Program Evaluation & Review Technique (PERT)

Time Estimates : Pert planner makes three kinds of time estimates, e.g.

1. The Optimistic time estimate
2. The Pessimistic time estimate
3. The Most Likely time estimate

The Optimistic time estimate : This is the shortest possible time in which an activity can be completed under ideal conditions. This time estimate is denoted by, t_o.

The Pessimistic time estimate : It is the best guess of the maximum time that would be required to complete the activity. It is the worst possible condition when the situation is adverse. This time estimate is denoted by, t_p.

The Most Likely time estimate : The most likely time or most probable time is the time that represents the time the activity would most often require if normal condition prevail. This time estimate is denoted by, t_L.

Expected Time : The average time taken for the completion of an activity or job, is defined as expected time and denoted by t_E.

$$t_E = \frac{t_o + 4t_L + t_p}{6}$$

Standard Deviation : $\sigma = \frac{t_p - t_o}{6}$

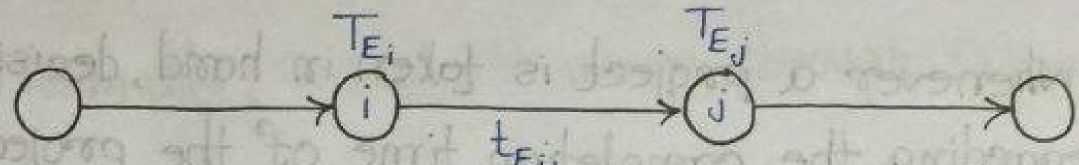
Variance : $\sigma^2 = \left\{\frac{(t_p - t_o)}{6}\right\}^2$

Earliest Expected Time (T_E) :

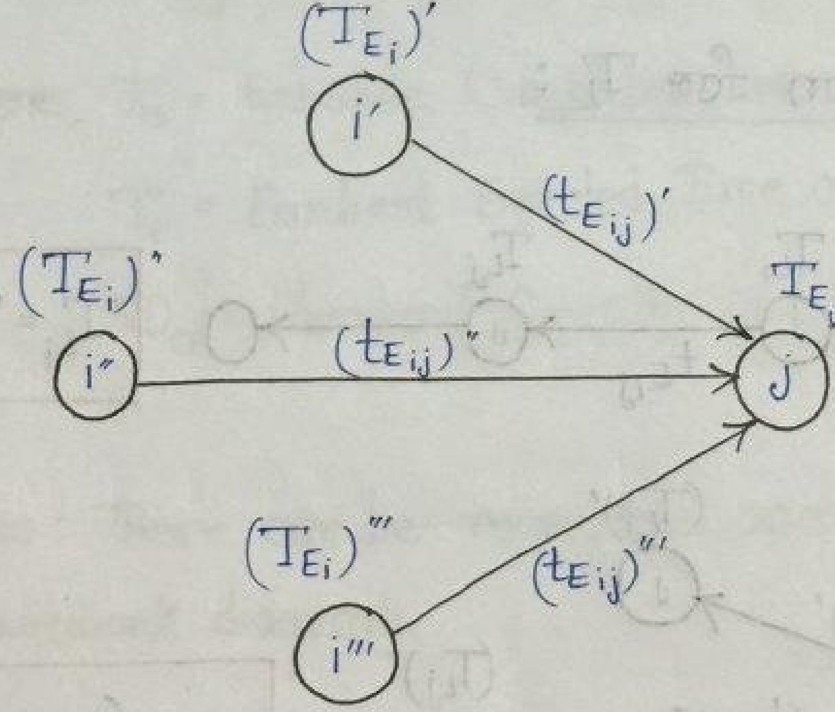
- The earliest expected time is the time when an event can be expected to occur.
- It is represented by T_E and appears above or below the node (event circle) in a network.
- The earliest expected time, T_E is computed by adding the expected times, t_E of all the activities along an activities path leading to that event.
- If more than one activity paths lead to that event,

then the maximum of the sum of t_E's along the various paths will give the earliest expected time.

Formulation for T_E:



$$\boxed{T_{E_j} = T_{E_i} + t_{E_{ij}}}$$

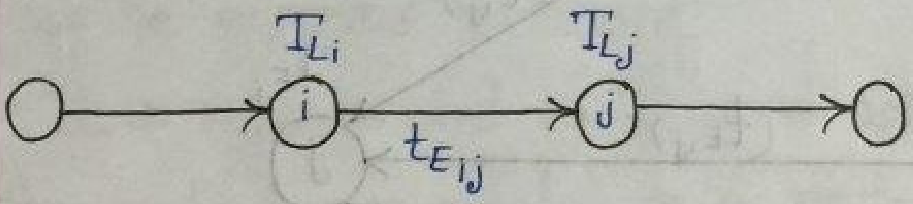


$$\boxed{T_{E_j} = (T_{E_i} + t_{E_{ij}})_{max}}$$

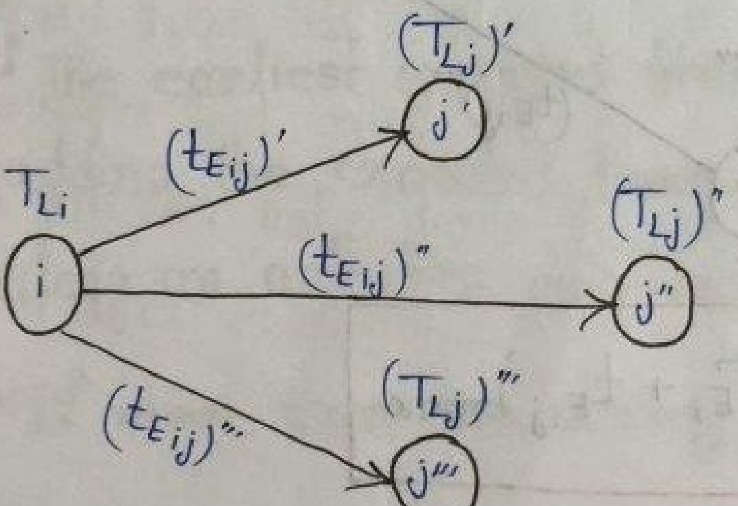
Latest Allowable Occurrence Time (T_L):

- The latest time, by which an event must occur, to keep the project on schedule is called the latest allowable occurrence time. It is denoted by T_L
- Whenever a project is taken in hand, decision is made regarding the completion time of the project and is called Schedule Completion Time and is denoted by T_S.
- Generally, $T_S = T_L$

Formulation for T_L:



$$\boxed{T_{L_i} = T_{L_j} - t_{E_{ij}}}$$



$$\boxed{T_{L_i} = \left(T_{L_j} - t_{E_{ij}}\right)_{min}}$$

Critical Path:

- The path leading to Maximum Expected Time, $T_{E(max)}$ as the summation of expected times of the events included in the path.
- Normally, Critical Path moves through Dummy Activity.

Z value for Standard Normal Distribution:

$$Z = \frac{T_S - T_E}{SD_{CP}}$$

Where, T_S = Schedule Completion time of Last Event

T_E = Earliest Expected Time of Last Event

SD_{CP} = Standard Deviation of Critical Path

Note: There can be more than one Critical Paths for a network diagram.

Extra

Example 1 : For a particular activity of a project, time estimates received from two engineers X and Y are as follows :

Engineer	Optimistic time	Most likely time	Pessimistic time
X	4	6	8
Y	3	5	8

State who is more certain about the time of completion of the job.

Soln: For engineer X, $\sigma_X^2 = \left(\frac{t_{P_X} - t_{O_X}}{6}\right)^2$

$$= \left(\frac{8-4}{6}\right)^2$$

$$= 0.4444$$

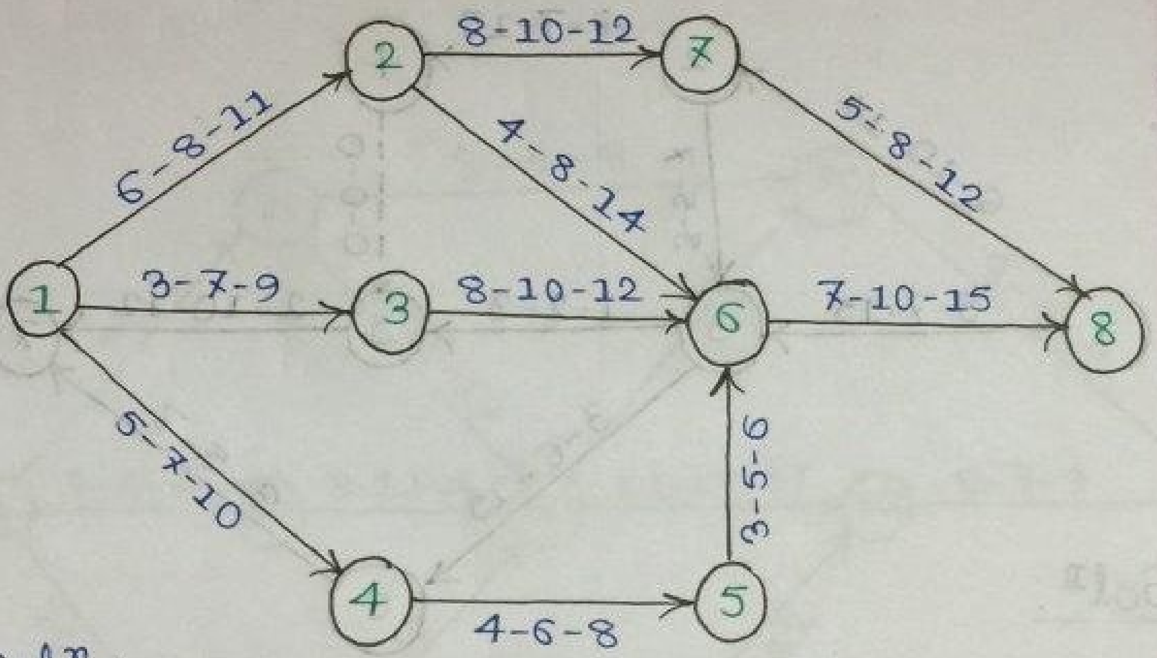
For engineer Y, $\sigma_Y^2 = \left(\frac{t_{P_Y} - t_{O_Y}}{6}\right)^2$

$$= \left(\frac{8-3}{6}\right)^2$$

$$= 0.6944$$

Since, $\sigma_X^2 < \sigma_Y^2$, Engineer X's time estimate has more certainty. (Ans:)

Example 2: The network for a certain project is shown in the following figure. Determine the expected time for each of the path. Which path is critical?

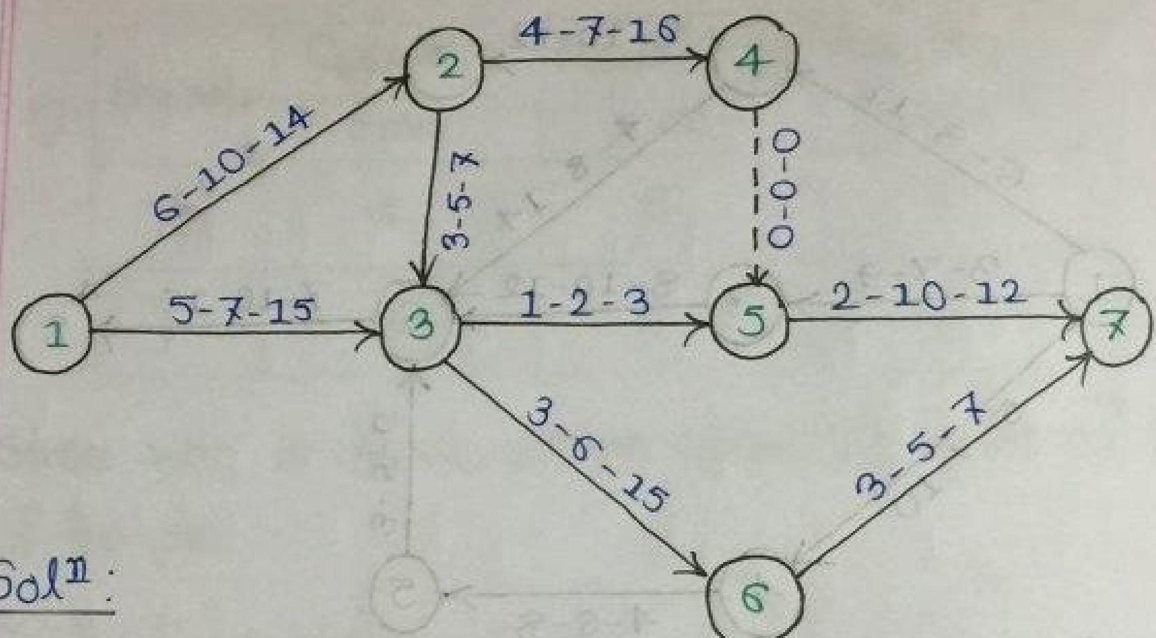


Soln:

Path	Optimistic time Σt_o	Most Likely time Σt_L	Pessimistic time Σt_p	Expected time $t_E = \frac{t_o+4t_L+t_p}{6}$
A(1-2-7-8)	19	26	35	26.3333
B(1-2-6-8)	17	26	40	26.8333
C(1-3-6-8)	18	27	36	27.0
D(1-4-5-6-8)	19	28	39	28.3333

So, Critical Path is 1-4-5-6-8 (Ans:)

Example 3: The network for a certain project shown in the following figure. Determine the expected time for each path. Which path is critical?



