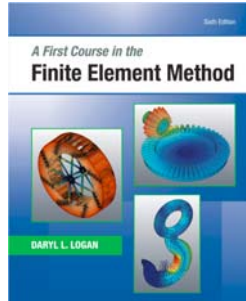


Chapter 2 – Introduction to the Stiffness (Displacement) Method



Learning Objectives

- To define the stiffness matrix
- To derive the stiffness matrix for a spring element
- To demonstrate how to assemble stiffness matrices into a global stiffness matrix
- To illustrate the concept of direct stiffness method to obtain the global stiffness matrix and solve a spring assemblage problem
- To describe and apply the different kinds of boundary conditions relevant for spring assemblages
- To show how the potential energy approach can be used to both derive the stiffness matrix for a spring and solve a spring assemblage problem

The Stiffness (Displacement) Method

This section introduces some of the basic concepts on which the **direct stiffness method** is based.

The linear spring is simple and an instructive tool to illustrate the basic concepts.

The steps to develop a finite element model for a linear spring follow our general 8 step procedure.

-
- 1. Discretize and Select Element Types** - Linear spring elements
 - 2. Select a Displacement Function** - Assume a variation of the displacements over each element.
 - 3. Define the Strain/Displacement and Stress/Strain Relationships** - use elementary concepts of equilibrium and compatibility.

The Stiffness (Displacement) Method

- 4. Derive the Element Stiffness Matrix and Equations** - Define the stiffness matrix for an element and then consider the derivation of the stiffness matrix for a linear-elastic spring element.

- 5. Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions** - We then show how the total stiffness matrix for the problem can be obtained by superimposing the stiffness matrices of the individual elements in a direct manner.

The term ***direct stiffness method*** evolved in reference to this method.

The Stiffness (Displacement) Method

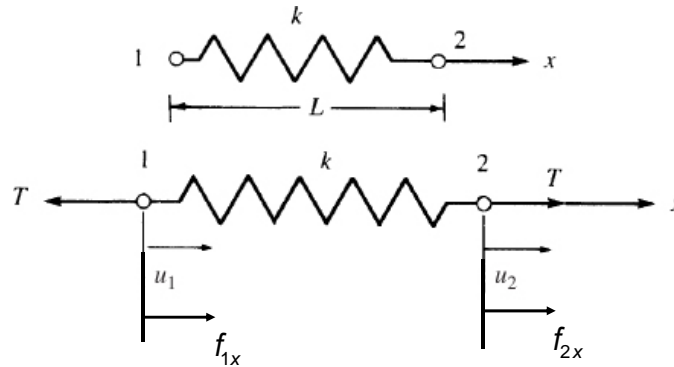
- 6. Solve for the Unknown Degrees of Freedom (or Generalized Displacements)** - Solve for the nodal displacements.

- 7. Solve for the Element Strains and Stresses** - The reactions and internal forces association with the bar element.

- 8. Interpret the Results**

The Stiffness (Displacement) Method

- Select Element Type** - Consider the linear spring shown below. The spring is of length L and is subjected to a nodal tensile force, T directed along the x -axis.



Note: Assumed sign conventions

The Stiffness (Displacement) Method

- Select a Displacement Function** - A displacement function $u(x)$ is assumed.

$$u = a_1 + a_2x$$

In general, the number of coefficients in the displacement function is equal to the total number of degrees of freedom associated with the element. We can write the displacement function in matrix forms as:

$$u = \begin{bmatrix} 1 & x \end{bmatrix}_{1 \times 2} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_{2 \times 1}$$

The Stiffness (Displacement) Method

We can express u as a function of the nodal displacements u_i by evaluating u at each node and solving for a_1 and a_2 .

$$\left. \begin{aligned} u(x=0) &= u_1 = a_1 \\ u(x=L) &= u_2 = a_2L + a_1 \end{aligned} \right\} \text{Boundary Conditions}$$

Solving for a_2 :

$$a_2 = \frac{u_2 - u_1}{L}$$

Substituting a_1 and a_2 into u gives:

$$u = \left(\frac{u_2 - u_1}{L} \right) x + u_1 = \left(1 - \frac{x}{L} \right) u_1 + \left(\frac{x}{L} \right) u_2$$

The Stiffness (Displacement) Method

In matrix form:

$$u = \left[\begin{pmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{pmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Or in another form:

