

Bar Elements Linear Static Analysis

By

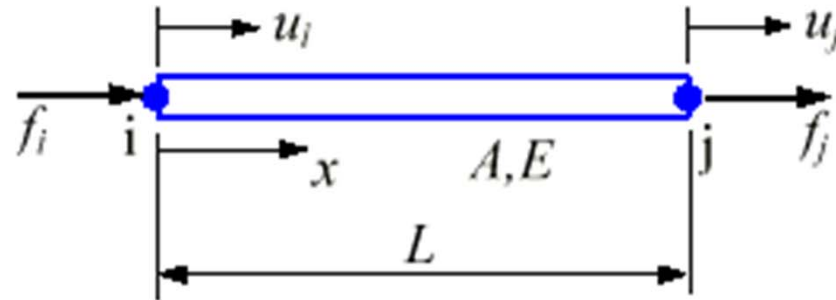
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I. Linear Static Analysis

- Most structural analysis problems can be treated as *linear static* problems, based on the following assumptions
 - **Small deformations** (loading pattern is not changed due to the deformed shape)
 - **Elastic materials** (no plasticity or failures)
 - **Static loads** (the load is applied to the structure in a slow or steady fashion) Linear analysis can provide most of the information about the behavior of a structure, and can be a good approximation for many analyses. It is also the bases of nonlinear analysis in most of the cases.

II. Bar Element

Consider a uniform prismatic bar:



L	length	$u = u(x)$	displacement
A	cross-sectional area	$\varepsilon = \varepsilon(x)$	strain
E	elastic modulus	$\sigma = \sigma(x)$	stress

Strain-displacement relation:

$$\varepsilon = \frac{du}{dx}$$

(1)

Stiffness Matrix - Direct Method

Stress-strain relation:

$$\sigma = E\varepsilon$$

Assuming that the displacement u is *varying linearly* along the axis of the bar, i.e.,

$$u(x) = \left(1 - \frac{x}{L}\right)u_i + \frac{x}{L}u_j$$

we have

$$\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L} \quad \text{where } (\Delta = \text{elongation})$$

$$\sigma = E\varepsilon = \frac{E\Delta}{L}$$

Stiffness Matrix - Direct Method

We also have

$$\sigma = \frac{F}{A} \quad \text{where} \quad (F = \text{force in bar})$$

Thus,

$$F = \frac{EA}{L} \Delta = k\Delta$$

where $k = \frac{EA}{L}$ is the stiffness of the bar.

The bar is acting like a spring in this case and we conclude that element stiffness matrix is

Stiffness Matrix - Direct Method

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \quad \text{or} \quad \mathbf{k} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

This can be verified by considering the equilibrium of the force at the two nodes.

Element equilibrium equation is

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

Stiffness Matrix - Direct Method

Degree of Freedom (dof)

Number of components of the displacement vector at a node.

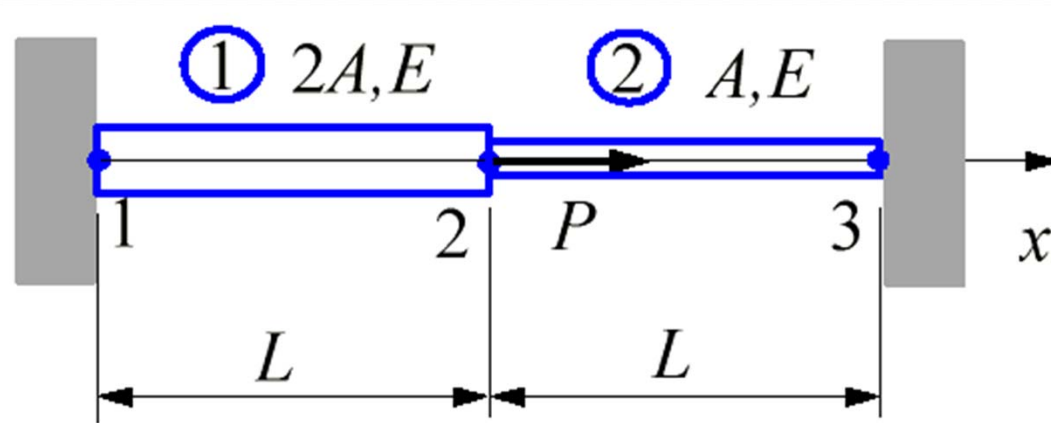
For 1-D bar element: one dof at each node.

Physical Meaning of the Coefficients in \mathbf{k}

The j th column of \mathbf{k} (here $j = 1$ or 2) represents the forces applied to the bar to maintain a deformed shape with unit displacement at node j and zero displacement at the other node.

Example 2.1

Problem: Find the stresses in the two bar assembly which is loaded with force P , and constrained at the two ends, as shown in the figure.



Example 2.1: Solution

Solution: Use two 1-D bar elements.