Sol^n:

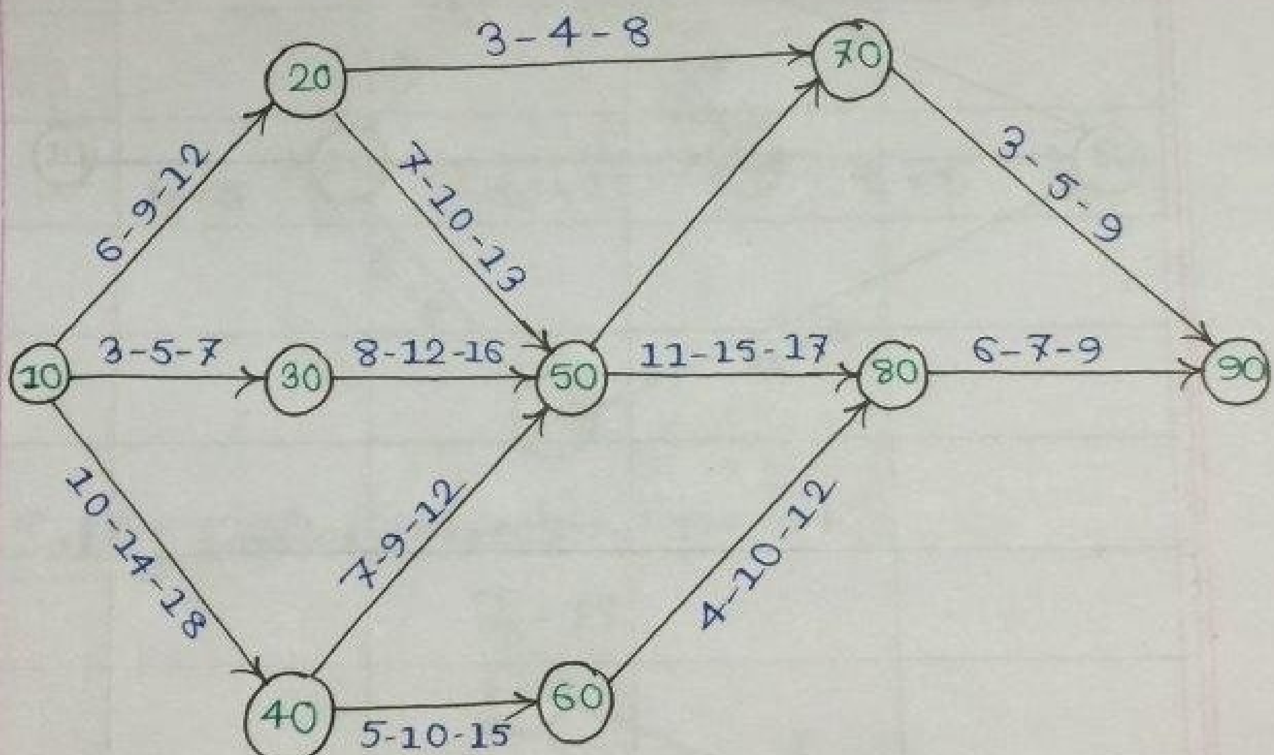
Path	Optimistic time Σt_o	Most Likely time Σt_L	Pessimistic time Σt_p	Expected time $t_E = \frac{t_o + 4t_L + t_p}{6}$
A(1-2-4-5-7)	12	27	42	27
B(1-2-3-5-7)	12	27	36	26
C(1-2-3-6-7)	15	26	43	27
D(1-3-5-7)	8	19	30	19
E(1-3-6-7)	11	18	37	20

So, Critical Paths are both 1-2-4-5-7 & 1-2-3-6-7

(Ans:)

Extra

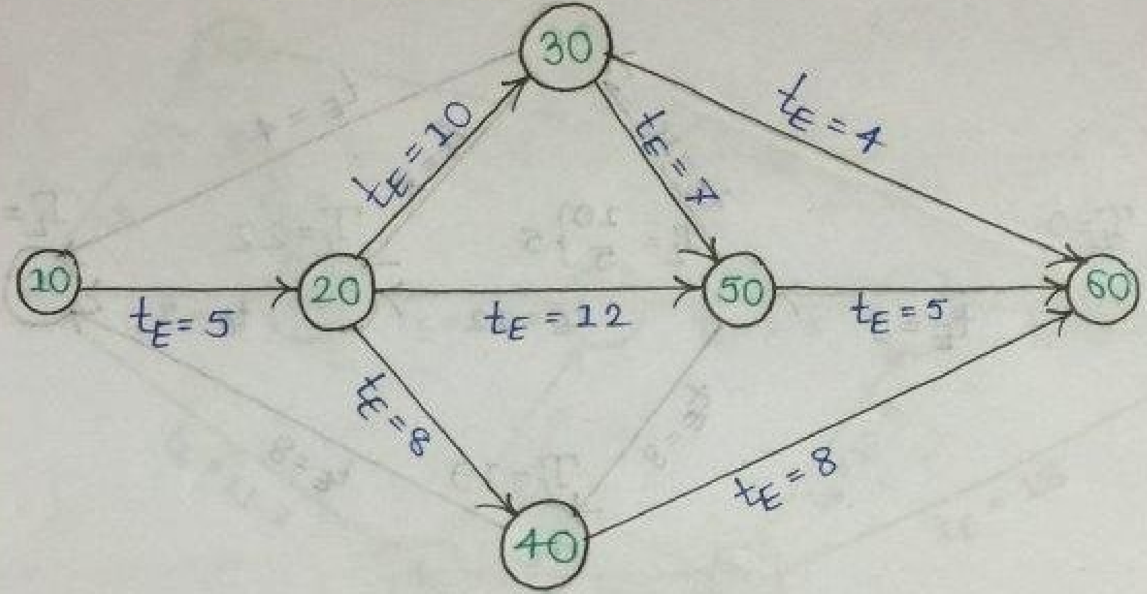
Example 4 : The network for a certain project shown in the following figure. Determine the expected time for each path. Which path is critical ?



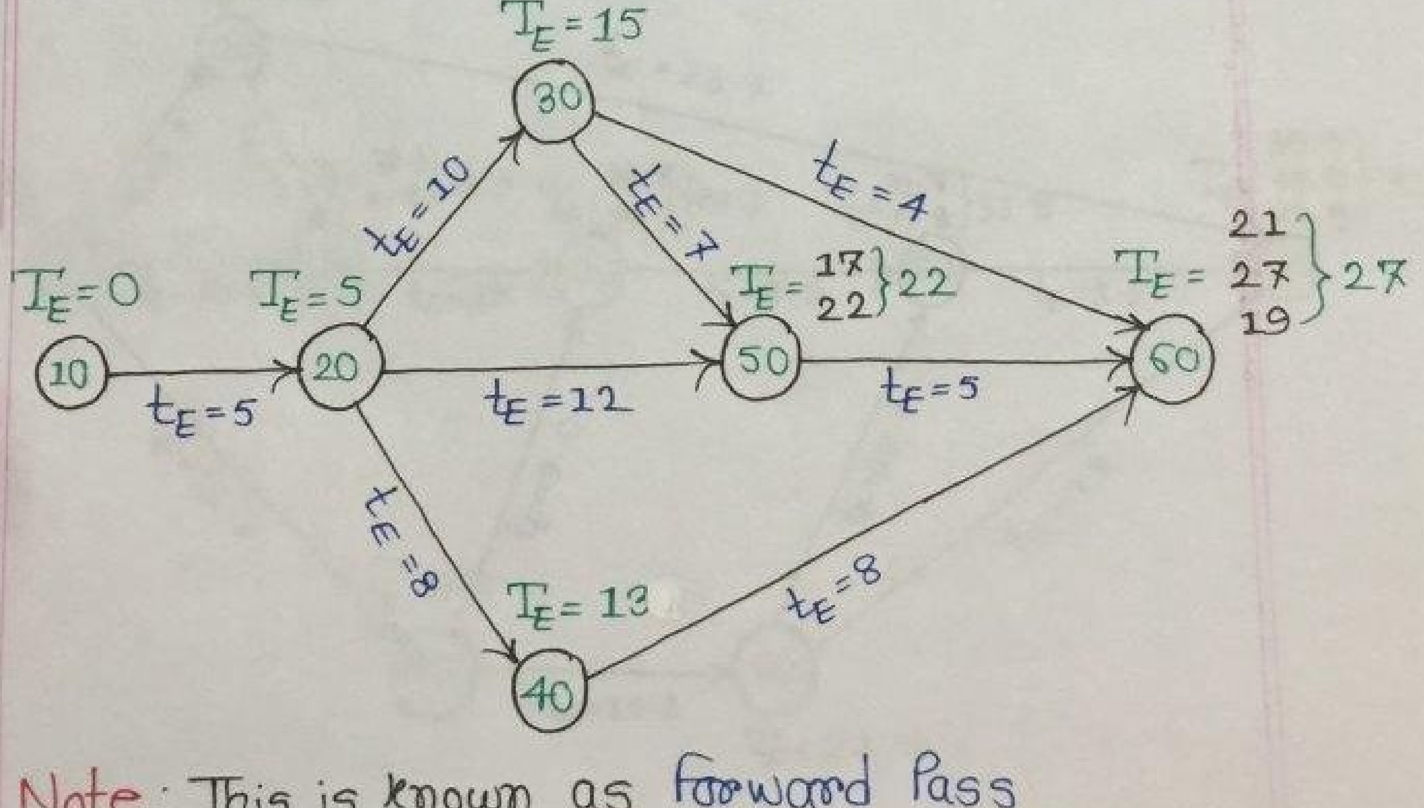
Soln:

Path	Optimistic time	Most Likely time	Pessimistic time	Expected time

Example 5: Compute the earliest expected time and latest allowable occurrence time of each event of the following network. Use $T_S = T_E$ (of last event).

