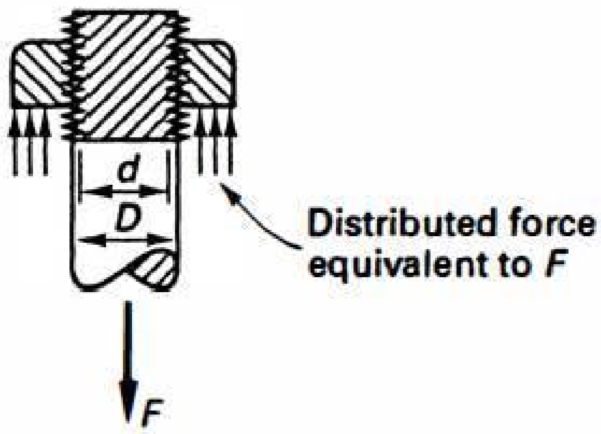
$$\sum M_B = 0 \curvearrowright + \qquad 10(2.5 + 1) - R_{Cy} \times 1 = 0 \qquad R_{Cy} = 35 \text{ kN} \uparrow$$

$$\sum M_c = 0 \curvearrowright + \qquad 10 \times 2.5 - R_{By} \times 1 = 0 \qquad R_{By} = 25 \text{ kN} \uparrow$$

$$\textit{Check:} \quad \sum F_y = 0 \uparrow + \qquad -25 + 35 - 10 = 0$$



(b)



(d)

Cross-sectional area of one 20-mm bolt: $A = \pi 10^2 = 314\ \text{mm}^2$. This is not the minimum area of a bolt; threads reduce it.

The cross-sectional area of one 20-mm bolt at the root of the threads is

$$A_{net} = \pi\, 8^2 = 201\ \text{mm}^2$$

Maximum normal tensile stress[11] in each of the two bolts BD:

$$\sigma_{max} = \frac{R_{By}}{2A} = \frac{25 \times 10^3}{2 \times 201} = 62\ \text{N/mm}^2 = 62\ \text{MPa}$$

Tensile stress in the shank of the bolts BD:

$$\sigma = \frac{25 \times 10^3}{2 \times 314} = 39.8\ \text{N/mm}^2 = 39.8\ \text{MPa}$$

Contact area at C:

$$A = 200 \times 200 = 40 \times 10^3\ \text{mm}^2$$

Bearing stress at C:

$$\sigma_b = \frac{R_{Cy}}{A} = \frac{35 \times 10^3}{40 \times 10^3} = 0.875\ \text{N/mm}^2 = 0.875\ \text{MPa}$$

Example 1-2

The concrete pier shown in Fig. 1-19(a) is loaded at the top with a uniformly distributed load of 20 kN/m^2. Investigate the state of stress at a level 1 m above the base. Concrete weighs approximately 25 kN/m^3.

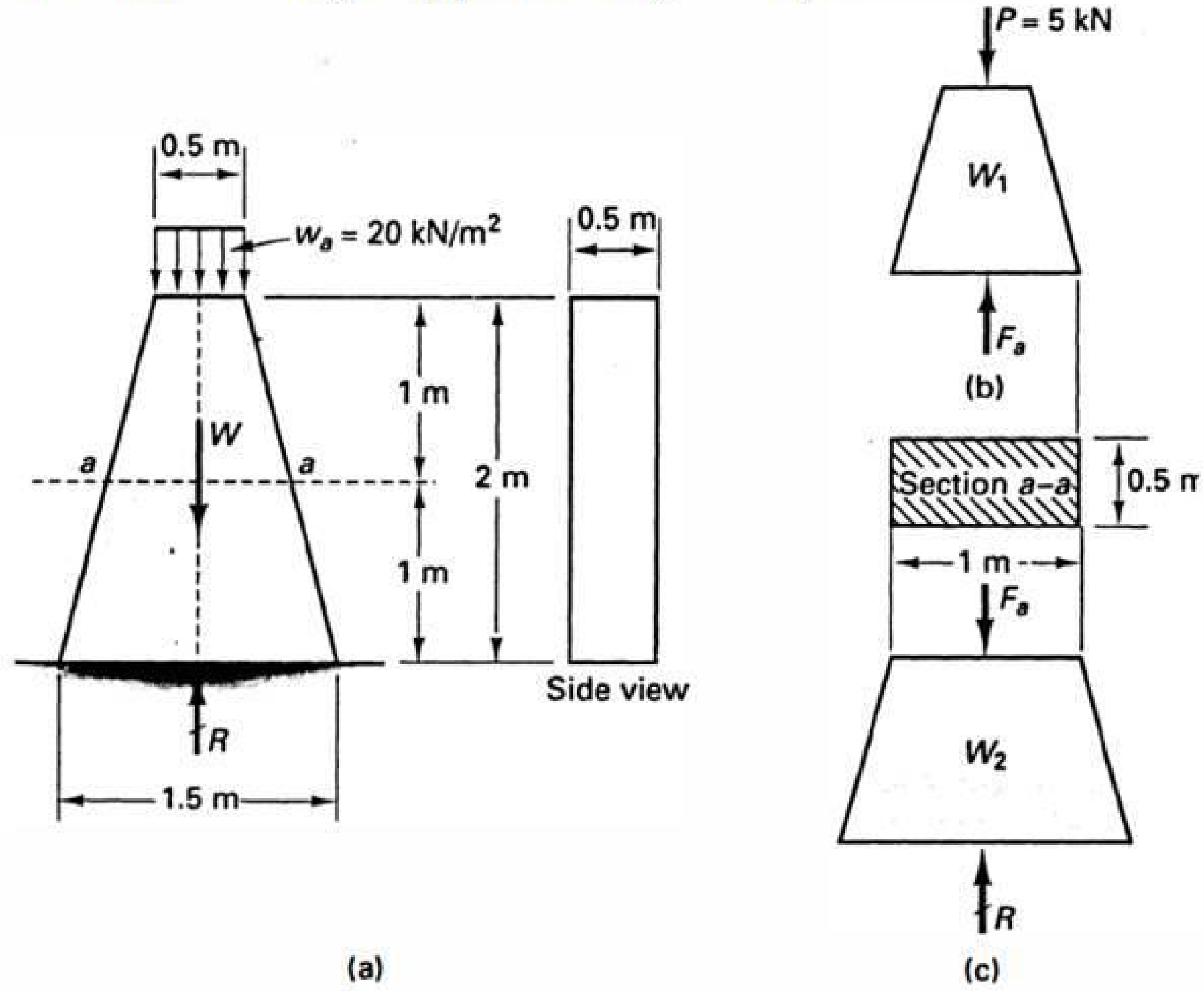


Fig. 1-19

In this problem, the weight of the structure itself is appreciable and must be included in the calculations.

Weight of the whole pier:

$$W = [(0.5 + 1.5)/2] \times 0.5 \times 2 \times 25 = 25 \text{ kN}$$

Total applied force:

$$P = 20 \times 0.5 \times 0.5 = 5 \text{ kN}$$

From $\Sigma F_y = 0$, reaction at the base:

$$R = W + P = 30 \text{ kN}$$

Using the upper part of the pier as a free body, Fig. 1-19(b), the weight of the pier above the section:

$$W_1 = (0.5 + 1) \times 0.5 \times 1 \times 25/2 = 9.4 \text{ kN}$$

From $\sum F_y = 0$, the force at the section:

$$F_a = P + W_1 = 14.4 \text{ kN}$$

Hence, using Eq. 1-6, the normal stress at the level a–a is

$$\sigma_a = \frac{F_a}{A} = \frac{14.4}{0.5 \times 1} = 28.8 \text{ kN/m}^2$$

This stress is compressive as F_a acts on the section.

Using the lower part of the pier as a free body, Fig. 1-19(c), the weight of the pier below the section:

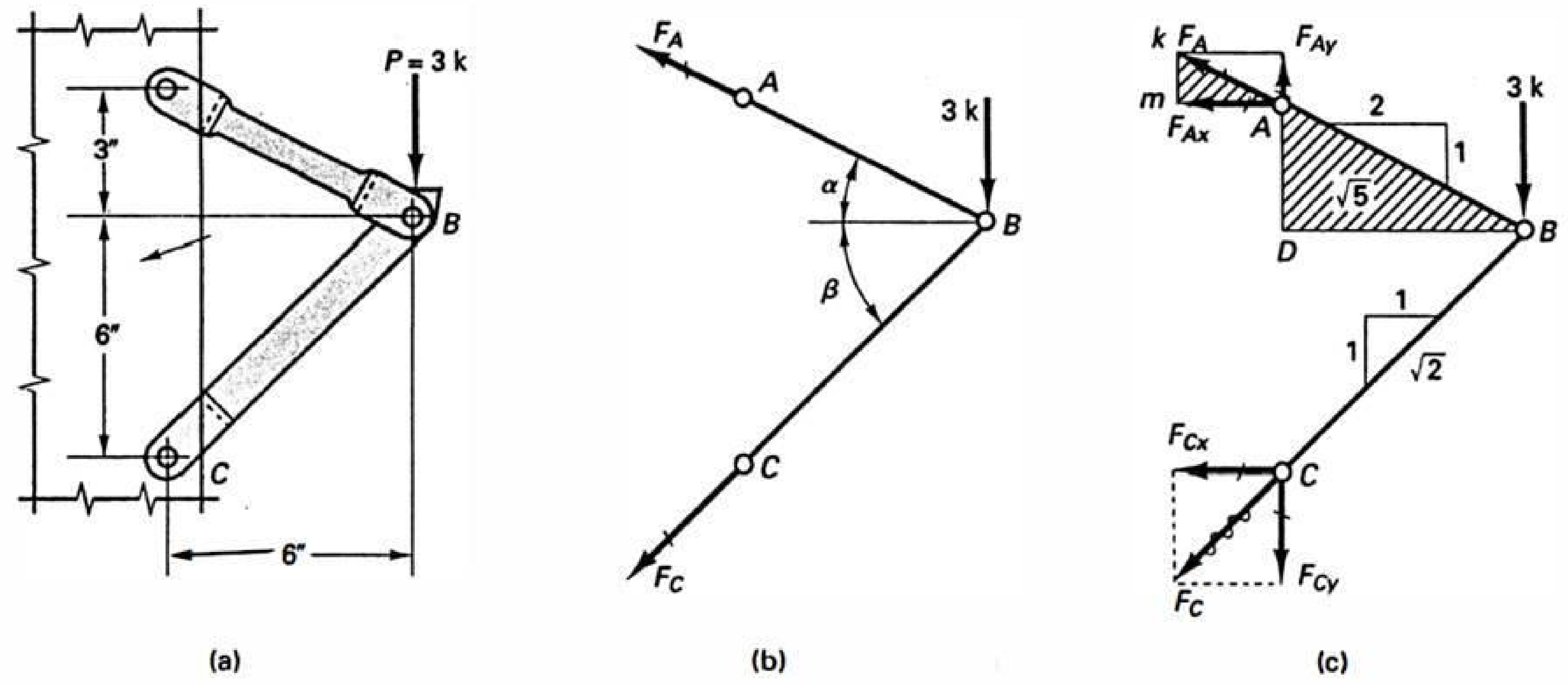
$$W_2 = (1 + 1.5) \times 0.5 \times 1 \times 25/2 = 15.6 \text{ kN}$$

From $\sum F_y = 0$, the force at the section:

$$F_a = R - W_2 = 14.4 \text{ kN}$$

Example 1-3

A bracket of negligible weight shown in Fig. 1-20(a) is loaded with a vertical force P of 3 kips. For interconnection purposes, the bar ends are clevised (forked). Pertinent dimensions are shown in the figure. Find the axial stresses in members AB and BC and the bearing and shear stresses for pin C. All pins are 0.375 in in diameter.



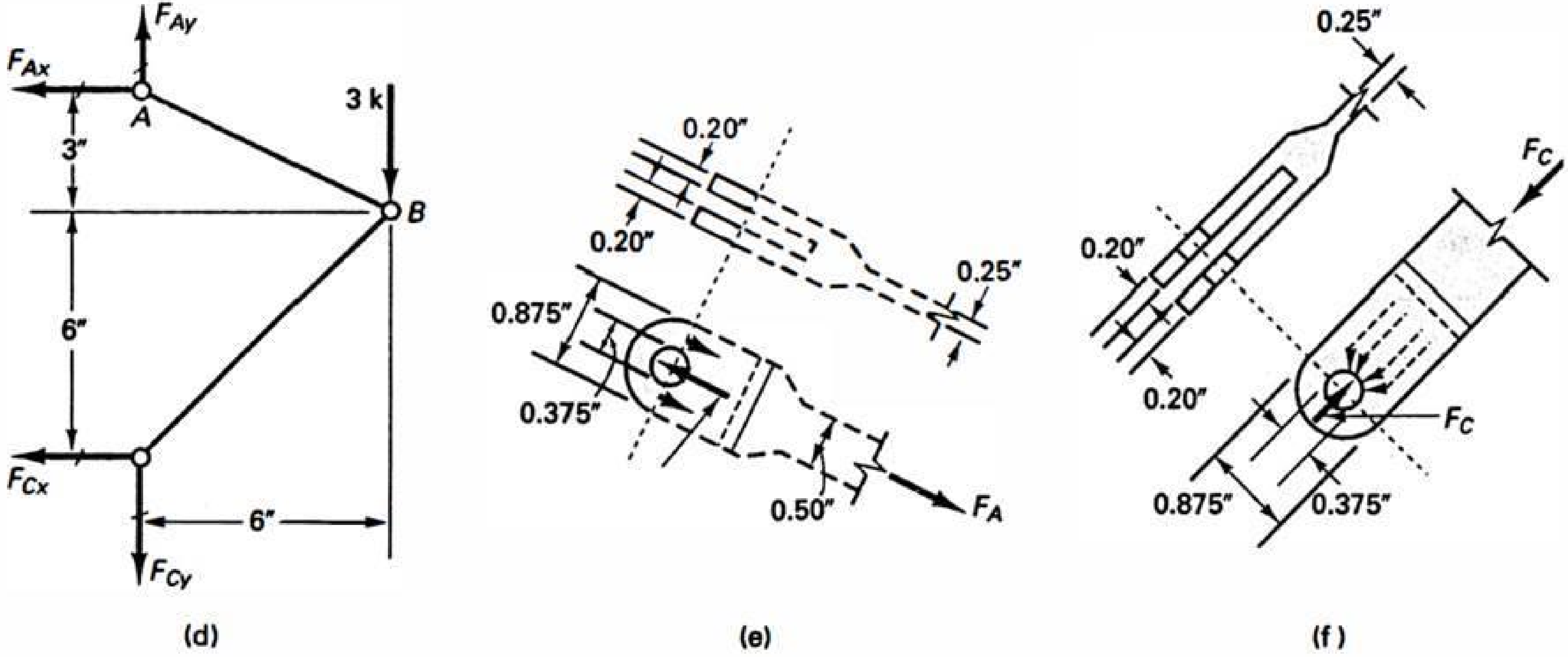


Fig. 1-20

$$\sum M_C = 0 \curvearrowleft + \qquad + F_{Ax}(3 + 6) - 3(6) = 0 \qquad F_{Ax} = +2\text{ k}$$

$$F_{Ay} = F_{Ax}/2 = 2/2 = +1\text{ k}$$

$$F_A = 2(\sqrt{5}/2) = +2.23\text{ k}$$

$$\sum M_A = 0 \curvearrowright + \qquad + 3(6) + F_{Cx}(9) = 0, \qquad F_{Cx} = -2\text{ k}$$

$$F_{Cy} = F_{Cx} = -2\text{ k}$$

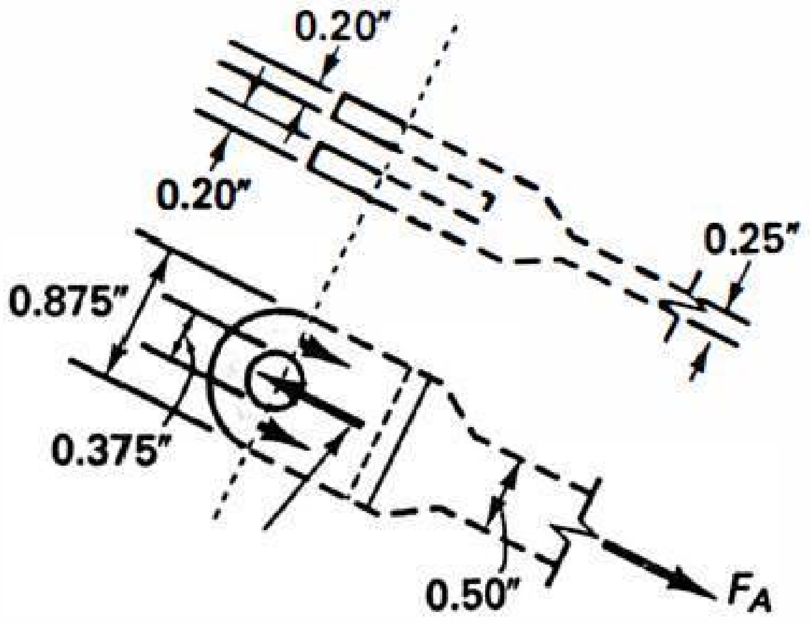
$$F_C = \sqrt{2}(-2) = -2.83\text{ k}$$

Tensile stress in main bar AB:

$$\sigma_{AB} = \frac{F_A}{A} = \frac{2.23}{0.25 \times 0.50} = 17.8 \text{ ksi}$$

Tensile stress in clevis of bar AB, Fig. 1-20(e):

$$(\sigma_{AB})_{\text{clevis}} = \frac{F_A}{A_{\text{net}}} = \frac{2.23}{2 \times 0.20 \times (0.875 - 0.375)} = 11.2 \text{ ksi}$$

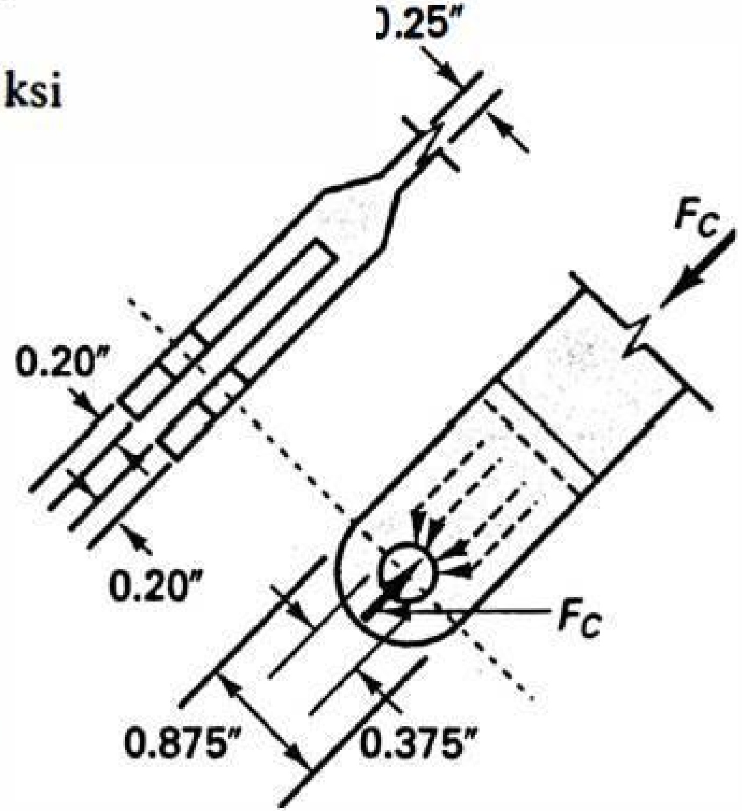


Compressive stress in main bar BC:

$$\sigma_{BC} = \frac{F_C}{A} = \frac{2.83}{0.875 \times 0.25} = 12.9 \text{ ksi}$$

In the compression member, the net section at the clevis need not be investigated; see Fig. 1-20(f) for the transfer of forces. The bearing stress at the pin is more critical. Bearing between pin C and the clevis:

$$\sigma_b = \frac{F_C}{A_{\text{bearing}}} = \frac{2.83}{0.375 \times 0.20 \times 2} = 18.8 \text{ ksi}$$



Bearing between the pin C and the main plate:

$$\sigma_b = \frac{F_C}{A} = \frac{2.83}{0.375 \times 0.25} = 30.2 \text{ ksi}$$

Double shear in pin C:

$$\tau = \frac{F_C}{A} = \frac{2.83}{2\pi(0.375/2)^2} = 12.9 \text{ ksi}$$

1-11. Deterministic Design of Members: Axially Loaded Bars

allowable and ultimate stresses may be converted into the allowable and ultimate forces or "loads," respectively, that a member can resist. Also a significant ratio may be formed:

$$\frac{\text{ultimate load for a member}}{\text{allowable load for a member}}$$

This is the basic definition of the *factor of safety,* F.S. This ratio must always be greater than unity. Traditionally this factor is recast in terms of stresses as

$$\text{F.S.} = \frac{\text{maximum useful material strength (stress)}}{\text{allowable stress}}$$

and is widely used not only for axially loaded members, but also for any type of member and loading conditions. As will become apparent from subsequent reading, whereas this definition of F.S. in terms of elastic stresses is satisfactory for some cases, it can be misleading in others.

In the aircraft industry, the term *factor of safety* is replaced by another, defined as

$$\frac{\text{ultimate load}}{\text{design load}} - 1$$

The application of the ASD approach for axially loaded members is both simple and direct. From Eq. 1-6, it follows that the required net area A of a member is

$$A = \frac{P}{\sigma_{\text{allow}}} \tag{1-15}$$

where P is the applied axial force and σ_{allow} is the allowable stress. Equation 1-15 is generally applicable to tension members and short compression blocks. *For slender compression members, the question of their stability arises and the methods discussed in Chapter 16 must be used.*

Example 1-5

Reduce the size of bar *AB* in Example 1-3 by using a better material such as chrome-vanadium steel. The ultimate strength of this steel is approximately 120 ksi. Use a factor of safety of 2.5.

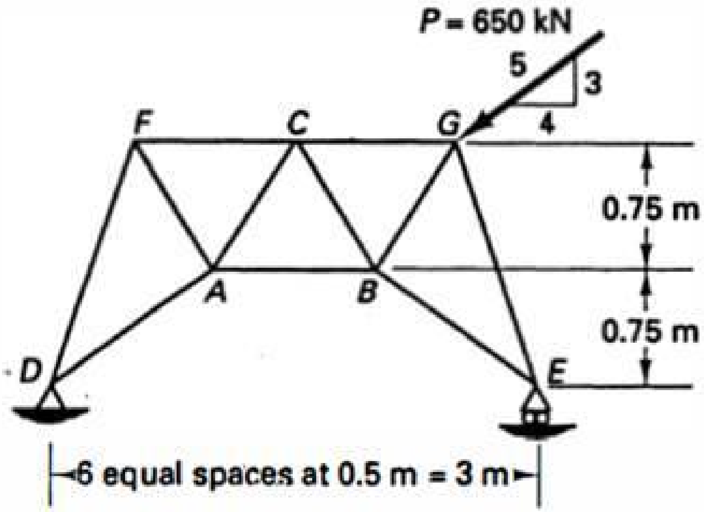
SOLUTION

$\sigma_{allow} = 120/2.5 = 48$ ksi. From Example 1-3, the force in the bar *AB*: $F_A = +2.23$ kips. Required area: $A_{net} = 2.23/48 = 0.0464$ in^2. Adopt: 0.20-in by 0.25-in bar. This provides an area of $(0.20)(0.25) = 0.050$ in^2, which is slightly in excess of the required area. Many other proportions of the bar are possible.

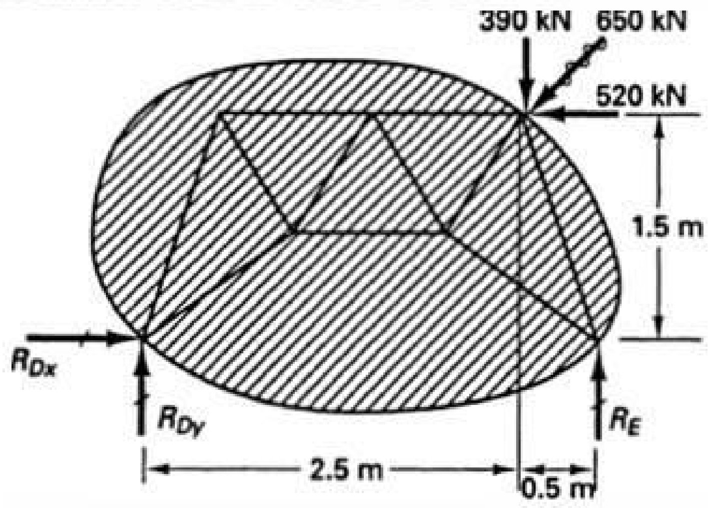
With the cross-sectional area selected, the actual or working stress is somewhat below the allowable stress: $\sigma_{actual} = 2.23/(0.050) = 44.6$ ksi. The actual factor of safety is $120/(44.6) = 2.69$, and the actual margin of safety is 1.69.

Example 1-6

Select members *FC* and *CB* in the truss of Fig. 1-23(a) to carry an inclined force *P* of 650 kN. Set the allowable tensile stress at 140 MPa.



(a)



(b)

Using the free-body diagram in Fig. 1-23(b),

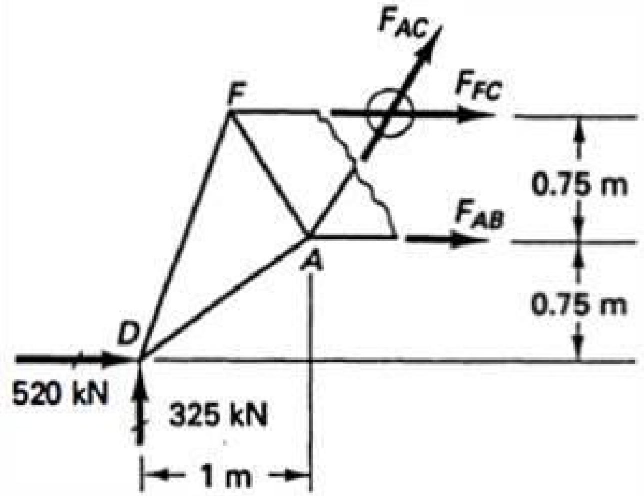
$$\sum F_x = 0 \qquad R_{Dx} - 520 = 0 \qquad R_{Dx} = 520 \text{ kN}$$

$$\sum M_E = 0 \curvearrowleft + \qquad R_{Dy} \times 3 - 390 \times 0.5 - 520 \times 1.5 = 0$$

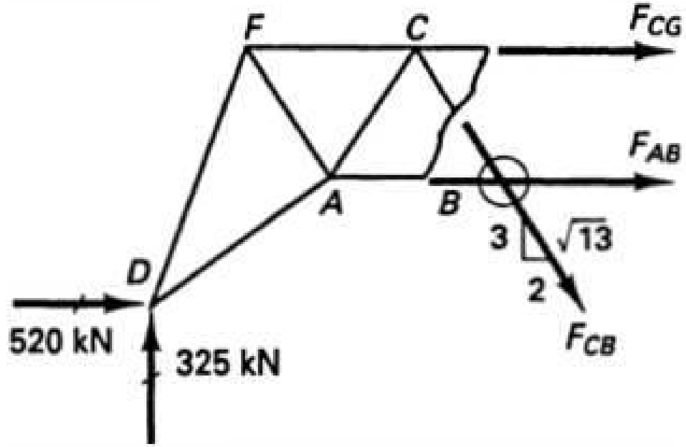
$$R_{Dy} = 325 \text{ kN}$$

$$\sum M_D = 0 \curvearrowleft + \qquad R_E \times 3 + 520 \times 1.5 - 390 \times 2.5 = 0$$

$$R_E = 65 \text{ kN}$$



(c)



(d)

Using the free-body diagram in Fig. 1-23(c),

$$\sum M_A = 0 \curvearrowright + \qquad F_{FC} \times 0.75 + 325 \times 1 - 520 \times 0.75 = 0$$

$$F_{FC} = +86.7 \text{ kN}$$

$$A_{FC} = Ff_{FC}/\sigma_{\text{allow}} = 86.7 \times 10^3/140 = 620 \text{ mm}^2$$

(use 12.5 × 50-mm bar)

Using the free-body diagram in Fig. 1-23(d),

$$\sum F_y = 0 \qquad -(F_{CB})_y + 325 = 0 \qquad (F_{CB})_y = +325 \text{ kN}$$

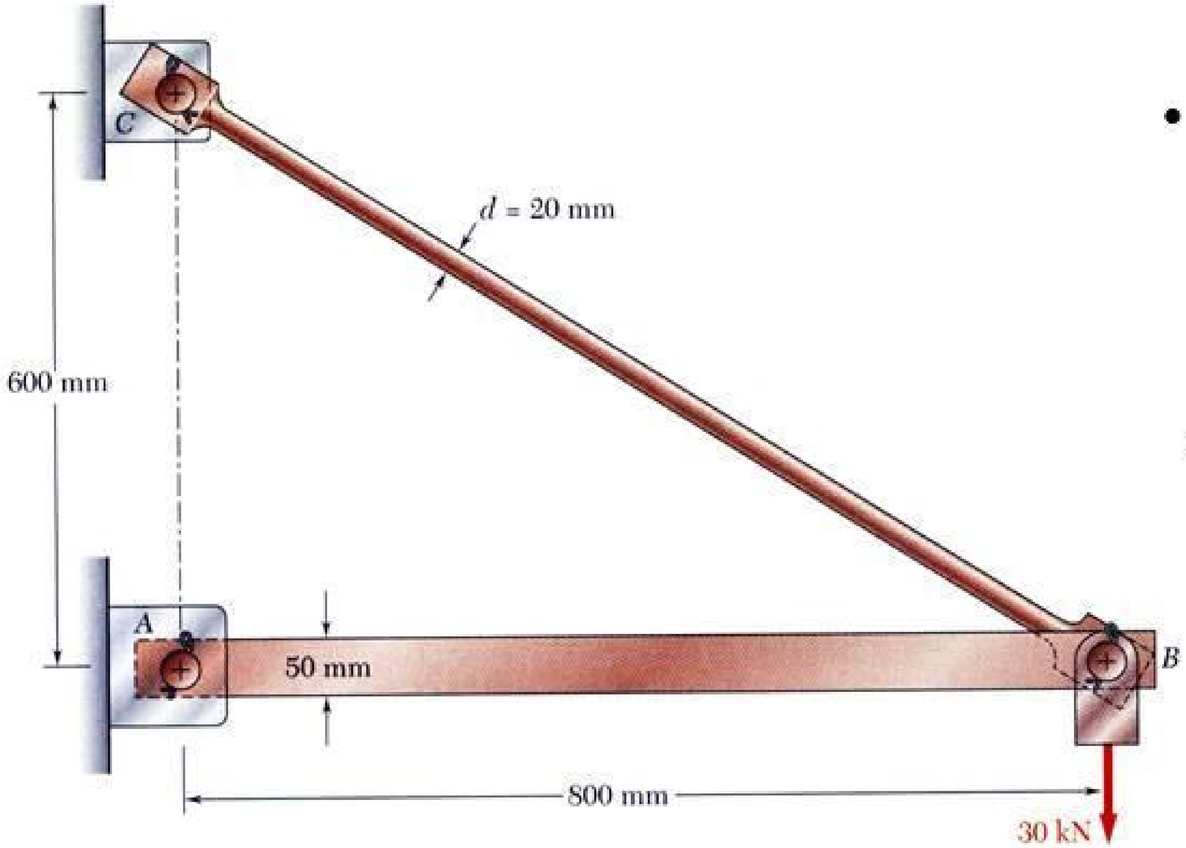
$$F_{CB} = \sqrt{13}(F_{CB})_y/3 = +391 \text{ kN}$$

Problems for Solution from Popov, Chapter-1

Solve following problems in addition to solved problems of different text books mentioned:

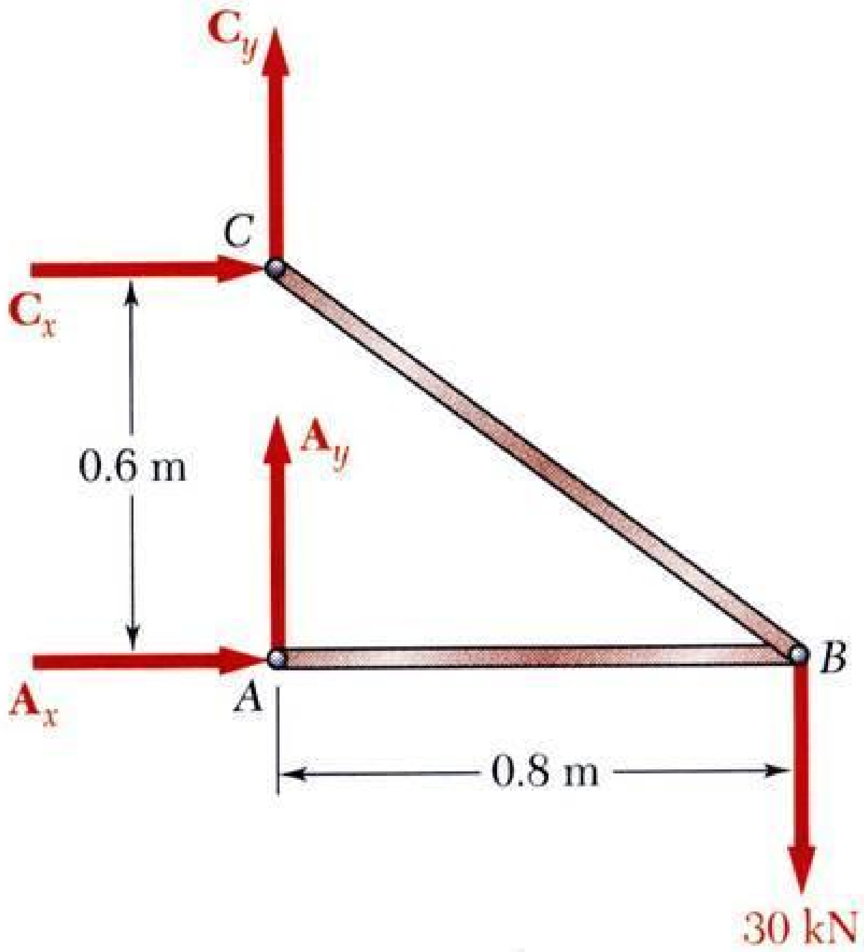
9-12, 23, 27-30, 32-36, 38, 39, 44, 47, 50, 52, 53, 55-58.

Review of Statics



- The structure is designed to support a 30 kN load
- The structure consists of a boom and rod joined by pins (zero moment connections) at the junctions and supports
- Perform a static analysis to determine the internal force in each structural member and the reaction forces at the supports

Structure Free-Body Diagram



- Structure is detached from supports and the loads and reaction forces are indicated
- Conditions for static equilibrium:

$$\sum M_C = 0 = A_x(0.6\,\text{m}) - (30\,\text{kN})(0.8\,\text{m})$$

$$A_x = 40\,\text{kN}$$

$$\sum F_x = 0 = A_x + C_x$$

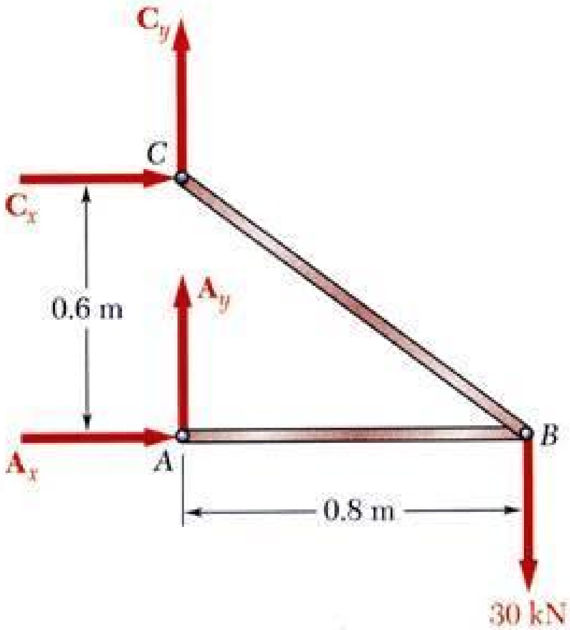
$$C_x = -A_x = -40\,\text{kN}$$

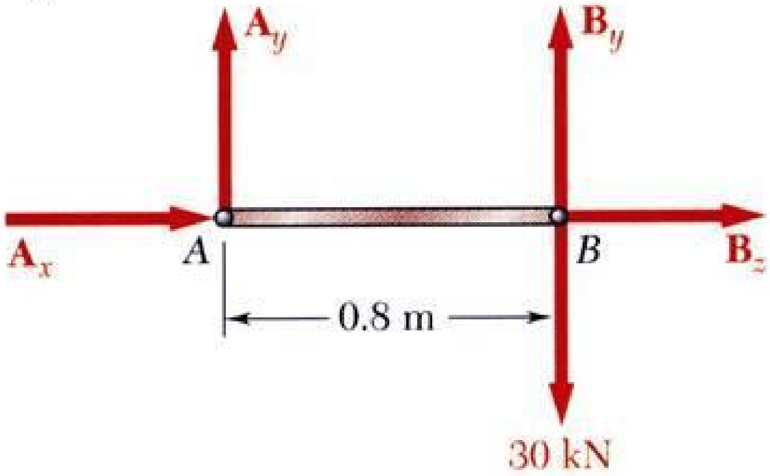
$$\sum F_y = 0 = A_y + C_y - 30\,\text{kN} = 0$$

$$A_y + C_y = 30\,\text{kN}$$

- A_y and C_y can not be determined from these equations

Component Free-Body Diagram





- In addition to the complete structure, each component must satisfy the conditions for static equilibrium
- Consider a free-body diagram for the boom:

  $$\sum M_B = 0 = -A_y(0.8\,\text{m})$$

  $$A_y = 0$$

  substitute into the structure equilibrium equation

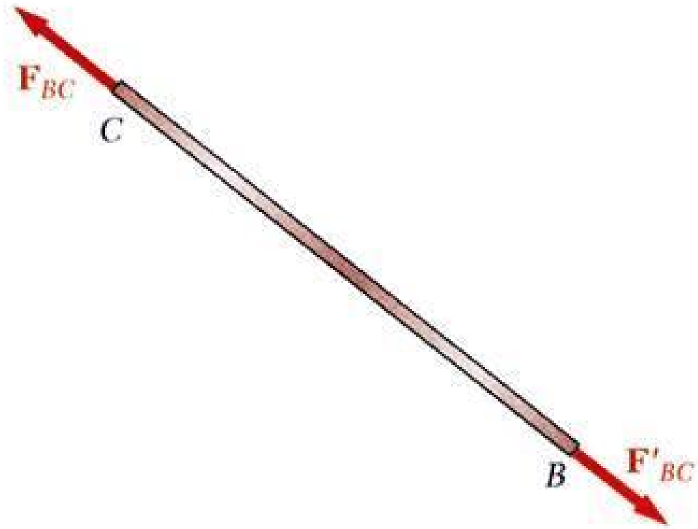
  $$C_y = 30\,\text{kN}$$

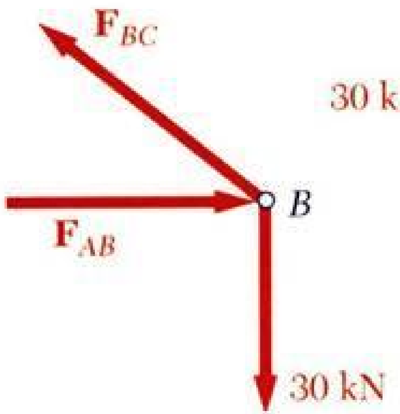
- Results:

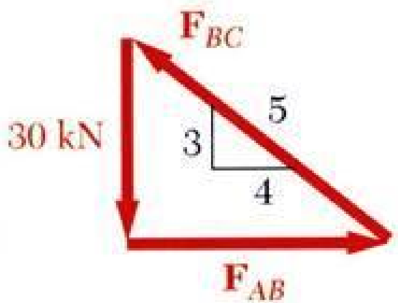
  $$A = 40\,\text{kN} \rightarrow \quad C_x = 40\,\text{kN} \leftarrow \quad C_y = 30\,\text{kN} \uparrow$$

  Reaction forces are directed along boom and rod

Method of Joints







- The boom and rod are 2-force members, i.e., the members are subjected to only two forces which are applied at member ends

- For equilibrium, the forces must be parallel to to an axis between the force application points, equal in magnitude, and in opposite directions

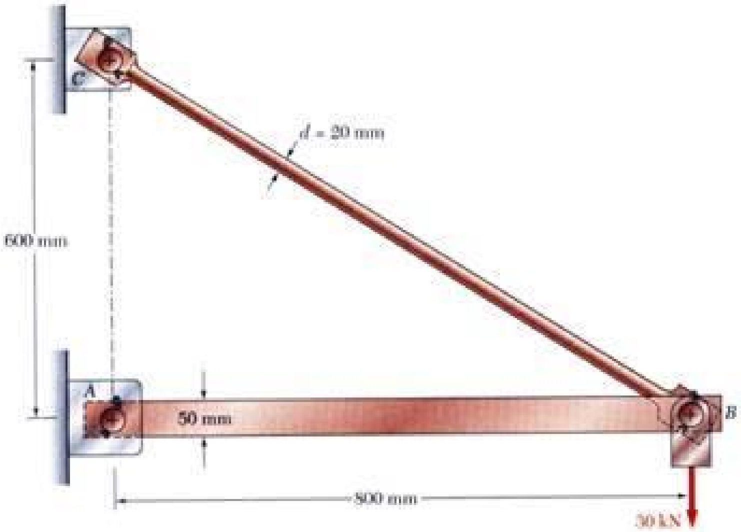
- Joints must satisfy the conditions for static equilibrium which may be expressed in the form of a force triangle:

$$\sum \vec{F}_B = 0$$

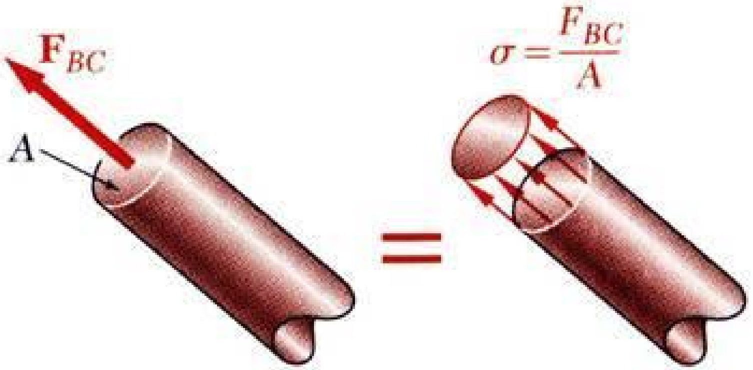
$$\frac{F_{AB}}{4} = \frac{F_{BC}}{5} = \frac{30\,\text{kN}}{3}$$

$$F_{AB} = 40\,\text{kN} \qquad F_{BC} = 50\,\text{kN}$$

Stress Analysis



$d_{BC} = 20 \text{ mm}$



Can the structure safely support the 30 kN load?

- From a statics analysis

  $F_{AB} = 40 \text{ kN (compression)}$
  $F_{BC} = 50 \text{ kN (tension)}$

- At any section through member BC, the internal force is 50 kN with a force intensity or stress of

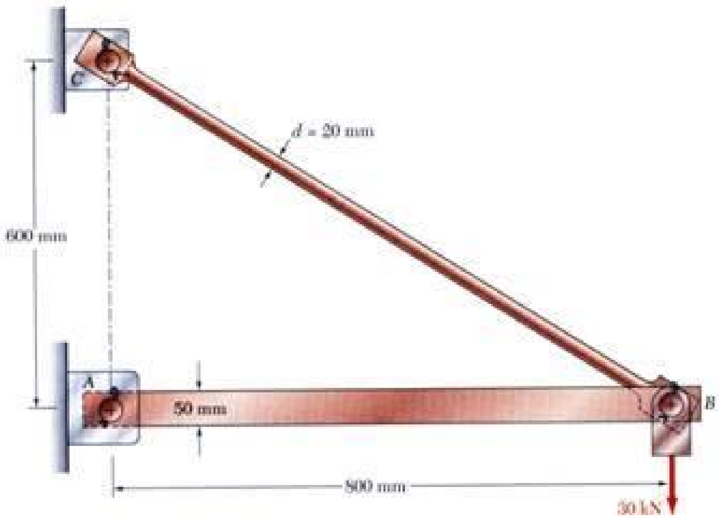
$$\sigma_{BC} = \frac{P}{A} = \frac{50\times10^{3}\,\text{N}}{314\times10^{-6}\,\text{m}^2} = 159\text{ MPa}$$

- From the material properties for steel, the allowable stress is

$$\sigma_{\text{all}} = 165\text{ MPa}$$

- Conclusion: the strength of member *BC* is adequate

Design



- Design of new structures requires selection of appropriate materials and component dimensions to meet performance requirements
- For reasons based on cost, weight, availability, etc., the choice is made to construct the rod from aluminum (σ_{all}= 100 MPa). What is an appropriate choice for the rod diameter?

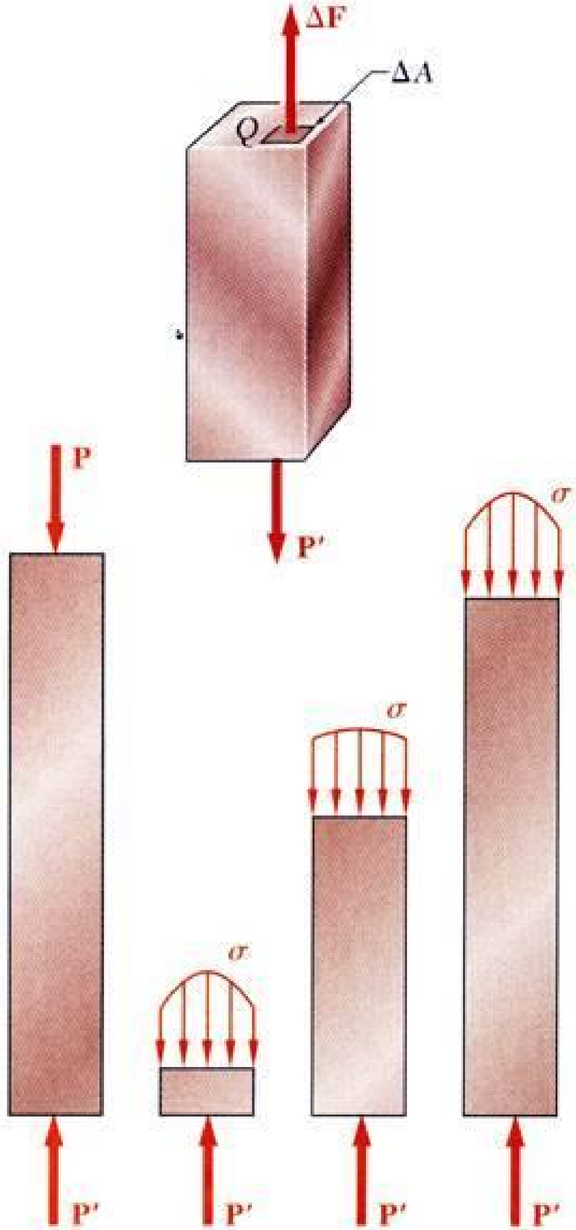
$$\sigma_{all} = \frac{P}{A} \qquad A = \frac{P}{\sigma_{all}} = \frac{50 \times 10^3 \,\text{N}}{100 \times 10^6 \,\text{Pa}} = 500 \times 10^{-6} \,\text{m}^2$$

$$A = \pi \frac{d^2}{4}$$

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4\left(500 \times 10^{-6} \,\text{m}^2\right)}{\pi}} = 2.52 \times 10^{-2} \,\text{m} = 25.2 \,\text{mm}$$

- An aluminum rod 26 mm or more in diameter is adequate

Axial Loading: Normal Stress



- The resultant of the internal forces for an axially loaded member is *normal* to a section cut perpendicular to the member axis.

- The force intensity on that section is defined as the normal stress.

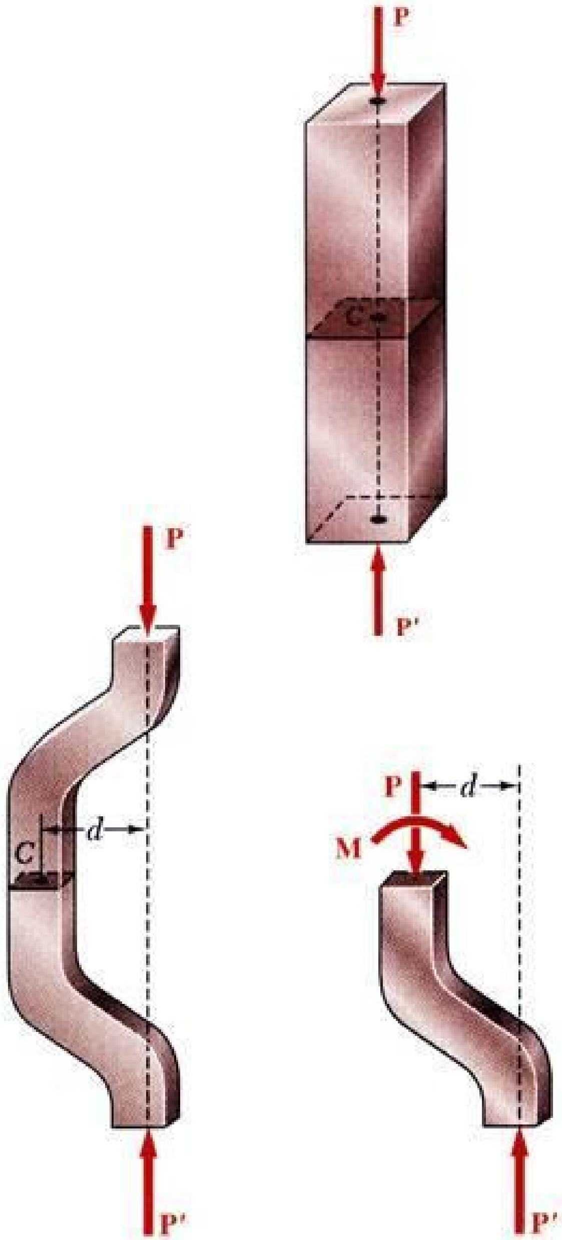
$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \qquad \sigma_{ave} = \frac{P}{A}$$

- The normal stress at a particular point may not be equal to the average stress but the resultant of the stress distribution must satisfy

$$P = \sigma_{ave} A = \int dF = \int_A \sigma \, dA$$

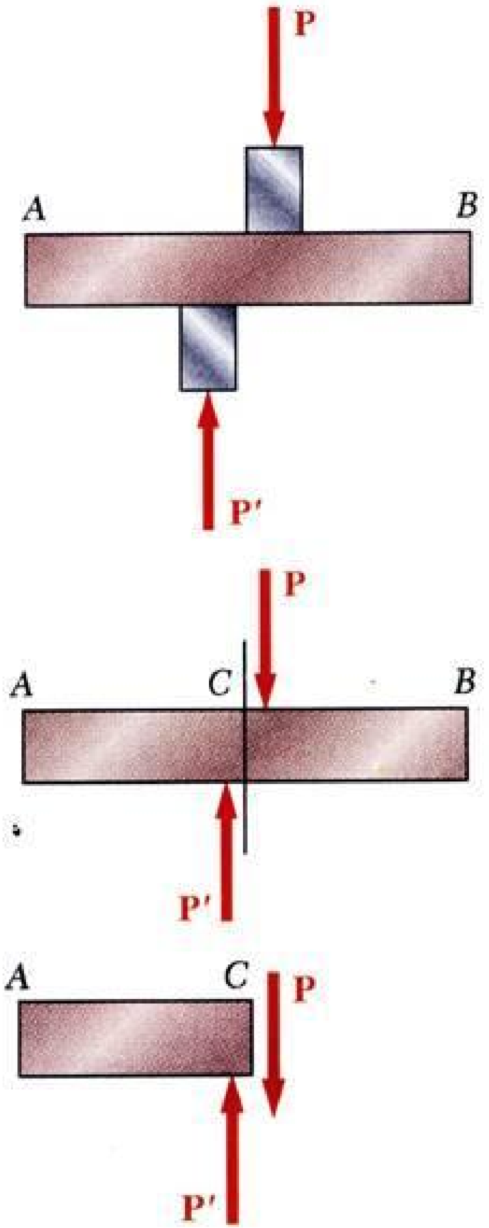
- The detailed distribution of stress is statically indeterminate, i.e., can not be found from statics alone.

Centric & Eccentric Loading



- A uniform distribution of stress in a section infers that the line of action for the resultant of the internal forces passes through the centroid of the section.
- A uniform distribution of stress is only possible if the concentrated loads on the end sections of two-force members are applied at the section centroids. This is referred to as *centric loading*.
- If a two-force member is *eccentrically loaded*, then the resultant of the stress distribution in a section must yield an axial force and a moment.
- The stress distributions in eccentrically loaded members cannot be uniform or symmetric.

Shearing Stress



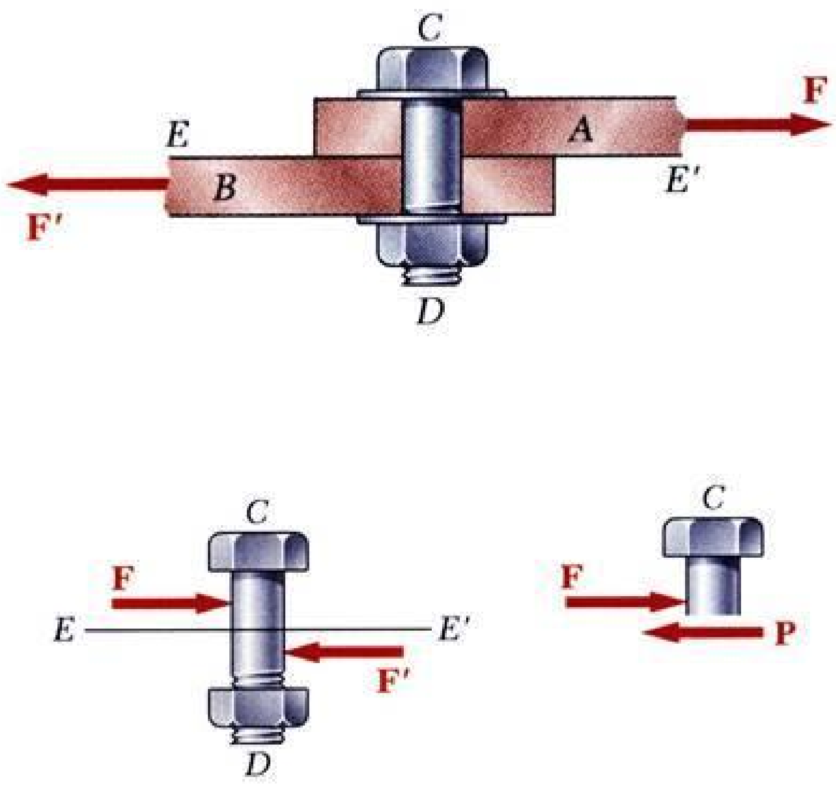
- Forces *P* and *P'* are applied transversely to the member *AB*.
- Corresponding internal forces act in the plane of section *C* and are called *shearing* forces.
- The resultant of the internal shear force distribution is defined as the *shear* of the section and is equal to the load *P*.
- The corresponding average shear stress is,

$$\tau_{\text{ave}} = \frac{P}{A}$$

- Shear stress distribution varies from zero at the member surfaces to maximum values that may be much larger than the average value.
- The shear stress distribution cannot be assumed to be uniform.