Element

$$\mathbf{k}_1 = \frac{2EA}{L} \begin{bmatrix} & u_1 & u_2 \\ 1 & & -1 \\ -1 & & 1 \end{bmatrix}$$

Element 2,

$$\mathbf{k}_2 = \frac{EA}{L} \begin{bmatrix} & u_2 & u_3 \\ 1 & & -1 \\ -1 & & 1 \end{bmatrix}$$

Imagine a frictionless pin at node 2, which connects the two elements. We can assemble the global FE equation as follows,

$$\frac{EA}{L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Example 2.1: Solution

Load and boundary conditions (BC) are,

$$u_1 = u_3 = 0, \quad F_2 = P$$

FE equation becomes,

$$\frac{EA}{L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Deleting the 1st row and column, and the 3rd row and column, we obtain,

$$\frac{EA}{L} [3] \{u_2\} = \{P\} \quad \rightarrow \quad u_2 = \frac{PL}{3EA}$$

Example 2.1: Solution

and

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{3EA} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Stress in element 1 is

$$\begin{aligned} \sigma_1 &= E\varepsilon_1 = E\mathbf{B}_1\mathbf{u}_1 = E\begin{bmatrix} -1/L & 1/L \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ &= E \frac{u_2 - u_1}{L} = \frac{E}{L} \left(\frac{PL}{3EA} - 0 \right) = \frac{P}{3A} \end{aligned}$$

This indicates that bar 1 is in Tension.

Stress in element 2 is

$$\begin{aligned} \sigma_2 &= E\varepsilon_2 = E\mathbf{B}_2\mathbf{u}_2 = E\begin{bmatrix} -1/L & 1/L \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \\ &= E \frac{u_3 - u_2}{L} = \frac{E}{L} \left(0 - \frac{PL}{3EA} \right) = -\frac{P}{3A} \end{aligned}$$

This indicates that bar 2 is in compression.

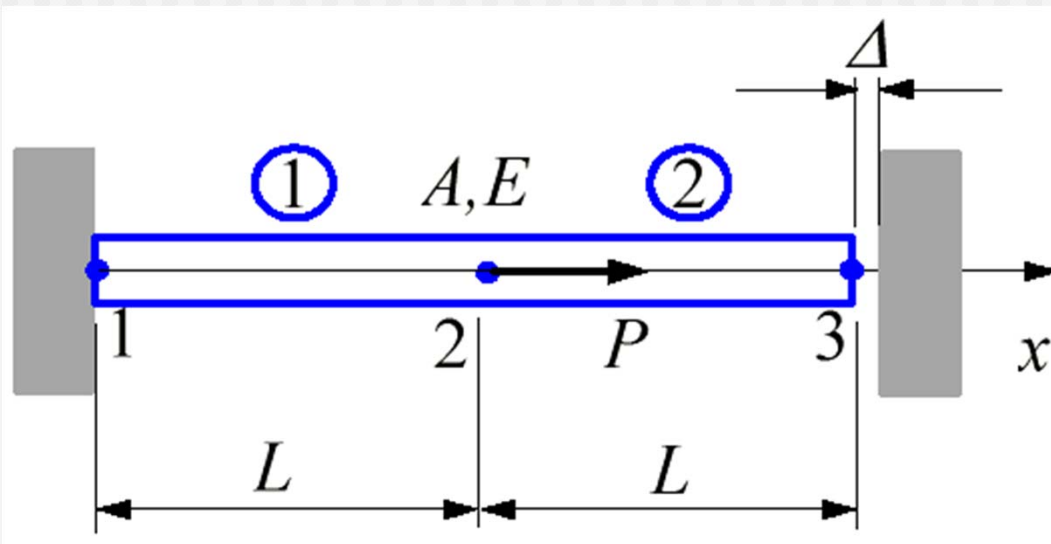
Notes about bar elements

- In this case, the calculated stresses in elements 1 and 2 are exact within the linear theory for 1-D bar structures. It will not help if we further divide element 1 or 2 into smaller finite elements.
- For tapered bars, averaged values of the cross-sectional areas should be used for the elements.
- We need to find the displacements first in order to find the stresses, since we are using the *displacement based FEM*.