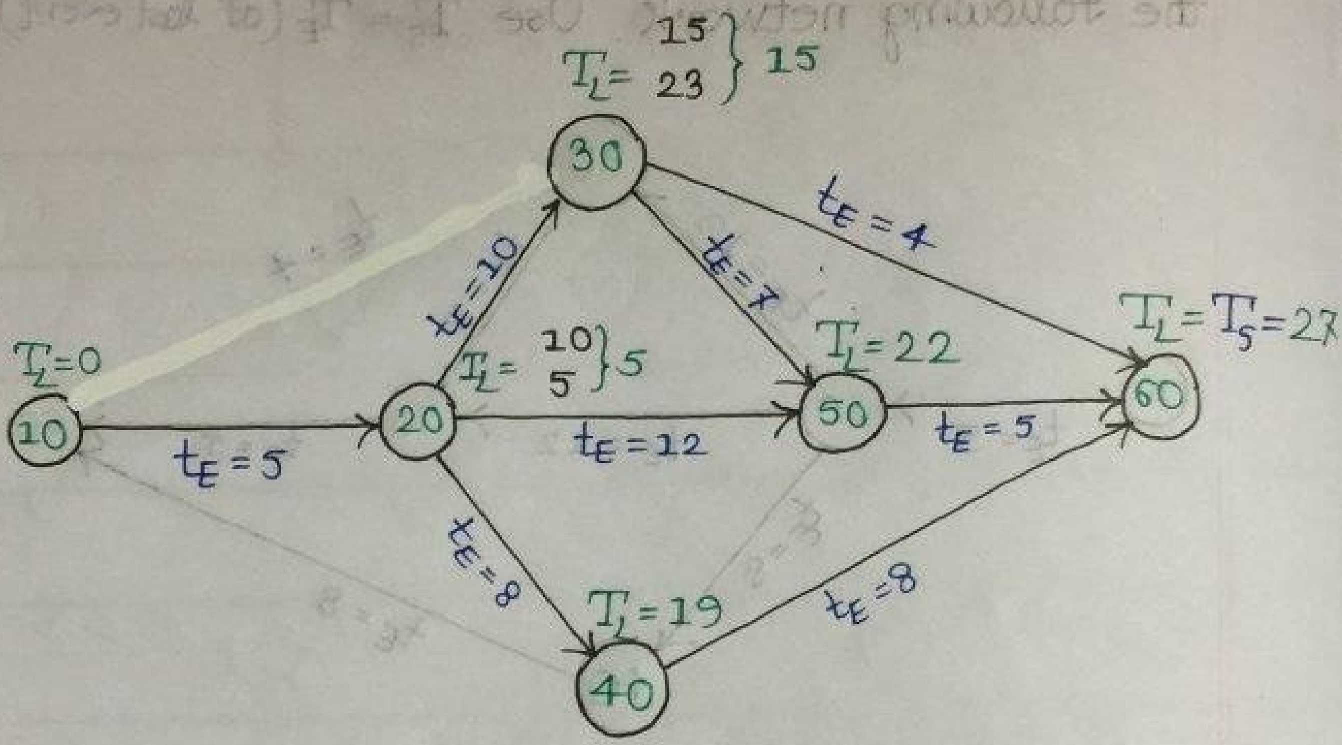


Sol^n: Earliest Expected Time (T_E),



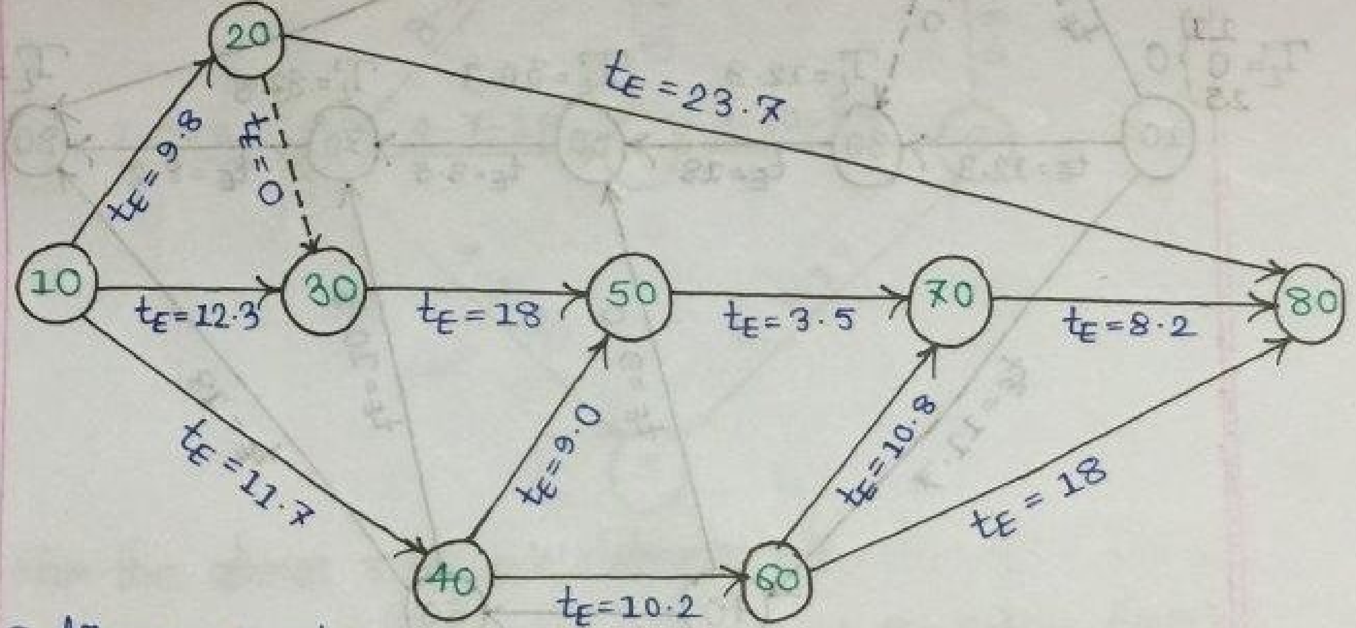
Note: This is known as Forward Pass

Latest Allowable Occurrence Time (T_L),

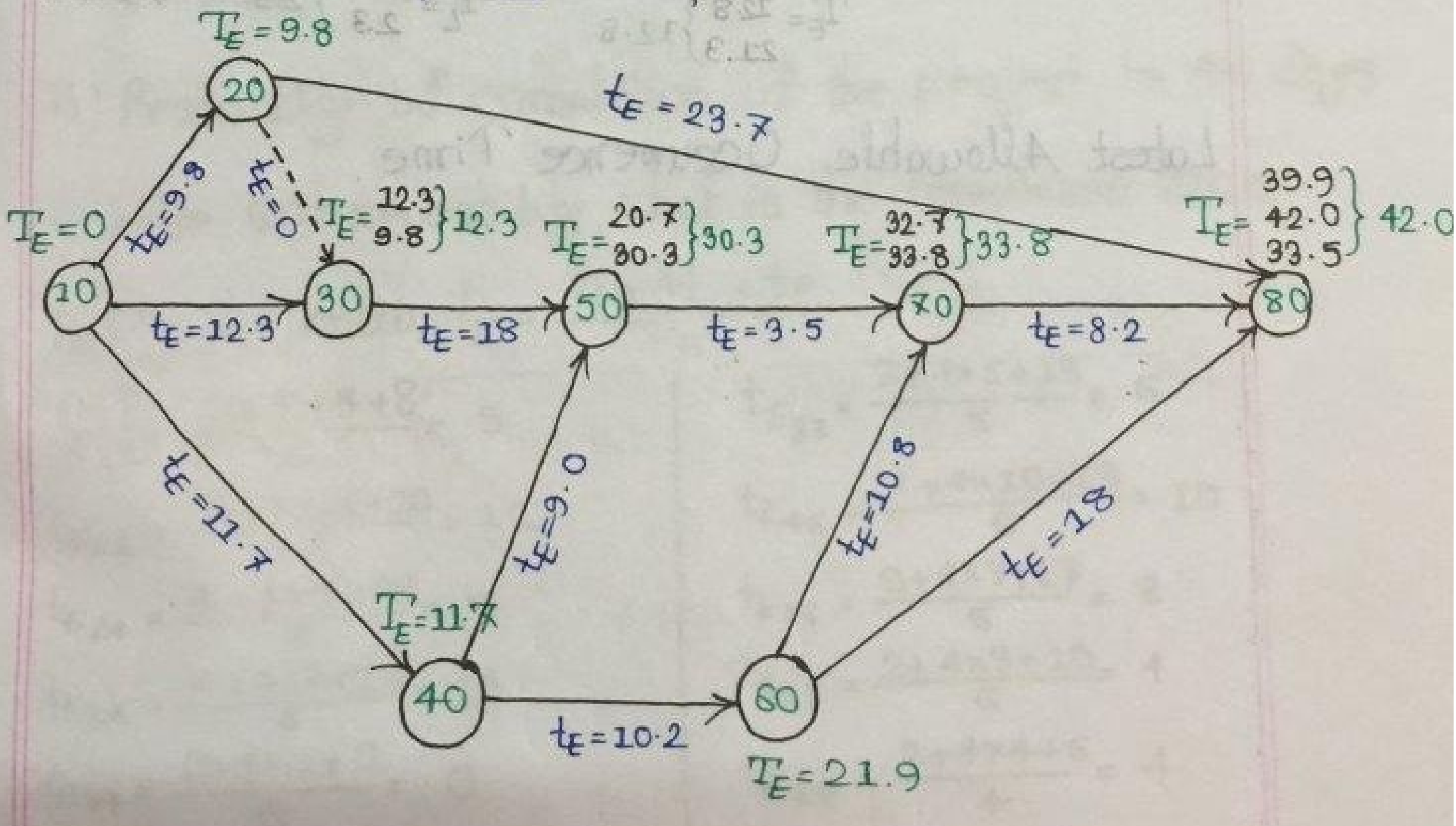


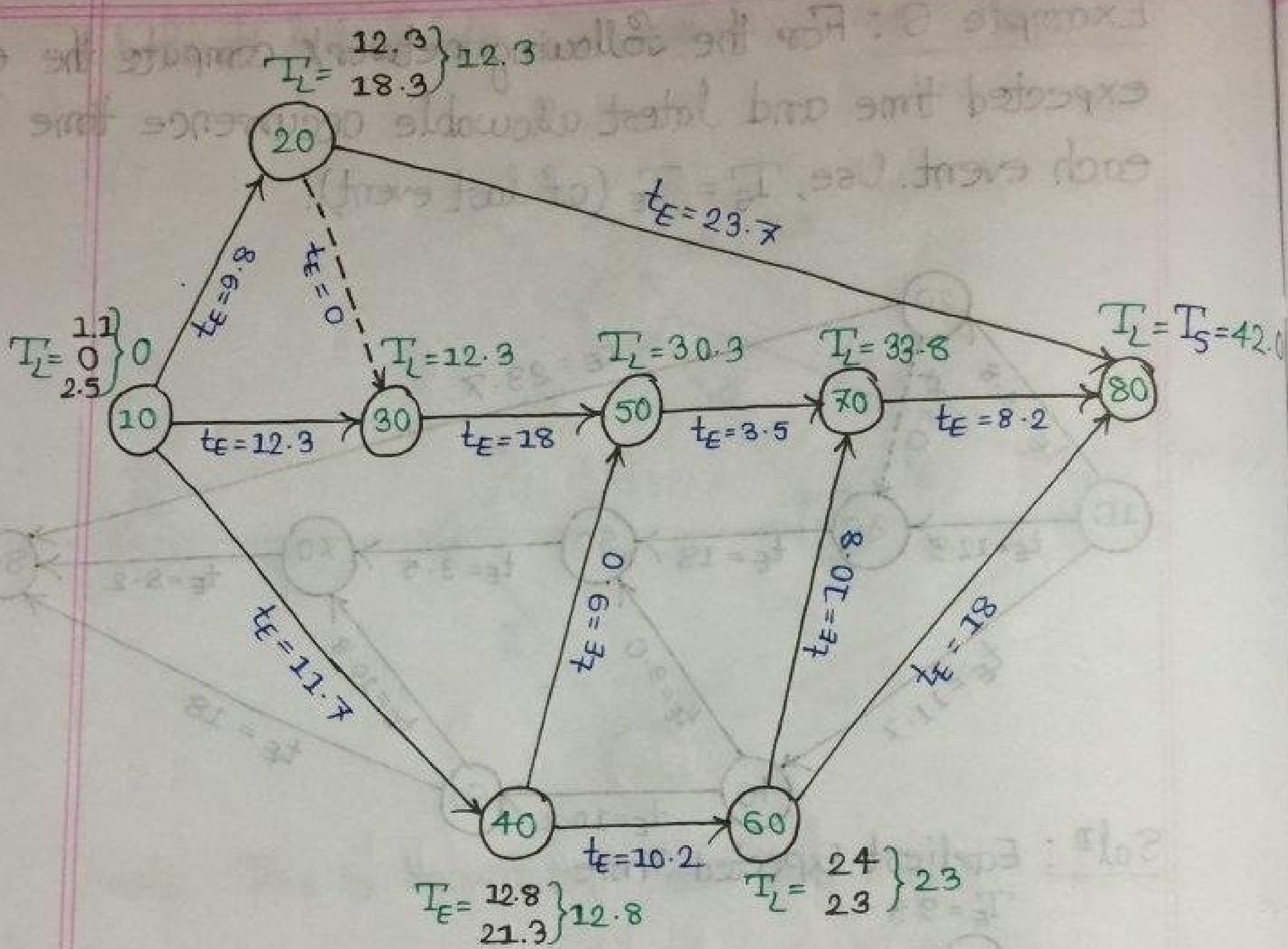
Note : This is known as Backward Pass

Example 6 : For the following network compute the earliest expected time and latest allowable occurrence time of each event. Use, $T_S = T_E$ (of last event).



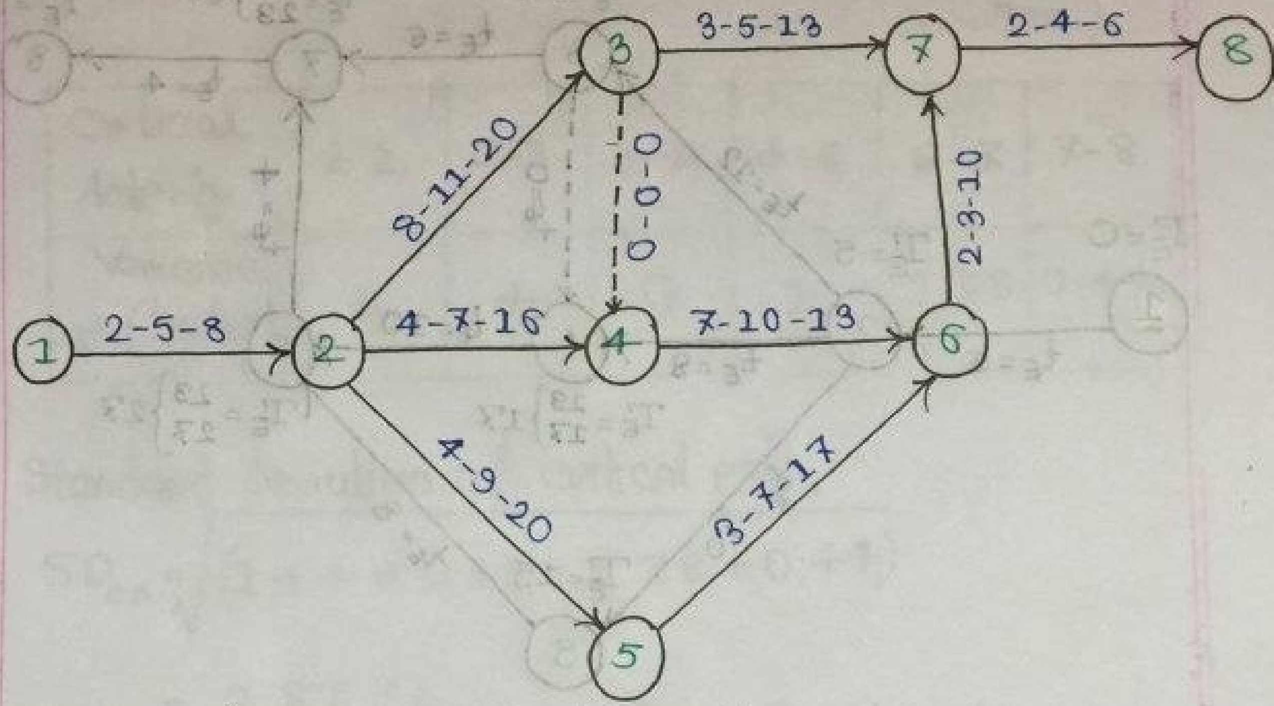
Sol^n : Earliest Expected Time,





Latest Allowable Occurence Time

Example 7:



