Finite Element Method

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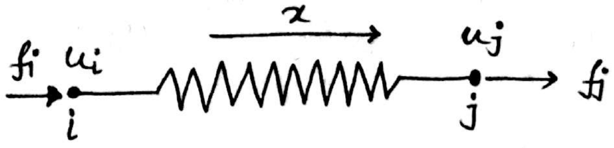
Chapter 1
Spring Element

Sudipta Dey Tirtha

One spring Element:



Two nodes : i and j

Nodal displacement: u_i and u_j (in, mm, m)

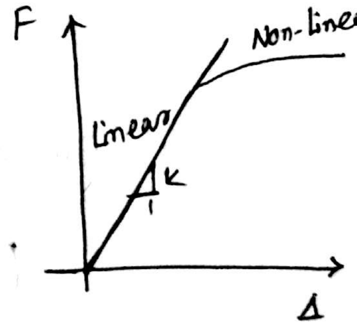
Nodal Force : f_i and f_j (lb, Newton)

Spring const. : k (lb/in, N/mm, N/m)

Now, Spring force-displacement relationship:

$F = k\Delta$ Here, Displacement, $\Delta = u_j - u_i$

Now, Spring const., $k = \frac{F}{\Delta}$ is the force required for a unit stretch.



Now,

$$f_i = -F = -k(u_j - u_i) = ku_i - ku_j$$

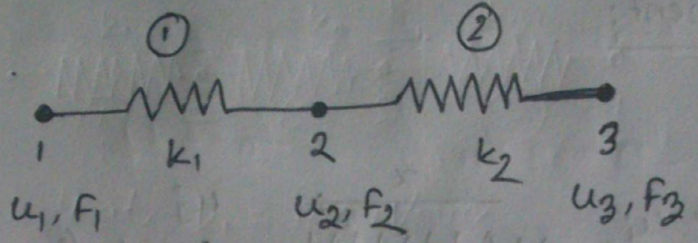
$$f_j = F = k(u_j - u_i) = -ku_i + ku_j$$

$$\therefore \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \times \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

So, $\underline{\underline{k}}\,\underline{\underline{u}} = \underline{\underline{f}}$

$\underline{\underline{k}}$ = (Element) stiffness Matrix.
$\underline{\underline{u}}$ = (Element nodal) displacement Matrix.
$\underline{\underline{f}}$ = (Nodal) force matrix.

Spring System:



For element 1,

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix}$$

For element 2,

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \times \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 \\ f_2^2 \end{Bmatrix}$$

NOW, $F_1 = f_1^1$

$F_2 = f_2^1 + f_1^2$

$F_3 = f_2^2$

By super-positioning,

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Boundary and Load condition:

Assuming, $u_1 = 0$, $F_2 = F_3 = P$

Now,
$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \times \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ P \end{Bmatrix}$$

From above relationship,

$$F_1 = -k_1 u_2$$

And,
$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \times \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ P \end{Bmatrix}$$

So,
$$k_1 u_2 + k_2 u_2 - k_2 u_3 = P \longrightarrow (1)$$
$$-k_2 u_2 + k_2 u_3 = P \longrightarrow (2)$$

(1) + (2) $\Rightarrow$
$$k_1 u_2 + k_2 u_2 - k_2 u_2 = 2P$$
$$\therefore u_2 = \frac{2P}{k_1}$$

From (2),

$$-k_2 \times \frac{2P}{k_1} + k_2 u_3 = P$$
$$\Rightarrow -\frac{2P}{k_1} + u_3 = \frac{P}{k_2}$$
$$\therefore u_3 = \frac{P}{k_2} + \frac{2P}{k_1}$$

$$\therefore F_1 = -k_1 \times \frac{2P}{k_1} = -2P$$

$$\therefore \boxed{F_1 = -2P}$$

$$\underline{\underline{U}} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{2P}{k_1} \\ \frac{2P}{k_1} + \frac{P}{k_2} \end{Bmatrix}.$$

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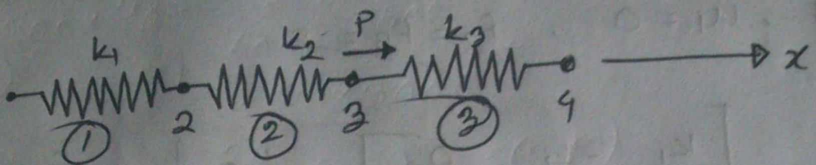
Example 1.1:



Given that, $k_1 = 100$ N/mm

$k_2 = 200$ N/mm

$k_3 = 100$ N/mm

$P = 500$ N

$u_1 = u_4 = 0$

find:

a. Global Stiffness Matrix: $[k] = ?$

b. Displacement at node 2 & 3, $u_2 = ?$ and $u_3 = ?$

c. Reaction forces at nodes 1 and 4, $f_1 = ?$, $f_2 = ?$

d. The force in spring ~~the~~ 2, $F = ?$

Solution:

(a) For element 1,

$$\begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix}$$

For element 2,

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \times \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 \\ f_2^2 \end{Bmatrix}$$

For element 3,

$$\begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \times \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1^3 \\ f_2^3 \end{Bmatrix}$$

Now,

$F_1 = f_1^1 = \cancel{0}$

$F_2 = f_2^1 + f_1^2 = 0$

$F_3 = f_2^2 + f_1^3 = 0 + P = P$

$F_4 = f_2^3 = \cancel{0}$

$$\begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 200 & -200 & 0 \\ 0 & -200 & 200 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \\ 0 & 0 & -100 \end{bmatrix}$$

By super-positioning,

$$\begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ P \\ F_4 \end{Bmatrix}$$

$$\therefore \underline{\underline{k}} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}$$ (Ans).

(b) As $u_1 = 0$ and $u_4 = 0$

$$\begin{bmatrix} 300 & -200 \\ -200 & 300 \end{bmatrix} \times \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

So, $300u_2 - 200u_3 = 0 \longrightarrow (1)$

And $-200u_2 + 300u_3 = P \longrightarrow (2)$

$(1) \times 2 + (2) \times 3$

$$\begin{aligned} 600u_2 - 400u_3 &= 0 \\ -600u_2 + 900u_3 &= 3P \\ \hline 500u_3 &= 3P \end{aligned}$$

$\therefore u_3 = \frac{3}{500} P$ (mm)

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From ①,

$$300 \times u_2 - 200 \times \frac{3P}{500} = 0$$

$$\Rightarrow u_2 = \frac{600P}{500 \times 300}$$

$$\therefore u_2 = \frac{2P}{500} (mm) = 2(mm)$$

$$\therefore \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \frac{2P}{500} \\ 3P/500 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

(Ans.)

ⓒ

$$F_1 = 100 \times 0 - 100 \times u_2 + 0 \times u_3 + 0 \times u_4$$

$$= -100 u_2$$

$$= -100 \times \frac{2P}{500}$$

$$= -100 \times \frac{2 \times 500}{500}$$

$$= -200 \text{ N}$$

$$F_4 = -100 u_3$$

$$= -100 \times 3$$

$$= -300 \text{ N}$$

ⓓ

Force in spring 2, $F = k_2 \Delta$

$$= k_2(u_3 - u_2)$$

$$= 200 \times (3-2)$$

$$= 200 \text{ N}$$

(Ans)

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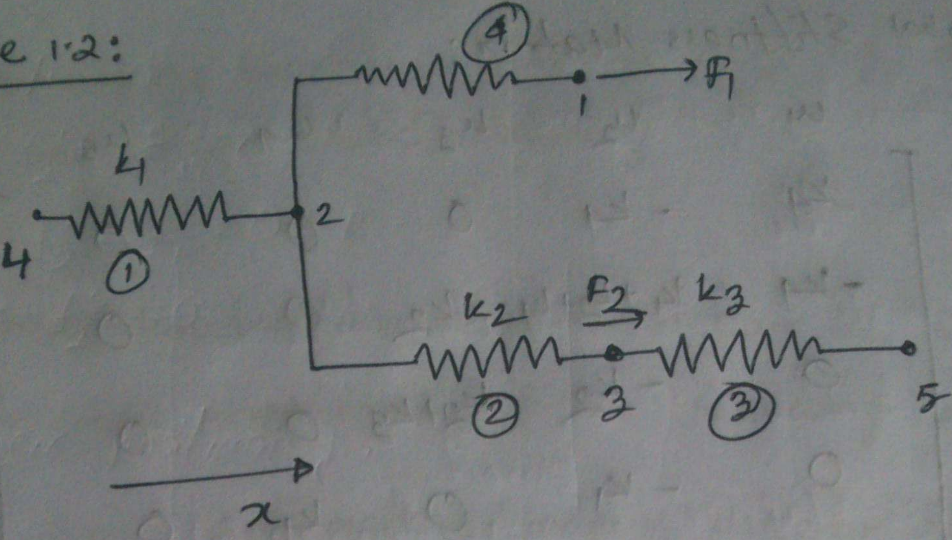
Example 1·2:



Find Global stiffness Matrix.

Solution:

Element connectivity table:

Element	Node i	Node j
1	4	2
2	2	3
3	3	5
4	2	1

For element 1, $\underline{\underline{k_1}} = \begin{array}{ccc} u_4 & u_2 & \\ \left[\begin{array}{cc} k_1 & -k_1 \\ -k_1 & k_1 \end{array}\right] & & \begin{array}{c} u_4 \\ u_2 \end{array} \end{array}$

For " 2, $\underline{\underline{k_2}} = \begin{array}{ccc} u_2 & u_3 & \\ \left[\begin{array}{cc} k_2 & -k_2 \\ -k_2 & k_2 \end{array}\right] & & \begin{array}{c} u_2 \\ u_3 \end{array} \end{array}$

For " 3, $\underline{\underline{k_3}} = \begin{array}{ccc} u_3 & u_5 & \\ \left[\begin{array}{cc} k_3 & -k_3 \\ -k_3 & k_3 \end{array}\right] & & \begin{array}{c} u_3 \\ u_5 \end{array} \end{array}$

" " 4, $k_4 = \begin{array}{ccc} u_2 & u_1 & \\ \left[\begin{array}{cc} k_4 & -k_4 \\ & k_4 \end{array}\right] & & \begin{array}{c} u_2 \\ u_1 \end{array} \end{array}$

So, Global Stiffness Matrix:

$$
\underline{\underline{k}} = \begin{array}{c}
\begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\
\begin{bmatrix}
k_1 & -k_1 & 0 & 0 & 0 \\
-k_1 & k_1+k_2+k_4 & -k_2 & 0 & 0 \\
0 & -k_2 & k_2+k_3 & 0 & 0 \\
0 & -k_4 & 0 & k_4 & 0 \\
0 & 0 & -k_3 & 0 & k_3
\end{bmatrix}
\end{array}
$$

Chapter 2
Bars & Beam Element

☐ What are the basic assumptions in analyzing a structure based on linear static analysis?

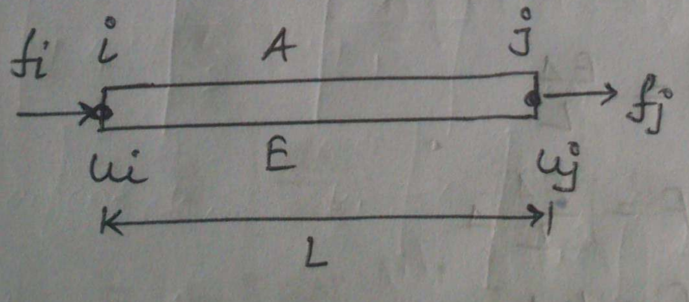
Ans:

Most structural analysis problem can be treated as linear static problem based on following assumptions:

1. Small deformation (Loading pattern does not change due to deformed shape).
2. Elastic material (No plasticity or failure).
3. Static Loads (The load is applied slowly or stea[dy] fashion).

☐ **Bar Element:**

Let, consider a prismatic bar element.



L ⟶ Length of bar element

A ⟶ x-sectional area of bar element

E ⟶ Modulus of elasticity

$u = u(x)$ displacement

$\varepsilon = \varepsilon(x)$ Strain

$\sigma = \sigma(x)$ Stress

Now,

Strain- displacement relationship:

$$\varepsilon = \frac{du}{dx}$$

Strain- stress relationship:

$$\sigma = E\varepsilon$$

Stiffness Matrix (~~Formal~~ Direct Approach):

$$u = \left(1 - \frac{x}{L}\right) u_i + \frac{x}{L} u_j$$

Now, $\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L}$

So, stress, $\sigma = E\varepsilon$

$$= \frac{E\Delta}{L}$$

Again, $\sigma = \frac{F}{A}$

So, $\frac{F}{A} = \frac{E\Delta}{L}$

$$\Rightarrow F = \frac{EA}{L}\Delta$$

$$\therefore F = k\Delta$$

k = stiffness of the element $= \frac{EA}{L}$

Similar to spring element:

$$\underset{=}{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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Element equilibrium equation:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

Stiffness Matrix (Formal Approach):

Let, consider two shape functions.

$N_i(\xi) = 1-\xi$ and $N_j(\xi) = \xi$

Here, $\xi = \frac{x}{L}$

And, $u = N_i(\xi) u_i + N_j(\xi) u_j$

$$\therefore \underline{u} = [N_i \quad N_j] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \underline{N} \, \underline{u}$$

Now, $\varepsilon = \frac{du}{dx}$

$$= \begin{bmatrix} \frac{dN_i}{dx} & \frac{dN_j}{dx} \end{bmatrix} \underline{u}$$

$$= \begin{bmatrix} \frac{dN_i}{d\xi} & \frac{dN_j}{d\xi} \end{bmatrix} \frac{d\xi}{dx} \underline{u}$$

$$= \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \underline{u}$$

Now, strain-displacement vector, $\underline{B} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$

$$\therefore \varepsilon = \underline{B} \, \underline{u}$$

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So, $\sigma = E\varepsilon$

$= E \underline{\underline{B}}\, \underline{\underline{u}}$

Now, Strain energy,

$$U = \frac{1}{2}\int_V \sigma^T \varepsilon \, dv$$

$$= \frac{1}{2}\int_V E \underline{\underline{B}}^T \underline{\underline{u}}^T \underline{\underline{B}}\, \underline{\underline{u}} \, dv$$

$$= \frac{1}{2} \underline{\underline{u}}^T \times \left[\int_V E B^T B \, dv \right] \underline{\underline{u}}$$

Again, Work done by two nodal forces,

$$W = \frac{1}{2} f_i u_i + \frac{1}{2} f_j u_j$$

$$= \frac{1}{2} \underline{\underline{u}}^T \underline{\underline{f}}$$

So, $U = W$

$$\Rightarrow \left[\int_V B^T E B \, dv \right] \underline{\underline{u}} = \underline{\underline{f}}$$

$$\therefore \underline{\underline{k}}\, \underline{\underline{u}} = \underline{\underline{f}}$$

So, $$\underline{\underline{k}} = \int_V B^T E B \, dv$$

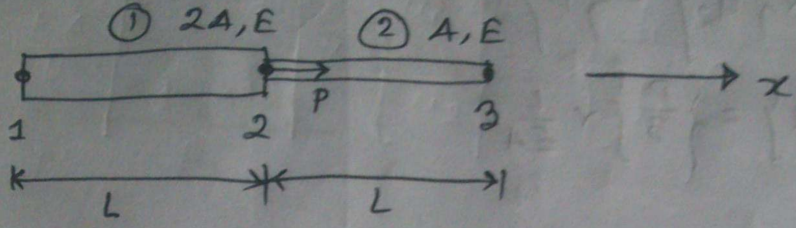
$$= \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \times E \times \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} A\,dx$$

$$= EA \times \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \times L$$

$$= \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore \boxed{U = \frac{1}{2} \underline{\underline{u}}^T \underline{\underline{k}}\, \underline{\underline{u}}}$$

Example 2.1:



① Find the stresses in two linear bar and ② constrained at the two ends.

Solution:

for element ①, $k_1 = \frac{2AE}{L} \begin{array}{cc} u_1 & u_2 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

For element ②, $k_2 = \frac{AE}{L} \begin{array}{cc} u_2 & u_3 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

∴ Global stiffness matrix:

$$K = \frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

NOW,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} * \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} * \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

$$So, \frac{EA}{L}[3] \times \{u_2\} = \{P\}$$

$$\therefore \{u_2\} = \left\{\frac{P}{3}\right\} \times \frac{L}{EA}$$

Now,

Stress at element 1, $\sigma_1 = \cancel{k_1 \left(\frac{u_2 - u_1}{L}\right)}$ $F/\cancel{A}$

$$\cancel{= \frac{2AE}{L}\left(\frac{P}{3}\right) \times \frac{L}{AE} \times \frac{1}{L}}$$

$$= k_1 (u_2 - u_1) \frac{1}{2A}$$

$$= \frac{2AE}{L}\left(\frac{P}{3AE}\right) \times \frac{1}{2A}$$

$$= \frac{P}{3A}$$

" " " 2, $\sigma_2 = \frac{F_2}{A}$

$$= k_1 (u_3 - u_2) \times \frac{1}{A}$$

$$= \frac{AE}{L}\left(0 - \frac{P}{3AE}\right) \times \frac{1}{A}$$

$$= -\frac{P}{3A}$$

Or,

$$\sigma_1 = \cancel{\ell}\, E\varepsilon = E\underline{\underline{B}}\,\underline{\underline{u}}$$

$$= E\left[-\frac{1}{L} \quad \frac{1}{L}\right] \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= E\,\frac{-u_1 + u_2}{L}$$

$$= E \times \frac{u_2}{L}$$

$$= E \times \frac{PL}{3AE} \times \frac{1}{L}$$

$$= \frac{P}{3A}$$ Ans.

$$\sigma_2 = E \underline{\underline{B}}\, \underline{u}$$

$$= E \times \left[-\frac{1}{L} \quad \frac{1}{L} \right] \times \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= E \times \frac{-u_2 + u_3}{L}$$

$$= -E \frac{u_2}{L}$$

$$= -\frac{P}{3A}$$ (Ans.)

NOW, $\{F_1\} = \frac{EA}{L} \{-2\} \times \{u_2\}$

$$= \frac{EA}{L} \left\{ -\frac{2PL}{3EA} \right\}$$

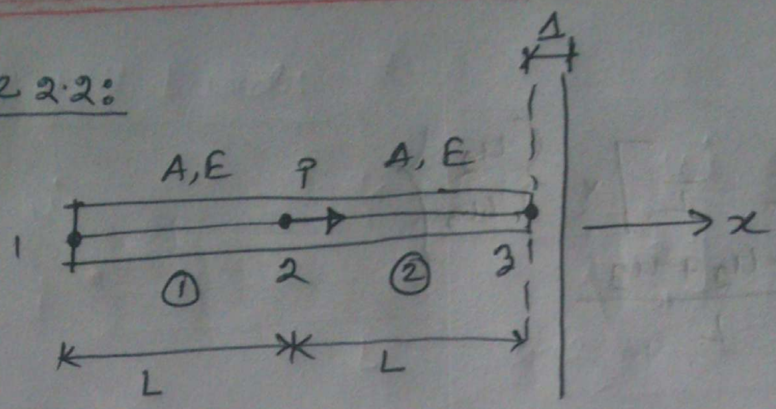
$$= \left\{ -\frac{2P}{3} \right\}$$ (Ans.)

$$\{F_2\} = \frac{EA}{L} \{-1\} \times \{u_2\}$$

$$= \frac{EA}{L} \left\{ -\frac{PL}{3EA} \right\}$$

$$= \left\{ -\frac{P}{3} \right\}$$ (Ans.)

Example 2.2:



Determine support Reaction.

Solution:

Given that, $P = 6.0 \times 10^4 \ N$

$E = 2.0 \times 10^4 \ N/mm^2$

$A = 250 \ mm^2$

$L = 150 \ mm$

$\Delta = 1.20 \ mm$

Check wheather contact occurs or not.

Now, $\Delta = \frac{PL}{AE} = \frac{6.0 \times 10^4 \times \cancel{2 \times 10^4}\ 150}{250 \times 2.0 \times 10^4}$

$= 1.80 \ mm > 1.20 \ mm$

Thus, contact occurs. So, $u_3 = 1.20 \ mm$ and

Now, $k_1 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$k_2 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\therefore \underline{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

So, $$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{Bmatrix} 0 \\ u_2 \\ 1.20 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

So, $$\frac{EA}{L}\begin{bmatrix} 2 & -1 \end{bmatrix} \times \begin{Bmatrix} u_2 \\ 1.20 \end{Bmatrix} = \{P\}$$

So, $$\frac{2.0\times10^4\times250}{150}\times(2u_2 - 1.20) = 6.0\times10^4$$

$$\Rightarrow 66.67\times10^3\, u_2 - 40000 = 6.0\times10^4$$

$$\Rightarrow \boxed{u_2 = 1.50\text{ mm}}$$

$$\therefore F_1 = -u_2 \times \frac{EA}{L} = -1.50\times\frac{2.0\times10^4\times250}{150}$$

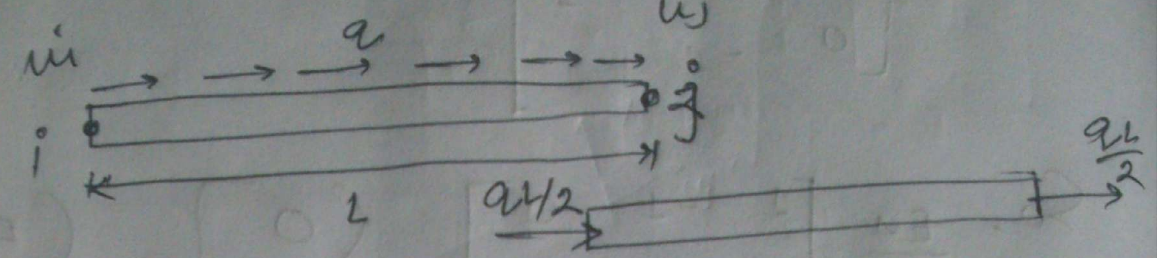
$$= -5\times10^4\text{ N} \quad \text{(Ans.)}$$

$$F_3 = \begin{bmatrix} -1 & 1 \end{bmatrix}\times\frac{EA}{L}\times\begin{Bmatrix} 1.50 \\ 1.20 \end{Bmatrix}$$

$$= \cancel{66}\ 33.33\times10^3\times(-1.50+1.20)$$

$$= -1\times10^4\text{ N} \quad \text{(Ans.)}$$

Distributed Load:



If an uniform load q (lb/in, N/m, N/mm) acts on a bar element, we can convert it to $\frac{qL}{2}$ nodal forces. This can be proved by the work done by the load, q.

