Finite Element Method –An Introduction
(One Dimensional Problems)

by

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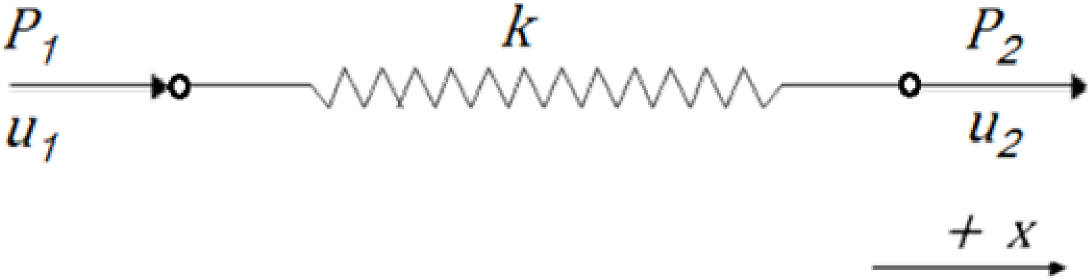
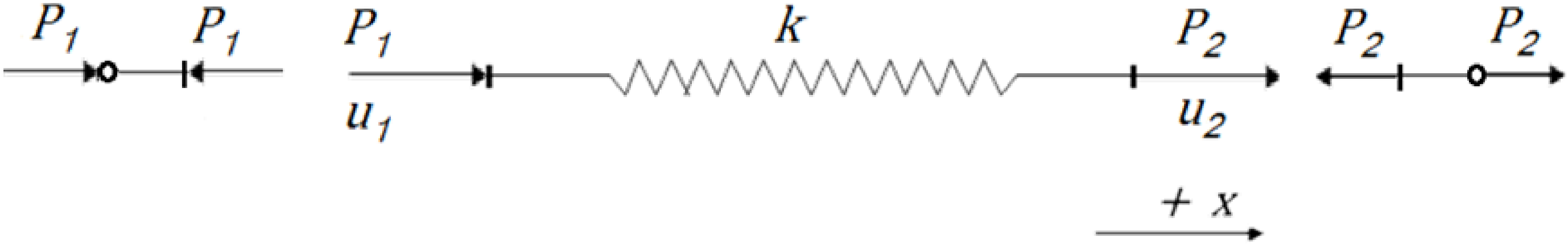
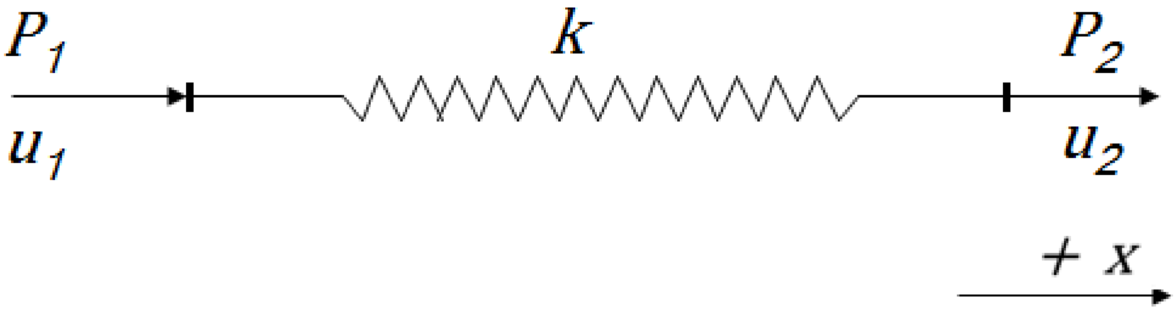
Elastic Spring





Elastic Spring



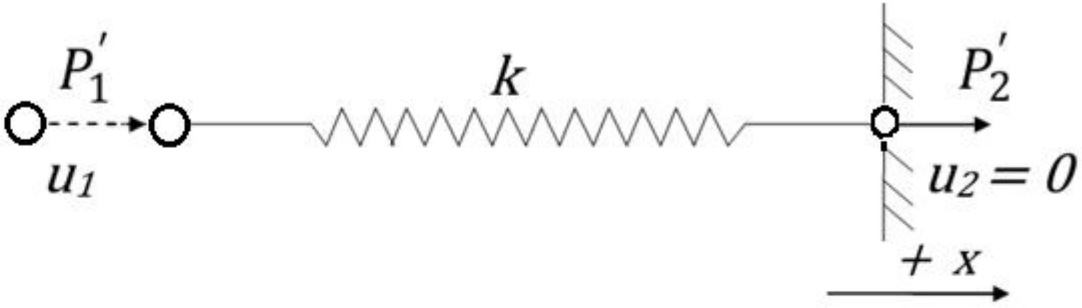
One Spring: 2 degrees of freedom (dofs)

We wish to establish a relationship between nodal forces and nodal displacements as:

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Elastic Spring

A.



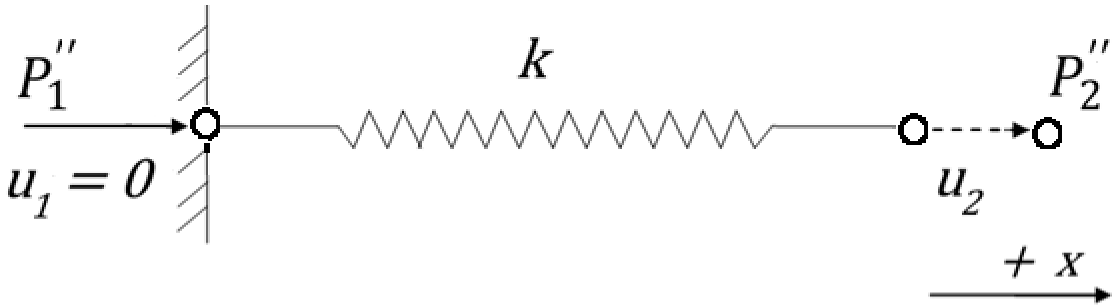
$$P_1' + P_2' = 0$$ Equilibrium eq.

$$P_1' = ku_1$$ Force-displacement rel.

$$\therefore P_2' = -P_1' = -ku_1$$

Elastic Spring

B.



$$P_1'' + P_2'' = 0$$ Equilibrium eq.

$$P_2 = ku_2$$ Force-displacement rel.