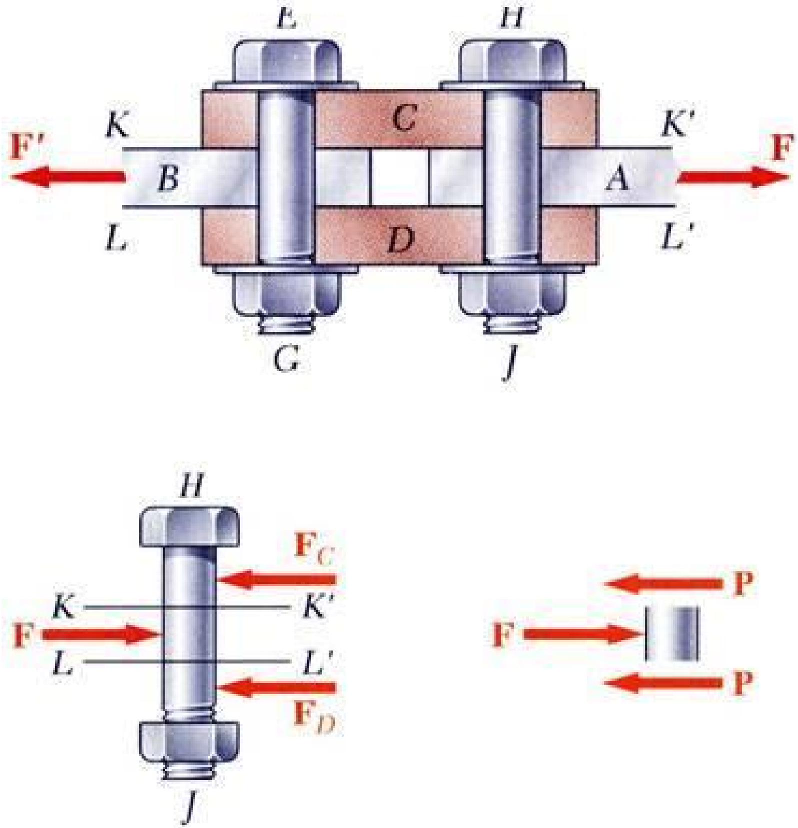
Shearing Stress Examples

Single Shear



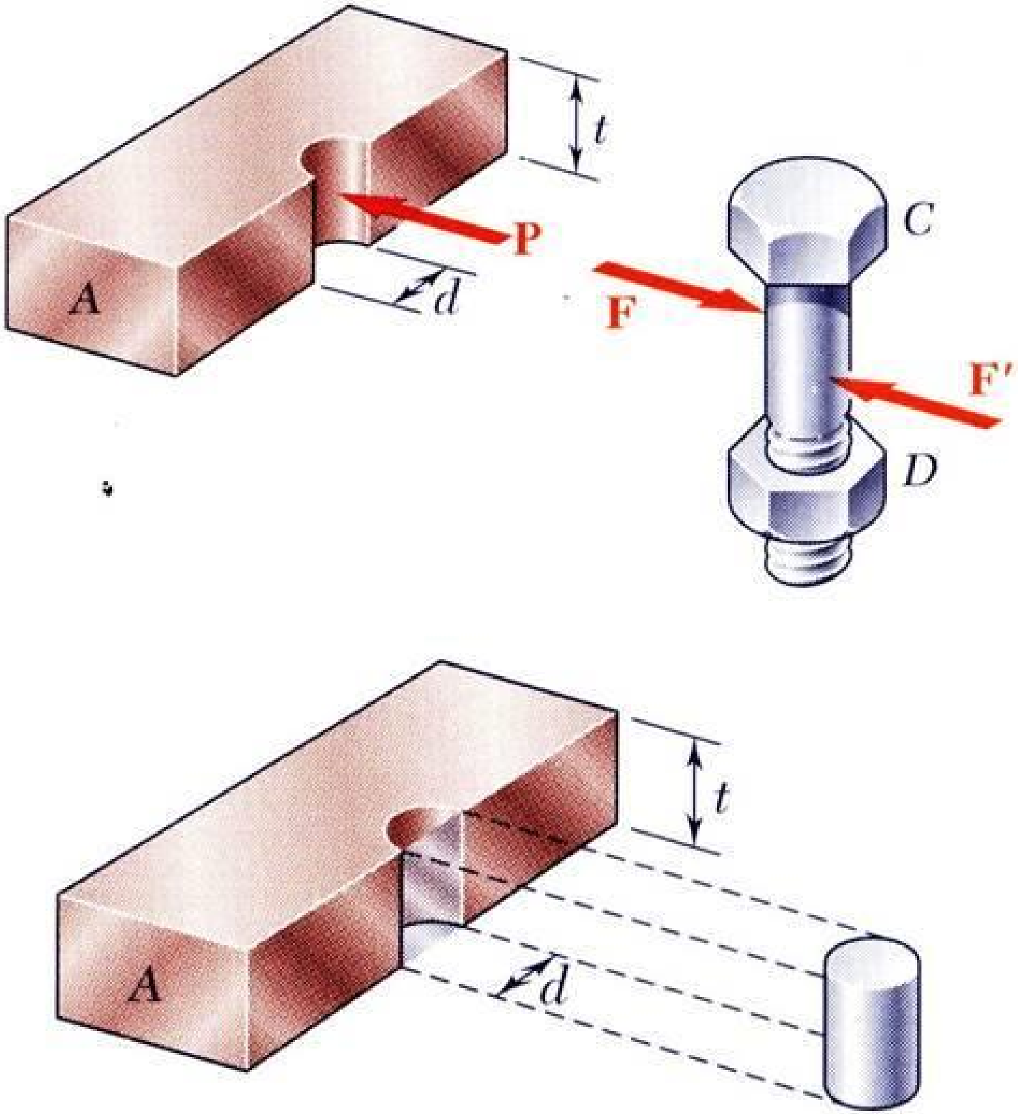
$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A}$$

Double Shear



$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{2A}$$

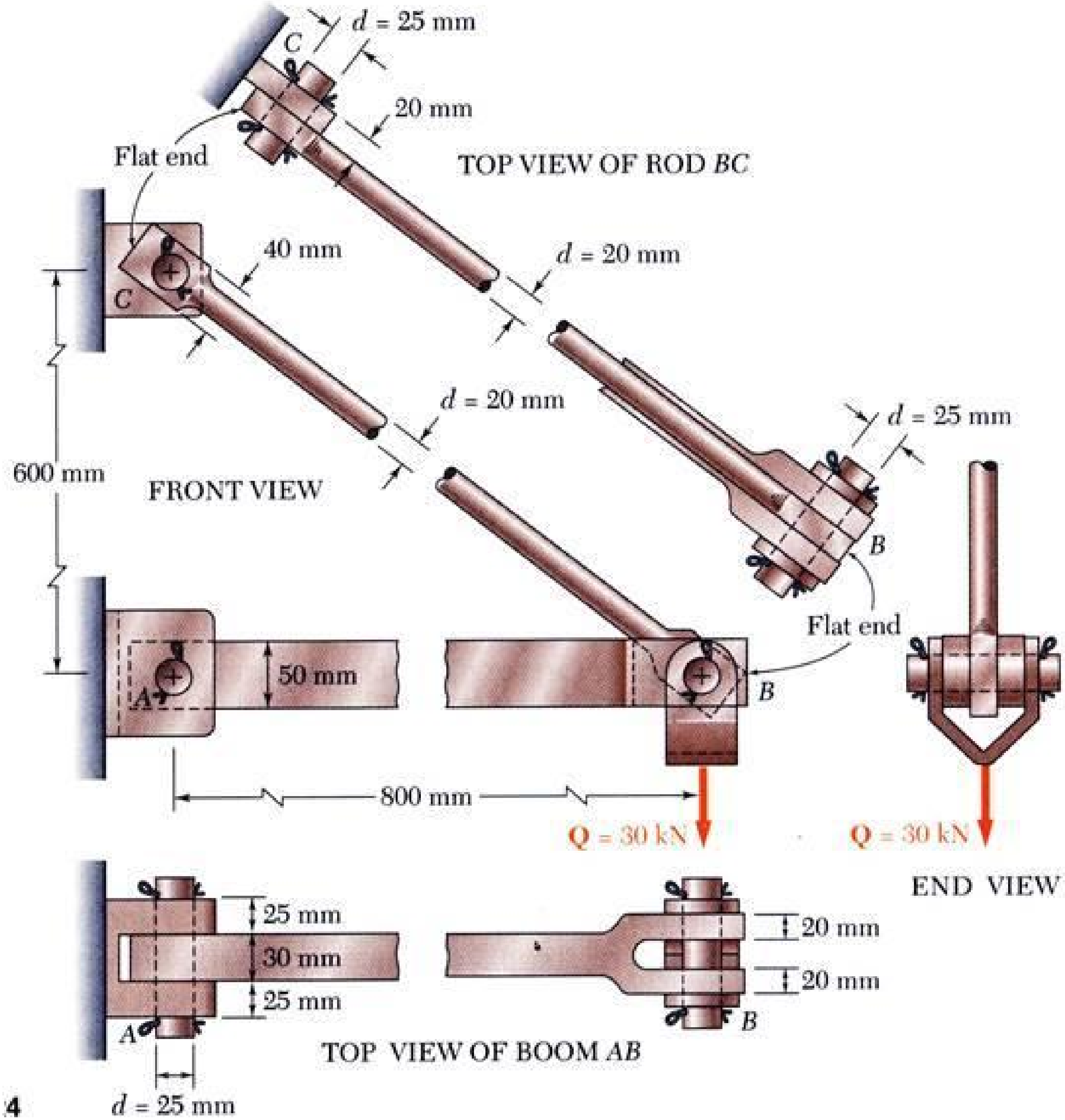
Bearing Stress in Connections



- Bolts, rivets, and pins create stresses on the points of contact or *bearing surfaces* of the members they connect.
- The resultant of the force distribution on the surface is equal and opposite to the force exerted on the pin.
- Corresponding average force intensity is called the bearing stress,

$$\sigma_b = \frac{P}{A} = \frac{P}{t\,d}$$

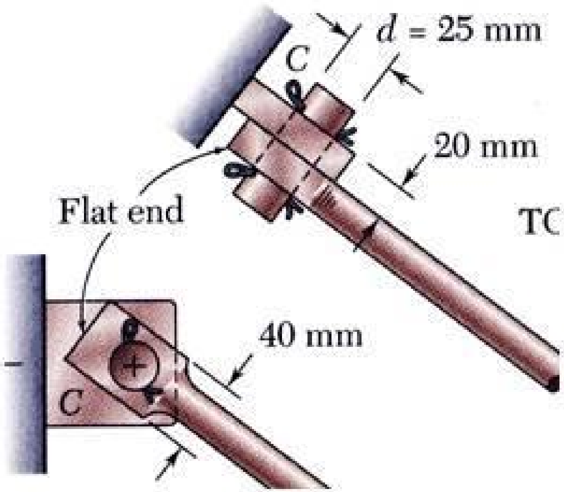
Stress Analysis & Design Example



- Would like to determine the stresses in the members and connections of the structure shown.
- From a statics analysis:
  $F_{AB} = 40$ kN (compression)
  $F_{BC} = 50$ kN (tension)
- Must consider maximum normal stresses in *AB* and *BC*, and the shearing stress and bearing stress at each pinned connection

4

Rod & Boom Normal Stresses



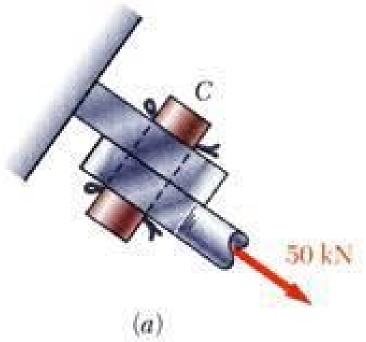
- The rod is in tension with an axial force of 50 kN.
- At the rod center, the average normal stress in the circular cross-section ($A = 314\text{x}10^{-6}\text{m}^2$) is $\sigma_{BC} = +159$ MPa.
- At the flattened rod ends, the smallest cross-sectional area occurs at the pin centerline,

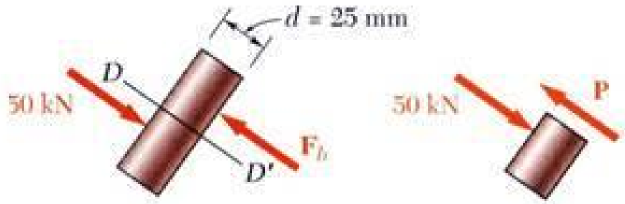
$$A = (20\text{ mm})(40\text{ mm} - 25\text{ mm}) = 300 \times 10^{-6}\text{ m}^2$$

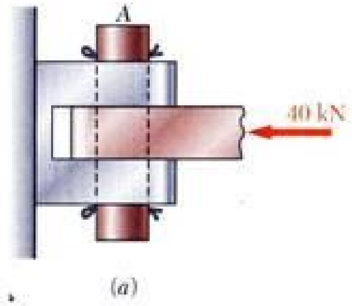
$$\sigma_{BC,end} = \frac{P}{A} = \frac{50 \times 10^3 N}{300 \times 10^{-6}\text{ m}^2} = 167\text{ MPa}$$

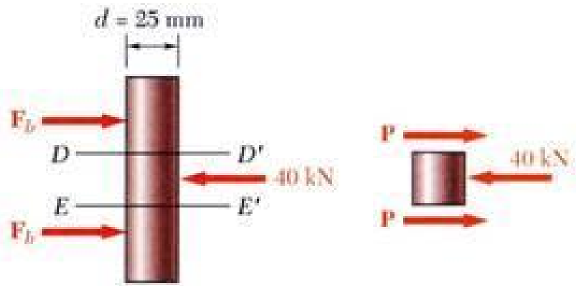
- The boom is in compression with an axial force of 40 kN and average normal stress of –26.7 MPa.
- The minimum area sections at the boom ends are unstressed since the boom is in compression.

Pin Shearing Stresses









- The cross-sectional area for pins at A, B, and C,

$$A = \pi r^2 = \pi\left(\frac{25\,\text{mm}}{2}\right)^2 = 491\times10^{-6}\,\text{m}^2$$

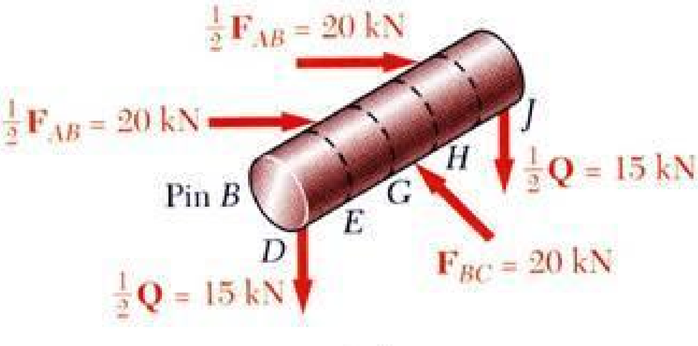
- The force on the pin at C is equal to the force exerted by the rod BC,

$$\tau_{C,ave} = \frac{P}{A} = \frac{50\times10^3\,\text{N}}{491\times10^{-6}\,\text{m}^2} = 102\,\text{MPa}$$

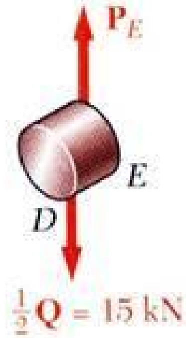
- The pin at A is in double shear with a total force equal to the force exerted by the boom AB,

$$\tau_{A,ave} = \frac{P}{A} = \frac{20\,\text{kN}}{491\times10^{-6}\,\text{m}^2} = 40.7\,\text{MPa}$$

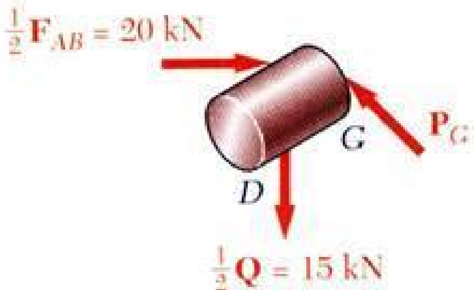
Pin Shearing Stresses



(a)



(b)



- Divide the pin at B into sections to determine the section with the largest shear force,

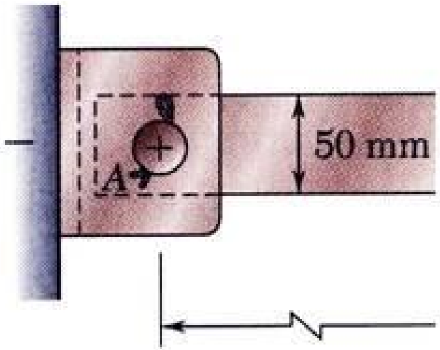
$$P_E = 15\,\text{kN}$$

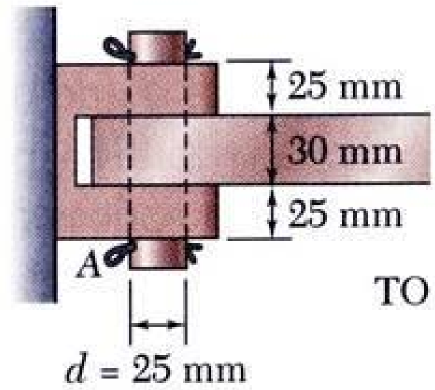
$$P_G = 25\,\text{kN} \quad \text{(largest)}$$

- Evaluate the corresponding average shearing stress,

$$\tau_{B,ave} = \frac{P_G}{A} = \frac{25\,\text{kN}}{491\times10^{-6}\,\text{m}^2} = 50.9\,\text{MPa}$$

Pin Bearing Stresses





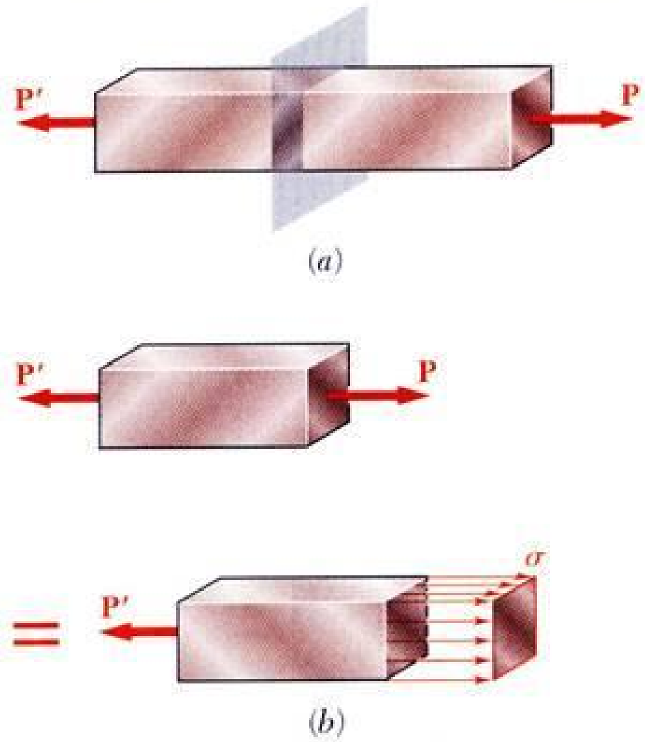
- To determine the bearing stress at *A* in the boom *AB*, we have $t = 30$ mm and $d = 25$ mm,

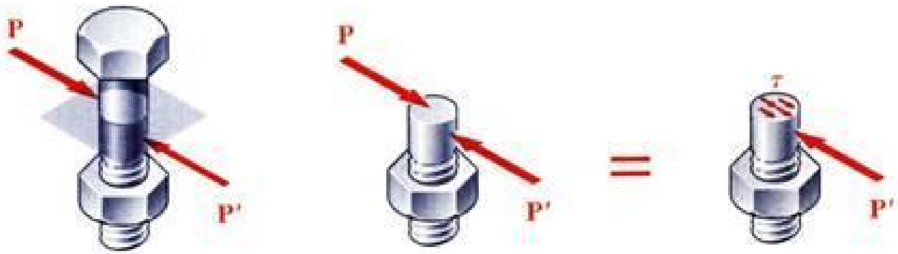
$$\sigma_b = \frac{P}{td} = \frac{40\,\text{kN}}{(30\,\text{mm})(25\,\text{mm})} = 53.3\,\text{MPa}$$

- To determine the bearing stress at *A* in the bracket, we have $t = 2(25\text{ mm}) = 50$ mm and $d = 25$ mm,

$$\sigma_b = \frac{P}{td} = \frac{40\,\text{kN}}{(50\,\text{mm})(25\,\text{mm})} = 32.0\,\text{MPa}$$

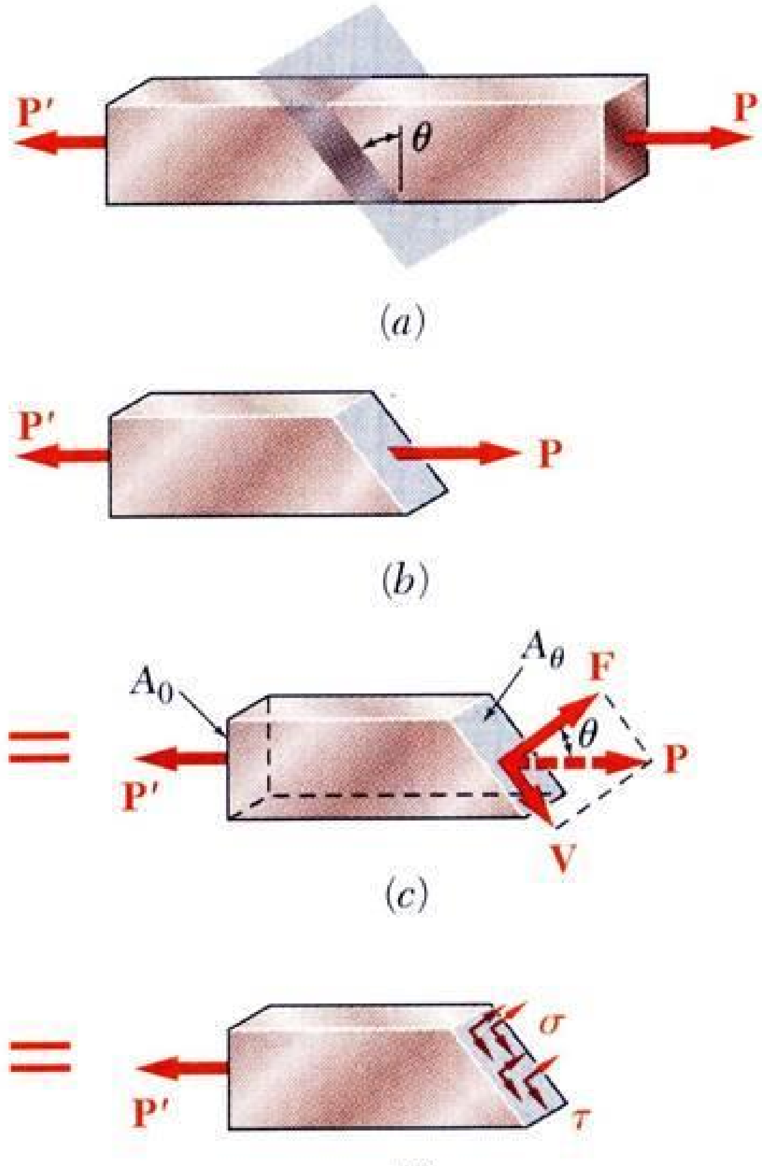
Stress in Two Force Members





- Axial forces on a two force member result in only normal stresses on a plane cut perpendicular to the member axis.
- Transverse forces on bolts and pins result in only shear stresses on the plane perpendicular to bolt or pin axis.
- Will show that either axial or transverse forces may produce both normal and shear stresses with respect to a plane other than one cut perpendicular to the member axis.

Stress on an Oblique Plane



(a) (b) (c) (d)

- Pass a section through the member forming an angle θ with the normal plane.
- From equilibrium conditions, the distributed forces (stresses) on the plane must be equivalent to the force P.
- Resolve P into components normal and tangential to the oblique section,

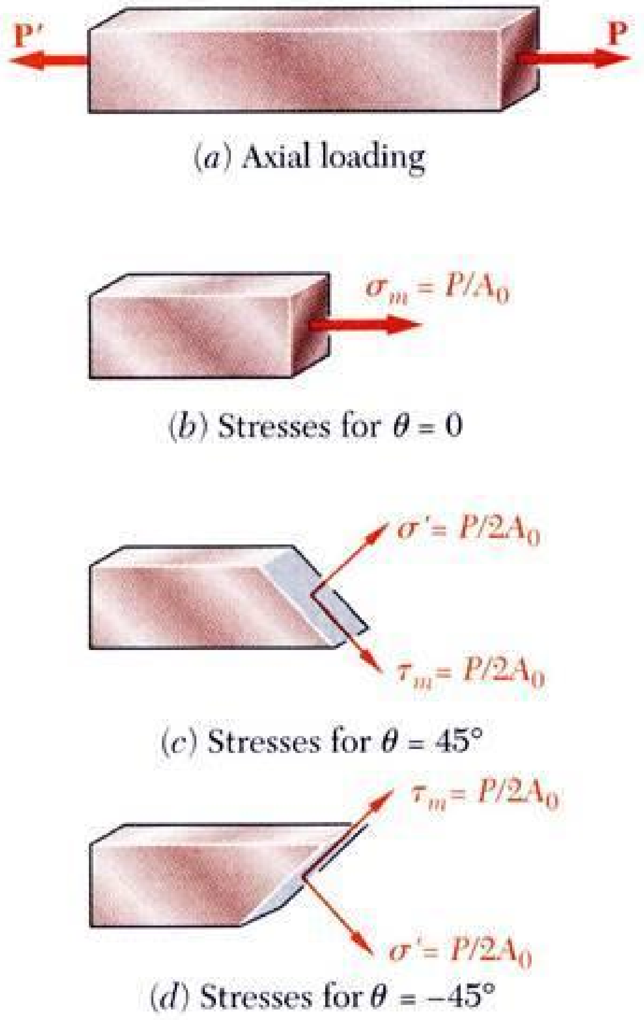
$$F = P\cos\theta \qquad V = P\sin\theta$$

- The average normal and shear stresses on the oblique plane are

$$\sigma = \frac{F}{A_\theta} = \frac{P\cos\theta}{A_0/\cos\theta} = \frac{P}{A_0}\cos^2\theta$$

$$\tau = \frac{V}{A_\theta} = \frac{P\sin\theta}{A_0/\cos\theta} = \frac{P}{A_0}\sin\theta\cos\theta$$

Maximum Stresses



- Normal and shearing stresses on an oblique plane

$$\sigma = \frac{P}{A_0}\cos^2\theta \quad \tau = \frac{P}{A_0}\sin\theta\cos\theta$$

- The maximum normal stress occurs when the reference plane is perpendicular to the member axis,

$$\sigma_{\mathrm{m}} = \frac{P}{A_0} \quad \tau' = 0$$

- The maximum shear stress occurs for a plane at $\pm$ 45° with respect to the axis,

$$\tau_m = \frac{P}{A_0}\sin 45\,\cos 45 = \frac{P}{2A_0} = \sigma'$$

Factor of Safety

Structural members or machines must be designed such that the working stresses are less than the ultimate strength of the material.

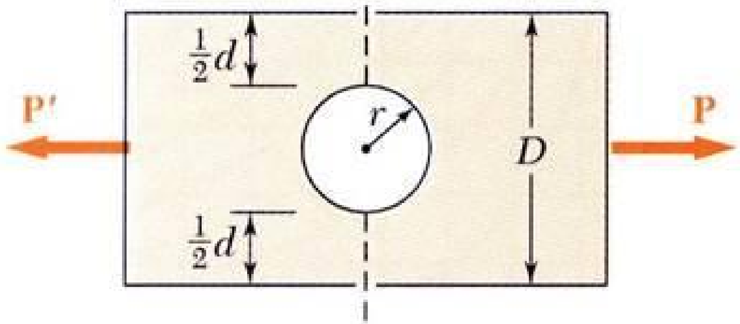
$$FS = \text{Factor of safety}$$

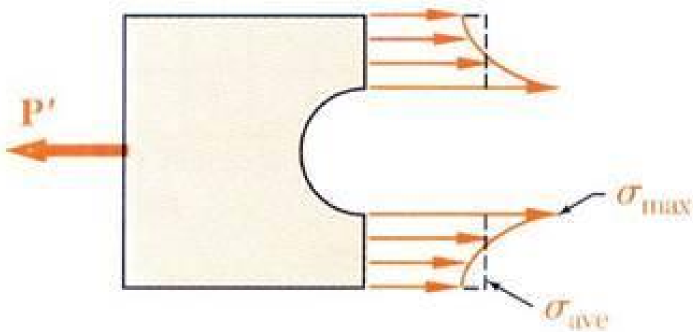
$$FS = \frac{\sigma_u}{\sigma_{all}} = \frac{\text{ultimate stress}}{\text{allowable stress}}$$

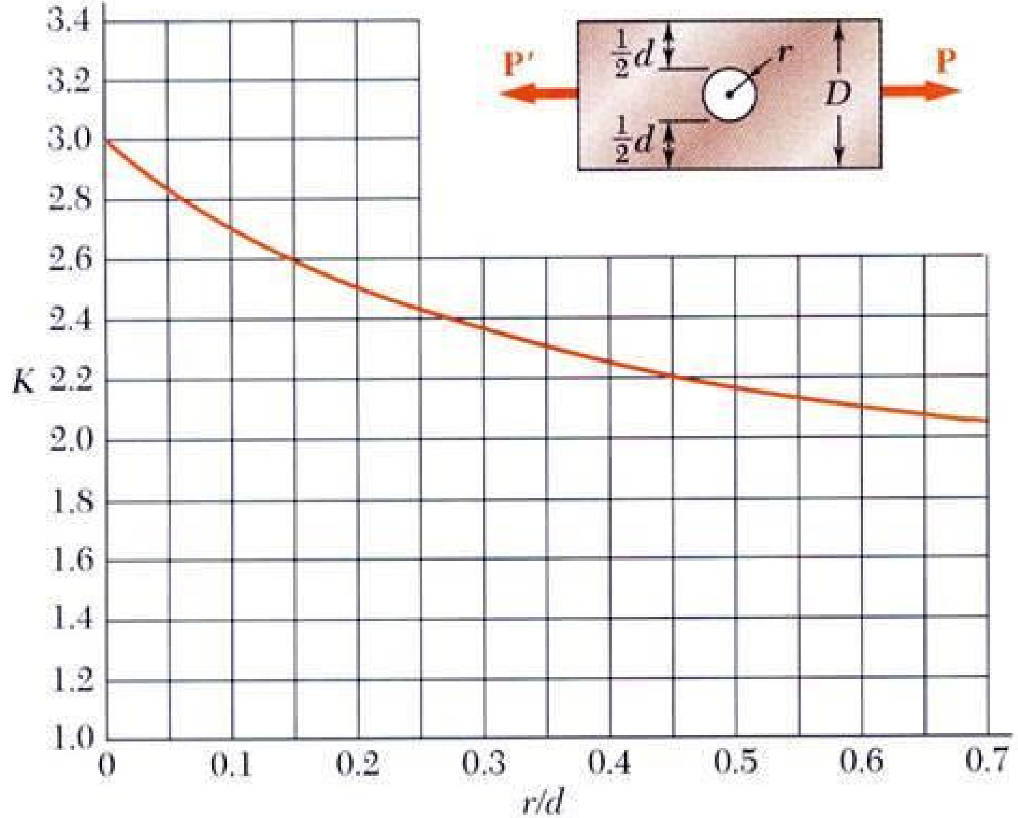
Factor of safety considerations:

- uncertainty in material properties
- uncertainty of loadings
- uncertainty of analyses
- number of loading cycles
- types of failure
- maintenance requirements and deterioration effects
- importance of member to structures integrity
- risk to life and property
- influence on machine function

Stress Concentration: Hole





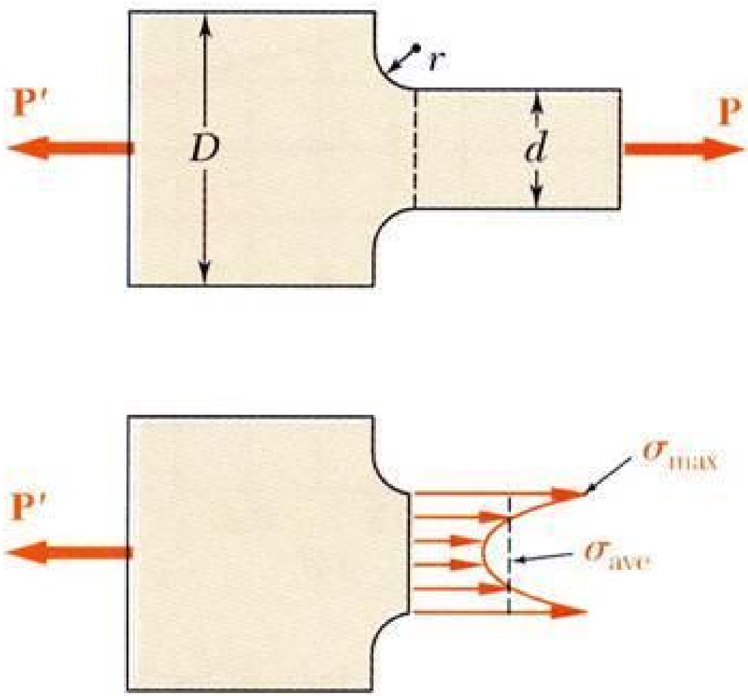


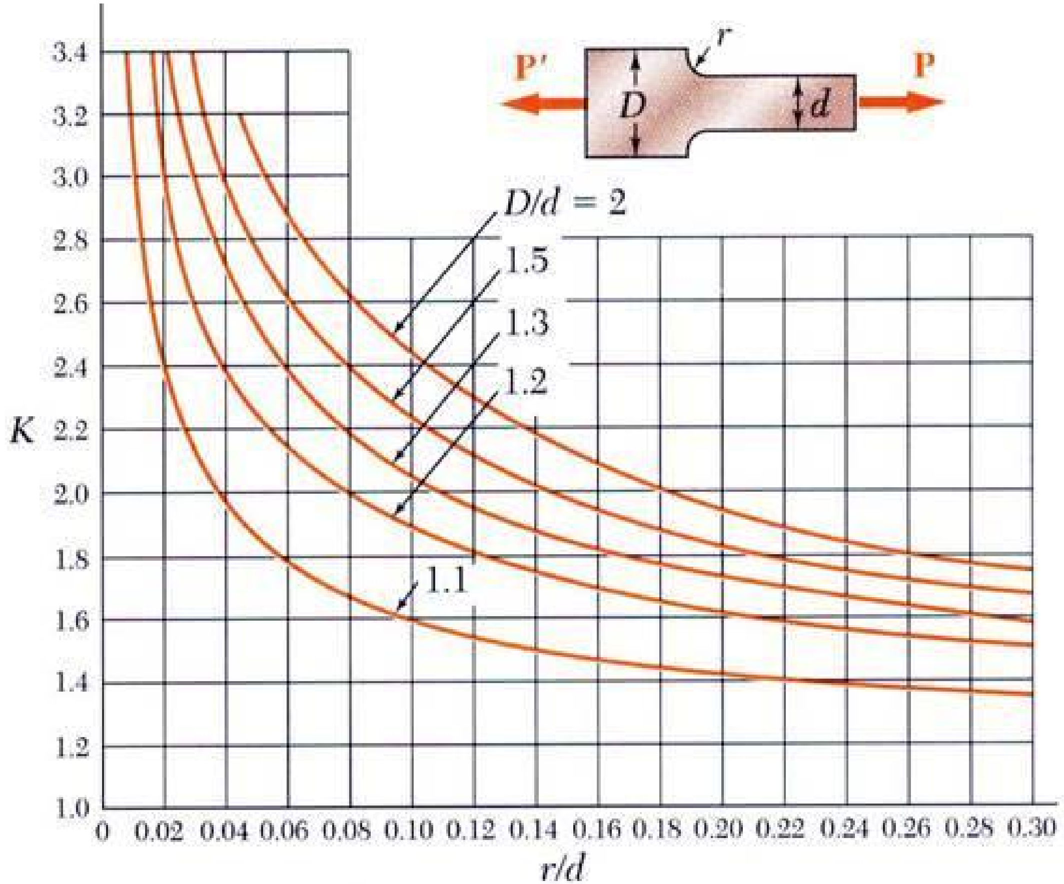
(*a*) Flat bars with holes

Discontinuities of cross section may result in high localized or *concentrated* stresses.

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

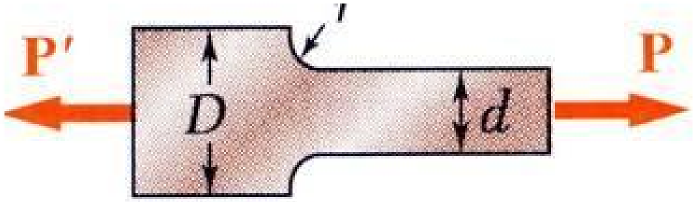
Stress Concentration: Fillet





(*b*) Flat bars with fillets

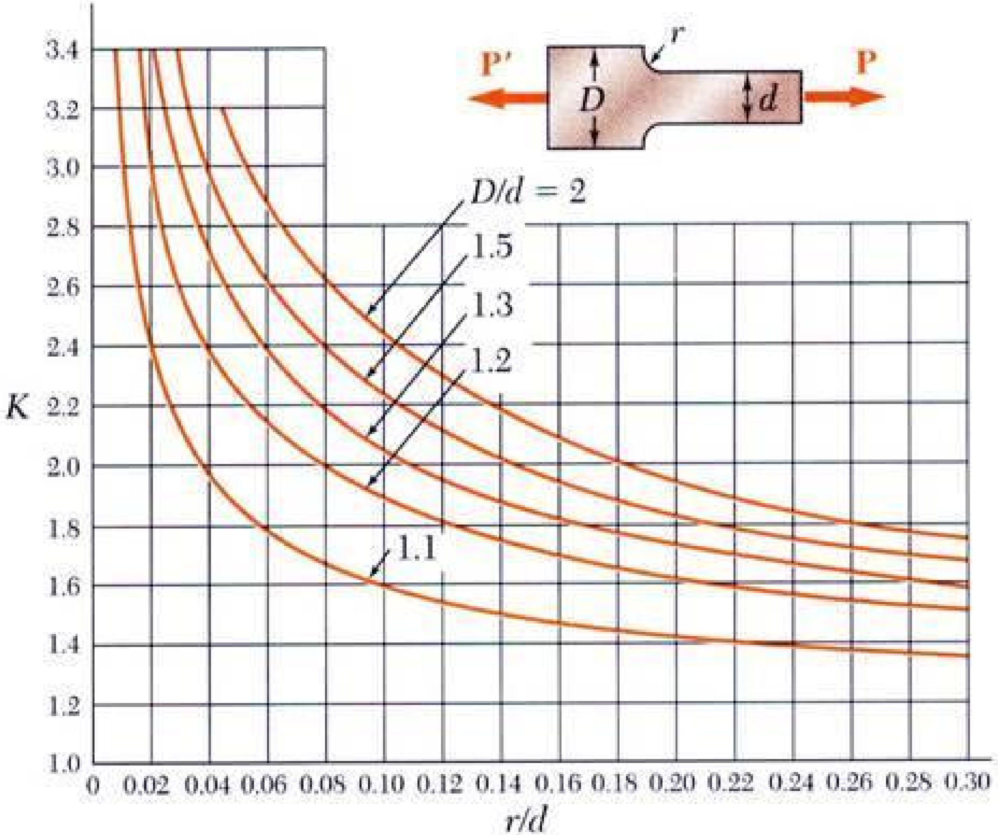
Example 2.12



Determine the largest axial load P that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius $r = 8$ mm. Assume an allowable normal stress of 165 MPa.

SOLUTION:

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.
- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.
- Apply the definition of normal stress to find the allowable load.



(*b*) Flat bars with fillets

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.

$$\frac{D}{d}=\frac{60\,\text{mm}}{40\,\text{mm}}=1.50 \qquad \frac{r}{d}=\frac{8\,\text{mm}}{40\,\text{mm}}=0.20$$

$$K=1.82$$

- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

$$\sigma_{ave}=\frac{\sigma_{max}}{K}=\frac{165\,\text{MPa}}{1.82}=90.7\,\text{MPa}$$

- Apply the definition of normal stress to find the allowable load.

$$P=A\sigma_{ave}=(40\,\text{mm})(10\,\text{mm})(90.7\,\text{MPa})$$

$$=36.3\times10^{3}\,\text{N}$$

$$\boxed{P=36.3\,\text{kN}}$$

CHAPTER

2
3
4
5

ENGINEERING MECHANICS OF SOLIDS

Strain

Stress & Strain: Axial Loading

- Suitability of a structure or machine may depend on the deformations in the structure as well as the stresses induced under loading. Statics analyses alone are not sufficient.
- Considering structures as deformable allows determination of member forces and reactions which are statically indeterminate.
- Determination of the stress distribution within a member also requires consideration of deformations in the member.
- This Chapter is concerned with deformation of a structural member under axial loading. Later chapters will deal with torsional and pure bending loads.

Strain (Linear): Strain (Linear) is defined as the deformation (elongation or contraction) per unit length at any point. To designate strain notation ε is used.

Normal Strain

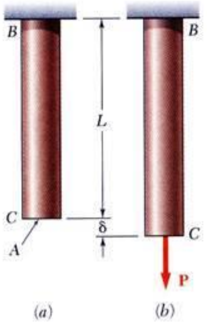


Fig. 2.1

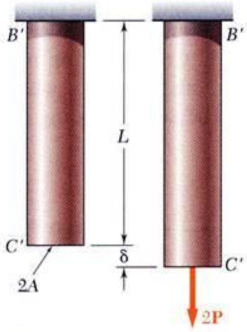


Fig. 2.3

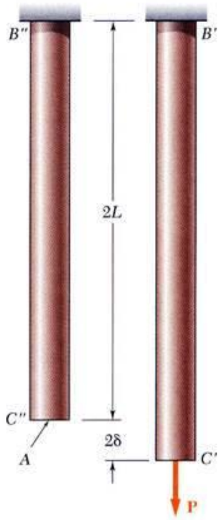


Fig. 2.4

$$\sigma = \frac{P}{A} = \text{stress}$$
$$\varepsilon = \frac{\delta}{L} = \text{normal strain}$$

$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$
$$\varepsilon = \frac{\delta}{L}$$

$$\sigma = \frac{P}{A}$$
$$\varepsilon = \frac{2\delta}{2L} = \frac{\delta}{L}$$

Stress-Strain Test

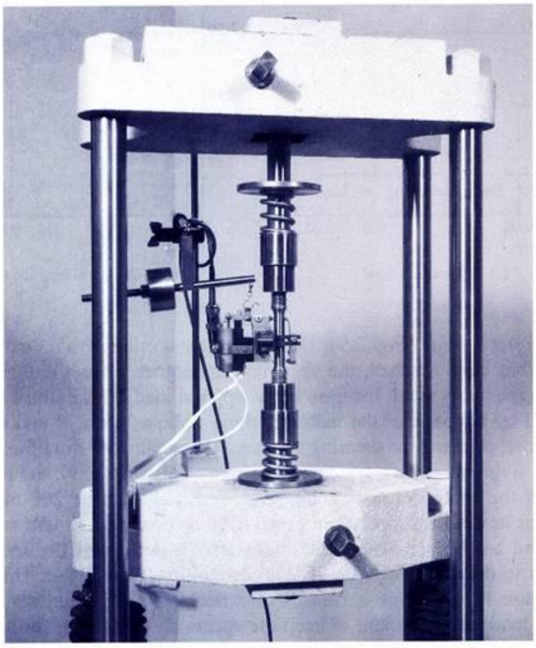


Fig. 2.7 This machine is used to test tensile test specimens, such as those shown in this chapter.

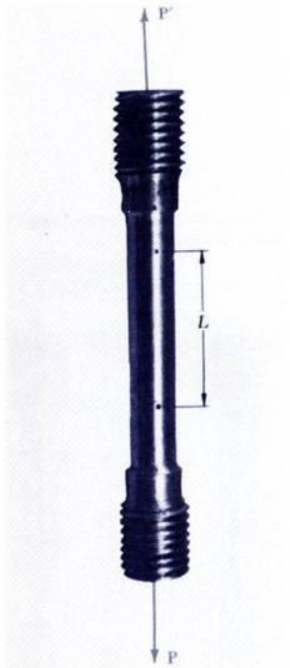
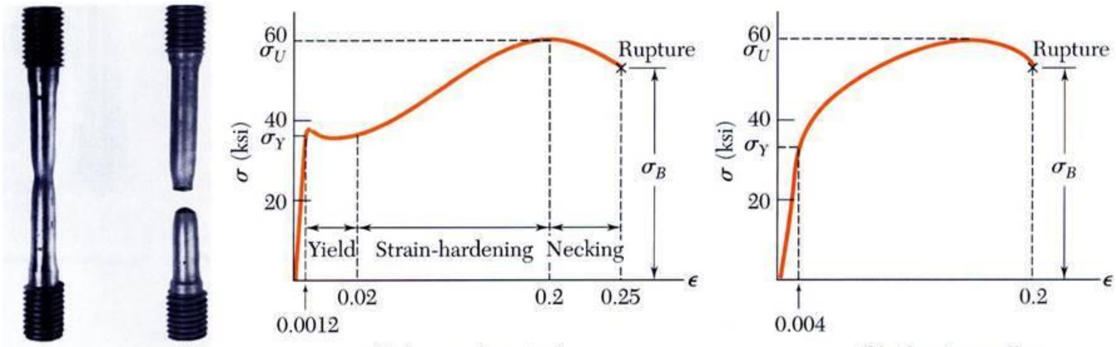


Fig. 2.8 Test specimen with tensile load.

Stress-Strain Diagram: Ductile Materials



(*a*) Low-carbon steel (*b*) Aluminum alloy

Stress-Strain Diagram for mild steel

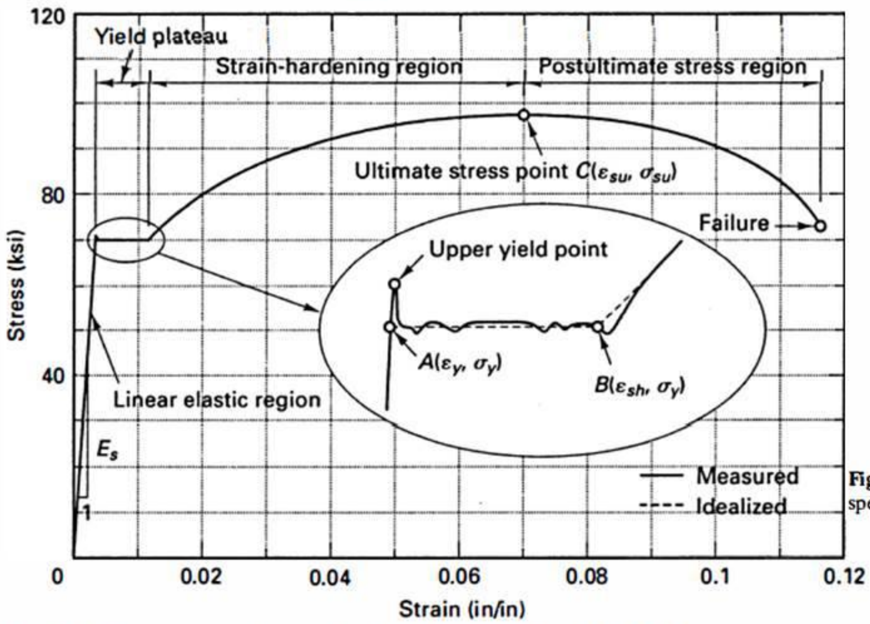


Fig. 2-5 Stress-strain diagram for ductile steel. (After Ref. 11.)

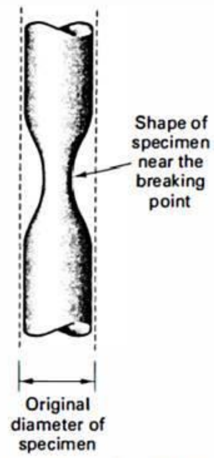


Fig. 2-6 Necking of ductile steel specimen.

Engineering and True Stress-Strain diagrams for mild steel

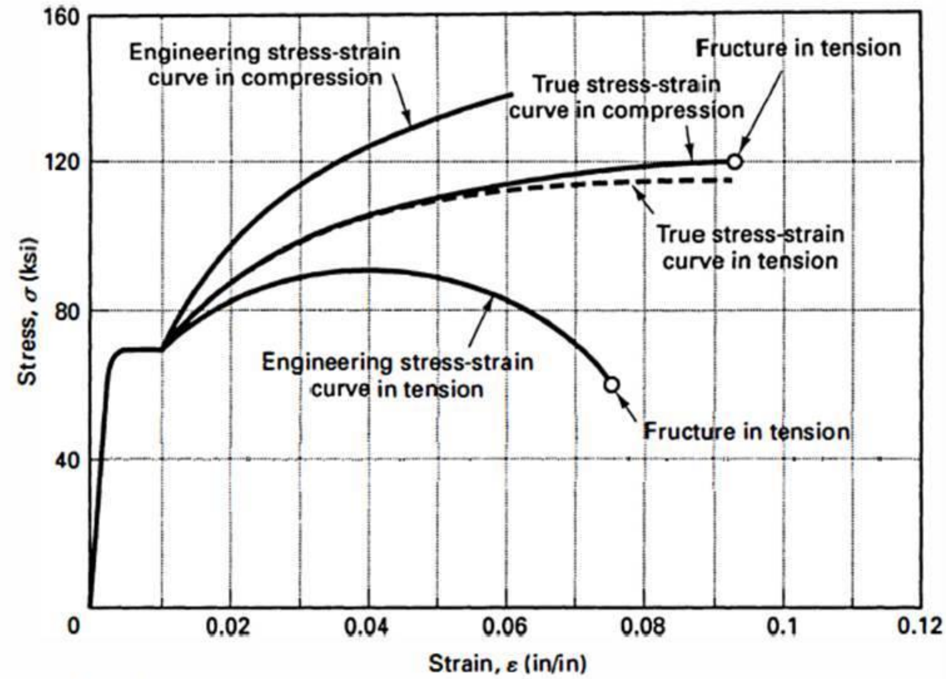


Fig. 2-7 Comparison of tension and compression monotonic stress-strain diagrams. (After Ref. 11).

Stress-Strain Diagram: Brittle Materials

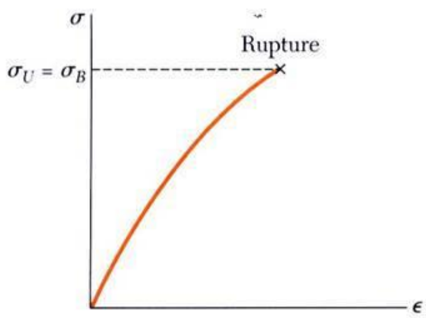


Fig. 2.11 Stress-strain diagram for a typical brittle material.

Stress-Strain Diagram: Different Materials

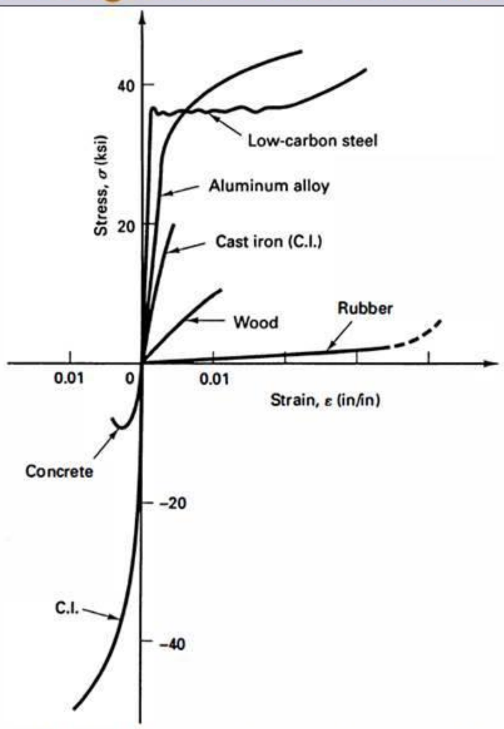


Fig. 2-9 Typical stress-strain diagrams for different materials.

ENGINEERING MECHANICS OF SOLIDS

Hooke's Law: Modulus of Elasticity

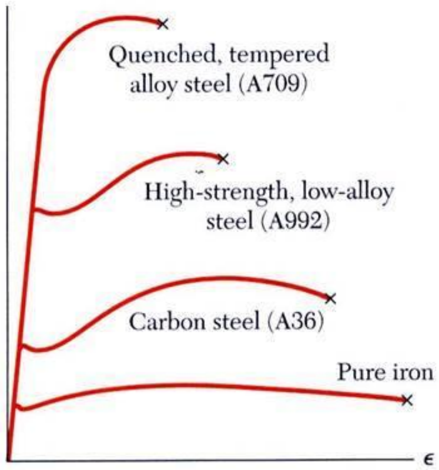


Fig. 2.16 Stress-strain diagrams for iron and different grades of steel.

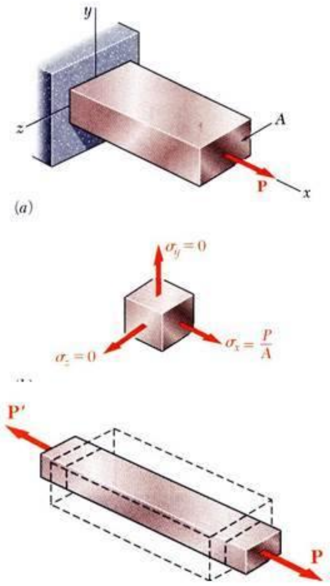
- Below the yield stress

$$\sigma = E\varepsilon$$

$$E = \text{Youngs Modulus or Modulus of Elasticity}$$

- Strength is affected by alloying, heat treating, and manufacturing process but stiffness (Modulus of Elasticity) is not.

Poisson's Ratio



- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$

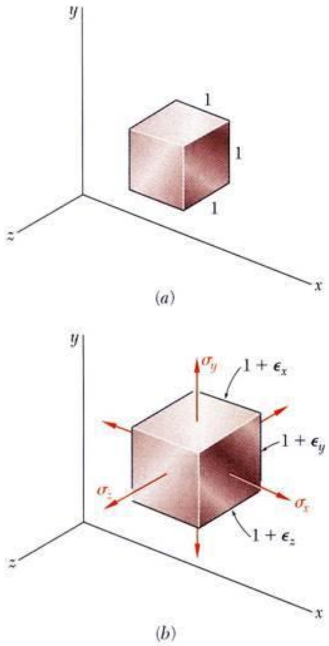
- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

- Poisson's ratio is defined as

$$\nu = \left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Generalized Hooke's Law



- For an element subjected to multi-axial loading, the normal strain components resulting from the stress components may be determined from the *principle of superposition*. This requires:
  1) strain is linearly related to stress
  2) deformations are small

- With these restrictions:

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

Example 2-1

Consider a carefully conducted experiment where an aluminum bar of 50-mm diameter is stressed in a testing machine, as shown in Fig. 2-16. At a certain instant the applied force P is 100 kN, while the measured elongation of the rod is 0.219 mm in a 300-mm gage length, and the diameter's dimension is decreased by 0.01215 mm. Calculate the constant ν of the material.

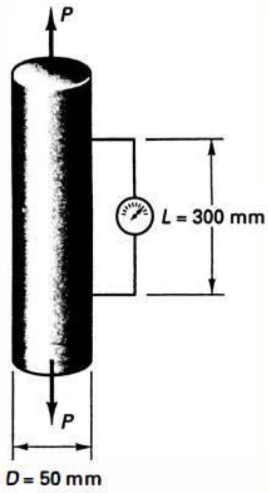


Fig. 2-16

SOLUTION

Transverse or lateral strain:

$$\varepsilon_t = \frac{\Delta_t}{D} = -\frac{0.01215}{50} = -0.000243 \text{ mm/mm}$$

In this case, the lateral strain ε_t is negative, since the diameter of the bar *decreases* by Δ_t.

Axial strain:

$$\varepsilon_a = \frac{\Delta}{L} = +\frac{0.219}{300} = 0.00073 \text{ mm/mm}$$

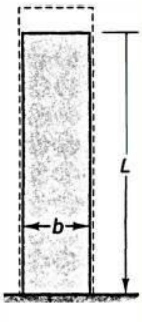
Poisson's ratio:

$$\nu = -\frac{\varepsilon_t}{\varepsilon_a} = -\frac{(-0.000243)}{0.00073} = 0.333$$

2-7. Thermal Strain and Deformation

With changes in temperature, solid bodies change their dimensions. If the temperature increases, generally a body expands, whereas if the temperature decreases, a solid body will contract. Ordinarily, over a limited range of temperature change this expansion or contraction is linearly related to the temperature increase or decrease that occurs. If the body material is homogeneous and isotropic, it has been found that the thermal strain ε_T caused by a change in temperature ΔT, measured in degrees Celsius (°C) or Fahrenheit (°F), can be expressed as

$$\boxed{\varepsilon_T = \alpha\,\Delta T} \qquad (2\text{-}10)$$



where α is a property of the material, referred to as the *coefficient of linear thermal expansion.* The units of α measure strain per degree of temperature. They are 1/°F in the U.S. customary system of units, and 1/°C in the SI system.

$$\boxed{\Delta_T = \alpha\,\Delta T\,L_0} \qquad (2\text{-}11)$$

For a decrease in temperature, ΔT assumes negative values.

An illustration of the thermal effect on deformation of a square and round specimen, due to an increase of temperature is shown in Fig. 2-17.

2-8. Other Idealizations of Constitutive Relations

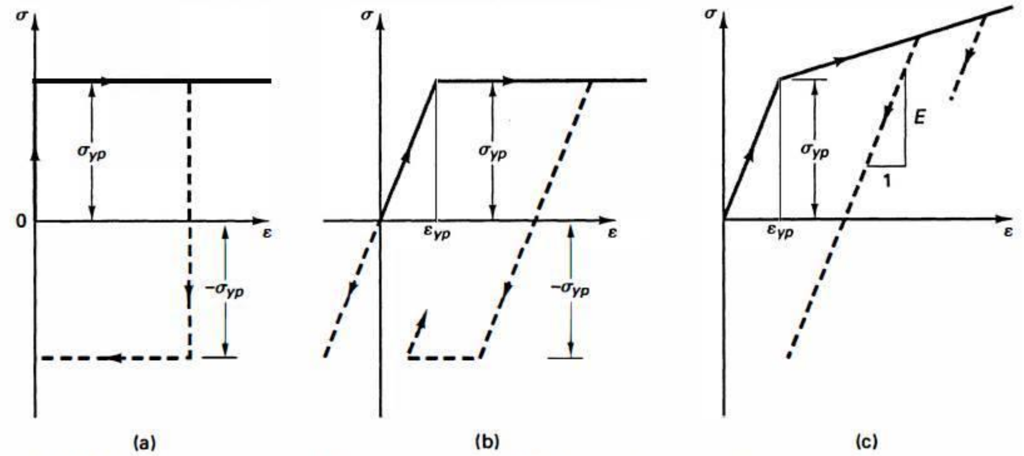


Fig. 2-18 Idealized stress-strain diagrams: (a) rigid perfectly plastic material, (b) elastic perfectly plastic material, and (c) elastic linearly hardening material.

Under rapid or impact loading, two additional material parameters have relevance: *resilience* and *toughness*. Resilience defines the ability of material to absorb energy without suffering plastic strain. The area in the elastic region under a stress-strain diagram, as will be shown in Section 3-5, represents the density of strain energy that can be absorbed without any permanent damage to the material. This area is called the *modulus of resilience* U_R and is equivalent to the shaded triangular area shown in Fig. 2-20(a).

Toughness defines the ability of material to absorb energy prior to fracture. It can be shown (see Section 3-5) that the area under the stress-strain

diagram represents the density of strain energy absorbed by material prior to fracture. The area under the complete monotonic stress-strain diagram is called *the modulus of toughness* U_T. Figure 2-20(b) illustrates the modulus of toughness for brittle and ductile materials. The figure shows that a brittle material, even of greater ultimate strength, generally absorbs much less energy from impact loading than a ductile material.