For the above network, determine,

i) Critical Path and its Standard Deviation (SD)

ii) Probability of completion of the project in 40 days

iii) For 60% probability, what is the completion time

Sol^n: We know, $t_E = \dfrac{t_o + 4t_L + t_P}{6}$

$t_{E_{12}} = \dfrac{2+4\times5+8}{6} = 5$

$t_{E_{23}} = \dfrac{8+4\times11+20}{6} = 12$

$t_{E_{24}} = \dfrac{4+4\times7+16}{6} = 8$

$t_{E_{25}} = \dfrac{4+4\times9+20}{6} = 10$

$t_{E_{34}} = \dfrac{0+4\times0+0}{6} = 0$

$t_{E_{37}} = \dfrac{3+4\times5+13}{6} = 6$

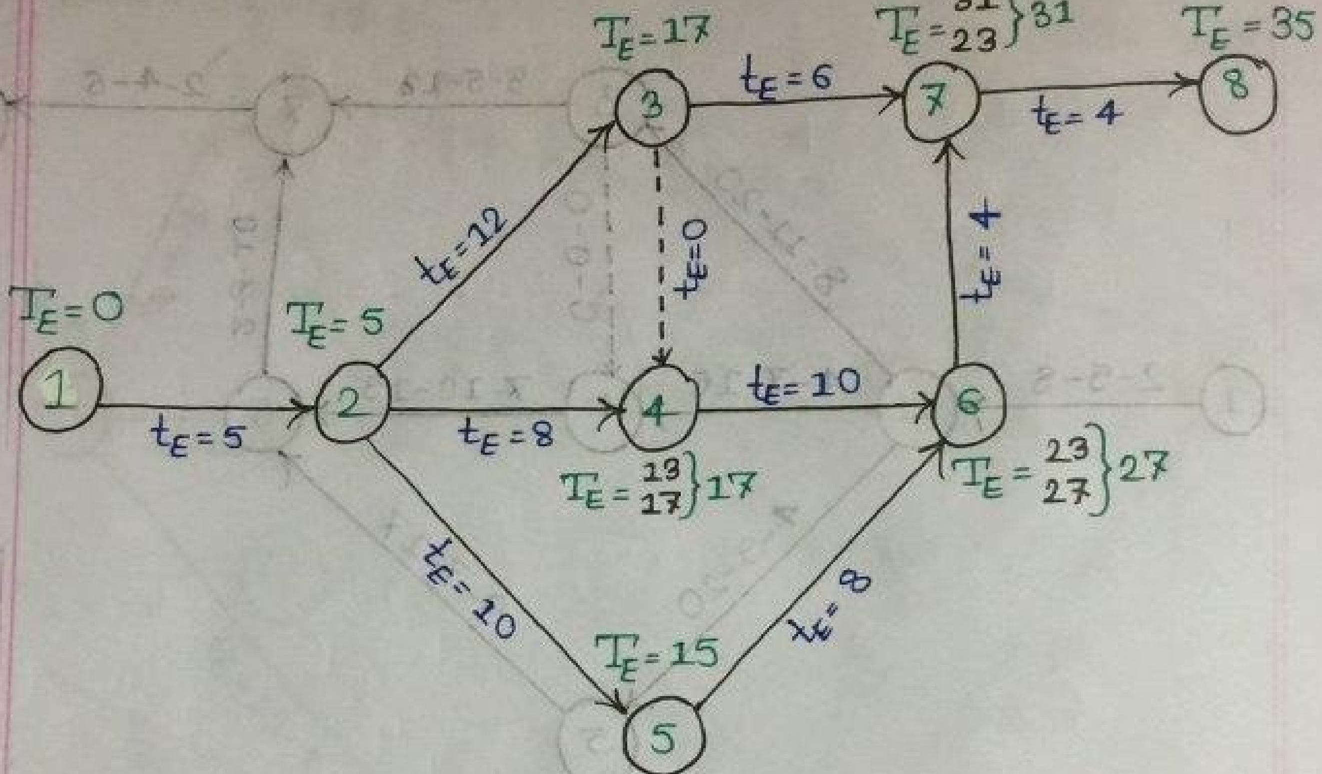
$t_{E_{46}} = \dfrac{7+4\times10+13}{6} = 10$

$t_{E_{56}} = \dfrac{3+4\times7+17}{6} = 8$

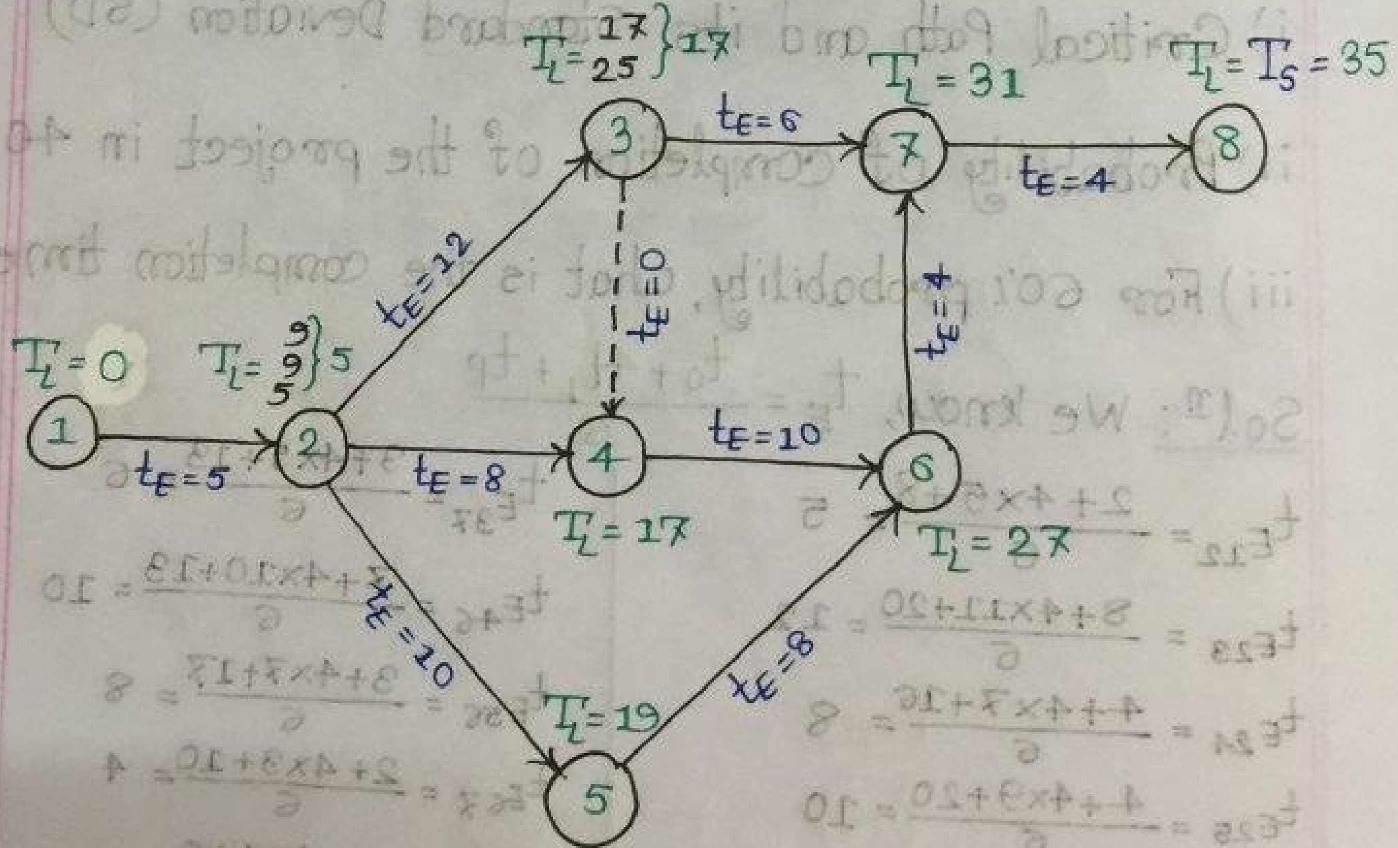
$t_{E_{67}} = \dfrac{2+4\times3+10}{6} = 4$

$t_{E_{78}} = \dfrac{2+4\times4+6}{4} = 4$

Earliest Expected Time,



Latest Allowable Occurrence Time,



(i) Here, Critical Path : 1-2-3-4-6-7-8 ($\Sigma t_E = T_S = 35$)
(Ans:)

Critical Activity	1-2	2-3	3-4	4-6	6-7	7-8
Variance, $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$	1	4	0	1	1.78	0.44

Standard deviation of critical path,

$$SD_{cp} = \sqrt{(1+4+0+1+1.78+0.44)}$$

$$= 2.87 \text{ (Ans:)}$$

(ii) We know,

$$Z = \frac{T_S - T_E}{SD_{cp}}$$

$$= \frac{40-35}{2.87}$$

$$= 1.74$$

From Probability Table, when $Z = 1.7$; $P_n = 95.54\%$

& when $Z = 1.8$; $P_n = 96.41\%$

So, for $Z = 1.74$; $P_n = \left[\left(\frac{96.41 - 95.54}{1.8 - 1.7} \times 0.04\right) + 95.54\right]$

$$= 95.888\% \text{ (Ans:)}$$

(iii) For 60% Probability,

From Probability Table,

when, $P_n = 57.93$; $Z = +0.2$

when, $P_n = 61.79$; $Z = +0.3$

So, for $P_n = 60.0$, $Z = \left[\frac{0.3-0.2}{61.79-57.93} \times (60-57.93) + 0.2\right]$

$= 0.2536$

Now, $Z = \frac{T_S - T_E}{SD_{CP}}$

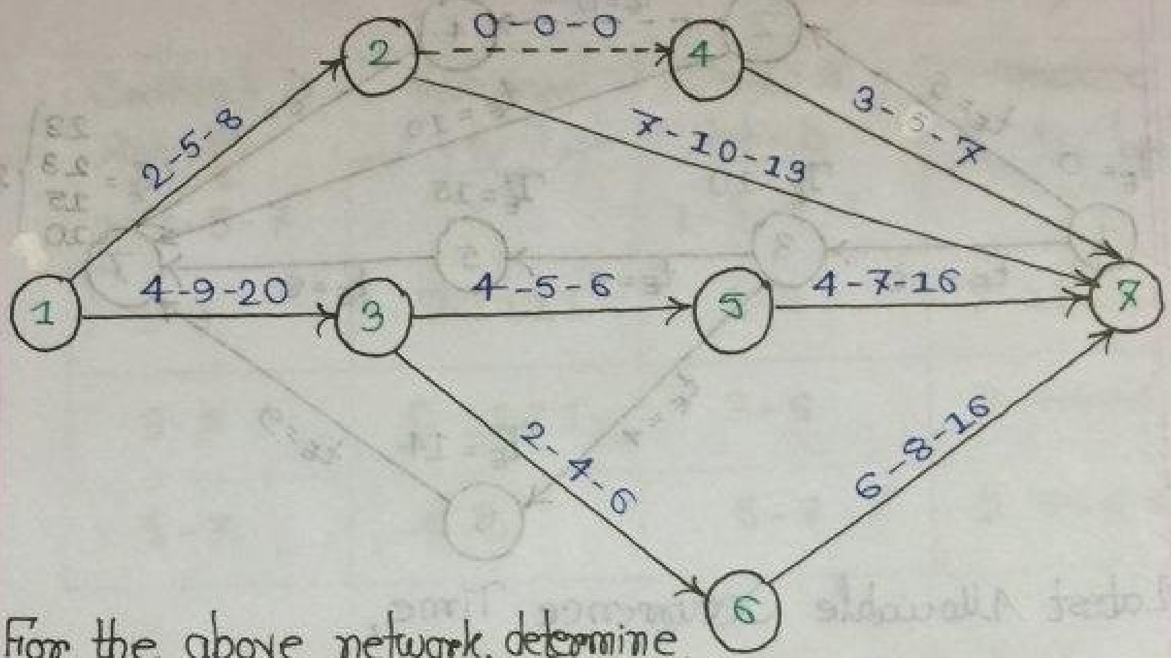
$\Rightarrow 0.2536 = \frac{T_S - 35}{1.74}$

$\therefore T_S = 35.44$

≈ 35 days (lower rounding)

(Ans:)

Extra **Example 8:**



For the above network, determine,

(i) Critical Path and its SD.