Example 2.2

Problem: Determine the support reaction forces at the two ends of the bar shown above, given the following,

$$P = 6.0 \times 10^4 \text{ N}, \quad E = 2.0 \times 10^4 \text{ N/mm}^2,$$
$$A = 250 \text{ mm}^2, \quad L = 150 \text{ mm}, \quad \Delta = 1.2 \text{ mm}$$



Example 2.2

Solution:

We first check to see if or not the contact of the bar with the wall on the right will occur. To do this, we imagine the wall

on the right is removed and calculate the displacement at the

right end,
$$\Delta_0 = \frac{PL}{EA} = \frac{(6.0 \times 10^4)(150)}{(2.0 \times 10^4)(250)} = 1.8 \text{ mm} > \Delta = 1.2 \text{ mm}$$

Thus, contact occurs. The global FE equation is found to be,

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Example 2.2

The load and boundary conditions are,

$$F_2 = P = 6.0 \times 10^4 \text{ N}$$

$$u_1 = 0, \quad u_3 = \Delta = 1.2 \text{ mm}$$

FE equation becomes,

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ \Delta \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

The 2nd equation gives,

Example 2.2

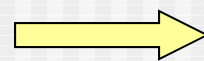
$$\frac{EA}{L} \begin{bmatrix} 2 & -1 \\ & \end{bmatrix} \begin{Bmatrix} u_2 \\ \Delta \end{Bmatrix} = \{P\}$$

that is,

$$\frac{EA}{L} [2] \{u_2\} = \left\{ P + \frac{EA}{L} \Delta \right\}$$

Solving this, we obtain

$$u_2 = \frac{1}{2} \left(\frac{PL}{EA} + \Delta \right) = 1.5 \text{ mm}$$



$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.5 \\ 1.2 \end{Bmatrix} \text{ (mm)}$$

Example 2.2

To calculate the support reaction forces, we apply the 1st and 3rd equations in the global FE equation.

The 1st equation gives,

$$F_1 = \frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{EA}{L} (-u_2) = -5.0 \times 10^4 \text{ N}$$

and the 3rd equation gives,

$$F_3 = \frac{EA}{L} \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{EA}{L} (-u_2 + u_3) = -1.0 \times 10^4 \text{ N}$$

Stiffness Matrix – Another Approach

We derive the same stiffness matrix for the bar using a formal approach which can be applied to many other more complicated situations.

Define two *linear shape functions* as follows

$$u(x) = \left(1 - \frac{x}{L}\right)u_i + \frac{x}{L}u_j$$

$$u(x) = u(\xi) = N_i(\xi)u_i + N_j(\xi)u_j$$

Shape Function

$$\xi = \frac{x}{L},$$

$$0 \leq \xi \leq 1$$

$$N_i(\xi) = 1 - \xi,$$

$$N_j(\xi) = \xi$$

Stiffness Matrix – Another Approach

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$$u = \begin{bmatrix} N_i & N_j \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \mathbf{N} \mathbf{u}$$

now

$$\varepsilon = \frac{du}{dx} = \left[\frac{d}{dx} \mathbf{N} \right] \mathbf{u} = \mathbf{B} \mathbf{u}$$

where \mathbf{B} is the element *strain-displacement matrix*, which is

$$\mathbf{B} = \frac{d}{dx} \begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} = \frac{d}{d\xi} \begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \bullet \frac{d\xi}{dx}$$

→ $\mathbf{B} = \begin{bmatrix} -1/L & 1/L \end{bmatrix}$

Stiffness Matrix – Another Approach

Stress can be written as $\sigma = E\varepsilon = E\mathbf{B}\mathbf{u}$

Consider the *strain energy* stored in the bar

$$U = \frac{1}{2} \int_V \sigma^T \varepsilon dV = \frac{1}{2} \int_V (\mathbf{u}^T \mathbf{B}^T E \mathbf{B} \mathbf{u}) dV = \frac{1}{2} \mathbf{u}^T \left[\int_V (\mathbf{B}^T E \mathbf{B}) dV \right] \mathbf{u}$$

The *work* done by the two nodal forces is

$$W = \frac{1}{2} f_i u_i + \frac{1}{2} f_j u_j = \frac{1}{2} \mathbf{u}^T \mathbf{f}$$

For conservative system, we state that $U = W$

Stiffness Matrix – Another Approach

which gives

$$\frac{1}{2} \mathbf{u}^T \left[\int_V (\mathbf{B}^T E \mathbf{B}) dV \right] \mathbf{u} = \frac{1}{2} \mathbf{u}^T \mathbf{f} \quad \Rightarrow \quad \left[\int_V (\mathbf{B}^T E \mathbf{B}) dV \right] \mathbf{u} = \mathbf{f}$$

or $\mathbf{k} \mathbf{u} = \mathbf{f}$ where $\mathbf{k} = \int_V (\mathbf{B}^T E \mathbf{B}) dV$ (**)

element stiffness matrix

Expression (**) is a general result which can be used for the construction of other types of elements. This expression can also be derived using other more rigorous approaches, such as the *Principle of Minimum Potential Energy*, or the *Galerkin's Method*.

Stiffness Matrix – Another Approach

Now, we evaluate stiffness for the bar element

$$\mathbf{k} = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} E [-1/L \quad 1/L] A dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which is the same as we derived using the direct method.

Note that, the strain energy in the element can be written as

$$U = \frac{1}{2} \mathbf{u}^T \mathbf{k} \mathbf{u}$$