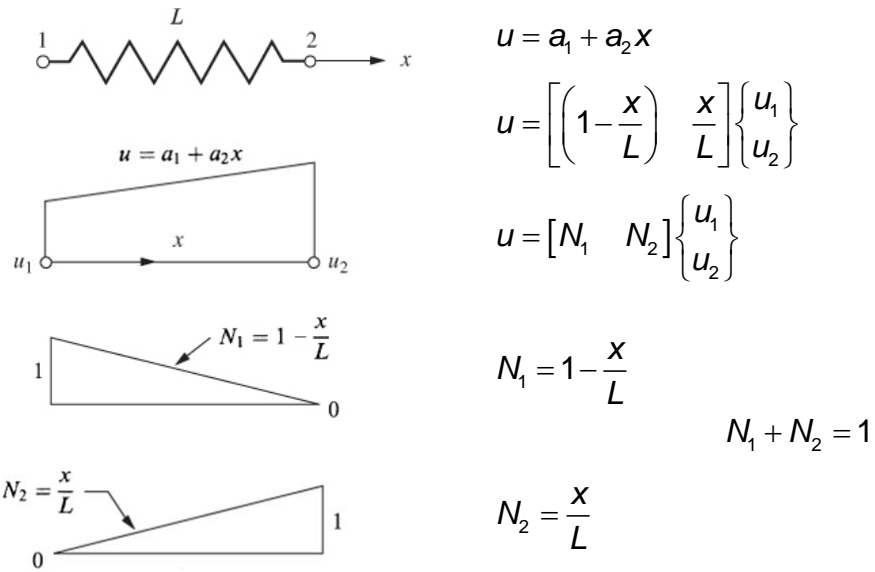
$$u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Where N_1 and N_2 are defined as:

$$N_1 = 1 - \frac{x}{L} \quad N_2 = \frac{x}{L}$$

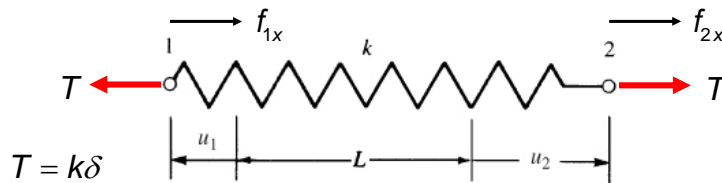
The functions N_i are called ***interpolation functions*** because they describe how the assumed displacement function varies over the domain of the element. In this case the interpolation functions are linear.

The Stiffness (Displacement) Method



The Stiffness (Displacement) Method

3. Define the Strain/Displacement and Stress/Strain Relationships - Tensile forces produce a total elongation (deformation) δ of the spring. For linear springs, the force T and the displacement u are related by Hooke's law:



where deformation of the spring δ is given as:

$$\delta = u(L) - u(0) \qquad \delta = u_2 - u_1 \qquad \begin{matrix} f_{1x} = -T \\ f_{2x} = T \end{matrix}$$

The Stiffness (Displacement) Method

- 4. Step 4 - Derive the Element Stiffness Matrix and Equations** - We can now derive the spring element stiffness matrix as follows:

Rewrite the forces in terms of the nodal displacements:

$$T = -f_{1x} = k(u_2 - u_1) \Rightarrow f_{1x} = k(u_1 - u_2)$$

$$T = f_{2x} = k(u_2 - u_1) \Rightarrow f_{2x} = k(-u_1 + u_2)$$

We can write the last two force-displacement relationships in matrix form as:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

The Stiffness (Displacement) Method

This formulation is valid as long as the spring deforms along the x axis. The coefficient matrix of the above equation is called the ***local stiffness matrix k*** :

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

- 5. Step 4 - Assemble the Element Equations and Introduce Boundary Conditions**

The ***global stiffness matrix*** and the ***global force vector*** are assembled using the nodal force equilibrium equations, and force/deformation and compatibility equations.

$$\mathbf{K} = [\mathbf{K}] = \sum_{e=1}^N \mathbf{k}^{(e)} \quad \mathbf{F} = \{\mathbf{F}\} = \sum_{e=1}^N \mathbf{f}^{(e)}$$

where \mathbf{k} and \mathbf{f} are the element stiffness and force matrices expressed in global coordinates.