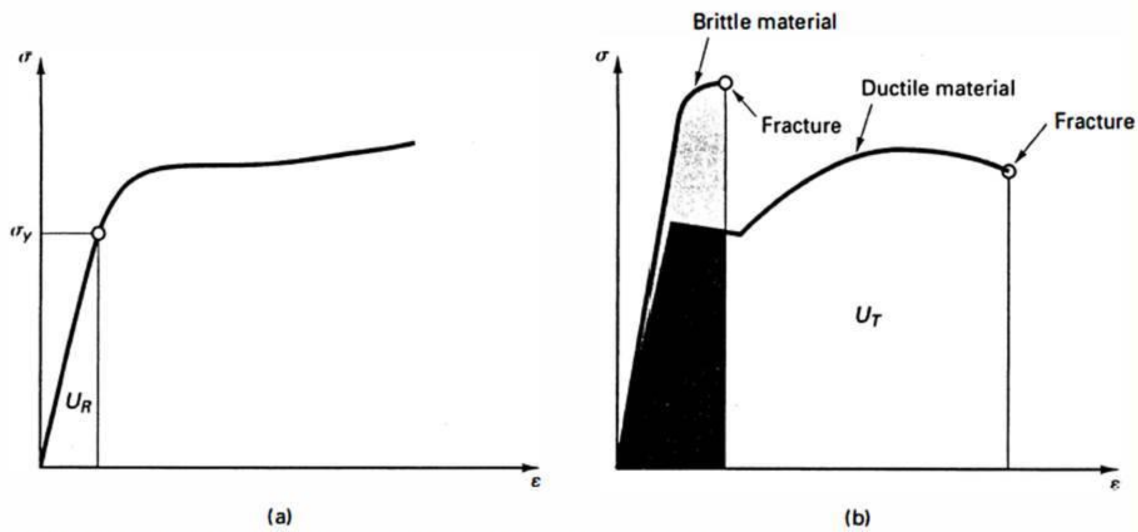


Fig. 2-20 (a) Modulus of resilience U_R. (b) modulus of toughness U_T.

In order to formulate the relation, Eq. 2-1 for the normal strain is recast for a differential element dx. Thus the normal strain ε_x in the x direction is

$$\boxed{\varepsilon_x = \frac{du}{dx}} \qquad (3\text{-}1)$$

where, due to the applied forces, u is the absolute displacement of a point on a bar from an initial fixed position in space, and du is the axial deformation of the infinitesimal element. This is the governing differential equation for axially loaded bars.

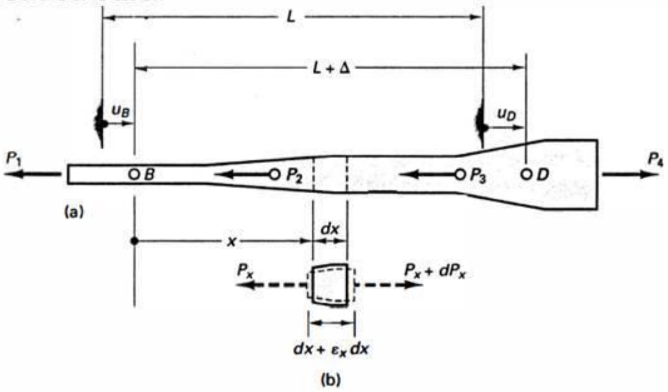


Fig. 3-1 An axially loaded bar.

Rearranging Eq. 3-1 as $du = \varepsilon_x\, dx$, assuming the origin of x at B, and integrating,

$$\int_0^L du = u(L) - u(0) = \int_0^L \varepsilon_x\, dx$$

where $u(L) = u_D$ and $u(0) = u_B$ are the absolute or global displacements of points D and B, respectively. As can be seen from the figure, $u(0)$ is a *rigid-body axial translation* of the bar. The difference between these displacements is the change in length Δ between points D and B. Hence

$$\boxed{\Delta = \int_0^L \varepsilon_x\, dx} \qquad (3\text{-}2)$$

Any appropriate constitutive relations can be used to define ε_x.

For linearly elastic materials, according to Hooke's law, $\varepsilon_x = \sigma_x/E$, Eq. 2-8, where $\sigma_x = P_x/A_x$, Eq. 1-6 or 1-8. By substituting these relations into Eq. 3-2 and simplifying,

$$\boxed{\Delta = \int_0^L \frac{P_x\, dx}{A_x E_x}} \qquad (3\text{-}3)$$

ENGINEERING MECHANICS OF SOLIDS

Example 3-1

Consider bar BC of constant cross-sectional area A and of length L shown in Fig. 3-2(a). Determine the deflection of the free end, caused by the application of a concentrated force P. The elastic modulus of the material is E.

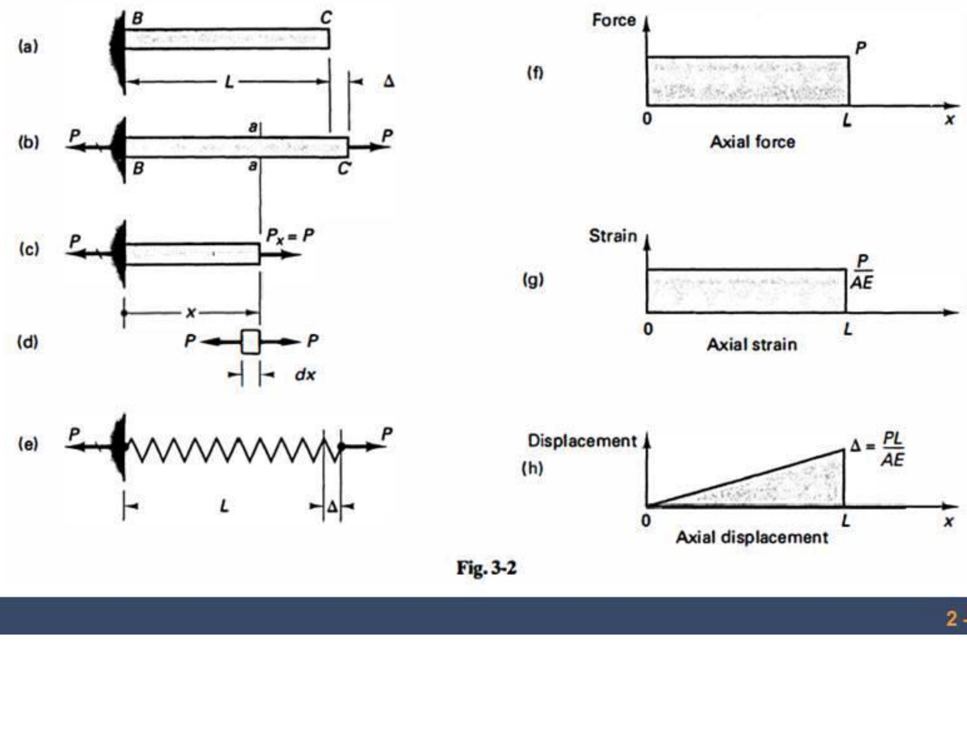


Fig. 3-2

A free-body diagram for an isolated part of the loaded bar to the left of an arbitrary section a–a is shown in Fig. 3-2(c). From this diagram, it can be concluded that the axial force P_x is the same everywhere along the bar and is equal to P. It is given that $A_x = A$, a constant. By applying Eq. 3-3,

$$\Delta = \int_A^B \frac{P_x\,dx}{A_x E} = \frac{P}{AE}\int_0^L dx = \frac{P}{AE}\Big|x\Big|_0^L = \frac{PL}{AE}$$

Hence

$$\boxed{\Delta = \frac{PL}{AE}} \tag{3-4}$$

Since Eq. 3-4 frequently occurs in practice, it is meaningful to recast it into the following form:

$$P = (AE/L)\Delta \tag{3-5}$$

This equation is related to the familiar definition for the *spring constant* or *stiffness* k reading

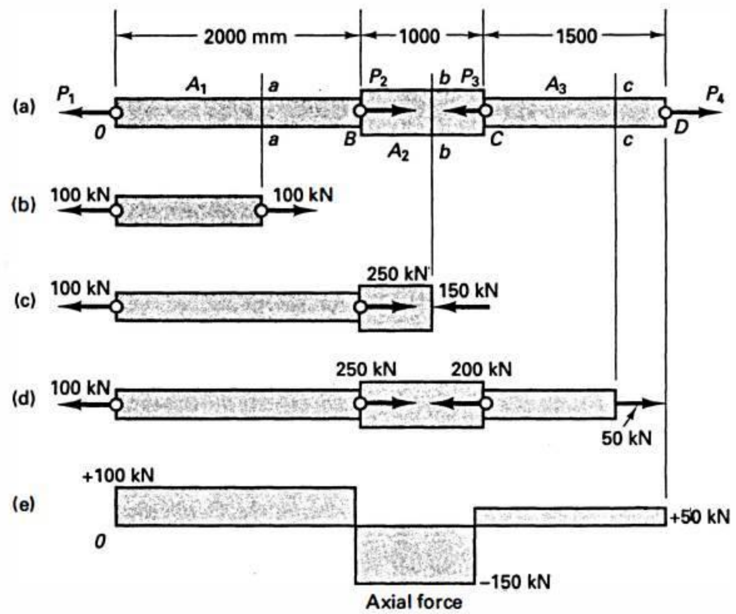
$$k = P/\Delta \quad [\text{lb/in}] \text{ or } [\text{N/m}] \tag{3-6}$$

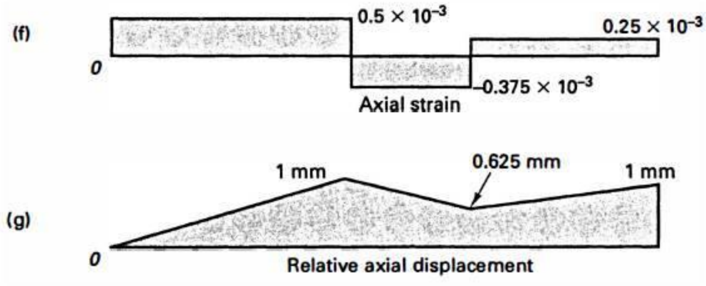
This constant represents the force required to produce a unit deflection (i.e., $\Delta = 1$). Therefore, for an axially loaded ith bar or bar segment of length L_i and constant cross section,

$$\boxed{k_i = \frac{A_i E_i}{L_i}} \tag{3-7}$$

Example 3-2

Determine the relative displacement of point D from O for the elastic steel bar of variable cross section shown in Fig. 3-3(a) caused by the application of concentrated forces $P_1 = 100$ kN and $P_3 = 200$ kN acting to the left, and $P_2 = 250$ kN and $P_4 = 50$ kN acting to the right. The respective areas for bar segments OB, BC, and CD are 1000, 2000, and 1000 mm². Let $E = 200$ GPa.





$$\Delta = \sum_i \frac{P_i L_i}{A_i E} = \frac{P_{OB} L_{OB}}{A_{OB} E} + \frac{P_{BC} L_{BC}}{A_{BC} E} + \frac{P_{CD} L_{CD}}{A_{CD} E}$$

where the subscripts identify the segments.

Using this relation, the relative displacement between O and D is

$$\Delta = +\frac{100 \times 10^3 \times 2000}{1000 \times 200 \times 10^3} - \frac{150 \times 10^3 \times 1000}{2000 \times 200 \times 10^3} + \frac{50 \times 10^3 \times 1500}{1000 \times 200 \times 10^3}$$

$$= +1.000 - 0.375 + 0.375 = +1.000 \text{ mm}$$

ENGINEERING MECHANICS OF SOLIDS

Example 3-3

Determine the deflection of free end B of elastic bar OB caused by its own weight w lb/in; see Fig. 3-4. The constant cross-sectional area is A. Assume that the constant E is given.

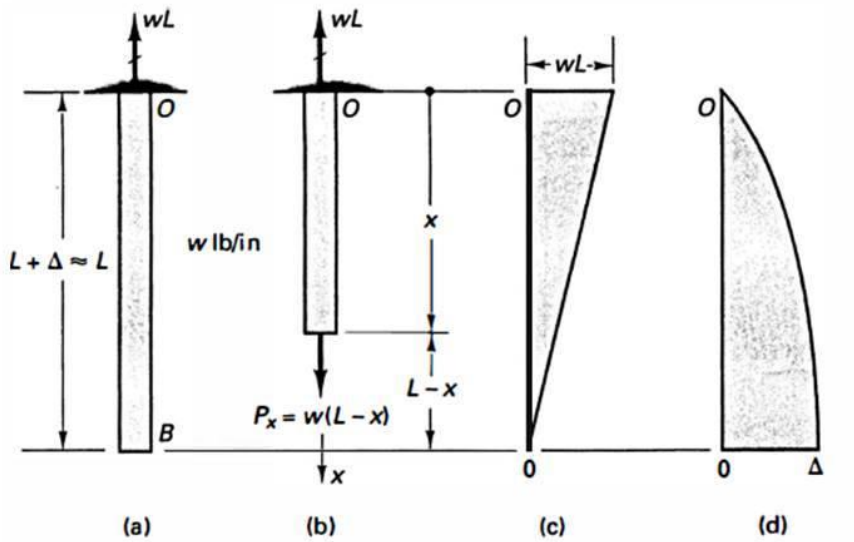


Fig. 3-4

SOLUTION

The free-body diagrams of the bar and its truncated segment are shown, respectively, in Figs. 3-4(a) and (b). These two steps are essential in the solution of such problems. The graph for the axial force $P_x = w(L - x)$ is in Fig. 3-4(c). By applying Eq. 3-3, the change in bar length $\Delta(x)$ at a generic point x,

$$\Delta(x) = \int_0^x \frac{P_x\,dx}{A_x E} = \frac{1}{AE}\int w(L-x)\,dx = \frac{w}{AE}\left(Lx - \frac{x^2}{2}\right)$$

A plot of this function is shown in Fig. 3-4(d), with its maximum as B.

The deflection of B is

$$\Delta = \Delta(L) = \frac{w}{AE}\left(L^2 - \frac{L^2}{2}\right) = \frac{wL^2}{2AE} = \frac{WL}{2AE}$$

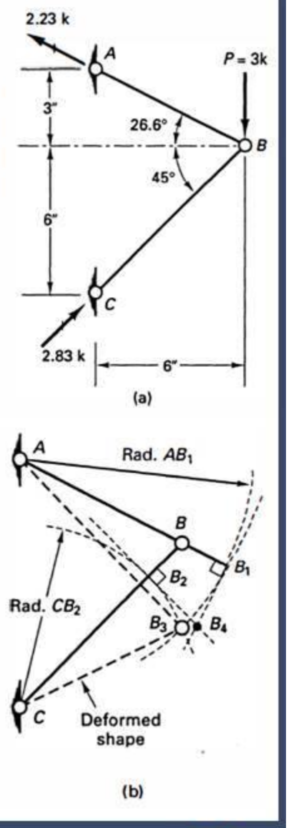
where $W = wL$ is the *total* weight of the bar.

If a concentrated force P, in *addition* to the bar's own weight, were acting on bar OB at end B, the total deflection due to the *two causes* would be obtained by *superposition* as

$$\Delta = \frac{PL}{AE} + \frac{WL}{2AE} = \frac{[P + (W/2)]L}{AE}$$

Example 3-4

For the bracket analyzed for stresses in Example 1-3, determine the deflection of point B caused by the applied vertical force $P = 3$ kips. Also determine the vertical stiffness of the bracket at B. Assume that the members are made of 2024-T4 aluminum alloy and that they have constant cross-sectional areas (i.e., neglect the enlargements at the connections). See the idealization in Fig. 3-6(a).



SOLUTION

As found in Example 1-3, the axial stresses in the bars of the bracket are $\sigma_{AB} = 17.8$ ksi and $\sigma_{BC} = 12.9$ ksi. The length of member AB is 6.71 in and that of BC is 8.49 in. Per Table 1A in the Appendix, for the specified material, $E = 10.6 \times 10^3$ ksi. Therefore, according to Eq. 3-4, the individual member length changes are

$$\Delta_{AB} = \left[\frac{PL}{AE}\right]_{AB} = \left[\sigma\frac{L}{E}\right]_{AB} = \frac{17.8 \times 6.71}{10.6 \times 10^3} = 11.3 \times 10^{-3}\text{ in} \quad \text{(elongation)}$$

$$\Delta_{BC} = -\frac{12.9 \times 8.29}{10.6 \times 10^3} = -10.3 \times 10^{-3}\text{ in} \quad \text{(contraction)}$$

If Δ is the deflection or displacement of point B to position B_4, Fig. 3-6(c), and changes in bar lengths $\Delta_{BC} = BB_2$ and $\Delta_{AB} = BB_1$,

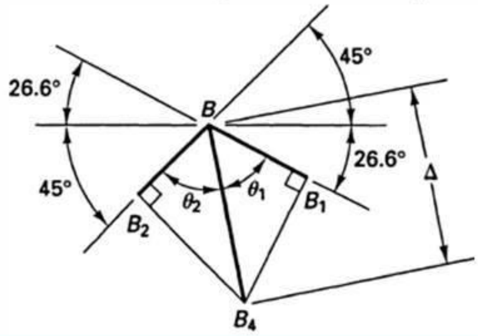
$$\Delta_{BC} = \Delta \cos\theta_2 \quad \text{and} \quad \Delta_{AB} = \Delta \cos\theta_1$$

On forming equal ratios for both sides of these equations, substituting the numerical values for Δ_{BC} and Δ_{AB} found earlier, and simplifying, one obtains

$$\frac{\cos\theta_2}{\cos\theta_1} = \frac{\Delta_{BC}}{\Delta_{AB}} = \frac{10.3 \times 10^{-3}}{11.3 \times 10^{-3}} = 0.912$$

However, since

$$\theta_2 = 180° - 45° - 26.6° - \theta_1 = 108.4° - \theta_1$$



Example 3-5

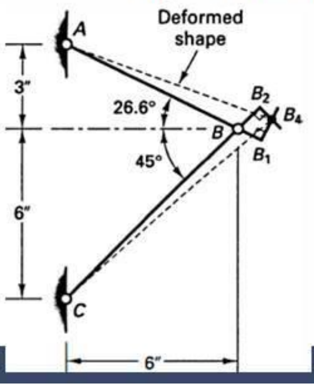
Determine the displacement of point B in Example 3-4 caused by an increase in temperature of 100°F. See Fig. 3-7(a).

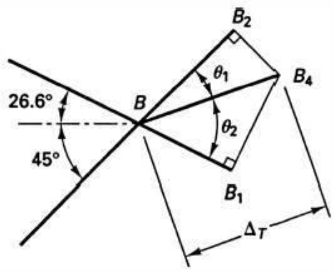
SOLUTION

Determining the deflection at point B due to an increase in temperature is similar to the solution of Example 3-4 for finding the deflection of the same point caused by stress. Per Table 1A in the Appendix, the coefficient of thermal expansion for 2024-T4 aluminum alloy is 12.9×10^{-6} per °F. Hence, from Eq. 2-11, and using the lengths of members given in Example 3-4,

$$\Delta_{AB} = 12.9 \times 10^{-6} \times 100 \times 6.71 = 8.656 \times 10^{-3} \text{ in}$$

$$\Delta_{BC} = 12.9 \times 10^{-6} \times 100 \times 8.49 = 10.95 \times 10^{-3} \text{ in}$$





Here the displacement Δ_T of point B to position B_4, Fig. 3-7(b), caused by a change in temperature, is related to the bar elongations in the following manner:

$$\Delta_T \cos\theta_2 = \Delta_{AB} \quad \text{and} \quad \Delta_T \cos\theta_1 = \Delta_{BC}$$

Forming equal ratios for both sides of these equations, substituting numerical values for Δ_{AB} and Δ_{BC}, and simplifying leads to the following result:

$$\frac{\cos\theta_2}{\cos\theta_1} = \frac{\Delta_{AB}}{\Delta_{BC}} = \frac{8.656 \times 10^{-3}}{10.95 \times 10^{-3}} = 0.7905$$

Here, however, $\theta_2 = 45° + 26.6° - \theta_1 = 71.6° - \theta_1$; therefore,

$$\cos\theta_2 = \cos 71.6° \cos\theta_1 + \sin 71.6° \sin\theta_1$$

and

$$\frac{\cos\theta_2}{\cos\theta_1} = \cos 71.6° + \sin 71.6° \tan\theta_1 = 0.7905$$

Hence,

$$\tan\theta_1 = 0.500 \quad \text{and} \quad \theta_1 = 26.6°$$

Based on this result,

$$\Delta_T = \Delta_{BC}/\cos\theta_1 = 12.2 \times 10^{-3} \text{ in}$$

forming an angle of $45° - \theta_1 = 18.4°$ with the horizontal.

It is interesting to note that the small displacement Δ_T is of comparable order of magnitude to that found due to the applied vertical force P in Example 3-4.

ENGINEERING MECHANICS OF SOLIDS

3-3. Saint-Venant's Principle and Stress Concentrations

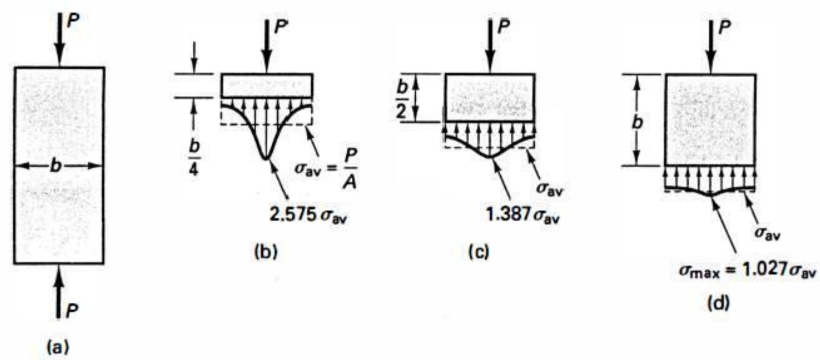


Fig. 3-9 Stress distribution near a concentrated force in a rectangular elastic plate.

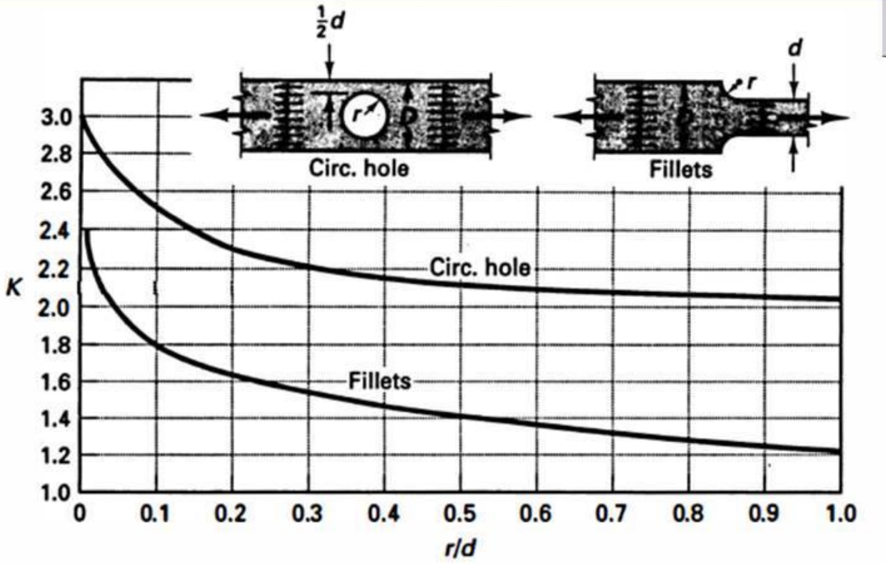


Fig. 3-11 Stress-concentration factors for flat bars in tension.

$$\sigma_{max} = K\sigma_{av} = K\frac{P}{A} \qquad (3\text{-}11)$$

where K is an appropriate stress-concentration factor, and P/A is the average stress per Eq. 1-6.

ENGINEERING MECHANICS OF SOLIDS

Example 3-7

Find the maximum stress in member AB in the forked end A in Example 1-3.

SOLUTION

Geometrical proportions:

$$\frac{\text{radius of the hole}}{\text{net width}} = \frac{3/16}{1/2} = 0.375$$

From Fig. 3-11:[7] $K \approx 2.15$ for $r/d = 0.375$.
Average stress from Example 1-3: $\sigma_{av} = P/A_{net} = 11.2$ ksi.
Maximum stress, Eq. 3.11: $\sigma_{max} = K\sigma_{av} = 2.15 \times 11.2 = 24.1$ ksi.

This answer indicates that a large local increase in stress occurs at this hole, a fact that may be highly significant.

3-5. Elastic Strain Energy for Uniaxial Stress

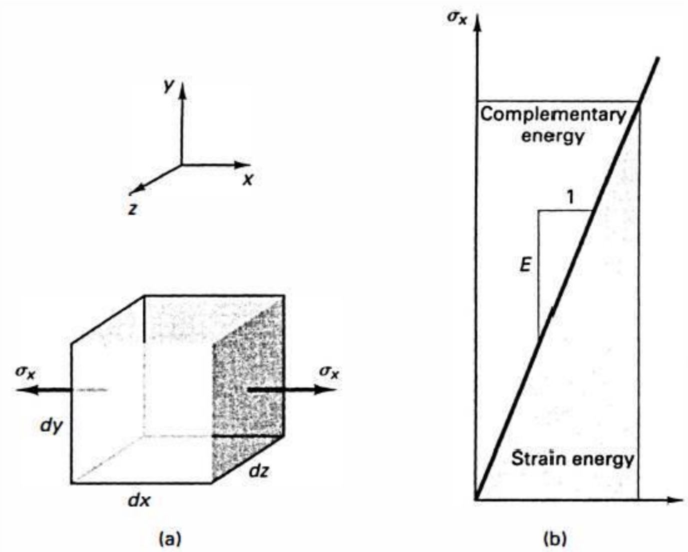


Fig. 3-17 (a) An element in uniaxial tension and (b) a Hookean stress-strain diagram.

ENGINEERING MECHANICS OF SOLIDS

Consider an infinitesimal element, such as shown in Fig. 3-17(a), subjected to a normal stress σ_x. The force acting on the right or the left face of this element is $\sigma_x\, dy\, dz$, where $dy\, dz$ is an infinitesimal area of the element. Because of this force, the element elongates an amount $\varepsilon_x\, dx$, where ε_x is normal strain in the x direction. If the element is made of a linearly elastic material, stress is proportional to strain; Fig. 3-17(b). Therefore, if the element is initially free of stress, the force that finally acts on the element increases linearly from zero until it attains its full value. The average force acting on the element while deformation is taking place is $\frac{1}{2}\sigma_x\, dy\, dz$. This average force multiplied by the distance through which it acts is the work done on the element. For a perfectly elastic body, no energy is dissipated and the work done on the element is stored as recoverable internal strain energy. Thus, the internal elastic strain energy U for an infinitesimal element subjected to uniaxial stress is

$$dU = \underbrace{\underbrace{\tfrac{1}{2}\sigma_x\, dy\, dz}_{\text{average force}} \times \underbrace{\varepsilon_x\, dx}_{\text{distance}}}_{\text{work}} = \tfrac{1}{2}\sigma_x\varepsilon_x\, dx\, dy\, dz = \tfrac{1}{2}\sigma_x\varepsilon_x\, dV \qquad (3\text{-}12)$$

where dV is the volume of the element.

By recasting Eq. 3-12, one obtains the strain energy stored in an elastic body per *unit volume* of the material, or its *strain-energy density* U_o. Thus,

$$U_o = \frac{dU}{dV} = \frac{\sigma_x \varepsilon_x}{2} \tag{3-13}$$

This expression may be graphically interpreted as an area under the inclined line on the stress-strain diagram; Fig. 3-17(b). The corresponding area enclosed by the inclined line and the vertical axis is called the *complementary energy,* a concept to be used in Chapter 18. For linearly elastic materials, the two areas are equal. Expressions analogous to Eq. 3-13 apply to the normal stresses σ_y and σ_z and to the corresponding normal strains ε_y and ε_z.

Since in the elastic range Hooke's law applies, $\sigma_x = E\varepsilon_x$, Eq. 3-13 may be written as

$$U_o = \frac{dU}{dV} = \frac{E\varepsilon_x^2}{2} = \frac{\sigma_x^2}{2E} \tag{3-14}$$

or

$$U = \int_{\text{vol}} \frac{\sigma_x^2}{2E}\, dV \tag{3-15}$$

ENGINEERING MECHANICS OF SOLIDS

Example 3-8

Two elastic bars, whose proportions are shown in Fig. 3-19, are to absorb the same amount of energy delivered by axial forces at the free end. Neglecting stress concentrations, compare the stresses in the two bars. The cross-sectional area of the left bar is A, and that of the right bar is A and $2A$ as shown.

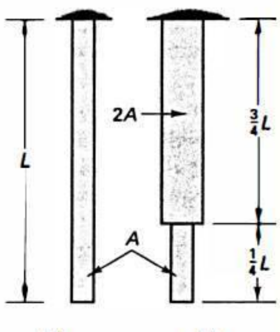


Fig. 3-19

SOLUTION

The bar shown in Fig. 3-19(a) is of uniform cross-sectional area; therefore, the normal stress σ_1 is constant throughout. Using Eq. 3-15 and integrating over the volume V of the bar, one can write the total energy for the bar as

$$U_1 = \int_V \frac{\sigma_1^2}{2E} dV = \frac{\sigma_2^2}{2E}\int_V dV = \frac{\sigma_1^2}{2E}(AL)$$

where A is the cross-sectional area of the bar and L is its length.

The bar shown in Fig. 3-19(b) is of variable cross section. Therefore, if the stress σ_2 acts in the lower part of the bar, the stress in the upper part is $\frac{1}{2}\sigma_2$. Again, by using Eq. 3-15 and integrating over the volume of the bar, it is found that the total energy that this bar will absorb in terms of the stress σ_2 is

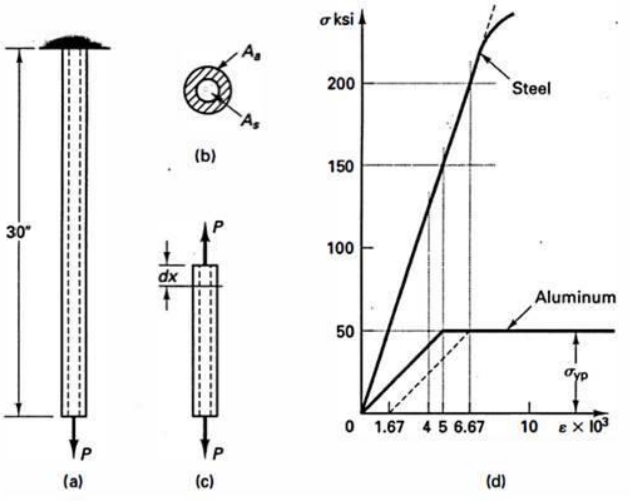
$$U_2 = \int_V \frac{\sigma^2}{2E} dV = \frac{\sigma_2^2}{2E}\int_{\text{lower part}} dV + \frac{(\sigma_2/2)^2}{2E}\int_{\text{upper part}} dV$$
$$= \frac{\sigma_2^2}{2E}\left(\frac{AL}{4}\right) + \frac{(\sigma_2/2)^2}{2E}\left(2A\frac{3L}{4}\right) = \frac{\sigma_2^2}{2E}\left(\frac{5}{8}AL\right)$$

If both bars are to absorb the same amount of energy, $U_1 = U_2$ and

$$\frac{\sigma_1^2}{2E}(AL) = \frac{\sigma_2^2}{2E}\left(\frac{5}{8}AL\right) \quad \text{or} \quad \sigma_2 = 1.265\sigma_1$$

Example 4-8

A 30-in-long aluminum rod is enclosed within a steel-alloy tube; see Figs. 4-14(a) and (b). The two materials are bonded together. If the stress-strain diagrams for the two materials can be idealized as shown, respectively, in Fig. 4-14(d), what end deflection will occur for $P_1 = 80$ kips and for $P_2 = 125$ kips? The cross-sectional areas of steel A_s and of aluminum A_a are the same and equal to 0.5 in^2.



From equilibrium:

$$P_a + P_s = P_1 \text{ or } P_2$$

From compatibility:

$$\Delta_a = \Delta_s \qquad \text{or} \qquad \varepsilon_a = \varepsilon_s$$

From material properties:

$$\varepsilon_a = \sigma_a / E_a \qquad \text{and} \qquad \varepsilon_s = \sigma_s / E_s$$

By noting that $\sigma_a = P_a/A_a$ and $\sigma_s = P_s/A_s$, one can solve the three equations. From the diagram the elastic moduli are $E_s = 30 \times 10^6$ psi and $E_a = 10 \times 10^6$ psi. Thus,

$$\varepsilon_a = \varepsilon_s = \frac{\sigma_a}{E_a} = \frac{\sigma_s}{E_s} = \frac{P_a}{A_a E_a} = \frac{P_s}{A_s E_s}$$

Hence, $P_s = [A_s E_s/(A_a E_a)]P_a = 3P_a$, and $P_a + 3P_a = P_1 = 80$ k; therefore, $P_a = 20$ k, and $P_s = 60$ k.

By applying Eq. 3-4 to either material, the tip deflection for 80 kips will be

ENGINEERING MECHANICS OF SOLIDS

From equilibrium:

$$P_a + P_s = P_1 \text{ or } P_2$$

From compatibility:

$$\Delta_a = \Delta_s \qquad \text{or} \qquad \varepsilon_a = \varepsilon_s$$

From material properties:

$$\varepsilon_a = \sigma_a / E_a \qquad \text{and} \qquad \varepsilon_s = \sigma_s / E_s$$

By noting that $\sigma_a = P_a/A_a$ and $\sigma_s = P_s/A_s$, one can solve the three equations. From the diagram the elastic moduli are $E_s = 30 \times 10^6$ psi and $E_a = 10 \times 10^6$ psi. Thus,

$$\varepsilon_a = \varepsilon_s = \frac{\sigma_a}{E_a} = \frac{\sigma_s}{E_s} = \frac{P_a}{A_a E_a} = \frac{P_s}{A_s E_s}$$

Hence, $P_s = [A_s E_s/(A_a E_a)]P_a = 3P_a$, and $P_a + 3P_a = P_1 = 80$ k; therefore, $P_a = 20$ k, and $P_s = 60$ k.

By applying Eq. 3-4 to either material, the tip deflection for 80 kips will be

$$\Delta = \frac{P_s L}{A_s E_s} = \frac{P_a L}{A_a E_a} = \frac{20 \times 10^3 \times 30}{0.5 \times 10 \times 10^6} = 0.120 \text{ in}$$

This corresponds to a strain of $0.120/30 = 4 \times 10^{-3}$ in/in. In this range, both materials respond elastically, which satisfies the material-property assumption made at the beginning of this solution. In fact, as may be seen from Fig. 4-14(d), since for the linearly elastic response the strain can reach 5×10^{-3} in/in for both materials, by direct proportion, the applied force P can be as large as 100 kips.

At $P = 100$ kips, the stress in aluminum reaches 50 ksi. According to the idealized stress-strain diagram, no higher stress can be resisted by this material, although the strains may continue to increase. Therefore, beyond $P = 100$ kips, the aluminum rod can be counted upon to resist only $P_a = A_a \sigma_{yp} = 0.5 \times 50 = 25$ kips. The remainder of the applied load must be carried by the steel tube. Therefore for $P_2 = 125$ kips, 100 kips must be carried by the steel tube. Hence, $\sigma_s = 100/0.5 = 200$ ksi. At this stress level, $\varepsilon_s = 200/(30 \times 10^3) = 6.67 \times 10^{-3}$ in/in. Therefore, the tip deflection

$$\Delta = \varepsilon_s L = 6.67 \times 10^{-3} \times 30 = 0.200 \text{ in}$$

ENGINEERING MECHANICS OF SOLIDS

Note that it is not possible to determine Δ from the strain in aluminum, since no unique strain corresponds to the stress beyond 50 ksi, which is all that the aluminum rod can carry. However, in this case, the elastic steel tube constrains the plastic flow. Therefore, since the strains in both materials are the same—that is, $\varepsilon_s = \varepsilon_a = 6.67 \times 10^{-3}$ in/in; see Fig. 4-14(d).

If the applied force $P_2 = 125$ kips were removed, both materials in the rod would rebound elastically. Thus, if one imagines the bond between the two materials broken, the steel tube would return to its initial shape. But a permanent set (stretch) of $(6.67 - 5) \times 10^{-3} = 1.67 \times 10^{-3}$ in/in would occur in the aluminum rod. This incompatibility of strain cannot develop if the two materials are bonded together. Instead, residual stresses develop, which maintain the same axial deformation in both materials. In this case, the aluminum rod remains slightly compressed and the steel tube is slightly stretched. The procedure for the solution of this kind of problem is

Solve following problems in addition to solved problems of different text books mentioned (CHAPTER-3, Popov):

11-14, 17-19, 25, 26, 30, 33 & 49

Elastic vs. Plastic Behavior

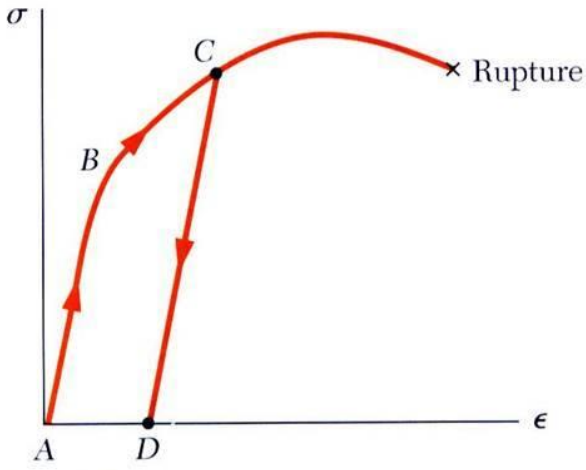


Fig. 2.18

- If the strain disappears when the stress is removed, the material is said to behave *elastically*.
- The largest stress for which this occurs is called the *elastic limit*.
- When the strain does not return to zero after the stress is removed, the material is said to behave *plastically*.

Fatigue

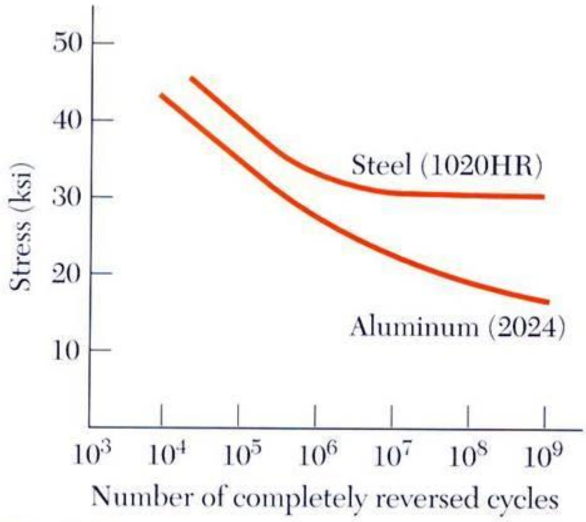


Fig. 2.21

- Fatigue properties are shown on S-N diagrams.
- A member may fail due to *fatigue* at stress levels significantly below the ultimate strength if subjected to many loading cycles.
- When the stress is reduced below the *endurance limit*, fatigue failures do not occur for any number of cycles.

ENGINEERING MECHANICS OF SOLIDS

Deformations Under Axial Loading

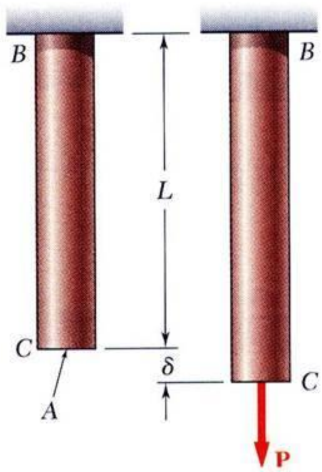


Fig. 2.22

- From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

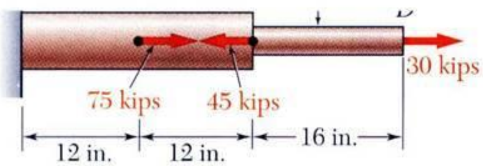
- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Example 2.01



$E = 29 \times 10^{-6}$ psi

$D = 1.07$ in. $\quad d = 0.618$ in.

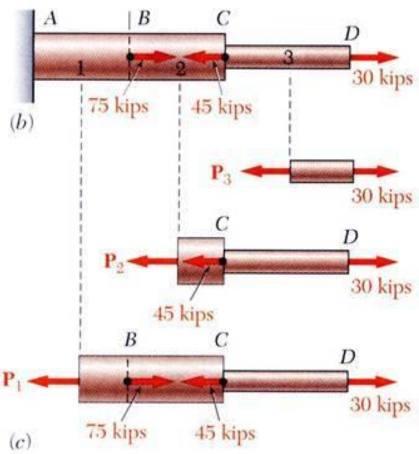
Determine the deformation of the steel rod shown under the given loads.

SOLUTION:

- Divide the rod into components at the load application points.
- Apply a free-body analysis on each component to determine the internal force
- Evaluate the total of the component deflections.

SOLUTION:

- Divide the rod into three components:



$L_1 = L_2 = 12$ in. $\quad L_3 = 16$ in.

$A_1 = A_2 = 0.9\ \text{in}^2 \quad A_3 = 0.3\ \text{in}^2$

- Apply free-body analysis to each component to determine internal forces,

$$P_1 = 60 \times 10^3\ \text{lb}$$

$$P_2 = -15 \times 10^3\ \text{lb}$$

$$P_3 = 30 \times 10^3\ \text{lb}$$

- Evaluate total deflection,

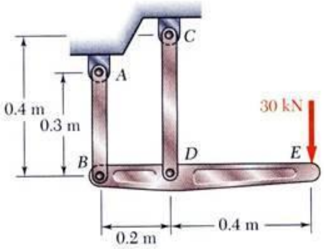
$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$

$$= \frac{1}{29 \times 10^6}\left[\frac{\left(60 \times 10^3\right)12}{0.9} + \frac{\left(-15 \times 10^3\right)12}{0.9} + \frac{\left(30 \times 10^3\right)16}{0.3}\right]$$

$$= 75.9 \times 10^{-3}\ \text{in.}$$

$$\boxed{\delta = 75.9 \times 10^{-3}\ \text{in.}}$$

Sample Problem 2.1



The rigid bar *BDE* is supported by two links *AB* and *CD*.

Link *AB* is made of aluminum (E = 70 GPa) and has a cross-sectional area of 500 mm^2. Link *CD* is made of steel (E = 200 GPa) and has a cross-sectional area of (600 mm^2).

For the 30-kN force shown, determine the deflection a) of *B*, b) of *D*, and c) of *E*.

SOLUTION:

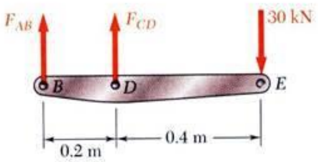
- Apply a free-body analysis to the bar *BDE* to find the forces exerted by links *AB* and *DC*.
- Evaluate the deformation of links *AB* and *DC* or the displacements of *B* and *D*.
- Work out the geometry to find the deflection at E given the deflections at B and D.

ENGINEERING MECHANICS OF SOLIDS

Sample Problem 2.1

SOLUTION:

Free body: Bar *BDE*



$$\sum M_B = 0$$

$$0 = -(30\,\text{kN} \times 0.6\,\text{m}) + F_{CD} \times 0.2\,\text{m}$$

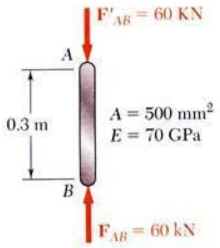
$$F_{CD} = +90\,\text{kN} \quad \textit{tension}$$

$$\sum M_D = 0$$

$$0 = -(30\,\text{kN} \times 0.4\,\text{m}) - F_{AB} \times 0.2\,\text{m}$$

$$F_{AB} = -60\,\text{kN} \quad \textit{compression}$$

Displacement of *B*:



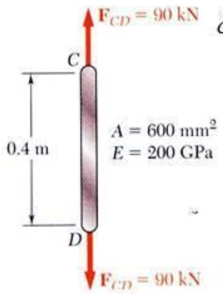
$$\delta_B = \frac{PL}{AE}$$

$$= \frac{(-60 \times 10^3\,\text{N})(0.3\,\text{m})}{(500 \times 10^{-6}\,\text{m}^2)(70 \times 10^9\,\text{Pa})}$$

$$= -514 \times 10^{-6}\,\text{m}$$

$$\boxed{\delta_B = 0.514\,\text{mm} \uparrow}$$

Displacement of *D*:



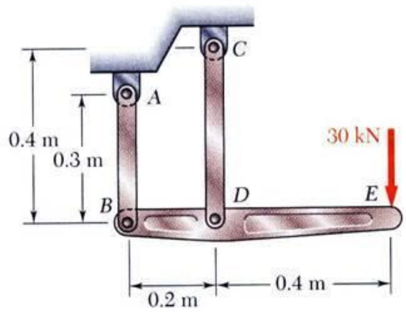
$$\delta_D = \frac{PL}{AE}$$

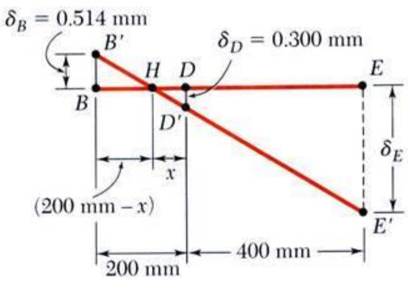
$$= \frac{(90 \times 10^3\,\text{N})(0.4\,\text{m})}{(600 \times 10^{-6}\,\text{m}^2)(200 \times 10^9\,\text{Pa})}$$

$$= 300 \times 10^{-6}\,\text{m}$$

$$\boxed{\delta_D = 0.300\,\text{mm} \downarrow}$$

Sample Problem 2.1





Displacement of D:

$$\frac{BB'}{DD'} = \frac{BH}{HD}$$

$$\frac{0.514 \text{ mm}}{0.300 \text{ mm}} = \frac{(200 \text{ mm}) - x}{x}$$

$$x = 73.7 \text{ mm}$$

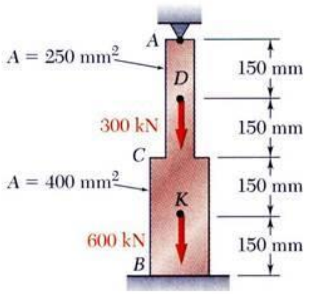
$$\frac{EE'}{DD'} = \frac{HE}{HD}$$

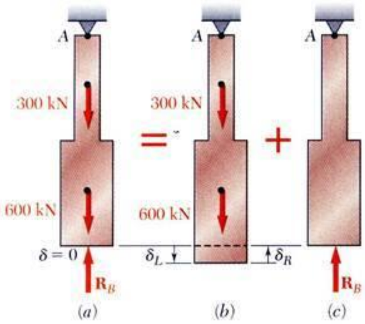
$$\frac{\delta_E}{0.300 \text{ mm}} = \frac{(400 + 73.7)\text{mm}}{73.7 \text{ mm}}$$

$$\delta_E = 1.928 \text{ mm}$$

$$\boxed{\delta_E = 1.928 \text{ mm} \downarrow}$$

Static Indeterminacy

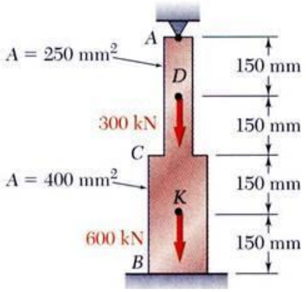


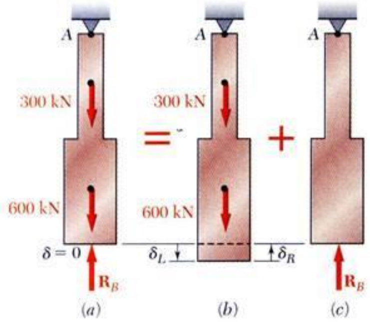


- Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.
- A structure will be statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.
- Redundant reactions are replaced with unknown loads which along with the other loads must produce compatible deformations.
- Deformations due to actual loads and redundant reactions are determined separately and then added or *superposed*.

$$\delta = \delta_L + \delta_R = 0$$

Example 2.04



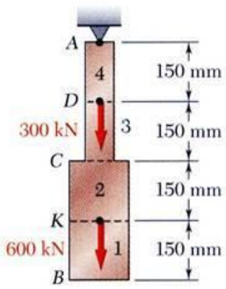


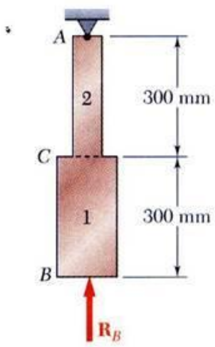
Determine the reactions at A and B for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.

SOLUTION:

- Consider the reaction at B as redundant, release the bar from that support, and solve for the displacement at B due to the applied loads.
- Solve for the displacement at B due to the redundant reaction at B.
- Require that the displacements due to the loads and due to the redundant reaction be compatible, i.e., require that their sum be zero.
- Solve for the reaction at A due to applied loads and the reaction found at B.

Example 2.04





SOLUTION:

- Solve for the displacement at B due to the applied loads with the redundant constraint released,

$$P_1 = 0 \quad P_2 = P_3 = 600\times10^3\,\text{N} \quad P_4 = 900\times10^3\,\text{N}$$

$$A_1 = A_2 = 400\times10^{-6}\,\text{m}^2 \quad A_3 = A_4 = 250\times10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150\ \text{m}$$

$$\delta_L = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1.125\times10^9}{E}$$

- Solve for the displacement at B due to the redundant constraint,

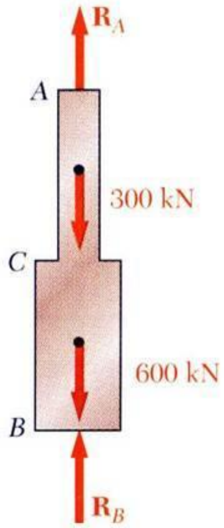
$$P_1 = P_2 = -R_B$$

$$A_1 = 400\times10^{-6}\,\text{m}^2 \quad A_2 = 250\times10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = 0.300\ \text{m}$$

$$\delta_R = \sum_i \frac{P_i L_i}{A_i E_i} = -\frac{\left(1.95\times10^3\right)R_B}{E}$$

Example 2.04



- Require that the displacements due to the loads and due to the redundant reaction be compatible,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3)R_B}{E} = 0$$

$$R_B = 577 \times 10^3 \text{ N} = 577 \text{ kN}$$

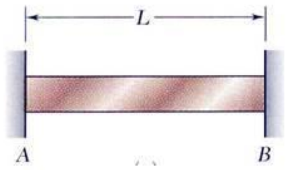
- Find the reaction at A due to the loads and the reaction at B

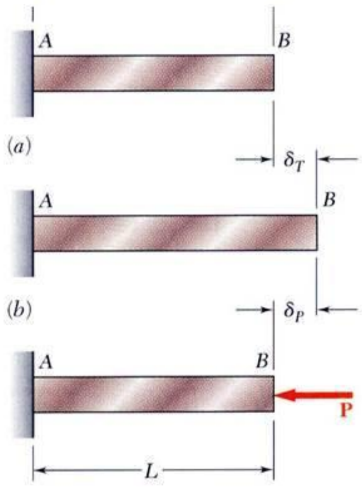
$$\sum F_y = 0 = R_A - 300 \text{ kN} - 600 \text{ kN} + 577 \text{ kN}$$

$$R_A = 323 \text{ kN}$$

$$\boxed{\begin{aligned} R_A &= 323 \text{ kN} \\ R_B &= 577 \text{ kN} \end{aligned}}$$

Thermal Stresses





- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.
- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$\alpha =$ thermal expansion coef.

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

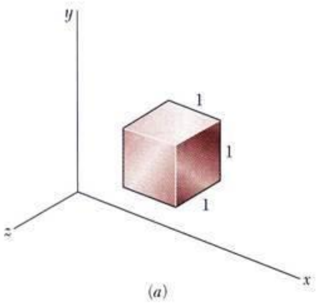
$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$\delta = \delta_T + \delta_P = 0$$

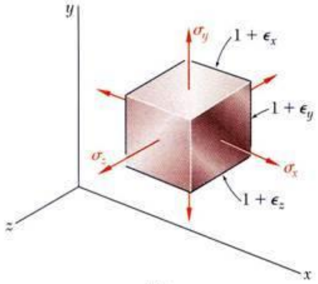
$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Dilatation: Bulk Modulus



(a)



(b)

- Relative to the unstressed state, the change in volume is

$$e = 1 - \left[(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\right] = 1 - \left[1+\varepsilon_x+\varepsilon_y+\varepsilon_z\right]$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$= \frac{1-2\nu}{E}\left(\sigma_x+\sigma_y+\sigma_z\right)$$

$$= \text{dilatation (change in volume per unit volume)}$$

- For element subjected to uniform hydrostatic pressure,

$$e = -p\frac{3(1-2\nu)}{E} = -\frac{p}{k}$$

$$k = \frac{E}{3(1-2\nu)} = \text{bulk modulus}$$

- Subjected to uniform pressure, dilatation must be negative, therefore

$$0 < \nu < \tfrac{1}{2}$$

ENGINEERING MECHANICS OF SOLIDS

Shearing Strain

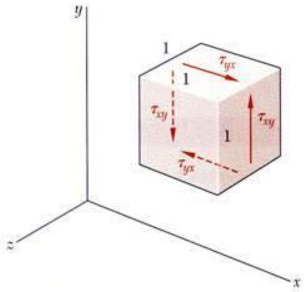


Fig. 2.46

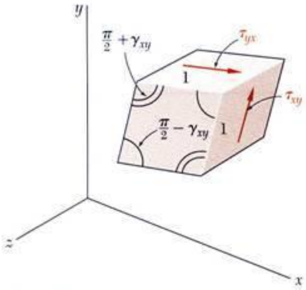


Fig. 2.47

- A cubic element subjected to a shear stress will deform into a rhomboid. The corresponding *shear strain* is quantified in terms of the change in angle between the sides,

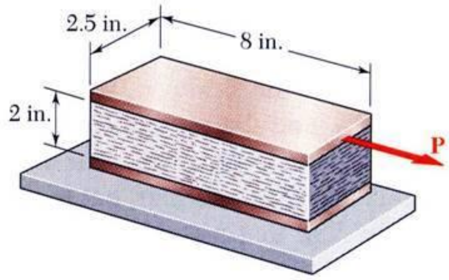
$$\tau_{xy} = f(\gamma_{xy})$$

- A plot of shear stress vs. shear strain is similar the previous plots of normal stress vs. normal strain except that the strength values are approximately half. For small strains,

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

where G is the modulus of rigidity or shear modulus.

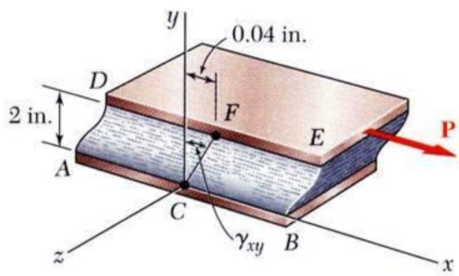
Example 2.10



A rectangular block of material with modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force P. Knowing that the upper plate moves through 0.04 in. under the action of the force, determine a) the average shearing strain in the material, and b) the force P exerted on the plate.

SOLUTION:

- Determine the average angular deformation or shearing strain of the block.
- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.
- Use the definition of shearing stress to find the force P.



- Determine the average angular deformation or shearing strain of the block.

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04\,\text{in.}}{2\,\text{in.}} \qquad \gamma_{xy} = 0.020\,\text{rad}$$

- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

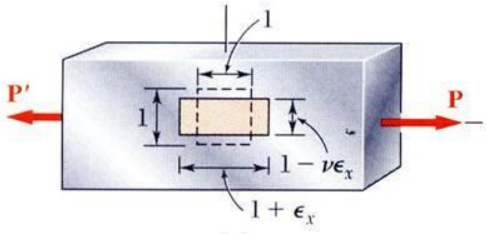
$$\tau_{xy} = G\gamma_{xy} = \left(90\times 10^3\,\text{psi}\right)\left(0.020\,\text{rad}\right) = 1800\,\text{psi}$$

- Use the definition of shearing stress to find the force P.

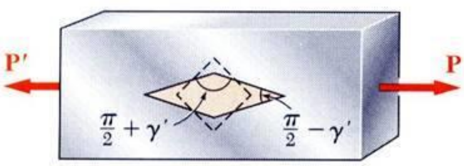
$$P = \tau_{xy}A = \left(1800\,\text{psi}\right)\left(8\,\text{in.}\right)\left(2.5\,\text{in.}\right) = 36\times 10^3\,\text{lb}$$

$$\boxed{P = 36.0\,\text{kips}}$$

Relation Among *E*, *ν*, and *G*



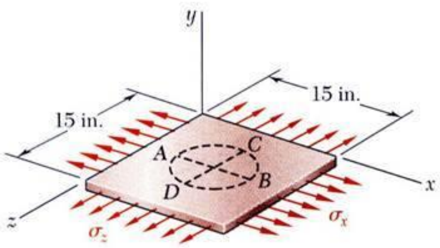
(*a*)



- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1+\nu)$$

Sample Problem 2.5



A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = 3/4$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi.

For $E = 10\text{x}10^6$ psi and $\nu = 1/3$, determine the change in:

a) the length of diameter *AB*,

b) the length of diameter *CD*,

c) the thickness of the plate, and

d) the volume of the plate.

SOLUTION:

- Apply the generalized Hooke's Law to find the three components of normal strain.