(ii) Probability of completion of the project within 25 days.

Sol^n: We know, $t_E = \dfrac{t_o + 4t_L + t_p}{6}$

$$t_{E_{12}} = \frac{2+4\times5+8}{6} = 5$$

$$t_{E_{13}} = \frac{4+4\times9+20}{6} = 10$$

$$t_{E_{24}} = \frac{0+4\times0+0}{6} = 0$$

$$t_{E_{27}} = \frac{7+4\times10+13}{6} = 10$$

$$t_{E_{35}} = \frac{4+4\times5+6}{8} = 5$$

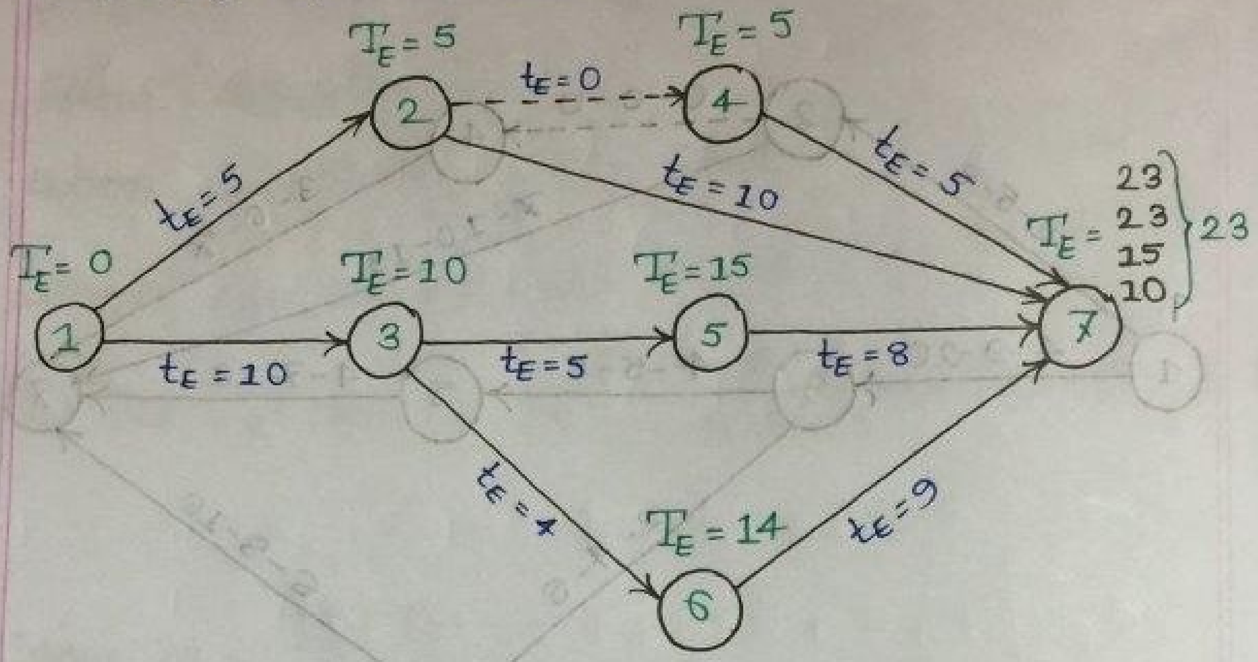
$$t_{E_{36}} = \frac{2+4\times4+6}{4} = 4$$

$$t_{E_{57}} = \frac{4+4\times7+16}{6} = 8$$

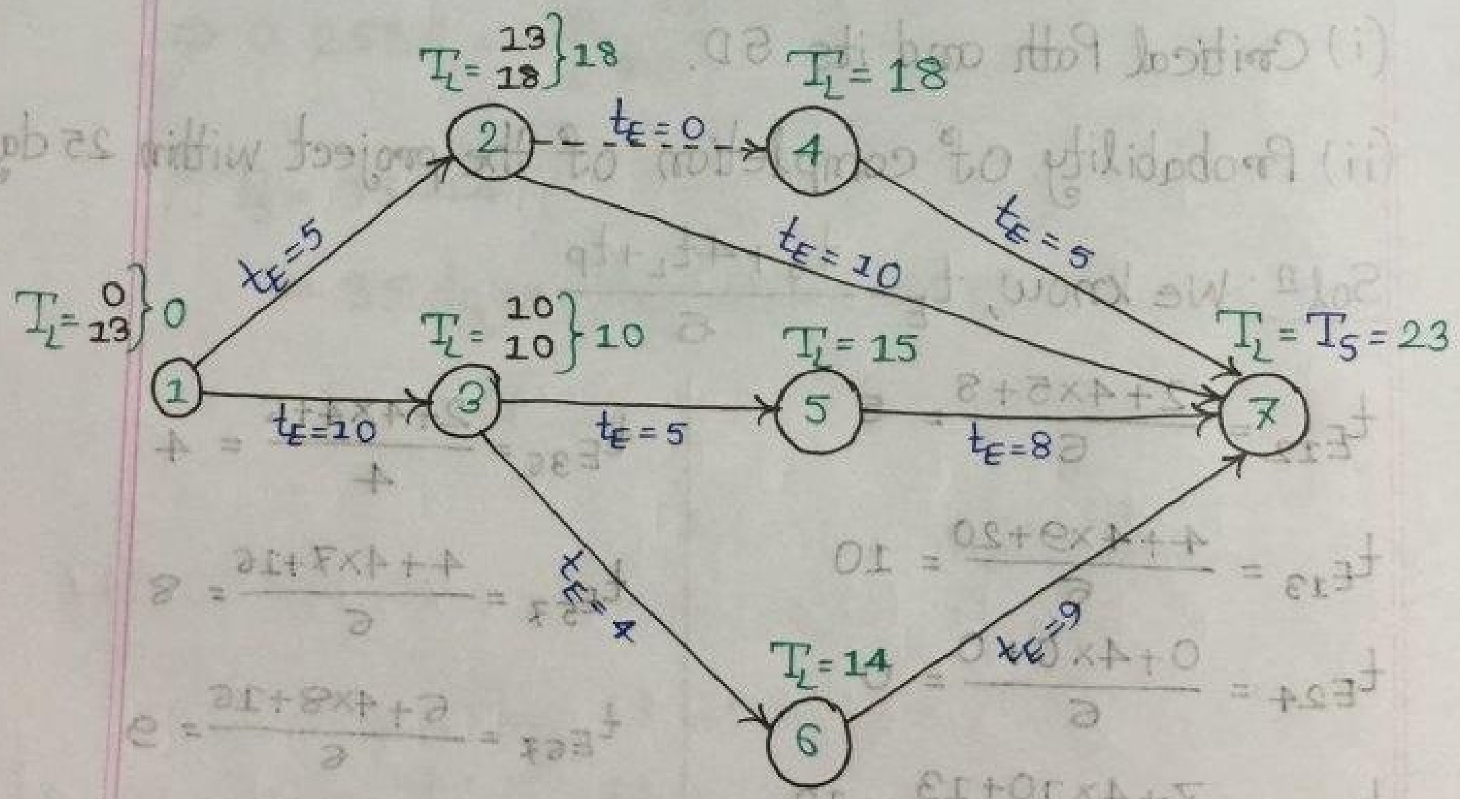
$$t_{E_{67}} = \frac{6+4\times8+16}{6} = 9$$

$$t_{E_{47}} = \frac{3+4\times5+7}{6} = 5$$

Earliest Expected Time,



Latest Allowable Occurrence Time,



(i) Here, Critical Path : 1-3-5-7 ($\Sigma t_E = T_S = 23$)

1-3-6-7 ($\Sigma t_E = T_S = 23$) (Ans:)

Critical Activity 1-3-5-7	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$	Critical Activity 1-3-6-7	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1-3	7.1111	1-3	7.1111
3-5	0.1111	3-6	0.4444
5-7	4.0	6-7	2.7778

Standard deviation for Critical Path 1-3-5-7,

$$SD_{CP(1-3-5-7)} = \sqrt{(7.11 + 0.11 + 4)} = 3.35 \text{ (Ans:)}$$

Standard deviation for Critical Path 1-3-6-7,

$$SD_{CP(1-3-6-7)} = \sqrt{(7.11 + 0.44 + 2.78)} = 3.21 \text{ (Ans:)}$$

(ii) For critical path 1-3-5-7,

$$Z = \frac{T_S - T_E}{SD_{CP}}$$

$$= \frac{25-23}{3.35}$$

$$= 0.597$$

From Probability Table,

when $Z = +0.5$; $P_n = 69.15\%$.

when $Z = +0.6$; $P_n = 72.57\%$

So, for $Z = +0.597$; $P_n = \frac{72.57 - 69.15}{0.6 - 0.5} \times (0.597 - 0.5) + 69.15$

$= 72.4674\%$

For critical path 1-3-6-7,

$Z = \frac{T_S - T_E}{SD_{CP}}$

$= \frac{25 - 23}{3.21}$

$= 0.623$

From Probability Table,

when $Z = +0.6$; $P_n = 72.57\%$

when $Z = +0.7$; $P_n = 75.80\%$

So, for $Z = +0.623$; $P_n = \frac{75.80 - 72.57}{0.7 - 0.6} \times (0.623 - 0.6) + 72.57$

$= 73.3129\%$

Min^{m} Probability governs.

Thus, $P_n = 72.4674\%$ (Ans:)

Critical Path Method (CPM) : Network Analysis

- CPM Networks are generally activity-oriented.

Start and Finish Times of Activity :

For CPM networks following activity times are useful for network computations :

1. Earliest Start Time (EST)
2. Earliest Finish Time (EFT)
3. Latest Start Time (LST)
4. Latest Finish Time (LFT)

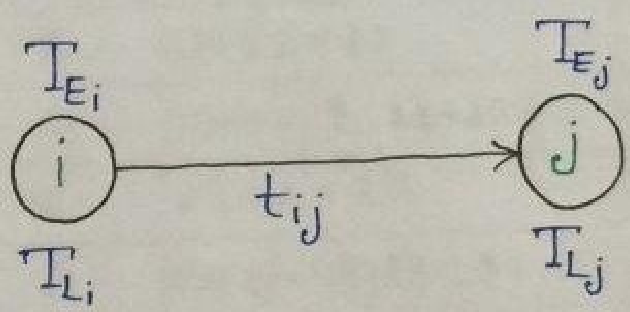
Formula :

EST = Earliest event time at its tail (T_{E_i})

EFT = EST + Activity Duration (t_{ij})

LFT = Latest event time at its head (T_{L_j})

LST = LFT - Activity Duration (t_{ij})



FLOAT: Float denotes the range within which an activity start time or its finish time may fluctuate without affecting the completion of the project.

There are four types of Float in CPM Networks:

Total Float: $F_T = (T_{L_j} - T_{E_i}) - t_{ij}$

or, $LST - EST$

or, $LFT - EFT$

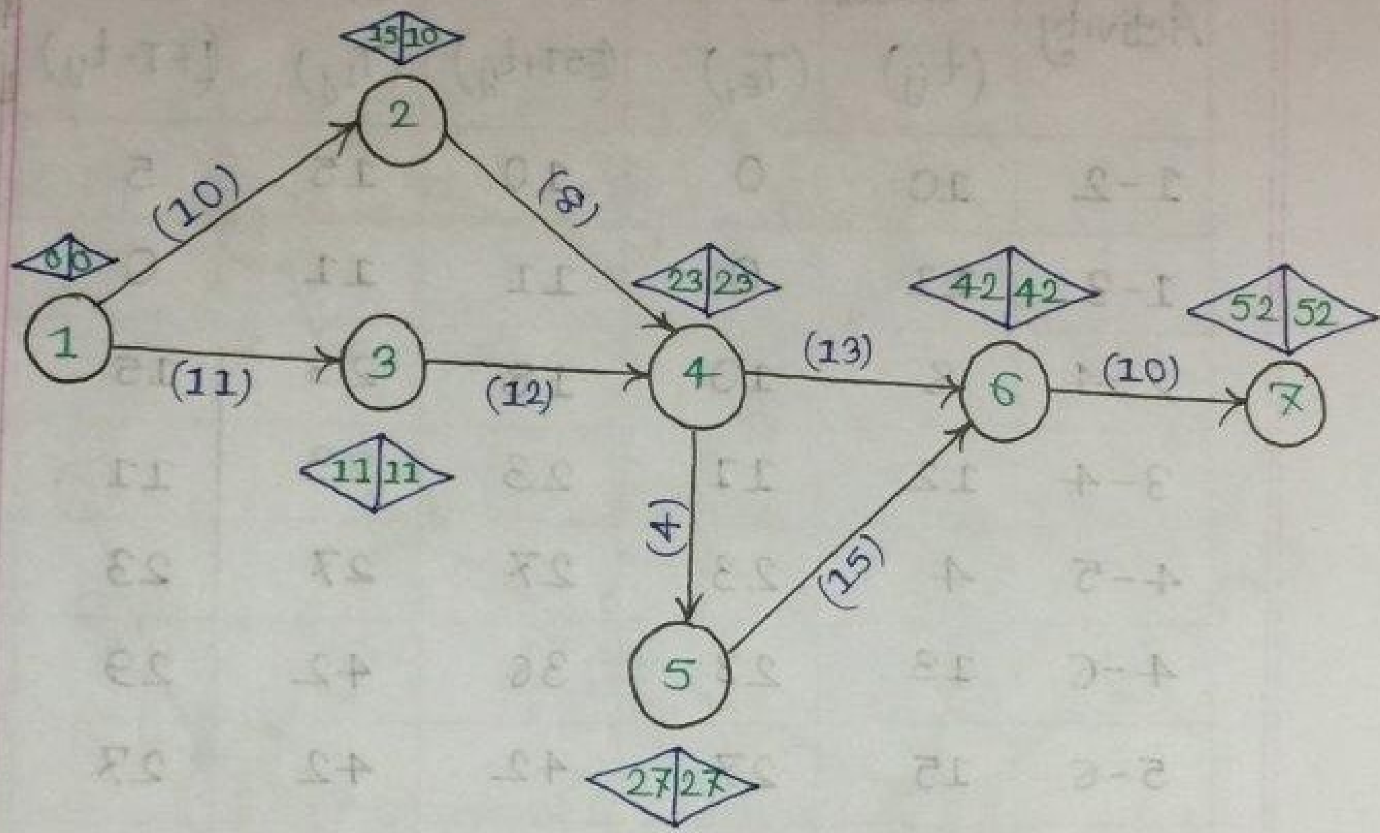
Note:
Total Float = 0
means
Critical Activity