The Stiffness (Displacement) Method

6. Step 6 - Solve for the Nodal Displacements

Solve the displacements by imposing the boundary conditions and solving the following set of equations:

$$\{F\} = [K]\{d\} \quad \Rightarrow \quad \mathbf{F} = \mathbf{Kd}$$

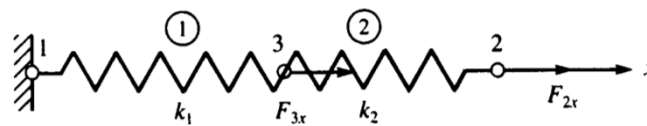
7. Step 7 - Solve for the Element Forces

Once the displacements are found, the forces in each element may be calculated from:

$$T = k\delta = k(u_2 - u_1)$$

The Stiffness Method – Spring Example 1

Consider the following two-spring system shown below:



where the element axis x coincides with the global axis x .

For element 1:

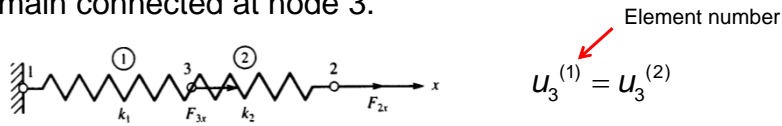
$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}$$

For element 2:

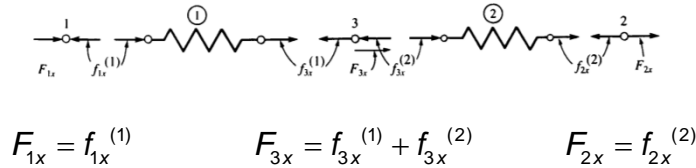
$$\begin{Bmatrix} f_{3x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix}$$

The Stiffness Method – Spring Example 1

Both continuity and compatibility require that both elements remain connected at node 3.



We can write the nodal equilibrium equation at each node as:



The Stiffness Method – Spring Example 1

Therefore the force-displacement equations for this spring system are:

$$F_{1x} = k_1 u_1 - k_1 u_3 \qquad F_{2x} = -k_2 u_3 + k_2 u_2$$

$$F_{3x} = \overset{\text{Element 1}}{(-k_1 u_1 + k_1 u_3)} + \overset{\text{Element 2}}{(k_2 u_3 - k_2 u_2)}$$

In matrix form the above equations are:

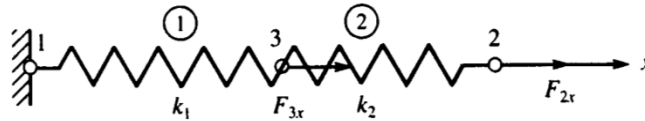
$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \mathbf{F} = \mathbf{Kd}$$

where **F** is the **global nodal force vector**, **d** is called the **global nodal displacement vector**, and **K** is called the **global stiffness matrix**.

The Stiffness Method – Spring Example 1

Assembling the Total Stiffness Matrix by Superposition

Consider the spring system defined in the last example:



The elemental stiffness matrices may be written for each element.

For element 1:

$$\mathbf{k}^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{matrix} u_1 \\ u_3 \end{matrix}$$

For element 2:

$$\mathbf{k}^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{matrix} u_3 \\ u_2 \end{matrix}$$

The Stiffness Method – Spring Example 1

Write the stiffness matrix in global format for element 1 as follows:

$$k_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix}$$

For element 2:

$$k_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix}$$

The Stiffness Method – Spring Example 1

Apply the force equilibrium equations at each node.

$$\begin{Bmatrix} f_{1x}^{(1)} \\ 0 \\ f_{3x}^{(1)} \end{Bmatrix} + \begin{Bmatrix} 0 \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

The above equations give:

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

The Stiffness Method – Spring Example 1

To avoid the expansion of the each elemental stiffness matrix, we can use a more direct, shortcut form of the stiffness matrix.

$$\mathbf{k}^{(1)} = \begin{matrix} & u_1 & u_3 \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & u_1 \\ & & u_3 \end{matrix} \quad \mathbf{k}^{(2)} = \begin{matrix} & u_3 & u_2 \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & u_3 \\ & & u_2 \end{matrix}$$

The global stiffness matrix may be constructed by directly adding terms associated with the degrees of freedom in $\mathbf{k}^{(1)}$ and $\mathbf{k}^{(2)}$ into their corresponding locations in the \mathbf{K} as follows:

$$\mathbf{K} = \begin{matrix} & u_1 & u_2 & u_3 \\ \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} & u_1 \\ & & & u_2 \\ & & & u_3 \end{matrix}$$

The Stiffness Method – Spring Example 1

Boundary Conditions

In order to solve the equations defined by the global stiffness matrix, we must apply some form of constraints or supports or the structure will be free to move as a rigid body.