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$
$$= \frac{1}{10\times10^6\,\text{psi}}\left[(12\,\text{ksi}) - 0 - \frac{1}{3}(20\,\text{ksi})\right]$$
$$= +0.533\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$
$$= -1.067\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$
$$= +1.600\times10^{-3}\,\text{in./in.}$$

- Evaluate the deformation components.

$$\delta_{B/A} = \varepsilon_x d = (+0.533\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$
$$\boxed{\delta_{B/A} = +4.8\times10^{-3}\,\text{in.}}$$

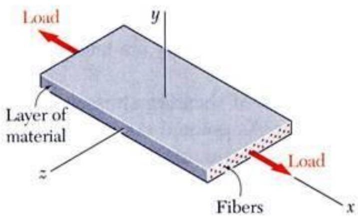
$$\delta_{C/D} = \varepsilon_z d = (+1.600\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$
$$\boxed{\delta_{C/D} = +14.4\times10^{-3}\,\text{in.}}$$

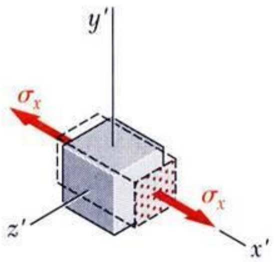
$$\delta_t = \varepsilon_y t = (-1.067\times10^{-3}\,\text{in./in.})(0.75\,\text{in.})$$
$$\boxed{\delta_t = -0.800\times10^{-3}\,\text{in.}}$$

- Find the change in volume

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1.067\times10^{-3}\,\text{in}^3/\text{in}^3$$
$$\Delta V = eV = 1.067\times10^{-3}(15\times15\times0.75)\,\text{in}^3$$
$$\boxed{\Delta V = +0.187\,\text{in}^3}$$

Composite Materials





- *Fiber-reinforced composite materials* are formed from *lamina* of fibers of graphite, glass, or polymers embedded in a resin matrix.
- Normal stresses and strains are related by Hooke's Law but with directionally dependent moduli of elasticity,

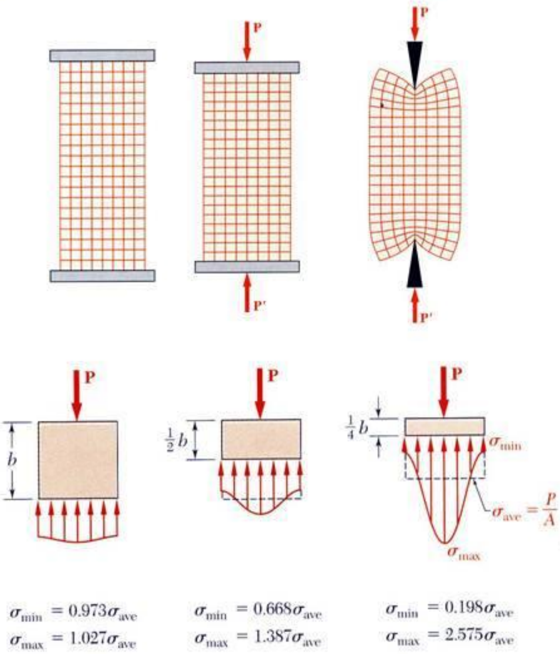
$$E_x = \frac{\sigma_x}{\varepsilon_x} \quad E_y = \frac{\sigma_y}{\varepsilon_y} \quad E_z = \frac{\sigma_z}{\varepsilon_z}$$

- Transverse contractions are related by directionally dependent values of Poisson's ratio, e.g.,

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} \quad \nu_{xz} = -\frac{\varepsilon_z}{\varepsilon_x}$$

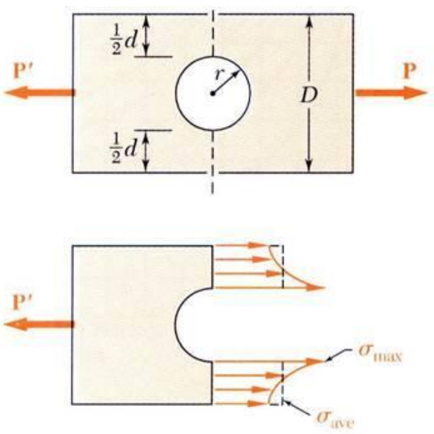
- Materials with directionally dependent mechanical properties are *anisotropic*.

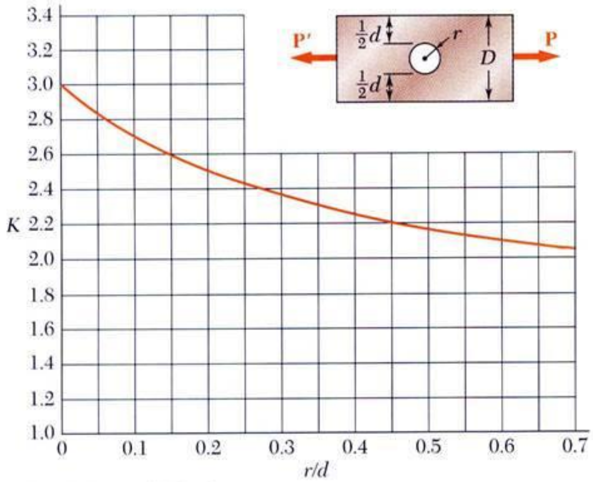
Saint-Venant's Principle



- Loads transmitted through rigid plates result in uniform distribution of stress and strain.
- Concentrated loads result in large stresses in the vicinity of the load application point.
- Stress and strain distributions become uniform at a relatively short distance from the load application points.
- **Saint-Venant's Principle:** Stress distribution may be assumed independent of the mode of load application except in the immediate vicinity of load application points.

Stress Concentration: Hole



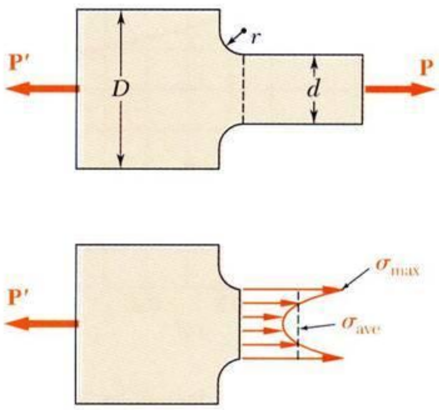


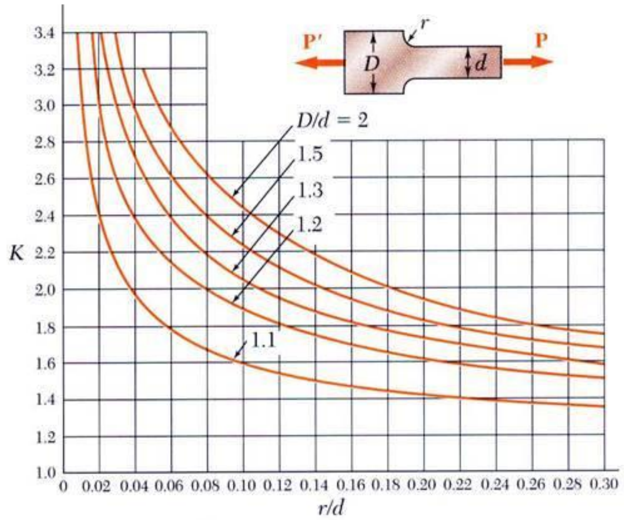
(*a*) Flat bars with holes

Discontinuities of cross section may result in high localized or *concentrated* stresses.

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

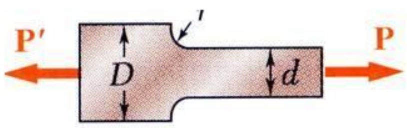
Stress Concentration: Fillet





(b) Flat bars with fillets

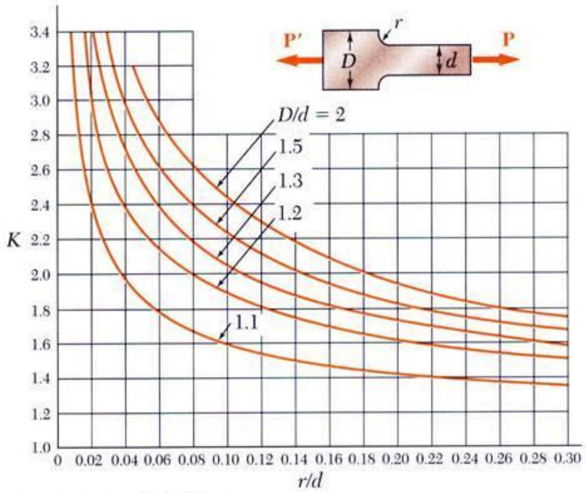
Example 2.12



Determine the largest axial load P that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius r = 8 mm. Assume an allowable normal stress of 165 MPa.

SOLUTION:

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.
- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.
- Apply the definition of normal stress to find the allowable load.



(*b*) Flat bars with fillets

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.

$$\frac{D}{d}=\frac{60\,\text{mm}}{40\,\text{mm}}=1.50 \qquad \frac{r}{d}=\frac{8\,\text{mm}}{40\,\text{mm}}=0.20$$

$$K=1.82$$

- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

$$\sigma_{ave}=\frac{\sigma_{max}}{K}=\frac{165\,\text{MPa}}{1.82}=90.7\,\text{MPa}$$

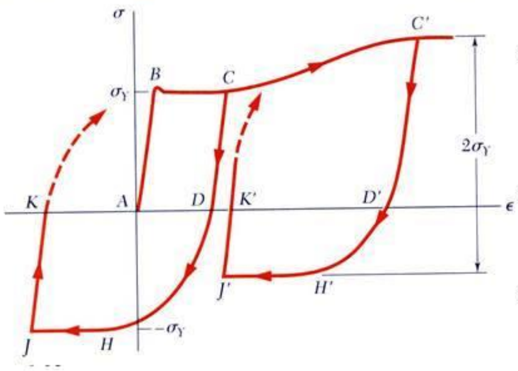
- Apply the definition of normal stress to find the allowable load.

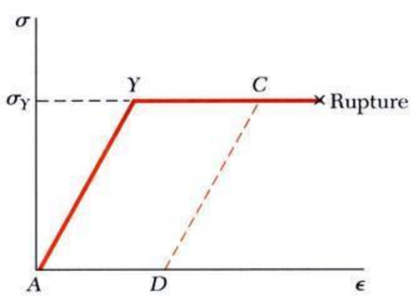
$$P=A\sigma_{ave}=(40\,\text{mm})(10\,\text{mm})(90.7\,\text{MPa})$$

$$=36.3\times10^{3}\,\text{N}$$

$$\boxed{P=36.3\,\text{kN}}$$

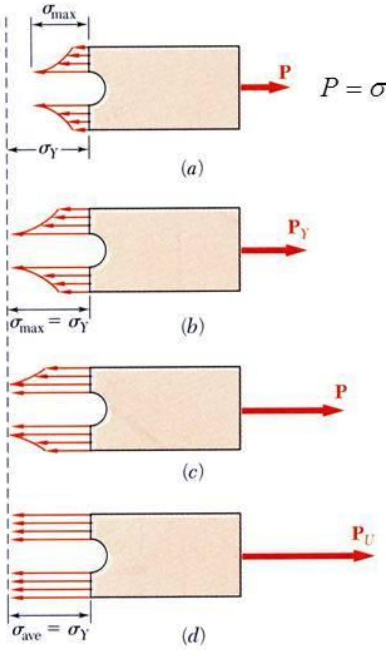
Elastoplastic Materials





- Previous analyses based on assumption of linear stress-strain relationship, i.e., stresses below the yield stress
- Assumption is good for brittle material which rupture without yielding
- If the yield stress of ductile materials is exceeded, then plastic deformations occur
- Analysis of plastic deformations is simplified by assuming an idealized *elastoplastic material*
- Deformations of an elastoplastic material are divided into elastic and plastic ranges
- Permanent deformations result from loading beyond the yield stress

Plastic Deformations



$$P = \sigma_{ave} A = \frac{\sigma_{max} A}{K}$$

- Elastic deformation while maximum stress is less than yield stress

$$P_Y = \frac{\sigma_Y A}{K}$$

- Maximum stress is equal to the yield stress at the maximum elastic loading

- At loadings above the maximum elastic load, a region of plastic deformations develop near the hole

$$P_U = \sigma_Y A = K P_Y$$

- As the loading increases, the plastic region expands until the section is at a uniform stress equal to the yield stress

Residual Stresses

- When a single structural element is loaded uniformly beyond its yield stress and then unloaded, it is permanently deformed but all stresses disappear. This is not the general result.
- *Residual stresses* will remain in a structure after loading and unloading if
  - only part of the structure undergoes plastic deformation
  - different parts of the structure undergo different plastic deformations
- Residual stresses also result from the uneven heating or cooling of structures or structural elements

Example 2.14, 2.15, 2.16

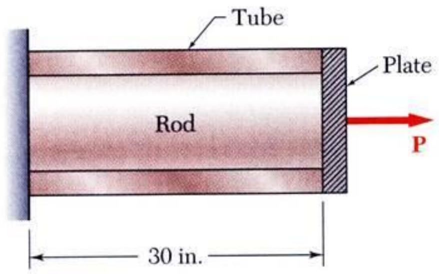
A cylindrical rod is placed inside a tube of the same length. The ends of the rod and tube are attached to a rigid support on one side and a rigid plate on the other. The load on the rod-tube assembly is increased from zero to 5.7 kips and decreased back to zero.

a) draw a load-deflection diagram for the rod-tube assembly

b) determine the maximum elongation

c) determine the permanent set

d) calculate the residual stresses in the rod and tube.

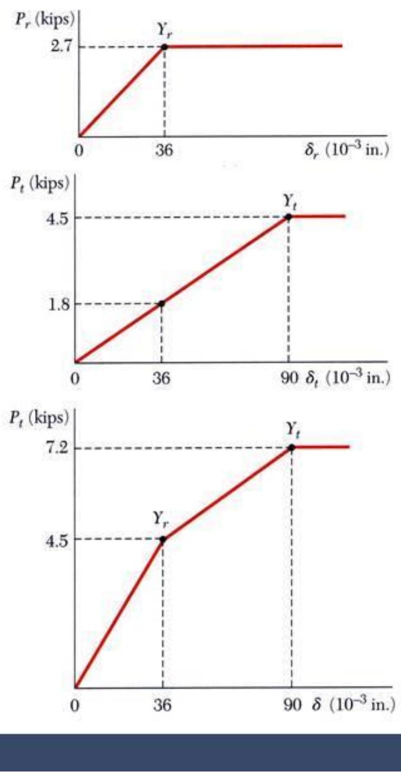


$A_r = 0.075\,\text{in.}^2$ $\qquad A_t = 0.100\,\text{in.}^2$

$E_r = 30\times10^6\,\text{psi}$ $\qquad E_t = 15\times10^6\,\text{psi}$

$\sigma_{Y,r} = 36\,\text{ksi}$ $\qquad \sigma_{Y,t} = 45\,\text{ksi}$

Example 2.14, 2.15, 2.16



a) draw a load-deflection diagram for the rod-tube assembly

$$P_{Y,r} = \sigma_{Y,r}A_r = (36\,\text{ksi})(0.075\,\text{in}^2) = 2.7\,\text{kips}$$

$$\delta_{Y,r} = \varepsilon_{Y,r}L = \frac{\sigma_{Y,r}}{E_{Y,r}}L = \frac{36\times10^3\,\text{psi}}{30\times10^6\,\text{psi}}30\,\text{in.} = 36\times10^{-3}\,\text{in.}$$

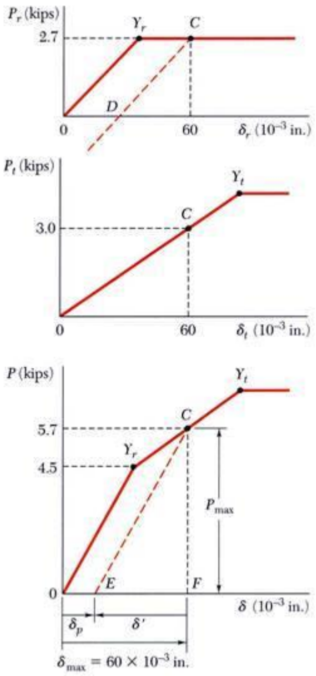
$$P_{Y,t} = \sigma_{Y,t}A_t = (45\,\text{ksi})(0.100\,\text{in}^2) = 4.5\,\text{kips}$$

$$\delta_{Y,t} = \varepsilon_{Y,t}L = \frac{\sigma_{Y,t}}{E_{Y,t}}L = \frac{45\times10^3\,\text{psi}}{15\times10^6\,\text{psi}}30\,\text{in.} = 90\times10^{-3}\,\text{in.}$$

$$P = P_r + P_t$$

$$\delta = \delta_r = \delta_t$$

Example 2.14, 2.15, 2.16



- at a load of $P = 5.7$ kips, the rod has reached the plastic range while the tube is still in the elastic range

$$P_r = P_{Y,r} = 2.7\,\text{kips}$$

$$P_t = P - P_r = (5.7 - 2.7)\,\text{kips} = 3.0\,\text{kips}$$

$$\sigma_t = \frac{P_t}{A_t} = \frac{3.0\,\text{kips}}{0.1\,\text{in}^2} = 30\,\text{ksi}$$

$$\delta_t = \varepsilon_t L = \frac{\sigma_t}{E_t} L = \frac{30\times10^3\,\text{psi}}{15\times10^6\,\text{psi}} 30\,\text{in.} \qquad \boxed{\delta_{max} = \delta_t = 60\times10^{-3}\,\text{in.}}$$

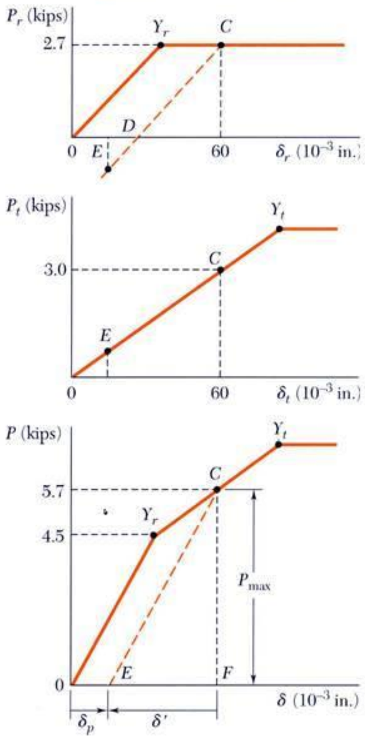
- the rod-tube assembly unloads along a line parallel to $0Y_r$

$$m = \frac{4.5\,\text{kips}}{36\times10^{-3}\,\text{in.}} = 125\,\text{kips/in.} = \text{slope}$$

$$\delta' = -\frac{P_{max}}{m} = -\frac{5.7\,\text{kips}}{125\,\text{kips/in.}} = -45.6\times10^{-3}\,\text{in.}$$

$$\delta_p = \delta_{max} + \delta' = (60 - 45.6)\times10^{-3}\,\text{in.} \qquad \boxed{\delta_p = 14.4\times10^{-3}\,\text{in.}}$$

Example 2.14, 2.15, 2.16



- calculate the residual stresses in the rod and tube.

calculate the reverse stresses in the rod and tube caused by unloading and add them to the maximum stresses.

$$\varepsilon' = \frac{\delta'}{L} = \frac{-45.6\times10^{-3}\,\text{in.}}{30\,\text{in.}} = -1.52\times10^{-3}\,\text{in./in.}$$

$$\sigma'_r = \varepsilon' E_r = \left(-1.52\times10^{-3}\right)\left(30\times10^6\,\text{psi}\right) = -45.6\,\text{ksi}$$

$$\sigma'_t = \varepsilon' E_t = \left(-1.52\times10^{-3}\right)\left(15\times10^6\,\text{psi}\right) = -22.8\,\text{ksi}$$

$$\sigma_{residual,r} = \sigma_r + \sigma'_r = (36 - 45.6)\,\text{ksi} = -9.6\,\text{ksi}$$

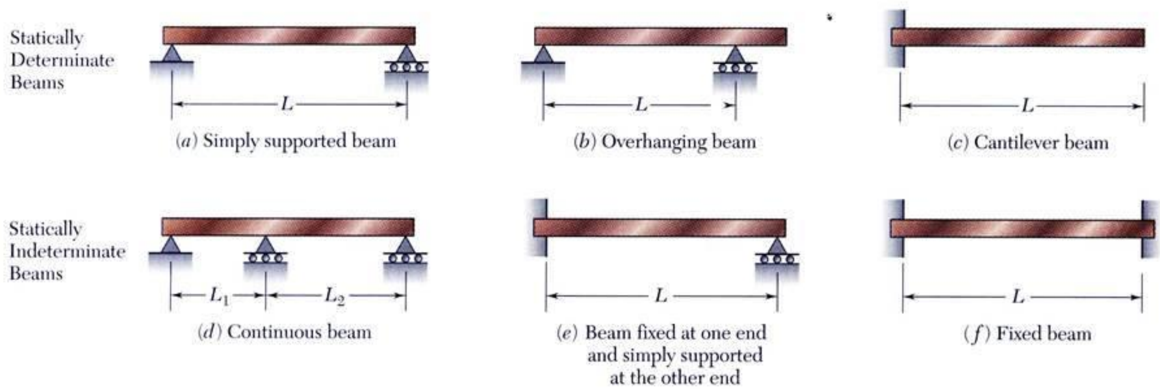
$$\sigma_{residual,t} = \sigma_t + \sigma'_t = (30 - 22.8)\,\text{ksi} = 7.2\,\text{ksi}$$

CHAPTER

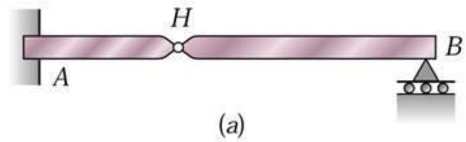
7 ENGINEERING MECHANICS OF SOLIDS

Shear Force, Axial Force and Bending Moment Diagrams

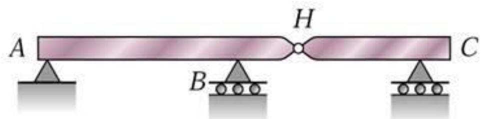
Classification of Beam Supports



Internal Hinge



(a)



(b)

7-3. Calculations of Beam Reactions

Example7-1

Find the reaction at the supports for a simple beam loaded as shown in Fig. 7-7(a). Neglect the weight of the beam.

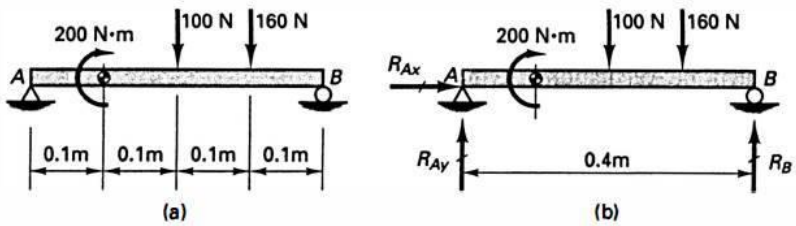


Fig. 7-7

$\sum F_x = 0$ $\quad R_{Ax} = 0$

$\sum M_A = 0 \curvearrowright + \quad 200 + 100 \times 0.2 + 160 \times 0.3 - R_B \times 0.4 = 0$

$R_B = +670 \text{ N} \uparrow$

$\sum M_B = 0 \curvearrowright + \quad R_{Ay} \times 0.4 + 200 - 100 \times 0.2 - 160 \times 0.1 = 0$

$R_{Ay} = -410 \text{ N} \downarrow$

Example7-2

Find the reactions for the partially loaded beam with a uniformly varying load shown in Fig. 7-8(a). Neglect the weight of the beam.

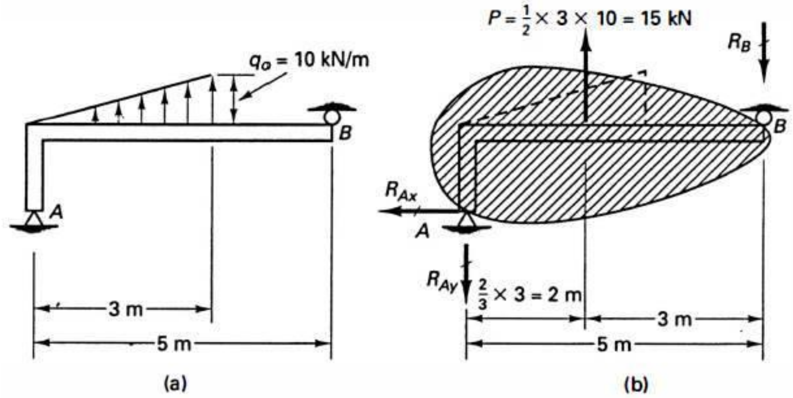


Fig. 7-8

$\Sigma F_x = 0$		$R_{Ax} = 0$
$\Sigma M_A = 0 \curvearrowleft +$	$+15 \times 2 - R_B \times 5 = 0$	$R_B = 6\ \text{kN} \downarrow$
$\Sigma M_B = 0 \curvearrowright +$	$-R_{Ay} \times 5 + 15 \times 3 = 0$	$R_{Ay} = 9\ \text{kN} \downarrow$

Example 7-3

Determine the reactions at A and B for the beam shown in Fig. 7-9(a) due to the applied force.

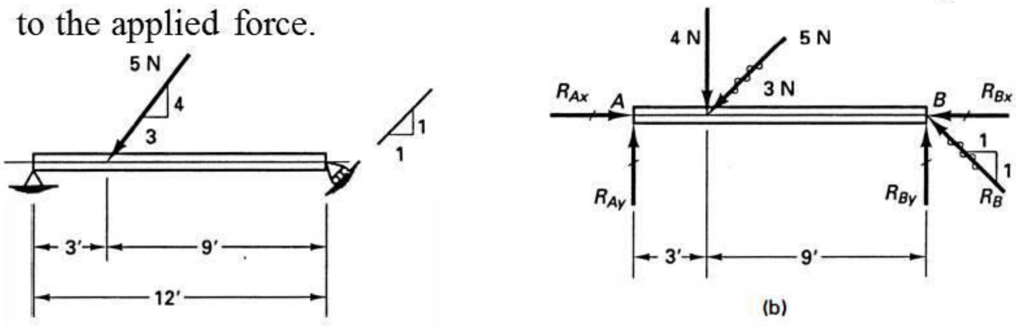
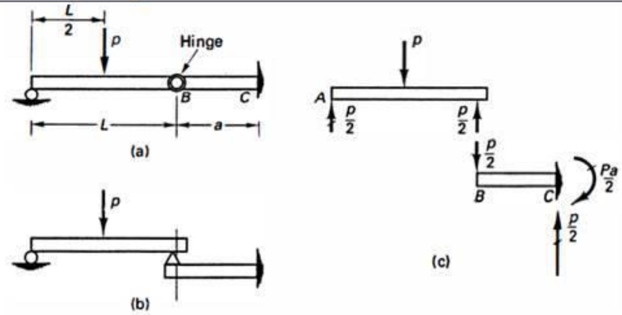


Fig. 7-9

$\Sigma M_A = 0 \curvearrowright +$	$4 \times 3 - R_{By} \times 12 = 0$	$R_{By} = 1\,\text{k} \uparrow = \lvert R_{Bx} \rvert$
$\Sigma M_B = 0 \curvearrowright +$	$R_{Ay} \times 12 - 4 \times 9 = 0$	$R_{Ay} = 3\,\text{k} \uparrow$
$\Sigma F_x = 0 \rightarrow +$	$R_{Ax} - 3 - 1 \quad = 0$	$R_{Ax} = 4\,\text{k} \rightarrow$
		$R_A = \sqrt{4^2 + 3^2} = 5\text{k}$
		$R_B = \sqrt{1^2 + 1^2} = \sqrt{2}\text{k}$

Check: $\Sigma F_y = 0 \uparrow +$ $\qquad + 3 - 4 + 1 = 0$

Hinges or pinned joints



Occasionally, hinges or pinned joints are introduced into beams and frames. A hinge is capable of transmitting only horizontal and vertical forces. No moment can be transmitted at a hinged joint. Therefore, the point where a hinge occurs is a particularly convenient location for "separation“ of the structure into parts for purposes of computing the reactions. This process is illustrated in Fig. 7-10. Each part of the beam so separated is treated independently. Each hinge provides an extra axis around which moments may be taken to determine reactions. The introduction of a hinge or hinges into a continuous beam in many cases makes the system statically determinate. The introduction of a hinge into a determinate beam results in a beam that is not stable. Note that the reaction at the hinge for one beam acts in an opposite direction on the other beam.

DIRECT APPROACH FOR P, V & M (Method of Sections)

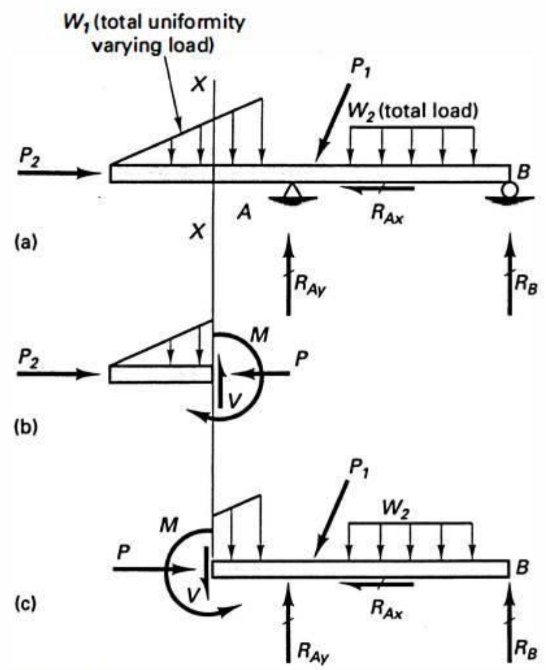


Fig. 7-11 An application of the method of sections to a statically determinate beam.

Definition of positive shear

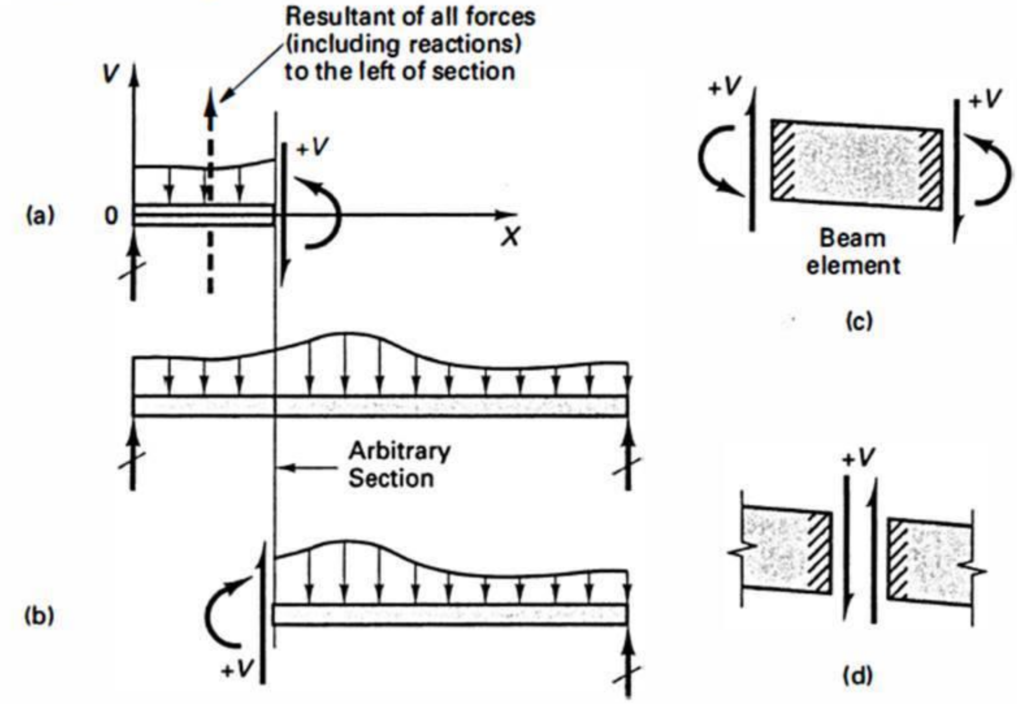


Fig. 7-12 Definition of positive shear.

Positive shear and Bending moment

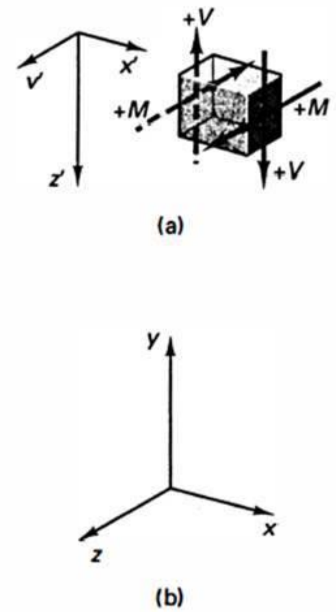


Fig. 7-13 Positive sense of shear and bending moment defined in (a) is used in this text with coordinates shown in (b).

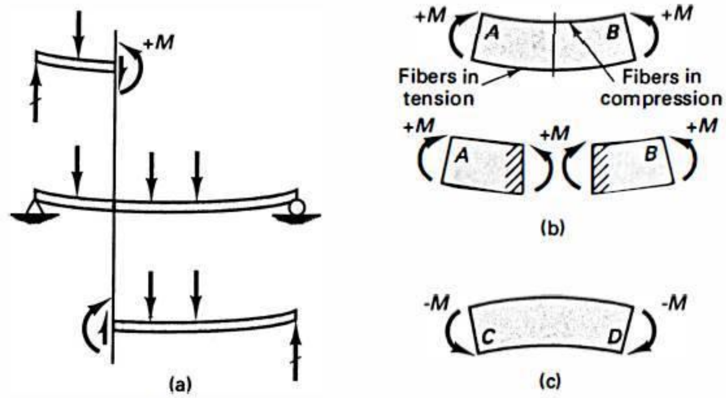


Fig. 7-14 Definition of bending moment signs.

Example7-4

Consider earlier Example 7-2 and determine the internal system of forces at sections a-a and b-b; see Fig. 7-15(a).

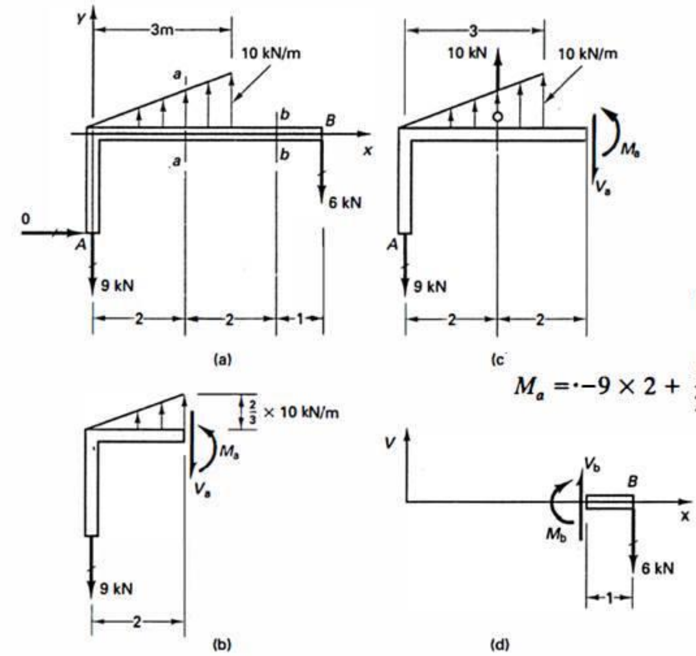


Fig. 7-15

$$V_a = -9 + \frac{1}{2} \times 2 \times \frac{2}{3} \times 10 = -2.33 \text{ kN}$$

$$M_a = -9 \times 2 + \frac{1}{2} \times 2 \times \frac{2}{3} \times 10 \times \frac{1}{3} \times 2 = -13.6 \text{ kN} \cdot \text{m}$$

$$V_b = +6 \text{ kN}$$

$$M_b = -6 \times 1 = -6 \text{ kN} \cdot \text{m}$$

P, V, and M Diagrams

Example 7-5

$V = +2\text{ k}$ for $0 < x < 5$

$V = -2\text{ k}$ for $5 < x < 10$

$M = +2x\text{ k-ft}$ for $0 \le x \le 5$

$M = +2x - 4(x - 5) = +20 - 2x\text{ k-ft}$ for $5 \le x \le 10$

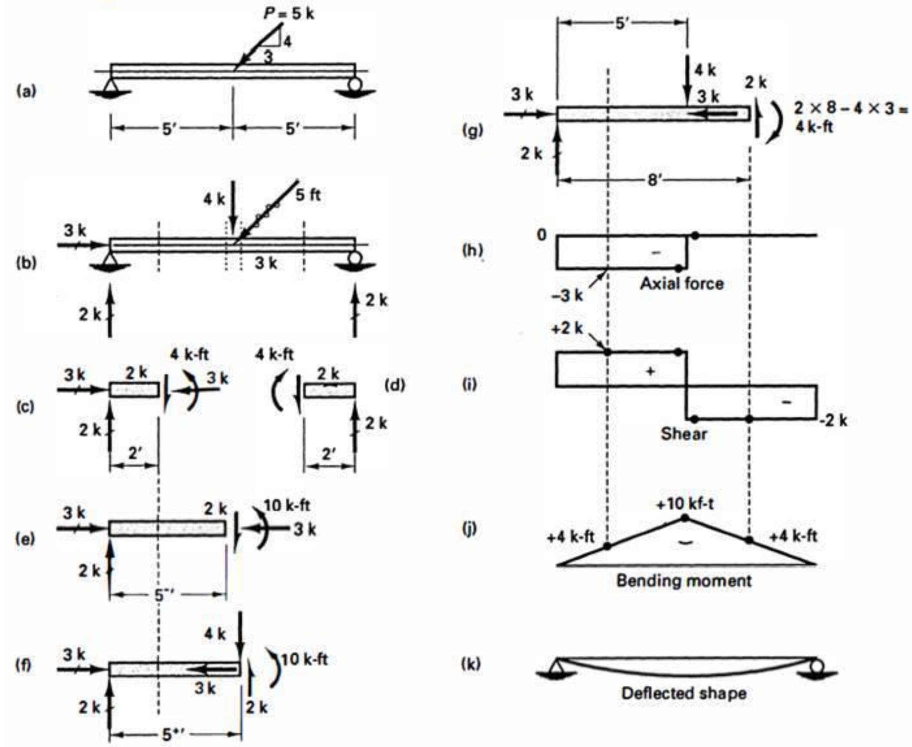


Fig. 7-16

Example7-6

Determine axial-force, shear, and bending-moment diagrams for the cantilever loaded with an inclined force at the end; see Fig. 7-17(a).

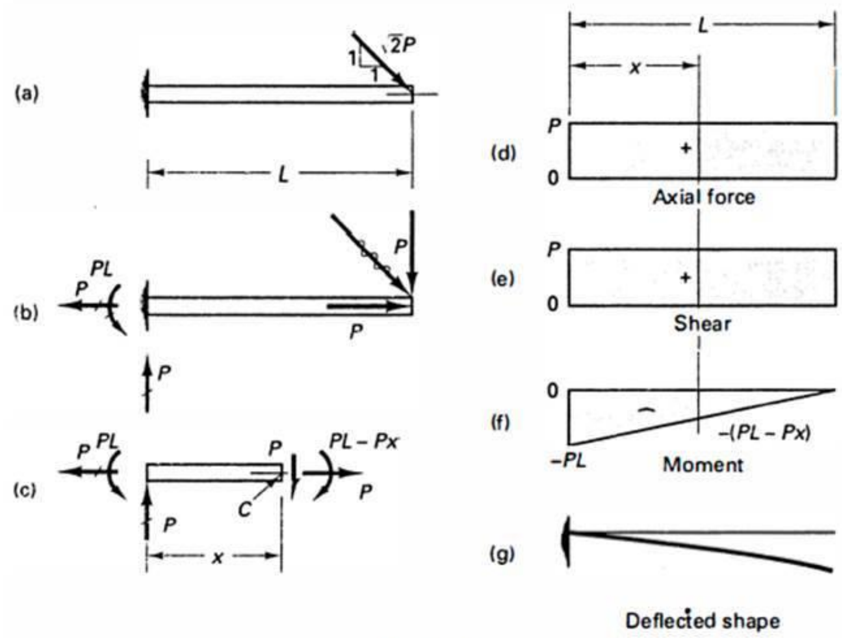


Fig. 7-17

Example7-7

Construct shear and bending-moment diagrams for the beam loaded with the forces shown in Fig. 7-18(a).

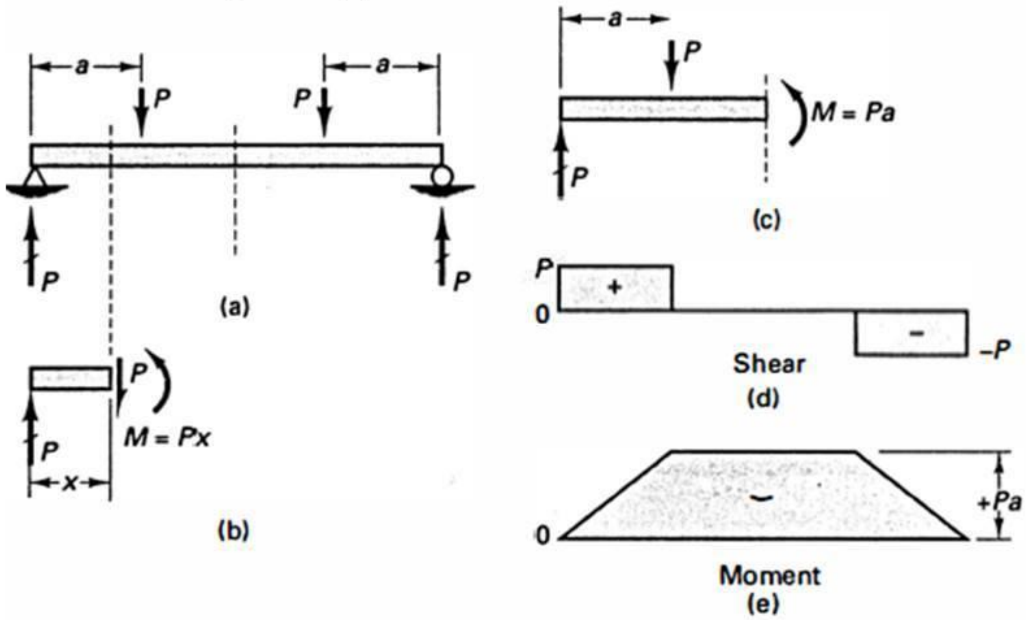


Fig. 7-18

Example7-8

Plot shear and a bending-moment diagram for a simple beam with a uniformly distributed load; see Fig. 7-19.

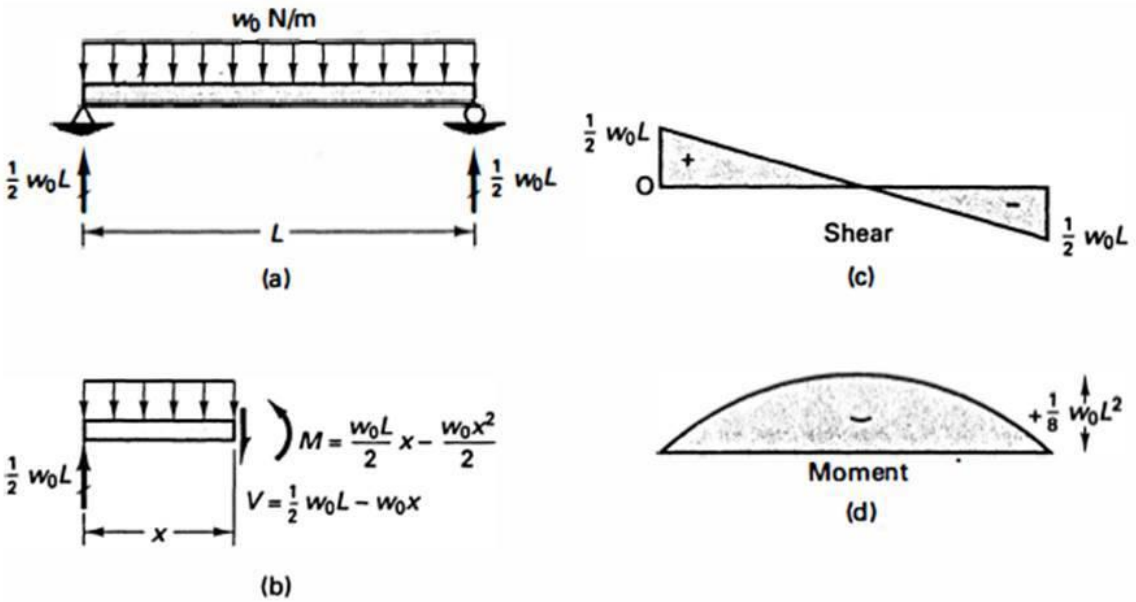


Fig. 7-19

Example7-9

For the beam in Example 7-4, shown in Fig. 7-20(a), express the shear V and the bending moment M as a function of x along the horizontal member.

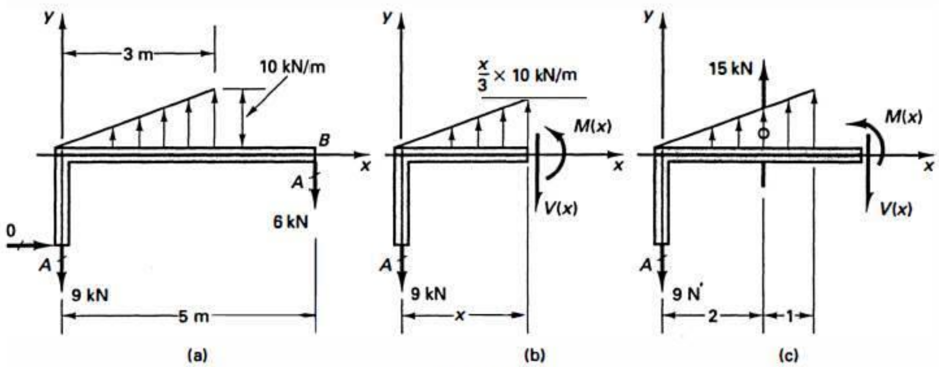


Fig. 7-20

der in Fig. 7-20(c). The required expressions for $0 < x < 3$ are

$$V(x) = -9 + \frac{1}{2}x\left(\frac{x}{3}\times 10\right) = -9 + \frac{5}{3}x^2 \text{ kN}$$

$$M(x) = -9x + \frac{1}{2}x\left(\frac{x}{3}\times 10\right)\left(\frac{x}{3}\right) = -9x + \frac{5}{9}x^3 \text{ kN}\cdot\text{m}$$

For $3 < x < 5$,

$$V(x) = -9 + 15 = +6 \text{ kN}$$

$$M(x) = -9x + 15(x-2) = 6x - 30 \text{ kN}\cdot\text{m}$$

Example 7-10

Write analytic expressions for V and M for the beam shown in Fig. 7-21.

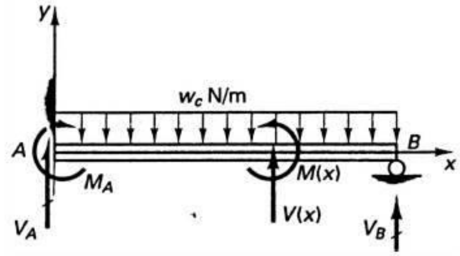


Fig. 7-21

SOLUTION

Unlike the preceding cases, this is a statically indeterminate problem to the first degree having one redundant reaction. There is no horizontal reaction at A. Except for carefully identifying the unknown reactions as V_A, V_B, and M_A, the procedure is the same as before, although numerical results cannot be obtained until the reactions are determined. On this basis, at a distance x away from the origin,

$$V(x) = V_A - w_o x$$

and

$$M(x) = M_A + V_A x - (w_o x)x/2$$
$$= M_A + V_A x - w_o x^2/2$$

Example 7-11

Consider a structural system of three interconnected straight bars, as shown in Fig. 7-22(a). At arbitrary sections, determine the internal forces P, V, and M in the members caused by the application of a vertical force P_1 at D.

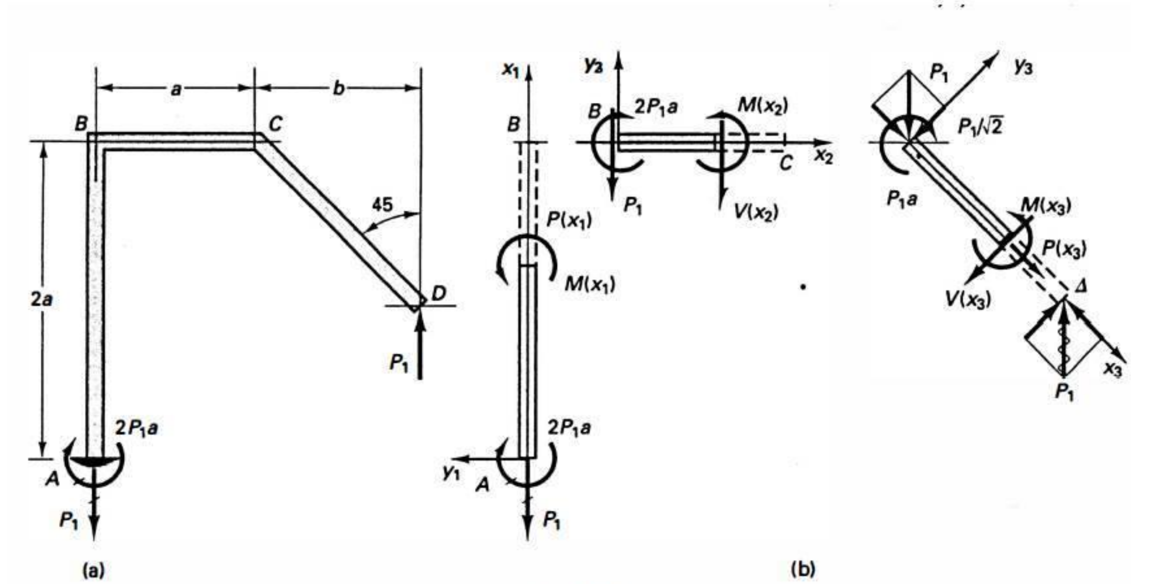


Fig. 7-22

SOLUTION

The solution begins by calculating the reaction at A, which is then shown on beam segment AB. At an arbitrary section through this beam, the internal forces are seen to be

$$P(x_1) = +P_1,\ V(x_1) = 0 \quad \text{and} \quad M(x_1) = +2P_1 a$$

These forces are constant throughout the length of the vertical bar and become the reactions at B for the beam segment BC. It is important to note that the axial force in member AB acts as shear in BC. After the reactions at B for BC are known, the usual procedure gives the following internal forces:

$$P(x_2) = 0,\ V(x_2) = -P_1, \quad \text{and} \quad M(x_2) = +2P_1 a - P_1 x_2$$

For member CD, except for the need for resolving the force P_1 at C, the procedure for determining the internal forces is the same as before, giving

$$P(x_3) = -P_1/\sqrt{2},\ V(x_3) = -P_1/\sqrt{2}, \quad \text{and} \quad M(x_3) = +P_1 a - P_1 x_3/\sqrt{2}$$

By substituting $x_3 = \sqrt{2}a$ into the last expression, it can be verified that the bending moment at D is zero, as it should be.

Shear and bending-moment diagrams for this structural system can be plotted directly on the outline of the frame.

V AND M BY INTEGRATION

7-9. Differential Equations of Equilibrium for a Beam Element

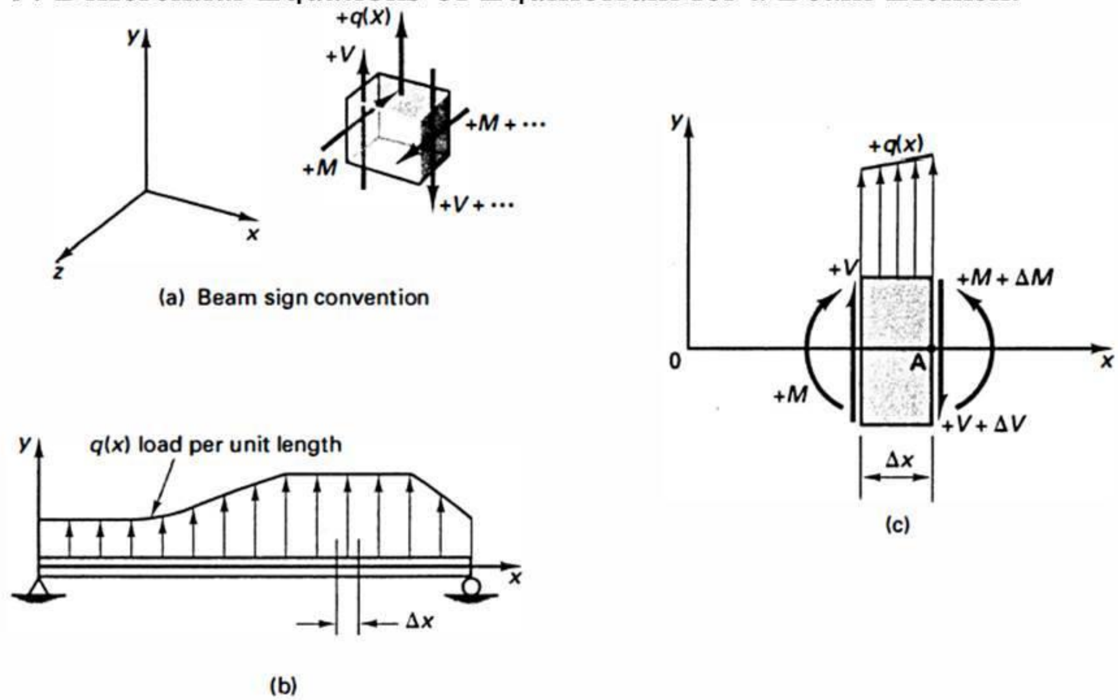


Fig. 7-24 Beam and beam elements between adjoining sections.

Consider a beam element Δx long, isolated by two adjoining sections taken perpendicular to its axis. Fig. 7-24(b). Such an element is shown as a free body in Fig. 7-24(c). All the forces shown acting on this element have positive sense. The positive sense of the distributed external force q is taken to coincide with the direction of the positive y axis. As the shear and the moment may each change from one section to the next, note that on the right side of the element, these quantities are, respectively, designated $V + \Delta V$ and $M + \Delta M$.

From the condition for equilibrium of vertical forces, one obtains[3]

$$\sum F_y = 0 \uparrow + \quad V + q\Delta x - (V + \Delta V) = 0$$

or

$$\frac{\Delta V}{\Delta x} = q \tag{7-1}$$

For equilibrium, the summation of moments around A also must be zero. So, upon noting that from point A the arm of the distributed force is $\Delta x/2$, one has

$$\sum M_A = 0 \curvearrowleft + \quad (M + \Delta M) - V\Delta x - M - (q\Delta x)(\Delta x/2) = 0$$

$$\frac{\Delta M}{\Delta x} = V + \frac{q\Delta x}{2} \qquad (7\text{-}2)$$

Equations 7-1 and 7-2 in the limit as $\Delta x \to 0$ yield the following two basic differential equations:

$$\boxed{\frac{dV}{dx} = q} \qquad (7\text{-}3)$$

and

$$\boxed{\frac{dM}{dx} = V} \qquad (7\text{-}4)$$

By substituting Eq. 7-4 into Eq. 7-3, another useful reaction is obtained:

$$\frac{d}{dx}\left(\frac{dM}{dx}\right) = \frac{d^2M}{dx^2} = q \qquad (7\text{-}5)$$

This differential equation can be used for determining reactions of statically determinate beams from the boundary conditions, whereas Eqs. 7-3 and 7-4 are very convenient for construction of shear and moment diagrams. These applications will be discussed next.

7-10. Shear Diagrams by Integration of the Load

$$\frac{dV}{dx} = q \qquad (7\text{-}3)$$

$$V = \int_0^x q\,dx + C_1 \qquad (7\text{-}6)$$

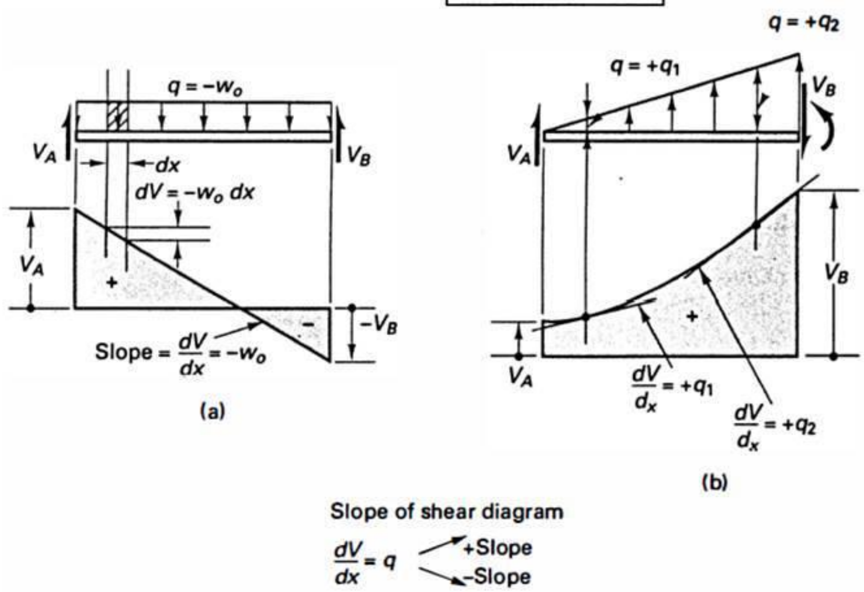


Fig. 7-25 Shear diagrams for (a) a uniformly distributed load intensity, and (b) a uniformly increasing load intensity.

By assigning definite limits to this integral, it is seen that the shear at a section is simply an integral (i.e., a sum) of the vertical forces along the beam from the left end of the beam *to the section in question* plus a constant of integration C_1. This constant is equal to the shear on the left-hand end. Between any two definite sections of a beam, the shear changes by the amount of the vertical force included *between* these sections. If no force occurs between any two sections, no change in shear takes place. If a concentrated force comes into the summation, a discontinuity, or a "jump," in the value of the shear occurs. The continuous summation process remains valid nevertheless, since a concentrated force may be thought of as being a distributed force extending for an infinitesimal distance along the beam.

On the basis of the preceding reasoning, a shear diagram can be established by the summation process. For this purpose, *the reactions must always be determined first.* Then the vertical components of forces *and reactions* are successively summed *from the left end* of the beam to preserve the mathematical sign convention for shear adopted in Fig. 7-12. The shear at a section is simply equal to the sum of *all* vertical forces to the left of the section.

When the shear diagram is constructed from the load diagram by the summation process, two important observations can be made regarding its shape. First, the sense of the applied load determines the sign of the slope of the shear diagram. If the applied load acts upward, the slope of the shear diagram is positive, and vice versa. Second, this slope is equal to the corresponding applied load intensity. For example, consider a segment of a beam with a uniformly distributed downward load w_o and known shears at both ends, as shown in Fig. 7-25(a). Since here the applied load intensity w_o is *negative* and *uniformly distributed* (i.e., $q = -w_o =$ constant), the slope of the shear diagram exhibits the same characteristics. Alternatively, the linearly varying load intensity acting upward on a beam segment with known shears at the ends, shown in Fig. 7-25(b), gives rise to a differently shaped shear diagram. Near the left end of this segment, the locally applied *upward* load q_1 is *smaller* than the corresponding one q_2 near the right end. Therefore, the *positive* slope of the shear diagram on the left is *smaller* than it is on the right, and the shear diagram is concave upward.