$$\therefore P_1'' = -P_2'' = -ku_2$$

Linear response – superpose two independent solutions

$$P_1 = P_1' + P_1'' = ku_1 - ku_2$$
$$P_2 = P_2' + P_2'' = -ku_1 + ku_2$$

In matrix form,

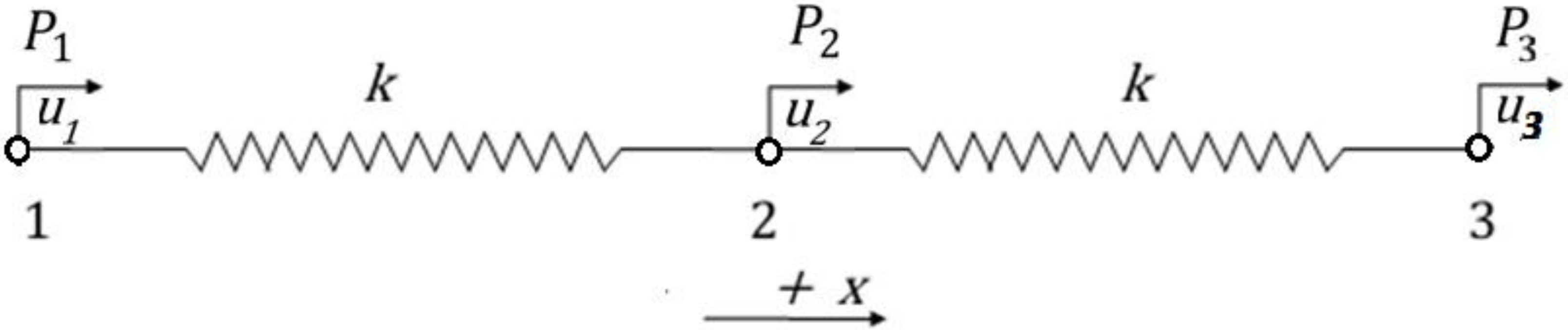
$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\vec{P} = \underline{K}\vec{d}$$

$\vec{P}$: Nodal force vector;

$\vec{d}$: Nodal displacement vector

Two Springs

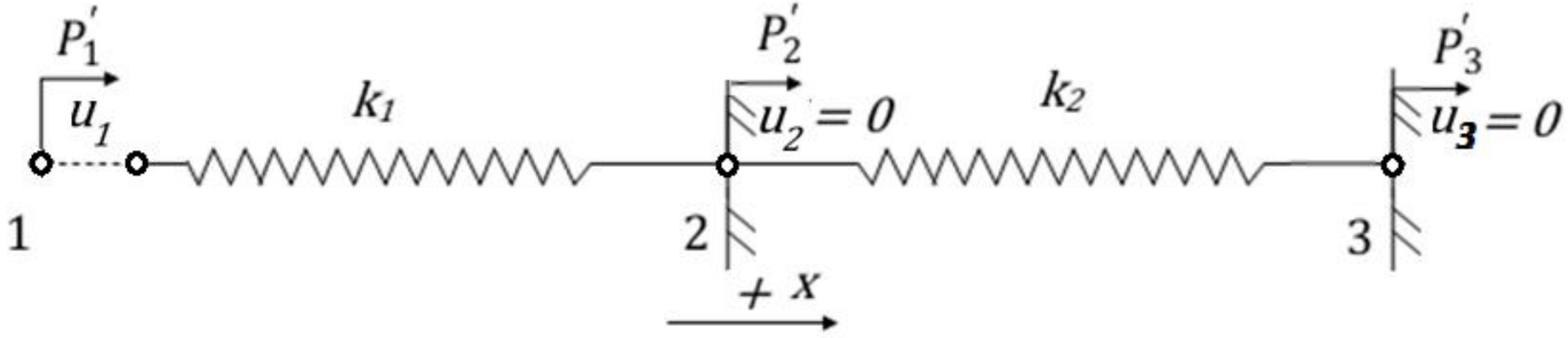


3 dofs

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Two Springs

1.



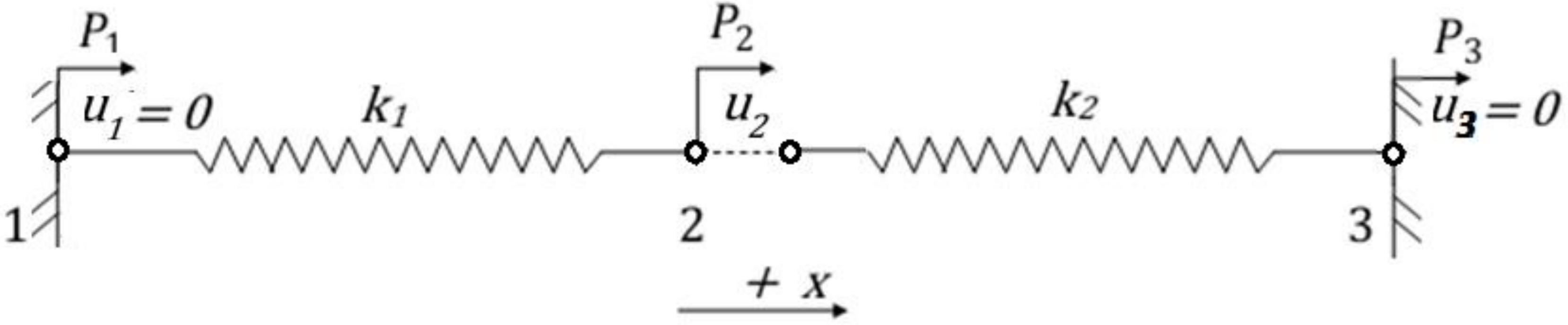
$$P_1' + P_2' + P_3' = 0$$ Equilibrium

$$P_1' = k_1 u_1$$ Force-Displacement rel.

$$P_3' = 0$$

Two Springs

2.