Free Float: $F_F = (T_{E_j} - T_{E_i}) - t_{ij}$

Independent Float: $F_{ID} = (T_{E_j} - T_{L_i}) - t_{ij}$

Interfering Float: $F_{IN} = F_T - F_F$

Example 1 :



For the above network diagram, determine Event Time (T_E, T_L), Activity Time (EST, EFT, LFT, LST). Find the critical path.

Sol^n :

Event No.	T_E	T_L
1	0	15-10=5 ; 11-11 = 0
2	0+10=10	23-8=15
3	0+11=11	23-12=11
4	10+8=18 ; 11+12 = 23	42-13=29 ; 27-4=23
5	23+4 = 27	42-15=27
6	23+13=36 ; 27+15 = 42	52-10=42
7	42+10=52	52

Activity	Duration (t_{ij})	EST (T_{E_i})	EFT ($EST+t_{ij}$)	LFT (T_{L_j})	LST ($LFT-t_{ij}$)	Total Float LST-EST or, LFT-EFT
1-2	10	0	10	15	5	5
1-3	11	0	11	11	0	0
2-4	8	10	18	23	15	5
3-4	12	11	23	23	11	0
4-5	4	23	27	27	23	0
4-6	13	23	36	42	29	6
5-6	15	27	42	42	27	0
6-7	10	42	52	52	42	0

Total float = 0 means Critical Activity

So, Critical Path : 1 - 3 - 4 - 5 - 6 - 7 (Ans:)

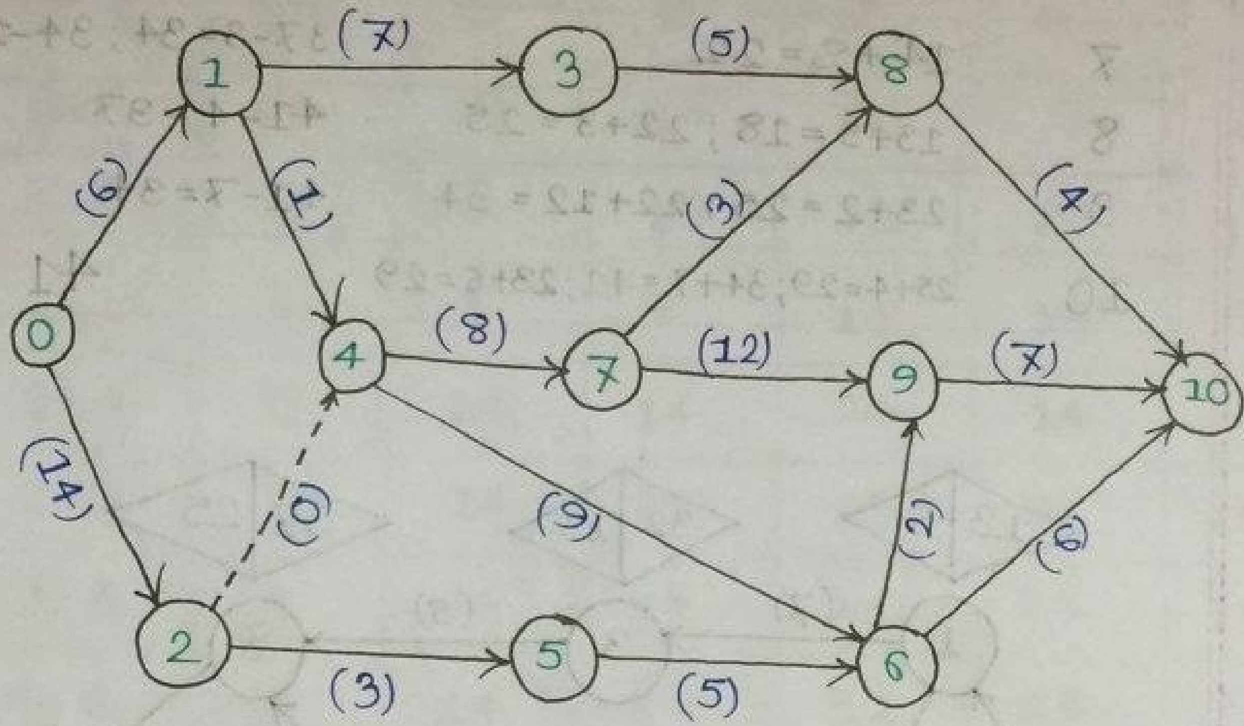
Summation of t_{ij} for Critical Path

$$\Sigma t_{ij} = 11 + 12 + 4 + 15 + 10$$

$$= 52$$

$= T_E$ or T_L of Last Event (Checked)

Example 2 :

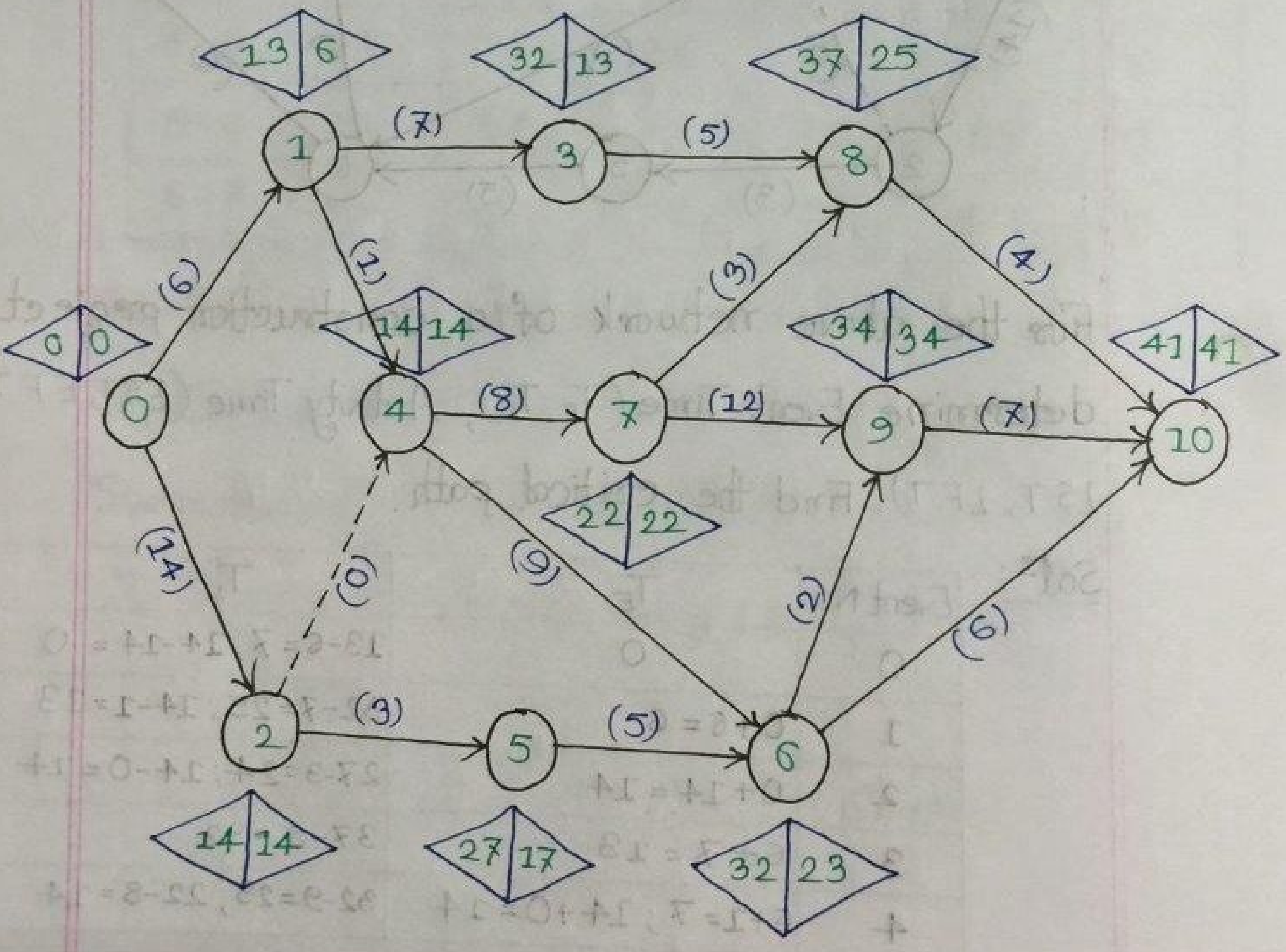


For the above network of a construction project, determine Event Time (T_E, T_L), Activity Time (EST, EFT, LST, LFT). Find the critical path.

Sol^n:

Event No.	T_E	T_L
0	0	13-6 = 7 ; 14-14 = 0
1	0+6 = 6	32-7 = 25 ; 14-1 = 13
2	0+14 = 14	27-3 = 24 ; 14-0 = 14
3	6+7 = 13	37-5 = 32
4	6+1 = 7 ; 14+0 = 14	32-9 = 23 ; 22-8 = 14

Event No.	T_E	T_L
5	14+3=17	32−5=27
6	17+5=22 ; 14+9=23	41-6=35 ; 34-2=32
7	14+8=22	37-3=34 ; 34−12=22
8	13+5=18 ; 22+3=25	41−4=37
9	23+2=25 ; 22+12=34	41-7=34
10	25+4=29 ; 34+7=41 ; 23+6=29	41



Activity	Duration (t_{ij})	EST (T_{E_i})	EFT ($EST+t_{ij}$)	LFT (T_{L_j})	LST ($LFT-t_{ij}$)	Total Float LST-EST or, LFT-EFT
0-1	6	0	6	13	7	7
0-2	14	0	14	14	0	0
1-3	7	6	13	32	25	19
1-4	1	6	7	14	13	7
2-4	0	14	14	14	14	0
2-5	3	14	17	27	24	10
3-8	5	13	18	37	32	19
4-6	9	14	23	32	23	9
4-7	8	14	22	22	14	0
5-6	5	17	22	32	27	10
6-9	2	23	25	34	32	9
6-10	6	23	29	41	35	12
7-8	3	22	25	37	34	12
7-9	12	22	34	34	22	0
8-10	4	25	29	41	37	12
9-10	7	34	41	41	34	0

Total float = 0 means Critical Activity.