Now, $W_q = \int_0^L \frac{1}{2} u q \, dx$

$$= \int_0^L [N_i \quad N_j] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \times q \times (L\, d\xi)$$

$$= \frac{qL}{2} \int_0^1 [1-\xi \quad \xi] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} d\xi$$

$$= \frac{1}{2} \boxed{\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}} \times \begin{bmatrix} \frac{qL}{2} & \frac{qL}{2} \end{bmatrix} \times \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \frac{1}{2} \{u_i \quad u_j\} \times \begin{bmatrix} \frac{qL}{2} \\ \frac{qL}{2} \end{bmatrix}$$

$$= \frac{1}{2} \underline{u}^T \underline{f}_q \qquad \text{Here, } f_q = \begin{bmatrix} qL/2 \\ qL/2 \end{bmatrix}$$

Now, $U = W$

$$\Rightarrow \frac{1}{2} \underline{u}^T \underline{K}\, \underline{u} = \frac{1}{2} \underline{u}^T \underline{f} + \frac{1}{2} \underline{u}^T \underline{f}_q$$

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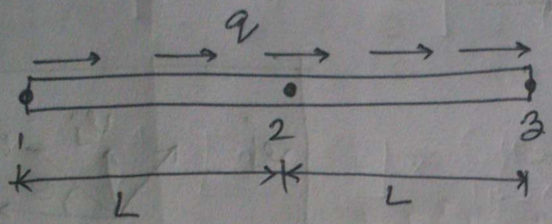
So, $\underline{\underline{K}}\,\underline{\underline{u}} = \underline{\underline{f}} + \underline{\underline{f_q}}$

So, New Nodal force vector $= \begin{Bmatrix} f_i + \frac{qL}{2} \\ f_j + \frac{qL}{2} \end{Bmatrix}$

In assembly of bar,



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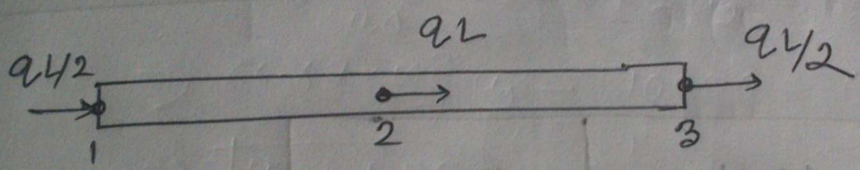
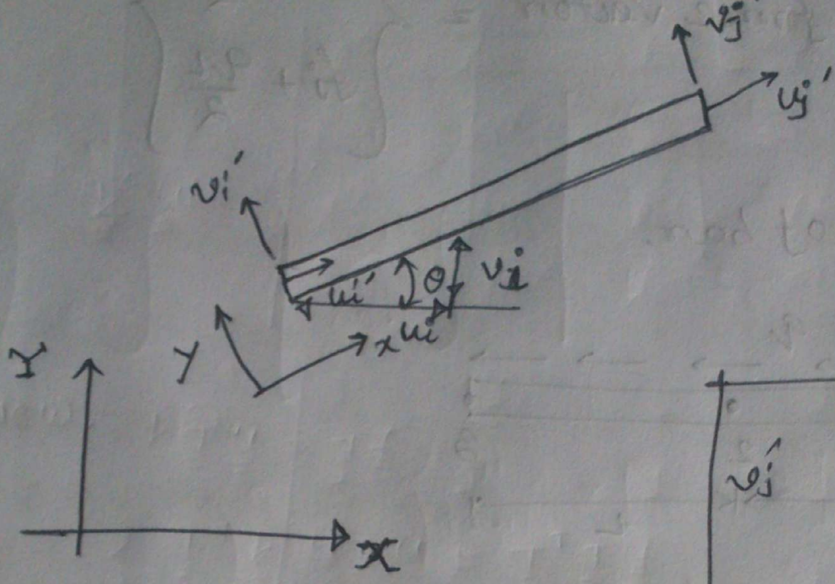
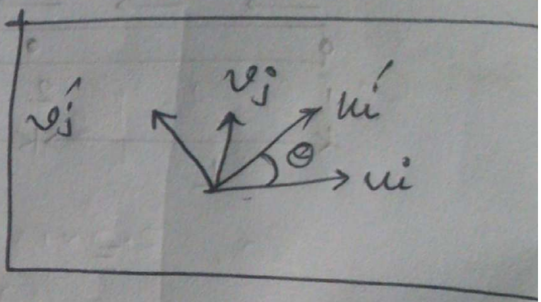


Bar Elements in 2D and 3-D Space:





Local	Global
x, y	X, Y
u_i', v_i'	u_i, v_i
1 dof at a node	2 dof at a node

Now,

$$u_i' = u_i\cos\theta + v_i\sin\theta = l u_i + m v_i$$

$$v_i' = -u_i\sin\theta + v_i\cos\theta = -m u_i + l v_i$$

$$\therefore \begin{Bmatrix} u_i' \\ v_i' \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$\therefore u_i' = \tilde{T} u$$

Here, $\tilde{\underline{T}} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix}$

For Both nodes,

$$\begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \times \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

Here, $\underline{T} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}$

$\therefore \quad \underline{u}' = \underline{T}\,\underline{u}$

Similarly, for nodal force,

$f' = \underline{T}\,\underline{f}$

<u>Stiffness Matrix for 2D space:</u>

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix} = \begin{Bmatrix} f_i' \\ f_j' \end{Bmatrix}$$

$$\Rightarrow \frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \times \begin{Bmatrix}u_i'\\ v_i'\\ u_j'\\ v_j'\end{Bmatrix} = \begin{Bmatrix}f_i'\\ 0\\ f_j'\\ 0\end{Bmatrix}$$

$$\Rightarrow \quad \underline{\underline{k}}'\,\underline{\underline{u}}' = \underline{\underline{f}}'$$

NOW, $\underline{\underline{k}}'\,\underline{\underline{T}}\,\underline{\underline{u}} = \underline{\underline{T}}\,\underline{\underline{f}}$

$$\Rightarrow \underline{\underline{T}}^T\,\underline{\underline{k}}'\,\underline{\underline{T}}\,\underline{\underline{u}} = \underline{\underline{T}}^T\,\underline{\underline{T}}\,\underline{\underline{f}}$$

$$\Rightarrow (\underline{\underline{T}}^T\,\underline{\underline{k}}'\,\underline{\underline{T}})\,\underline{\underline{u}} = \underline{\underline{f}}$$

So,

$$\underline{\underline{k}} = \underline{\underline{T}}^T\,\underline{\underline{k}}'\,\underline{\underline{T}}$$

So,

$$\underline{\underline{k}} = \frac{EA}{L}\begin{bmatrix}l^2 & lm & -l^2 & -lm\\ lm & m^2 & -lm & -m^2\\ -l^2 & -lm & l^2 & lm\\ -lm & -m^2 & lm & m^2\end{bmatrix}$$

Here,

$$l = \frac{x_j - x_i}{L}$$

$$m = \frac{Y_j - Y_i}{L}$$

NOW, Stress, $\sigma = E\varepsilon = E\underline{\underline{B}}\,\underline{\underline{u}}$

$$= E\left[-\frac{1}{L}\ \ \frac{1}{L}\right] \times \begin{Bmatrix}u_i'\\ u_j'\end{Bmatrix}$$

$$= E\left[-\frac{1}{L} \quad \frac{1}{L}\right] \times \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \times \begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix}$$

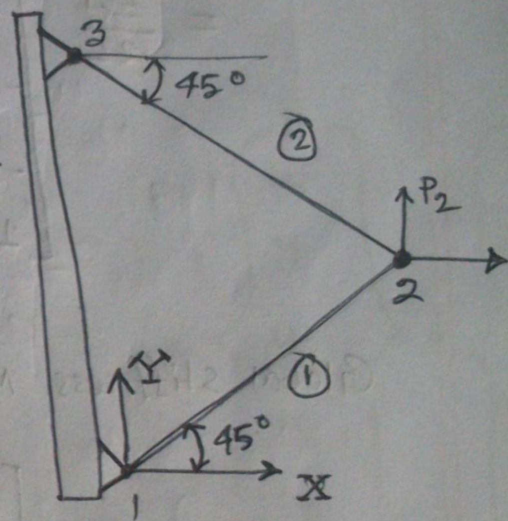
$$\sigma = \frac{E}{L}\left[-l \quad -m \quad l \quad m\right] \begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix}$$

(Ans.)

Example 2.3:

A simple plane truss is made of two identical bars (with E, A and L), and loaded as shown in the figure. Find

1) displacement of node 2;
2) Stress in each bar.



Solution:

$\boxed{1}$

$$k_1' = k_2' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \frac{EA}{L}$$

For element ①, $l = \cos\theta = \cos 45° = \frac{1}{\sqrt{2}}$

$m = \sin\theta = \sin 45° = \frac{1}{\sqrt{2}}$

$$\therefore \underline{k}_1 = \frac{EA}{2L} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

For element ②,

$$l = \cos\theta = \cos 135^\circ = -\frac{1}{\sqrt{2}}$$

$$m = \sin\theta = \sin 135^\circ = \frac{1}{\sqrt{2}}$$

$$\underline{\underline{k}}_2 = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{EA}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Global stiffness Matrix

$$\underline{\underline{k}} = \frac{EA}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 & -1 & 1 \\ \cancel{2}\,{}^{-1} & -1 & & & & \end{bmatrix}$$

Global stiffness Matrix:

$$\underline{\underline{k}} = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Now,

$$\frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1Y} \\ P_1 \\ P_2 \\ F_{3x} \\ F_{3Y} \end{Bmatrix}$$

$$\Rightarrow \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1Y} \\ P_1 \\ P_2 \\ F_{3x} \\ F_{3Y} \end{Bmatrix}$$

So,

$$\frac{EA}{2L}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Now, $\frac{EA}{2L}(2u_2 - 0) = P_1$

$\therefore u_2 = \frac{P_1 L}{EA}$

And, $\frac{EA}{2L}(0 - 2v_2) = P_2$

$\therefore v_2 = \frac{P_2 L}{EA}$

$$\therefore \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} \frac{P_1 L}{EA} \\ \frac{P_2 L}{EA} \end{Bmatrix} = \frac{L}{EA}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$ (Ans.)

$\boxed{2}$

Stress at bar 1,

$$\sigma_1 = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= E/L \times \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ P_1 L/EA \\ P_2 L/EA \end{Bmatrix}$$

$$= \frac{E}{\sqrt{2}L} \times \frac{L}{EA} \times \begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ P_1 \\ P_2 \end{Bmatrix}$$

$$= \frac{1}{\sqrt{2}A} \times (0 + 0 + P_1 + P_2)$$

$$= \frac{1}{\sqrt{2}A}(P_1 + P_2)$$ (Ans.)

Stress at bar 2,

$$\sigma_2 = E/L \times \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{Bmatrix} \frac{P_1 L}{EA} \\ \frac{P_2 L}{EA} \\ 0 \\ 0 \end{Bmatrix}$$

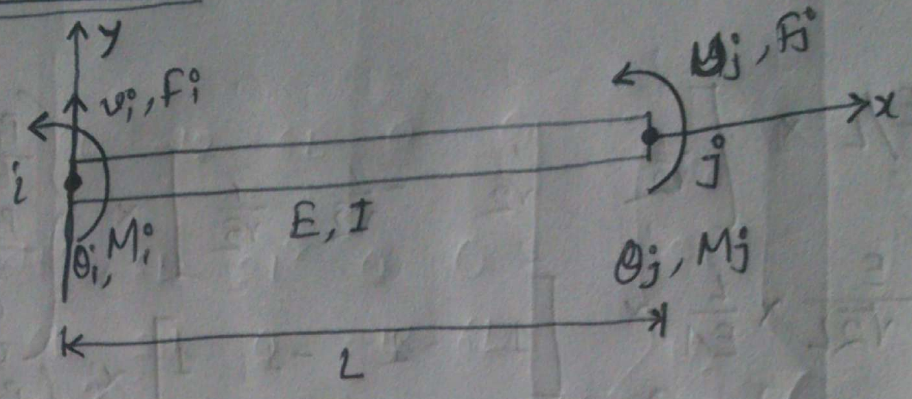
$$= \frac{E}{\sqrt{2}L} \times \frac{L}{EA} \times \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \times \begin{Bmatrix} P_1 \\ P_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \frac{1}{\sqrt{2}A} \times \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \times \begin{Bmatrix} P_1 \\ P_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \frac{1}{\sqrt{2}A} (P_1 - P_2 - 0 + 0)$$

$$= \frac{1}{\sqrt{2}A} (P_1 - P_2)$$ Ans.)

Beam Element:



L → Length

E → Modulus of Elasticity

I → Moment of Inertia

$v = v(x)$ deflection of the neutral axis.

$F = F(x)$ Shear force

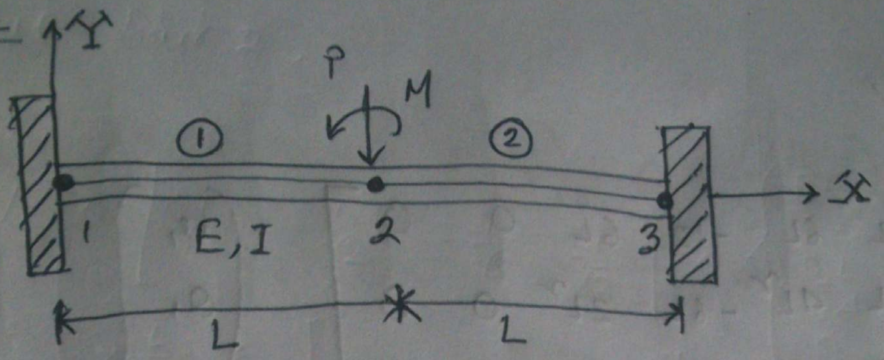
$M = M(x)$ Moment about z-axis.

Dirrect Method:

Element stiffness Matrix:

$$k = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_i & \theta_i & v_j & \theta_j \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Example 2.5:



Find: the deflection and rotation at the center node and the reaction forces and moments at the two nodes.

Solution:

$$\underline{k}_1 = \underline{k}_2 = \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \times \frac{EI}{L^3}$$

Global stiffness Matrix:

$$\underline{k} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccccc} u_1 & \theta_1 & u_2 & \theta_2 & u_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Now,

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & 6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \times \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ M_{1x} \\ -P \\ M \\ F_3 \\ M_3 \end{Bmatrix}$$

Now, $v_1 = v_3 = 0$ and $\theta_1 = \theta_3 = 0$

So, $$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \times \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ M \end{Bmatrix}$$

Now, $$\frac{EI}{L^3}\left(24v_2 + 0\right) = -P$$

$$\therefore \quad v_2 = -\frac{PL^3}{24EI}$$

And $$\frac{EI}{L^3}\left(0 + 8L^2\theta_2\right) = M$$

$$\Rightarrow \quad \theta_2 = +\frac{ML}{8EI}$$

$$\therefore \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{24EI}\begin{Bmatrix} -PL^2 \\ 3M \end{Bmatrix}$$

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From ①,

$$F_1 Y = (12 v_2 + 6L\theta_2) \times EI/L^3 = \left[-12 \times \left(-\frac{PL^3}{24EI}\right) + 6L \times \left(\frac{ML}{8EI}\right)\right] \times \frac{EI}{L^3}$$

$$= \left(\frac{PL^3}{2EI} + \frac{3ML^2}{4EI}\right) \times \frac{EI}{L^3}$$

$$= \frac{P}{2} + \frac{3M}{4L}$$

$$M_1 = \frac{EI}{L^3}\left[-6L\left(-\frac{PL^3}{24EI}\right) + 2L^2\left(\frac{ML}{8EI}\right)\right]$$

$$= \frac{EI}{L^3}\left[\frac{PL^4}{\cancel{24}\,4EI} + \frac{ML^3}{4EI}\right]$$

$$= \frac{PL}{4} + \frac{M}{4}$$

$$F_3 = \frac{EI}{L^3}\left[-12\left(-\frac{PL^3}{24EI}\right) - 6L\left(\frac{ML}{8EI}\right)\right]$$

$$= \frac{EI}{L^3}\left[\frac{PL^3}{2EI} - \frac{3ML^2}{4EI}\right]$$

$$= \frac{P}{2} - \frac{3M}{4L}$$

$$M_3 = \frac{EI}{L^3}\left[6L\left(\frac{-PL^3}{24EI}\right) + 2L^2\left(\frac{ML}{8EI}\right)\right]$$

$$= \frac{EI}{L^3}\left[-\frac{PL^4}{4EI} + \frac{ML^3}{4EI}\right]$$

$$= -\frac{PL}{4} + \frac{M}{4}$$

$$\therefore \begin{Bmatrix} F_1 \\ M_1 \\ F_3 \\ M_3 \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} \left(2P + \frac{3M}{L}\right) \\ (PL + M) \\ \left(2P - \frac{3M}{L}\right) \\ (-PL + M) \end{Bmatrix}$$

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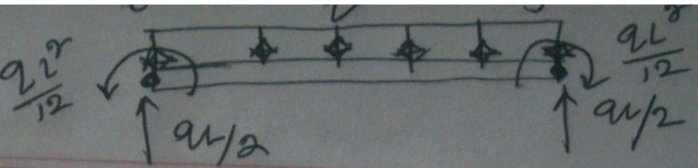
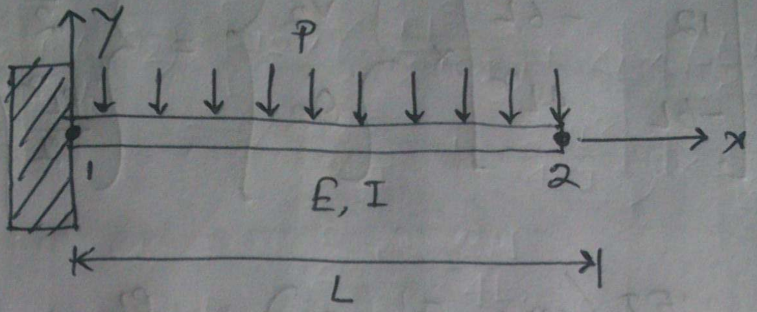


New Nodal force vector

$$= \begin{Bmatrix} f_i + qL/2 \\ M_i + qL^2/12 \\ f_j + qL/2 \\ M_j + qL^2/12 \end{Bmatrix}$$

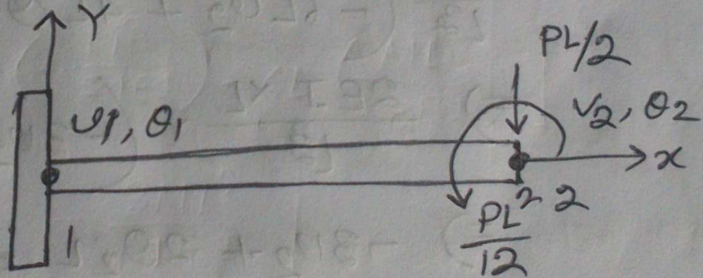
Example 2.6:



Given that, A cantilever beam with distributed lateral load p as shown above.

Find: The deflection and rotation at the end, the reaction force and moment at the end.

Solution:



$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\text{So, } \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ -\frac{PL}{2} \\ PL^2/12 \end{Bmatrix}$$

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Now,

$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} \times \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{PL}{2} \\ \frac{PL^2}{12} \end{Bmatrix}$$

Now,

$$\frac{EI}{L^3}(12v_2 - 6L\theta_2) = -\frac{PL}{2}$$

$$\Rightarrow \frac{6EI}{L^3}(2v_2 - \theta_2 L) = -\frac{PL}{2}$$

$$\Rightarrow 2v_2 - \theta_2 L = -\frac{PL}{2} \times \frac{L^3}{6EI}$$

$$\Rightarrow 2v_2 - \theta_2 L = -\frac{PL^4}{12EI} \longrightarrow (1)$$

And

$$\frac{EI}{L^3}(-6Lv_2 + 4L^2\theta_2) = \frac{PL^2}{12}$$

$$\Rightarrow \frac{2EI \times L}{L^3} \times (-3v_2 + 2L\theta_2) = \frac{PL^2}{12}$$

$$\Rightarrow -3v_2 + 2\theta_2 L = \frac{PL^2}{12} \times \frac{L^2}{2EI}$$

$$\therefore -3v_2 + 2\theta_2 L = \frac{PL^4}{24EI} \longrightarrow (2)$$

$(1) \times 2 + (2) \Rightarrow$

$$4v_2 - 2\theta_2 L = -\frac{PL^4}{6EI}$$

$$-3v_2 + 2\theta_2 L = \frac{PL^4}{24EI}$$

$(+)$, $$v_2 = -\frac{PL^4}{6EI} + \frac{PL^4}{24EI}$$

$$= \frac{PL^4(-4+1)}{24EI}$$

$$= -\frac{PL^4}{8EI}$$

from ①,

$$-\frac{PL^4}{4EI} - \theta_2 L = -\frac{PL^4}{12EI}$$

$$\Rightarrow \theta_2 L = \frac{PL^4}{12EI} - \frac{PL^4}{4EI}$$

$$\Rightarrow \theta_2 L = \frac{PL^4(1-3)}{12EI}$$

$$\Rightarrow \theta_2 L = -\frac{PL^4}{6EI}$$

$$\therefore \theta_2 = -\frac{PL^3}{6EI}$$

$$\therefore \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{PL^4}{8EI} \\ -\frac{PL^3}{6EI} \end{Bmatrix}$$ (Ans.)