Boundary conditions are of two general types:

1. **homogeneous boundary conditions** (the most common) occur at locations that are completely prevented from movement;
2. **nonhomogeneous boundary conditions** occur where finite non-zero values of displacement are specified, such as the settlement of a support.

The Stiffness Method – Spring Example 1

Consider the equations we developed for the two-spring system. We will consider node 1 to be fixed $u_1 = 0$. The equations describing the elongation of the spring system become:

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

Expanding the matrix equations gives:

$$\left. \begin{aligned} F_{1x} &= -k_1 u_3 \\ F_{2x} &= -k_2 u_3 + k_2 u_2 \\ F_{3x} &= -k_2 u_2 + (k_1 + k_2) u_3 \end{aligned} \right\} \text{Solve for } u_2 \text{ and } u_3$$

The Stiffness Method – Spring Example 1

The second and third equation may be written in matrix form as:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix}$$

Once we have solved the above equations for the unknown nodal displacements, we can use the first equation in the original matrix to find the support reaction.

$$F_{1x} = -k_1 u_3$$

For homogeneous boundary conditions, we can delete the row and column corresponding to the zero-displacement degrees-of-freedom.

The Stiffness Method – Spring Example 1

Let's again look at the equations we developed for the two-spring system.

However, this time we will consider a nonhomogeneous boundary condition at node 1: $u_1 = \delta$.

The equations describing the elongation of the spring system become:

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \delta \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

Expanding the matrix equations gives:

$$F_{1x} = k_1 \delta - k_1 u_3 \qquad F_{2x} = -k_2 u_3 + k_2 u_2$$

$$F_{3x} = -k_1 \delta + k_1 u_3 + k_2 u_3 - k_2 u_2$$

The Stiffness Method – Spring Example 1

By considering the second and third equations because they have known nodal forces we get:

$$F_{2x} = -k_2 u_3 + k_2 u_2 \qquad F_{3x} = -k_1 \delta + k_1 u_3 + k_2 u_3 - k_2 u_2$$

In matrix form the above equations are:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ F_{3x} + k_1 \delta \end{Bmatrix}$$

For nonhomogeneous boundary conditions, we must transfer the terms from the stiffness matrix to the right-hand-side force vector before solving for the unknown displacements.

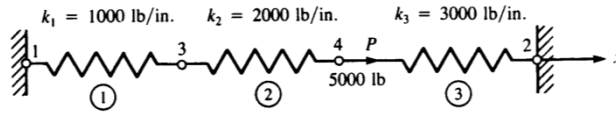
The Stiffness Method – Spring Example 1

Once we have solved the above equations for the unknown nodal displacements, we can use the first equation in the original matrix to find the support reaction.

$$F_{1x} = k_1 \delta - k_1 u_3$$

The Stiffness Method – Spring Example 2

Consider the following three-spring system:



The elemental stiffness matrices for each element are:

$$\mathbf{k}^{(1)} = 1000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 1 & 3 \\ 3 & 1 \end{matrix} \quad \mathbf{k}^{(2)} = 2000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 3 & 4 \\ 4 & 3 \end{matrix}$$

$$\mathbf{k}^{(3)} = 3000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 4 & 2 \\ 2 & 4 \end{matrix}$$

The Stiffness Method – Spring Example 2

Using the concept of superposition (the direct stiffness method), the global stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} \text{Element 1} & & & \\ 1000 & 0 & -1000 & 0 \\ 0 & \text{Element 3} & 0 & -3000 \\ -1000 & 0 & \text{Element 2} & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix}$$

The global force-displacement equations are:

$$\begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix}$$

The Stiffness Method – Spring Example 2

We have homogeneous boundary conditions at nodes 1 and 2 ($u_1 = 0$ and $u_2 = 0$).

The global force-displacement equations reduce to:

$$\begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix}$$

The Stiffness Method – Spring Example 2

Substituting for the known force at node 4 ($F_{4x} = 5,000 \text{ lb}$) gives:

$$\begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5,000 \end{Bmatrix}$$

Solving for u_3 and u_4 gives:

$$u_3 = \frac{10}{11} \text{ in} \quad u_4 = \frac{15}{11} \text{ in}$$

The Stiffness Method – Spring Example 2

To obtain the global forces, substitute the displacement in the force-displacement equations.