$P_1'' + P_2'' + P_3'' = 0$ Equilibrium

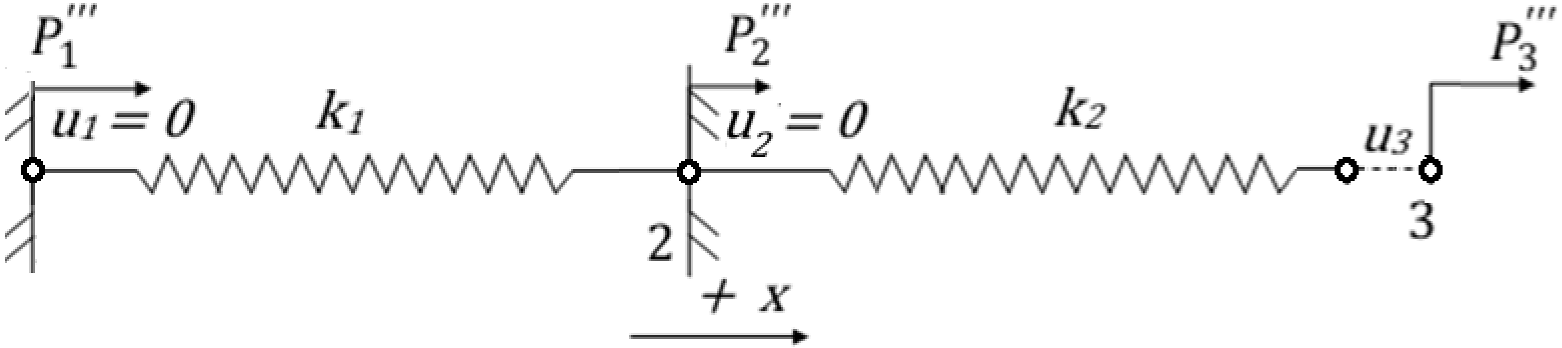
$P_2'' = (k_1+k_2)u_2$ Force-Displacement rel.

$P_3'' = -k_2 u_2$

$P_1'' = -k_1 u_2$

Two Springs

3.



$$P_1''' + P_2''' + P_3''' = 0$$ Equilibrium

$$P_3''' = k_2 u_3$$ Force-Displacement rel.

$$P_1''' = 0$$

Two Springs

4. For all dofs to be active, we obtain by superposition

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

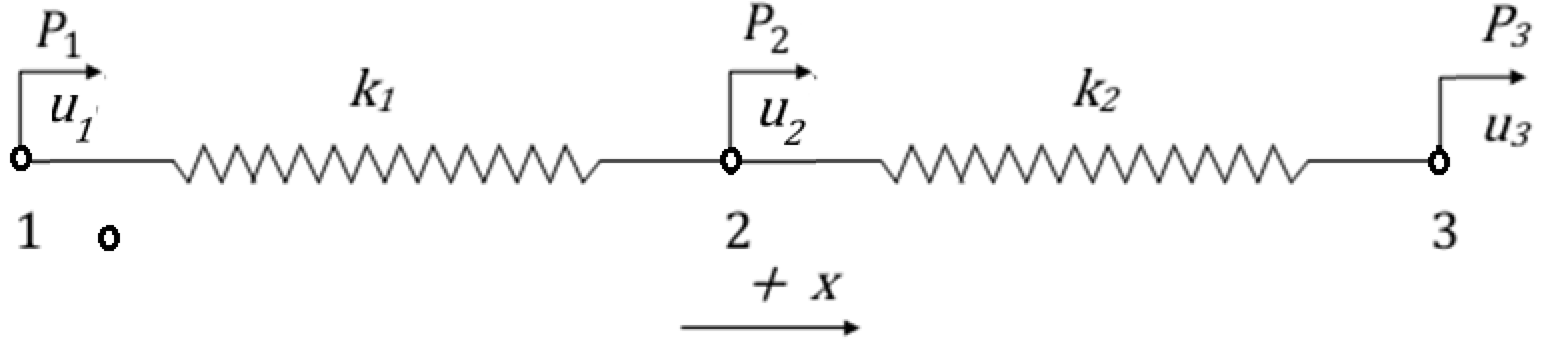
$$\vec{P} = \underline{K}\vec{d}$$

Some Properties of $\underline{K}$

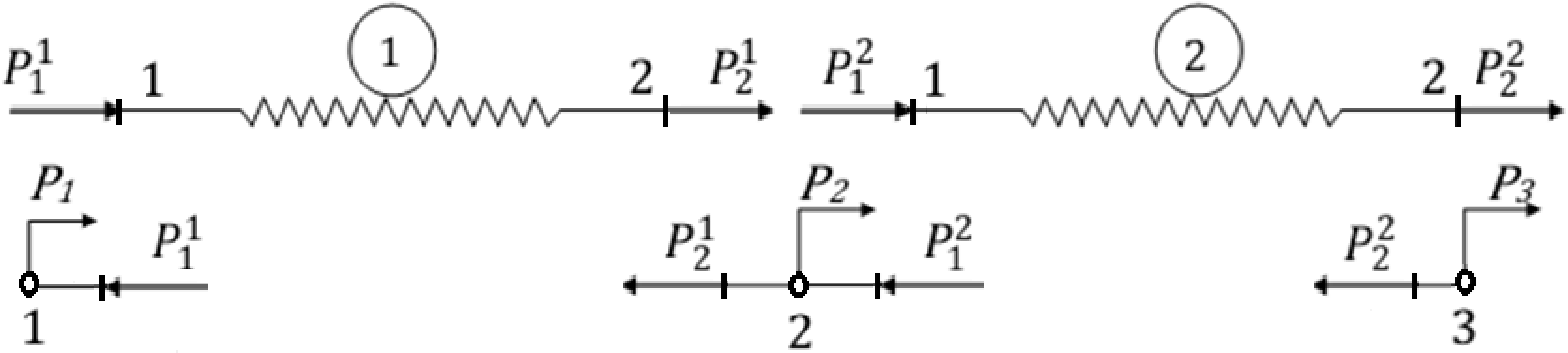
1. Sum of elements in any column is 0 equilibrium
2. $\underline{K}$ is symmetric
3. singular : no BC's
4. All terms on main diagonal positive.

   If this were not so, a +ive nodal force P_i *could produce a corresponding –ive* u_i*.*
5. If proper node numbering is done, $\underline{K}$ is banded.