$-12v_2 + 6L\theta_2$

$= -12 \times \left(-\frac{PL^4}{8EI}\right) + 6L \times \left(\frac{-PL^3}{6EI}\right)$

$= \frac{3PL^4}{2EI} - \frac{PL^4}{EI}$

$= \frac{(3-2)PL^4}{2EI} \times \frac{EI}{L^3}$

$= \frac{P\cancel{L^4}}{2\cancel{EI}} \times \frac{\cancel{EI}}{\cancel{L^3}} = \frac{PL}{2}$

$-\cancel{6}L \times \left(-\frac{PL^4}{\cancel{8}EI}\right) + 2L^2\left(-\frac{PL^3}{6EI}\right)$

$= \frac{3PL^5}{4EI} - \frac{PL^5}{3EI}$

$= \frac{(9-4)PL^5}{12EI}$

$= \frac{5P\cancel{L^5}}{12\cancel{EI}} \times \frac{\cancel{EI}}{\cancel{L^3}}$

$= \frac{5PL^2}{12}$

NOW,

$$\begin{Bmatrix} F_1 \\ M_1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} -12 & 6L \\ -6L & 2L^2 \end{bmatrix} \times \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{PL}{2} \\ \frac{5PL^2}{12} \end{Bmatrix}$$ (Ans)

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Chap1 Introduction

☐ Describe the concept of FEM brietly.

Ans: In civil engineering problems, there are some basic unknowns. When these unknowns are solved, the characteristics of entire structure can be pridected. These basic unknowns or field variables are displacement in solid mechanics, velocity in fluid mechanics etc.

In the continuum, the number of the field variabl is infinite. Finite element method divides a structur into a finite number of elements and field variable are expressed in form of an assumed approximate functions. These functions are defined in terms of field variable of nodal points.

When elements and nodal variables are defi it is necessary to know the element properties an assemble it. For example, force-displacement mat is a property of an element.

$$[k]_e \{\delta\}_e = \{F\}_e$$

Here, $[k]_e$ = stiffness matrix

$\{\delta\}_e$ = displacement "

$\{F\}$ = nodal force "

After that, element property matrix needs to be assembled to get global property matrix of the entire structure that is $[k]\{\delta\}=\{F\}$. Then, the boundary condition is imposed and system of equ^n is solved to get nodal variables known. Then, using nodal variables, other required values of stress, moment, strain etc. can be calculated.

So, the basic steps are:-

1. Select the elements and nodal variables.
2. Discritise the continua.
3. Select the interpolation function.
4. Define element properties.
5. Assemble global ~~elem~~ property vector.
6. Impose boundary condition.
7. Solve equations to get nodal unknown
8. Require values are calculated.

□ Clearly point out the advantages of FEM over classical method. / Compare FEM and classical method of analysis.

Ans:

1. Classical methods analyze a structure by exact equation and exact solution is obtained. In FEM, exact equation is used but solution is approximate.

2. Classical methods are applicable in case of some ideal situations. But FEM is applicable for all kind of problems.

3. Classical method makes drastic assumptions when the following difficulties are faced:-

i. Shape
ii. Loading
iii. Boundary condition.

FEM solves a problem taking it as it is.

4. When material property is not same over the whole structure, solution by classical method is difficult. But FEM can solve a structure of anisotropic material easily.

5. When different materials are used in a single structure, solution is difficult in classical method. But FEM can easily deal with that.

6. When geometric and material non-linearity is faced, FEM is preferable over classical method.

□ What are the advantages of FEM over FDM (Finite Difference Method)?

Ans: 1. FDM makes point-wise approximation. So, ... is continuity at nodal points but ~~not~~ no continuity ... edge of grid lines.

But FEM makes piece-wise approximation. So, there is a continuity of both nodal points ... sides of a structure.

2. Basic values like displacement etc. can ... be calculated at any point except nodal points. But in FEM, basic values can be calculated ... each and every points.

3. FDM makes strain type approximation ... sloping or curved boundaries as shown ... following figure. But FEM can easily def...

Sloped / curved boundaries

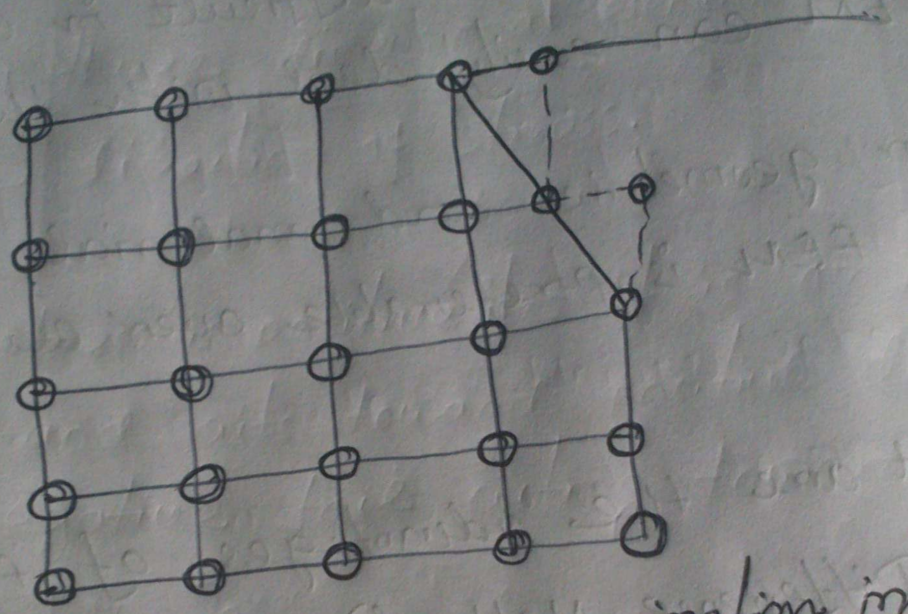


Fig.: Stair type approximation in FDM.

5. for same accuracy, FDM required greater number of nodes than that required by FEM.

6. FEM is more appropriate for complex structures than FDM.

□ Why FEM study is much important?

Ans: There are number of user frieldly FEM packages available in the market. So, one may ask "Why ~~E~~ will one study FEM."

The above arguement is not sound at all. knowledge of FEM makes good engineers but lack of knowledge may produce more dangerous ~~[illegible]~~ results. To use FEM packages properly, the user must know the followings:-

1. Which elements need to be used to solve the problem.

2. How to discritise the structure to get good result.

3. How to impose boundary condition properly.

4. How to define element property properly.

5. How is the display in pre and post proce[ssing] to know the limitation.

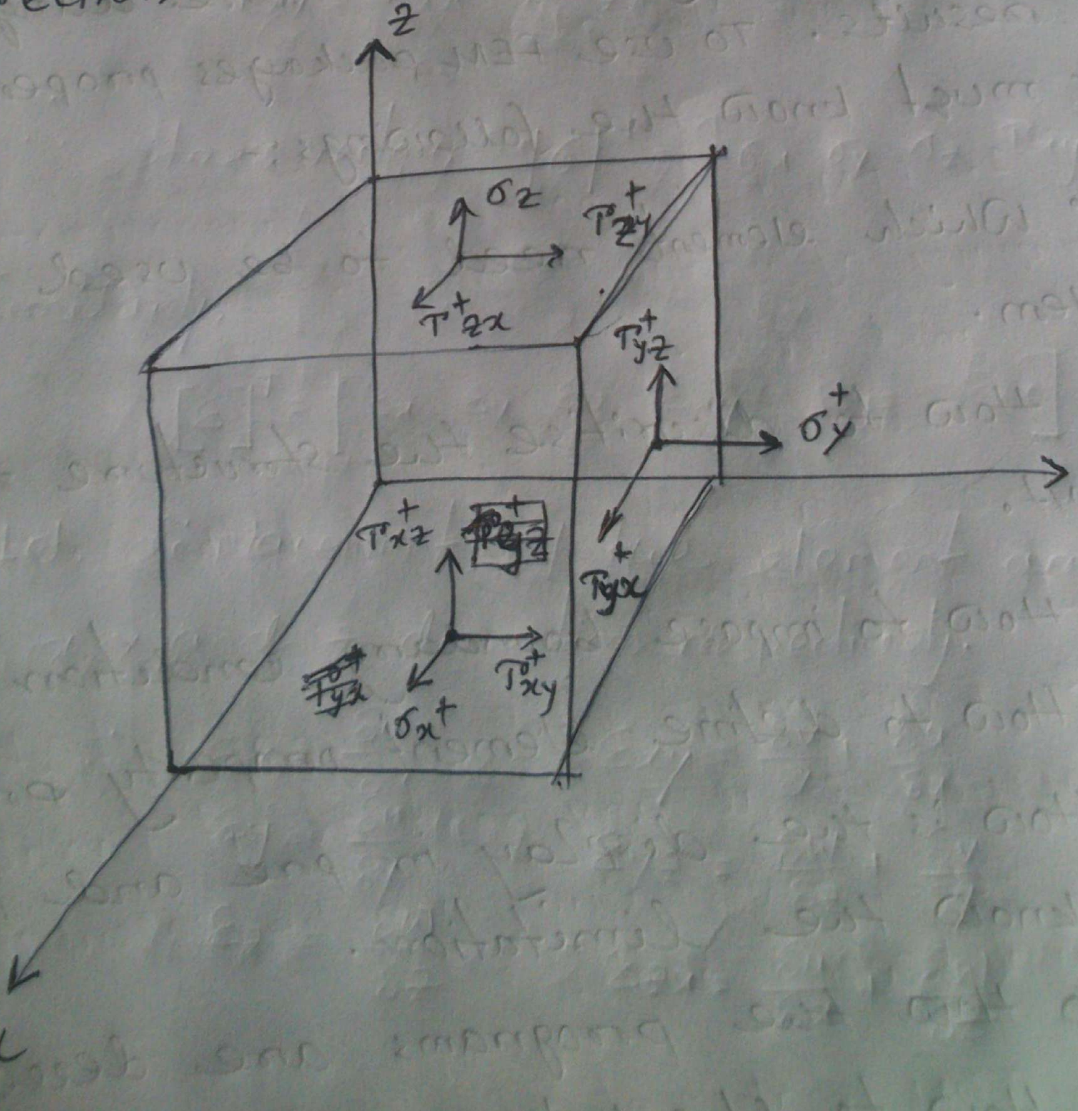
6. ~~To~~ How the programs are developed and How to check the FEM package by classical solutions.

Chapter 2

Basic Equations in Elasticity

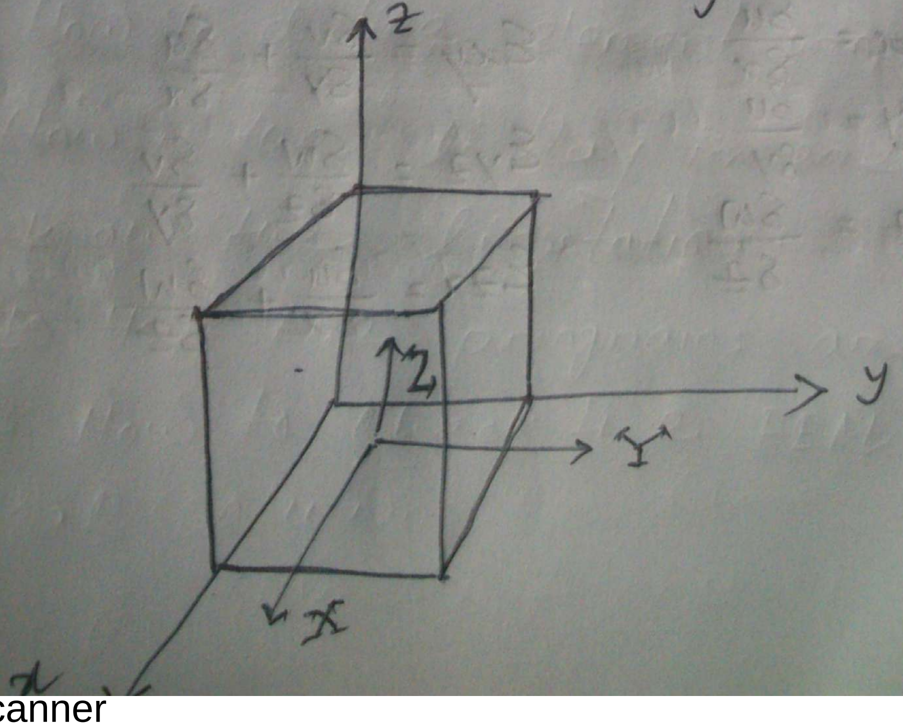
□ Draw a three dimensional element and indicate the positive direction of stress.

Ans: A stress is positive if it is on positive face of element and acts towards positive direction or it is on negative face acting towards negative direction.



face	Stress on -ve face	Stress on +ve face
x	σ_x	$\sigma_x + \frac{\delta \sigma_x}{\delta x} dx$
	τ_{xy}	$\tau_{xy} + \frac{\delta \tau_{xy}}{\delta x} dx$
	τ_{xz}	$\tau_{xz} + \frac{\delta \tau_{xz}}{\delta x} dx$
y	σ_y	$\sigma_y + \frac{\delta \sigma_y}{\delta y} dy$
	τ_{yx}	$\tau_{yx} + \frac{\delta \tau_{yx}}{\delta y} dy$
	τ_{yz}	$\tau_{yz} + \frac{\delta \tau_{yz}}{\delta y} dy$
z	σ_z	$\sigma_z + \frac{\delta \sigma_z}{\delta z} dz$
	τ_{zx}	$\tau_{zx} + \frac{\delta \tau_{zx}}{\delta z} dz$
	τ_{zy}	$\tau_{zy} + \frac{\delta \tau_{zy}}{\delta z} dz$

Let, intensity of body forces are X, Y and Z
and z direction respectively.



Now,

$$\frac{\delta \sigma_x}{\delta x} + \frac{\delta \tau_{xy}}{\delta x} + \frac{\delta \tau_{xz}}{\delta x} + X = 0$$

$$\frac{\delta \sigma_y}{\delta y} + \frac{\delta \tau_{yx}}{\delta y} + \frac{\delta \tau_{yz}}{\delta y} + Y = 0$$

$$\frac{\delta \sigma_z}{\delta z} + \frac{\delta \tau_{zy}}{\delta z} + \frac{\delta \tau_{zx}}{\delta z} + Z = 0$$

And, $\tau_{xy} = \tau_{yx}$, $\tau_{zx} = \tau_{xz}$ and $\tau_{yz} = \tau_{zy}$

$\therefore$ Stress vector, $|\sigma|^T = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}]$

Similarly,

$$[\varepsilon]^T = [\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \varepsilon_{xy} \ \varepsilon_{yz} \ \varepsilon_{zx}]$$

Let, displacements of the element are u, v, w in x, y and z direction respectively.

$\therefore$ $\varepsilon_x = \frac{\delta u}{\delta x}$ $\qquad$ $\varepsilon_{xy} = \frac{\delta v}{\delta y} + \frac{\delta u}{\delta x}$

$\varepsilon_y = \frac{\delta v}{\delta y}$ $\qquad$ $\varepsilon_{yz} = \frac{\delta w}{\delta z} + \frac{\delta v}{\delta y}$

$\varepsilon_z = \frac{\delta w}{\delta z}$ $\qquad$ $\varepsilon_{zx} = \frac{\delta u}{\delta x} + \frac{\delta w}{\delta z}$

$\sigma_x = D_{11}\varepsilon_x + D_{12}\varepsilon_y + D_{13}\varepsilon_z$

$\Rightarrow \frac{\sigma_x}{E_x} = \varepsilon_x + \frac{D_{12}}{E_x}\varepsilon_y$

+

□ Explain Anisotropic, Orthrotopic and Isotropic material.

Ans:

Constitutive law expresses relationship between stress and strain. Linear constitutive law defines a linear relationship between stress-strain and constant of proportionality is young modulus of elasticity. It is very well known as Hooke's law.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \times \begin{Bmatrix} e_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

D is a 6x6 matrix of constant of elasticity. If D is symmetric matrix, there are 21 material properties for Anisotropic material.

In some material, there is a symmetry with respect to faces within the element. These material is called orthotropic material with 9 material properties.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 & 0 \\ & & D_{33} & 0 & 0 & 0 \\ & & & D_{44} & 0 & 0 \\ & & & & D_{55} & 0 \\ & & & & & D_{66} \end{bmatrix} \times \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Now,

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\mu_{yx}\sigma_y}{E_y} - \frac{\mu_{zx}\sigma_z}{E_z}$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \mu_{xy}\frac{\sigma_x}{E_x} - \mu_{zy}\frac{\sigma_z}{E_z}$$

$$\varepsilon_z = \frac{\sigma_z}{E_z} - \mu_{xz}\frac{\sigma_x}{E_x} - \mu_{yz}\frac{\sigma_y}{E_y}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \quad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \quad \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}}$$

From above equations, there are 12 material properties.

As, $\frac{E_y}{\mu_{yx}} = \frac{E_x}{\mu_{xy}}$, $\frac{E_z}{\mu_{zx}} = \frac{E_x}{\mu_{xz}}$ and $\frac{E_z}{\mu_{zy}} = \frac{E_y}{\mu_{yz}}$

So, there are 9 material properties.

For Isotropic material, further simplification is made assuming that material properties are same in all directions.

$$E_x = E_y = E_z = E \quad \text{and} \quad \mu_{xy} = \mu_{yx} = \mu_{zx} = \mu_{xz} = \mu_{yz} = \mu_{zy} = \mu$$

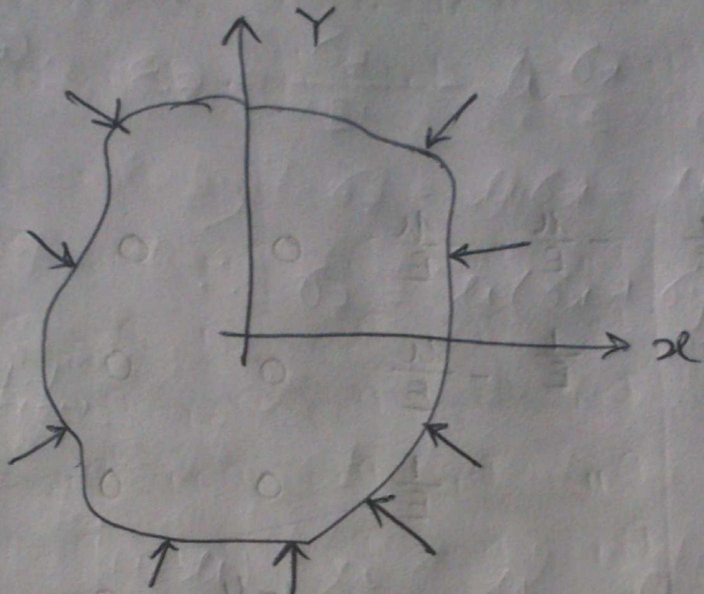
So,

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\mu}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{1-\mu}{2} & 0 & 0 \\ & & & & \frac{1-\mu}{2} & 0 \\ & & & & & \frac{1-\mu}{2} \end{bmatrix} \times \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

NOW,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ & (1-\mu) & \mu & 0 & 0 & 0 \\ & & (1-\mu) & 0 & 0 & 0 \\ & & & \frac{1-2\mu}{2} & 0 & 0 \\ & & & & \frac{1-2\mu}{2} & 0 \\ & & & & & \frac{1-2\mu}{2} \end{bmatrix} \frac{E}{(1-2\mu)(1+\mu)} \times \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Plane-stress problem: If intensity of force/stress acts only over the plane of a thin plate, it is called plane stress problem. So, there will be no stress towards z direction.



$\cancel{\varepsilon_x - \mu \varepsilon_y}$ σ_y

According to the definition,

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

As. $\tau_{zx} = \tau_{zy} = 0$, $\gamma_{zx} = \gamma_{yz} = 0$

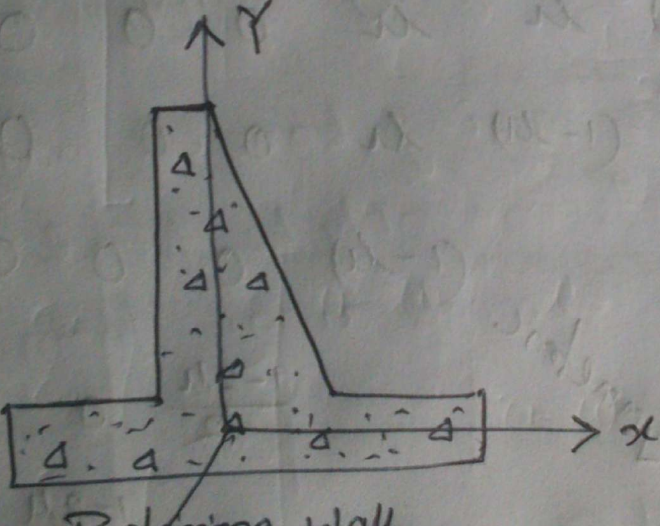
And,
$$\sigma_z = \frac{E\mu}{\cancel{\quad}}(\varepsilon_x + \varepsilon_y) + (1-\mu)E\varepsilon_z = 0$$

$$\therefore \varepsilon_z = -\frac{\mu(\varepsilon_x + \varepsilon_y)}{(1-\mu)}$$

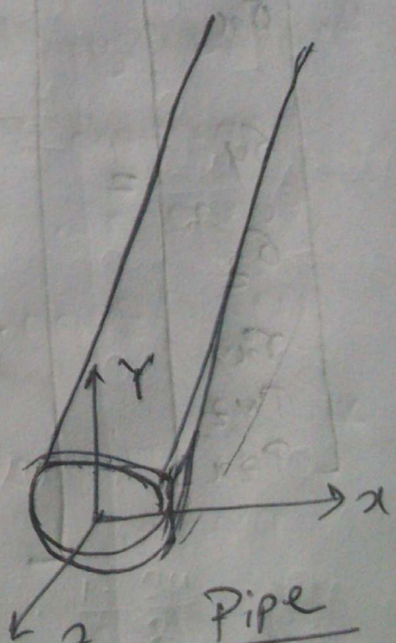
Now, from stress-strain Relationship for isot materials and substituting above values.

$$\therefore \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \times \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

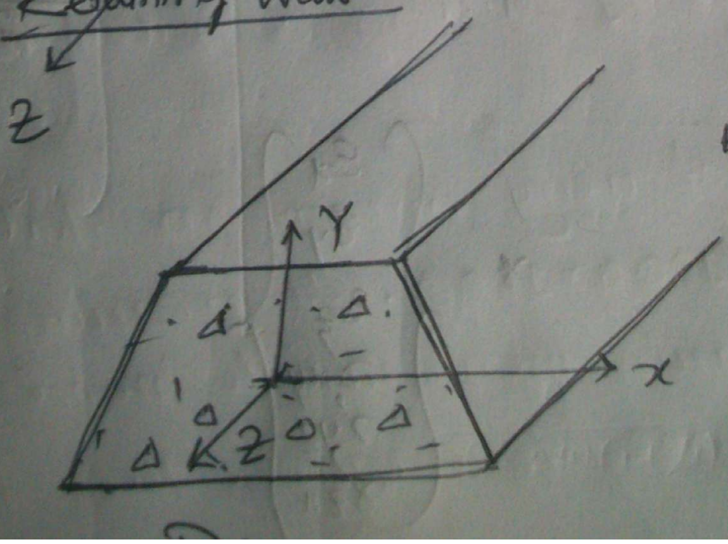
□ **Plane-Strain Problem:** If a structure withstands significant amount of lateral load but negligle amount of longitudinal load/stress, the problem called plane-strain problem. Underground pipe, dam, retaining wall etc. are the examples of such problems.