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{10}{11} \\ \frac{15}{11} \end{Bmatrix}$$

Solving for the forces gives:

$$F_{1x} = -\frac{10,000}{11} \text{ lb} \quad F_{2x} = -\frac{45,000}{11} \text{ lb}$$

$$F_{3x} = 0 \quad F_{4x} = \frac{55,000}{11} \text{ lb} = 5,000 \text{ lb}$$

The Stiffness Method – Spring Example 2

Next, use the local element equations to obtain the force in each spring.

For element 1: $\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{10}{11} \end{Bmatrix}$

The local forces are: $f_{1x} = -\frac{10,000}{11} \text{ lb} \quad f_{3x} = \frac{10,000}{11} \text{ lb}$

A free-body diagram of the spring element 1 is shown below.



The Stiffness Method – Spring Example 2

Next, use the local element equations to obtain the force in each spring.

For element 2:

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} \frac{10}{11} \\ \frac{15}{11} \end{Bmatrix}$$

The local forces are:

$$f_{3x} = -\frac{10,000}{11} \text{ lb} \quad f_{4x} = \frac{10,000}{11} \text{ lb}$$

A free-body diagram of the spring element 2 is shown below.



The Stiffness Method – Spring Example 2

Next, use the local element equations to obtain the force in each spring.

For element 3:

$$\begin{Bmatrix} f_{4x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} \frac{15}{11} \\ 0 \end{Bmatrix}$$

The local forces are:

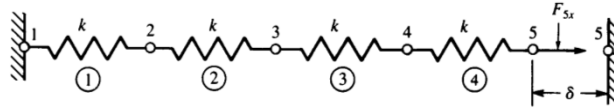
$$f_{4x} = \frac{45,000}{11} \text{ lb} \quad f_{2x} = -\frac{45,000}{11} \text{ lb}$$

A free-body diagram of the spring element 3 is shown below.



The Stiffness Method – Spring Example 3

Consider the following four-spring system:



The spring constant $k = 200 \text{ kN/m}$ and the displacement $\delta = 20 \text{ mm}$.

Therefore, the elemental stiffness matrices are:

$$\mathbf{k}^{(1)} = \mathbf{k}^{(2)} = \mathbf{k}^{(3)} = \mathbf{k}^{(4)} = 200 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ kN/m}$$

The Stiffness Method – Spring Example 3

Using superposition (the direct stiffness method), the global stiffness matrix is:

	Element 1	Element 2	Element 3	Element 4	
$\mathbf{K} =$	200	-200	0	0	0
	-200	400	-200	0	0
	0	-200	400	-200	0
	0	0	-200	400	-200
	0	0	0	-200	200

The Stiffness Method – Spring Example 3

The global force-displacement equations are:

$$\begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \\ F_{5x} \end{Bmatrix}$$

Applying the boundary conditions ($u_1 = 0$ and $u_5 = 20 \text{ mm}$) and the known forces (F_{2x} , F_{3x} , and F_{4x} equal to zero) gives:

$$\begin{bmatrix} 400 & -200 & 0 & 0 \\ -200 & 400 & -200 & 0 \\ 0 & -200 & 400 & -200 \\ 0 & 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ 0.02 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F_{5x} \end{Bmatrix}$$

The Stiffness Method – Spring Example 3

Rearranging the first three equations gives:

$$\begin{bmatrix} 400 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 400 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 4 \end{Bmatrix}$$

Solving for u_2 , u_3 , and u_4 gives:

$$u_2 = 0.005 \text{ m} \quad u_3 = 0.01 \text{ m} \quad u_4 = 0.015 \text{ m}$$

Solving for the forces F_{1x} and F_{5x} gives:

$$F_{1x} = -200(0.005) = -1.0 \text{ kN}$$

$$F_{5x} = -200(0.015) + 200(0.02) = 1.0 \text{ kN}$$

The Stiffness Method – Spring Example 3

Next, use the local element equations to obtain the force in each spring.

$$\text{For element 1: } \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \end{Bmatrix}$$

$$f_{1x} = -1.0kN \quad f_{2x} = 1.0kN$$

$$\text{For element 2: } \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.005 \\ 0.01 \end{Bmatrix}$$

$$f_{2x} = -1.0kN \quad f_{3x} = 1.0kN$$

The Stiffness Method – Spring Example 3

Next, use the local element equations to obtain the force in each spring.

$$\text{For element 3: } \begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.01 \\ 0.015 \end{Bmatrix}$$