An Alternative Procedure



Superposing the $\underline{K}_s$ of individual elements,



where,

1, 2, 3 :Global and Local node numbers

①,② :Element numbers

$$\begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix}; \begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix}$$

Our aim is to compute a 3x3 $\underline{K}$ matrix of the two spring assemblage

Continuity (Compatibility) conditions

$$u_1 = u_1^1$$
$$u_2 = u_2^1 = u_1^2$$
$$u_3 = u_2^2$$

Equilibrium of nodal forces

$$P_1 = P_1^1$$
$$P_2 = P_2^1 + P_1^2$$
$$P_3 = P_2^2$$

$$P_1 = [k_1 \quad -k_1]\begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = [k_1 \quad -k_1]\binom{u_1}{u_2}$$

$$= [k_1 \quad -k_1 \quad 0]\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$P_2 = [-k_1 \quad k_1]\begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} + [k_2 \quad -k_2]\begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix}$$

$$= [-k_1 \quad k_1]\binom{u_1}{u_2} + [k_2 \quad -k_2]\binom{u_2}{u_3}$$

$$= \begin{bmatrix} -k_1 & k_1 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{bmatrix} 0 & k_2 & -k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= \begin{bmatrix} -k_1 & (k_1 + k_2) & -k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$P_3 = \begin{bmatrix} -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = \begin{bmatrix} -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

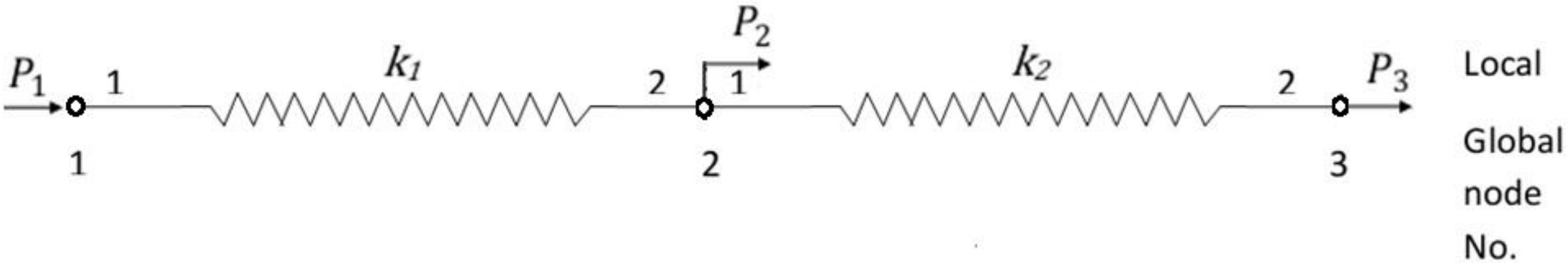
$$= \begin{bmatrix} 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\vec{P} = \underline{K}\vec{d}$$

Direct Stiffness Method -Assembly



$$\begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

We expand the local matrices into 3x3 for global assembly

$$\begin{pmatrix} P_1^1 \\ P_2^1 \\ 0 \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

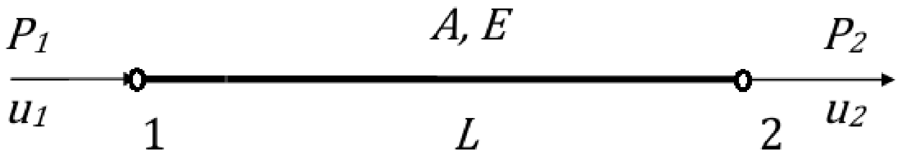
$$\begin{pmatrix} 0 \\ P_1^2 \\ P_2^2 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

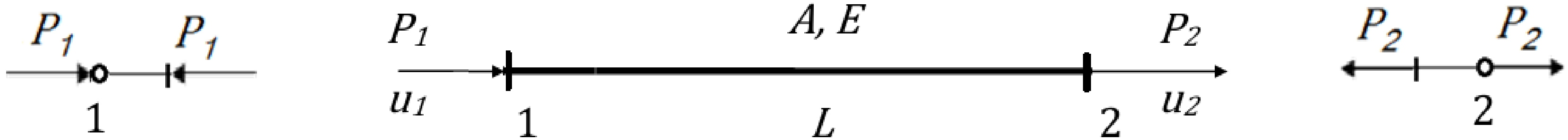
Superpose,

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} <= \begin{pmatrix} P_1^1 \\ P_2^1 + P_1^2 \\ P_2^2 \end{pmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

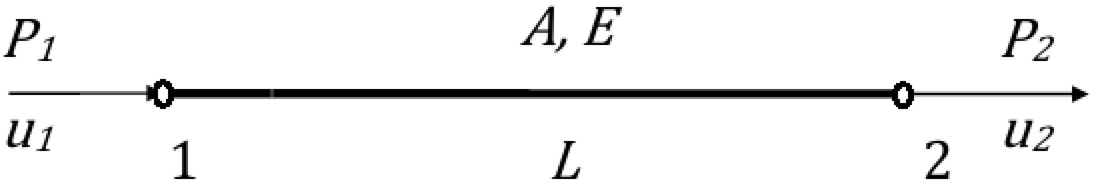
$$\vec{P} = \sum_e \overrightarrow{P^e} \qquad \underline{K} = \sum_e \underline{K}^e$$

Axial Rod





Axial Rod



$$k = \frac{AE}{L}$$

Where k = stiffness of spring

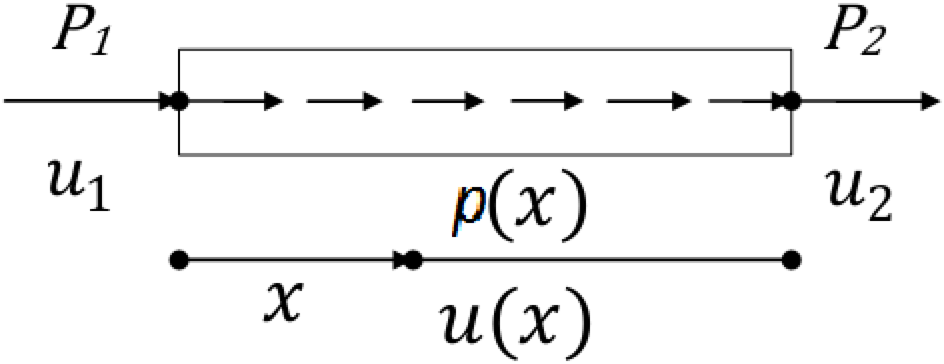
$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Direct Method – Simple 'discrete' elements

Variational Method
-Energy Method

$$d = \binom{u_1}{u_2}$$

$$u \equiv u(x)$$



Relate $u(x)$ to nodal displacements$\binom{u_1}{u_2}$.