So, Critical Path : 0 - 2 - 4 - 7 - 9 - 10 (Ans:)

Summation of t_{ij} for Critical Path,

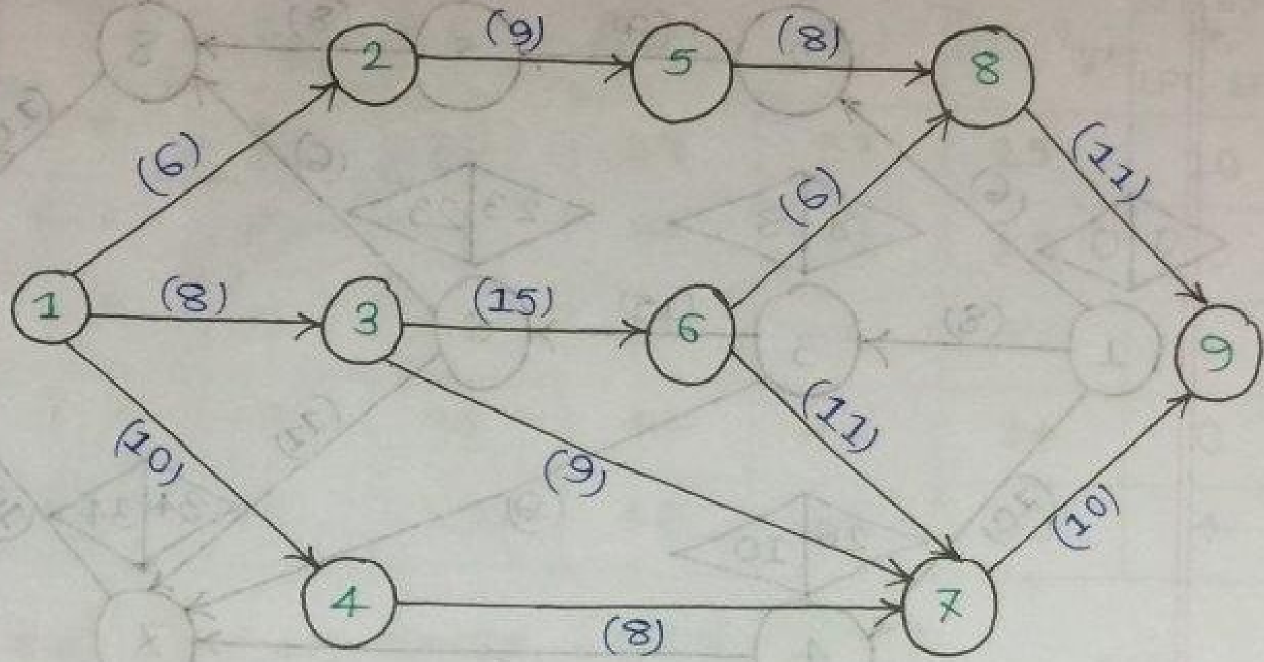
$$\sum t_{ij} = 14 + 0 + 8 + 12 + 7$$

$$= 41$$

$= T_E$ or T_L of Last event (Checked)

Extra

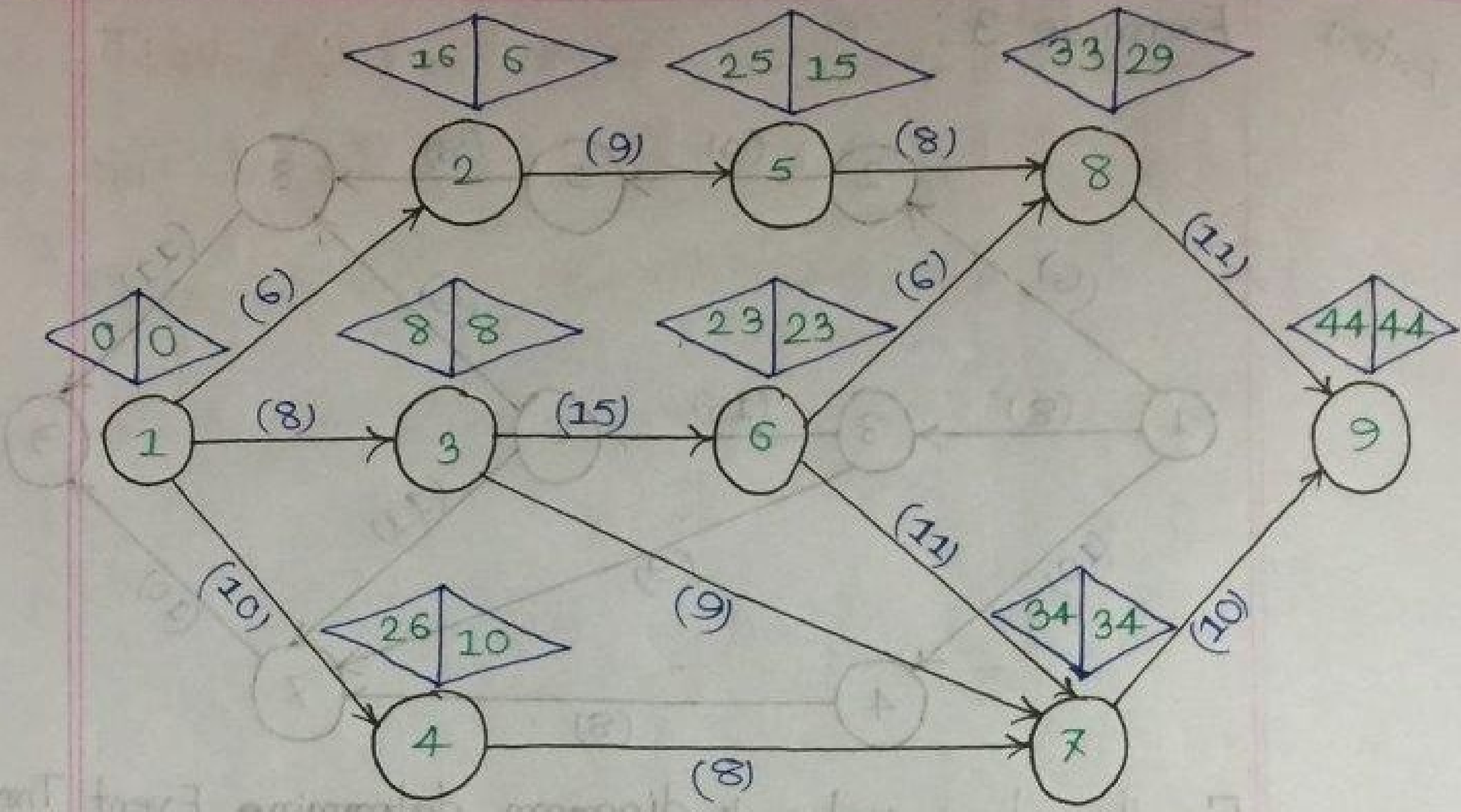
Example 3 :



For the above network diagram, determine Event Time (T_E, T_L), Activity Time (EST, EFT, LFT, LST). Find the critical path.

Sol^n:

Event No.	T_E	T_L
1	0	26-10=16; 16-6=10; 8-8= 0
2	0+6=6	25-9 = 16
3	0+8=8	34-9=25; 23-15 = 8
4	0+10=10	34-8 = 26
5	6+9=15	33-8 = 25
6	8+15=23	33-6=27; 34-11=23
7	10+8=18; 8+9=17; 23+11=34	44-10 = 34
8	15+8=23; 23+6=29	44-11=33
9	29+11=40; 34+10=44	44



Activity	Duration (t_{ij})	EST (T_{E_i})	EFT $(EST+t_{ij})$	LFT (T_{L_j})	LST $(LFT-t_{ij})$	Total Float LST-EST or, LFT-EFT
1-2	6	0	6	16	10	10
1-3	8	0	8	8	0	0
1-4	10	0	10	26	16	16
2-5	9	6	15	25	16	10
3-6	15	8	23	23	8	0
3-7	9	8	17	34	25	17
4-7	8	10	18	34	26	16

Activity	Duration (t_{ij})	EST (T_{E_i})	EFT ($EST+t_{ij}$)	LFT (T_{L_j})	LST ($LFT-t_{ij}$)	Total Float LST- EST or, LFT - EFT
5-8	8	15	23	33	25	10
6-7	11	23	34	34	23	0
6-8	6	23	29	33	27	4
7-9	10	34	44	44	34	0
8-9	11	29	40	44	33	4

Total float = 0 means Critical Activity.

So, Critical Path : 1-3-6-7-9 (Ans:)

Summation of t_{ij} for Critical Path,

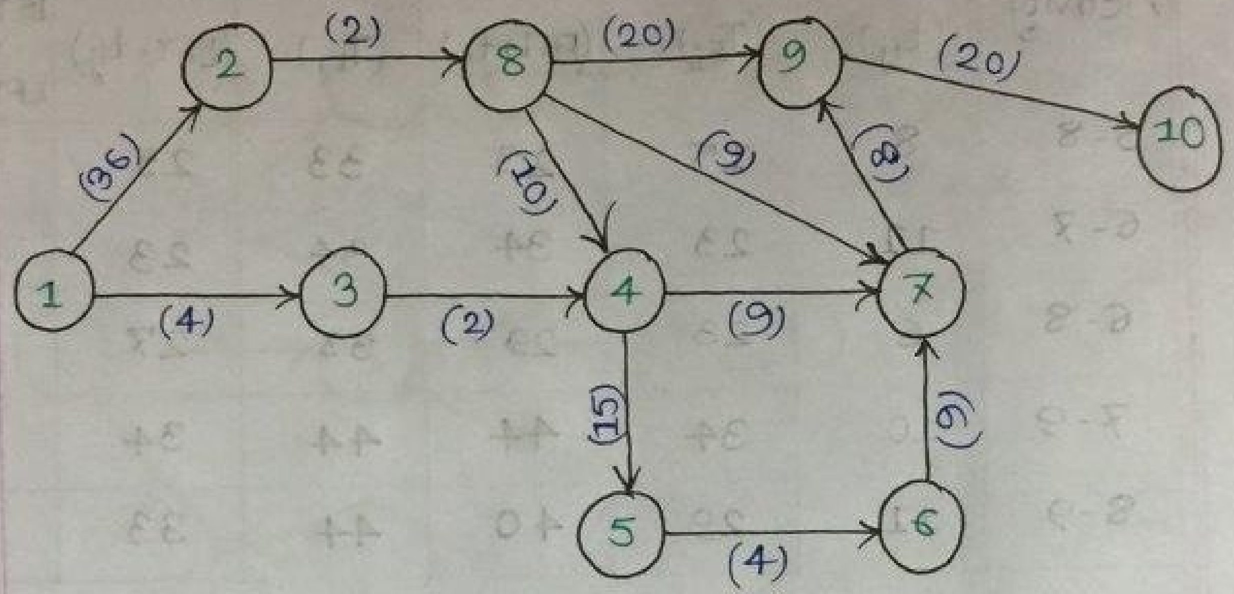
$$\sum t_{ij} = 8+15+11+10$$

$$= 44$$

$$= T_E \text{ or } T_L \text{ of Last Event}$$

(Checked)

Example 4 :



For the above network diagram, compute Event Time (T_E, T_L), Activity Time (EST, EFT, LFT, LST) and find the critical path & locate it on the diagram.

Sol^n:

"Linear Programming"

Example 1: Following information is available on different type of Machine and products.

Machine Type	Available Time (Machine)
M1	200
M2	100
M3	50

Productivity (machine hours per unit)

Machine	Product P1	Product P2	Product P3
M1	8	2	3
M2	4	3	1
M3	2	1	1

The sales department indicates that the sales potential for product P1 and P2 exceeds the maximum production rate and the sales potential for product P3 is 20 units/week. The unit profit would be Taka 20, 6 and 8 respectively on P1, P2 and P3.

Formulate a LP model for determining how much of each product the company should produce in order to maximize profit.

$\underline{Sol^n}$: Objective Function,

Maximize $Z = 20P1 + 6P2 + 8P3$ (Ans:)

Subject to :

$$8P1 + 2P2 + 3P3 \le 200$$

$$4P1 + 3P2 + P3 \le 100$$

$$2P1 + P2 + P3 \le 50$$

$$P3 \le 20$$

$$P1, P2, P3 \ge 0$$

(Ans:)

Example 2: A small electronic company produces two products - a high quality cable TV tuners and a high quality receiver. Both products go through the same two department production (PR) and inspection and testing (IT). The production manager's job is to produce the optimal combination of tuners (T) and receiver (R) that maximize profit margin and use the time in (PR) and (IT) most efficiently.

Unit profit of T and R is USD 30 and 20 respectively. Production capacity per day (in hours) of PR and IT is 150 and 80 respectively. For PR department number of hours required per unit of T and R are 10 and 6 respectively. On the other hand for IT department number of hours required per unit of T and R are 4 and 4 respectively.