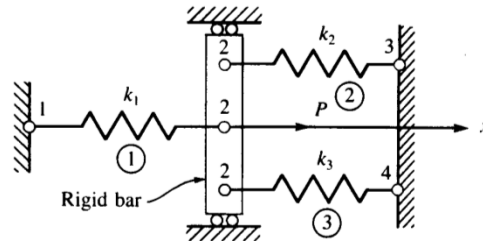
$$f_{3x} = -1.0kN \quad f_{4x} = 1.0kN$$

$$\text{For element 4: } \begin{Bmatrix} f_{4x} \\ f_{5x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.015 \\ 0.02 \end{Bmatrix}$$

$$f_{4x} = -1.0kN \quad f_{5x} = 1.0kN$$

The Stiffness Method – Spring Example 4

Consider the following spring system:



The boundary conditions are: $u_1 = u_3 = u_4 = 0$

The compatibility condition at node 2 is:

$$u_2^{(1)} = u_2^{(2)} = u_2^{(3)} = u_2$$

The Stiffness Method – Spring Example 4

Using the direct stiffness method: the elemental stiffness matrices for each element are:

$$\mathbf{k}^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \quad \mathbf{k}^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \quad \mathbf{k}^{(3)} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{matrix} 2 \\ 4 \end{matrix}$$

Using the concept of superposition (the direct stiffness method), the global stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Potential Energy Approach to Derive Spring Element Equations

One of the alternative methods often used to derive the element equations and the stiffness matrix for an element is based on the ***principle of minimum potential energy***.

This method has the advantage of being more general than the methods involving nodal and element equilibrium equations, along with the stress/strain law for the element.

The principle of minimum potential energy is more adaptable for the determination of element equations for complicated elements (those with large numbers of degrees of freedom) such as the plane stress/strain element, the axisymmetric stress element, the plate bending element, and the three-dimensional solid stress element.

Total Potential Energy

The total potential energy π_p is defined as the sum of the internal strain energy U and the potential energy of the external forces Ω :

$$\pi_p = U + \Omega$$

Strain energy is the capacity of the internal forces (or stresses) to do work through deformations (strains) in the structure.

The potential energy of the external forces Ω is the capacity of forces such as body forces, surface traction forces, and applied nodal forces to do work through the deformation of the structure.

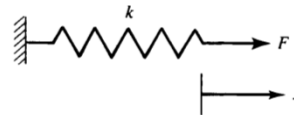
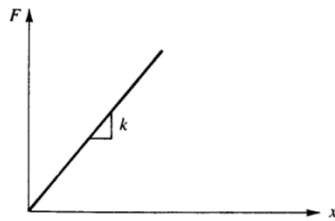
Total Potential Energy

Recall the force-displacement relationship for a linear spring:

$$F = kx$$

The differential internal work (or strain energy) dU in the spring is the internal force multiplied by the change in displacement which the force moves through:

$$dU = Fdx = (kx) dx$$

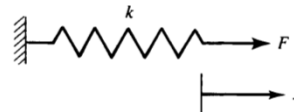
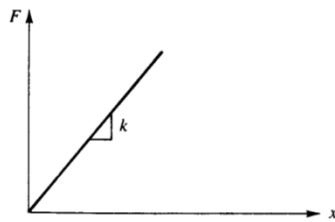


Total Potential Energy

The total strain energy is: $U = \int_L dU = \int_0^x (kx) dx = \frac{1}{2} kx^2$

The strain energy is the area under the force-displacement curve. The potential energy of the external forces is the work done by the external forces:

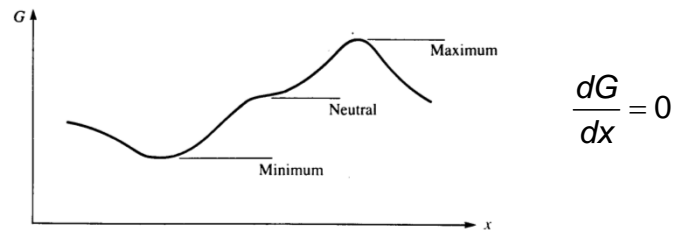
$$\Omega = -Fx$$



Total Potential Energy

Therefore, the total potential energy is: $\pi_p = \frac{1}{2}kx^2 - Fx$

The concept of a **stationary value** of a function G is shown below:



The function G is expressed in terms of x .

To find a value of x yielding a stationary value of $G(x)$, we use differential calculus to differentiate G with respect to x and set the expression equal to zero.