Retaining Wall



Pipe



In plane-strain problem,

$$\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$$

As ~~γ~~ $\gamma_{zx} = \gamma_{zy} = 0$, $\tau_{xz} = \tau_{yz} = 0$

NOW, $\varepsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} = 0$

$$\Rightarrow \sigma_z - \mu\sigma_x - \mu\sigma_y = 0$$

$$\therefore \sigma_z = \mu(\sigma_x + \sigma_y)$$

~~Substitu~~

NOW,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ & (1-\mu) & \mu & 0 & 0 & 0 \\ & & (1-\mu) & 0 & 0 & 0 \\ & \text{Symmetric} & & \frac{1-2\mu}{2} & 0 & 0 \\ & & & & \frac{1-2\mu}{2} & 0 \\ & & & & & \frac{1-2\mu}{2} \end{bmatrix} \times \frac{E}{(1+\mu)(}$$

$$\times \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma \end{Bmatrix}$$

Substituting,

$\gamma_{zx} = \gamma_{yz} = 0$ and $\sigma_z = \mu(\sigma_x + \sigma_y)$,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \times \frac{E}{(1+\mu)(1-2\mu)} \times \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

(Ans.)

□ Axi-Symmetric problem: If a structure is generated by rotating a line or a curve around an axis line, it is called axi-symmetric. Cylinder is an axi-symmetric structure.

The stress-strain Relationship:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \times \begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ & (1-\mu) & \mu & 0 \\ \text{Symmetric} & & (1-\mu) & 0 \\ & & & \frac{1-2\mu}{2} \end{bmatrix} \times \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

* What is stiffness matrix? What are the special features?

⇒ Stiffness matrix is a matrix which defines the geometric and material properties of an element.

The special features are:-

1. The matrix is having diagonal dominance and is positive definite. So, It is not necessary to rearrange the matrix.

2. The matrix is symmetric. There are only upper & lower triangle of elements and other elements can be found by using symmetry.

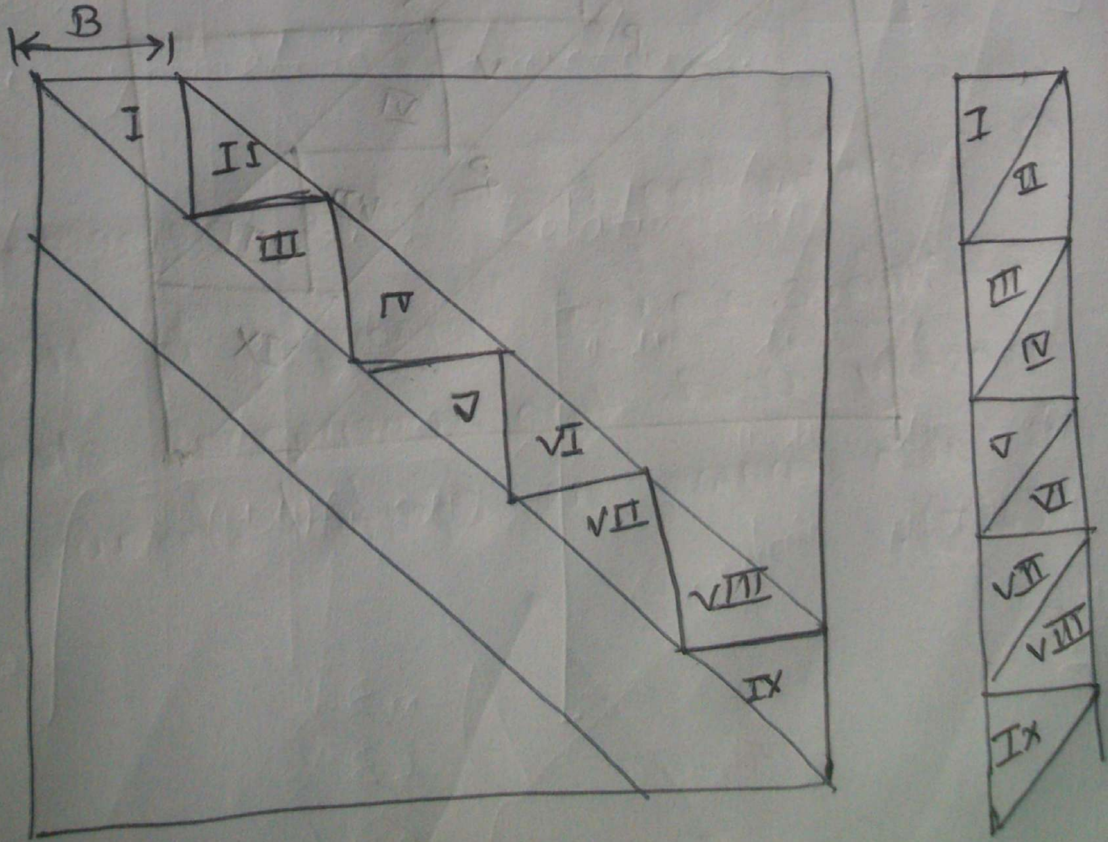
3. The sysmtem is having banded nature. It means non-zero elements can only be found near the diagonal. The elements away from the diagonal are zero.

* Techniques of Saving computer memory requirements.

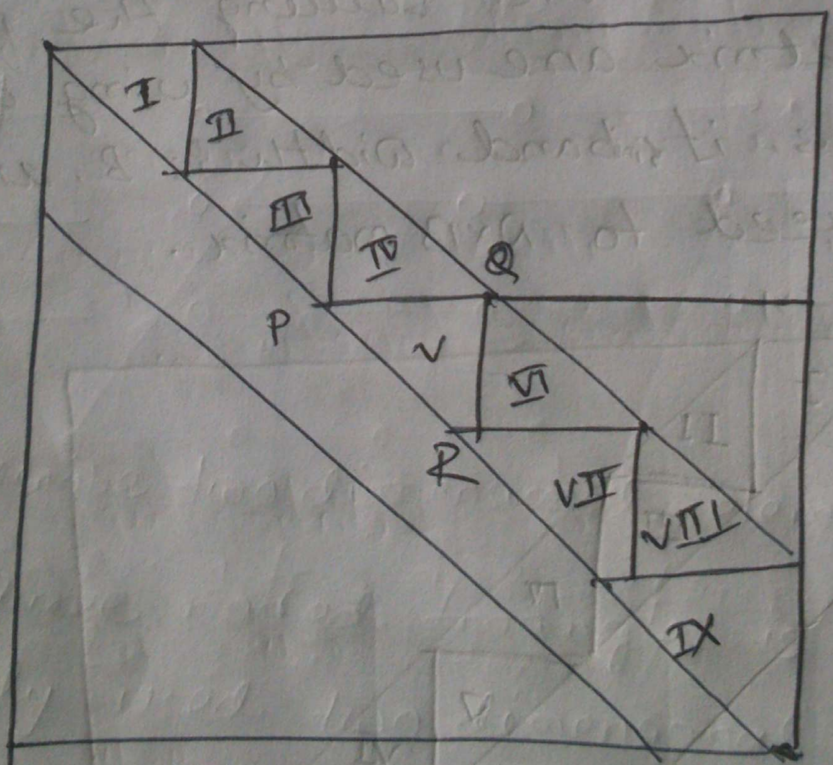
⇒ It is not very uncommon that stiffness matr is size of 1000×1000. It requires large memory requirement which needs to be redu by using some techniques. The techniques ar

1. Use of symmetry and banded nature.
2. Partioning of matrix (Front solution)
3. Skyline storage.

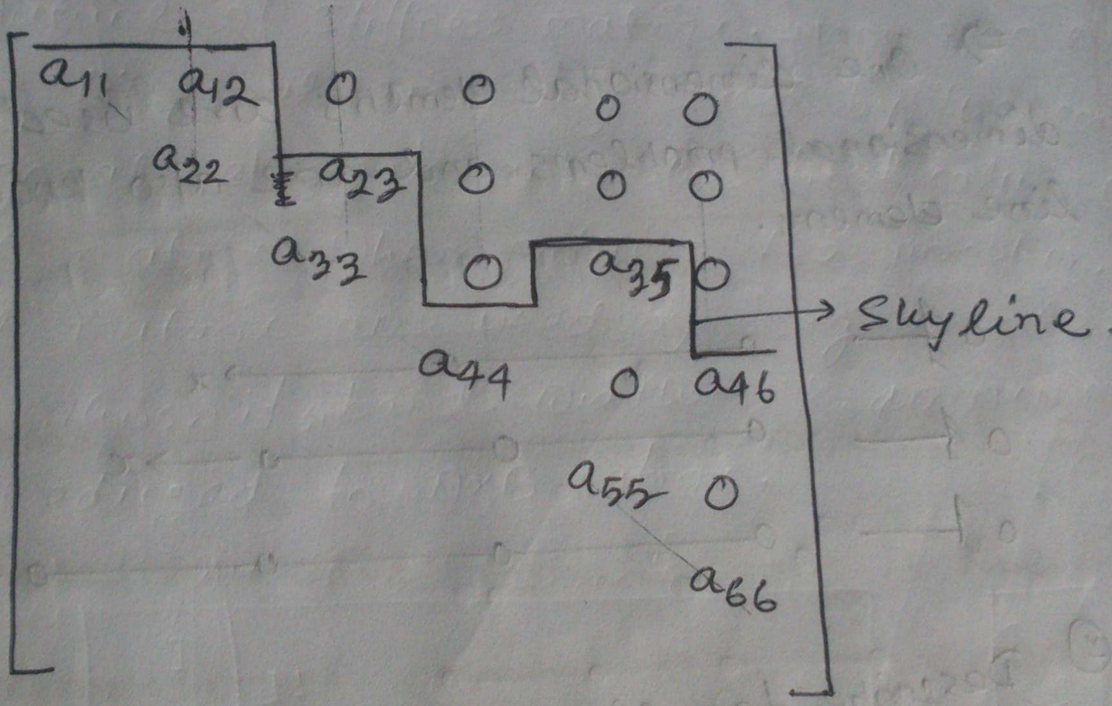
1. <u>Use of symmetry and banded nature</u>: Memory requirements can be reduced by use of symmetory and banded nature of stiffness matrix. Non-zero elements near the diagonal are only saved by the computer. The diagonal elements are the first column of modified matrix. When solving the problem, the modified matrix are used by using a coded program. Thus, if band width is B, an N×N matrix can be reduced to N×B matrix.



2. Partitioning of matrix: Using symmetry is not much beneficiary for large problems. In that case, Partitioning of matrix can be applied. Few trianges of elements are only stored in core and others are in the periphery of shortage like hard-disk. It should be noted that reduction of a row affects the triangle below it. For example, reduction of row PQ only affects the triangle PQR in the following figure.



3. Skyline Storage: Another technique of reducing memory requirement is skyline storage. In this technique, the columns starting with zeroelements, are saved forom the non-zero elements. The line saparating nonzero element from zero elements are called skyline.



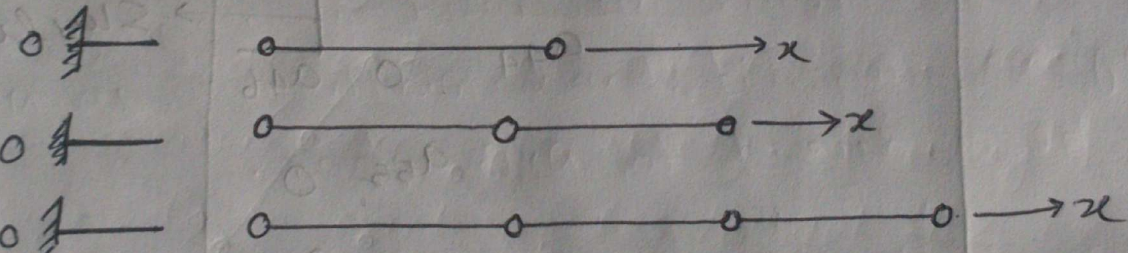
Chapte 4
Element Shapes, Node and Co-ordinate System

* Element shapes:-

1. One dimensional Elements,
2. Two dimensional Elements,
3. Axi-symmetric Elements,
4. Three dimensional Elements.

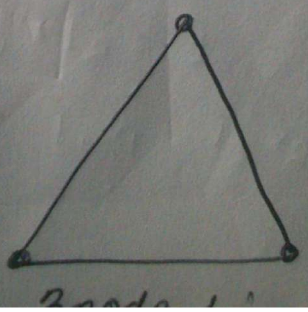
(x) Desribe one dimensional Element.

=> One dimensional elements are used in one dimensional problems and are also known as line element.

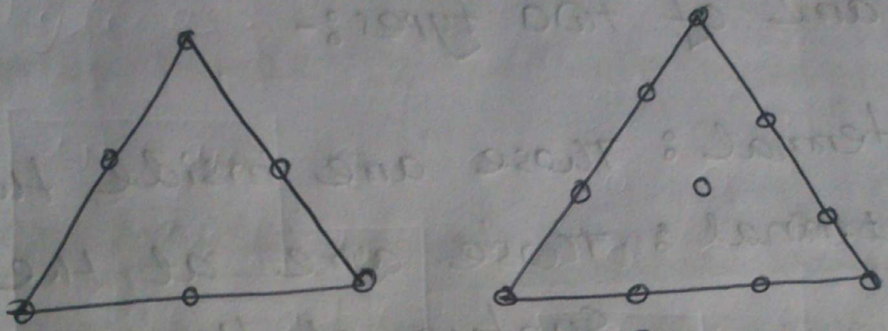


(x) Describe two dimensional element, CST, LST, QST.

=> Three noded triangle is the basic and mostly used two dimensional element which is also known as constant strain triangle (CST).



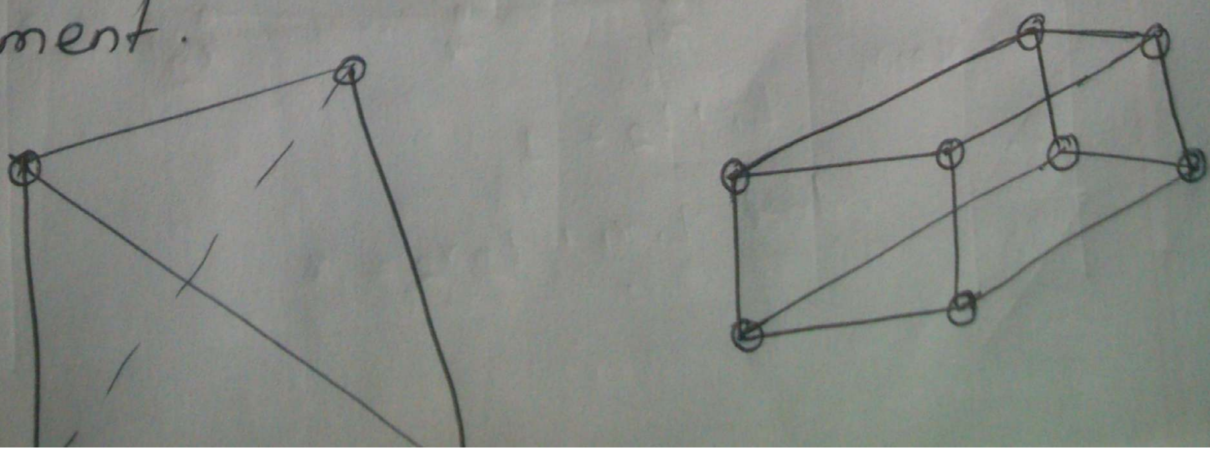
Six node and ten noded triangles are also used as two dimensional element. Six noded triangle elemenent is called linear strain triangle (LST) and ten noded triangles are called Quadratic strain traiangle (QST)



LST QST

⊗ Describe three dimensional elements in FEM?

⇒ like triangles, tetrahedron is the basic three dimensional element with four node, one each corner. ~~Six~~ Eight noded elements are hexahed and rectangular prism which is a form of hexahedron. Rectangular prism is also called brick element.

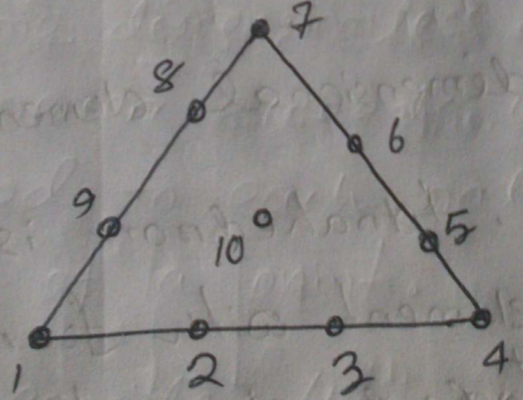


(x) Describe node?

⇒ Nodes are finite points in which basic unknowns are to be determined. At any point in element, unknowns are determined by using approximated/shape/interpolation fuctions.

Nodes are of two types:-

1. Internal: Those are inside the elemen
2. External: Those are at the edge o surface of the elements.



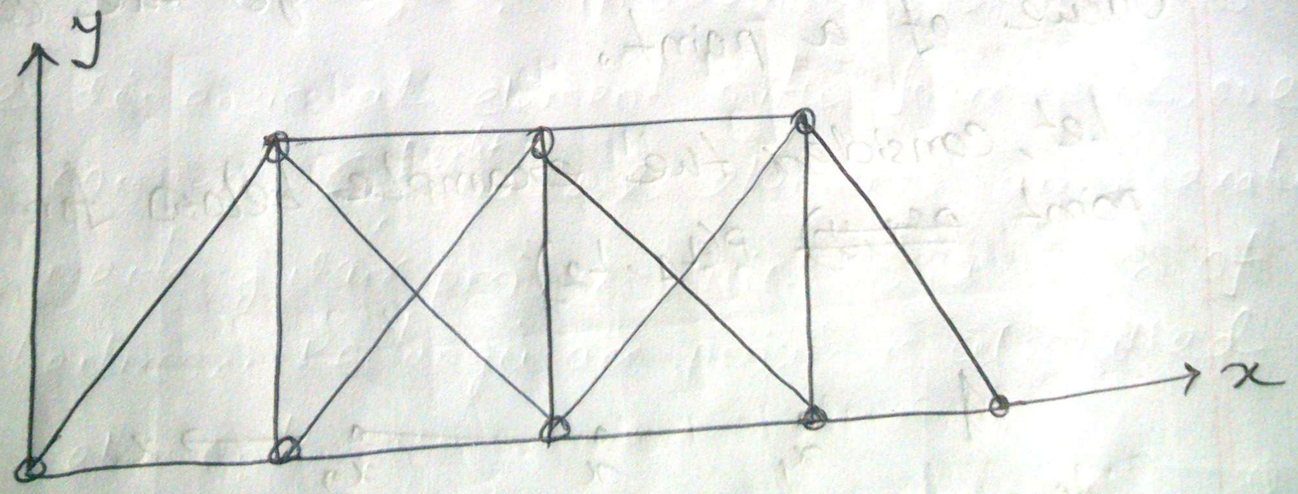
e.g. Nodes 1 to 9 are external, Node 10 is internal.

④ Type of Coordinate system:

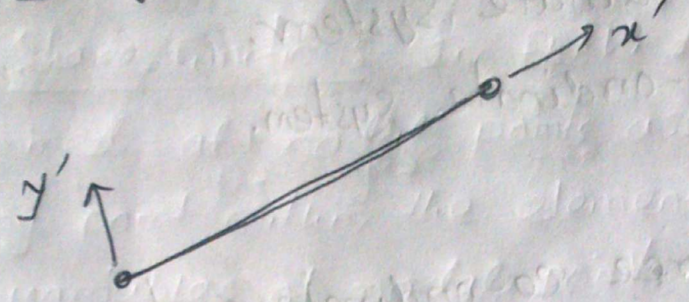
1. Global co-ordinate System,
2. Local Co-ordinate System,
3. Natural co-ordinate systen.

④ Describe three co-ordinate systems.