Let $u(x)$ vary linearly within the element

$$u(x) \cong a + bx$$

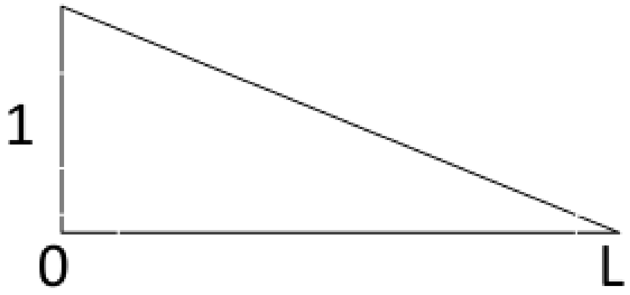
At $x = 0$ $\qquad u(0) = u_1 = a$

At $x = L$ $\qquad u(L) = u_2 = a + bL = u_1 + bL$

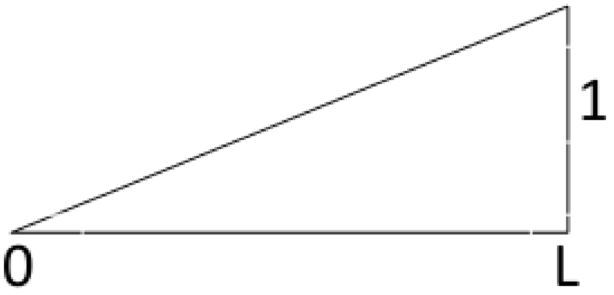
$$\therefore b = \frac{u_2 - u_1}{L}$$

$$u(x) \cong u_1 + \frac{u_2 - u_1}{L} x \cong \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2$$

$$\left(1 - \frac{x}{L}\right) \equiv N_1(x)$$



$$\left(\frac{x}{L}\right) \equiv N_2(x)$$



$$\therefore\ u(x) \cong N_1(x).u_1 + N_2(x).u_2$$

$$\cong \sum_{i=1}^{2} N_i(x).u_i$$

where N_i = Nodal Shape Functions

or $u(x) \cong [N_1(x) \quad N_2(x)] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$\boldsymbol{N} = [N_1(x) \quad N_2(x)]$ Shape Function Matrix

$u(x) \cong \boldsymbol{Nd}$ ← Discretization

If $\hat{u}(x)$ *approximates* $u(x)$,

$$Error = \hat{u}(x) - u(x)$$

Strains

$$\begin{aligned}
\varepsilon_x &= \frac{du(x)}{dx} \\
&\cong \frac{d}{dx}\hat{u}(x) \\
&\cong \frac{d}{dx}\left[N_1(x) \quad N_2(x)\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&\cong \boldsymbol{LNd} \\
&\cong \boldsymbol{B}(x)\boldsymbol{d} \quad \left[\boldsymbol{B}(x) \rightarrow \text{Strain Matrix}\right] \\
&\cong \left[\frac{dN_1(x)}{dx} \quad \frac{dN_2(x)}{dx}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\end{aligned}$$

$$\varepsilon \cong \boldsymbol{Bd}$$

Constitutive Relation

$\sigma_x = E\varepsilon_x$

In general $\boldsymbol{\sigma} = \boldsymbol{E}\,\boldsymbol{\varepsilon}$

Minimum Potential Energy

The Total Potential Energy pi is given by:

$\Pi = \mathrm{U} - \mathrm{W}$

$\delta\Pi = \delta(\mathrm{U}) - \delta(\mathrm{W}) = 0$ (Minimizing)

where, **U** is strain energy stored in the body during deformation and

W is the work done by the external loads

We have established

$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{d} \quad \boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{d} \quad \boldsymbol{\sigma} = \boldsymbol{E}\,\boldsymbol{\varepsilon}$

In the above, we have discretized displacement and strain expressions

Variational Statement

$$\delta\Pi = \int_V (\delta\varepsilon)^t . \sigma . dv \; - \int_V (\delta u)^t . p . dv \; = 0$$

Continuum expression

$$= \int_V \left\{ (B.\delta d)^t . E . (Bd) \right\} dv \; - \int_V (N.\delta d)^t . p . dv$$

$$= (\delta d)^t \left(\int_V B^t . E . B \, dv \right) d - (\delta d)^t \left(\int_V N^t . p . dv \right)$$

$$= (\delta d)^t \left[\left(\int_V B^t . E . B \, dv \right) d - \left(\int_V N^t . p . dv \right) \right]$$

$$\because \delta d \neq 0; \left(\int_V B^t . E . B \, dv \right) d - \left(\int_V N^t . p . dv \right) = 0$$

$$\therefore \quad \boldsymbol{kd} = \boldsymbol{P}$$

$$\boldsymbol{k} = \int_V \boldsymbol{B}^t . \boldsymbol{E} . \boldsymbol{B} dv \; ; \; \boldsymbol{P} = \int_V \boldsymbol{N}^t . \boldsymbol{p} \; dv$$

$$N_1 = \left(1 - \frac{x}{L}\right); \; N_2 = \frac{x}{L}$$

$$\frac{dN_1}{dx} = -\frac{1}{L}; \quad \frac{dN_2}{dx} = \frac{1}{L}, \quad dv = A.dx$$

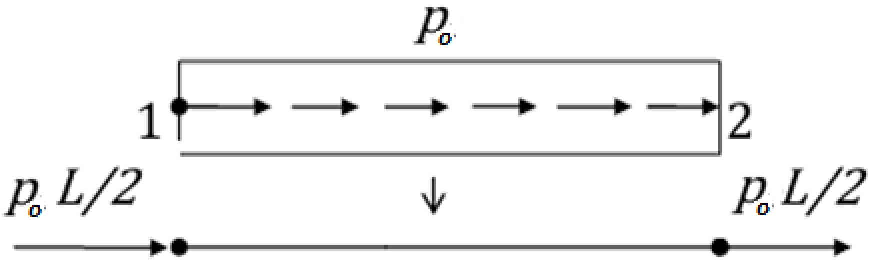
$$\boldsymbol{k} = \int_0^L \left\{ \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} AE \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \right\} dx = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix}$$