7-11. Moment Diagrams by Integration of the Shear

$$M = \int_0^x V\,dx + C_2 \qquad (7\text{-}7)$$

$$\frac{dM}{dx} = V \qquad (7\text{-}4)$$

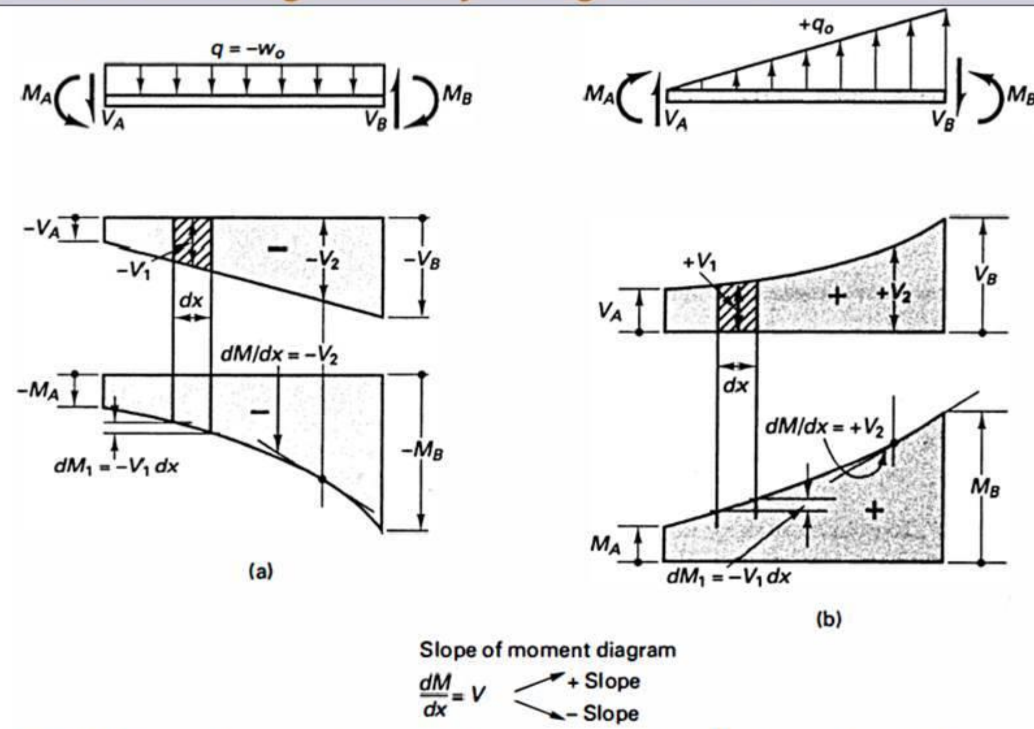


Fig. 7-26 Shear and moment diagrams for (a) a uniformly distributed load intensity, and (b) a uniformly increasing load intensity.

where C_2 is a constant of integration corresponding to boundary conditions at $x = 0$. This equation is analogous to Eq. 7-6 developed for the construction of shear diagrams. The meaning of the term $V\,dx$ is shown graphically by the hatched areas of the shear diagrams in Fig. 7-26. The summation of these areas between definite sections through a beam corresponds to an evaluation of the definite integral. If the ends of a beam are on rollers, pin-ended, or free, the starting and the terminal moments are zero. If the end is built-in (fixed against rotation), in statically determinate beams, the end moment is known from the reaction calculations. If the fixed end of a beam is on the left, this moment with the proper[4] sign is the *initial constant of integration* C_2.

By proceeding *continuously along the beam from the left-hand end* and summing up the areas of the shear diagram with due regard to their sign, the moment diagram is obtained. This process of obtaining the moment diagram from the shear diagram by summation is exactly the same as that employed earlier to go from loading to shear diagrams. *The change in moment in a given segment of a beam is equal to the area of the corresponding shear diagram.* Qualitatively, the shape of a moment diagram can be easily established from the slopes at some selected points along the beam. These slopes have the same sign and magnitude as the corresponding shears on the shear diagram, since according to Eq. 7-4, $dM/dx = V$. Alternatively, the change of moment $dM = V\,dx$ can be studied along the beam. Examples are shown in Fig. 7-26. According to these principles, variable shears cause nonlinear variation of the moment. A constant shear produces a uniform change in the bending moment, resulting in a straight line in the moment diagram. If no shear occurs along a certain portion of a beam, *no change in moment* takes place.

Since $dM/dx = V$, according to the fundamental theorem of calculus, the *maximum or minimum moment occurs where the shear is zero.*

In a bending-moment diagram obtained by summation, *at the right-hand end* of the beam, an invaluable check on the work is available again. *The terminal conditions for the moment must be satisfied.* If the end is free or pinned, the computed sum must equal zero. If the end is built-in, the end moment computed by summation equals the one calculated initially for the reaction. These are the boundary conditions and must always be satisfied.

Example 7-13

Construct shear and moment diagrams for the symmetrically loaded beam shown in Fig. 7-27(a) by the integration process.

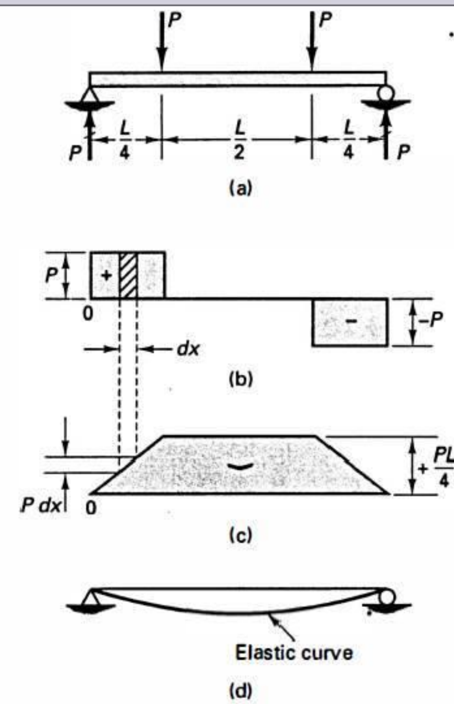


Fig. 7-27

Example 7-14

Consider a simple beam with a uniformly increasing load intensity from an end, as shown in Fig. 7-28(a). The total applied load is W. (a) Construct shear and moment diagrams with the aid of the integration process.
(b) Derive expressions for V and M using Eq. 7-5.

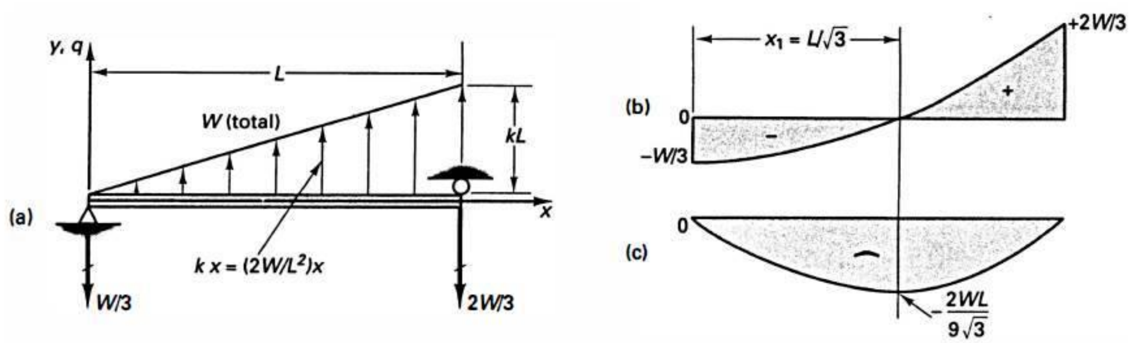


Fig. 7-28

SOLUTION

(a) Since the total load $W = kL^2/2, k = 2W/L^2$. For the given load distribution, the downward reactions are $W/3$ and $2W/3$, as shown in Fig. 7-28(a). Therefore, the shear diagram given in Fig. 7-28(b) begins and ends as shown. Since the rate of applied load is smaller on the left end than on the right, the shear diagram is concave upward. The point of zero shear occurs where the reaction on the left is balanced by the applied load; that is,

$$\frac{W}{3} = \frac{1}{2}x_1 \frac{2W}{L^2} x_1 \quad \text{hence, } x_1 = \frac{L}{\sqrt{3}}$$

At x_1, the bending moment is maximum; therefore,

$$M_{\max} = M\left(\frac{L}{\sqrt{3}}\right) = -\frac{W}{3}\frac{L}{\sqrt{3}} + \frac{1}{2}\frac{L}{\sqrt{3}}\frac{2W}{L^2}\frac{L}{\sqrt{3}}\left(\frac{1}{3}\frac{L}{\sqrt{3}}\right) = -\frac{2WL}{9\sqrt{3}}$$

By following the rules given in Fig. 7-26, the moment diagram has the shape shown in Fig. 7-28(c).

Although the shear and bending moment diagrams could be sketched qualitatively, it was necessary to supplement the results analytically for determining the critical values.

(b) Applying Eq. 7-5 and integrating it twice, one has

$$\frac{d^2M}{dx^2} = q = +kx = +\frac{2W}{L^2}x$$

$$\frac{dM}{dx} = \frac{kx^2}{2} + C_1 \quad \text{and} \quad M = \frac{kx^3}{6} + C_1x + C_2$$

However, the boundary conditions require that the moments at $x = 0$ and $x = L$ be zero; that is, $M(0) = 0$ and $M(L) = 0$. Therefore, since

$$M(0) = 0 \quad C_2 = 0$$

and, similarly, since $M(L) = 0$,

$$\frac{kL^3}{6} + C_1L = 0 \quad \text{or} \quad C_1 = -\frac{kL^2}{6}$$

With these constants,

$$V = \frac{dM}{dx} = \frac{kx^2}{2} - \frac{kL^2}{6} = \frac{Wx^2}{L^2} - \frac{W}{3}$$

and

$$M = \frac{kx^2}{6} - \frac{kL^2x}{6} = \frac{Wx^3}{3L^2} - \frac{Wx}{3}$$

Example 7-15

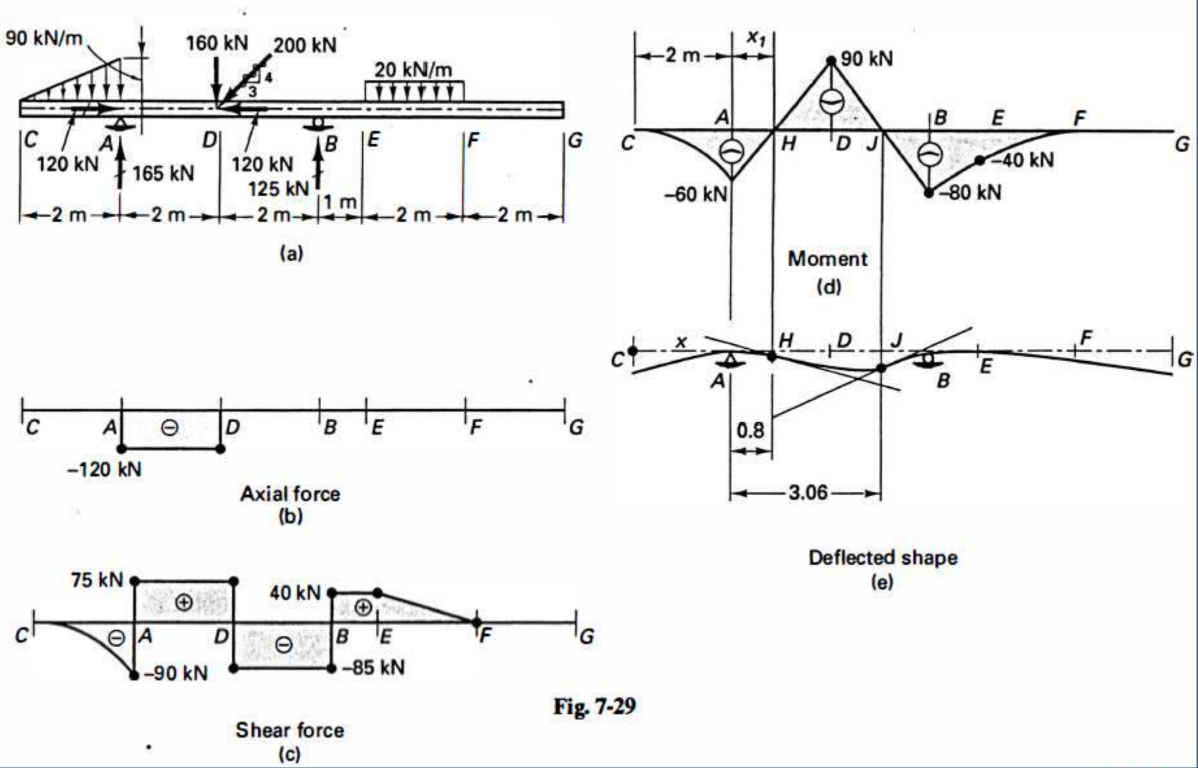


Fig. 7-29

7-12. Effect of Concentrated Moment on Moment Diagrams

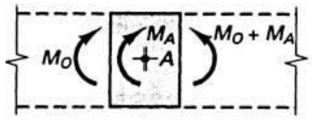


Fig. 7-30 An external concentrated moment acting on an element of a beam.

In the derivation for moment diagrams by summation of shear-diagram areas, no external concentrated moment acting on the infinitesimal element was included, yet such a moment may actually be applied. Hence, the summation process derived applies only up to the point of application of an external moment. At a section just beyond an externally applied moment, a different bending moment is required to maintain the segment of a beam in equilibrium. For example, in Fig. 7-30 an external clockwise moment M_A is acting on the element of the beam at A. Then, if the internal clockwise moment on the left is M_0, for equilibrium of the element, the resisting counterclockwise moment on the right must be $M_0 + M_A$. At the point of the externally applied moment, a discontinuity, or a "jump," equal to the concentrated moment appears in the moment diagram. Hence, in applying the summation process, due regard must be given the concentrated moments as their effect is not apparent in the shear diagram.

Example 7-16

Construct the bending-moment diagram for the horizontal beam loaded as shown in Fig. 7-31(a).

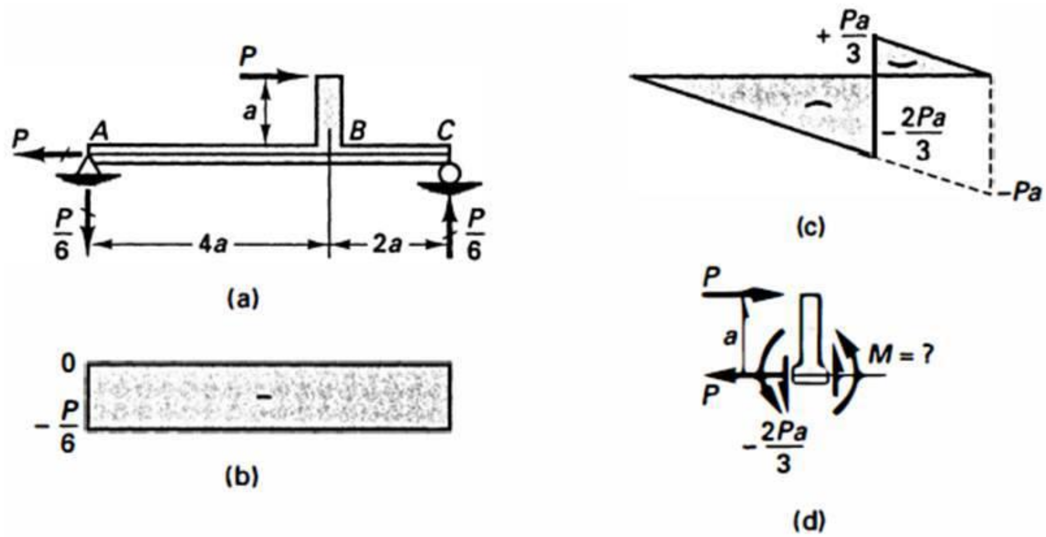


Fig. 7-31

Example 7-17

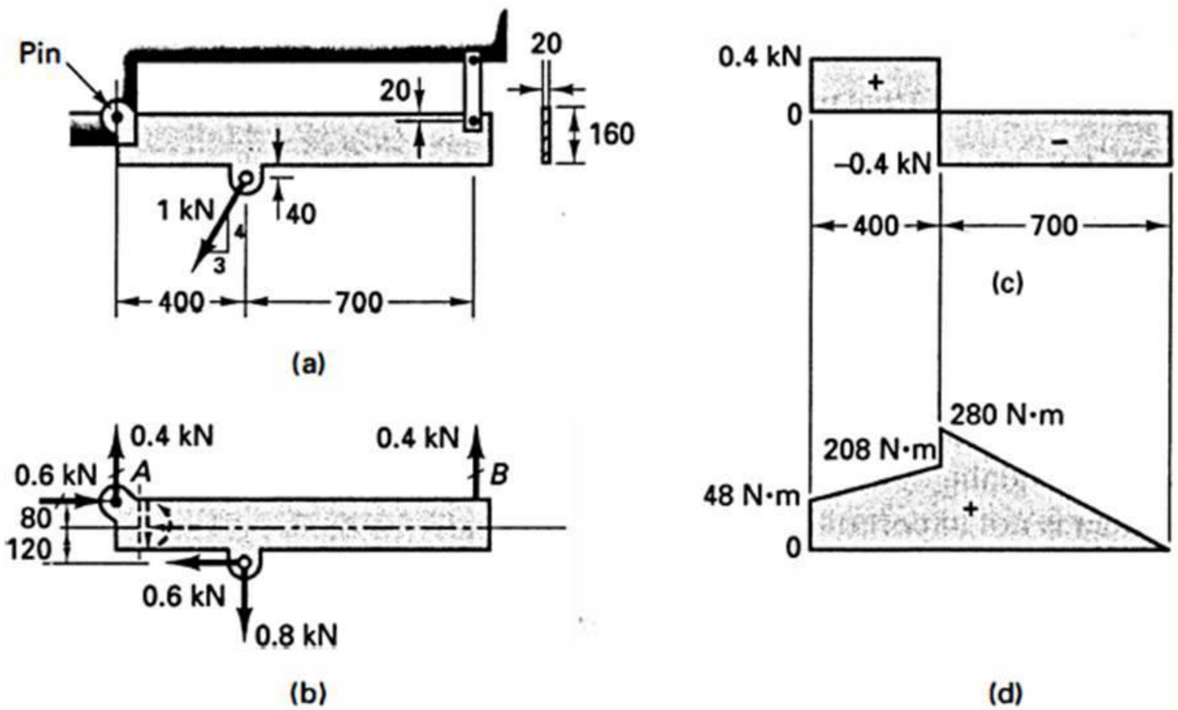
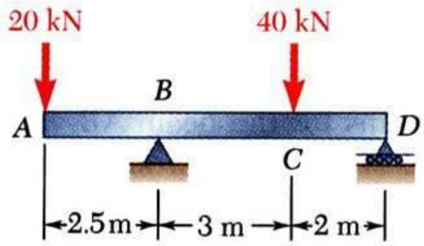


Fig. 7-32

Problems for solution from Popov

In addition to other books mentioned, solve following problems from Popov:
3-10, 13-19, 24-28, 40-66, 71-73, 77-83 & 88-93

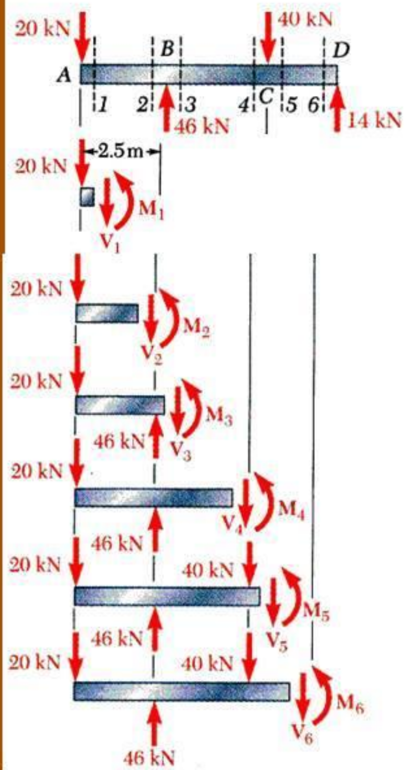
Sample Problems



Draw the shear and bending moment diagrams for the beam and loading shown.

SOLUTION:

- Taking entire beam as a free-body, calculate reactions at *B* and *D*.
- Find equivalent internal force-couple systems for free-bodies formed by cutting beam on either side of load application points.
- Plot results.



SOLUTION:

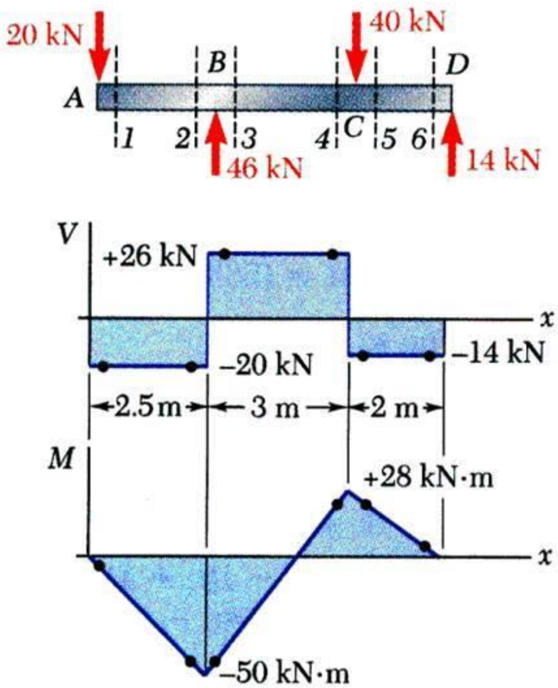
- Taking entire beam as a free-body, calculate reactions at B and D.
- Find equivalent internal force-couple systems at sections on either side of load application points.

$$\sum F_y = 0: \quad -20\,\text{kN} - V_1 = 0 \qquad \boxed{V_1 = -20\,\text{kN}}$$

$$\sum M_2 = 0: \quad (20\,\text{kN})(0\,\text{m}) + M_1 = 0 \qquad \boxed{M_1 = 0}$$

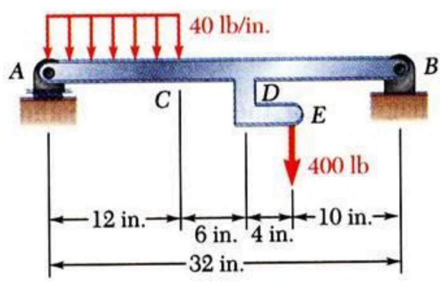
Similarly,

$$\boxed{\begin{array}{ll} V_3 = 26\text{kN} & M_3 = -50\text{kN}\cdot\text{m} \\ V_4 = 26\text{kN} & M_4 = -50\text{kN}\cdot\text{m} \\ V_5 = 26\text{kN} & M_5 = -50\text{kN}\cdot\text{m} \\ V_6 = 26\text{kN} & M_6 = -50\text{kN}\cdot\text{m} \end{array}}$$



- Plot results.

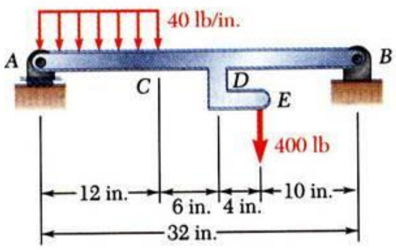
  Note that shear is of constant value between concentrated loads and bending moment varies linearly.

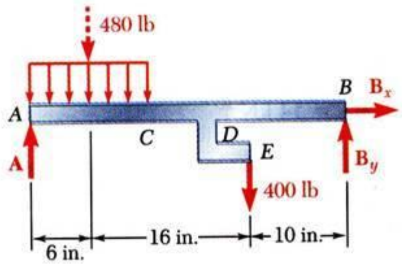


Draw the shear and bending moment diagrams for the beam *AB*. The distributed load of 40 lb/in. extends over 12 in. of the beam, from *A* to *C*, and the 400 lb load is applied at *E*.

SOLUTION:

- Taking entire beam as free-body, calculate reactions at *A* and *B*.
- Determine equivalent internal force-couple systems at sections cut within segments *AC*, *CD*, and *DB*.
- Plot results.





SOLUTION:

- Taking entire beam as a free-body, calculate reactions at A and B.

$$\sum M_A = 0:$$

$$B_y(32\,\text{in.}) - (480\,\text{lb})(6\,\text{in.}) - (400\,\text{lb})(22\,\text{in.}) = 0$$

$$\boxed{B_y = 365\,\text{lb}}$$

$$\sum M_B = 0:$$

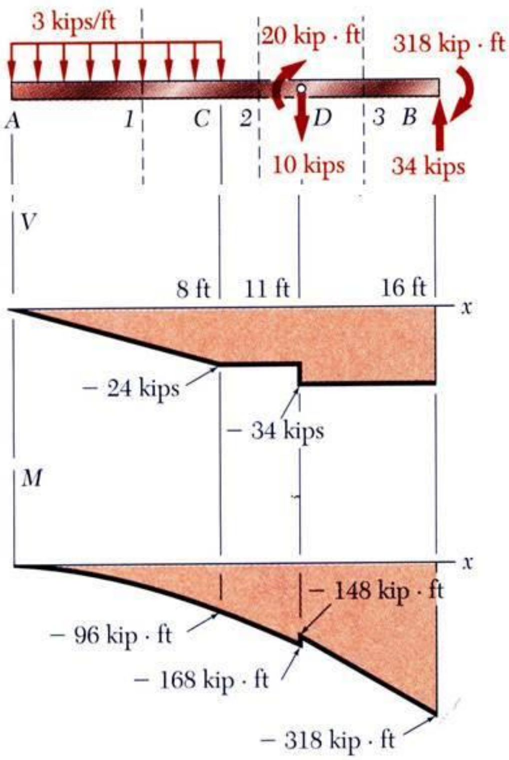
$$(480\,\text{lb})(26\,\text{in.}) + (400\,\text{lb})(10\,\text{in.}) - A(32\,\text{in.}) = 0$$

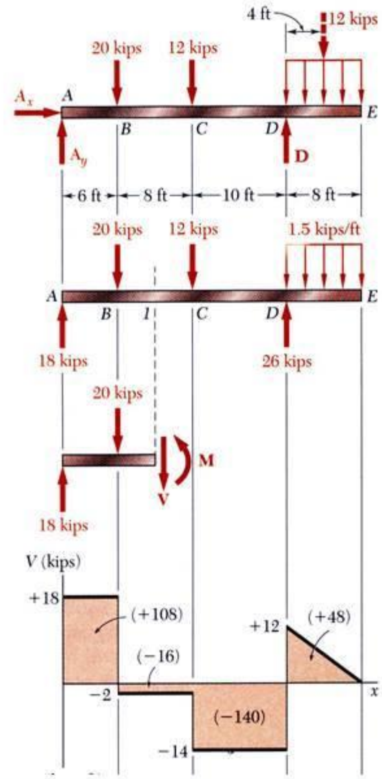
$$\boxed{A = 515\,\text{lb}}$$

$$\sum F_x = 0: \qquad \boxed{B_x = 0}$$

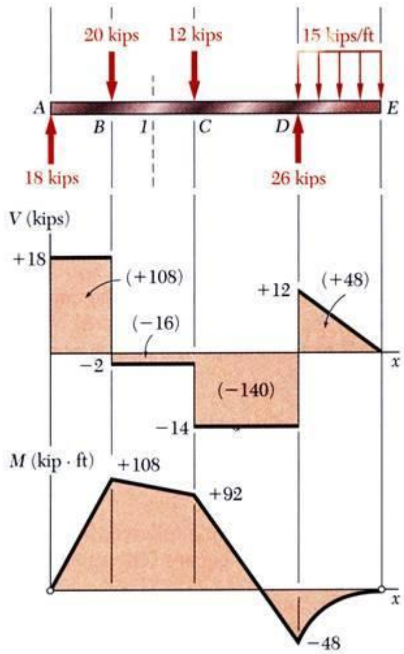
- Note: The 400 lb load at E may be replaced by a 400 lb force and 1600 lb-in. couple at D.

ENGINEERING MECHANICS OF SOLIDS

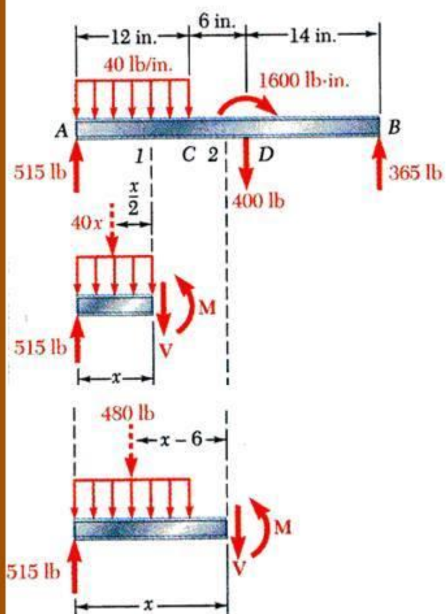




ENGINEERING MECHANICS OF SOLIDS



ENGINEERING MECHANICS OF SOLIDS



- Evaluate equivalent internal force-couple systems at sections cut within segments *AC*, *CD*, and *DB*.

From A to *C*:

$$\sum F_y = 0: \quad 515 - 40x - V = 0$$

$$\boxed{V = 515 - 40x}$$

$$\sum M_1 = 0: \quad -515x - 40x\left(\tfrac{1}{2}x\right) + M = 0$$

$$\boxed{M = 515x - 20x^2}$$

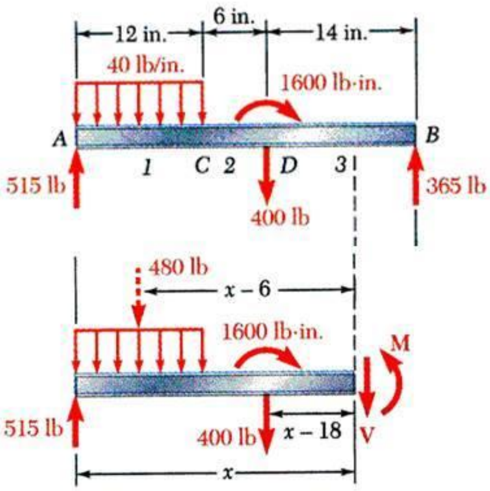
From *C* to *D*:

$$\sum F_y = 0: \quad 515 - 480 - V = 0$$

$$\boxed{V = 35\,\text{lb}}$$

$$\sum M_2 = 0: \quad -515x + 480(x - 6) + M = 0$$

$$\boxed{M = (2880 + 35x)\,\text{lb} \cdot \text{in.}}$$



- Evaluate equivalent internal force-couple systems at sections cut within segments *AC*, *CD*, and *DB*.

From *D* to *B*:

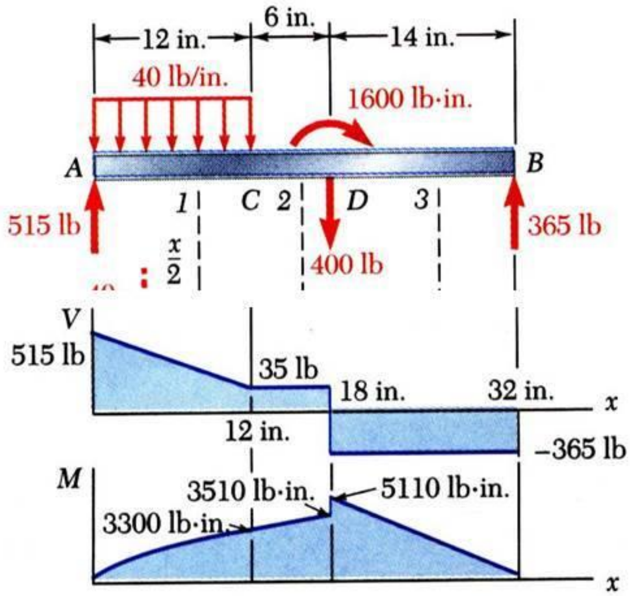
$$\sum F_y = 0: \qquad 515 - 480 - 400 - V = 0$$

$$\boxed{V = -365\,\text{lb}}$$

$$\sum M_2 = 0:$$

$$-515x + 480(x-6) - 1600 + 400(x-18) + M = 0$$

$$\boxed{M = (11{,}680 - 365x)\,\text{lb}\cdot\text{in.}}$$



- Plot results.

From A to C: $V = 515 - 40x$

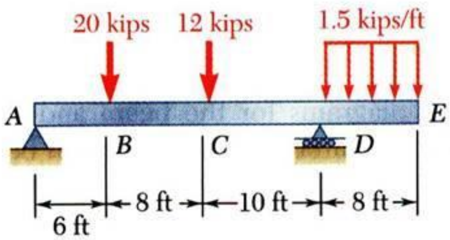
$M = 515x - 20x^2$

From C to D: $V = 35\,\text{lb}$

$M = (2880 + 35x)\,\text{lb}\cdot\text{in.}$

From D to B: $V = -365\,\text{lb}$

$M = (11{,}680 - 365x)\,\text{lb}\cdot\text{in.}$

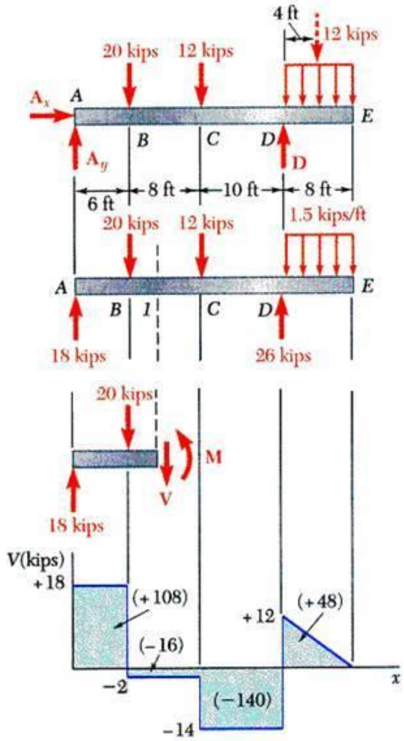


Draw the shear and bending-moment diagrams for the beam and loading shown.

SOLUTION:

- Taking entire beam as a free-body, determine reactions at supports.
- Between concentrated load application points, $dV/dx = -w = 0$ and shear is constant.
- With uniform loading between D and E, the shear variation is linear.
- Between concentrated load application points, $dM/dx = V = \text{constant}$. The change in moment between load application points is equal to area under shear curve between points.
- With a linear shear variation between D and E, the bending moment diagram is a parabola.

ENGINEERING MECHANICS OF SOLIDS



SOLUTION:

- Taking entire beam as a free-body, determine reactions at supports.

$$\sum M_A = 0:$$

$$D(24\,\text{ft}) - (20\,\text{kips})(6\,\text{ft}) - (12\,\text{kips})(14\,\text{ft}) - (12\,\text{kips})(28\,\text{ft}) = 0$$

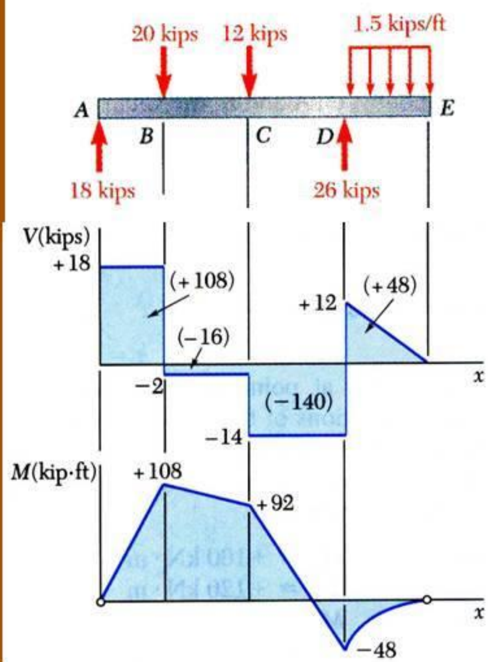
$$\boxed{D = 26\,\text{kips}}$$

$$\sum F_y = 0:$$

$$A_y - 20\,\text{kips} - 12\,\text{kips} + 26\,\text{kips} - 12\,\text{kips} = 0$$

$$\boxed{A_y = 18\,\text{kips}}$$

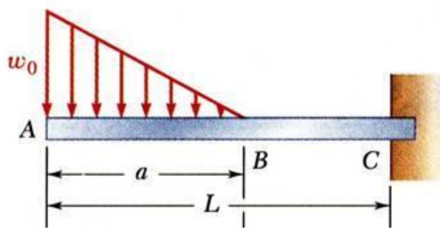
- Between concentrated load application points, $dV/dx = -w = 0$ and shear is constant.
- With uniform loading between D and E, the shear variation is linear.



- Between concentrated load application points, $dM/dx = V =$ constant. The change in moment between load application points is equal to area under the shear curve between points.

$$M_B - M_A = +108 \quad M_B = +108\,\text{kip}\cdot\text{ft}$$
$$M_C - M_B = -16 \quad M_C = +92\,\text{kip}\cdot\text{ft}$$
$$M_D - M_C = -140 \quad M_D = -48\,\text{kip}\cdot\text{ft}$$
$$M_E - M_D = +48 \quad M_E = 0$$

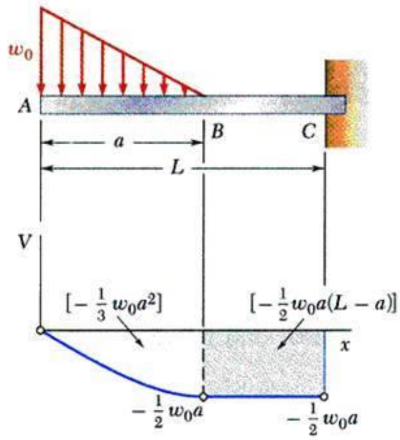
- With a linear shear variation between D and E, the bending moment diagram is a parabola.



Sketch the shear and bending-moment diagrams for the cantilever beam and loading shown.

SOLUTION:

- The change in shear between A and B is equal to the negative of area under load curve between points. The linear load curve results in a parabolic shear curve.
- With zero load, change in shear between B and C is zero.
- The change in moment between A and B is equal to area under shear curve between points. The parabolic shear curve results in a cubic moment curve.
- The change in moment between B and C is equal to area under shear curve between points. The constant shear curve results in a linear moment curve.



SOLUTION:

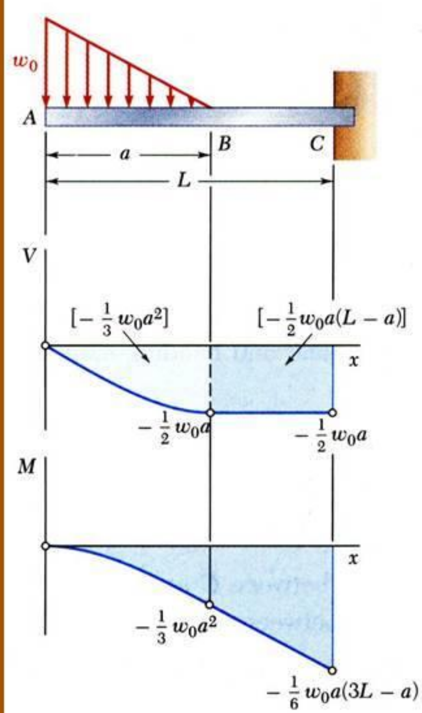
- The change in shear between A and B is equal to negative of area under load curve between points. The linear load curve results in a parabolic shear curve.

$$\text{at } A, \quad V_A = 0, \quad \frac{dV}{dx} = -w = -w_0$$

$$V_B - V_A = -\tfrac{1}{2}w_0 a \qquad\qquad V_B = -\tfrac{1}{2}w_0 a$$

$$\text{at } B, \quad \frac{dV}{dx} = -w = 0$$

- With zero load, change in shear between B and C is zero.



- The change in moment between A and B is equal to area under shear curve between the points. The parabolic shear curve results in a cubic moment curve.

$$\text{at } A, \quad M_A = 0, \quad \frac{dM}{dx} = V = 0$$

$$M_B - M_A = -\tfrac{1}{3} w_0 a^2 \qquad M_B = -\tfrac{1}{3} w_0 a^2$$

$$M_C - M_B = -\tfrac{1}{2} w_0 a(L-a) \quad M_C = -\tfrac{1}{6} w_0 a(3L-a)$$

- The change in moment between B and C is equal to area under shear curve between points. The constant shear curve results in a linear moment curve.

CHAPTER

6

ENGINEERING MECHANICS OF SOLIDS

Torsion

Example 6.1

Find the internal torque at section K-K for the shaft shown in Fig. 6-l(a) and acted upon by the three torques indicated.

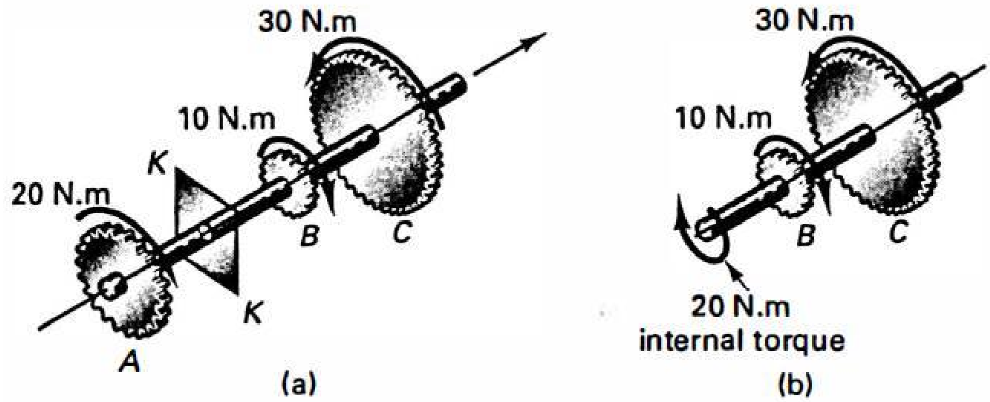


Fig. 6-1

SOLUTION

The 30 N · m torque at *C* is balanced by the two torques of 20 and 10 N · m at *A* and *B*, respectively. Therefore, the body as a whole is in equilibrium. Next, by passing a section *K–K* perpendicular to the axis of the rod *anywhere* between *A* and *B*, a free body of a part of the shaft, shown in Fig. 6-1(b), is obtained. Whereupon, from $\sum M_x = 0$, or

$$\text{externally applied torque} = \text{internal torque}$$

the conclusion is reached that the internal or resisting torque developed in the shaft between A and B is 20 N · m. Similar considerations lead to the conclusion that the internal torque resisted by the shaft between B and C is 30 N · m.

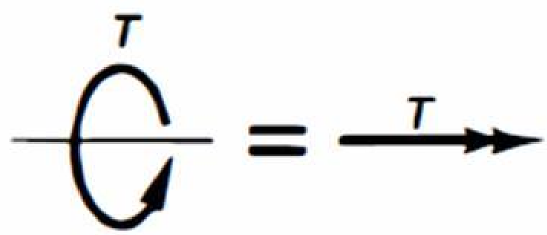


Fig. 6-2 Alternative representations of torque.

6-3. Basic Assumptions for Circular Members

To establish a relation between the internal torque and the stresses it sets up in members with *circular solid and tubular cross sections,* it is necessary to make two assumptions, the validity of which will be justified later. These, in addition to the homogeneity of the material, are as follows:

1. A plane section of material perpendicular to the axis of a circular member remains *plane* after the torques are applied (i.e., no *warpage* or distortion of parallel planes normal to the axis of a member takes place).[1]

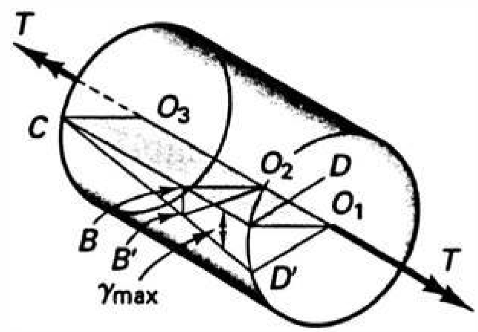


Fig. 6-3 Variation of strain in circular member subjected to torque.

2. In a circular member subjected to torque, *shear strains* γ *vary linearly from the central axis,* reaching γ_{max} at the periphery. This assumption is illustrated in Fig. 6-3 and means that an imaginary plane such as DO_1O_3C moves to $D'O_1O_3C$ when the torque is applied. Alternatively, if an imaginary radius O_3C is considered fixed in direction, similar radii initially at O_2B and O_1D rotate to the respective new positions O_2B' and O_1D'. These radii remain straight.

It must be emphasized that these assumptions *hold only for circular solid and tubular members.* For this class of members, these assumptions work so well that they *apply beyond the limit of the elastic behavior of a material.* These assumptions will be used again in Section 6-13, where stress distribution beyond the proportional limit is discussed.

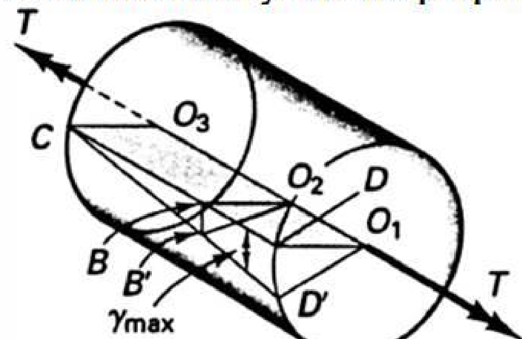


Fig. 6-3 Variation of strain in circular member subjected to torque.

3. If attention is confined to the linearly *elastic* material, Hooke's law applies, and it follows that shear stress is proportional to shear strain. For this case complete agreement between experimentally determined and computed quantities is found with the derived stress and deformation formulas based on these assumptions. Moreover, their validity can be rigorously demonstrated by the methods of the mathematical theory of elasticity.

6-4. The Torsion Formula

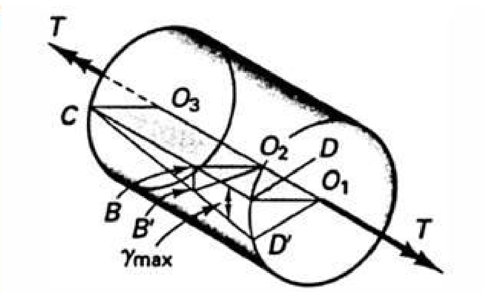


Fig. 6-3 Variation of strain in circular member subjected to torque.

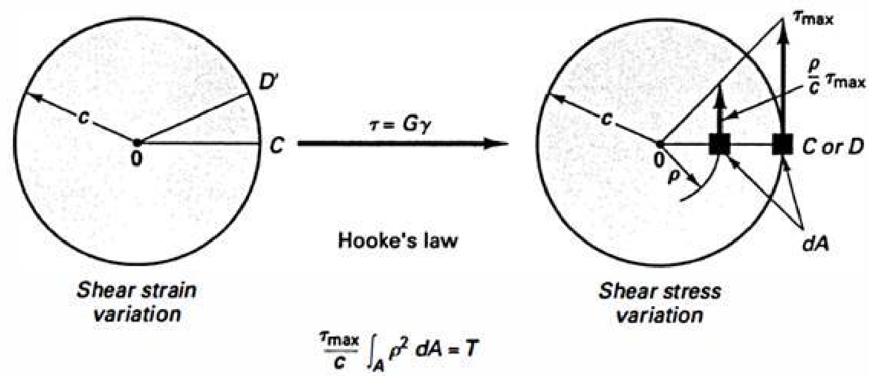


Fig. 6-4 Shear strain assumption leading to elastic shear stress distribution in a circular member.

In the elastic case, on the basis of the previous assumptions, since stress is proportional to strain, and the latter varies linearly from the center, stresses vary linearly from the central axis of a circular member. The stresses induced by the assumed distortions are shear stresses and lie in the plane parallel to the section taken normal to the axis of a rod. The variation of the shear stress follows directly from the shear-strain assumption and the use of Hooke's law for shear. This is illustrated in Fig. 6-4.

Unlike the case of an axially loaded rod, this stress is not of uniform intensity. The maximum shear stress occurs at points most remote from the center 0 and is designated T_{max}· These points, such as points C and D in Figs 6-3 and 6-4, lie at the periphery of a section at a distance c from the center. For linear shear stress variation, at any arbitrary point at a distance ρ from 0, the shear stress is $\frac{\rho}{c}T_{max}$·

The resisting torque can be expressed in terms of stress once the stress distribution at a section is established. For equilibrium this internal resisting torque must equal the externally applied torque *T*. Hence,

$$\int_A \underbrace{\underbrace{\underbrace{\frac{\rho}{c}\tau_{max}}_{\text{Stress}} \; \underbrace{dA}_{\text{area}}}_{\text{force}} \quad \underbrace{\rho}_{\text{arm}}}_{\text{torque}} = T$$

where the integral sums up all torques developed on the cut by the infinitesimal forces acting at a distance ρ from a member's axis, *O* in Fig. 6-4, over the whole area *A* of the cross section, and where *T* is the resisting torque.

At any given section, τ_{max} and *c* are constant; hence, the previous relation can be written as

$$\frac{\tau_{max}}{c}\int_A \rho^2\, dA = T \qquad (6\text{-}1)$$

However, $\int_A \rho^2\, dA$, *the polar moment of inertia* of a cross-sectional area, is also a constant for a particular cross-sectional area. It will be designated by I_p in this text. For a circular section, $dA = 2\pi\rho\, d\rho$, where $2\pi\rho$ is the circumference of an annulus[2] with a radius ρ of width $d\rho$. Hence,

$$I_P = \int_A \rho^2\, dA = \int_0^c 2\pi\rho^3\, d\rho = 2\pi \left|\frac{\rho^4}{4}\right|_0^c = \frac{\pi c^4}{2} = \frac{\pi d^4}{32} \qquad (6\text{-}2)$$

That is,

$$\boxed{I_p = \frac{\pi c^4}{2} = \frac{\pi d^4}{32}} \qquad (6\text{-}2)$$

where d is the diameter of a solid circular shaft. If c or d is measured in millimeters, I_p has the units of mm^4; if in inches, the units become in^4.

By using the symbol I_p for the polar moment of inertia of a circular area, Eq. 6-1 may be written more compactly as

$$\tau_{max} = \frac{Tc}{I_p} \qquad (6\text{-}3)$$

This equation is the well-known *torsion formula*[3] for circular shafts that expresses the maximum shear stress in terms of the resisting torque and the dimensions of a member. In applying this formula, the internal torque T can be expressed in newton-meters, N · m, or inch-pounds, c in meters or inches, and I_p in m^4 or in^4. Such usage makes the units of the torsional shear stress

$$\frac{[\text{N} \cdot \text{m}][\text{m}]}{[\text{m}^4]} = \left[\frac{\text{N}}{\text{m}^2}\right]$$

or *pascals* (Pa) in SI units, or

where d is the diameter of a solid circular shaft. If c or d is measured in millimeters, I_p has the units of mm^4; if in inches, the units become in^4.

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$$\frac{[\text{N} \cdot \text{m}][\text{m}]}{[\text{m}^4]} = \left[\frac{\text{N}}{\text{m}^2}\right]$$

or *pascals* (Pa) in SI units, or

$$\frac{[\text{in-lb}][\text{in}]}{[\text{in}^4]} = \left[\frac{\text{lb}}{\text{in}^2}\right]$$

A more general relation that Eq. 6-3 for a shear stress, τ, at *any* point a distance ρ from the center of a section is

$$\tau = \frac{\rho}{c}\tau_{\max} = \frac{T\rho}{I_p} \tag{6-4}$$

Equations 6-3 and 6-4 *are applicable* with equal rigor *to circular tubes*, since the same assumptions as used in the previous derivation apply. It is necessary, however, to modify I_p. For a tube, as may be seen from Fig. 6-5, the limits of integration for Eq. 6-2 extend from b to c. Hence, for a *circular tube*,

$$I_p = \int_A \rho^2\, dA = \int_b^c 2\pi\rho^3\, d\rho = \frac{\pi c^4}{2} - \frac{\pi b^4}{2} \tag{6-5}$$

or, stated otherwise, I_p for a circular tube equals $+I_p$ for a solid shaft using the outer diameter and $-I_p$ for a solid shaft using the inner diameter.

For very *thin* tubes, if b is nearly equal to c, and $c - b = t$, the thickness of the tube, I_p reduces to a simple approximate expression:

$$I_p \approx 2\pi R_{av}^3 t \tag{6-6}$$

ENGINEERING MECHANICS OF SOLIDS

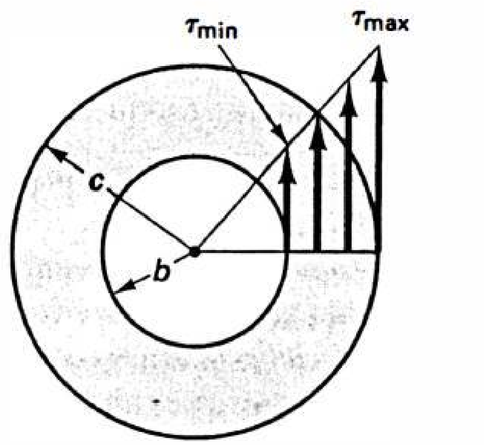


Fig. 6-5 Variation of stress in an elastic circular tube.

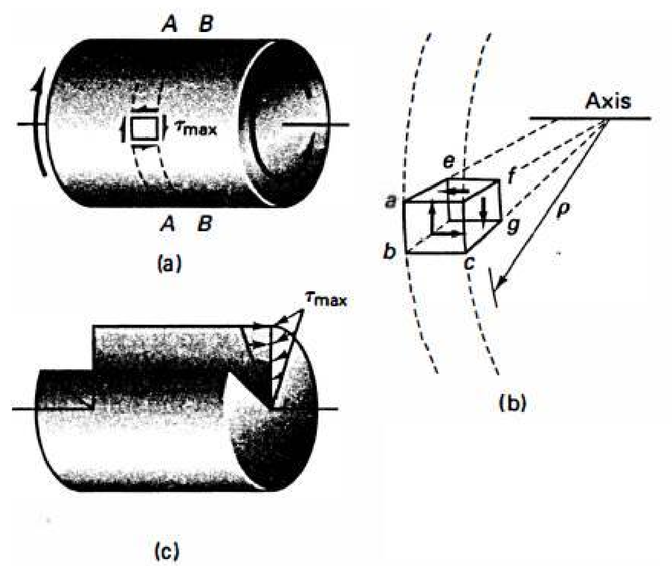


Fig. 6-7 Existence of shear stresses on mutually perpendicular planes in a circular shaft subjected to torque.

Procedure Summary For the torsion problem of circular shafts the *three basic concepts* of engineering mechanics of solids as used previously may be summarized in the following manner:

1. *Equilibrium conditions* are used for determining the internal resisting torques at a section.
2. *Geometry of deformation* (kinematics) is postulated such that shear strain varies linearly from the axis of a shaft.
3. *Material properties* (constitutive relations) are used to relate shear strains to shear stresses and permit calculation of shear stresses at a section.

Fracture of brittle materials subjected to torque

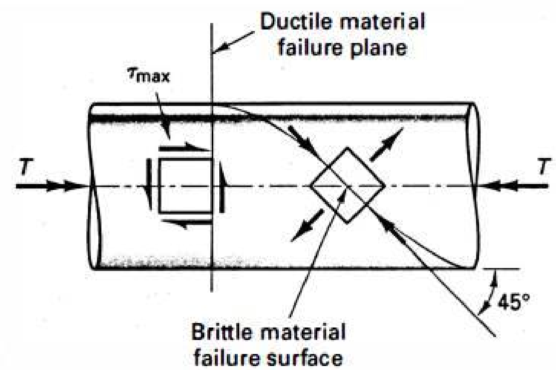


Fig. 6-8 Potential torsional failure surfaces in ductile and brittle materials.

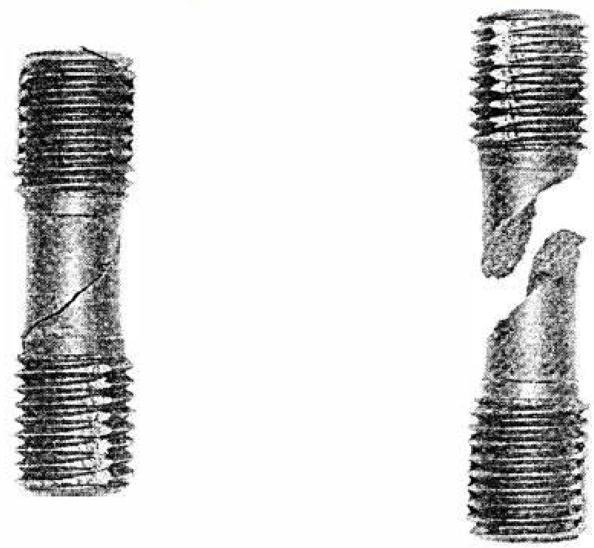


Fig. 6-10 Fractured cast iron specimen in torsion. The photograph on the right shows the specimen more widely separated.

Example 6-2

Find the maximum torsional shear stress in shaft *AC* shown in Fig. 6-1(a). Assume the shaft from *A* to *C* is 10 mm in diameter.

SOLUTION

From Example 6-1, the maximum internal torque resisted by this shaft is known to be 30 N · m. Hence, $T = 30$ N · m, and $c = d/2 = 5$ mm. From Eq. 6-2,

$$I_p = \frac{\pi d^4}{32} = \frac{\pi \times 10^4}{32} = 982 \text{ mm}^4$$

and from Eq. 6-3,

$$\tau_{max} = \frac{Tc}{I_p} = \frac{30 \times 10^3 \times 5}{982} = 153 \text{ MPa}$$

Example 6-3

Consider a long tube of 20 mm outside diameter, d_0, and of 16 mm inside diameter, d_i, twisted about its longitudinal axis with a torque T of 40 N · m. Determine the shear stresses at the outside and the inside of the tube; see Fig. 6-12.