1. Global cordinate System: Global co-ordinate system is one which defines all the points in a structure. Typical Global co-ordinate system is shown in below:

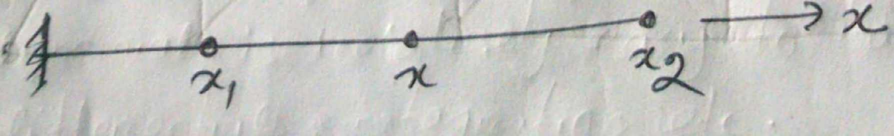


2. Local co-ordinate system: To assemble element properties, sometimes local co-ordinate system is used for individual elements. For example:



3. Natural co-ordinate system: In natural co-ordinate system, a point is specified by some dimensionless ~~sub~~ numbers. This numbers ~~are~~ never exceeds unity. A weightage is assigned to each nodes to get the co-ordinate value of a point.

Let, consider the example below for any point ~~P(x, y)~~ $P(L_1, L_2)$



Now, $L_1 + L_2 = 1$

And $L_1 x_1 + L_2 x_2 = x$

So,

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \times \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \times \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix}^T \times \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \times \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1} \times \begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \frac{x_2 - x}{L} \\ \frac{x - x_1}{L} \end{Bmatrix}$$

[$x_2 - x_1$ = Length of the element = L]

Chapter 5
Shape Function

* Define shape function. What are the ~~assumption~~ advantages of using polynomial as shape function?

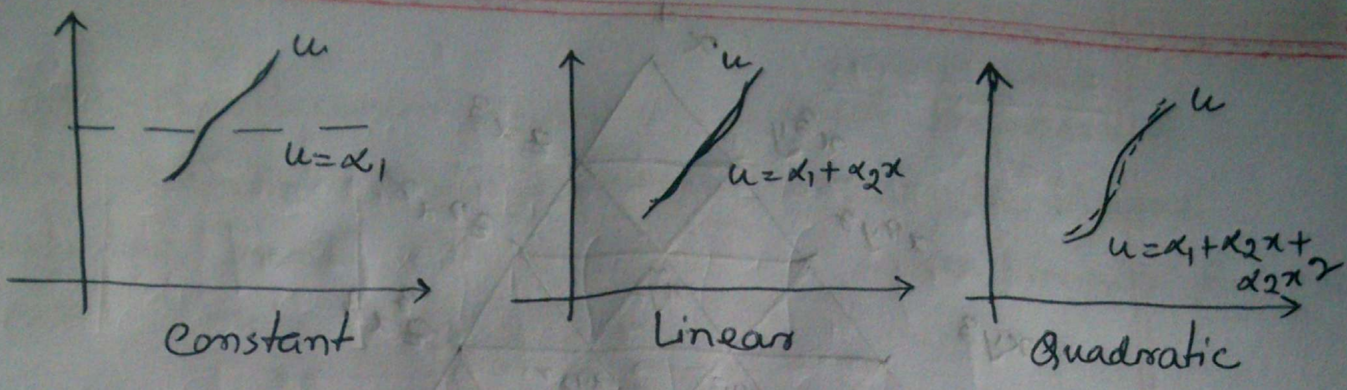
=) **Shape Function:** In FEM, field variables are determined at nodal points assuming field variables at any point within the element is a function of field variables of nodal points. The function used for relating field variable at any point with the field variable of nodal point is called shape function. Let, Consider a point $P(x,y)$ within an element. For point P:

$$u = \sum N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = \sum N_i v_i = N_1 v_i + N_2 v_2 + N_3 v_3$$

So,
$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

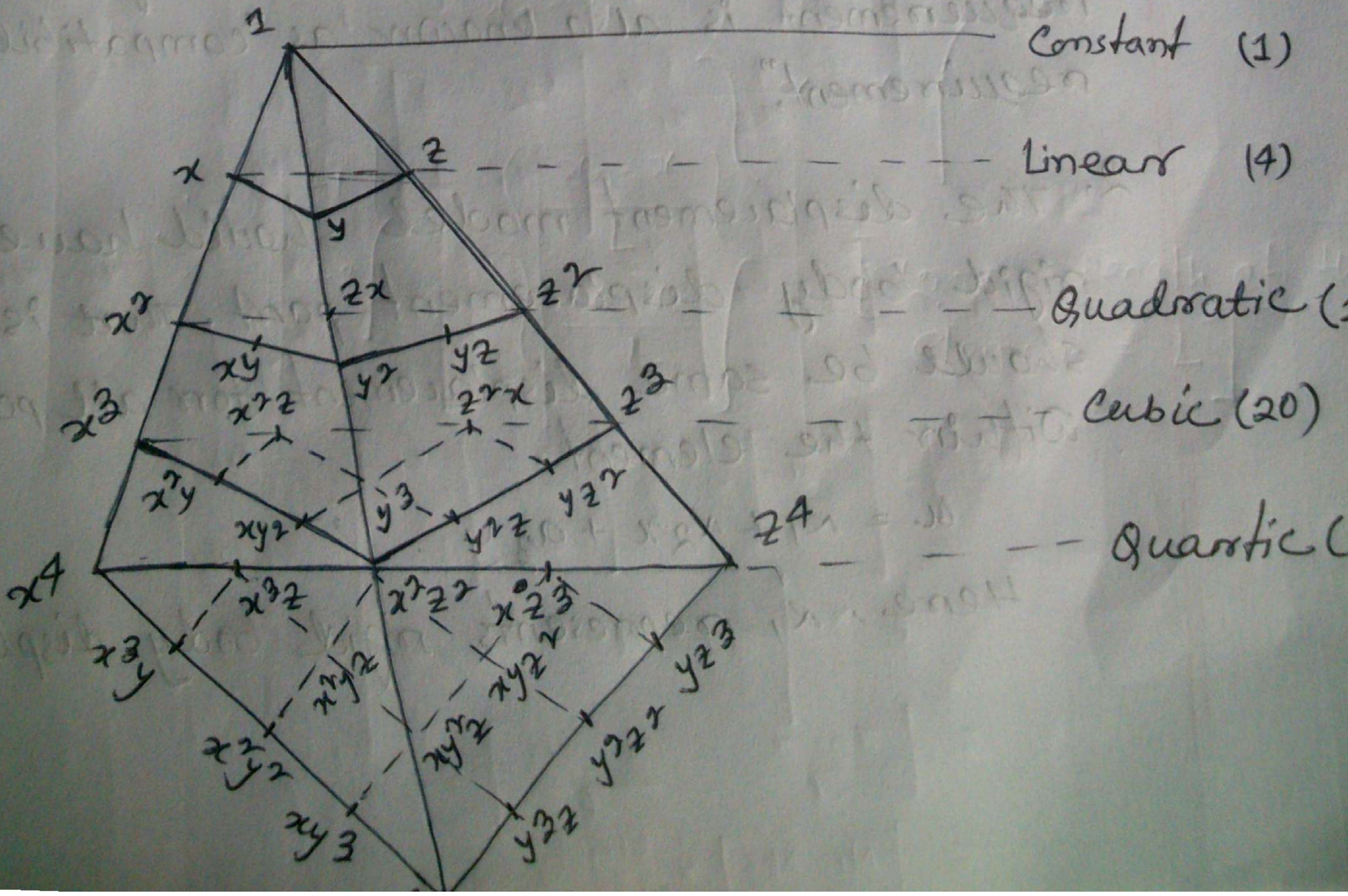
Advantages of using polynomial:

1. Polynomial is relatively easy to handle mathematically e.g. differential, integration etc.
2. Using polynomial, a function can be easily approximated. A highly non-linear function can be approximated by higher order polynomial.



* What are the convergence requirements of shape function?

⟹ Numerical method is an approximate method. In Finite element method, displacement estimated is less than actual. To get the result converged, finite element analysis mesh needs to be ~~rede~~ refined a following diagram. Also shape function should fulfill the following requirements:



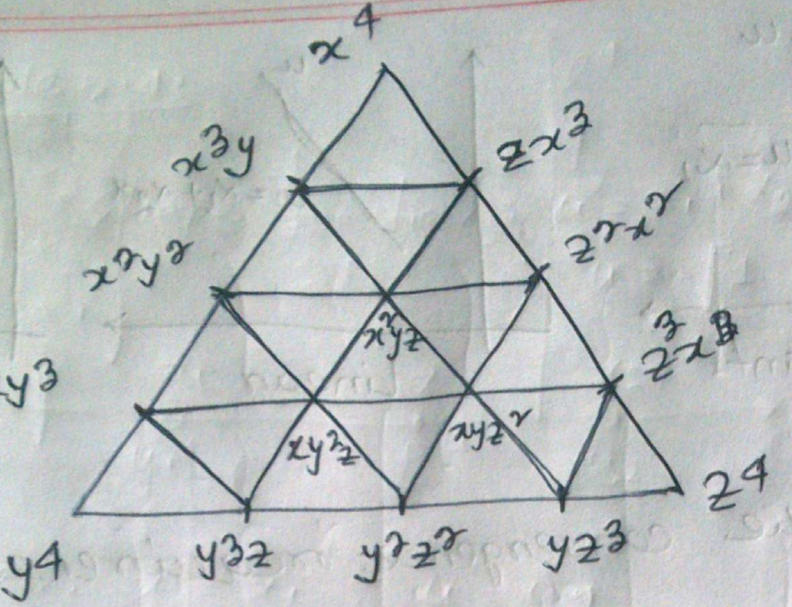


Fig: Terms at base of quadratic tetrahedron of polynomial

1. The displacement model should be continuous within the element and displacement should be compatible between the elements. The second part implies that adjacent elements should deform without causing opening, overlapping etc. This requirement is also known as "compatibility requirement."

2. The displacement model should have a rigid body displacement part. That is there should be same displacement for all points within the element.

$$u = \alpha_1 + \alpha_2 x + a_3 y$$

Here, α_1 represents rigid body displacement.

3. The displacement model should represent constant strain state of element. that is, there should be part that represents same strain f all points within the element.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + a_5 y^2 + ----$$

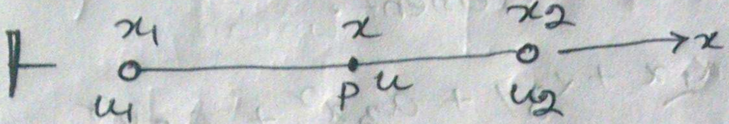
α_2 provides const. strain, ε_x

α_3 provides const strain, ε_y

Example 5·4:

Let, shape function: $u = \alpha_1 + \alpha_2 x$

$$\therefore u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$



When, $x = x_1$, $u = u_1$

$$\therefore u_1 = \alpha_1 + \alpha_2 x_1$$

When, $x = x_2$, $u = u_2$

$$u_2 = \alpha_1 + \alpha_2 x_2$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1} \times \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad [x_2 - x_1 = l]$$

$$\therefore u = \begin{bmatrix} 1 & x \end{bmatrix} \times \frac{1}{l} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

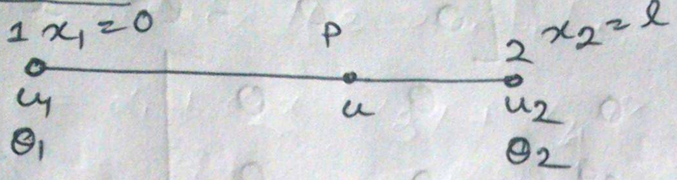
$$= \frac{1}{l} \times \begin{bmatrix} x_2 - x & x - x_1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{x_2 - x}{l} & \frac{x - x_1}{l} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = N_1 u_1 + N_2 u_2$$

$$\therefore \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{bmatrix} \frac{x_2 - x}{l} \\ \frac{x - x_1}{l} \end{bmatrix}$$

Example 5.5:



Let, Shape function: $u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$

$$\therefore \theta = \frac{\delta u}{\delta x} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2$$

When $x = x_1 = 0$, ~~at~~ $u = u_1$, $\theta = \theta_1$

$$\therefore u_1 = \alpha_1 + \alpha_2 \times 0 + \alpha_3 \times 0 + \alpha_4 \times 0$$

$$\theta_1 = \alpha_1 \times 0 + \alpha_2 + 2\alpha_3 \times 0 + 3\alpha_4 \times 0$$

When $x = x_2 = l$, $u = u_2$ and $\theta = \theta_2$

$$u_2 = \alpha_1 + \alpha_2 l + \alpha_3 l^2 + \alpha_4 l^3$$

$$\theta_2 = \alpha_1 \times 0 + \alpha_2 + \alpha_3 \times 2l + \alpha_4 \times 3l^2$$

$$\therefore \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \frac{1}{3l^4 - 2l^4} \times \begin{bmatrix} l^4 & 0 & -3l^2 & 2l \\ 0 & l^4 & -2l^3 & l^2 \\ 0 & 0 & 3l^2 & -2l \\ 0 & 0 & -l^3 & l^2 \end{bmatrix}^T \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

$$= \frac{1}{l^4} \begin{bmatrix} l^4 & 0 & 0 & 0 \\ 0 & l^4 & 0 & 0 \\ -3l^2 & -2l^3 & 3l^2 & -l^3 \\ 2l & l^2 & -2l & l^2 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -\frac{2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & -\frac{2}{l^3} & \frac{1}{l^2} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

NOW,

$$u = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -\frac{2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ 2/l^3 & \frac{1}{l^2} & -\frac{2}{l^3} & \frac{1}{l^2} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

$$\cancel{u = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \times}$$

$$u = \begin{bmatrix} 1+0-\dfrac{3x^2}{l^2}+\dfrac{2x^3}{l^3} & x-\dfrac{2x^2}{l}+\dfrac{x^3}{l^2} & \dfrac{3x^2}{l^2}-\dfrac{2x^3}{l^3} & -\dfrac{x^2}{l}+\dfrac{x^3}{l^2} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

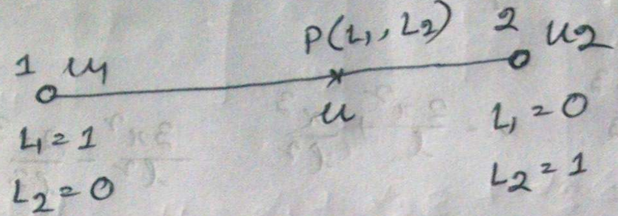
$$= \begin{bmatrix} \left(1-\dfrac{3x^2}{l^2}+\dfrac{2x^3}{l^3}\right) & \left(x-\dfrac{2x^2}{l}+\dfrac{x^3}{l^2}\right) & \left(\dfrac{3x^2}{l^2}-\dfrac{2x^3}{l^3}\right) & \left(\dfrac{x^3}{l^2}-\dfrac{x^2}{l}\right) \end{bmatrix} \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_1 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} = \begin{bmatrix} 1-\dfrac{3x^2}{l^2}+\dfrac{2x^3}{l^3} \\ x-\dfrac{2x^2}{l}+\dfrac{x^3}{l^2} \\ \dfrac{3x^2}{l^2}-\dfrac{2x^3}{l^3} \\ \dfrac{x^3}{l^2}-\dfrac{x^2}{l} \end{bmatrix}$$

(Ans).

Example 5.7:



$$u = \alpha_1 L_1 + \alpha_2 L_2 \qquad \therefore u = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\therefore u_1 = \alpha_1 \times 1 + \alpha_2 \times 0$$

$$u_2 = \alpha_1 \times 0 + \alpha_2 * 1$$

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{1} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{T} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

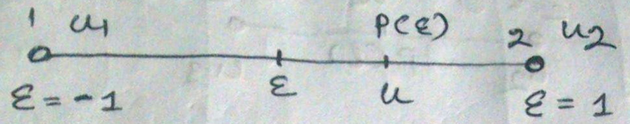
$$\therefore u = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} L_1 & L_2 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \therefore \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} \text{ (Ans)}$$

$$= N_1 u_1 + N_2 u_2$$

Example 5.8:



$u = \alpha_1 + \alpha_2 \varepsilon \quad \therefore \{u\} = \begin{bmatrix} 1 & \varepsilon \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$

If $\varepsilon = -1$, $u = u_1$

$\therefore u_1 = \alpha_1 - \alpha_2$

If $\varepsilon = 1$, $u_2 = u$

$\therefore u_2 = \alpha_1 + \alpha_2$

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{+} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore u = \begin{bmatrix} 1 & \varepsilon \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

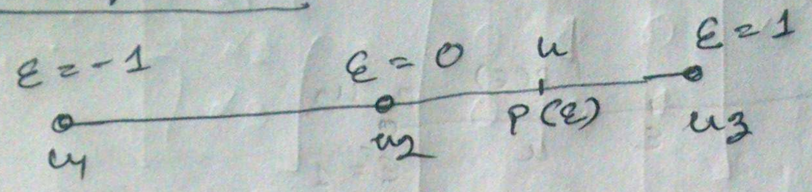
$$= \begin{bmatrix} \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) & \left(\frac{1}{2} + \frac{\varepsilon}{2}\right) \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \therefore \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1-\varepsilon}{2} \\ \frac{1+\varepsilon}{2} \end{Bmatrix} \text{ (Ans)}$$

$$= N_1 u_1 + N_2 u_2$$

Example 5.9:



Let, $u = \alpha_1 + \alpha_2 \varepsilon + \alpha_3 \varepsilon^2$

$$\therefore u = \begin{bmatrix} 1 & \varepsilon & \varepsilon^2 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

When $\varepsilon = -1$, $u = u_1$

$$\therefore u_1 = \alpha_1 - \alpha_2 + \alpha_3$$

When $\varepsilon = 0$, $u = u_2$

$$\therefore u_2 = \alpha_1 + \alpha_2 \times 0 + \alpha_3 \times 0$$

When $\varepsilon = 1$, $u = u_3$

$$\therefore u_3 = \alpha_1 + \alpha_2 + \alpha_3$$

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{T} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

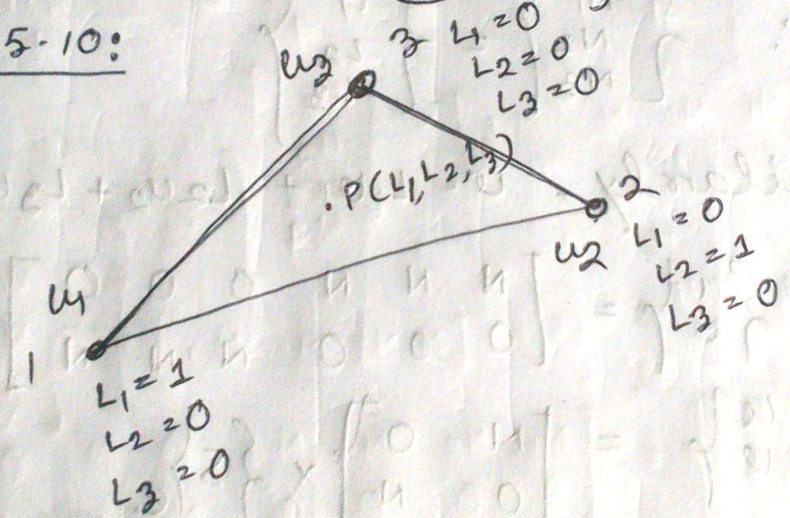
$$\therefore u = \begin{bmatrix} 1 & \varepsilon & \varepsilon^2 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2} \times \begin{bmatrix} \varepsilon^2 - \varepsilon & 2 - 2\varepsilon^2 & \varepsilon + \varepsilon^2 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{\varepsilon^2 - \varepsilon}{2} & 1 - \varepsilon^2 & \frac{\varepsilon + \varepsilon^2}{2} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} \frac{\varepsilon^2 - \varepsilon}{2} \\ 1 - \varepsilon^2 \\ \frac{\varepsilon + \varepsilon^2}{2} \end{Bmatrix}$$ (Ans).

Example 5.10:



Let, $u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 \quad \therefore u = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$

Now,
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\therefore \{u\} = [L_1 \quad L_2 \quad L_3] \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [L_1 \quad L_2 \quad L_3] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= L_1 u_1 + L_2 u_2 + L_3 u_2$$

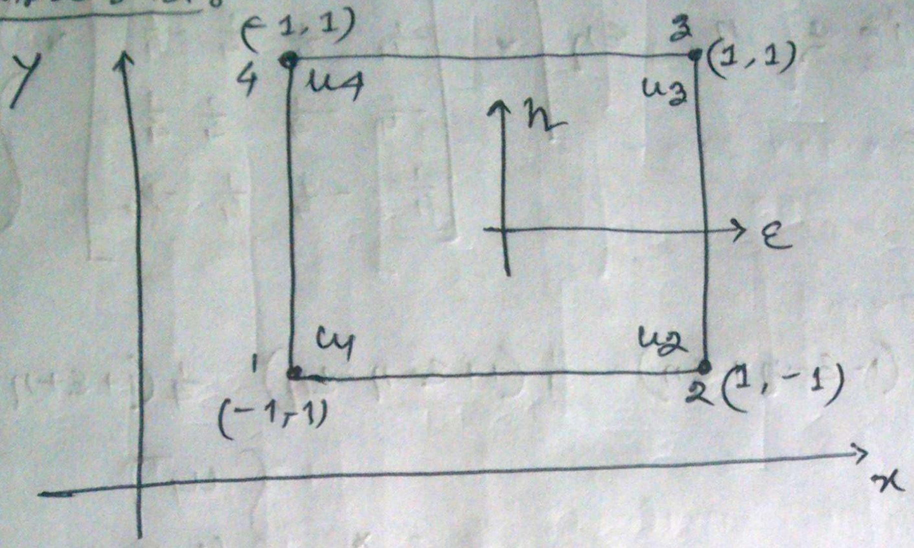
$$\therefore \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}$$

Similarly, $v = L_1 v_1 + L_2 v_2 + L_3 v_3$

$$\therefore \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N & N & N & 0 & 0 & 0 \\ 0 & 0 & 0 & N & N & N \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \times \{\delta\}_e$$

Example 5-12:



Let, $u = \alpha_1 + \alpha_2\varepsilon + \alpha_3\eta + \alpha_4\varepsilon\eta$

$$\therefore u = \begin{bmatrix} 1 & \varepsilon & \eta & \varepsilon\eta \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

Now,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\therefore u = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix} \times \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(1-\xi-\eta+\xi\eta) & \frac{1}{4}(1+\xi-\eta-\xi\eta) & \frac{1}{4}(1+\xi+\eta+\xi\eta) & \frac{1}{4}(1-\xi+\eta-\xi\eta) \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{(1-\xi)(1-\eta)}{4} & \frac{(1+\xi)(1-\eta)}{4} & \frac{(1+\xi)(1+\eta)}{4} & \frac{(1-\xi)(1+\eta)}{4} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Similarly,

$$v = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \times \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

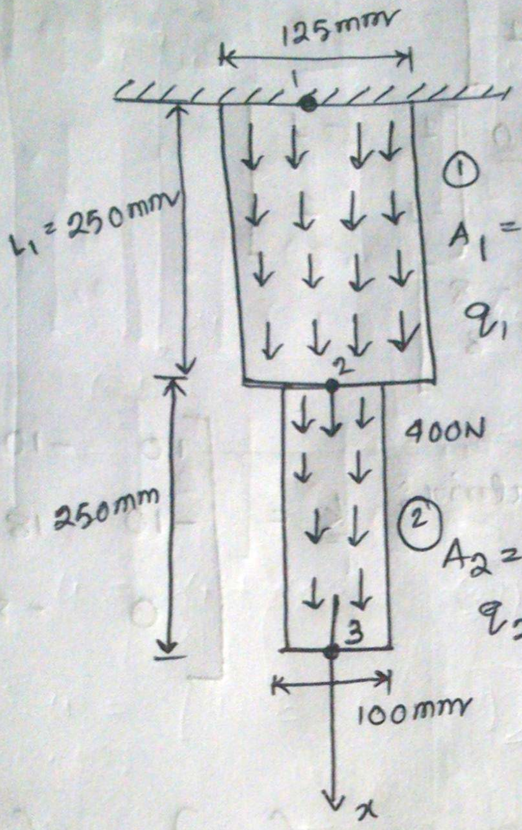
$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \times \{\delta\}_e$$

(Ans).