Uniform Load

$$\boldsymbol{P} = \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \underbrace{(\overset{\text{Per unit volume}}{p}.A)}_{\text{Per unit length}} dx$$

$$= \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} p_o \, dx$$

$$= \begin{pmatrix} \frac{p_o L}{2} \\ \frac{p_o L}{2} \end{pmatrix}$$



Axial Rod

To determine Governing Equation,

1. Equilibrium equation: $\frac{d\sigma_x}{dx} + B_x = 0$
2. Strain-displacement relation: $\varepsilon_x = \frac{du}{dx}$
3. Constitutive relation: $\sigma_x = E\varepsilon_x$

Using the above relations, we have

$$\sigma_x = E\frac{du}{dx}$$

$$\frac{d}{dx}\left(E\frac{du}{dx}\right) + B_x = 0$$

If E = constant; $E\frac{d^2u}{dx^2} + B_x = 0$

If p(x) is prescribed as per unit length, then

$$\frac{p(x)}{A} = B_x$$

$$\therefore E\frac{d^2u}{dx^2} + \frac{p(x)}{A} = 0$$

$EA\frac{d^2u}{dx^2} + p(x) = 0$ is the governing equation

On adding Boundary Conditions, the Boundary Value Problem (BVP) is defined

In 2D, the above equation takes the form

$\nabla^2 u = p$ also known as **Poisson Equation**

Weighted Residual Method

$$EA\frac{d^2u}{dx^2} + p = 0; \quad x_1 < x < x_2$$

Let $\hat{u}(x) \cong u(x)$

then $EA\frac{d^2u}{dx^2} + p = R(x) \leftarrow$ Residue in DE

We need to minimize the weighted residue in order to develop our appropriate method

$$\int_{x_1}^{x_2} WR(x).\,dx = 0$$

$$0 = \int W\left(EA\frac{d^2u}{dx^2} + p\right)dx$$

$$= \int_{x_1}^{x_2} W\,EA\frac{d^2u}{dx^2}dx + \int_{x_1}^{x_2} W.pdx$$

This is known as ***Strong*** Weighted Residue (WR) statement

$$= \left(W.EA.\frac{d\hat{u}}{dx}\right)_{x_2}^{x_1} - \int_{x_1}^{x_2}\frac{dW}{dx}EA\frac{d\hat{u}}{dx}dx + \int_{x_1}^{x_2} W.pdx$$

This is known as ***Weak*** WR statement

$$= W(x_2)P_2 - W(x_1)P_1 - \int \frac{dW}{dx}EA\frac{d\hat{u}}{dx}dx + \int W.pdx$$

$\because AE\frac{d\hat{u}}{dx} = P \longleftarrow$ axial force at the section

Substituting previously derived expansion for $u(x)$

$u(x) = N_1(x).\, u_1 + N_2(x)$

And substituting W as N_1 and N_2 in W-R Statement, we get

$$0 = -\int_{x_1}^{x_2} \frac{dN_1}{dx} EA \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 \right) dx + \int_{x_1}^{x_2} N_1 p.\, dx$$

$$0 = -\int_{x_1}^{x_2} \frac{dN_2}{dx} EA \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 \right) dx + \int_{x_1}^{x_2} N_2 p.\, dx$$

$$\binom{0}{0} = -\begin{bmatrix} \int N_1' EA N_1' dx & \int N_1' EA N_2' dx \\ \int N_2' EA N_1' dx & \int N_2' EA N_2' dx \end{bmatrix} \binom{u_1}{u_2} + \begin{pmatrix} \int N_1 p dx \\ \int N_1 p dx \end{pmatrix}$$

$$\begin{pmatrix}0\\0\end{pmatrix} = -\begin{bmatrix}\frac{EA}{L} & -\frac{EA}{L}\\ -\frac{EA}{L} & \frac{EA}{L}\end{bmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} + \begin{pmatrix}P_1\\P_2\end{pmatrix}$$

$$0 = -kd + P$$

Thus, we obtain the discrete governing equation.

$$P = kd$$

Transformation

- Local Coordinate

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{bmatrix} \frac{AE}{L} & \frac{-AE}{L} \\ \frac{-AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

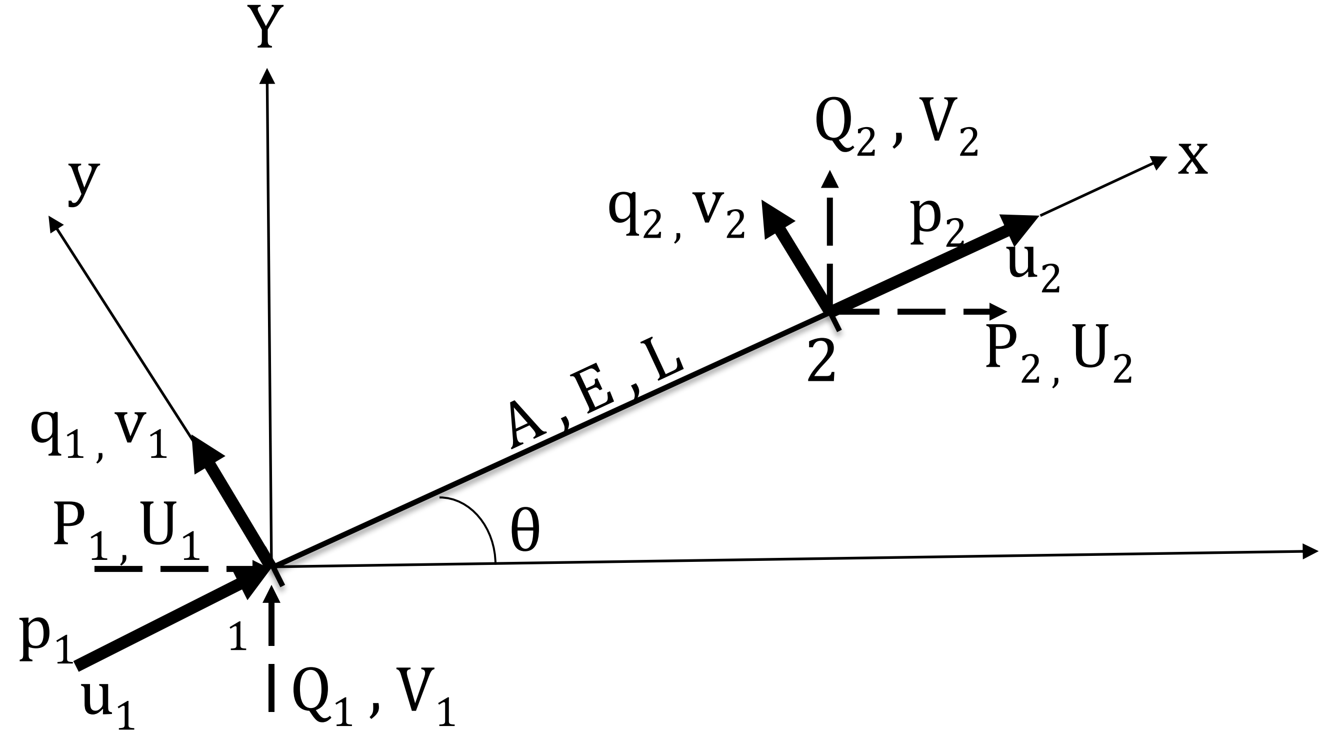
Transformation (contd.)