SOLUTION

From Eq. 6-5,

$$I_p = \frac{\pi(c^4 - b^4)}{2} = \frac{\pi(d_o^4 - d_i^4)}{32} = \frac{\pi(20^4 - 16^4)}{32} = 9270 \text{ mm}^4$$

and from Eq. 6-3,

$$\tau_{max} = \frac{Tc}{I_p} = \frac{40 \times 10^3 \times 10}{9270} = 43.1 \text{ MPa}$$

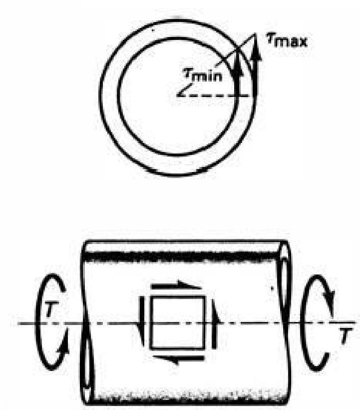


Fig. 6-12

Similarly, from Eq. 6-4,

$$\tau_{min} = \frac{T\rho}{I_p} = \frac{40 \times 10^3 \times 8}{9270} = 34.5 \text{ MPa}$$

6-6. Design of Circular Members in Torsion for Strength

After the torque to be transmitted by a shaft is determined and the maximum allowable shear stress is selected, according to Eq. 6-3. the proportions of a member are given as: $\frac{I_p}{c} = \frac{T}{\tau_{max}}$ (6-8)

where $\frac{I_p}{c} = \frac{T}{\tau_{max}}$ is the parameter on which the elastic strength of a shaft depends. For an axially loaded rod. such a parameter is the cross-sectional area of a member. For a solid shaft, $\frac{I_p}{c} = \frac{\pi c^3}{2}$, where c is the outside radius. By using this expression and Eq. 6-8. the required radius of a shaft can be determined. Any number of tubular shafts can be chosen to satisfy Eq. 6-8 by varying the ratio of the outer radius to the inner radius. c/b. to provide the required value of $\frac{I_p}{c}$.

By definition, I hp does the work of 745.7 N · m/s. One N · m/s is conveniently referred to as a watt (W) in the SI units. Thus 1 hp can be converted into 745.7 W. Likewise, it will be recalled that power is equal to torque multiplied by the angle, measured in radians, through which the shaft rotates per unit of time. For a shaft rotating with a frequency of f =Hz, the angle is 2πf rad/s. Hence, if a shaft were transmitting a constant torque T measured in N · m, it would do (2πf T) N · m of work per second. Equating this to the horsepower supplied, hp x 745.7 2πf T [N · m/s]

$$T = \frac{119\, hp}{f} \text{ [N.m]} \qquad (6\text{-}9)$$

$$T = \frac{159\, kW}{f} \text{ [N.m]} \qquad (6\text{-}10)$$

In the U.S. customary system of units. 1 hp does work of 550 ft-lb per second, or 550 x 12 x 60 in-lb per minute. If the shaft rotates at N rpm (revolutions per minute), an equation similar to those just given can be obtained: $T = \frac{63000\, hp}{N}$ [in-lb] (6-11)

Example 6-4

Select a solid shaft for a 10-hp motor operating at 30 Hz. The maximum shear stress is limited to 55 MPa.

SOLUTION

From Eq. 6-9,

$$T = \frac{119 \times \text{hp}}{f} = \frac{119 \times 10}{30} = 39.7\ \text{N} \cdot \text{m}$$

and from Eq. 6-8,

$$\frac{I_p}{c} = \frac{T}{\tau_{max}} = \frac{39.7 \times 10^3}{55} = 722\ \text{mm}^3$$

$$\frac{I_p}{c} = \frac{\pi c^3}{2} \quad \text{or} \quad c^3 = \frac{2}{\pi}\frac{I_p}{c} = \frac{2 \times 722}{\pi} = 460\ \text{mm}^3$$

Hence, $c = 7.72$ mm or $d = 2c = 15.4$ mm.

For practical purposes, a 16-mm shaft would probably be selected.

Example 6-5

Select solid shafts to transmit 150 kW each without exceeding a shear stress of 70 MPa. One of these shafts operates at a frequency of 0.30 Hz and the other at a frequency of 300 Hz.

SOLUTION

Subscript 1 applies to the low-speed shaft and 2 to the high-speed shaft. From Eq. 6-10,

$$T_1 = \frac{159 \times \text{kW}}{f_1} = \frac{159 \times 150}{0.30} = 79{,}500 \text{ N} \cdot \text{m}$$

Similarly,

$$T_2 = 79.5 \text{ N} \cdot \text{m}$$

From Eq. 6-8,

$$\frac{I_{p1}}{c} = \frac{T_1}{\tau_{\max}} = \frac{79{,}500}{70} = 1.14 \times 10^6 \text{ mm}^3$$

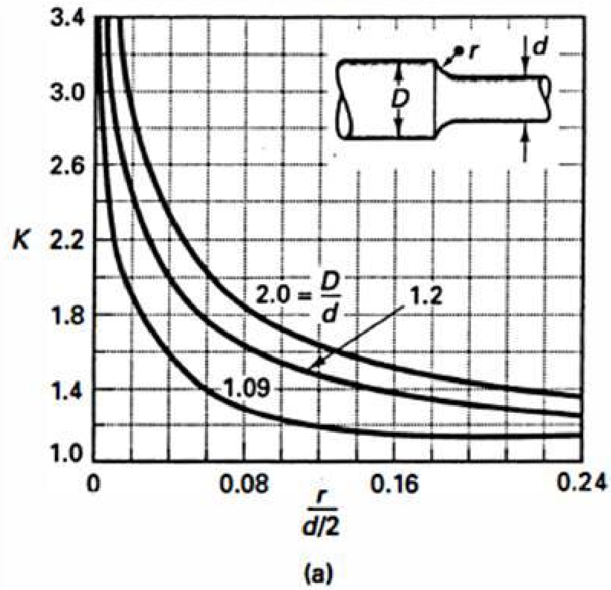
$$\frac{I_{p1}}{c} = \frac{\pi d_1^3}{16} \qquad \text{or} \qquad d_1^3 = \frac{16}{\pi}(1.14 \times 10^6) = 5.81 \times 10^6 \text{ mm}^3$$

Hence,

$$d_1 = 180 \text{ mm} \qquad \text{and} \qquad d_2 = 18 \text{ mm}$$

This example illustrates the reason for the modern tendency to use high speed machines in mechanical equipment. The difference in size of the two shafts is striking. Further savings in the weight of the material can be effected by using hollow tubes.

6-7. Stress Concentrations



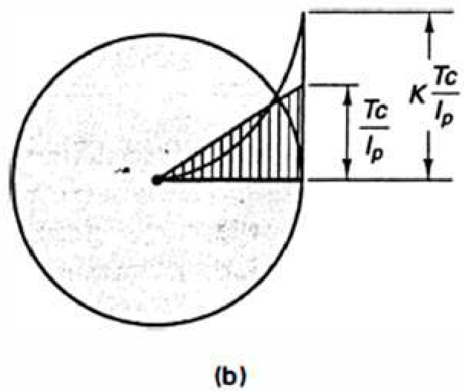


Fig. 6-13(a) Torsional stress-concentration factors in circular shafts of two diameters. (b) Stress increase at a fillet.

$$\tau_{max} = K\frac{Tc}{I_p} \tag{6-12}$$

where the shear stress Tc/I_p is determined for the smaller shaft.

6-8. Angle of Twist of Circular Members

According to assumption 1 stated in Section 6-3, planes perpendicular to the axis of a circular rod do not warp. The elements of a shaft undergo deformation of the type shown in Fig. 6-15(b). The shaded element is shown in its undistorted form in Fig. 6-15(a). From such a shaft, a typical element of length *dx* is shown isolated in Fig. 6-16 similar to Fig. 6-3.

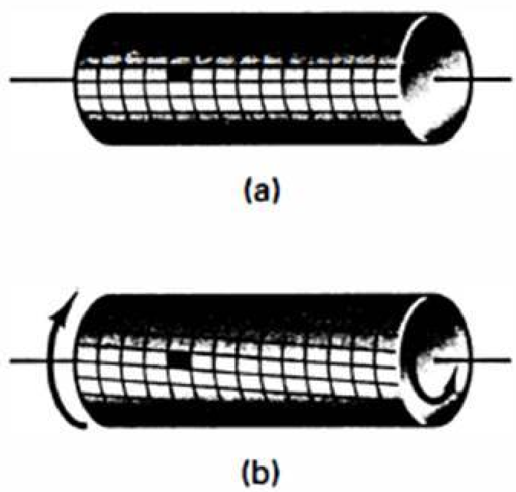


Fig. 6-15 Circular shaft (a) before and (b) after torque is applied.

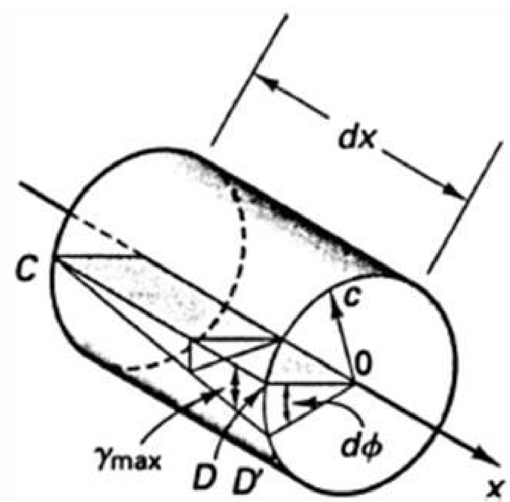


Fig. 6-16 Deformation of a circular bar element due to torque.

In the element shown, a line on its surface such as CD is initially parallel to the axis of the shaft. After the torque is applied. it assumes a new position CD'. At the same time, by virtue of earlier assumption 2, radius OD remains straight and rotates through a small angle $d\phi$ to a new position OD'.

Denoting the small angle DCD' by γ_{max}, from geometry, one has two alternative expressions for the arc DD':

$$\text{arc } DD' = \gamma_{max}\, dx \qquad \text{or} \qquad \text{arc } DD' = d\phi\, c$$

where both angles are small and are measured in radians. Hence,

$$\gamma_{max}\, dx = d\phi\, c \tag{6-13}$$

The γ_{max} applies only in the zone of an infinitesimal "tube" of constant maximum shear stress τ_{max}. Limiting attention to linearly elastic response makes Hooke's law applicable. Therefore, according to Eq. 5-1, the angle γ_{max} is proportional to τ_{max} (i.e., $\gamma_{max} = \tau_{max}/G$). Moreover, by Eq. 6-3, $\tau_{max} = Tc/I_p$. Hence, $\gamma_{max} = Tc/(I_p G)$.[10] By substituting the latter expression into Eq. 6-13 and simplifying, the governing differential equation for the angle of twist is obtained:

$$\frac{d\phi}{dx} = \frac{T}{I_p G} \qquad \text{or} \qquad d\phi = \frac{T\, dx}{I_p G} \tag{6-14}$$

This gives the relative angle of twist of two adjoining sections of infinitesimal distance dx apart. To find the total angle of twist ϕ between any two sections A and B on a shaft a finite distance apart, the rotations of all elements must be summed. Hence, a general expression for the angle of twist between any two sections of a shaft of a linearly elastic material is

$$\phi = \phi_B - \phi_A = \int_A^B d\phi = \int_A^B \frac{T_x \, dx}{I_{px} G} \tag{6-15}$$

where ϕ_B and ϕ_A are, respectively, the global shaft rotations at ends B and A. The rotation at A may not necessarily be zero. In this equation, the internal torque T_x, the polar moment of inertia I_{px}, as well as G, may vary along the length of a shaft. In such cases, $T_x = T(x)$, $I_{px} = I_p(x)$, and $G = G(x)$. The direction of the angle of twist ϕ coincides with the direction of the applied torque T.

Example 6-6

Find the relative rotation of section *B–B* with respect to section *A–A* of the solid elastic shaft shown in Fig. 6-17 when a constant torque *T* is being transmitted through it. The polar moment of inertia of the cross-sectional area I_p is constant.

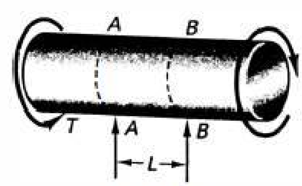


Fig. 6-17

SOLUTION

In this case, $T_x = T$ and I_p is constant; hence, from Eq. 6-15,

$$\phi = \int_A^B \frac{T_x\,dx}{I_pG} = \int_O^L \frac{T\,dx}{I_pG} = \frac{T}{I_pG}\int_O^L dx = \frac{TL}{I_pG}$$

That is,

$$\boxed{\phi = \frac{TL}{I_pG}} \qquad (6\text{-}16)$$

In applying Eq. 6-16, note particularly that the angle ϕ is expressed in *radians*. Also observe the great similarity of this relation to Eq. 3-4, $\Delta = PL/AE$, for axially loaded bars. Here $\phi \Leftrightarrow \Delta$, $T \Leftrightarrow P$, $I_p \Leftrightarrow A$, and $G \Leftrightarrow E$. Analogous to Eq. 3-4, Eq. 6-16 can be recast to express the *torsional spring constant*, or *torsional stiffness*, k_t as

$$\boxed{k_t = \frac{T}{\phi} = \frac{I_pG}{L}} \left[\frac{\text{in-lb}}{\text{rad}}\right] \quad \text{or} \quad \left[\frac{\text{N}\cdot\text{m}}{\text{rad}}\right] \qquad (6\text{-}17)$$

Example 6-7

Consider the stepped shaft shown in Fig. 6-19(a) rigidly attached to a wall at E, and determine the angle of twist of the end A when the two torques at B and at D are applied. Assume the shear modulus G to be 80 GPa, a typical value for steels.

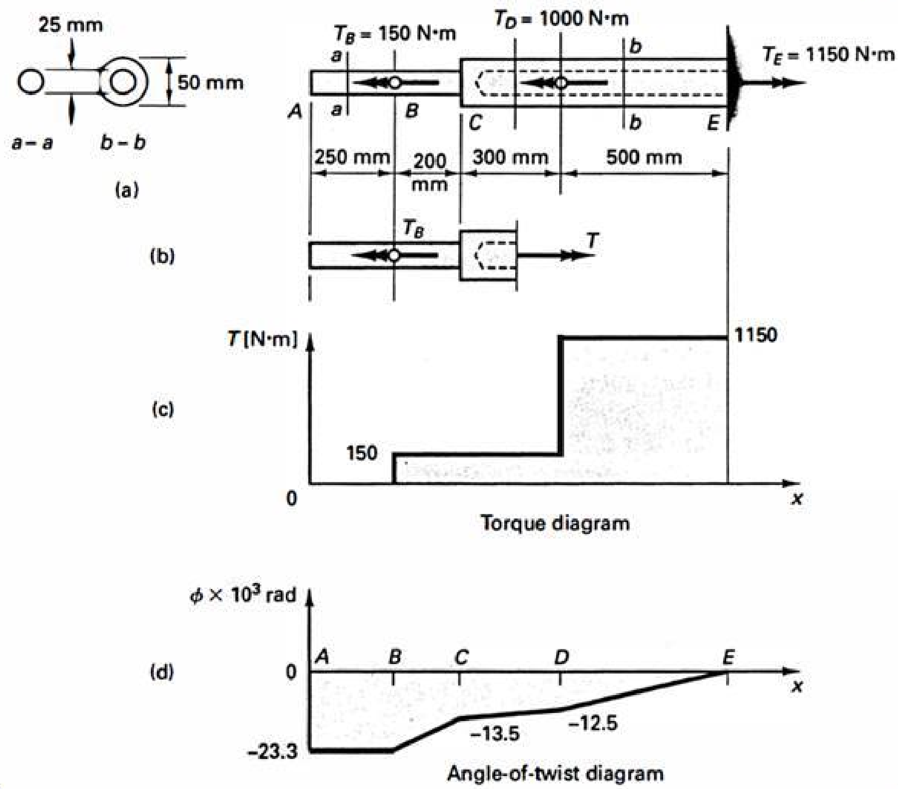


Fig. 6-19

$$(I_p)_{AB} = (I_p)_{BC} = \frac{\pi d^4}{32} = \frac{\pi \times 25^4}{32} = 38.3 \times 10^3 \text{ mm}^4$$

$$(I_p)_{CD} = (I_p)_{DE} = \frac{\pi}{32}(d_0^4 - d_i^4) = \frac{\pi}{32}(50^4 - 25^4) = 575 \times 10^3 \text{ mm}^4$$

$$\phi = \int_A^E \frac{T_x\,dx}{I_{px}G} = \int_A^B \frac{T_{AB}\,dx}{(I_p)_{AB}G} + \int_B^C \frac{T_{BC}\,dx}{(I_p)_{BC}G} + \int_C^D \frac{T_{CD}\,dx}{(I_p)_{CD}G} + \int_D^E \frac{T_{DE}\,dx}{(I_p)_{DE}G}$$

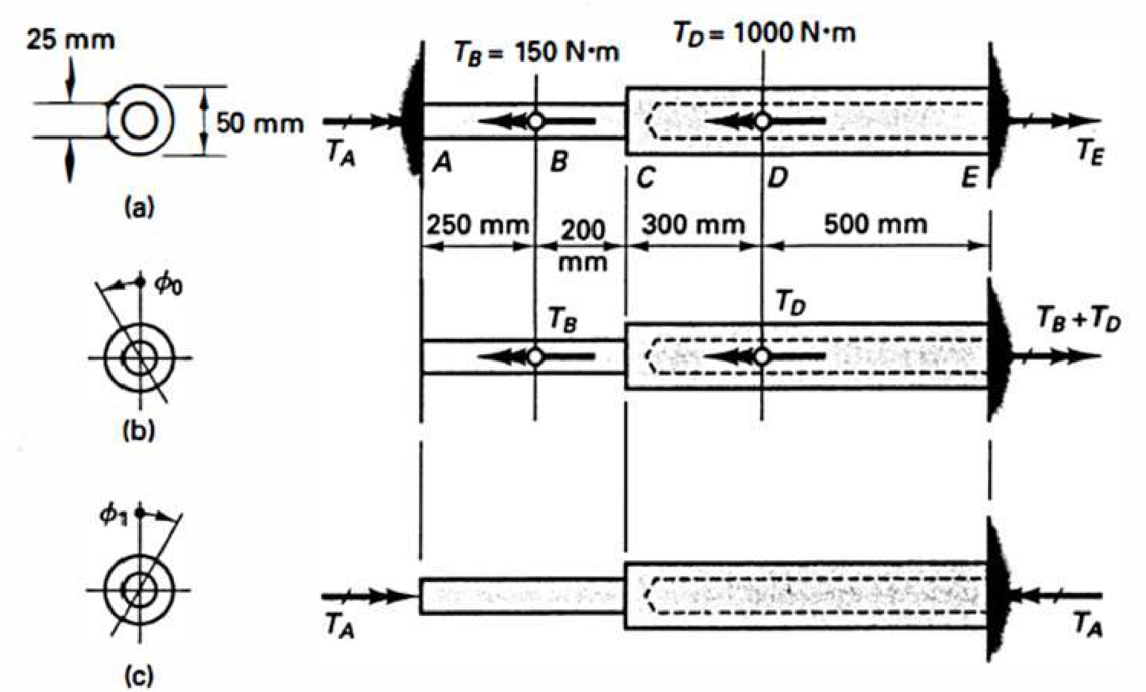
$$\phi = \sum_i \frac{T_i L_i}{(I_p)_i G_i} = \frac{T_{AB}L_{AB}}{(I_p)_{AB}G} + \frac{T_{BC}L_{BC}}{(I_p)_{BC}G} + \frac{T_{CD}L_{CD}}{(I_p)_{CD}G} + \frac{T_{DE}L_{DE}}{(I_p)_{DE}G}$$

$$= 0 + \frac{150 \times 10^3 \times 200}{38.3 \times 10^3 \times 80 \times 10^3} + \frac{150 \times 10^3 \times 300}{575 \times 10^3 \times 80 \times 10^3} + \frac{1150 \times 10^3 \times 500}{575 \times 10^3 \times 80 \times 10^3}$$

$$= 0 + 9.8 \times 10^{-3} + 1.0 \times 10^{-3} + 12.5 \times 10^{-3} = 23.3 \times 10^{-3} \text{ rad}$$

Example 6-9

Assume that the stepped shaft of Example 6-7, while loaded in the same manner is now built-in at both ends, as shown in Fig. 6-22. Determine the end reactions and plot the torque diagram for the shaft. Apply the force method.



Example 7-14

$$-23.3 \times 10^{-3} + 164 \times 10^{-6} T_A = 0$$

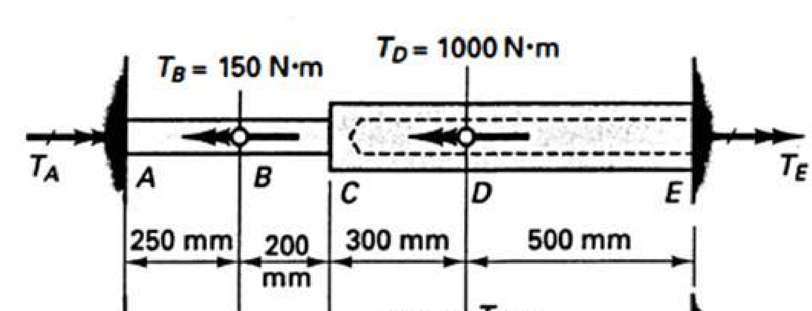
Hence,

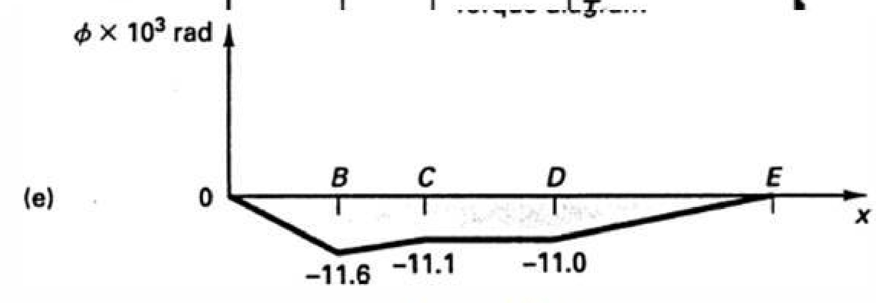
$$T_A = 142\ \text{N}\cdot\text{m} \quad \text{and} \quad T_B = 1150 - 142 = 1008\ \text{N}\cdot\text{m}$$

$$\phi_1 = \sum_i \frac{T_i L_i}{(I_p)_i G_i}$$

$$= T_A \times 10^3 \left(\frac{450}{38.3 \times 10^3 \times 80 \times 10^3} + \frac{800}{575 \times 10^3 \times 80 \times 10^3} \right)$$

$$= (147 \times 10^{-6} + 17 \times 10^{-6}) T_A = 164 \times 10^{-6} T_A \text{ rad}$$





Angle-of-twist diagram

6-12. Shaft Couplings

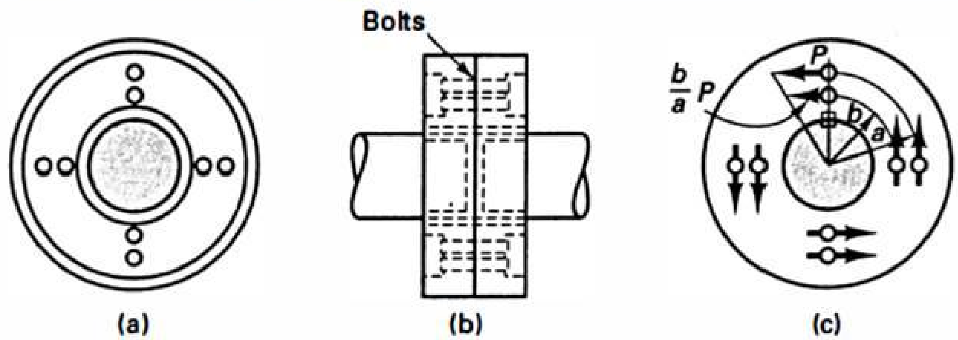


Fig. 6-26 Flanged shaft coupling.

Example 6-12

Estimate the torque-carrying capacity of a steel coupling forged integrally with the shaft, shown in Fig. 6-27, as controlled by an allowable shear stress of 40 MPa in the eight bolts. The bolt circle is diameter 240 mm.

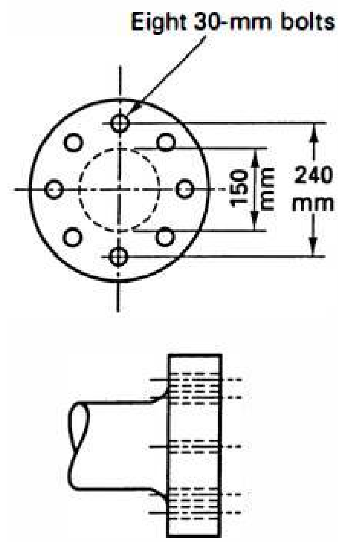


Fig. 6-27

SOLUTION

Area of one bolt:

$$A = (1/4)\pi(30)^2 = 706 \text{ mm}^2$$

Allowable force for one bolt:

$$P_{\text{allow}} = A\tau_{\text{allow}} = 706 \times 40 = 28.2 \times 10^3 \text{ N}$$

Since eight bolts are available at a distance of 120 mm from the central axis,

$$T_{\text{allow}} = 28.2 \times 10^3 \times 120 \times 8 = 27.1 \times 10^6 \text{ N} \cdot \text{mm} = 27.1 \times 10^3 \text{ N} \cdot \text{m}$$

TORSION OF SOLID NONCIRCULAR MEMBERS

6-14. Solid Bars of Any Cross Section

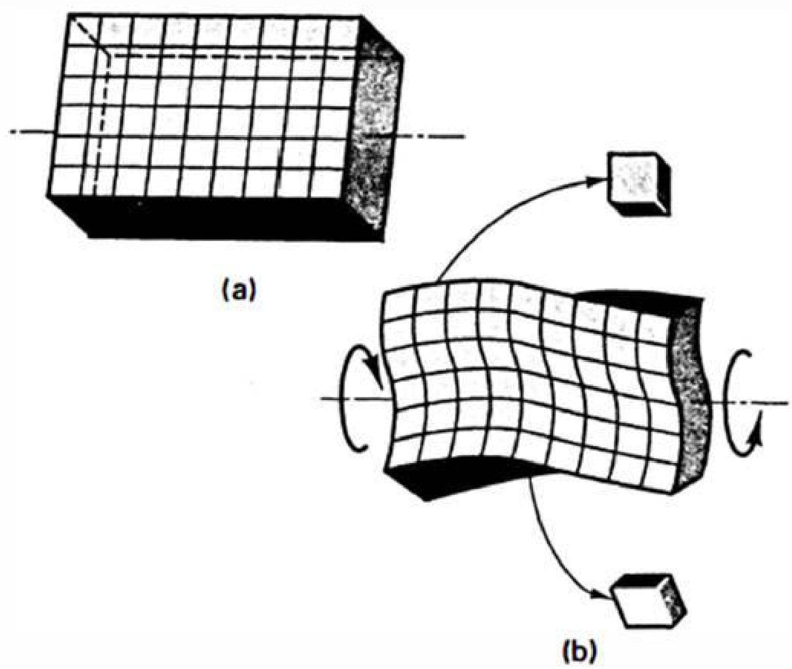


Fig. 6-32 Rectangular bar (a) before and (b) after a torque is applied.

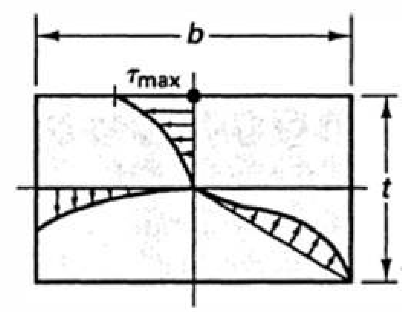


Fig. 6-33 Shear stress distribution in a rectangular shaft subjected to a torque.

$$\tau_{max} = \frac{T}{\alpha b t^2} \quad \text{and} \quad \phi = \frac{TL}{\beta b t^3 G} \qquad (6\text{-}30)$$

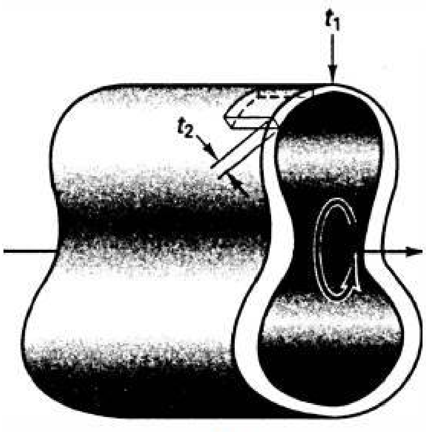
Table of Coefficients for Rectangular Bars

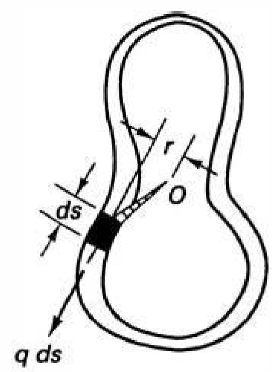
b/t	1.00	1.50	2.00	3.00	6.00	10.0	∞
α	0.208	0.231	0.246	0.267	0.299	0.312	0.333
β	0.141	0.196	0.229	0.263	0.299	0.312	0.333

$$k_t = \frac{T}{\phi} = \beta b t^3 \frac{G}{L} \qquad (6\text{-}31)$$

TORSION OF THIN-WALLED/TUBULAR MEMBERS

6-16. Thin-Walled Hollow Members





one can imagine a constant quantity of water steadily circulating in this channel. In this arrangement, the quantity of water flowing through a plane across the channel is constant. Because of this analogy, the quantity q has been termed shear flow.

Next consider the cross section of the tube as shown in Fig. 6-40(c). The force per unit distance of the perimeter of this tube, by virtue of the previous argument, is constant and is the shear flow q. This shear flow multiplied by the length ds of the perimeter gives a force q ds per differential length. The product of this infinitesimal force q ds and r around some convenient point such as 0, Fig. 6-40(c), gives the contribution of an element to the resistance of applied torque T. Adding or integrating this,

$$T = \oint rq\, ds \qquad (6\text{-}32)$$

where the integration process is carried around the tube along the center line of the perimeter. Since for a tube. q is a constant. this equation may be written as

$$T = q \oint r\, ds \qquad (6\text{-}33)$$

Instead of carrying out the actual integration. a simple interpretation of the integral is available. It can be seen from Fig. 6-40(c) that r ds is twice the value of the shaded area of an infinitesimal triangle of altitude r and base ds. Hence the complete integral is twice the whole area bounded by the center line of :he perimeter of the tube. Defining this area by a special Symbol Ⓐ one obtains

$$T = 2Ⓐq \quad \text{or} \quad q = \frac{T}{2Ⓐ} \qquad (6\text{-}34)$$

This equation[24] applies only to *thin-walled* tubes. The area Ⓐ is approximately an average of the two areas enclosed by the inside and the outside surfaces of a tube, or, as noted, it is an area enclosed by the center line of the wall's contour. Equation 6-34 is not applicable at all if the tube is slit, when Eqs. 6-30 should be used.

Since for any tube, the shear flow q given in Eq. 6-34 is constant, from the definition of shear flow, the shear stress at any point of a tube where the wall thickness is t is

$$\boxed{\tau = \frac{q}{t}} \tag{6-35}$$

For linearly *elastic* materials, the angle of twist for a hollow tube can be found by applying the principle of conservation of energy, Eq. 3-16. In this derivation, it is convenient to introduce the angle of twist per unit length of the tube defined as $\theta = d\phi/dx$. The elastic shear strain energy for the tube should also be per unit length of the tube. Hence, Eq. 5-5 for the elastic strain energy here reduces to $U_{sh} = \int_{vol}(\tau^2/2G)\, dV$, where $dV = 1 \times t\, ds$. By substituting Eq. 6-35 and then Eq. 6-34 into this relation and simplifying,

$$\overline{U}_{sh} = \oint \frac{T^2}{8Ⓐ^2 Gt}\, ds = \frac{T^2}{8Ⓐ^2 G} \oint \frac{ds}{t} \tag{6-36}$$

Equating this relation to the external work per unit length of member expressed as $\overline{W}_e = T\theta/2$, the governing differential equation becomes

$$\theta = \frac{d\phi}{dx} = \frac{T}{4Ⓐ^2 G}\oint \frac{ds}{t} \tag{6-37}$$

Here again it is useful to recast Eq. 6-37 to express the torsional stiffness k_t for a thin-walled hollow tube. Since for a prismatic tube subjected to a constant torque, $\phi = \theta L$,

$$\boxed{k_t = \frac{T}{\phi} = \frac{4Ⓐ^2}{\oint ds/t}\frac{G}{L}} \tag{6-38}$$

Example 6-16

Rework Example 6-3 using Eqs. 6-34 and 6-35. The tube has outside and inside radii of 10 and 8 mm, respectively, and the applied torque is 40 N · m.

SOLUTION

The mean radius of the tube is 9 mm and the wall thickness is 2 mm. Hence,

$$\tau = \frac{q}{t} = \frac{T}{2 \textcircled{A} t} = \frac{40 \times 10^3}{2\pi \times 9^2 \times 2} = 39.3 \text{ MPa}$$

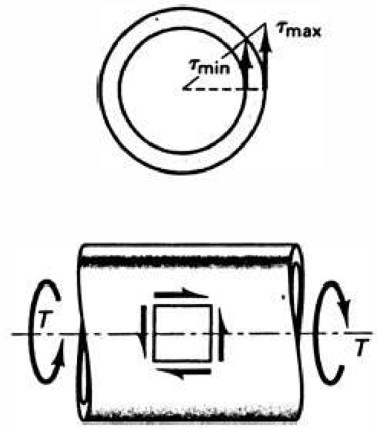


Fig. 6-12

Example 6-17

An aluminum extrusion has the cross section shown in Fig. 6-42. If torque $T = 300$ N · m is applied, (a) determine the maximum shear stresses that would develop in the three different parts of the member, and (b) find the torsional stiffness of the member. Neglect stress concentrations.

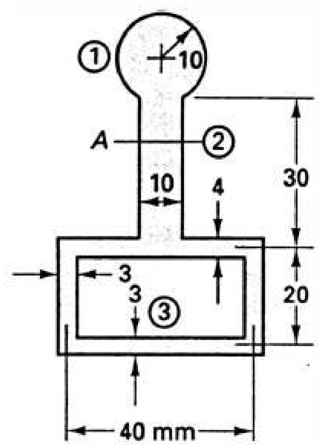


Fig. 6-42

$$(k_t)_1 = I_p \frac{G}{L} = \frac{\pi \times 10^4}{2}\frac{G}{L} = 1.57 \times 10^4 \frac{G}{L}$$

$$(k_t)_2 = \beta b t^3 \frac{G}{L} = 0.263 \times 30 \times 10^3 \frac{G}{L} = 0.789 \times 10^4 \frac{G}{L}$$

$$(k_t)_3 = \frac{4\textcircled{A}^2}{\oint ds/t}\frac{G}{L} = \frac{4 \times (40 \times 20)^2}{(40 + 2 \times 20)/3 + 40/4}\frac{G}{L} = 6.98 \times 10^4 \frac{G}{L}$$

By adding the stiffnesses for the parts, the member torsional stiffness $\Sigma (k_t)_i = 9.34 \times 10^4 G/L$

The applied torque is distributed among the three parts in a ratio of $(k_t)_i/\Sigma (k_t)_i$. On this basis, the torques are $300 \times (1.57 \times 10^4 G/L)$ $(9.34 \times 10^4 G/L) = 50.4$ N · m for the knob, 25.3 N · m for the bar, and 224 N · m for the box. The maximum stresses in each of the parts are determined using, respectively, Eqs. 6-3, 6-30, and 6-34.

$$\tau_{1\text{-max}} = \frac{Tc}{I_p} = \frac{50.4 \times 10^3 \times 10}{\pi \times 10^4/2} = 32.1 \text{ MPa}$$

$$\tau_{2\text{-max}} = \frac{T}{\alpha b t^2} = \frac{25.3 \times 10^3}{0.267 \times 30 \times 10^2} = 31.6 \text{ MPa}$$

$$\tau_{3\text{-max}} = \frac{T}{2 (A) t} = \frac{224 \times 10^3}{2 \times 40 \times 20 \times 3} = 46.7 \text{ MPa}$$

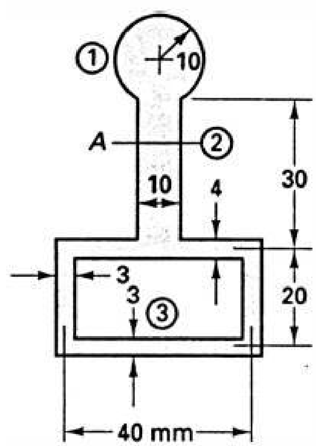


Fig. 6-42

Problems for solution

In addition to other books mentioned, solve following problems from Popov:
1, 4-6, 10, 11, 16, 19, 20, 24-30, 33, 35, 42, 43, 51, 52 & 58-60

CHAPTER

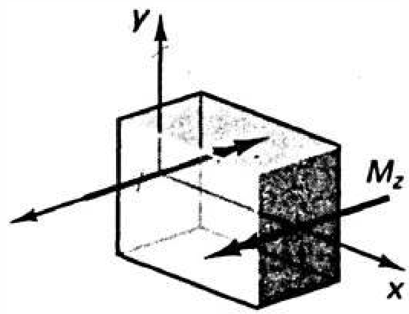
8

ENGINEERING MECHANICS OF SOLIDS

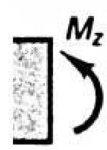
Symmetric Beam Bending

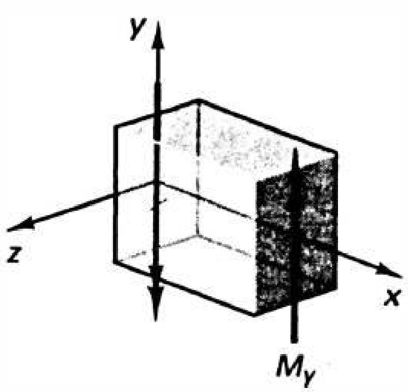
[Flexural stress in beams]

Definitions of positive moments



(a)

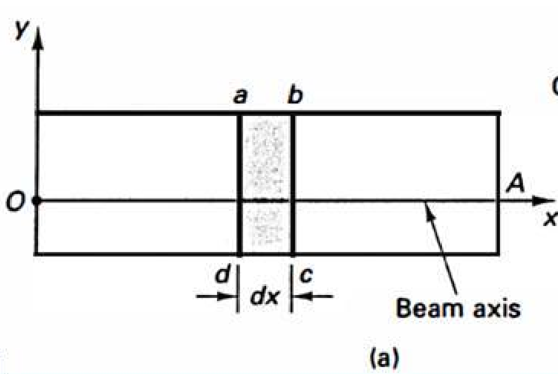




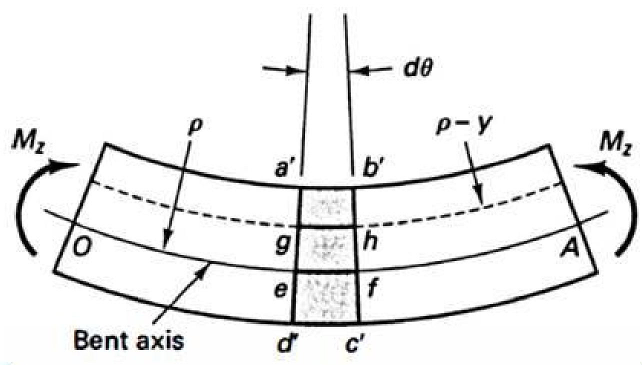
(b)

Basic Kinematic Assumption

1. Plane sections through a beam taken normal to its axis remain plane after the beam is subjected to bending.
2. Some fibers or filaments of the beam along a surface do not change length. These fibers free of stress and strain exist continuously over the whole length and width of the beam. These fibers lie in a surface called neutral surface of the beam. Intersection of the neutral surface with a right section through the beam is termed as neutral axis of the beam (location of zero stress and zero strain in a beam subjected to bending.
3. In a beam subjected to bending, strains in it's fibers vary linearly or directly as there respective distance from the neutral surface.

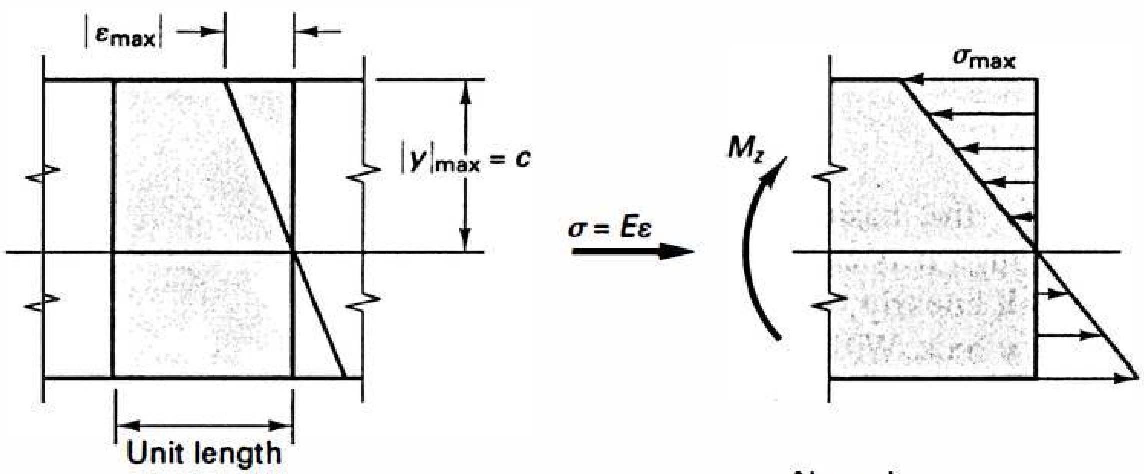


(a)

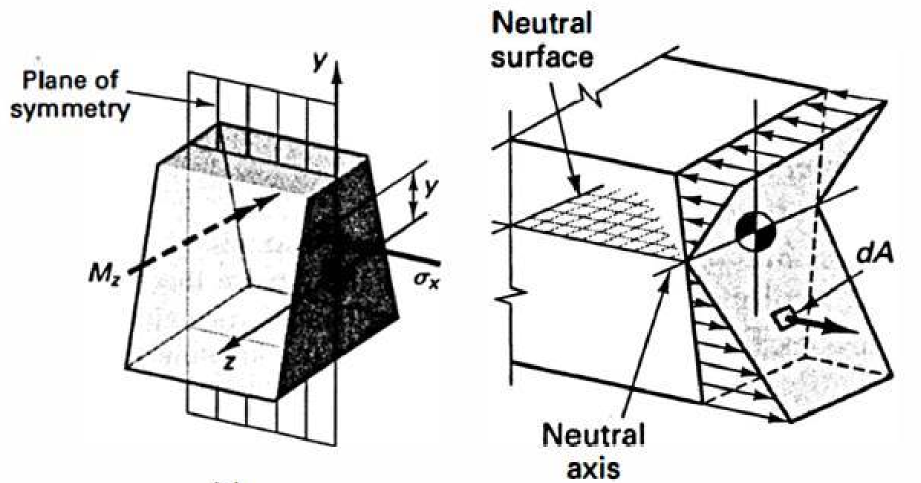


(b)

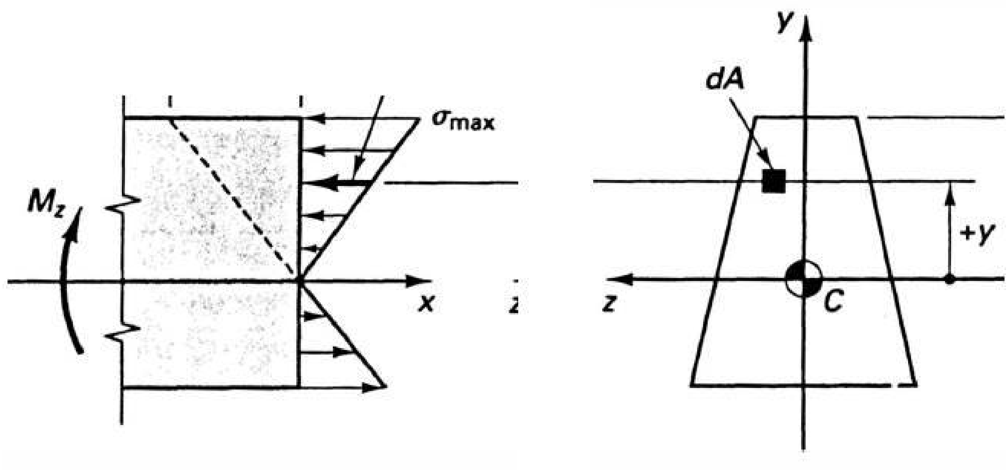
ENGINEERING MECHANICS OF SOLIDS



(c)

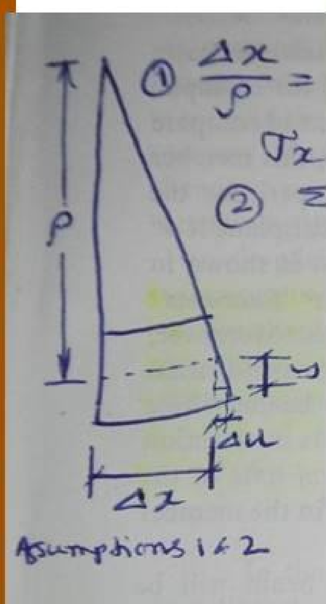


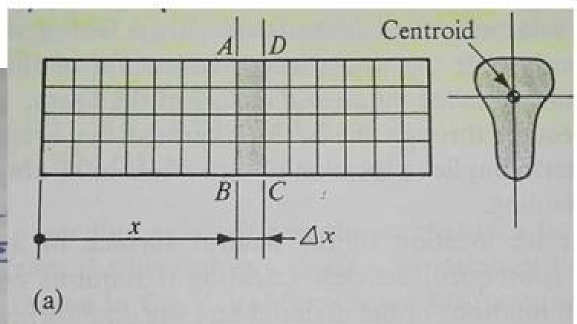
(e)

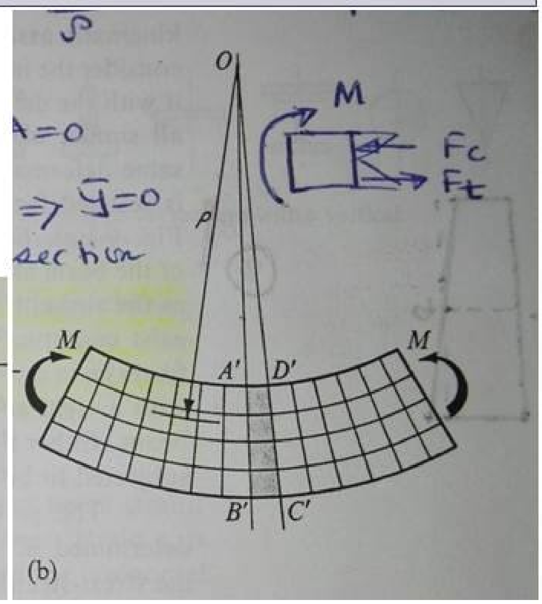


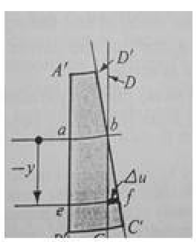
(a) (b)

ENGINEERING MECHANICS OF SOLIDS









$$\frac{\Delta x}{\rho} = \frac{\Delta u}{y} \quad \Rightarrow \quad \frac{\Delta u}{\Delta x} = \frac{y}{\rho}$$

$$\varepsilon_x = \frac{y}{\rho} \quad \Rightarrow \quad \sigma_x = E\varepsilon_x \quad \Rightarrow \sigma_x = E\frac{y}{\rho}$$

$$\sum F_x = 0$$

$$\oint dF = 0 \qquad \Rightarrow \quad \oint \sigma_x dA = 0$$

$$\Rightarrow \frac{E}{\rho}\oint y dA = 0 \quad \Rightarrow \quad \bar{y}A = 0 \qquad \Rightarrow \bar{y} = 0$$

NA passes through the centroid of the section.

$$\sum M_z = 0 \qquad \Rightarrow M_z + \oint \sigma_x dA.y = 0$$

$$M_z + \frac{E}{\rho}\oint y^2 dA = 0 \qquad \Rightarrow M_z + \frac{E}{\rho}\mathrm{I} = 0$$

$$M_z + \frac{\sigma_x}{y}\mathrm{I} = 0 \quad \Rightarrow \quad \sigma_x = -\frac{M_z y}{I}$$

6-4. The Torsion Formula

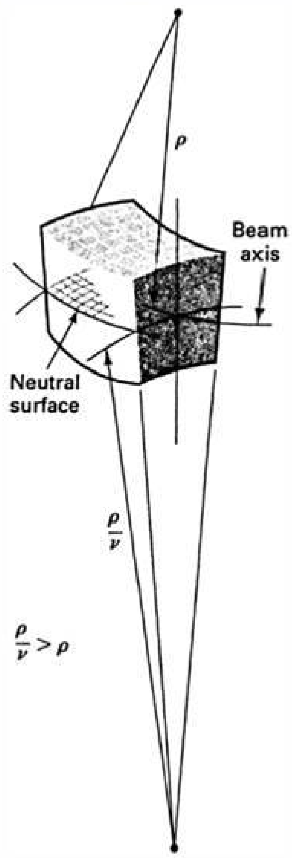


Figure 8-6 Segment of a bent beam.

Example8-1

Find the moment of inertia around the horizontal axis passing through the centroid for the rectangular area shown in Fig. 8-9.

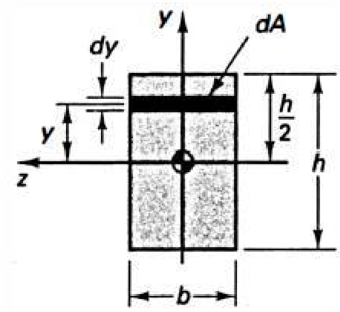


Figure 8-9

SOLUTION

The centroid of this section lies at the intersection of the two axes of symmetry. Here it is convenient to take dA as $b\,dy$. Hence,

$$I_z = I_o = \int_A y^2\,dA = \int_{-h/2}^{+h/2} y^2 b\,dy = b\left|\frac{y^3}{3}\right|_{-h/2}^{+h/2} = \frac{bh^3}{12}$$

Hence,

$$\boxed{I_z = \frac{bh^3}{12}} \quad \text{and} \quad \boxed{I_y = \frac{b^3h}{12}} \qquad (8\text{-}19)$$

Example 8-2

Find the moment of inertia about a diameter for a circular area of radius c; see Fig. 8-10.

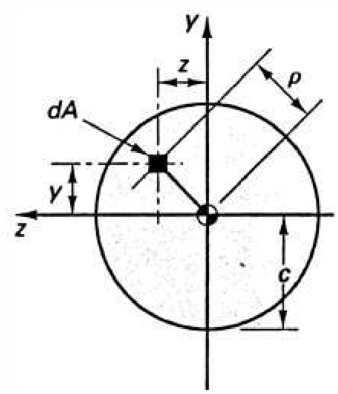


Figure 8-10

SOLUTION

To find I for a circle, first note that $\rho^2 = z^2 + y^2$, as may be seen from the figure. Then using the definition of I_p, noting the symmetry around both axes, and using Eq. 6-2,

$$I_p = \int_A \rho^2 \, dA = \int_A (y^2 + z^2) \, dA = \int_A y^2 \, dA + \int_A z^2 \, dA$$

$$= I_z + I_y = 2I_z$$

$$\boxed{I_z = I_y = \frac{I_p}{2} = \frac{\pi c^4}{4}} \qquad (8\text{-}20)$$

Example 8-3

Determine the moment of inertia I around the horizontal axis for the area shown in mm in Fig. 8-11 for use in the flexure formula.

Area	A (mm^2)	y (mm) *(from bottom)*	Ay
Entire area	$40 \times 60 = 2400$	30	72 000
Hollow interior	$-20 \times 30 = -600$	35	−21 000
	$\Sigma A = 1800$ mm^2		$\Sigma Ay = 51\,000$ mm^3

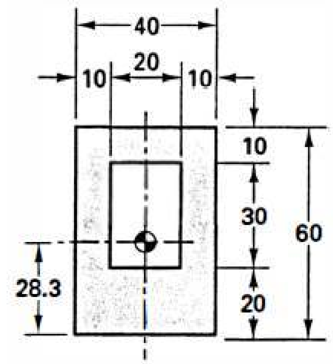


Figure 8-11

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{51\,000}{1800} = 28.3 \text{ mm from bottom}$$

For the entire area:

$$I_{z_c} = \frac{bh^3}{12} = \frac{40 \times 60^3}{12} = 72 \times 10^4 \text{ mm}^4$$

$$Ad^2 = 2400(30 - 28.3)^2 = \underline{0.69 \times 10^4 \text{mm}^4}$$

$$I_z = 72.69 \times 10^4 \text{ mm}^4$$

For the hollow interior:

$$I_z = \frac{bh^3}{12} = \frac{20 \times 30^3}{12} = 4.50 \times 10^4 \text{ mm}^4$$

$$Ad^2 = 600(35 - 28.3)^2 = \underline{2.69 \times 10^4 \text{ mm}^4}$$

$$I_z = 7.19 \times 10^4 \text{ mm}^4$$

For the composite section:

$$I_z = (72.69 - 7.19)10^4 = 65.50 \times 10^4 \text{ mm}^4$$

Example 8-4

A 300-by-400-mm wooden cantilever beam weighing 0.75 kN/m carries an upward concentrated force of 20 kN at the end, as shown in Fig. 8-12(a). Determine the maximum bending stresses at a section 2 m from the free end.

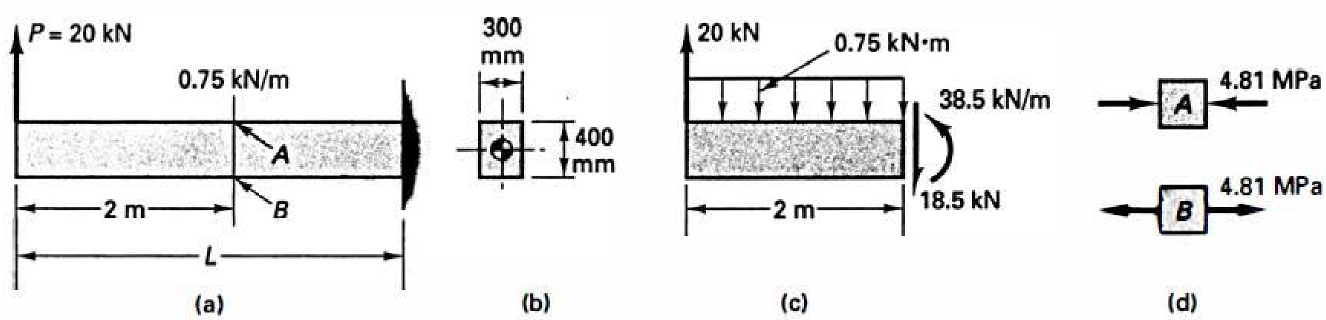


Figure 8-12

From Eq. 8-19,

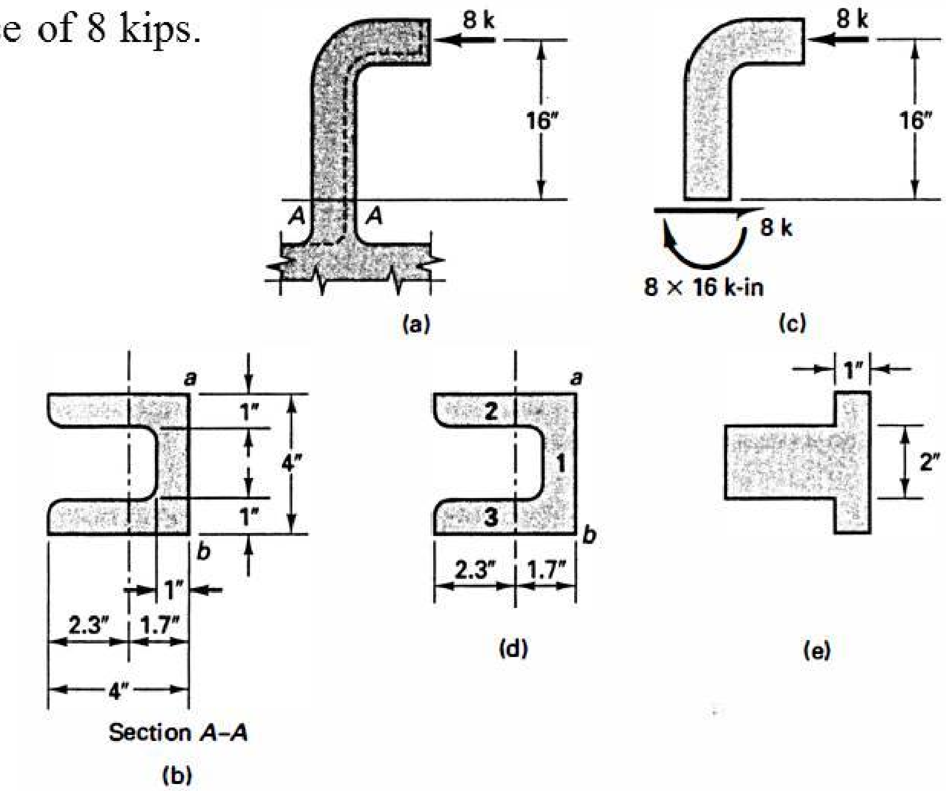
$$I_z = \frac{bh^3}{12} = \frac{300 \times 400^3}{12} = 16 \times 10^8 \text{ mm}$$

From Eq. 8-13,

$$\sigma_{max} = \frac{Mc}{I} = \frac{38.5 \times 10^6 \times 200}{16 \times 10^8} = \pm 4.81 \text{ MPa}$$

Example 8-5

Find the maximum tensile and compressive stresses acting normal to section A-A of the machine bracket shown in Fig. 8-13(a) caused by the applied force of 8 kips.



$$\bar{y} = \frac{\sum Ay}{\sum A} = \frac{17.0}{10.0} = 1.70 \text{ in} \quad \text{from line } ab$$

$$I = \sum (I_o + Ad^2) = \frac{4 \times 1^3}{12} + 4 \times 1.2^2$$

$$+ \frac{2 \times 1 \times 3^3}{12} + 2 \times 3 \times 0.8^2 = 14.43 \text{ in}^4$$

$$\sigma_{max} = \frac{Mc}{I} = \frac{8 \times 16 \times 2.3}{14.43} = 20.4 \text{ ksi} \quad \text{(compression)}$$

$$\sigma_{max} = \frac{Mc}{I} = \frac{8 \times 16 \times 1.7}{14.43} = 15.1 \text{ ksi} \quad \text{(tension)}$$

8-6. Stress Concentrations

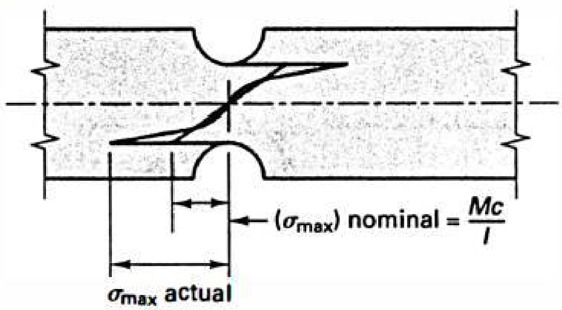


Figure 8-15 Meaning of stress-concentration factor in bending.

$$K = \frac{(\sigma_{max})\ \text{actual}}{(\sigma_{max})\ \text{nominal}}$$

$$\boxed{(\sigma_{max})_{actual} = K\frac{Mc}{I}} \qquad (8\text{-}23)$$

ENGINEERING MECHANICS OF SOLIDS

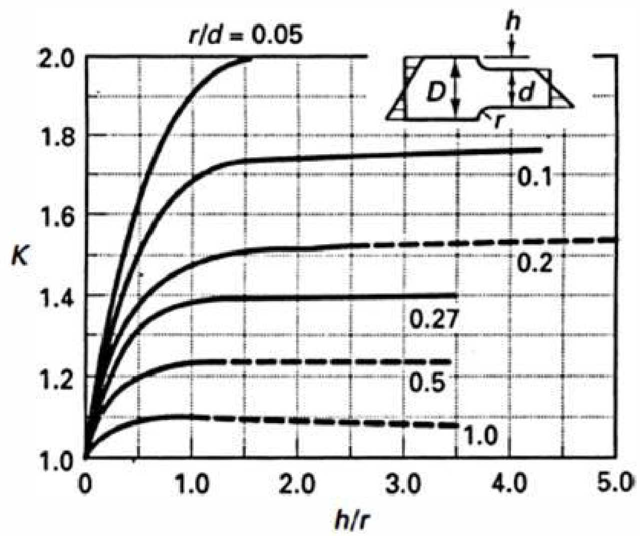


Figure 8-16 Stress-concentration factors in pure b nding for flat bars with various fillets.