Chapter → 11

Barr & Truss Element

Example 11·1:

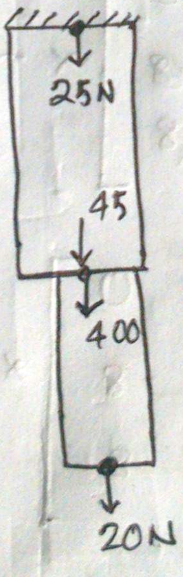


① $A_1 = 125 \times 20 = 2500\ mm^2$

$q_1 = \rho \times 125 = 0.80 \times 10^{-4} \times 125 \times 20 = 0.20\ N/mm$

② $A_2 = 100 \times 20 = 2000\ mm^2$

$q_2 = 0.80 \times 10^{-4} \times 100 \times 20 = \cancel{0.008\ N/mm}$ $0.16\ N/mm$



Now, for element 1, $k_1 = \frac{EA_1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \frac{2 \times 10^5 \times 2500}{250}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 2 \times 10^5 \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}$$

$$k_2 = \frac{EA_2}{L_2} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2 \times 10^5 \times 2000}{250} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 2 \times 10^5 \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

$\therefore$ Global stiffness matrix: $\underline{\underline{k}} = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 18 & -8 \\ 0 & -8 & 8 \end{bmatrix}$

Now,

$$\underline{\underline{k}}\ \underline{\underline{u}} = \underline{\underline{f}}$$

$$\Rightarrow 2 \times 10^5 \begin{bmatrix} 10 & -10 & 0 \\ -10 & 18 & -8 \\ 0 & -8 & 8 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 + 25 \\ 445 \\ 20 \end{Bmatrix}$$

$$\Rightarrow 2 \times 10^5 \begin{bmatrix} 10 & -10 & 0 \\ -10 & 18 & -8 \\ 0 & -8 & 8 \end{bmatrix} \times \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 + 25 \\ 445 \\ 20 \end{Bmatrix}$$

So, $2 \times 10^5 \begin{bmatrix} 18 & -8 \\ -8 & 8 \end{bmatrix} \times \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 445 \\ 20 \end{Bmatrix}$

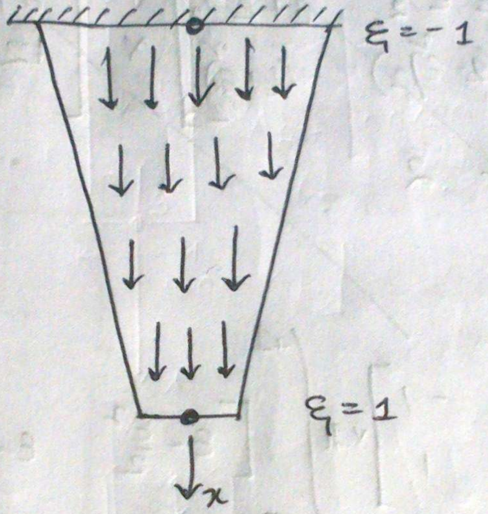
So, $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \frac{46.5}{2 \times 10^5} \\ 49/2 \times 10^5 \end{Bmatrix} = \begin{Bmatrix} 2.325 \times 10^{-4} \\ 2.45 \times 10^{-4} \end{Bmatrix}$

NOW, $\begin{Bmatrix} R_1 \\ +25 \end{Bmatrix} = \begin{bmatrix} 10 & -10 & 0 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 2.325 \times 10^{-4} \\ 2.45 \times 10^{-4} \end{Bmatrix} \times 2 \times 105$

$= \{-465\} - 25$

$= \{-490\,N\}$ (Ans.)

Example 11.2:



NOW, $~~\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = [~~$ $\quad [A] = [N_1 \quad N_2] \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$

$= \begin{bmatrix} \frac{1-\varepsilon}{\ell} & \frac{1+\varepsilon}{\ell} \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$

NOW, $B = \frac{1}{\ell}[-1 \quad 1]$

And $\xi = \dfrac{x - x_c}{\ell/2}$

$\Rightarrow d\varepsilon = \frac{2}{\ell} dx$

$\frac{-\frac{\ell}{2} - 0}{\ell}$

$- \frac{\ell/2}{\ell/2}$

NOW, $[k] = \iiint_V [B]^T [D] [B] \, dV$

$= \int_0^{\ell} \frac{1}{\ell} \begin{bmatrix} \cancel{1}\, -1 \\ 1 \end{bmatrix} E \times \frac{1}{\ell} [-1 \quad 1] \, A\, dx$

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$$= \frac{E}{4l} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times 2 \int_{-1}^{+1} \begin{bmatrix} \varepsilon - \frac{\varepsilon^2}{2} & \varepsilon + \frac{\varepsilon^2}{2} \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} d\varepsilon$$

$$= \frac{E}{4l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 2 \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} d\varepsilon$$

$$= \frac{E}{2l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$= \frac{E}{2l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \cancel{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}} \begin{bmatrix} A_1 + A_2 \end{bmatrix}$$

$$= \frac{E}{l} \times \frac{A_1 + A_2}{2} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{E\bar{A}}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Nodal Force:

$$\{F\} = \iiint_V [N]^T \rho \, dv$$

$$= \int_0^l [N]^T \rho \times [A] \, dx$$

$$= \int_0^l [N]^T \rho \times [N] \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} dx$$

$$\cancel{= \frac{\rho l}{2} \int_{-1}^{1} \begin{bmatrix} \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} \end{bmatrix} \times \begin{bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} d\xi}$$

$$\cancel{= \frac{\rho L}{2} \int_{-1}^{1}}$$

$$= \frac{\rho L}{2} \int_{-1}^{1} \begin{bmatrix} \frac{1-\xi}{2} \\ \frac{1+\varepsilon}{2} \end{bmatrix} \times \begin{bmatrix} \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} d\xi$$

$$= \frac{\rho l}{2} \int_{-1}^{1} \begin{bmatrix} \frac{(1-\varepsilon)^2}{4} & \frac{1-\varepsilon^2}{4} \\ \frac{1-\varepsilon^2}{4} & \frac{(1+\varepsilon)^2}{4} \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} d\xi$$

$1 - \frac{1}{3} + 1 - \frac{2}{3} = \frac{6-}{3}$
$= 2 - 2/3 =$
$2 + \frac{2}{3} =$
$= \frac{8}{3}$

$$\int_{-1}^{1} \frac{(1-\varepsilon)^2}{4} d\xi = \frac{1}{4} \int_{-1}^{1} (1 - 2\xi + \varepsilon^2) d\xi = \frac{1}{4} \left[\xi - \varepsilon^2 + \frac{\varepsilon^3}{3} \right]_{-1}^{1}$$

$$= \frac{1}{4} \left(1 - 1 + \frac{1}{3} + 1 + 1 + \frac{1}{3}\right)$$

$$= \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$$

$$\int_{-1}^{1} \frac{1-\varepsilon^2}{4} = \frac{1}{4} \left[\varepsilon - \frac{\varepsilon^3}{3} \right]_{-1}^{1} = \frac{1}{4} \left[\varepsilon - \frac{\varepsilon^3}{3} \right]_{-1}^{1} = \frac{1}{4} \left[\left(1 - \frac{1}{3}\right) - (-1 \right.$$

$$= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$$\int_{-1}^{1} \frac{(1 + 2\varepsilon + \varepsilon^2)}{4} d\varepsilon = \cancel{\frac{1}{2} \left[\varepsilon + \varepsilon^2 + \frac{\varepsilon^3}{3} \right]_0^1 = \frac{1}{2} \times \left(1 + 1 + \frac{1}{3}\right) = \frac{1}{2} \times \frac{7}{3} = \frac{7}{6}}$$

$$= \frac{2}{3}$$

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$$= \frac{Pl}{2} \times \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \times \begin{bmatrix} A_1 & \cancel{A_2} \\ A_2 & \end{bmatrix}$$

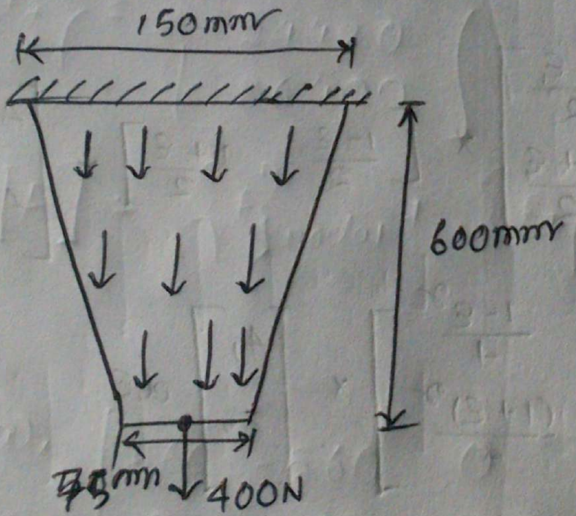
$$= \frac{PL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$= \frac{Pl}{6} \begin{bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{bmatrix}$$

$$\therefore \frac{E\bar{A}}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{Pl}{6} \begin{bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{bmatrix}$$

(Ans.)

Example 11.3:



$\bar{A} = \frac{A_1 + A_2}{2} = 2250m$

$A_1 = 150 \times 20 = 300$

$A_2 = 75 \times 20 = 1500$

$$\frac{E\bar{A}}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} = \frac{Pl}{6} \times \begin{bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{bmatrix}$$

$$\Rightarrow \frac{2\times10^5 \times 2250}{600} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 48 \end{Bmatrix}$$

So, $\frac{2\times10^5 \times 2250}{600} \cancel{\{u_2\}} \times [1] \times \{u_2\} = \{48\}$

$\therefore \{u_2\} = \{6.4 \times 10^{-5}\}$ mm (Ans.)

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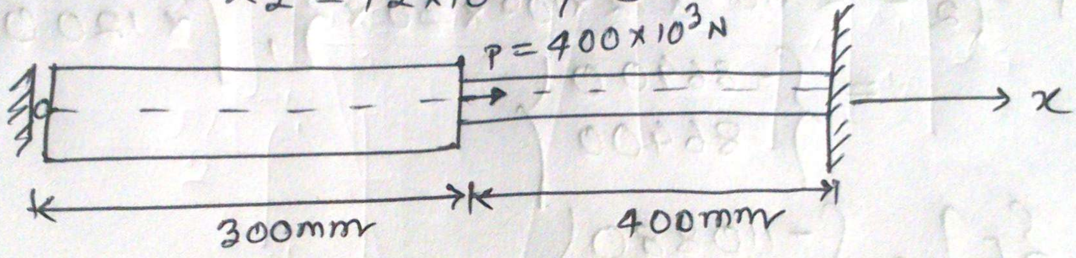
xxx Example 11.4:

Given that,
$A_1 = 2400\ mm^2$
$A_2 = 1200\ mm^2$
$l_1 = 300\ mm$
$l_2 = 400\ mm$
$E_1 = 0.7 \times 10^5\ N/mm^2$
$E_2 = 2 \times 10^5\ N/mm^2$
$\alpha_1 = 22 \times 10^{-6}\ /°C$
$\alpha_2 = 12 \times 10^{-6}\ /°C$



NOW,
$$k_1 = \frac{E_1A_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{0.7 \times 10^5 \times 2400}{300} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= 10^3 \times \begin{bmatrix} 560 & -560 \\ -560 & 560 \end{bmatrix}$$

$$k_2 = \frac{E_2A_2}{L_2} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= 10^3 \times \begin{bmatrix} 600 & -600 \\ -600 & 600 \end{bmatrix}$$

$$\therefore k = \begin{bmatrix} \cancel{1160}\ 560 & -560 & 0 \\ -560 & 1160 & -600 \\ 0 & -600 & 0 \end{bmatrix} \times 10^3$$

$$\{F\}_{nodal} = \begin{Bmatrix} 0 \\ 400000 \\ 0 \end{Bmatrix}$$

$$\{F_T\}_1 = E_1 \alpha_1 \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 0.70 \times 105 \times 22 \times 10^{-6} \times 30 \times \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \times A_1$$

$$= \begin{Bmatrix} -110880 \\ 110880 \end{Bmatrix}$$

$$\{F_T\}_2 = 2 \times 10^5 \times 12 \times 10^{-6} \times 30 \times \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \times 1200$$

$$= \begin{Bmatrix} -86400 \\ 86400 \end{Bmatrix}$$

$$\therefore \{F_T\} = \begin{Bmatrix} -110880 \\ 24480 \\ 86400 \end{Bmatrix}$$

$$\therefore [F] = \{F\}_{nodal} + \{F_T\} = \begin{Bmatrix} -110880 \\ 424480 \\ 86400 \end{Bmatrix}$$

NOW,

$$\begin{bmatrix} 560 & -560 & 0 \\ 0 & 1160 & -600 \\ 0 & -600 & 600 \end{bmatrix} \times 10^3 \times \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -110880 \\ 424480 \\ 86400 \end{Bmatrix}$$

$$\therefore \{u_2\} = \{0.36593\} \text{ mm}$$

NOW,

$$\sigma_1 = E \underline{\underline{B}}\, \underline{\underline{u}} - E \alpha_1 \Delta T$$

$$= 0.7 \times 10^5 \times \frac{1}{300} [-1 \quad 1] \times \begin{Bmatrix} 0 \\ 0.366 \end{Bmatrix} - (0.7 \times 10^5 \times 22 \times 10^{-6} \times$$

$$= 39.2 \text{ N/mm}^2$$

$$\sigma_2 = E\underline{\underline{B}}\,\underline{\underline{u}} - E\alpha_2 \Delta T$$

$$= 2\times10^5 \times \cancel{12}\ \frac{1}{400}\times\begin{bmatrix}-1 & 1\end{bmatrix}\times\begin{Bmatrix}0.366\\ 0\end{Bmatrix} - 2\times10^5 \times 12\times10^{-6}\times 30$$

$$= -255\,N/mm^2$$

$$R_1 = -560\times 0.366\times 10^3 + 110880 = -94080\,N$$

$$R_3 = -600\times 0.366\times10^3 - 86400 = -306000\,N$$ Ans.

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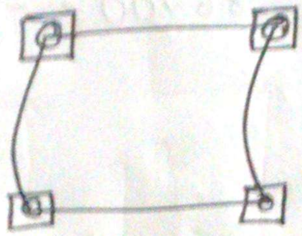
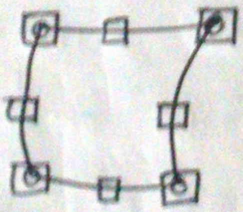
Chapter-13
Isoparametric Element

* What is isoparametric, superparametric and subparametr element.

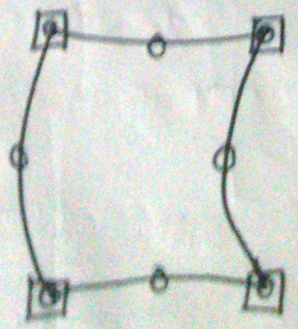
Isoparametric: If the nodal points representing geometry or boundary also represent displacement, the element is defined as an isoparametric element.



Superparametric: If number of element representing geometry is greater than the number of element representing displacement, the element is called superparametric.



Subparametric: If no. of element representing displ is greater than the no. of element representing geo the element is said to subparametric.



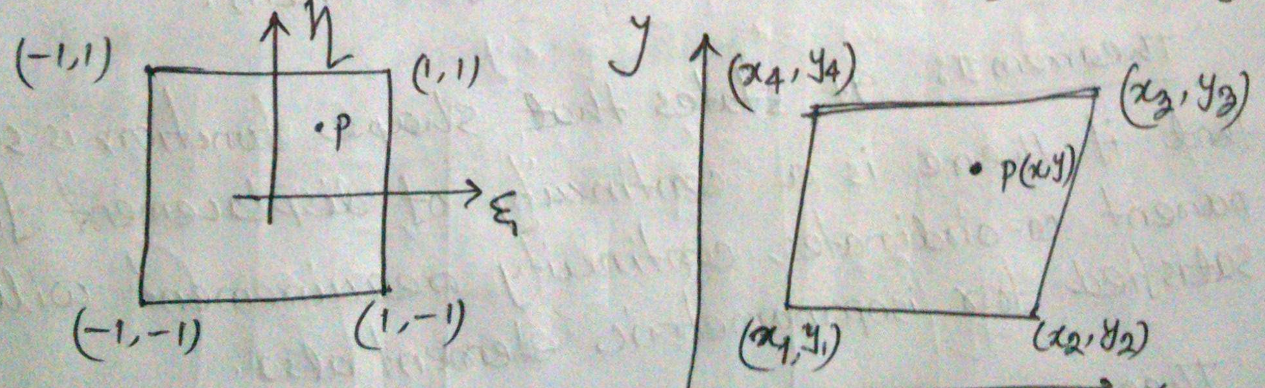
* Show the process of coordinate transformation in FEM.

⟹ In FEM co-ordinate can be transferred through following equations:-

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$
$$y = N_2 y_1 + N_2 y_2 + N_3 y_3$$
$$z = N_1 z_1 + N_2 z_2 + N_3 z_3$$
$$\therefore \{x\} = \{N\}\{x\}_e$$

Here, $\{x\}$ is the matrix ordinate of a point, $\{N\}$ is shape function and $\{x\}_e$ is the coordinate of nodal points.

Let, consider a rectangular parent element with coordinates of 1(−1,−1), 2(1,−1), 3(1,1) and 4(−1,1)



Now, $N_1 = \frac{(1-\xi)(1-\eta)}{4}$ $\quad N_2 = \frac{(1+\xi)(1-\eta)}{4}$

$N_3 = \frac{(1+\xi)(1+\eta)}{4}$ $\quad N_4 = \frac{(1-\xi)(1+\eta)}{4}$

Now, co-ordinates of any point $P(x,y)$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

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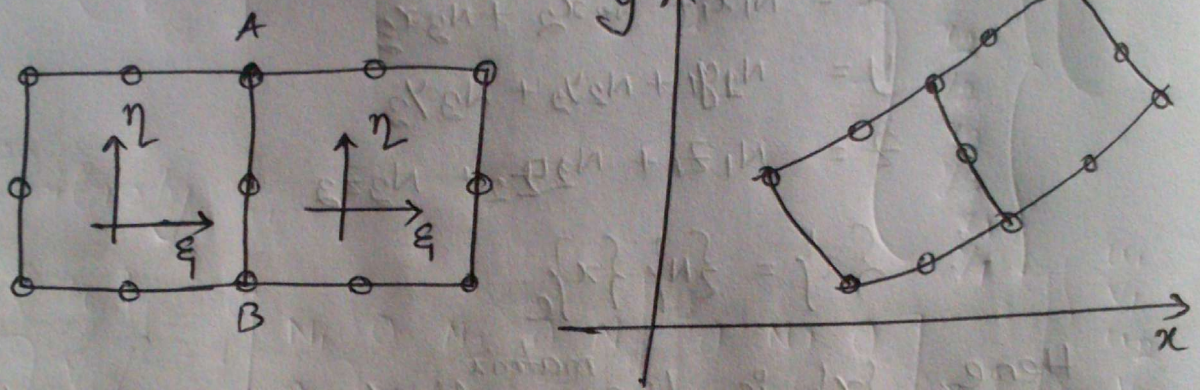
* Explain the theorems of Isoparametric element.

⇒ Theorem I: If two adjacent elements are generated using shape function, there will be a continuity at the common edge.



It can proved that for any point on edge AB $N_i = 0$ for nodal points not on the edge and N_i has any value for nodes on the edge. so, the final function will be same for edge if same co-ordinate value is used for them.

Theorem II: It states that shape function is such that if there is a continuity of displacement for parent co-ordinates, continuity requirement will be satisfied for isoparametric element also.