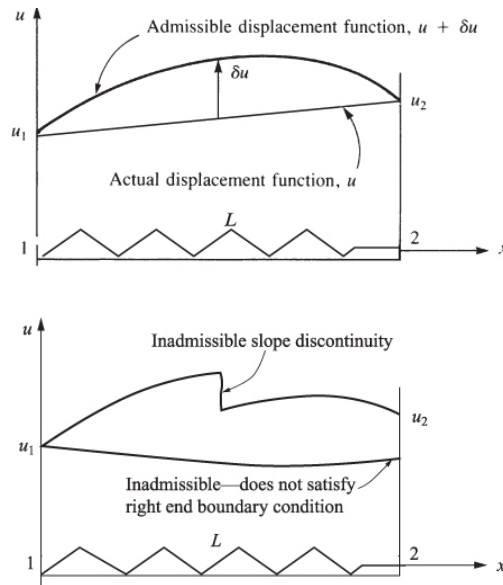
Total Potential Energy

We can replace G with the total potential energy π_p and the coordinate x with a discrete value d_i . To minimize π_p we first take the **variation** of π_p (we will not cover the details of variational calculus):

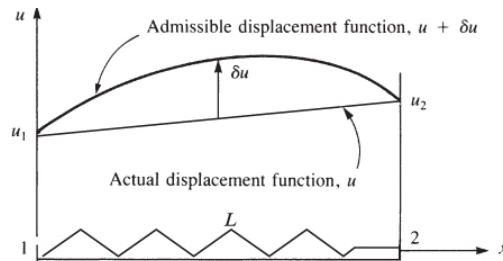
$$\delta\pi_p = \frac{\partial\pi_p}{\partial d_1} \delta d_1 + \frac{\partial\pi_p}{\partial d_2} \delta d_2 + \dots + \frac{\partial\pi_p}{\partial d_n} \delta d_n$$

The principle states that equilibrium exist when the d_i define a structure state such that $\delta\pi_p = 0$ for arbitrary admissible variations δd_i from the equilibrium state.

Total Potential Energy



Total Potential Energy



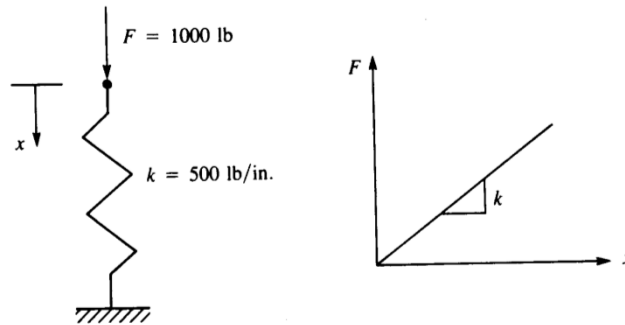
To satisfy $\delta\pi_p = 0$, all coefficients associated with δd_i must be zero independently, therefore:

$$\frac{\partial \pi_p}{\partial d_i} = 0 \quad i = 1, 2, \dots, n \quad \text{or} \quad \frac{\partial \pi_p}{\partial \{d\}} = 0$$

Total Potential Energy – Example 5

Consider the following linear-elastic spring system subjected to a force of 1,000 lb.

Evaluate the potential energy for various displacement values and show that the minimum potential energy also corresponds to the equilibrium position of the spring.



Total Potential Energy – Example 5

The total potential energy is defined as the sum of the internal strain energy U and the potential energy of the external forces Ω :

$$\pi_p = U + \Omega \quad U = \frac{1}{2} kx^2 \quad \Omega = -Fx$$

The variation of π_p with respect to x is: $\delta\pi_p = \frac{\partial\pi_p}{\partial x} \delta x = 0$

Since δx is arbitrary and might not be zero, then: $\frac{\partial\pi_p}{\partial x} = 0$

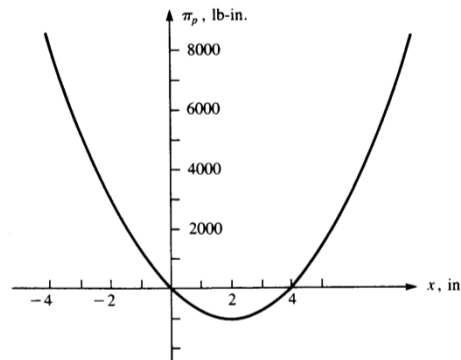
Total Potential Energy – Example 5

Using our express for π_p , we get:

$$\pi_p = \frac{1}{2}kx^2 - Fx = \frac{1}{2}500(\text{lb/in})x^2 - (1,000\text{lb})x$$