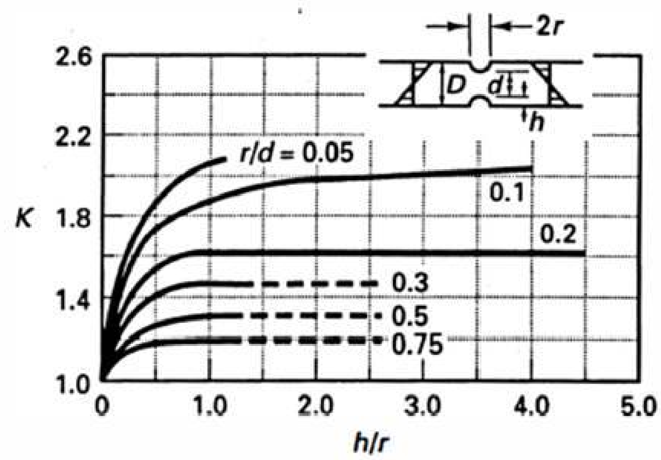


Figure 8-17 Stress-concentration factors in bending for grooved flat bars.

Example 8-12

Consider a composite beam of the cross-sectional dimensoins shown in Fig. 8-30(a). The upper 150-by-250-mm part is wood, Ew = 10 GPa; the lower 10-by-150-mm strap is steel, E, = 200 GPa. If this beam is subjected to a bending moment of 30 kN · m around a horizontal axis, what are the maximum stresses in the steel and wood?

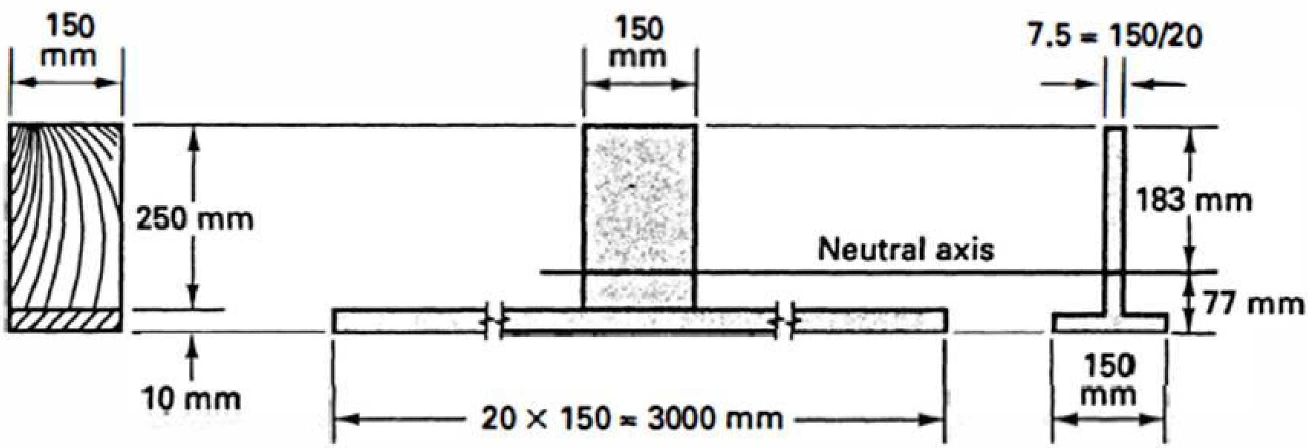


Figure 8-30

SOLUTION

Select E_w as E_{ref}. Then $n_s = E_s/E_w = 20$. Hence, the transformed cross section is as in Fig. 8-30(b) with the equivalent width of steel equal to $150 \times 20 = 3000$ mm. The centroid and moment of inertia around the centroidal axis for this transformed section are, respectively,

$$\bar{y} = \frac{150 \times 250 \times 125 + 10 \times 3000 \times 255}{150 \times 250 + 10 \times 3000} = 183 \text{ mm} \qquad \text{(from the top)}$$

$$I_z = \frac{150 \times 250^3}{12} + 150 \times 250 \times 58^2 + \frac{3000 \times 10^3}{12} + 10 \times 3000 \times 72^2$$

$$= 478 \times 10^6 \text{ mm}^4$$

The maximum stress in the wood is

$$(\sigma_w)_{max} = \frac{Mc}{I} = \frac{0.03 \times 10^9 \times 183}{478 \times 10^6} = 11.5 \text{ MPa}$$

The maximum stress in the steel is

$$(\sigma_s)_{max} = n\sigma_w = 20 \times \frac{0.03 \times 10^9 \times 77}{478 \times 10^6} = 96.7 \text{ MPa}$$

ALTERNATIVE SOLUTION

Select E_s as E_{ref}. Then $n_w = E_w/E_s = 1/20$, and the transformed section is as in Fig. 8-30(c).

$$\bar{y} = \frac{7.5 \times 250 \times 135 + 150 \times 10 \times 5}{7.5 \times 250 + 150 \times 10}$$

$$= 77 \text{ mm} \quad \text{(from the bottom)}$$

$$I_z = \frac{7.5 \times 250^3}{12} + 7.5 \times 250 \times 58^2 + \frac{150 \times 10^3}{12}$$

$$+ 150 \times 10 \times 72^2 = 23.9 \times 10^6 \text{ mm}^4$$

$$(\sigma_s)_{max} = \frac{0.03 \times 10^9 \times 77}{23.9 \times 10^6} = 96.7 \text{ Mpa}$$

$$(\sigma_w)_{max} = \frac{\sigma_s}{n} = \frac{1}{20} \times \frac{0.03 \times 10^9 \times 183}{23.9 \times 10^6} = 11.5 \text{ MPa}$$

Example 8-13

Determine the maximum stress in the concrete and the steel for a reinforced-concrete beam with the section shown in Fig. 8-31(a) if it is subjected to a positive bending moment of 50,000 ft-lb. The reinforcement consists to two #9 steel bars. (These bars are $1\frac{1}{8}$ in in diameter and have a cross-sectional area of 1 in^2.) Assume the ratio of E for steel to that of concrete to be 15 (i.e., $n = 15$).

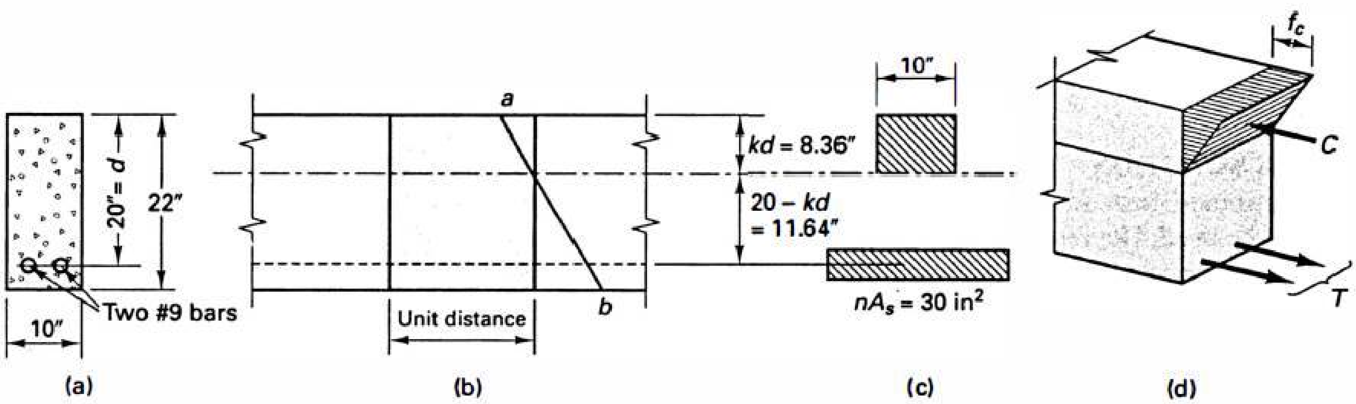


Figure 8-31

It is known that this axis coincides with the axis through the centroid of the transformed section. It is further known that the first (or statical) moment of the area on one side of a centroidal axis is equal to the first moment of the area on the other side. Thus, let kd be the distance from the top of the beam to the centroidal axis, as shown in Fig. 8-31(c), where k is the unknown ratio's and d is the distance from the top of the beam to the center of the steel. An algebraic restatement of the foregoing locates the neutral axis

$$\underbrace{10(kd)}_{\text{concrete area}}\ \underbrace{(kd/2)}_{\text{arm}} = \underbrace{30}_{\text{transformed steel area}}\ \underbrace{(20 - kd)}_{\text{arm}}$$

$$5(kd)^2 = 600 - 30(kd)$$

$$(kd)^2 + 6(kd) - 120 = 0$$

Hence,

$$kd = 8.36 \text{ in} \quad \text{and} \quad 20 - kd = 11.64 \text{ in}$$

$$I = \frac{10(8.36)^3}{12} + 10(8.36)\left(\frac{8.36}{2}\right)^2 + 0 + 30(11.64)^2 = 6020 \text{ in}^4$$

$$(\sigma_c)_{max} = \frac{Mc}{I} = \frac{50{,}000 \times 12 \times 8.36}{6020} = 833 \text{ psi}$$

$$\sigma_s = n\frac{Mc}{I} = \frac{15 \times 50{,}000 \times 12 \times 11.64}{6020} = 17{,}400 \text{ psi}$$

Problems for solution

In addition to other books mentioned, solve following problems from Popov:
11-15, 18, 25-26

CHAPTER

ENGINEERING MECHANICS OF SOLIDS

10

Shear Stresses in Beams

10-2 Preliminary Remarks

In deriving the torsion and the flexure formulas, the same sequence of reasoning was employed. First, a strain distribution was assumed across the section; next, properties of the material were brought in to relate these strains to stresses: and, finally, the equations of equilibrium were used to establish the desired relations. However, the development of the expression linking the shear force and the cross-sectional area of a beam to the stress follows a different path. The previous procedure cannot be employed, as no simple assumption for the strain distribution due to the shear force can be made. Instead, an indirect approach is used. The stress distribution caused by flexure, as determined in the preceding two chapters, is assumed, which, together with the equilibrium requirements, resolves the problem of the shear stresses.

$$\frac{dM}{dx} = V \qquad (10\text{-}1)$$

Equation 10-1 means that if shear V is acting at a section, there will be a change in the bending moment M on an adjoining section. The difference between the bending moments on the adjoining sections is equal to V dx. If no shear is acting, no change in the bending moment occurs. Alternatively, the rate of change in moment along a beam is equal to the shear. Thus, if a shear is (and a bending moment) present at one section through a beam, a different bending moment will exist at an adjoining section, although the shear may remain constant. This will lead to the establishment of the shear stresses on the imaginary longitudinal planes through the members that are parallel to its axis. Since at a point, equal shear stresses exist on the mutually perpendicular planes, the shear stresses whose direction is coincident with the shear force at a section will be determined.

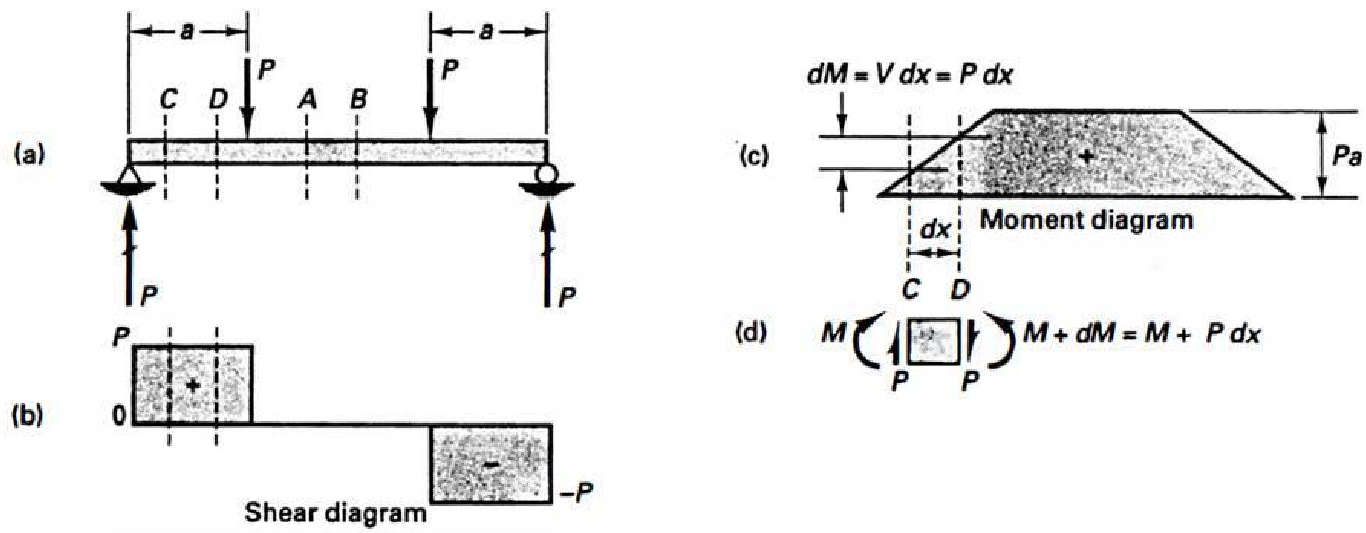


Fig. 10-1 Shear and bending moment diagrams for the loading shown.

Here at any two sections, such as A and B, taken through the beam anywhere between applied forces P, the bending moment is the same. No shear acts at these sections. On the other hand, between any two sections, such as C and D, near the support, a change in the bending moment does take place. Shear forces act at these sections. These shears are shown acting on an element of the beam in Fig. 10-1(d). Note that in this zone of the beam, the change in the bending moment in a distance dx is P dx, as shear V is equal to P.

Before a detailed analysis is given, a study of a sequence of photographs of a model (Fig. 10-2) may prove helpful. The model represents a segment of an I beam. In Fig. 10-2(a), in addition to the beam itself, blocks simulating stress distribution caused by bending moments may be seen. The moment on the right is assumed to be larger than the one on the left. This system of forces is in equilibrium providing vertical shears V (not seen in this view) also act on the beam segment. By separating the model along the neutral surface, one obtains two separate parts of the beam segment, as shown in Fig. 10-2(b). Again, either one of these parts alone must be in equilibrium. If the upper and the lower segments of Fig. 10-2(b) are connected by a dowel or a bolt in an actual beam, the axial forces on either the upper or the lower part caused by the bending-moment stresses must be maintained in equilibrium by a force in the dowel. The force that must be resisted can be evaluated by summing the forces in the axial direction caused by bending stresses. In performing such a calculation, either the upper or the lower part of the beam segment can be used.

The horizontal force transmitted by the dowel is the force needed to balance the net force caused by the bending stresses acting on the two adjoining sections. Alternatively, by subtracting the same bending stress on both ends of the segment, the same results can be obtained. This is shown schematically in Fig. 10-2(c), where, assuming a zero bending moment on the left, only the normal stresses due to the increment in moment within the segment need be shown acting on the right. If, initially, the I beam considered is one piece requiring no bolts or dowels, an imaginary longitudinal plane can be used to separate the beam segment into two parts; see Fig. 10-2(d). As before, the net force that must be developed across the cut area to maintain equilibrium can be determined. Dividing this force by the area of the imaginary horizontal cut gives average shear stresses acting in this plane.

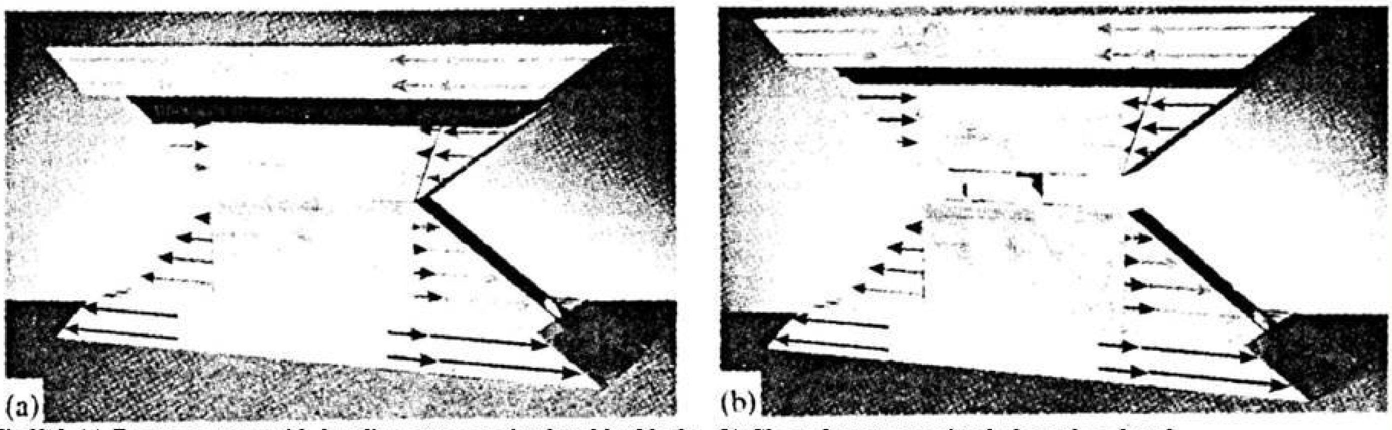
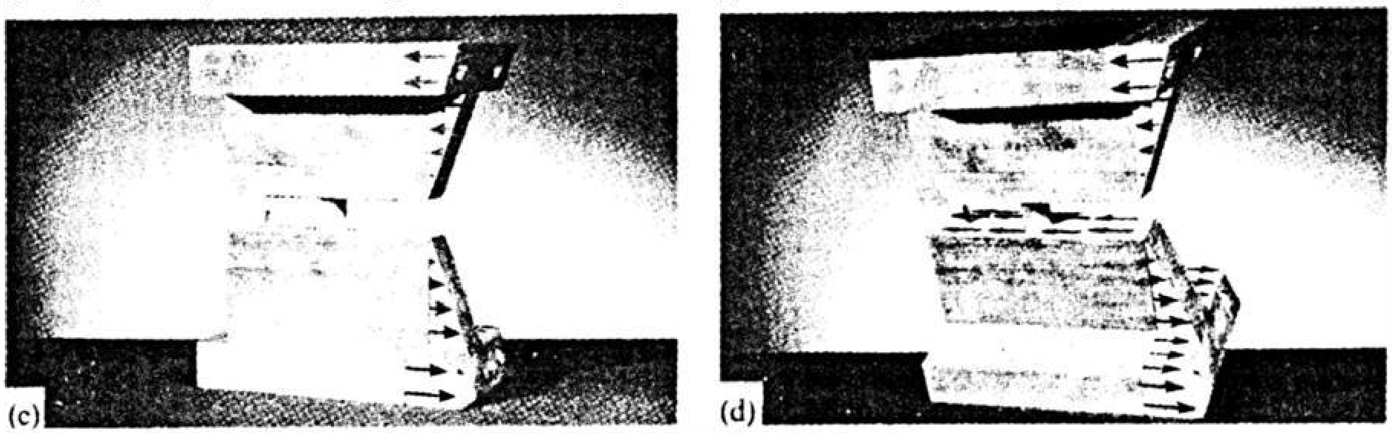


Fig.10·2 (a) Beam segment with bending stresses simulated by blocks. (b) Shear force transmitted through a dowel.



(c) For determining the force on a dowel, only a change in moment is needed. (d) The longitudinal shear force divided by the area of the imaginary cut yields shear stress.

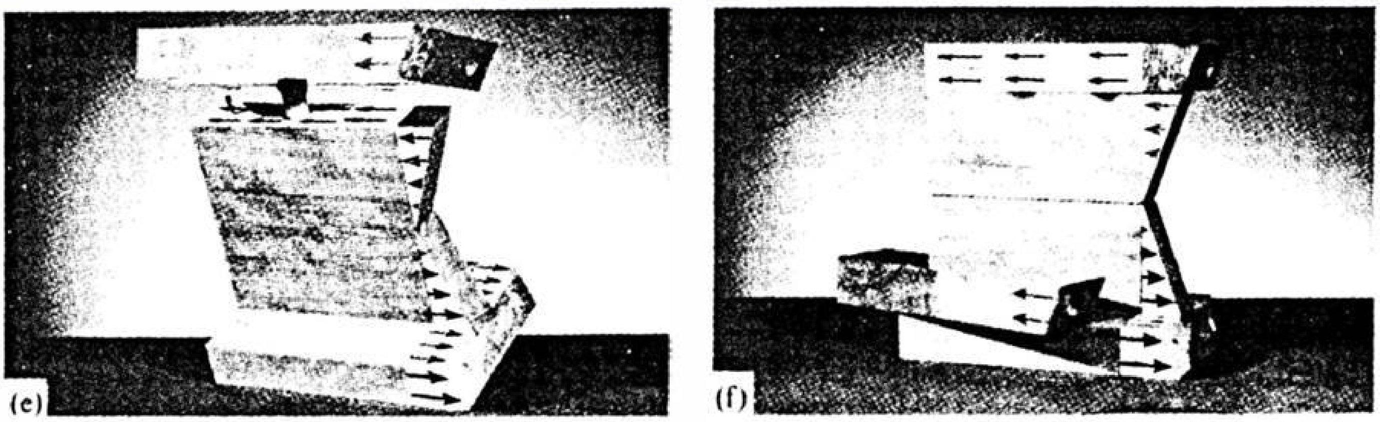


Fig.l0·2 (e) Horizontal cut below the flange for determining the shear stress.
(f) Vertical cut through the flange for determining the shear stress.

After the shear stresses on one of the planes are found [i.e., the horizontal one in Fig.I0-2(d)], shear stresses on mutually perpendicular planes of an infinitesimal element also become known since they must be numerically equal; see Eq. 1-2. This approach establishes the shear stresses in the plane of the beam section taken normal to its axis.

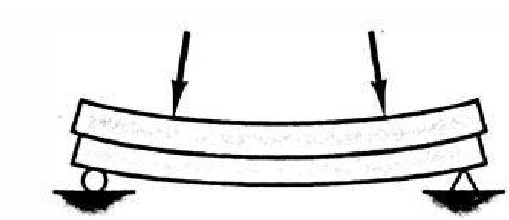


Fig. 10-3 Sliding between planks not fastened together.

Consider a wooden plank placed on top of another, as shown in Fig. 10-3. If these planks act as a beam and are not interconnected, sliding at the surfaces of their contact will take place. The interconnection of these planks with nails or glue is necessary to make them act as an integral beam.

10-3. Shear Flow

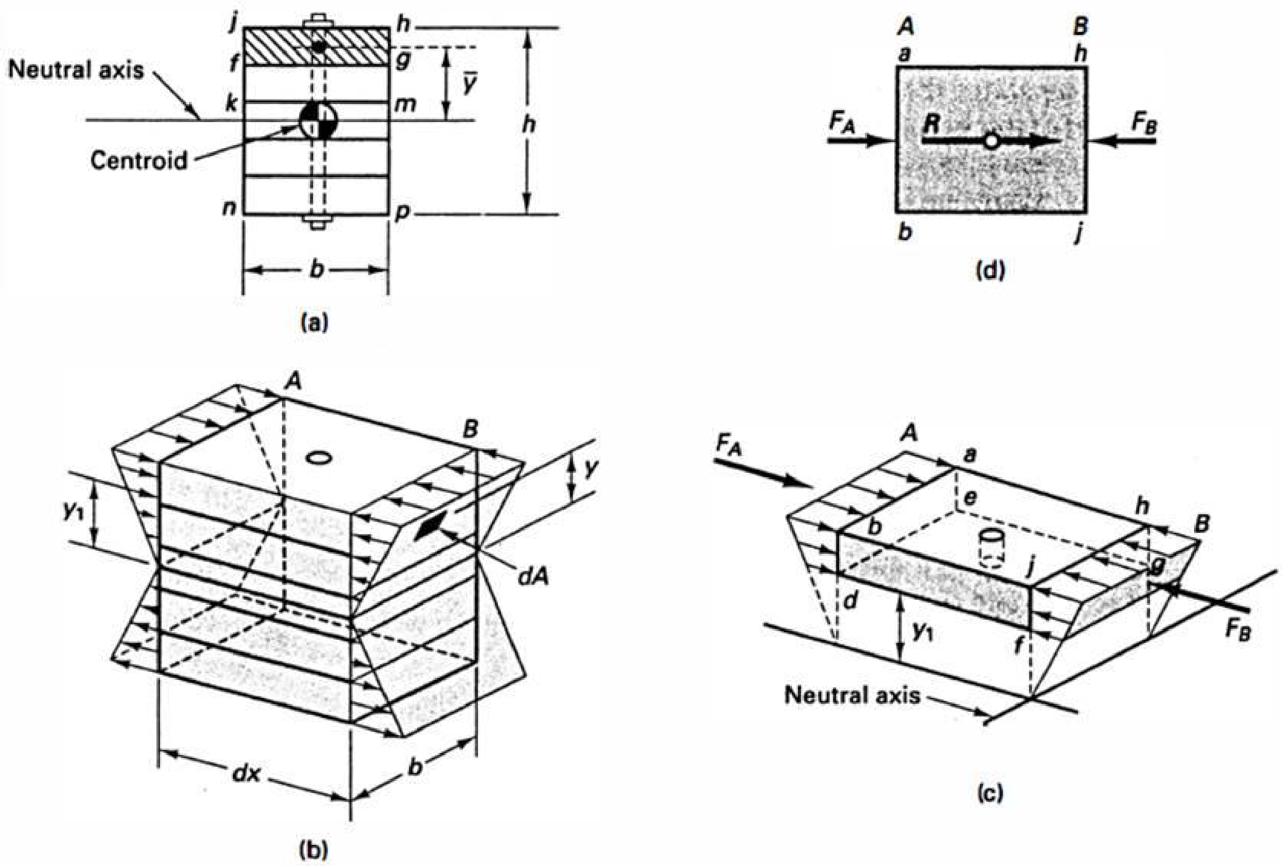


Fig. 10-4 Elements for deriving shear flow in a beam.

Consider an elastic beam made from several continuous longitudinal planks whose cross section is shown in Fig. 10-4(a). For simplicity, the beam has a rectangular cross section, but such a limitation is not necessary. To make this beam act as an integral member, it is assumed that the planks are fastened at intervals by vertical bolts. An element of this beam isolated by two parallel sections, both of which are perpendicular to the axis of the beam, is shown in Fig. 10-4(b).

If the element shown in Fig. 10-4(b) is subjected to a bending moment $+M_A$ at end A and to $+M_B$ at end B, bending stresses that act normal to the sections are developed. These bending stresses vary linearly from their respective neutral axes, and at any point at a distance y from the neutral axis are $-M_B y/I$ on the B end and $-M_A y/I$ on the A end.

From the beam element, Fig. 10-4(b), isolate the top plank, as shown in Fig. 10-4(c). The fibers of this plank nearest the neutral axis are located by the distance y_1. Then, since stress times area is equal to force, the forces acting perpendicular to ends A and B of this plank may be determined. At end B, the force acting on an infinitesimal area dA at a distance y from the neutral axis is $(-M_B y/I)\, dA$. The total force acting on the area $fghj$, A_{fghj}, is the sum, or the integral, of these elementary forces over this area. Denoting the total force acting normal to the area $fghj$ by F_B and remembering that, at section B, M_B, and I are constants, one obtains the following relation:

$$F_B = \int_{\substack{\text{area} \\ fghi}} -\frac{M_B y}{I}\, dA = -\frac{M_B}{I} \int_{\substack{\text{area} \\ fghi}} y\, dA = -\frac{M_B Q}{I} \qquad (10\text{-}2)$$

where

$$Q = \int_{\substack{\text{area}\\fghj}} y\, dA = A_{fghj}\bar{y} \qquad (10\text{-}3)$$

The integral defining Q is the first, or the statical, moment of area *fghj* around the neutral axis. By definition, $\bar{y}$ is the distance from the neutral axis to the centroid of A_{fghj}.[1] Illustrations of the manner of determining Q are in Fig. 10-5. Equation 10-2 provides a convenient means of calculating the longitudinal force acting normal to any selected part of the cross-sectional area.

Next consider end A of the element in Fig. 10-4(c). One can then express the total force acting normal to the area *abde* as

$$F_A = -\frac{M_A}{I}\int_{\substack{\text{area}\\abde}} y\, dA = -\frac{M_A Q}{I} \qquad (10\text{-}4)$$

where the meaning of Q is the same as that in Eq. 10-2 since for prismatic beams, an area such as *fghj* is equal to the area *abde*. Hence, if the moments at A and B were equal, it would follow that $F_A = F_B$, and the bolt shown in the figure would perform a nominal function of keeping the planks together and would not be needed to resist any known longitudinal forces.

On the other hand, if M_A is not equal to M_B, which is always the case when shears are present at the adjoining sections, F_A is not equal to F_B. More push (or pull) develops on one end of a "plank" than on the other, as different normal stresses act on the section from the two sides. Thus, if $M_A \neq M_B$, equilibrium of the horizontal forces in Fig. 10-4(c) may be attained only by developing a horizontal resisting force R in the bolt. If $M_B > M_A$, then $|F_B| > |F_A|$, and $|F_A| + R = |F_B|$, as in Fig. 10-4(d). The force $|F_B| - |F_A| = R$ tends to shear the bolt in the plane of the plank *edfg*.[2] If the shear force acting across the bolt at level *km*, Fig. 10-4(a), were to be investigated, the two upper planks should be considered as one unit.

If $M_A \neq M_B$ and the element of the beam is only dx long, the bending moments on the adjoining sections change by an infinitesimal amount. Thus, if the bending moment at A is M_A, the bending moment at B is $M_B = M_A + dM$. Likewise, in the same distance dx, the longitudinal forces F_A and F_B change by an infinitesimal force dF (i.e., $|F_B| - |F_A| = dF$). By substituting these relations into the expression for F_B and F_A found previously, with areas *fghj* and *abde* taken equal, one obtains an expression for the differential longitudinal push (or pull) dF:

$$dF = |F_B| - |F_A| = \left(\frac{M_A + dM}{I}\right)Q - \left(\frac{M_A}{I}\right)Q = \frac{dM}{I}Q$$

In the final expression for dF, the actual bending moments at the adjoining sections are eliminated. Only the difference in the bending moments dM at the adjoining sections remains in the equation.

Instead of working with a force dF, which is developed in a distance dx, it is more significant to obtain a similar force per unit of beam length. This quantity is obtained by dividing dF by dx. Physically, this quantity represents the difference between F_B and F_A for an element of the beam of unit length. The quantity dF/dx will be designated by q and will be referred to as the *shear flow*. Since force is measured in newtons or pounds, shear flow q has units of newtons per meter or pounds per inch. Then, recalling that $dM/dx = V$, one obtains the following expression for the shear flow in beams:

$$q = \frac{dF}{dx} = \frac{dM}{dx}\frac{1}{I}\int_{\substack{\text{area}\\ fghj}} y\,dA = \frac{VA_{fghj}\bar{y}}{I} = \frac{VQ}{I} \qquad (10\text{-}5)$$

Example 10-1

Two long wooden planks form a T section of a beam, as shown in mm in Fig. 10-6(a). If this beam transmits a constant vertical shear of 3000 N, find the necessary spacing of the nails between the two planks to make the beam act as a unit. Assume that the allowable shear force per nail is 700 N.

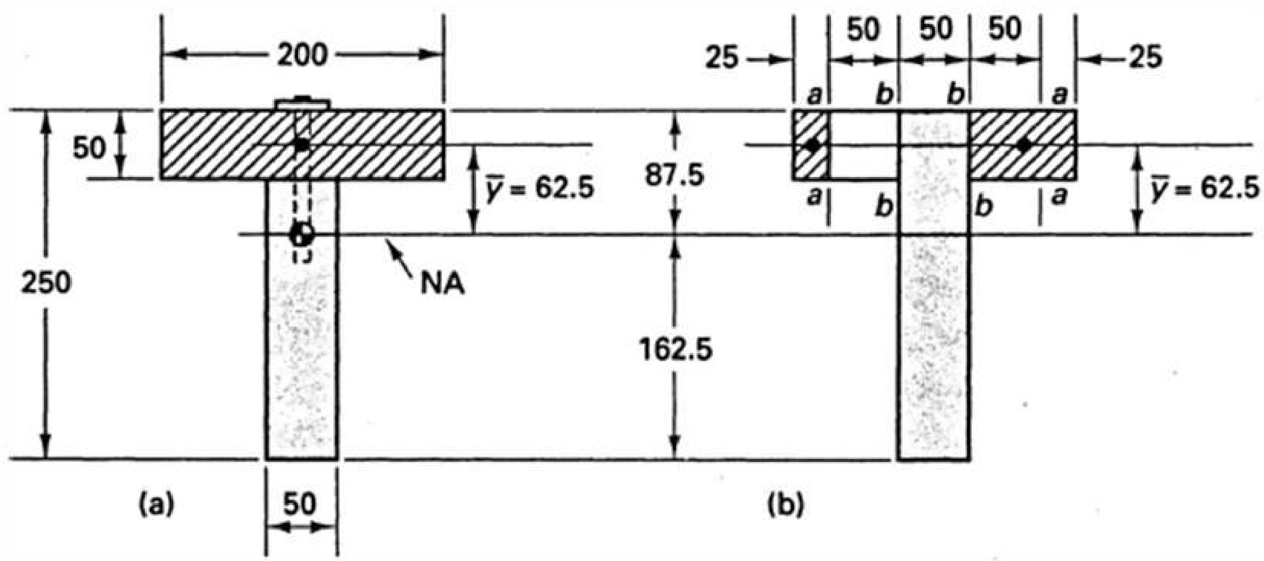


Fig. 10-6

$$y_c = \frac{50 \times 200 \times 25 + 50 \times 200 \times 150}{50 \times 200 + 50 \times 200} = 87.5 \text{ mm}$$

$$I = \frac{200 \times 50^3}{12} + 50 \times 200 \times 62.5^2 + \frac{50 \times 200^3}{12} + 50 \times 200 \times 62.5^2$$

$$= 113.54 \times 10^6 \text{ mm}^4$$

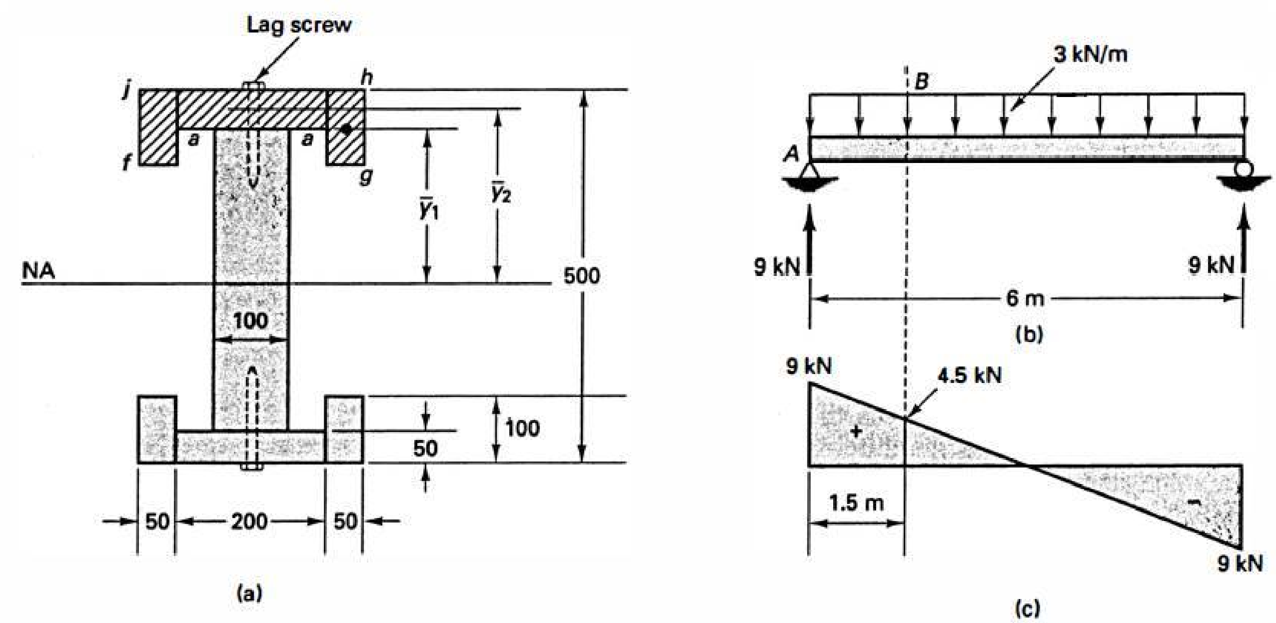
$$Q = A_{fghj}\bar{y} = 50 \times 200 \times (87.5 - 25) = 625 \times 10^3 \text{ mm}^3$$

$$q = \frac{VQ}{I} = \frac{3000 \times 625 \times 10^3}{113.54 \times 10^6} = 16.5 \text{ N/mm}$$

Thus, a force of 16.5 N/mm must be transferred from one plank to the other along the length of the beam. However, from the data given, each nail is capable of resisting a force of 700 N; hence, one nail is adequate for transmitting shear along 700/16.5 = 42 mm of the beam length. As shear remains constant at the consecutive sections of the beam, the nails should be spaced throughout at 42-mm intervals.

Example 10-2

A simple beam on a 6-m span carriers a load of 3 kN/m, including its own weight. The beam's cross section is to be made from several wooden pieces, as is shown in mm in Fig. 10-7(a). Specify the spacing of the 10-mm lag screws shown that is necessary to fasten this beam together. Assume that one 10-mm lag screw, as determined by laboratory tests, is good for 2 kN when transmitting lateral load parallel to the grain of the wood. For the *entire* section, I is equal to $2.36 \times 10^9\ mm^2$.



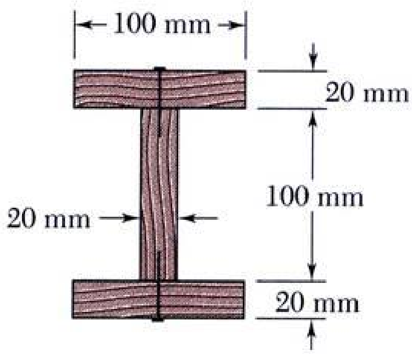
$$\int_{\substack{\text{area}\\ fghj}} y\,dA = Q$$

$$Q = A_{fghj}\bar{y} = 2A_1\bar{y}_1 + A_2\bar{y}_2$$
$$= 2 \times 50 \times 100 \times 200 + 50 \times 200 \times 225 = 4.25 \times 10^6 \text{ mm}^3$$

$$q = \frac{VQ}{I} = \frac{9 \times 4.25 \times 10^9}{2.36 \times 10^9} = 16.2 \text{ N/mm}$$

At the supports, the spacing of the lag screws must be $2 \times 10^3/16.2 = 123$ mm apart. This spacing of the lag screws applies only at a section where shear V is equal to 9 kN. Similar calculations for a section where $V = 4.5$ kN give $q = 8.1$ N/mm, and the spacing of the lag screws becomes $2 \times 10^3/8.1 = 246$ mm. Thus, it is proper to specify the use of 10-mm lag screws on 120-mm centers for a distance of 1.5 m nearest both of the supports and 240-mm spacing of the same lag screws for the middle half of the beam. A greater refinement in making the transition from one spacing of fastenings to another may be desirable in some problems. The *same* spacing of lag screws should be used at section *b–b* as at section *a–a*.

Example 6.01

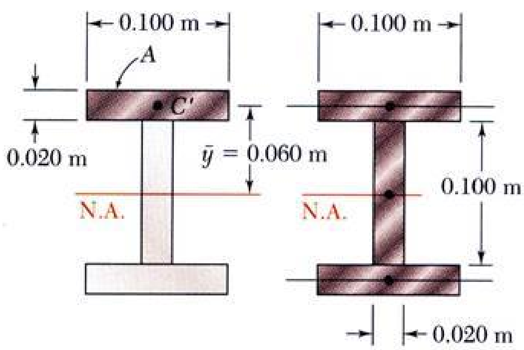


A beam is made of three planks, nailed together. Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is $V = 500$ N, determine the shear force in each nail.

SOLUTION:

- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.
- Calculate the corresponding shear force in each nail.

Example 6.01



$$Q = A\bar{y}$$
$$= (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})$$
$$= 120 \times 10^{-6}\,\text{m}^3$$

$$I = \tfrac{1}{12}(0.020\,\text{m})(0.100\,\text{m})^3$$
$$+ 2[\tfrac{1}{12}(0.100\,\text{m})(0.020\,\text{m})^3$$
$$+ (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})^2]$$
$$= 16.20 \times 10^{-6}\,\text{m}^4$$

SOLUTION:

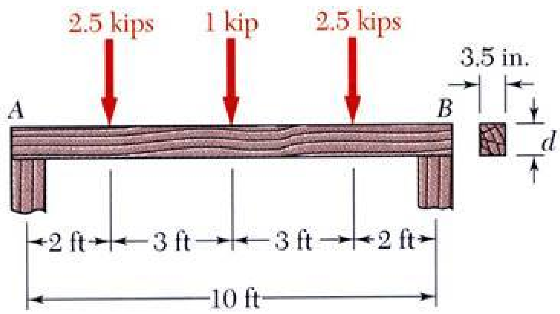
- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.

$$q = \frac{VQ}{I} = \frac{(500\text{N})(120 \times 10^{-6}\,\text{m}^3)}{16.20 \times 10^{-6}\,\text{m}^4}$$
$$= 3704\,\text{N/m}$$

- Calculate the corresponding shear force in each nail for a nail spacing of 25 mm.

$$F = (0.025\text{ m})q = (0.025\text{ m})(3704\ N/m$$
$$\boxed{F = 92.6\,\text{N}}$$

Sample Problem 6.2



A timber beam is to support the three concentrated loads shown. Knowing that for the grade of timber used,

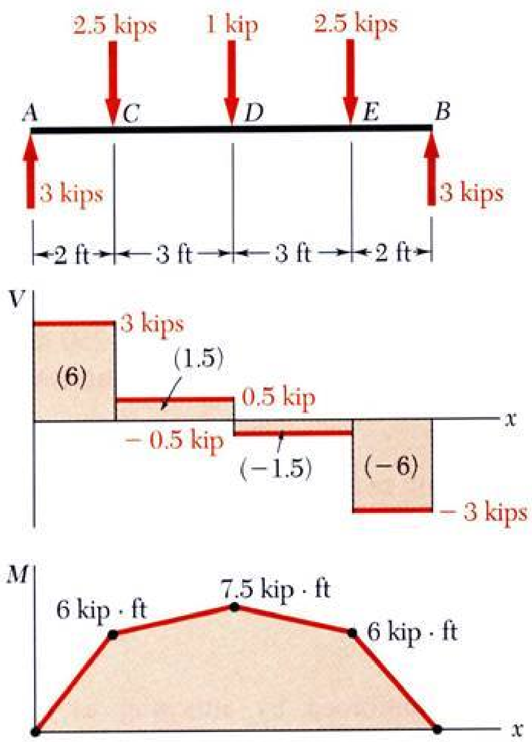
$$\sigma_{all} = 1800\,\text{psi} \qquad \tau_{all} = 120\,\text{psi}$$

determine the minimum required depth d of the beam.

SOLUTION:

- Develop shear and bending moment diagrams. Identify the maximums.
- Determine the beam depth based on allowable normal stress.
- Determine the beam depth based on allowable shear stress.
- Required beam depth is equal to the larger of the two depths found.

Sample Problem 6.2



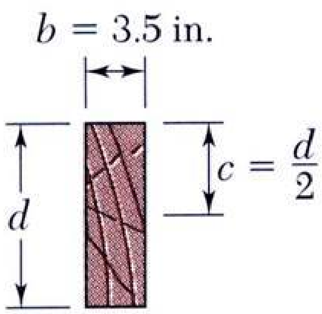
SOLUTION:

Develop shear and bending moment diagrams. Identify the maximums.

$$V_{max} = 3\,\text{kips}$$

$$M_{max} = 7.5\,\text{kip}\cdot\text{ft} = 90\,\text{kip}\cdot\text{in}$$

Sample Problem 6.2



$$I = \frac{1}{12} b d^3$$

$$S = \frac{I}{c} = \frac{1}{6} b d^2$$

$$= \frac{1}{6}(3.5\,\text{in.})d^2$$

$$= (0.5833\,\text{in.})d^2$$

- Determine the beam depth based on allowable normal stress.

$$\sigma_{all} = \frac{M_{\max}}{S}$$

$$1800\,\text{psi} = \frac{90 \times 10^3\,\text{lb} \cdot \text{in.}}{(0.5833\,\text{in.})d^2}$$

$$d = 9.26\,\text{in.}$$

- Determine the beam depth based on allowable shear stress.

$$\tau_{all} = \frac{3}{2}\frac{V_{\max}}{A}$$

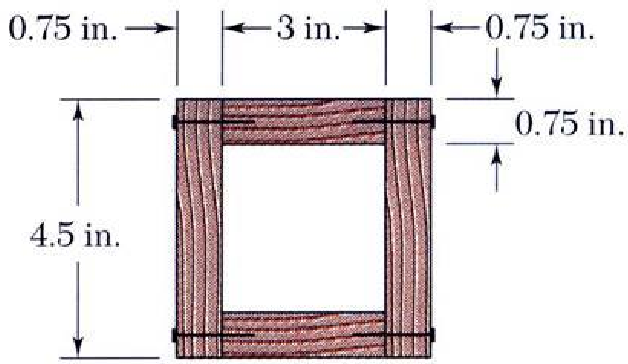
$$120\,\text{psi} = \frac{3}{2}\frac{3000\,\text{lb}}{(3.5\,\text{in.})d}$$

$$d = 10.71\,\text{in.}$$

- Required beam depth is equal to the larger of the two.

$$\boxed{d = 10.71\,\text{in.}}$$

Example 6.04

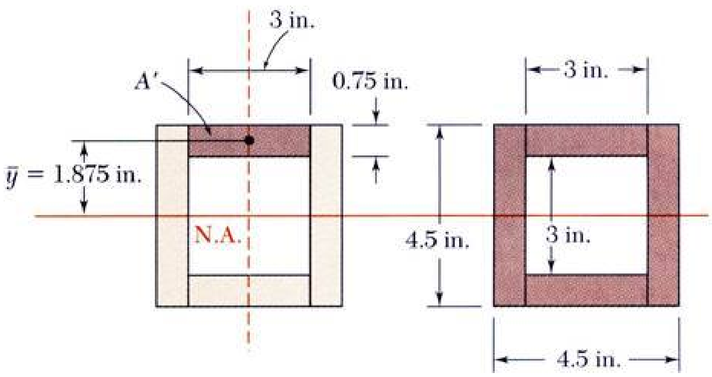


A square box beam is constructed from four planks as shown. Knowing that the spacing between nails is 1.5 in. and the beam is subjected to a vertical shear of magnitude $V = 600$ lb, determine the shearing force in each nail.

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.
- Based on the spacing between nails, determine the shear force in each nail.

Example 6.04



For the upper plank,

$$Q = A'y = (0.75\text{in.})(3\text{in.})(1.875\text{in.})$$
$$= 4.22\,\text{in}^3$$

For the overall beam cross-section,

$$I = \tfrac{1}{12}(4.5\,\text{in})^3 - \tfrac{1}{12}(3\,\text{in})^3$$
$$= 27.42\,\text{in}^4$$

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.

$$q = \frac{VQ}{I} = \frac{(600\,\text{lb})(4.22\,\text{in}^3)}{27.42\,\text{in}^4} = 92.3\frac{\text{lb}}{\text{in}}$$

$$f = \frac{q}{2} = 46.15\frac{\text{lb}}{\text{in}}$$

= edge force per unit length

- Based on the spacing between nails, determine the shear force in each nail.

$$F = f\ell = \left(46.15\frac{\text{lb}}{\text{in}}\right)(1.75\,\text{in})$$

$$\boxed{F = 80.8\,\text{lb}}$$

10-4. The Shear Stress Formula for Beams

$$\tau = \frac{dF}{dx\,t} = \frac{VQ}{It} = \frac{q}{t}$$

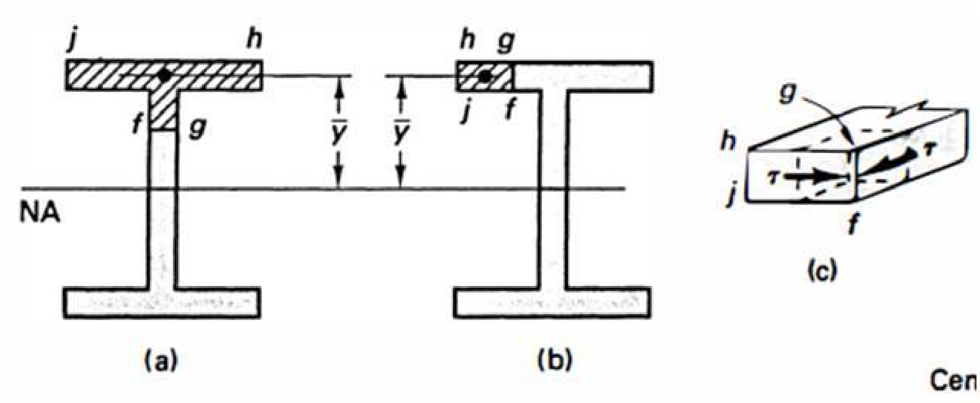


Fig. 10-10 Sectioning for partial areas of cross sections for computing shear stresses.

Procedure Summary

The same *three basic concepts* of engineering mechanics of solids as before are used in developing the formula for shear stresses in beams. However, their use is less direct.

1. *Equilibrium conditions* are used,
   (a) for determining the shear at a section
   (b) by using the relationship between the shear and the rate of change in bending moment along a span, and
   (c) by determining the force at a longitudinal section of a beam element for obtaining the average shear stress.
2. *Geometry of deformation*, as in pure bending, is assumed such that plane sections remain plane after deformation, leading to the conclusion that normal strains in a section vary linearly from the neutral axis. Since, due to shear, the cross sections do not remain plane, but warp, this assumption is less accurate than for pure bending. However, for small and moderate magnitudes of shear, and slender members, this assumption is satisfactory.
3. *Material properties* are considered to obey Hooke's law, although extension to other constitutive relations is possible for elementary solutions.

Example 10-3

Derive an expression for the shear stress distribution in a beam of solid rectangular cross section transmitting a vertical shear V.

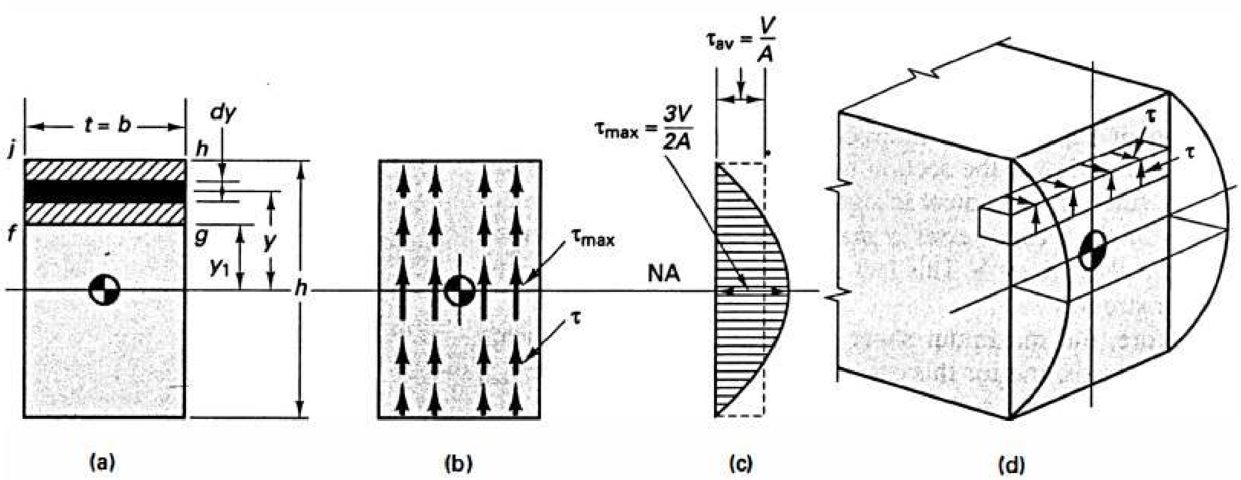


Fig. 10-11

Eq. 10-6, the horizontal shear stress is found *at* level y_1 of the beam. *At* the same cut, numerically equal vertical shear stresses act *in the plane of the cross section;* see Eq. 1-2. So,

$$\tau = \frac{VQ}{It} = \frac{V}{It}\int_{\substack{\text{area}\\ fghj}} y\, dA = \frac{V}{Ib}\int_{y1}^{h/2} by\, dy$$

$$= \frac{V}{I}\left|\frac{y^2}{2}\right|_{y1}^{h/2} = \frac{V}{2I}\left[\left(\frac{h}{2}\right)^2 - y_1^2\right] \qquad (10\text{-}7)$$

As noted before, the maximum shear stress in a rectangular beam occurs at the neutral axis, and for this case, the general expression for τ_{max} may be simplified by setting $y_1 = 0$:

$$\tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8bh^3/12} = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A} \qquad (10\text{-}8a)$$

From the property of the statical moments of areas around a centroidal axis, the maximum value of Q is obtained by considering one-half the cross-sectional area around the neutral axis of the beam. Hence, alternately,

$$\tau_{max} = \frac{VQ}{It} = \frac{V\left(\frac{bh}{2}\right)\left(\frac{h}{4}\right)}{\left(\frac{bh^3}{12}\right)b} = \frac{3}{2}\frac{V}{A} \qquad (10\text{-}8b)$$

Example 10-5

An I beam is loaded as shown in Fig. 10-16(a). If it has the cross section shown in Fig. 10-16(c), determine the shear stresses at the levels indicated. Neglect the weight of the beam.

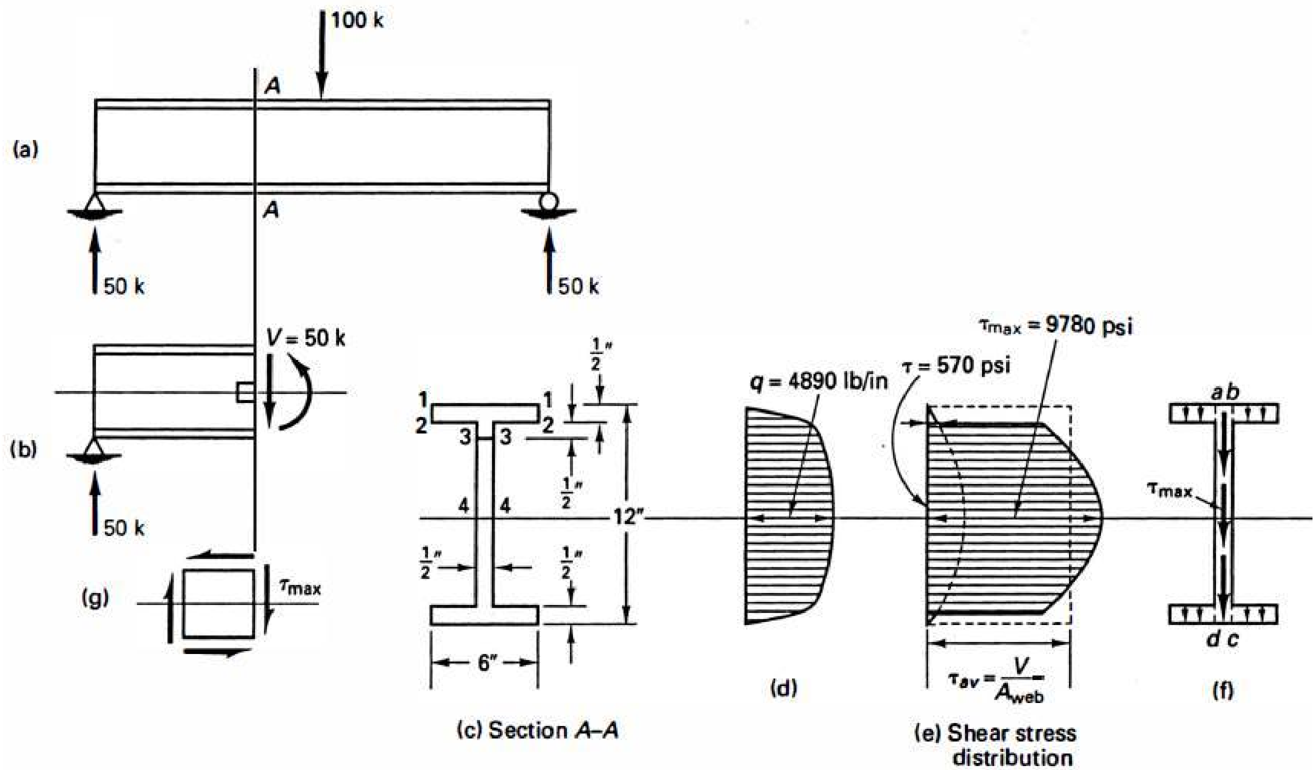


Fig. 10-16

Level	A_{fghj}[a]	y[b]	$Q = A_{fghj}y$	$q = VQ/I$	t	τ (psi)
1–1	0	6	0	0	6.0	0
					6.0	570
2–2	0.5 × 6 = 3.00	5.75	17.25	3400	0.5	6800
3–3	0.5 × 6 = 3.00 0.5 × 0.5 = 0.25	5.75 5.25	17.25 1.31 } 18.56	3650	0.5	7300
4–4	0.5 × 6 = 3.00 0.5 × 5.5 = 2.75	5.75 2.75	17.25 7.56 } 24.81	4890	0.5	9780

[a] A_{fghj} is the partial area of the cross section above a given level in in^2.