Theorem III: The condition of constant derivative and condition of rigid body will be satisfied if $\Sigma N_i = 1$.

proof:

Say, Displacement function: $u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i$

So, For any nodal point, i

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i$$

Now, $u = \sum N_i u_i$

$$= \sum N_i (\alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i)$$

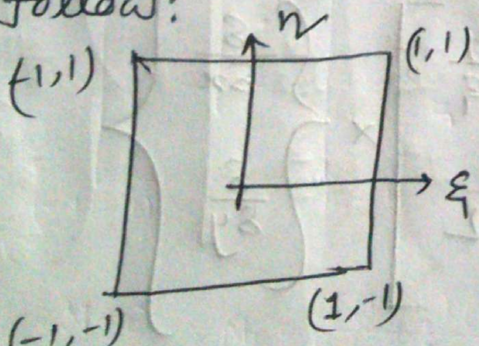
$$= \alpha_1 \sum N_i + \alpha_2 \sum N_i x_i + \alpha_3 \sum N_i y_i + \alpha_4 \sum N_i z_i$$

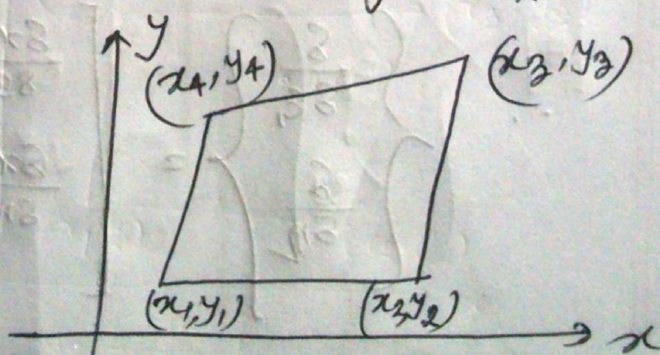
$$= \alpha_1 \sum N_i + \alpha_2 x + \alpha_3 y + a_4 z$$

So, $\boxed{\sum N_i = 1}$

(X) Assembling of Stiffness Matrix:

~~Assim~~ Assembling of stiffness matrix is of much importance for FEM problems. Let, consider a quadrilater element and a rectangular parent element as follow:





Now, $N_1 = \frac{(1-\xi)(1-\eta)}{4}$ $\quad N_2 = \frac{(1+\xi)(1-\eta)}{4}$

$N_3 = \frac{(1+\xi)(1+\eta)}{4}$ $\quad N_3 = \frac{(1-\xi)(1+\eta)}{4}$

So, $x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$

$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$

$$\therefore \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \times \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

Similarly,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Using formulas of partial differentiation,

$$\frac{\delta}{\delta\xi} = \frac{\delta x}{\delta\xi}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\xi}\frac{\delta}{\delta y}$$

$$\frac{\delta}{\delta\eta} = \frac{\delta x}{\delta\eta}\frac{\delta}{\delta x} + \frac{\delta y}{\delta\eta}\frac{\delta}{\delta y}$$

$$\therefore \begin{Bmatrix} \frac{\delta}{\delta\xi} \\ \frac{\delta}{\delta\eta} \end{Bmatrix} = \begin{bmatrix} \frac{\delta x}{\delta\xi} & \frac{\delta y}{\delta\xi} \\ \frac{\delta x}{\delta\eta} & \frac{\delta y}{\delta\eta} \end{bmatrix} \times \begin{Bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \end{Bmatrix}$$

Here,

Jacobian matrix: $[J] = \begin{bmatrix} \frac{\delta x}{\delta \xi} & \frac{\delta y}{\delta \xi} \\ \frac{\delta x}{\delta \eta} & \frac{\delta y}{\delta \eta} \end{bmatrix}$

Here, $J_{11} = \frac{\delta x}{\delta \xi}$ & $J_{12} = \frac{\delta y}{\delta \xi}$

$J_{21} = \frac{\delta x}{\delta \eta}$ $\quad J_{22} = \frac{\delta y}{\delta \eta}$

Now, $J_{11} = \frac{\delta x}{\delta \xi} = \frac{\delta}{\delta \xi}\left(\sum N_i x_i\right)$

$$= \sum \frac{\delta N_i}{\delta \xi} x_i$$

So, $J_{12} = \sum \frac{\delta N_i}{\delta \xi} y_i \quad J_{21} = \sum \frac{\delta N_i}{\delta \eta} x_i \quad J_{22} = \sum \frac{\delta N_i}{\delta \eta} y_i$

So, $[J] = \begin{bmatrix} \sum \frac{\delta N_i}{\delta \xi} x_i & \sum \frac{\delta N_i}{\delta \xi} y_i \\ \sum \frac{\delta N_i}{\delta \eta} x_i & \sum \frac{\delta N_i}{\delta \eta} y_i \end{bmatrix}$

Now, $\begin{Bmatrix} \frac{\delta}{\delta \xi} \\ \frac{\delta}{\delta \eta} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \times \begin{Bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \end{Bmatrix}$

$\Rightarrow \begin{Bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \times \begin{Bmatrix} \frac{\delta}{\delta \xi} \\ \frac{\delta}{\delta \eta} \end{Bmatrix}$

$\therefore \begin{Bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \times \begin{Bmatrix} \delta/\delta \xi \\ \delta/\delta \eta \end{Bmatrix}$

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According to above equation,

$$\begin{Bmatrix} \frac{\delta u}{\delta x} \\ \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} \\ \frac{\delta v}{\delta y} \end{Bmatrix} = \begin{bmatrix} J_{11}^{*} & J_{12}^{*} & 0 & 0 \\ J_{12}^{*} & J_{22}^{*} & 0 & 0 \\ 0 & 0 & J_{11}^{*} & J_{12}^{*} \\ 0 & 0 & J_{21}^{*} & J_{22}^{*} \end{bmatrix} \times \begin{Bmatrix} \frac{\delta u}{\delta \xi} \\ \frac{\delta u}{\delta \eta} \\ \frac{\delta v}{\delta \xi} \\ \frac{\delta u}{\delta \eta} \end{Bmatrix}$$

NOW, Strain,

$$[\varepsilon] = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \times \begin{Bmatrix} \frac{\delta u}{\delta x} \\ \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} \\ \frac{\delta v}{\delta y} \end{Bmatrix}$$

Replacing $\begin{Bmatrix} \delta u/\delta x \\ \delta u/\delta y \\ \delta v/\delta x \\ \delta u/\delta y \end{Bmatrix}$,

$$[\varepsilon] = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} J_{11}^{*} & J_{12}^{*} & 0 & 0 \\ 0 & 0 & J_{21}^{*} & J_{22}^{*} \\ J_{12}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*} \end{bmatrix} \times \begin{Bmatrix} \frac{\delta u}{\delta \xi} \\ \frac{\delta u}{\delta \eta} \\ \frac{\delta v}{\delta \xi} \\ \frac{\delta u}{\delta \eta} \end{Bmatrix}$$

NOW, $u = \sum N_i u_i$ and $v = \sum N_i v_i$

$$\therefore \begin{Bmatrix} \frac{\delta u}{\delta \xi} \\ \frac{\delta u}{\delta \eta} \\ \frac{\delta v}{\delta \xi} \\ \frac{\delta v}{\delta \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} & 0 \\ \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} & 0 \\ 0 & \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} \\ 0 & \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\therefore [B] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} & 0 \\ \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} & 0 \\ 0 & \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} \\ 0 & \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} \end{bmatrix}$$

We know,

$$[k] = \iiint_V [B]^T [D] B \, dV$$

$$= t \iint [B]^T [D] [B] \, \delta x \, \delta y$$

NOW, $\delta x \, \delta y = |J| \, \delta \xi \, \delta \eta$

$$\therefore [k] = t \iint [B]^T [D] [B] \, |J| \, \delta \xi \, \delta \eta$$ (Ans).

* Two Most Important Formula For Maths:

Jacobian Matrix:

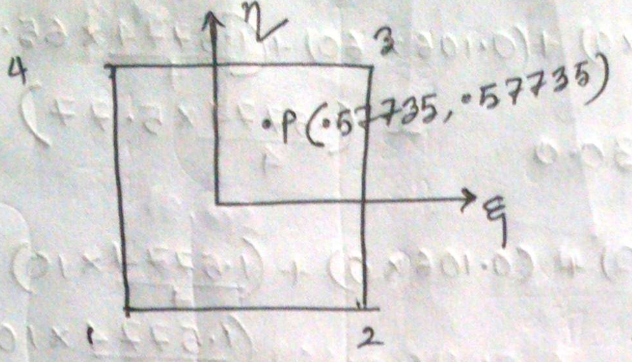
$$\underline{J} = \begin{bmatrix} \sum \frac{\delta N_i}{\delta \xi} x_i & \sum \frac{\delta N_i}{\delta \xi} y_i \\ \sum \frac{\delta N_i}{\delta \eta} x_i & \sum \frac{\delta N_i}{\delta \eta} y_i \end{bmatrix}$$

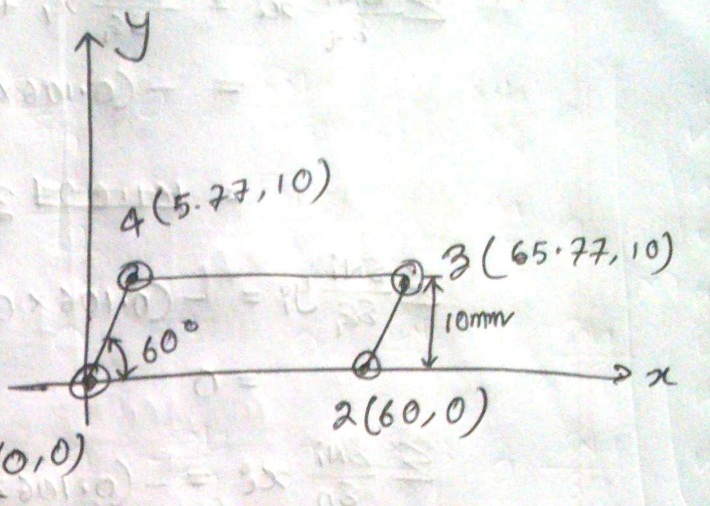
Strain Displacement matrix:

$$\underline{B} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times \begin{bmatrix} \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} & 0 \\ \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} & 0 \\ 0 & \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} \\ 0 & \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} \end{bmatrix}$$

Mathematical Problems:

Example 13.1:





Now,

Jacobian Matrix: $[J] = \begin{bmatrix} \sum \frac{\delta N_i}{\delta \xi} x_i & \sum \frac{\delta N_i}{\delta \xi} y_i \\ \sum \frac{\delta N_i}{\delta \eta} x_i & \sum \frac{\delta N_i}{\delta \eta} y_i \end{bmatrix}$

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}$$

$$\therefore \frac{\delta N_1}{\delta \xi} = -\frac{1-\eta}{4} = -\frac{1-0.57735}{4} = -0.106$$

$$\frac{\delta N_1}{\delta \eta} = -\frac{1-\xi}{4} = -0.106$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$\therefore \frac{\delta N_2}{\delta \xi} = \frac{1-\eta}{4} = 0.106$$

$$\frac{\delta N_2}{\delta \eta} = -\frac{1+\xi}{4} = -\frac{1.5774}{4} = -0.394$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4}$$

$$\therefore \frac{\delta N_3}{\delta \xi} = \frac{1+\eta}{4} = 1.5774/4 = 0.394$$

$$\frac{\delta N_3}{\delta \eta} = \frac{1+\xi}{4} = \frac{1.5774}{4} = 0.394$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

$$\therefore \frac{\delta N_4}{\delta \xi} = \frac{-(1+\eta)}{4} = -\frac{1.5774}{4} = -0.394$$

$$\frac{\delta N_4}{\delta \eta} = \frac{1-\xi}{4} = 0.106$$

$$\therefore \sum \frac{\delta N_i}{\delta \xi} x_i = \frac{\delta N_1}{\delta \xi} x_1 + \frac{\delta N_2}{\delta \xi} x_2 + \frac{\delta N_3}{\delta \xi} x_3 + \frac{\delta N_4}{\delta \xi} x_4$$

$$= -(0.106 \times 0) + (0.106 \times 60) + \left(\frac{1.5774}{4} \times 65.77\right) - \left(\frac{1.5774}{4} \times 5.77\right)$$

$$= \cancel{101.004}\ 30.0$$

$$\sum \frac{\delta N_i}{\delta \xi} y_i = -(0.106 \times 0) + (0.106 \times 0) + \left(\frac{1.5774}{4} \times 10\right) - \left(\frac{1.5774}{4} \times 10\right)$$

$$= 0$$

$$\sum \frac{\delta N_i}{\delta \eta} x_i = -(0.106 \times 0) - \left(\cancel{0.1}\frac{1.577}{4} \times 60\right) + \left(\frac{1.577}{4} \times 65.77\right) + (0.106 \times 5.77)$$

$$= \cancel{9.713}\ 2.89$$

$$\sum \frac{\delta N_i}{\delta \eta} y_i = -(0.106 \times 0) - \left(\frac{1.5774}{4} \times 0\right) + \left(\frac{1.5774}{4} \times 10\right) + (0.106 \times 10)$$

$$= \cancel{16.834}\ 5.0$$

$$\therefore [J] = \begin{bmatrix} 30 & 0 \\ 2.89 & 5.0 \end{bmatrix}$$

$$\therefore [J]^{-1} = \frac{1}{30 \times 5.0 - (2.89 \times 0)} \times \begin{bmatrix} 5.0 & -2.89 \\ 0 & 30 \end{bmatrix}$$

$$= \frac{1}{150} \begin{bmatrix} 5.0 & -2.89 \\ 0 & 30 \end{bmatrix}$$

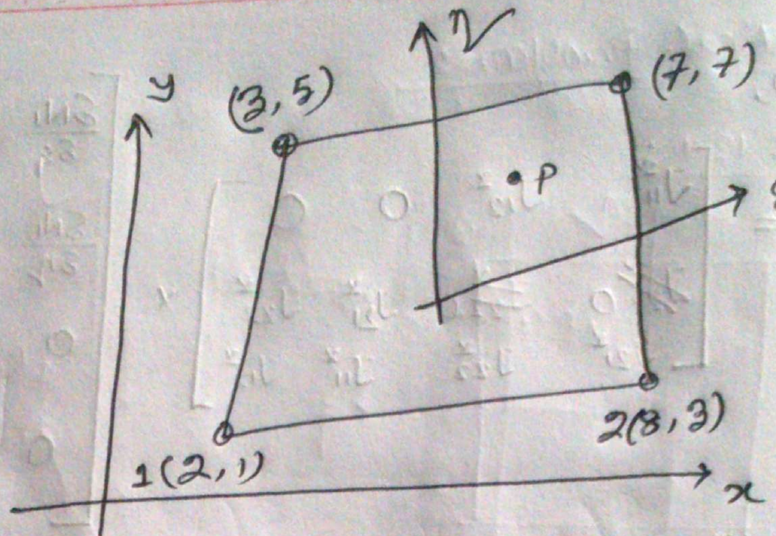
$$\therefore [J^*] = \begin{bmatrix} 0.033 & -0.0193 \\ 0 & 0.20 \end{bmatrix}$$

Now,

$$[B] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ \cancel{J_{21}} \; 0 & \cancel{J_{22}^*} \; 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times \begin{bmatrix} \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} & 0 \\ \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} & 0 \\ 0 & \frac{\delta N_1}{\delta \xi} & 0 & \frac{\delta N_2}{\delta \xi} & 0 & \frac{\delta N_3}{\delta \xi} & 0 & \frac{\delta N_4}{\delta \xi} \\ 0 & \frac{\delta N_1}{\delta \eta} & 0 & \frac{\delta N_2}{\delta \eta} & 0 & \frac{\delta N_3}{\delta \eta} & 0 & \frac{\delta N_4}{\delta \eta} \end{bmatrix}$$

$$= \begin{bmatrix} 0.033 & -0.193 & 0 & 0 \\ 0 & 0 & 0 & 0.20 \\ 0 & 0.20 & 0.033 & -0.193 \end{bmatrix} \times \begin{bmatrix} -0.106 & 0 & 0.106 & 0 & 0.394 & 0 & -0.394 & 0 \\ -0.106 & 0 & -0.394 & 0 & 0.394 & 0 & 0.106 & 0 \\ 0 & -0.106 & 0 & 0.106 & 0 & 0.394 & 0 & -0.394 \\ 0 & -0.106 & 0 & -0.394 & 0 & 0.394 & 0 & 0.106 \end{bmatrix}$$

Example 13.2:



So, $N_1 = \frac{(1-\xi)(1-\eta)}{4} = \frac{(1-0.50)(1-0.60)}{4} = 0.05$

$N_2 = \frac{(1+\xi)(1-\eta)}{4} = \frac{(1+0.50)(1-0.60)}{4} = 0.15$

$N_3 = \frac{(1+\xi)(1+\eta)}{4} = \frac{(1+0.50)(1+0.60)}{4} = 0.60$

$N_4 = \frac{(1-\xi)(1+\eta)}{4} = \frac{(1-0.50)(1+0.60)}{4} = 0.20$

$\therefore x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

$= (0.05 \times 2) + (0.15 \times 8) + (0.60 \times 7) + (0.20 \times 3)$

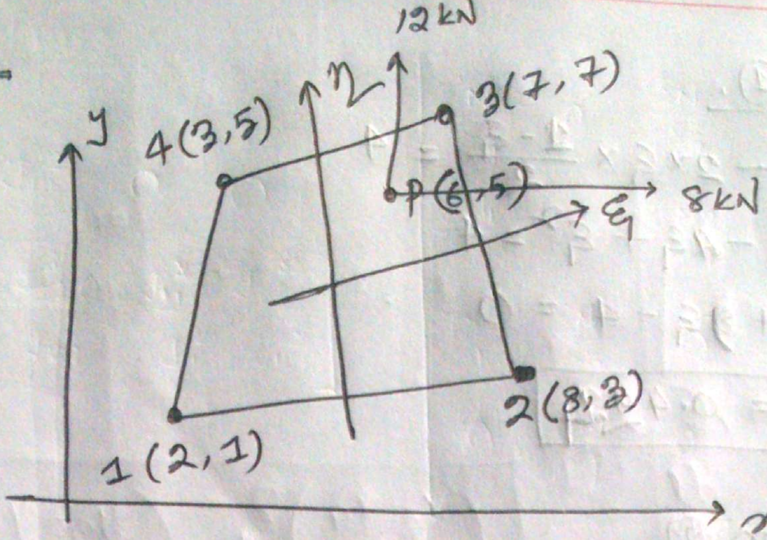
$= 6.1$

$y = (0.05 \times 1) + (0.15 \times 3) + (0.60 \times 7) + (0.20 \times 5)$

$= 5.7$

$\therefore$ Cartesian Co-ordinate $(6.1, 5.7)$ (Ans).

Example 13.3:



We know,

$$x = \sum N_i x_i$$

$$\Rightarrow 6 = N_1 \times 2 + N_2 \times 8 + N_3 \times 7 + N_4 \times 3$$

$$= \frac{2(1-\xi)(1-\eta)}{4} + \frac{8(1+\xi)(1-\eta)}{4} + \frac{7(1+\xi)(1+\eta)}{4} + \frac{3(1-\xi)(1+\eta)}{4}$$

$$\Rightarrow 24 = 2(1-\eta-\xi+\xi\eta) + 8(1-\eta+\xi-\xi\eta) + 7(1+\eta+\xi+\xi\eta) + 3(1+\eta-\xi-\xi\eta)$$

$$\Rightarrow 24 = (2+8+7+3) + (2\xi+8\xi+7\xi-3\xi) + (-2\eta-8\eta+7\eta+3\eta) + (2\xi\eta-8\xi\eta+7\xi\eta-3\xi\eta)$$

$$\Rightarrow 24 = 20 + 10\xi - 2\xi\eta$$

$$\Rightarrow 10\xi - 2\xi\eta = 4 \longrightarrow (1)$$

Again, $y = \sum N_i y_i$

$$\Rightarrow 5 = \frac{1\times(1-\xi)(1-\eta)}{4} + \frac{3(1+\xi)(1-\eta)}{4} + \frac{7(1+\xi)(1+\eta)}{4} + \frac{5(1-\xi)(1+\eta)}{4}$$

$$\Rightarrow 20 = (1-\eta-\xi+\xi\eta) + 3(1-\eta+\xi-\xi\eta) + 7(1+\eta+\xi+\xi\eta) + 5(1+\eta-\xi-\xi\eta)$$

$$\Rightarrow 20 = (1+3+7+5) + (-\xi+3\xi+7\xi-5\xi) + (-\eta-3\eta+7\eta+5\eta) + (1-3+7-5)\xi\eta$$

$$\Rightarrow 20 = 16 + 4\xi + 8\eta$$

$$\Rightarrow 4\xi + 8\eta = 4$$

$$\therefore \xi + 2\eta = 1 \quad \therefore \eta = \frac{1-\xi}{2} \longrightarrow (2)$$

From ①,

$$10\xi - 2\times\xi\times\frac{2-\xi}{2} = 4$$

$$\Rightarrow 10\xi - 2\xi + \xi^2 = 4$$

$$\Rightarrow \xi^2 + 8\xi - 4 = 0$$

$$\therefore \boxed{\xi = 0.424}$$

From ②,

$$\eta = \frac{1-\xi}{2}$$

$$\therefore \eta = \frac{1-0.424}{2}$$

$$\boxed{\eta = 0.2878}$$

$$\therefore N_1 = \frac{(1-\xi)(1-\eta)}{4} = \frac{(1-0.424)(1-0.2878)}{4} = 0.103$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4} = 0.254$$