In many instances it is convenient to introduce both <u>local</u> and <u>global</u> coordinates.

The local coordinates are always chosen to represent an individual element. Global coordinates on other hand, are chosen for the entire system.

Usually, applied loads, boundary conditions, etc. are described in global coordinates

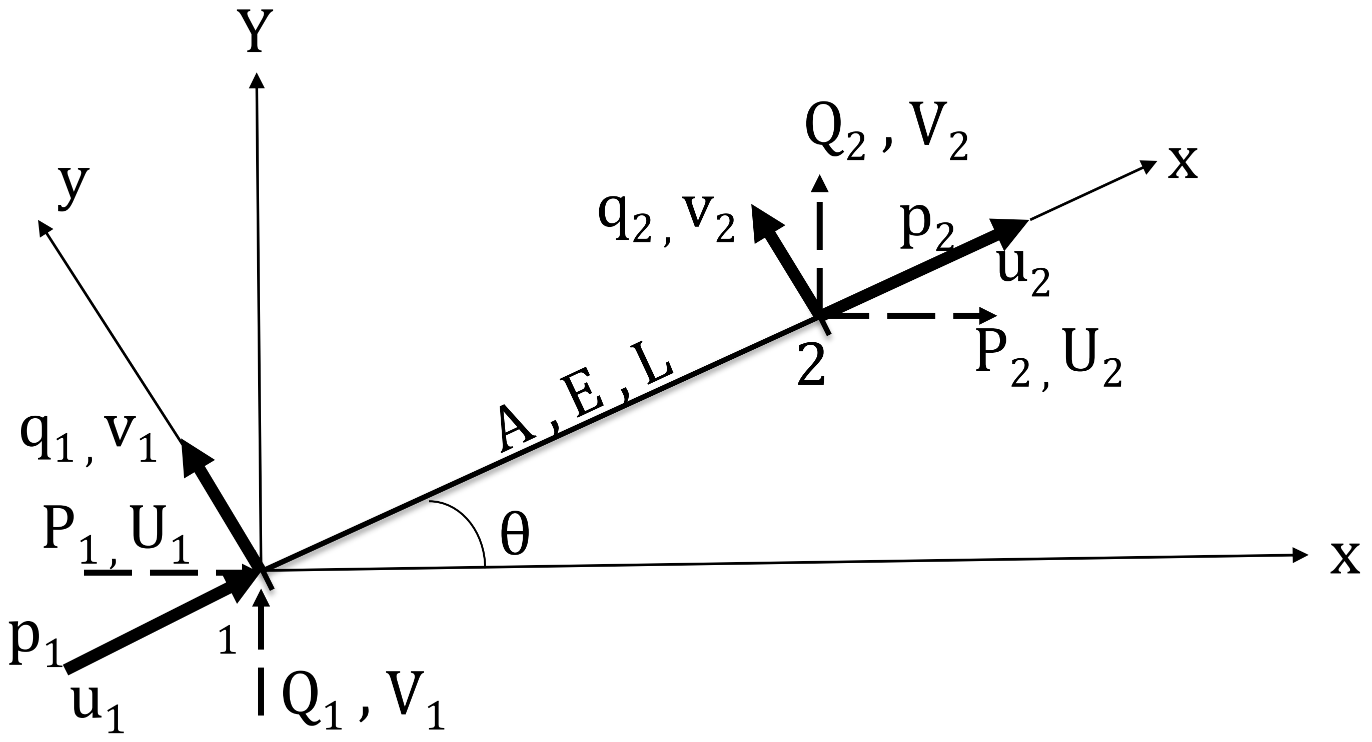
Stiffness Equation in Local Coordinates in Expanded Form



We have introduced q_1 and q_2 and v_1 and v_2

q_1 and q_2 do not exist since a truss element can not withstand a force normal to its axis.

Stiffness Equation in Local Coordinates in Expanded Form (Contd.)



$$\begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

Stiffness Equation in Local Coordinates in Expanded Form (Contd.)