[b] y is distance in mm from the neutral axis to the centroid of the partial area.

$$(\tau_{max})_{approx} = \frac{V}{A_{web}} \tag{10-10}$$

In the example considered, this gives

$$(\tau_{max})_{approx} = \frac{50{,}000}{0.5 \times 12} = 8330 \text{ psi}$$

10-6. Some Limitations of the Shear Stress Formula

The shear stress formula for beams is based on the flexure formula. Hence, all of the limitations imposed on the flexure formula apply. (i) The material is assumed to be elastic with the same elastic modulus in tension as in compression. (ii) The theory developed applies only to straight beams. (iii) Moreover, there are additional limitations that are not present in the flexure formula. Some of these are:

Consider a section through the I beam analyzed in Example 10-5. Some of the results of this analysis are reproduced in Fig. 10-17. The shear stresses computed earlier for level 1-1 apply to the infinitesimal element a. The vertical shear stress is zero for this element. Likewise, no shear stresses exist on the top plane of the beam. This is as it should be, since the top surface of the beam is a free surface. In mathematical phraseology. this means that the conditions at the boundary are satisfied.

ENGINEERING MECHANICS OF SOLIDS

A different condition is found when the shear stresses determined for the I beam at levels 2-2 are scrutinized. The shear stresses were found to be 570 psi for the elements such as b or c shown in the figure. This requires matching horizontal shear stresses on the inner surfaces of the flanges. However, the latter surfaces must be free of the shear stresses, as they are free boundaries of the beam. This leads to a contradiction that cannot be resolved by the methods of engineering mechanics of solids. The more advanced techniques of the mathematical theory of elasticity or three-dimensional finite element analysis must be used to obtain an accurate solution.

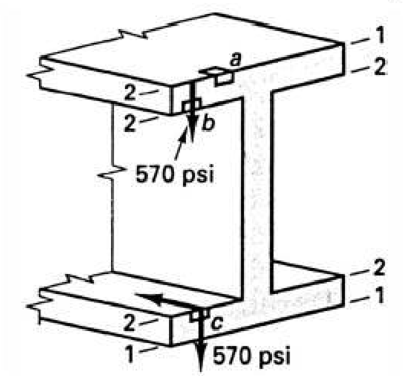


Fig.10-17 Boundary conditions are not satisfied at the levels 2-2.

ENGINEERING MECHANICS OF SOLIDS

In mechanical applications, circular shafts frequently act as beams. Hence, beams having a solid circular cross section form an important class. These beams are not "thin walled." An examination of the boundary conditions for circular members, Fig. 10.18(a). leads to the conclusion that when shear stresses are present. they must act parallel to the boundary. As no matching shear stress can exist on the free surface of a beam, no shear stress component can act normal to the boundary. However, according to Eq. 10-6, vertical shear stresses of equal intensity act at every level, such as ac in Fig. 10-18(b). This is incompatible with the boundary conditions for elements a and c, and the solution indicated by Eq. 10-6 is inconsistent. Fortunately, the maximum shear stresses occurring at the neutral axis satisfy the boundary conditions and are within about 5% of their true value.

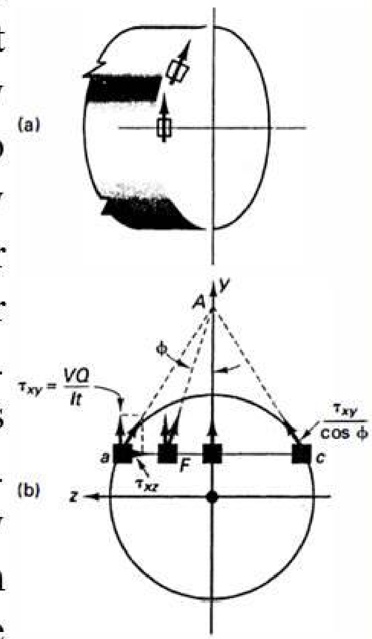


Fig. 10-18 Modification of shear stresses to satisfy the boundary conditions.

10-7. Shear Stress in Beam Flanges

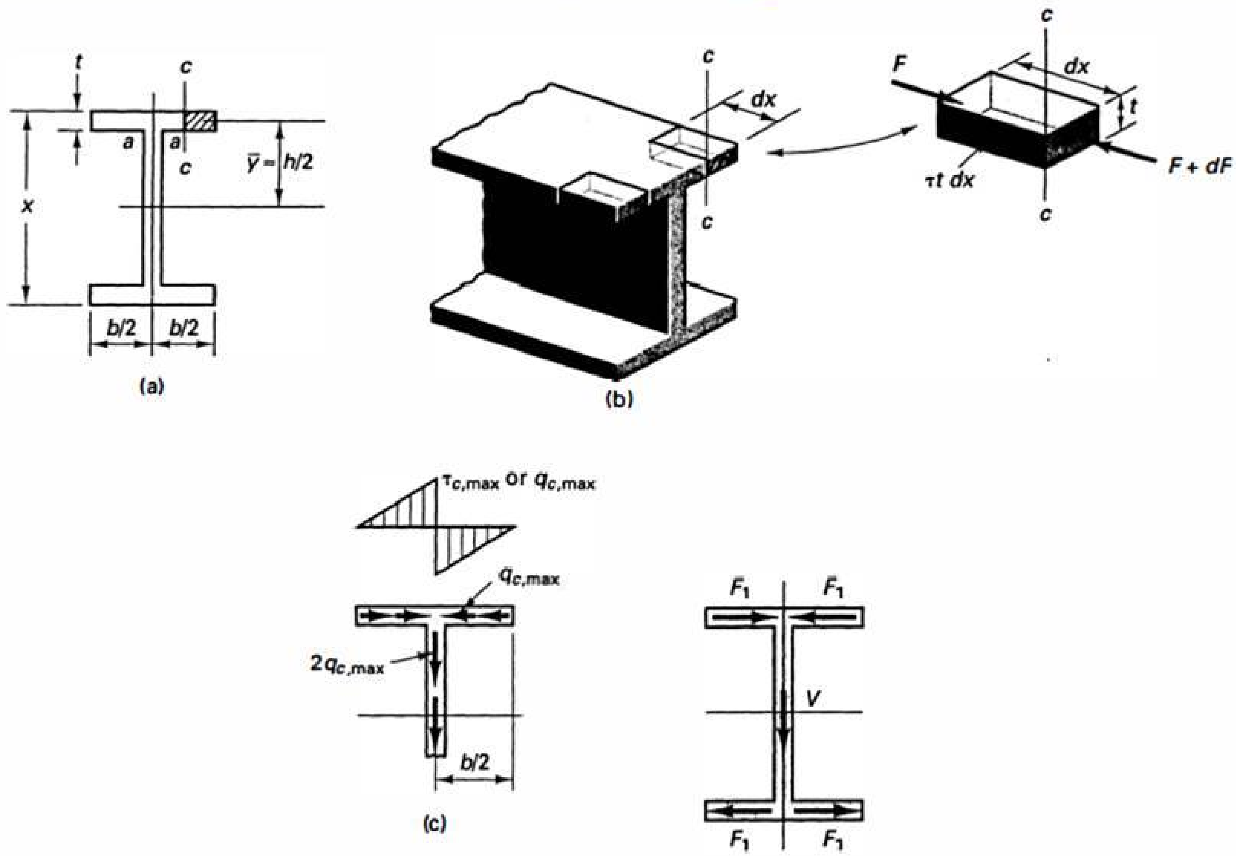


Fig. 10-19 Shear forces in the flanges of an I beam act perpendicularly to the axis of symmetry.

In an I beam, the existence of shear stresses acting in a vertical longitudinal cut, such as c-c in Fig. 10-19(a), was indicated in Fig. 10-2(f) and Section 10-4. These shear stresses act perpendicular to the plane of the paper. Their magnitude may be found by applying Eq. 10-6, and their sense follows by considering the bending moments at the adjoining sections through the beam. For example, if, for the beam shown in Fig. 10-19(b), positive bending moments increase toward the reader, larger normal forces act on the near section. For the elements shown, $\tau t dx$ or $q dx$ must aid the smaller force acting on the partial area of the cross section. This fixes the sense of the shear stresses in the longitudinal cuts. However, numerically equal shear stresses act on the mutually perpendicular planes of an infinitesimal element, and the shear stresses on such planes either meet or part with their directional arrowheads at a corner. Hence, the sense of the shear stresses in the plane of the section also becomes known.

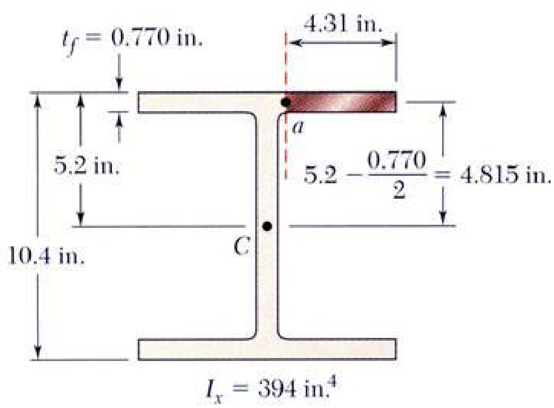
The magnitude of the shear stresses varies for the different vertical cuts. For example, if cut c–c in Fig. 10-19(a) is at the edge of the beam, the *hatched* area of the beam's cross section is zero. However, if the thickness of the flange is constant, and cut c–c is made progressively closer to the web, this area increases from zero at a linear rate. Moreover, as $\bar{y}$ remains constant for any such area, Q also increases linearly from zero toward the web. Therefore, since V and I are constant at any section through the beam, shear flow $q_c = VQ/I$ follows the same variation. If the thickness of the flange remains the same, the shear stress $\tau_c = VQ/It$ varies similarly. The same variation of q_c and τ_c applies on both sides of the axis of symmetry of the cross section. However, as may be seen from Fig. 10-19(b), these quantities in the plane of the cross section act in *opposite* directions on the two sides. The variation of these shear stresses or shear flows is represented in Fig. 10-19(c), where, for simplicity, it is assumed that the web has zero thickness.

$$F_1 = \left(\frac{\tau_{c-\max}}{2}\right)\left(\frac{bt}{2}\right) \quad \text{or} \quad F_1 = \left(\frac{q_{c-\max}}{2}\right)\left(\frac{b}{2}\right) \qquad (10\text{-}11)$$

If an I beam transmits a vertical shear, these horizontal forces act in the upper and lower flanges. However, because of the symmetry of the cross section, these equal forces occur in pairs and oppose each other and cause no apparent external effect.

The shear forces that act at a section of an I beam are shown in Fig. 10-19(d), and, for equilibrium, the applied vertical forces must act through the centroid of the cross-sectional area to be coincident with V. If the forces are so applied, no torsion of the member will occur. This is true for all sections having cross-sectional areas with an axis of symmetry. To avoid torsion of such members, the applied forces must act in the plane of symmetry of the cross section and the axis of the beam.

Sample Problem 6.3



Knowing that the vertical shear is 50 kips in a W10x68 rolled-steel beam, determine the horizontal shearing stress in the top flange at the point a.

SOLUTION:

- For the shaded area,

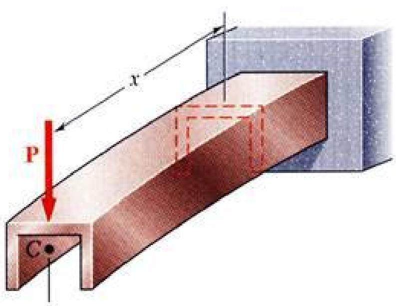
$$Q = (4.31\,\text{in})(0.770\,\text{in})(4.815\,\text{in})$$
$$= 15.98\,\text{in}^3$$

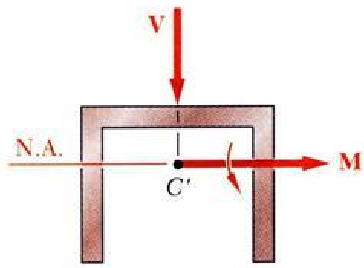
- The shear stress at a,

$$\tau = \frac{VQ}{It} = \frac{(50\,\text{kips})(15.98\,\text{in}^3)}{(394\,\text{in}^4)(0.770\,\text{in})}$$

$$\boxed{\tau = 2.63\,\text{ksi}}$$

Unsymmetric Loading of Thin-Walled Members

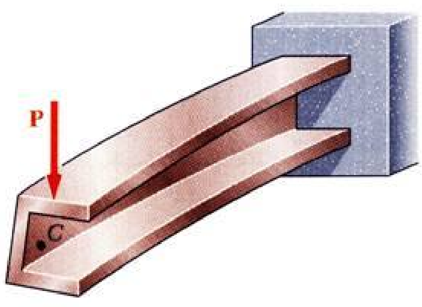


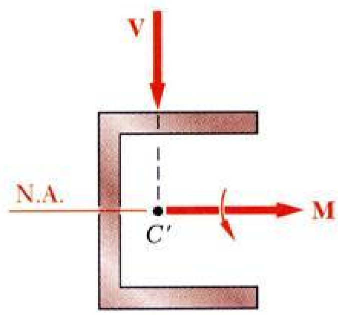


$(V = P, M = Px)$

- Beam loaded in a vertical plane of symmetry deforms in the symmetry plane without twisting.

$$\sigma_x = -\frac{My}{I} \qquad \tau_{ave} = \frac{VQ}{It}$$



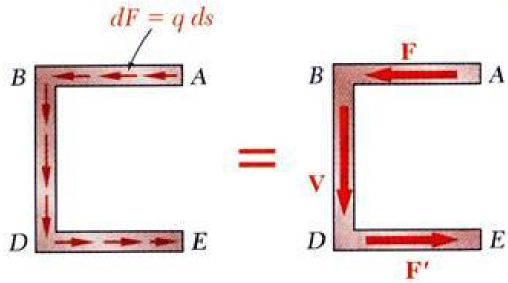


$(V = P, M = Px)$

- Beam without a vertical plane of symmetry bends and twists under loading.

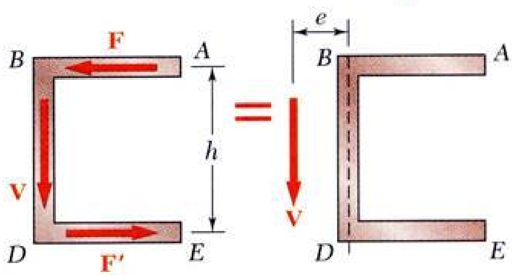
$$\sigma_x = -\frac{My}{I} \qquad \tau_{ave} \neq \frac{VQ}{It}$$

Unsymmetric Loading of Thin-Walled Members



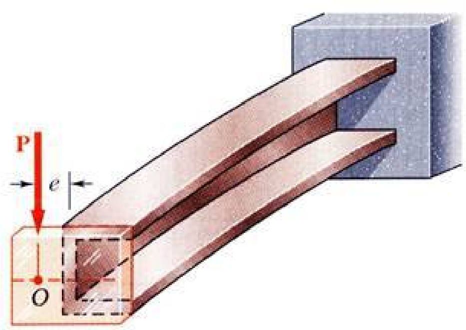
- If the shear load is applied such that the beam does not twist, then the shear stress distribution satisfies

$$\tau_{ave} = \frac{VQ}{It} \quad V = \int_B^D q\,ds \quad F = \int_A^B q\,ds = -\int_D^E q\,ds = -F'$$



- F and F' indicate a couple Fh and the need for the application of a torque as well as the shear load.

$$Fh = Ve$$



- When the force P is applied at a distance e to the left of the web centerline, the member bends in a vertical plane without twisting.

10-8. Shear Center

Consider a beam having the cross section of a channel; see Fig. 10-20(a). (i) The walls of this channel are assumed to be sufficiently thin that the computations may be based on centerline dimensions. (ii) Bending of this channel takes place around the horizontal axis, and although this cross section does not have a vertical axis of symmetry, it will be assumed that the bending stresses are given by the usual flexure formula. (iii) Assuming further that this channel resists a vertical shear, the bending moments will vary from one section through the beam to another.

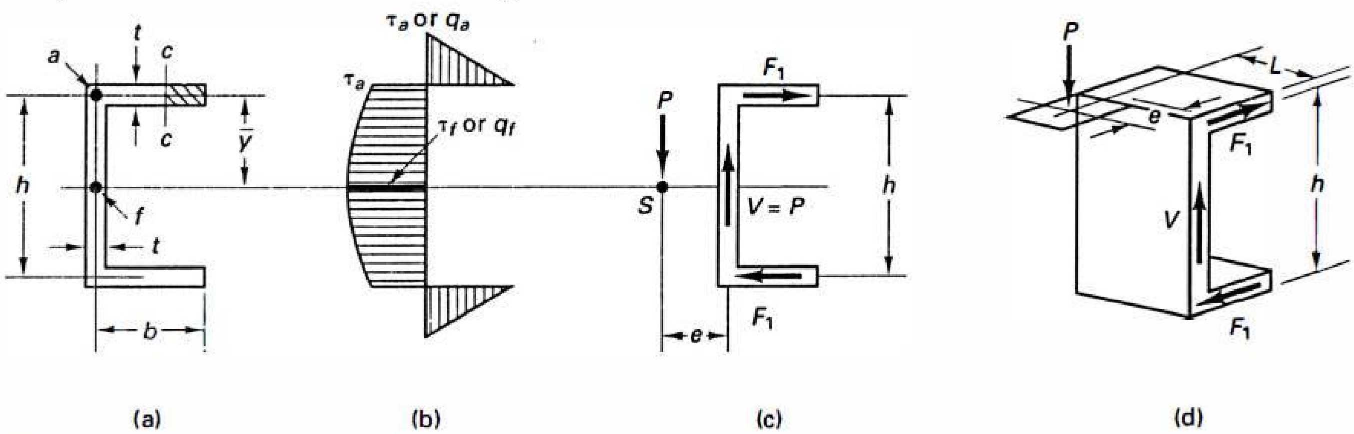


Fig. 10-20 Deriving location of shear center for a channel.

By taking an arbitrary vertical cut such as c--c in Fig. 10-20(a), q and, may be found in the usual manner. Along the horizontal legs of the channel, these quantities vary linearly from the free edge. just as they do for one side of the flange in an I beam. The variation of q and ,. is parabolic along the web. The variation of these quantities is shown in Fig. 10-20(b), where they are plotted along the centerline of the channel's section.

The *average* shear stress $\tau_a/2$ multiplied by the areas of the flange gives a force $F_1 = (\tau_a/2)bt$. and the sum of the vertical shear stresses over the area of the web is the shear $V = \int_{-h/2}^{+h/2} \tau t \, dy$.[12] These shear forces acting in the plane of the cross section are shown in Fig. 10-20(c) and indicate that a force *V and a couple* F_1h are developed at the section through the channel. Physically, there is a tendency for the channel to twist around some longitudinal axis. To prevent twisting and thus maintain the applicability of the initially assumed bending-stress distribution, the externally applied forces must be applied in such a manner as to *balance the internal couple* F_1h. For example, consider the segment of a cantilever beam of negligible weight, shown in Fig. 10-20(d), to which a vertical force P is applied parallel to the web at a distance e from the web's *centerline.* To maintain this applied force in equilibrium, an *equal and opposite* shear force V must be developed in the web. Likewise, to cause *no twisting of the channel,* couple Pe must *equal* couple F_1h. At the same section through the channel, bending moment PL is resisted by the *usual* flexural stresses (these are not shown in the figure).

An expression for distance e, locating the plane in which force P must be applied so as to cause *no twist* in the channel, may now be obtained. Thus, remembering that $F_1h = Pe$ and $P = V$,

$$e = \frac{F_1h}{P} = \frac{(1/2)\tau_a bth}{P} = \frac{bth}{2P}\frac{VQ}{It} = \frac{bth}{2P}\frac{Vbt(h/2)}{It} = \frac{b^2h^2t}{4I} \qquad (10\text{-}12)$$

Note that distance e is independent of the magnitude of applied force P, as well as of its location along the beam. Distance e is a property of a section and is measured outward from the *center* of the web to the applied force.

A similar investigation may be made to locate the plane in which the horizontal forces must be applied so as to cause no twist in the channel. However, for the channel considered, by virtue of symmetry, it may be seen that this plane coincides with the neutral plane of the former case. The intersection of these two mutually perpendicular planes with the plane of the cross section locates a point that is called the *shear center*.[13] The shear center is designated by the letter S in Fig. 10-20(c). The shear center for any cross section lies on a longitudinal line parallel to the axis of the beam. *Any transverse force applied through the shear center causes no torsion of the beam.* A detailed investigation of this problem shows that when a member of any cross-sectional area *is* twisted, the twist takes place around the shear center, which remains fixed. For this reason, the shear center is sometimes called the *center of twist*.

For cross-sectional areas having one axis of symmetry, the shear center is always located on the axis of symmetry. For those that have two axes of symmetry, the shear center coincides with the centroid of the cross-sectional area. This is the case for the I beam that was considered in the previous section.

The exact location of the shear center for unsymmetrical cross sections of thick materials is difficult to obtain and is known only in a few cases. If the material is *thin*, as has been assumed in the preceding discussion, relatively simple procedures may always be devised to locate the shear center of the cross section. The usual procedure consists of determining the shear forces, as F_1 and V before, at a section, and then finding the location of the external force necessary to keep these forces in equilibrium.

Example 10-6

Find the approximate location of the shear center for a beam with the cross section of the channel shown in Fig. 10-21.

SOLUTION

Instead of using Eq. 10-12 directly, we may make some further simplifications. The moment of inertia of a thin-walled channel around its neutral axis may be found with sufficient accuracy by neglecting the moment of inertia of the flanges *around their own axes* (only!). This expression for I may then be substituted into Eq. 10-12, and, after simplifications, a formula for e of channels is obtained:

$$I \approx I_{web} + (Ad^2)_{flanges} = th^3/12 + 2bt(h/2)^2 = th^3/12 + bth^2/2$$

$$e = \frac{b^2h^2t}{4I} = \frac{b^2h^2t}{4(bth^2/2 + th^3/12)} = \frac{b}{2 + h/3b} \tag{10-13}$$

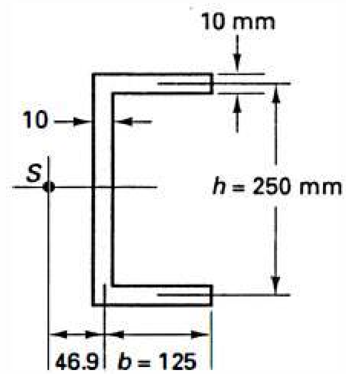


Fig. 10-21

Equation 10-13 shows that when the width of flanges b is very large, e approaches its maximum value of $b/2$. When h is very large, e approaches its minimum value of zero. Otherwise, e assumes an intermediate value between these two limits. For the numerical data given in Fig. 10-21,

$$e = \frac{125}{2 + 250/(3 \times 125)} = 46.9 \text{ mm}$$

Hence, the shear center S is $46.9 - 5.0 = 41.9$ mm from the outside vertical face of the channel.

Example 10-7

Find the approximate location of the shear center for the cross section of the I beam shown in Fig. 10-22(a). Note that the flanges are unequal.

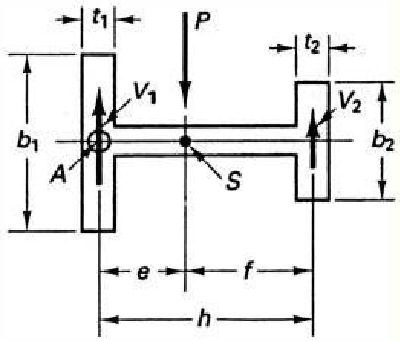
SOLUTION

This cross section has a horizontal axis of symmetry, and the shear center is located on it; where it is located remains to be answered. Applied force P causes significant bending and shear stresses *only in the flanges*, and the contribution of the web to the resistance of applied force P is negligible.

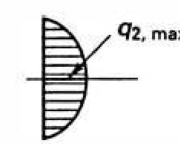
Let the shear force resisted by the left flange of the beam be V_1, and by the right flange V_2. For equilibrium, $V_1 + V_2 = P$. Likewise, to have no twist of the section, from $\sum M_A = 0$, $Pe = V_2 h$ (or $Pf = V_1 h$). Thus, only V_2 remains to be determined to solve the problem. This may be done by noting that the right flange is actually an ordinary rectangular beam. The shear stress (or shear flow) in such a beam is distributed parabolically, as seen in Fig. 10-22(b), and since the area of a parabola is two-thirds of the base times the maximum altitude, $V_2 = \frac{2}{3}b_2(q_2)_{\max}$. However, since the total shear $V = P$, by Eq. 10-5, $(q_2)_{\max} = VQ/I = PQ/I$, where Q is the statical moment of the *upper half of the right-hand flange* and I is the moment of inertia of the *whole* section. Hence,

$$Pe = V_2 h = \frac{2}{3}b_2(q_2)_{\max} h = \frac{\frac{2}{3}hb_2 PQ}{I} \tag{10-14}$$

$$e = \frac{2hb_2}{3I}Q = \frac{2hb_2}{3I}\frac{b_2 t_2}{2}\frac{b_2}{4} = \frac{h}{I}\frac{t_2 b_2^3}{12} = \frac{hI_2}{I}.$$



(a)



Shear flow in right flange

(b)

Fig. 10-22

Alternate Solution for Rectangular Beam

The distribution of shearing stresses in a rectangular section can be obtained by applying Eq. (5-4) to Fig. 5-25. For a layer at a distance y from the neutral axis, we have

$$\tau = \frac{V}{Ib} A'\bar{y} = \frac{V}{Ib}\left[b\left(\frac{h}{2} - y\right)\right]\left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right]$$

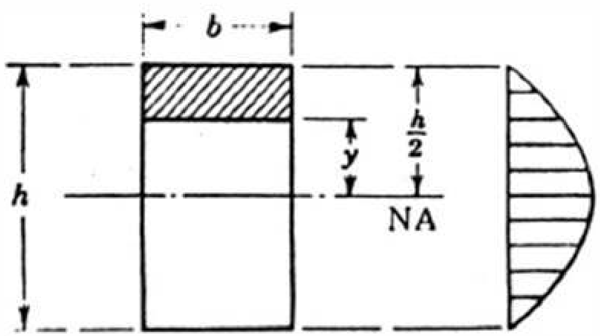
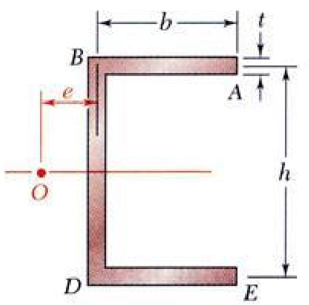


Figure 5-25 Shearing stress is distributed parabolically across a rectangular section.

which reduces to

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)$$

Example 6.05



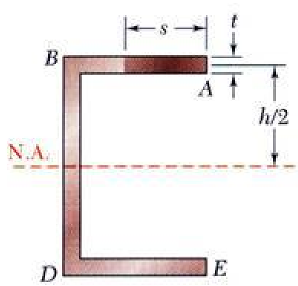
- Determine the location for the shear center of the channel section with $b = 4$ in., $h = 6$ in., and $t = 0.15$ in.

$$e = \frac{F\,h}{I}$$

- where

$$F = \int_0^b q\;ds = \int_0^b \frac{VQ}{I}\,ds = \frac{V}{I}\int_0^b st\,\frac{h}{2}\,ds$$

$$= \frac{Vthb^2}{4I}$$



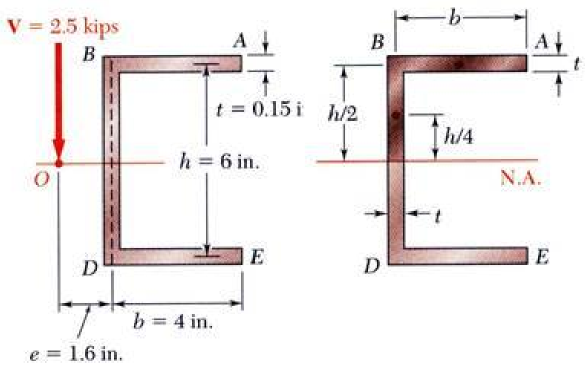
$$I = I_{web} + 2I_{flange} = \frac{1}{12}th^3 + 2\left[\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right]$$

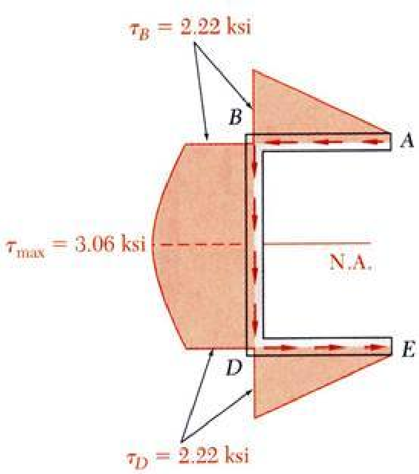
$$\cong \tfrac{1}{12}th^2(6b + h)$$

- Combining,

$$e = \frac{b}{2 + \dfrac{h}{3b}} = \frac{4\,\text{in.}}{2 + \dfrac{6\,\text{in.}}{3(4\,\text{in.})}} \qquad \boxed{e = 1.6\,\text{in.}}$$

Example 6.06





- Determine the shear stress distribution for $V = 2.5$ kips.

$$\tau = \frac{q}{t} = \frac{VQ}{It}$$

- Shearing stresses in the flanges,

$$\tau = \frac{VQ}{It} = \frac{V}{It}(st)\frac{h}{2} = \frac{Vh}{2I}s$$

$$\tau_B = \frac{Vhb}{2\left(\frac{1}{12}th^2\right)(6b+h)} = \frac{6Vb}{th(6b+h)}$$

$$= \frac{6(2.5\,\text{kips})(4\,\text{in})}{(0.15\,\text{in})(6\,\text{in})(6\times 4\,\text{in} + 6\,\text{in})} = 2.22\,\text{ksi}$$

- Shearing stress in the web,

$$\tau_{\max} = \frac{VQ}{It} = \frac{V\left(\frac{1}{8}ht\right)(4b+h)}{\frac{1}{12}th^2(6b+h)t} = \frac{3V(4b+h)}{2th(6b+h)}$$

$$= \frac{3(2.5\,\text{kips})(4\times 4\,\text{in} + 6\,\text{in})}{2(0.15\,\text{in})(6\,\text{in})(6\times 6\,\text{in} + 6\,\text{in})} = 3.06\,\text{ksi}$$

In addition to other books mentioned, solve following problems from Popov:
2-4, 6, 8-10, 13, 15, 18, 20-23, 28, 29, 38-44

CHAPTER

ENGINEERING MECHANICS OF SOLIDS

5

Thin Walled Pressure Vessels

ENGINEERING MECHANICS OF SOLIDS

Pressure Vessels

Pressure vessels are made in different shapes and sizes and are used in diverse applications. The applications range from air receivers in gasoline stations to nuclear reactors in submarines to heat exchangers in refineries.

ENGINEERING MECHANICS OF SOLIDS

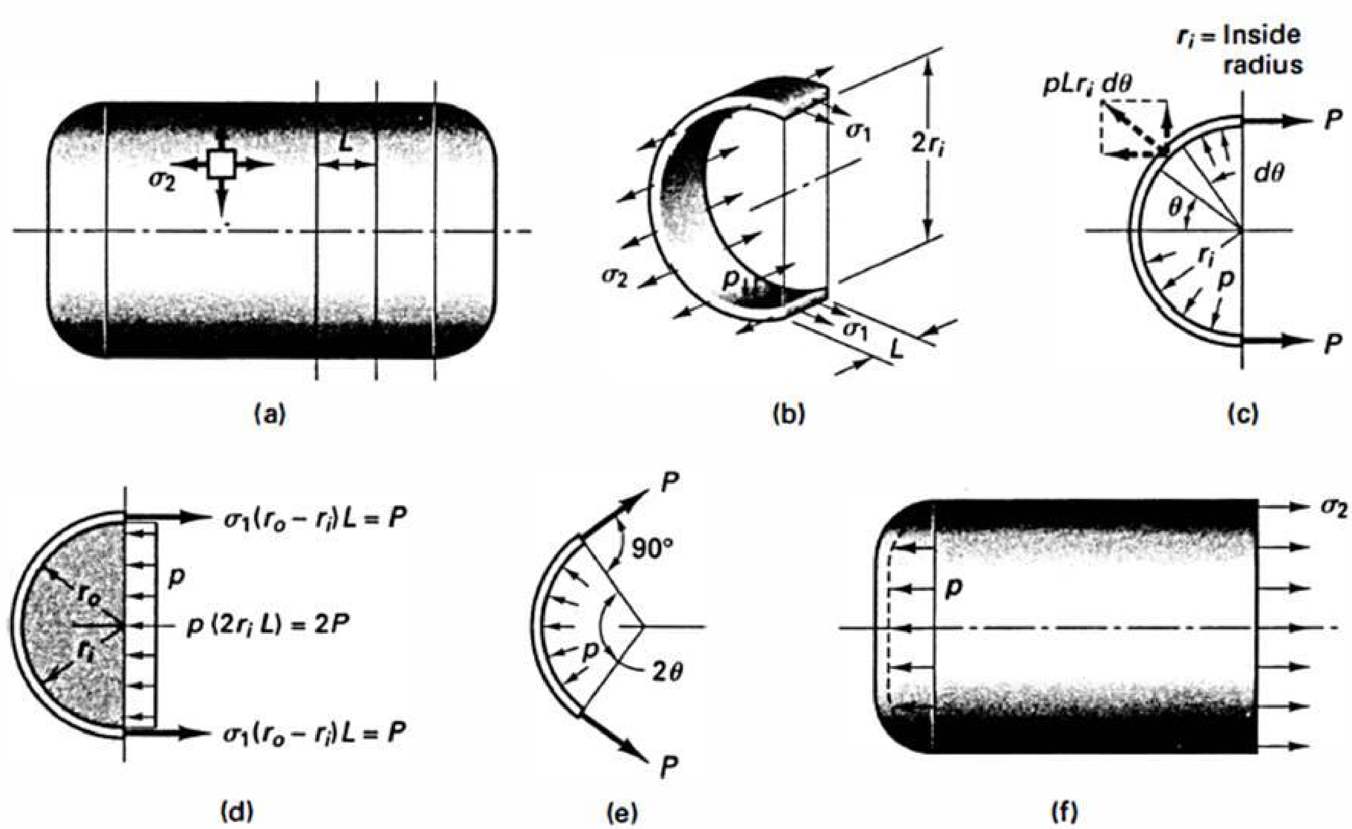


Fig. 5-13 Diagrams for analysis of thin-walled cylindrical pressure vessels.

The analysis of pressure vessels will begin by considering a cylindrical pressure vessel such as a boiler, as shown in Fig. 5-13(a). A segment is isolated from this vessel by passing two planes perpendicular to the axis of the cylinder and one additional longitudinal plane through the same axis, shown in Fig. 5-13(b). The conditions of symmetry exclude the presence of any shear stresses in the planes of the sections, as shear stresses would cause an incompatible distortion of the cylinder. Along the sections of the cylindrical free body there can be only normal stresses. The two that occur are the circumferential or hoop stresses σ_h and the longitudinal stresses σ_1, identified in Fig.5-13(b) $\sigma_1 = \sigma_h$ and $\sigma_2 = \sigma_l$.These stresses multiplied by their respective areas maintain the cylindrical element in equilibrium with the internal pressure. On this basis, by making reference to Fig. 5-13(d), the internal pressure p multiplied by the projected area $2r_iL$, where r_i is the inside radius, generates the force acting on the cylindrical element. This force is balanced by the two forces P developed by the hoop stresses $\sigma_1 = \sigma_h$ multiplied by their respective areas L(r_o - r_i). In this relation r_o is the outside radius of the cylinder. Equating the opposing forces, Fig. 5-13(d), to assure equilibrium of the horizontal forces, one has

$$p(2r_iL) = 2\sigma_l(r_o - r_i)L \tag{5-25}$$

and since $(r_o - r_i) = t$, the thickness of the cylinder, the basic expression for determining, the *circumferential or hoop stress* in a cylinder is

$$\boxed{\sigma_1 = \frac{pr_i}{t}} \tag{5-26}$$

This equation is valid only for thin-walled cylinders, as it gives the *average* stress in the hoop. However, as is shown in Example 5-6, the wall thickness can reach one-tenth of the internal radius and the error in applying Eq. 5-25 will still be small. Since this equation is used primarily for *thin-walled* vessels, where $r_i \approx r_o$, the subscript for the radius is usually omitted.

Equation 5-25 can also be derived by passing two longitudinal sections, as shown in Fig. 5-13(e). Because of the assumed membrane action, the forces P in the hoop must be considered acting tangentially to the cylinder. The horizontal components of the forces P maintain the horizontal component of the internal pressure in a state of static equilibrium.

The other normal stress σ_2 acting in a cylindrical pressure vessel acts *longitudinally,* Fig. 5-13(b), and it is determined by solving a simple axial-force problem. By passing a section through the vessel perpendicular to its axis, a free body as shown in Fig. 5-13(f) is obtained. The force developed by the internal pressure is $p\pi r_i^2$, and the force developed by the longitudinal stress σ_2 in the walls is $\sigma_2(\pi r_o^2 - \pi r_i^2)$. Equating these two forces and solving for σ_2,

$$p\pi r_i^2 = \sigma_2(\pi r_P^2 - \pi r_i^2)$$

$$\sigma_2 = \frac{pr_i^2}{r_o^2 - r_i^2} = \frac{pr_i^2}{(r_o + r_i)(r_o - r_i)}$$

However, as pointed out earlier, $r_o - r_i = t$, the thickness of the cylindrical wall, and since this development is restricted to *thin-walled* vessels, $r_o \approx r_i \approx r$; hence,

$$\boxed{\sigma_2 = \frac{pr}{2t}} \qquad (5\text{-}27)$$

Note that for *thin-walled cylindrical* pressure vessels, $\sigma_2 \approx \sigma_1/2$.

Example 5-3

Consider a closed cylindrical steel pressure vessel, as shown in Fig. 5-13(a). The radius of the cylinder is 1000 mm and its wall thickness is 10 mm. (a) Determine the hoop and the longitudinal stresses in the cylindrical wall caused by an internal pressure of 0.80 MPa. (b) Calculate the change in diameter of the cylinder caused by pressurization. Let $E = 200$ GPa and $\nu = 0.25$. Assume that $r_i \approx r_o \approx r$.

SOLUTION

The stresses follow by direct application of Eqs. 5-26 and 5-27:

$$\sigma_1 = \frac{pr}{t} = \frac{0.8 \times 1}{10 \times 10^{-3}} = 80 \text{ MPa}$$

and

$$\sigma_2 = \frac{pr}{2t} = \frac{0.8 \times 1}{2 \times 10 \times 10^{-3}} = 40 \text{ MPa}$$

The stress perpendicular to the cylinder wall, $\sigma_3 = p = 0.80$ MPa, on the inside decreases to zero on the outside. Being small, it can be neglected. Hence, on setting $\sigma_x = \sigma_1, \sigma_y = \sigma_2$, and $\sigma_z = 0$ in the first expression in Eq. 5-14, one obtains the hoop strain ε_1:

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{80}{200 \times 10^3} - \frac{40}{4 \times 200 \times 10^3}$$

$$= 0.35 \times 10^{-3}\ \text{mm/mm}$$

On pressurizing the cylinder, the radius r increases by an amount Δ. For this condition, the hoop strain ε_1 can be found by calculating the difference in the strained and the unstrained hoop circumferences and dividing this quantity by the initial hoop length. Therefore,

$$\varepsilon_1 = \frac{2\pi(r + \Delta) - 2\pi r}{2\pi r} = \frac{\Delta}{r} \qquad (5\text{-}29)$$

By recasting this expression and substituting the numerical value for ε_1 found earlier,

$$\Delta = \varepsilon_1 r = 0.35 \times 10^{-3} \times 10^3 = 0.35\ \text{mm}$$

Example 5-4

Consider a steel spherical pressure vessel of radius 1000 mm having a wall thickness of 10 mm. (a) Determine the maximum membrane stresses caused by an internal pressure of 0.80 MPa. (b) Calculate the change in diameter in the sphere caused by pressurization. Let $E = 200$ GPa and $\nu = 0.25$. Assume that $r_i \approx r_o \approx r$.

SOLUTION

The maximum membrane normal stresses follow directly from Eq. 5-28.

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{0.80 \times 1}{2 \times 10 \times 10^{-3}} = 40 \text{ MPa}$$

The same procedure as in the previous example can be used for finding the expansion of the sphere due to pressurization. Hence, if Δ is the increase in the radius r due to this cause, $\Delta = \varepsilon_1 r$, where ε_1 is the membrane strain on the great circle. From the first expression in Eq. 5-14, one has

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{40}{200 \times 10^3} - \frac{40}{4 \times 200 \times 10^3}$$

Hence,

$$\Delta = \varepsilon_1 r = 0.15 \times 10^{-3} \times 10^3 = 0.15 \text{ mm}$$

Example 5-5

For an industrial laboratory a pilot unit is to employ a pressure vessel of the dimensions shown in Fig. 5-16. The vessel will operate at an internal pressure of 0.7 MPa. If for this unit 20 bolts are to be used on a 650-mm bolt circle diameter, what is the required bolt diameter at the root of the threads? Set the allowable stress in tension for the bolts at 125 MPa; however, assume that at the root of the bolt threads the stress concentration factor is 2.

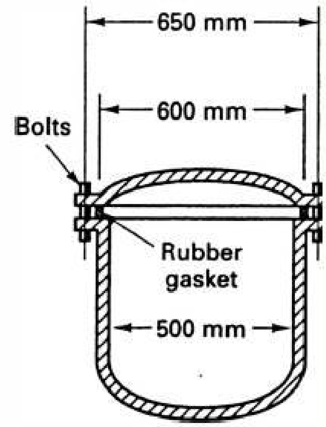


Fig. 5-16

SOLUTION

The vertical force F acting on the cover is caused by the internal pressure p of 0.7 MPa acting on the horizontal projected area within the self-sealing rubber gasket; that is,

$$F = 0.7 \times 10^6 \times \pi(600/2)^2 = 198 \times 10^9 \text{ N}$$

Assuming that this force is equally distributed among the 20 bolts, the force P per bolt is $198 \times 10^9/20 = 9.90 \times 10^9$ N. Using the given stress-concentration factor $K = 2$ and applying Eq. 3-11, the required bolt area A at the root of the threads

$$A = K\frac{P}{\sigma_{\text{allow}}} = \frac{2 \times 9.90 \times 10^9}{125 \times 10^6} = 158 \text{ mm}^2$$

Hence the required bolt diameter d at the root of the threads $d = 2\sqrt{A/\pi} = 14.2$ mm. Note from Example 4-11 that initial tightening of the bolts results in a relatively small increase in total bolt stress when the vessel is pressurized.

A large pipe, called a penstock in hydraulic work, is 1.5 m in diameter. Here it is composed of wooden staves bound together by steel hoops, each 300 mm^2 in cross-sectional area, and is used to conduct water from a reservoir to a powerhouse. If the maximum tensile stress permitted in the hoops is 130 MPa, what is the maximum spacing between hoops under a head of water of 30 m? (The mass density of water is 1000 kg/m^3.)

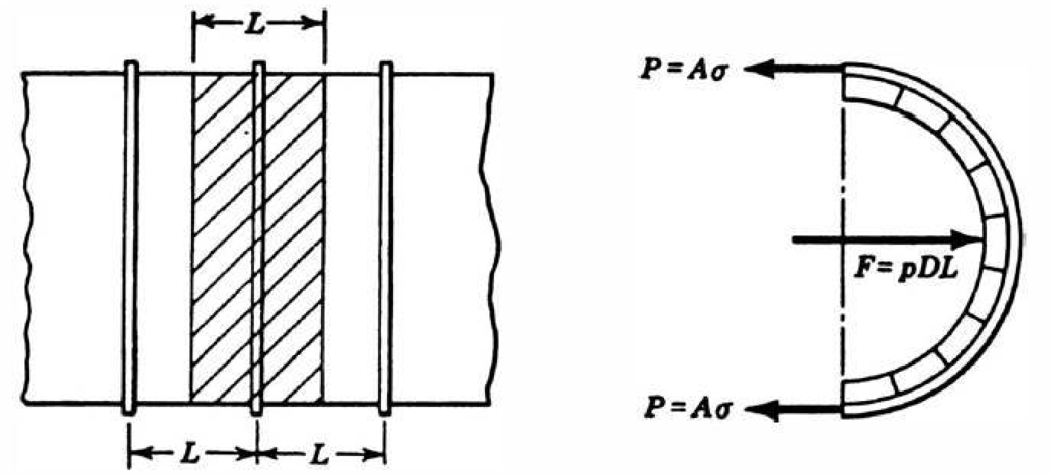


Figure 1-22 Spacing of hoops in a penstock.

Solution: The pressure corresponding to a head of water of 30 m is given by

$$[p = \rho g h] \qquad p = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(30\ \text{m})$$

$$= 294 \times 10^3\ \text{N/m}^2 = 294\ \text{kPa}$$

If the maximum spacing between hoops is denoted by L, then, as shown in Fig. 1-22, each hoop must resist the bursting force on the length L. Since the tensile force in a hoop is given by $P = A\sigma$, we obtain from the free-body diagram

$$[pDL = 2P]$$

$$(294 \times 10^3)(1.5)L = 2(300 \times 10^{-6})(130 \times 10^6)$$

which gives $\qquad L = 0.177\ \text{m} = 177\ \text{mm} \qquad$ *Ans.*

4

Pressure Containers; Riveted and Welded Joints

4-1. Introduction. Two topics are discussed in this chapter. First, a method is demonstrated for finding the forces caused by fluid pressure in a closed container or vessel. After the force is found the unit stress in the wall of the vessel is found by using $S = P/A$.

Second, the riveted, bolted, and welded joints used in making tanks, boilers, pipe, and many types of structures are described, and the stresses in these joints are considered.

The two topics are closely associated because it is necessary to determine the rupturing forces in a boiler, for instance, in order to determine the kind of riveted joints or welded joints needed to join the plates without exceeding allowable stresses.

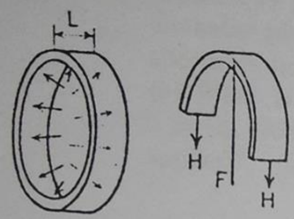
STRESSES CAUSED BY INTERNAL PRESSURE

4-2. Rupturing Forces in Pressure Containers. The pressure of a liquid or of any confined gas acts normally to the surface of the container in which the pressure exists. This normal pressure sets up stresses in the walls of the container and tends to rupture them. The design of such a container, or the investigation of the stresses set up in one by a given unit pressure, includes two distinct steps: first, the determination of the force which tends to rupture the container along the surface or surfaces where rupture is most likely to occur; second, the determination of the stresses which result from the action of this force.

As an example consider the pressure container shown in Fig. 4-1*a*. Suppose that the unit stress on the section AB is wanted. This unit stress may be found by considering the part of the container above the plane AB as a body in equilibrium, shown in Fig. 4-1*b*. The upward force F is the resultant in a direction *perpendicular* to the plane AB of all the force exerted on the interior surface by the fluid pressure. The downward forces shown represent forces due to the tensile stress in the wall of the container and the sum of these downward forces equals SA, where S is the unit tensile stress and A is the area of the cross-section of the wall cut by the plane. Evidently $S = F/A$.

$\frac{t}{r} < \frac{2}{10}$ thin walls

Solution: Imagine two transverse planes a short distance L apart and perpendicular to the axis of the cylinder. Between these two planes is a ring or "hoop" shown in Fig. 4-5*a*. Against the inner surface of this ring are normal forces due to the gas pressure in the cylinder. Next the hoop is cut into two equal parts by a horizontal plane, and the upper half is shown in Fig. 4-5*b*. This is a body in equilibrium with the vertical force F (the resultant of the gas pressure on the inner surface) and two forces H, due to the circumferential stress in the wall of the cylinder. Equilibrium requires that



(*a*) (*b*)

FIG. 4-5

$$2H = F$$

whence

$$2 \times 1 \times L \times S = 20L \times 80$$

from which

$$S = 1{,}600/2 = 800 \text{ psi}$$

Note that the longitudinal unit stress calculated in Example 2 is somewhat less than half the circumferential stress. For cylinders with thinner walls in proportion to the inside diameter, the longitudinal stress is more nearly equal to half the circumferential stress. The statement is often made that in "thin cylinders" the longitudinal stress is one-half the circumferential stress, and this is nearly true. For instance, if the wall thickness is one-fiftieth of the inside diameter, the ratio of longitudinal stress to circumferential stress is 0.49.

A formula for longitudinal stress in a thin-walled cylinder will now be derived. Let Fig. 4-4*c* be the end of a thin-walled cylinder with inside diameter of D in. and wall thickness of t in. The area of the metal cut by the cross-section equals the mean circumference times the thickness, or $\pi (D + t) t$; but, if t is small in comparison to D, a close approximation will be πDt. Equating the force F due to the internal pressure of R psi with the unit stress in the shell times the cut area,

$$(\pi/4)D^2R = \pi DtS$$

whence

$$S = RD/4t$$

Note that this formula also gives a value for the stress in a thin spherical shell with the same degree of approximation.

A formula for the "hoop tension" or circumferential stress in a thin-walled cylinder is derived as follows. Let Fig. 4-5*a* represent a ring cut from a thin-walled tank with internal fluid pressure of R psi. Let L be 1 in. Now consider half this ring, as shown in Fig. 4-5*b*, as a body in equilibrium. If t is the thickness of the shell, the force H equals $S \times 1 \times t = St$. The force $F = RD$. Equating the upward and downward forces, $2St = RD$, from which

$$S = RD/2t$$

Note that this is exactly twice the value of the longitudinal stress as given by the approximate formula derived just above.

Since the plane dividing the ring into two half rings (Fig. 4-5*b*) may cut the ring at any two opposite points, it follows that the total tension is the same at all cross-sections of the ring. If for any reason the area of the ring is not the same for all cross-sections, the maximum unit stress will occur at the cross-sections where the area is least.

Tanks and pipes are sometimes made of wooden staves held together by hoops. The construction is somewhat like that of a wooden barrel. The total stress on one hoop may be found much as the force H is found. The maximum unit stress is then determined by dividing the total stress by the minimum cross-section of the hoop. The distance L between the two transverse planes should be taken equal to the distance between the hoops. This same method applies to other types of fastenings which occur at intervals along a pipe or tank.

PROBLEMS

4-1. A fire extinguisher has a copper tank holding 2.5 gal. The inside diameter is 7 in., and the thickness of the shell of the tank is 0.12 in. The extinguisher was tested with a water pressure of 350 psi. What stress did this pressure cause in the shell of the tank? *Ans.* $S = 10{,}200$ psi.

4-2. A "blind flange" or cover is used to close the end of a 14-in. (outside diameter, see Table VIII) steam line which is subjected to a pressure of 600 psi at a temperature of 750°F. The "American standard" for this service requires that the flange be held on with twenty $1\frac{3}{8}$-in. alloy steel bolts. (*a*) What is the maximum stress in each bolt? (*b*) On this same pipe and flange a hydraulic (non-shock) pressure of 1,100 psi at ordinary temperature is permitted. What is the maximum bolt stress?

4-3. Specifications of the American Water Works Association provide that a 36-in.-diameter Class A (wall thickness 1.15 in.) cast-iron pipe must withstand a hydrostatic pressure of 200 psi. What circumferential unit stress does this pressure cause?

4-4. The inside diameter of a wood-stave pipe is 66 in. (Fig. 4-6). Hoops are steel rods 1 in. in diameter spaced 6 in. center to center. The ends of each rod are threaded so that the hoop can be tightened by turning a nut. What is the maximum unit stress in the hoop when the water pressure in the pipe is 50 psi? *Ans.* $S = 17{,}900$ psi.

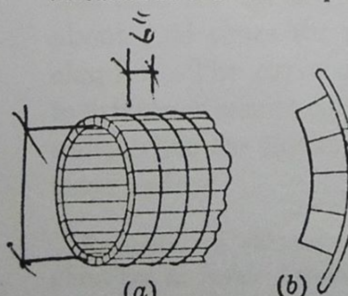


FIG. 4-6

4-5. A rectangular tank is 20 in. square by 40 in. long (interior dimensions). It is cast in two sections, as shown in Fig. 4-7. The two halves are bolted together with ten $1\frac{1}{2}$-in. bolts having an ultimate strength of 65,000 psi. Find the factor of safety for the bolts when the pressure in the tank is 210 psi. Is there any shearing stress on the bolt cross-sections? Slope of joint is 45°.

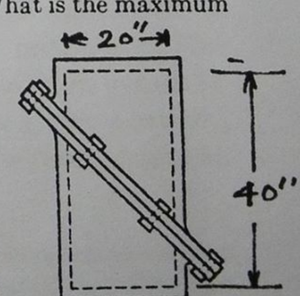


FIG. 4-7

braze= Solder with an alloy of brass a

4-6. An air chamber for a pump is shown in Fig. 4-8. For a pressure of 240 psi calculate the number of $\frac{7}{8}$-in.-diameter bolts required at A and also at B. Stress is not to exceed 6,000 psi. *Ans.* 10 bolts at A.

4-7. A small cylinder subject to high pressure is to be made by brazing a plug p into each end of a steel tube (inside diameter d of 2.00 in.) as shown in Fig. 4-9. Through one of the plugs there is an accurately drilled hole 0.60 in. in diameter. The pressure is produced by filling the tube with a liquid and forcing in the 0.60-in.-diameter plunger with a load F. (*a*) What pressure in pounds per square inch is allowable in the liquid if the allowable shearing stress in the brazing is 1,500 psi? (*b*) What is the load F to produce this pressure? (*c*) The tube is made of metal 0.20 in. thick. What is the maximum unit stress in the tube when this load is applied? (*d*) What is the longitudinal unit stress in the tube?

Ans. (*a*) $R = 3{,}600$ psi.

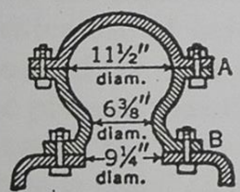


FIG. 4-8

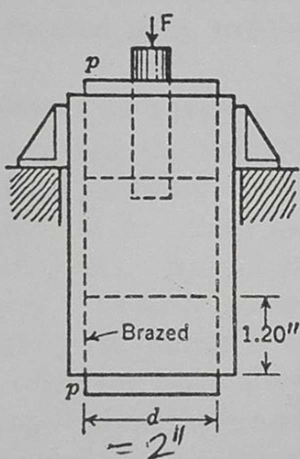


FIG. 4-9

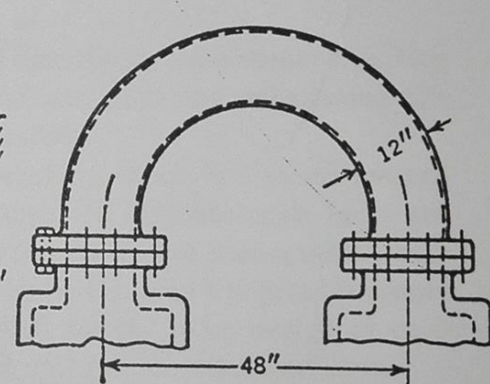


FIG. 4-10

4-8. A bent steam pipe connecting a valve chamber and a cylinder is shown in Fig. 4-10. The pipe is 12 in. in outside diameter and 11 in. in inside diameter. Steam pressure is 450 psi. Determine the total number of bolts required to attach the bent pipe to the two chambers. Bolts are $1\frac{1}{4}$ in. in diameter, and allowable stress in the bolts is 5,000 psi.

4-9. A water tank made of wood staves has an inside diameter of 12 ft and is 18 ft high. Hoops are flat steel bars 2 in. by $\frac{3}{8}$ in., spaced 10 in. center to center. (*a*) What is the unit stress in a hoop 10 in. above the bottom of the tank when the tank is full? (*b*) If instead of fresh water the tank is to hold brine (specific gravity = 1.20), to what should the hoop spacing be reduced for retention of the same factor of safety?

4-10. A welded steel water pipe used as a "siphon" in the Owyhee reclamation project in eastern Oregon has a diameter of 9 ft and is made of $\frac{13}{16}$-in. plate. After fabrication this pipe was tested under a water pressure of 220 psi. What circumferential stress was developed? *Ans.* $S = 14{,}600$ psi.

4-11. The Outardes hydroelectric project in Canada includes what is believed to be the largest wooden-stave pipe so far constructed. (See *Civil Engineering*, December, 1937.) This pipe has an internal diameter of 17 ft 6 in. and operates under a maximum head of 113.0 ft. The staves are held together by 1-in. steel bars, threaded. What maximum tensile unit stress does the water pressure cause in these bars where the head is 100 ft and the hoops are spaced 2.5 in. center to center?