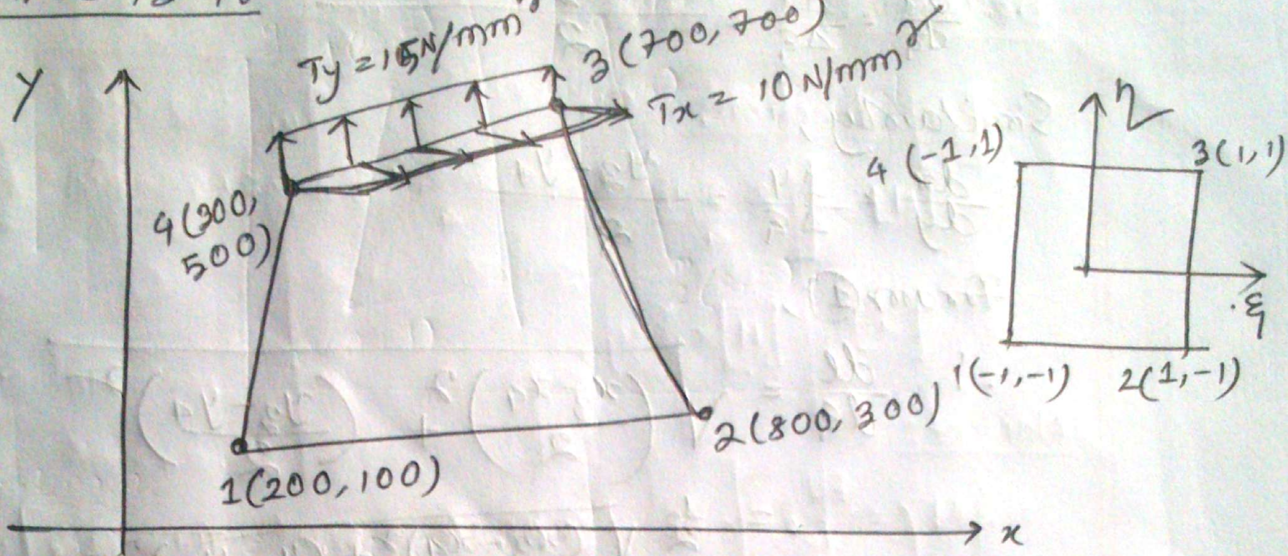
$$N_3 = \frac{(1+\xi)(1+\eta)}{4} = 0.458$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4} = 0.185$$

$$\therefore \begin{Bmatrix} F_{x1} \\ F_{x2} \\ F_{x3} \\ F_{x4} \end{Bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \{8\} = \begin{bmatrix} 0.824 \\ \cancel{0.2}\ 2.032 \\ 3.664 \\ 1.480 \end{bmatrix}$$

$$\therefore \begin{Bmatrix} F_{y1} \\ F_{y2} \\ F_{y3} \\ F_{y4} \end{Bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \times \{12\} = \begin{bmatrix} 1.236 \\ 3.048 \\ 5.496 \\ 2.220 \end{bmatrix} \text{(Ans).}$$

Example 13.4:



for edge 3-4, $\eta = 1$

$$\therefore N_1 = \frac{(1-\xi)(1-\eta)}{4} = \frac{(1-\xi)(1-1)}{4} = 0$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4} = 0$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} = \frac{2(1+\xi)}{4} = \frac{1+\xi}{2}$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4} = \frac{2(1-\xi)}{4} = \frac{1-\xi}{2}$$

Now,

$$\{F_x\} = \iint_A [N]^T \times \{T_x\} \times dA$$

$$= t \int [N]^T \{T_x\}\, dl$$

Now, $dl = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$\Rightarrow \frac{dl}{d\xi} = \sqrt{\left(\frac{\Delta x}{\Delta \xi}\right)^2 + \left(\frac{\Delta y}{\Delta \xi}\right)^2} \longrightarrow (1)$$

Now, $x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

$$= \frac{1+\xi}{2} x_3 + \frac{1-\xi}{2} x_4$$

$$= \frac{(1+\xi)x_3 + (1-\xi)x_4}{2}$$

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$$\therefore \frac{dx}{d\xi} = \frac{\Delta x}{\Delta \xi} = \frac{x_3 - x_4}{2}$$

Similarly,

$$\frac{dx}{dy} = \frac{\Delta y}{\Delta \xi} = \frac{y_3 - y_4}{2}$$

From (1),

$$\frac{dl}{d\xi} = \sqrt{\left(\frac{x_3 - x_4}{2}\right)^2 + \left(\frac{y_3 - y_4}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_3 - x_4)^2 + (y_3 - y_4)^2}$$

$$\Rightarrow \frac{dl}{d\xi} = \frac{1}{2} l_{34}$$

$$\therefore dl = \frac{1}{2} l_{34} \, d\xi$$

So,
$$\{F_x\} = t \int_{-1}^{1} [N]^T \{T_x\} \times \frac{1}{2} l_{34} \, d\xi$$

$$= \frac{l_{34} t}{2} \int_{-1}^{1} \begin{bmatrix} 0 \\ 0 \\ \frac{1+\xi}{2} \\ \frac{1-\xi}{2} \end{bmatrix} \times \{T_x\} \, d\xi$$

Now,
$$\int_{-1}^{1} \frac{1+\xi}{2} \cdot d\xi = 1$$

$$\int_{-1}^{1} \frac{1-\xi}{2} \, d\xi = 1$$

So,
$$\{F_x\} = \frac{l_{34} t}{2} \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \times \{T_x\}$$

NOW,

$$l_{34} = \sqrt{(700-300)^2 + (700-500)^2} = 447.21 mm$$

$$t = 20 mm$$

$$T_x = 10 N/mm^2$$

$$\therefore \frac{l_{34} t}{2} = 4472.10$$

$$\therefore \begin{Bmatrix} f_{x1} \\ f_{x2} \\ f_{x3} \\ f_{x3} \end{Bmatrix} = 4472.10 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \times \{10\}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 44721 \\ 44721 \end{bmatrix} \text{(Ans).}$$

Similarly,

$$\begin{Bmatrix} f_{y1} \\ f_{y2} \\ f_{y3} \\ f_{y4} \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \times 4472.10 \times \{15\}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 67081.5 \\ 67081.5 \end{bmatrix} \text{(Ans).}$$

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Chapter → 14
Analysis of Beam

Deflection:

Shape function:

$$[N] = \left[\frac{2-3\xi+\xi^3}{4} \quad \frac{l_e}{2}\cdot\frac{1-\xi-\xi^2+\xi^3}{4} \quad \frac{2+3\xi-\xi^3}{4} \quad \frac{l_e}{2}\,\frac{-1-\xi+\xi^2+\xi^3}{4} \right]$$

$$\therefore \quad [u] = [N] \times \{u\}_e$$

Moment:

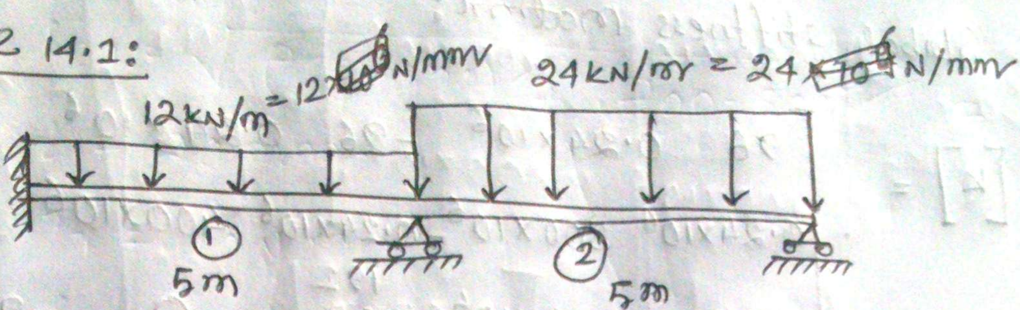
$$\{M\} = \frac{EI}{l_e^2}\left[6\xi \quad -(1-3\xi)l_e \quad -6\xi \quad (1+3\xi)l_e \right] \times \cancel{\{\sigma\}_e}\,\{u\}_e$$

Shear Force:

$$\{V\} = \frac{EI}{l_e^3}\left[12 \quad 6L \quad -12 \quad 6l \right] \times \cancel{\{V\}_e}\,\{u\}_e$$

Mathematical Problems:

Example 14.1:



NOW, $\dfrac{EI}{L^3} = \dfrac{2\times10^5 \times 5\times10^6}{(5000)^3} = 8.0$

For Element 1:

$$[k_1] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$= 8.0 \times \begin{bmatrix} 12 & 30000 & -12 & 3000 \\ 30000 & 100\times10^6 & -30000 & 50\times10^6 \\ -12 & -30000 & +12 & -30000 \\ 30000 & 50\times10^6 & -30000 & 100\times10^6 \end{bmatrix}$$

$$= \begin{bmatrix} 96 & 0.24\times10^6 & -96 & 0.24\times10^6 \\ 0.24\times10^6 & 800\times10^6 & -0.24\times10^6 & 400\times10^6 \\ -96 & -0.24\times10^6 & 96 & -0.24\times10^6 \\ 0.24\times10^6 & 400\times10^6 & -0.24\times10^6 & 800\times10^6 \end{bmatrix}$$

Similarly, $[k_2] = [k_1]$

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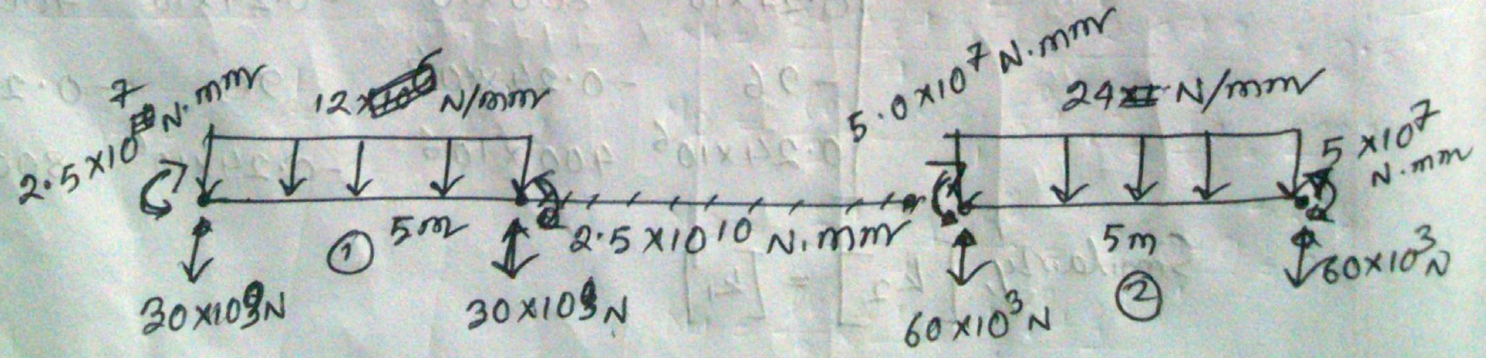
∴ Global stiffness matrix:

$$[K] = \begin{bmatrix} 96 & 0.24\times10^6 & -96 & 0.24\times10^6 & 0 & 0 \\ 0.24\times10^6 & 800\times10^6 & -0.24\times10^6 & 400\times10^6 & 0 & 0 \\ -96 & -0.24\times10^6 & \overset{192}{\cancel{108}} & 0 & -96 & 0.24\times10^6 \\ 0.24\times10^6 & 400\times10^6 & 0 & 1600\times10^6 & -0.24\times10^6 & 400\times10^6 \\ 0 & 0 & -96 & -0.24\times10^6 & 96 & -0.24\times10^6 \\ 0 & 0 & 0.24\times10^6 & 400\times10^6 & -0.24\times10^6 & 800\times10^6 \end{bmatrix}$$

Now,

$$\{F_1\} = \begin{bmatrix} -30\times10^3 \\ -2.5\times10^7 \\ -30\times10^3 \\ +2.5\times10^7 \end{bmatrix} \qquad \{F_2\} = \begin{bmatrix} -60\times10^3 \\ -5.0\times10^7 \\ -60\times10^3 \\ +5.0\times10^7 \end{bmatrix}$$



$$\therefore \{F\} = \begin{Bmatrix} -30\times10^3 \\ -2.5\times10^7 \\ -90\times10^3 \\ -2.5\times10^7 \\ -60\times10^3 \\ +5.0\times10^7 \end{Bmatrix}$$

Now,

$$[K]\times\{u\} = \{F\}$$

Now, $u_1 = u_2 = u_3 = u_5 = 0$

So,

$$\begin{bmatrix} 96 & 0.24\times10^6 & -96 & 0.24\times10^6 & 0 & 0 \\ 0.24\times10^6 & 800\times10^6 & -0.24\times10^6 & 400\times10^6 & 0 & 0 \\ -96 & -0.24\times10^6 & 192 & 0 & -96 & 0.24\times10^6 \\ 0.24\times10^6 & 400\times10^6 & 0 & 1600\times10^6 & -0.24\times10^6 & 400\times10^6 \\ 0 & 0 & -96 & -0.24\times10^6 & 96 & -0.24\times10^6 \\ 0 & 0 & 0.24\times10^6 & 400\times10^6 & -0.24\times10^6 & 800\times10^6 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ 0 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} -30\times10^3 \\ -2.5\times10^7 \\ -90\times10^3 \\ -2.5\times10^7 \\ -60\times10^3 \\ +5\times10^7 \end{Bmatrix}$$

So, $\begin{bmatrix} 1600\times10^6 & 400\times10^6 \\ 400\times10^6 & 800\times10^6 \end{bmatrix} \times \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} -2.5\times10^7 \\ +5.0\times10^7 \end{Bmatrix}$

$\therefore \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} = \begin{bmatrix} -0.0357 \\ +0.0804 \end{bmatrix}$ radian

1/28

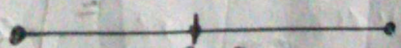
-9/112

Now,

$[R] = [k] \times \{u\} + \begin{Bmatrix} 30\times10^3 \\ 2.5\times10^7 \\ 90\times10^3 \\ 2.5\times10^7 \\ 60\times10^3 \\ -5\times10^7 \end{Bmatrix}$

$\begin{Bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \\ R_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 21432 \\ 10.72\times10^6 \\ 109296 \\ 40000 \\ 49272 \\ 0 \end{bmatrix}$

For ~~pei~~ element 1:



$\xi = -1$ $\xi = 0$ $\xi = 1$

$N_1 = \dfrac{2-3\xi+\xi^3}{4} = \dfrac{2-0+0}{4} = 0.50$

$N_2 = \dfrac{l_e}{2} \times \dfrac{1-\xi-\xi^2+\xi^3}{4} = \dfrac{l_e}{8} = \dfrac{5000}{8}$
$= 625$

$N_3 = \dfrac{2+3\xi-\xi^3}{4} = \dfrac{2+0-0}{4} = 0.50$

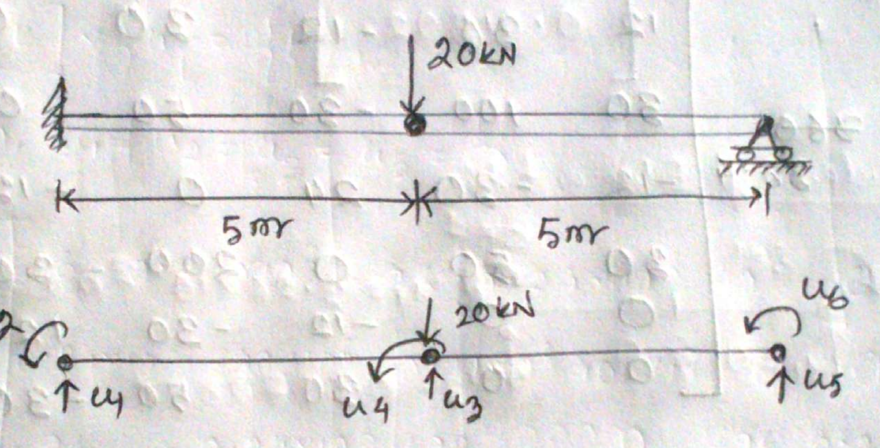
$N_4 = \dfrac{l_e}{2} \times \dfrac{-1-\xi+\xi^2+\xi^3}{4}$
$= -\dfrac{l_e}{8} = -625$

NOW,

$$\{u\} = \begin{bmatrix} 0.50 & 625 & 0.50 & -625 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0357 \end{Bmatrix}$$

$$\therefore \{u\} = \{22.3125\} \text{ mm} \quad \underline{(\text{Ans.})}$$

Example 14·2:



NOW, $E = 200\,GPa = 200 \times 10^3\ N/mm^2 = 200 \times 10^6\ kN/m^2$

$I = 24 \times 10^{-6}\ m^4$

$L = 5m$

$$\therefore \frac{EI}{L^3} = \cancel{360}\ 38.4$$

NOW, $$[k_1] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6 & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$= \cancel{360}\ 38.4 \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}$$

Similarly,

$$[K_2] = \cancel{960}\ 38.4 \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}$$

$\therefore$ Global stiffness matrix:

$$[K] = \overset{1}{\cancel{960}}\ 38.4 \begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix}$$

Now,

$$\{F_{x_1}\} = \begin{bmatrix} 0 \\ 0 \\ -20 \\ 0 \end{bmatrix} \qquad \{F_{x_2}\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \{F_x\} = \begin{bmatrix} 0 \\ 0 \\ -20 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \cancel{960}\ 38.4 \begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -20 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Now, $u_1 = u_2 = u_5 = 0$

So, $$\cancel{960}\ 38.4 \begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ 0 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -20 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

So, $$\cancel{960}\ 38.4 \begin{bmatrix} 24 & 0 & 30 \\ 0 & 200 & 50 \\ 30 & 50 & 100 \end{bmatrix} \times \begin{Bmatrix} u_3 \\ u_4 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} u_3 \\ u_4 \\ u_6 \end{Bmatrix} = \frac{1}{\cancel{960}\ 38.4} \begin{bmatrix} 24 & 0 & 30 \\ 0 & 200 & 50 \\ 30 & 50 & 100 \end{bmatrix}^{-1} \times \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u_3 \\ u_4 \\ u_6 \end{Bmatrix} = \begin{bmatrix} -1.4583 \\ -0.125 \\ 0.50 \end{bmatrix} \times \frac{1}{\cancel{960}\, 38.4}$$

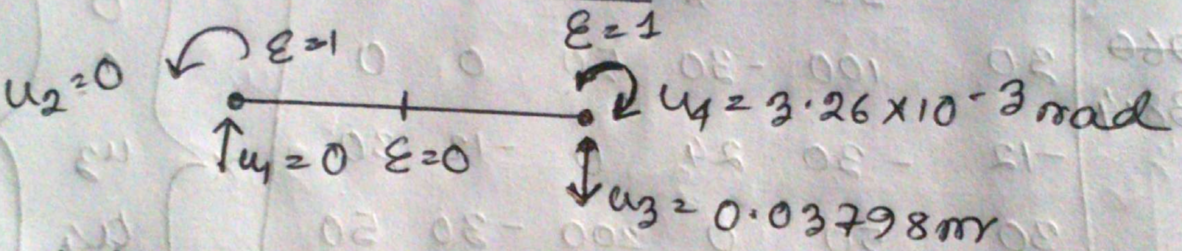
$-\frac{35}{24}$

$-\frac{1}{8}$

$\frac{1}{2}$

$$= \begin{bmatrix} -0.03798 \\ \cancel{130.21 \times 10^{-6}} \\ -3.26 \times 10^{-3} \\ 0.01302 \end{bmatrix}$$

Considering Element 1:



NOW,

$$V = \frac{EI}{l^3} \begin{bmatrix} 12 & +6le & -12 & 6l \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= 38.4 \times \begin{bmatrix} 12 & 30 & -12 & 30 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -0.03798 \\ -3.26 \times 10^{-3} \end{Bmatrix}$$

$$= 38.4 \times \{0.35796\}$$

$$= \{13.75\} \text{ kN}$$

NOW,

$$\frac{2 - 3\xi + \xi^3}{4} = \frac{2 - 3 + 1}{4} = 0$$

$$\frac{le}{2} \times \frac{1 - \xi - \xi^2 + \xi^3}{4} = \frac{5}{2} \times \frac{1 - 1 - 1 + 1}{4} = 0$$

$$\frac{2 + 3\xi - \xi^3}{4} = \frac{2 + 3 - 1}{4} = 1$$

$$\frac{le}{2} \times \frac{-1 - \xi + \xi^2 + \xi^3}{4} = \frac{5}{2} \times \frac{-1 - 1 + 1 + 1}{4} = 0$$

NOW,

$$\{M\} = \boxed{\cancel{\frac{EI}{le^2}}} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -1.4583 \\ -0.125 \end{Bmatrix} \times \frac{1}{38.4}$$

$$= \boxed{\cancel{\frac{200\times10^6 \times 24\times10^{-6}}{25\times 38.4\times 5}}} \times \{-1.4583\} \times \frac{1}{38.4}$$

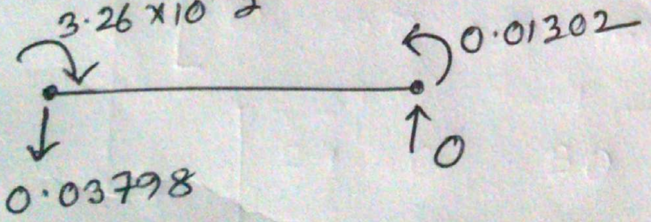
$$= \{-0.03798\}\ m$$

$$M = \frac{EI}{l^2} \begin{bmatrix} 6 & 2le & -6 & 4le \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -0.03798 \\ -0.00326 \end{Bmatrix}$$

$$= 192 \begin{bmatrix} 6 & 10 & -6 & 20 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -0.03798 \\ -0.00326 \end{Bmatrix}$$

$$= \{31.23\}\ kN \cdot m$$

<u>Considering Element 2:</u>



$$\text{NOW, } \{v\} = \frac{EI}{le^3} \begin{bmatrix} 12 & 6L & -12 & 6L \end{bmatrix} \times \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= 38.4 \times \begin{bmatrix} 12 & 30 & -12 & 30 \end{bmatrix} \times \begin{Bmatrix} -0.03798 \\ -3.26\times10^{-3} \\ 0 \\ 0.01302 \end{Bmatrix}$$

$$= \{-6.26\}\ kN$$

Ans: 1. $u_3 = 0.03798\ m = 37.98\ mm$

$u_4 = 3.26 \times 10^{-3}\ rad\ (\circlearrowleft)$

$u_6 = 0.01302\ rad\ (\circlearrowright)$

2. $V(Left) = 13.75\ kN$

$V(Right) = -6.26\ kN$

3.