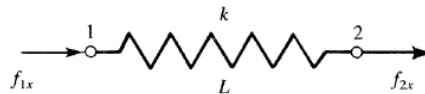
$$\frac{\partial \pi_p}{\partial x} = 0 = 500x - 1,000 \quad x = 2.0 \text{ in}$$

If we had plotted the total potential energy function π_p for various values of deformation we would get:



Total Potential Energy

Let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



The total potential energy π_p

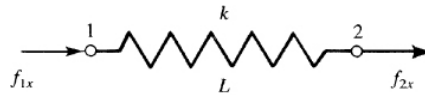
$$\pi_p = \frac{1}{2}k(u_2 - u_1)^2 - f_{1x}u_1 - f_{2x}u_2$$

Expanding the above express gives:

$$\pi_p = \frac{1}{2}k(u_2^2 - 2u_1u_2 + u_1^2) - f_{1x}u_1 - f_{2x}u_2$$

Total Potential Energy

Let's derive the spring element equations and stiffness matrix using the principle of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



Recall: $\frac{\partial \pi_p}{\partial d_i} = 0 \quad i = 1, 2, \dots, n \quad \text{or} \quad \frac{\partial \pi_p}{\partial \{d\}} = 0$

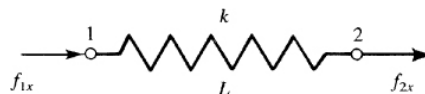
Therefore:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{k}{2}(-2u_2 + 2u_1) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{k}{2}(2u_2 - 2u_1) - f_{2x} = 0$$

Total Potential Energy

Let's derive the spring element equations and stiffness matrix using the principle of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



Therefore: $k(u_1 - u_2) = f_{1x}$

$$k(-u_1 + u_2) = f_{2x}$$

In matrix form the equations are:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Total Potential Energy – Example 6

Obtain the total potential energy of the spring system shown below and find its minimum value.



The potential energy π_p for element 1 is:

$$\pi_p^{(1)} = \frac{1}{2} k_1 (u_3 - u_1)^2 - f_{1x} u_1 - f_{3x} u_3$$

The potential energy π_p for element 2 is:

$$\pi_p^{(2)} = \frac{1}{2} k_2 (u_4 - u_3)^2 - f_{3x} u_3 - f_{4x} u_4$$

Total Potential Energy – Example 6

Obtain the total potential energy of the spring system shown below and find its minimum value.



The potential energy π_p for element 3 is:

$$\pi_p^{(3)} = \frac{1}{2} k_3 (u_2 - u_4)^2 - f_{2x} u_2 - f_{4x} u_4$$

The total potential energy π_p for the spring system is:

$$\pi_p = \sum_{e=1}^3 \pi_p^{(e)}$$

Total Potential Energy – Example 6

Minimizing the total potential energy π_p :

$$\frac{\partial \pi_p}{\partial u_1} = 0 = -k_1 u_3 + k_1 u_1 - f_{1x}^{(1)}$$

$$\frac{\partial \pi_p}{\partial u_2} = 0 = k_3 u_2 - k_3 u_4 - f_{2x}^{(3)}$$

$$\frac{\partial \pi_p}{\partial u_3} = 0 = k_1 u_3 - k_1 u_1 - k_2 u_4 + k_2 u_3 - f_{3x}^{(1)} - f_{3x}^{(2)}$$

$$\frac{\partial \pi_p}{\partial u_4} = 0 = k_2 u_4 - k_2 u_3 - k_3 u_2 + k_3 u_4 - f_{4x}^{(2)} - f_{4x}^{(3)}$$

Total Potential Energy – Example 6

In matrix form:

$$\begin{bmatrix} k_1 & 0 & k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x}^{(1)} + f_{3x}^{(2)} \\ f_{4x}^{(2)} + f_{4x}^{(3)} \end{Bmatrix}$$

Using the following force equilibrium equations:

$$f_{1x}^{(1)} = F_{1x}$$

$$f_{2x}^{(3)} = F_{2x}$$

$$f_{3x}^{(1)} + f_{3x}^{(2)} = F_{3x}$$

$$f_{4x}^{(2)} + f_{4x}^{(3)} = F_{4x}$$

Total Potential Energy – Example 6

The global force-displacement equations are:

$$\begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix}$$

The above equations are identical to those we obtained through the direct stiffness method.

Homework Problems

1. Do problems **2.4**, **2.10**, and **2.22** on pages 66 - 71 in your textbook "A First Course in the Finite Element Method" by D. Logan.

End of Chapter 2