Solⁿ: Objective Function,

Maximize $Z = 30T + 20R$

Subject to :

$$10T + 6R \leq 150$$

$$4T + 4R \leq 80$$

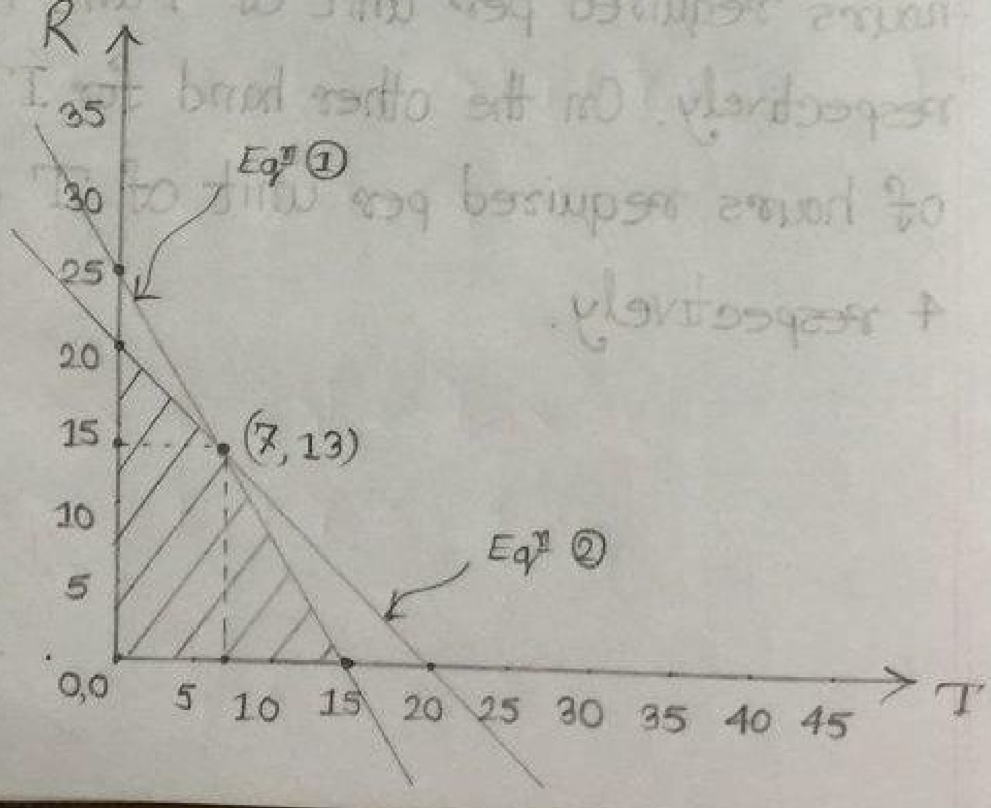
$$T, R \geq 0$$

From 1st eqⁿ,

When, $T = 0$; $R = 25$ & when $R = 0$; $T = 15$

From 2nd eqⁿ,

When, $T = 0$; $R = 20$ & when $R = 0$; $T = 20$



Using Graphical method, the optimal combination is found to be $T = 7$; $R = 13$

This combination also satisfies the constraints.

(Ans:)

Example 3: A manufacturer produces two products X & Y with two machines A and B. The cost of producing each unit of X is for machine A: 50 minutes and for machine B is 30 minutes. The cost of producing each unit of Y is for machine A: 24 minutes and for machine B: 33 minutes. Working plan for a particular week are: 40 hours of work on machine A and 35 hours of work on machine B. The week starts with a stock of 30 units of X and 90 units of Y and a demand of 75 units of X and 95 units of Y. How to plan the production in order to end the week with the maximum stock.

Sol^n: Objective Function,

Maximize $Z = (X + 30 - 75) + (Y + 90 - 95)$

or, $Z = (X - 45) + (Y - 5)$

Subject to:

$50X + 24Y \leq 40 \times 60$

or, $50X + 24Y \leq 2400$ . . . . . . (1)

$30X + 33Y \leq 35 \times 60$

or, $30X + 33Y \leq 2100$ ②

& $X \geq 75 - 30$

or, $X \geq 45$

$Y \geq 95 - 90$

or, $Y \geq 5$

Example 4 : A company produce two products A & B. Product A requires 2 machine hours for cutting, 1 machine hour for shapping and 3 machine hours for finishing. While product B requires 2 machine hours for cutting, 2 machine hours for shapping and 1 machine hour for finishing. Each day the company has available 140 machine hours for cutting, 120 machine hours for shapping, 150 machine hours for finishing. How many product A & B should the company manufacture each day in order to maximize product if product A yields a profit of \$10 & product B yields a profit of \$8 ?

Sol^n: Let, the amount of product A & B to be produced to be X & Y respectively.

Objective Function:

Maximize $Z = 10X + 8Y$

Subject to:

$$2X + 2Y \leq 140 \quad \ldots\ldots\ldots (1)$$

$$X + 2Y \leq 120 \quad \ldots\ldots\ldots (2)$$

$3X + Y \leq 150$ . . . . ③

$X, Y \geq 0$

From eq^n ①,

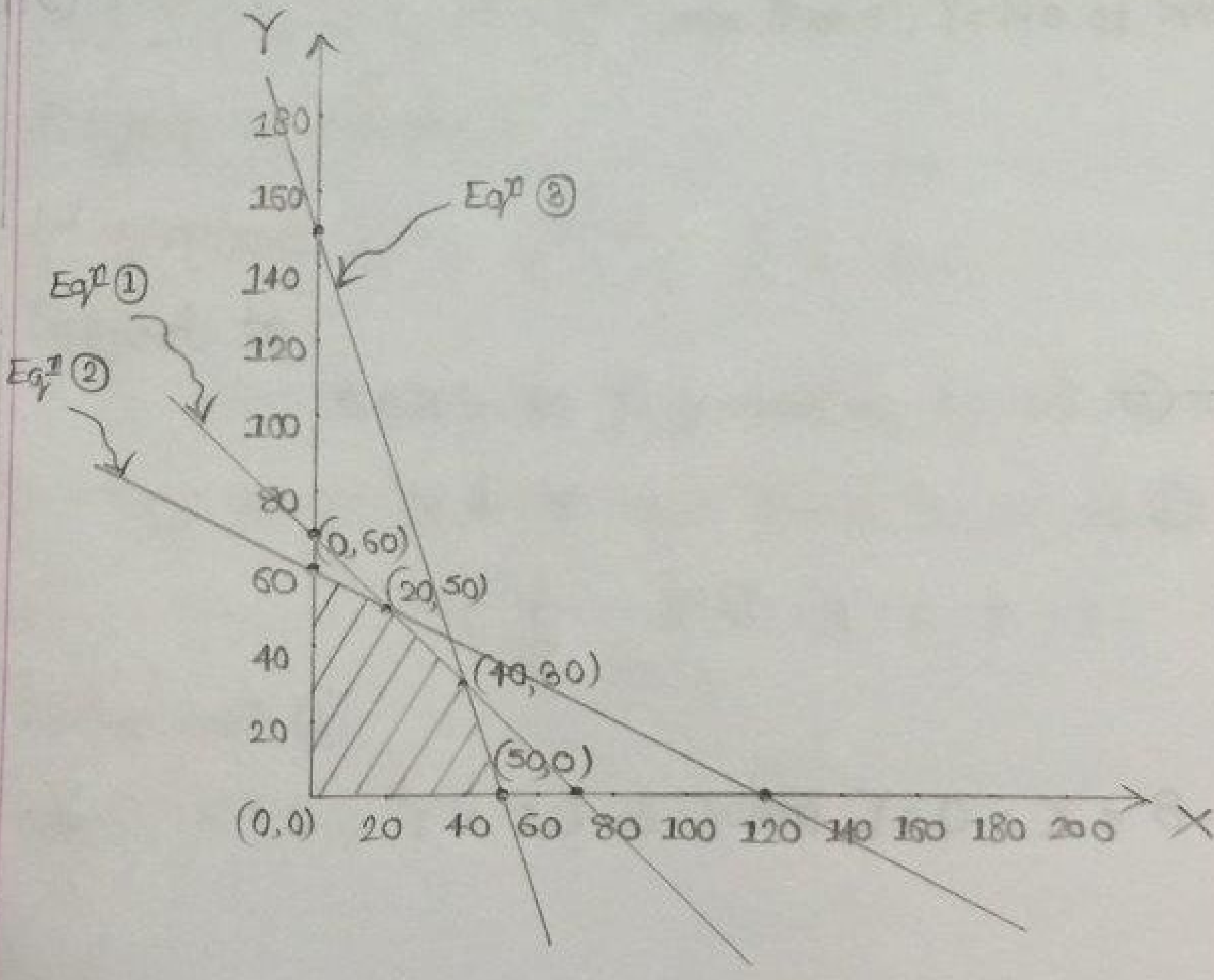
when, $X = 0$; $Y = 70$ & when $Y = 0$; $X = 70$

From eq^n ②,

when $X = 0$; $Y = 60$ & when $Y = 0$; $X = 120$

From eq^n ③,

when $X = 0$; $Y = 150$ & when $Y = 0$; $X = 50$



From the graph we get four probable solutions.

For $(0,60)$, $Z = 10\times 0 + 8\times 60 = 480$

For $(20,50)$, $Z = 10\times 20 + 8\times 50 = 600$

For $(40,30)$, $Z = 10\times 40 + 8\times 30 = 640$ (maximum)

For $(50,0)$, $Z = 10\times 50 + 8\times 0 = 500$

So, when $X = 40$ & $Y = 30$ we get maximum Z.

These values also satisfy the constraints

(Ans:)

Example 5: A school is preparing a trip for 400 students, the company which is providing the transportation has 10 buses of 50 seats each and 8 buses of 40 seats each but only has 9 drivers available. The rent cost for a large bus is \$800 and \$600 for the small bus. Calculate how many buses of each type should be used for the trip for the least possible cost using linear programming?

Sol^n: Let, X = No. of Large Buses, Y = No. of Small Buses

Objective Function:

Minimize $Z = 800X + 600Y$

Subject to:

$$50X + 40Y \geq 400 \quad \ldots\ldots (1)$$

$$X + Y \leq 9 \quad \ldots\ldots (2)$$

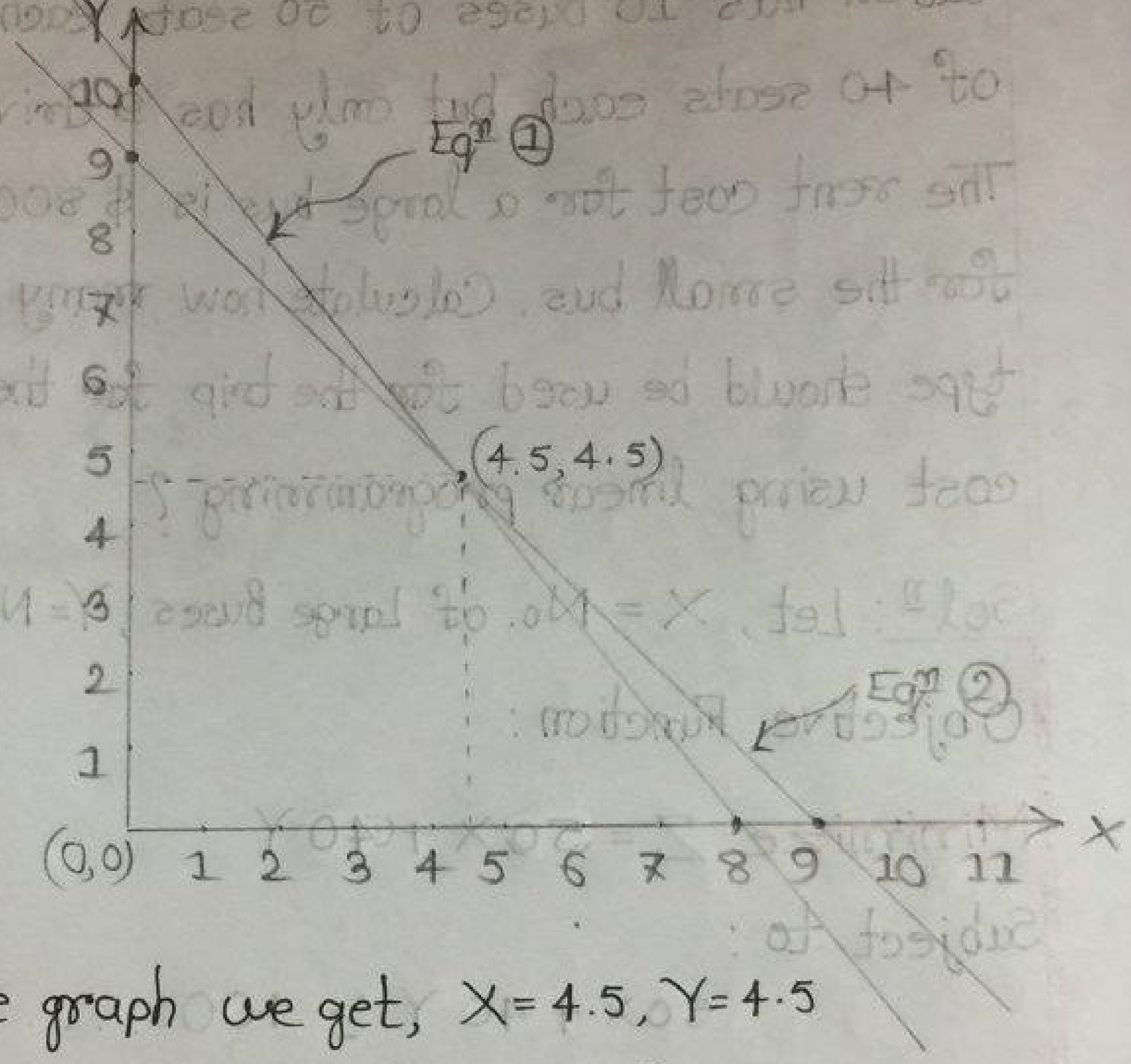
$$X, Y \geq 0$$

From eq^n (1),

when, $X = 0$; $Y = 10$ & when $Y = 0$; $X = 8$

From eq^n ②,

when, $X = 0$; $Y = 9$ & when $Y = 0$; $X = 9$



From the graph we get, $X = 4.5$, $Y = 4.5$

So, the no. of buses to be used of each type,

$X = 4$ & $Y = 5$

(Ans:)