In a compact form the matrix equation in a local coordinates can be expressed as

$$\boldsymbol{p} = \boldsymbol{k}\boldsymbol{u}$$

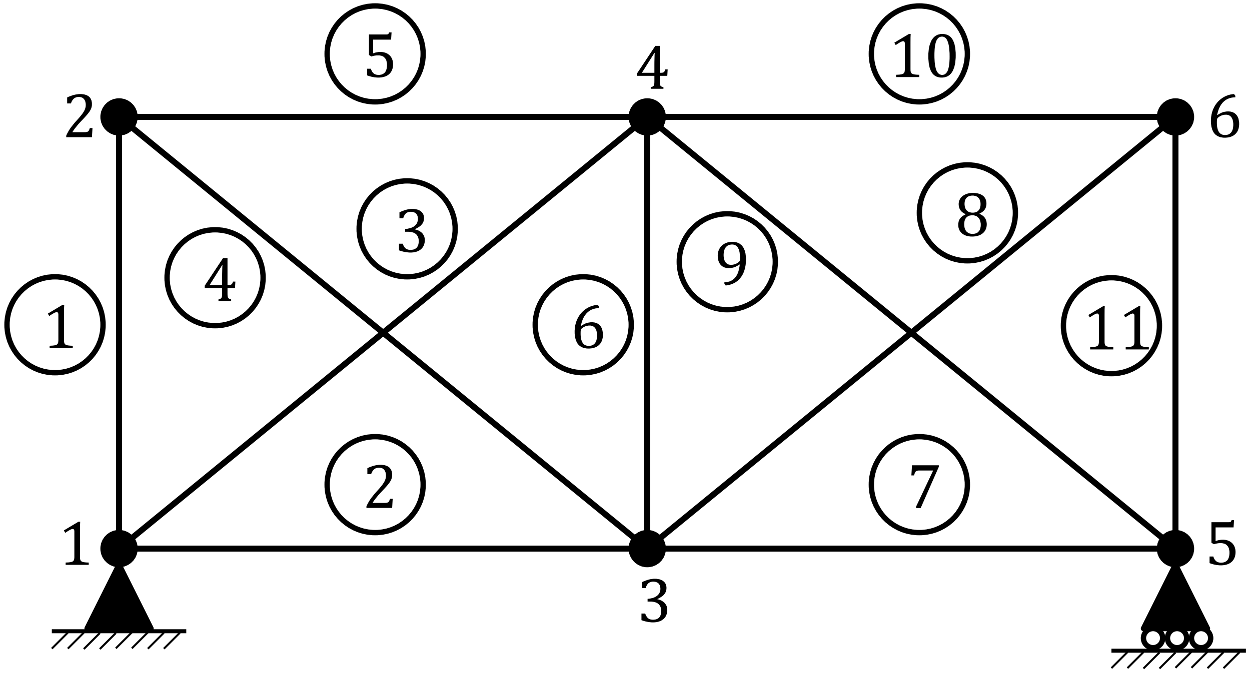
in which,

$$\boldsymbol{p} = \begin{pmatrix} p_1 & q_1 & p_2 & q_2 \end{pmatrix}^t$$

$$\boldsymbol{u} = \begin{pmatrix} u_1 & v_1 & u_2 & v_2 \end{pmatrix}^t$$

Transformation of Coordinates

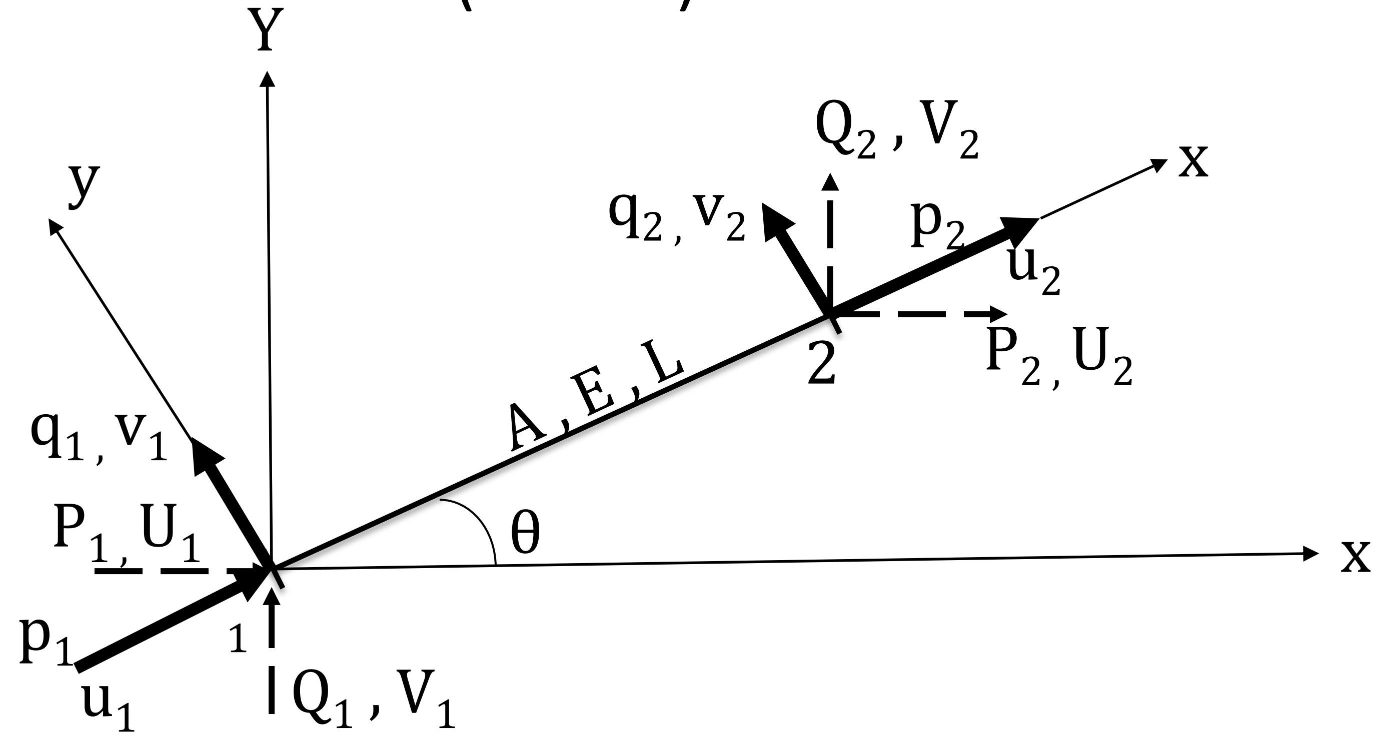
Transformation is required when local coordinates for description of elements change from element to element, e.g.,



6 – Nodes
11 – Elements

Transformation of Coordinates

(Contd.)



At Node-1

$$p_1 = P_1 \cos\theta + Q_1 \sin\theta$$

$$q_1 = -P_1 \sin\theta + Q_1 \cos\theta$$

At Node-2

$$p_2 = P_2 \cos\theta + Q_2 \sin\theta$$

$$q_2 = -P_2 \sin\theta + Q_2 \cos\theta$$

Transformation of Coordinates

(Contd.)

Let $\sin\theta = s$ *and* $\cos\theta = c$,

then combining the above equations, we can write,

$$\begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{pmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{pmatrix}$$

in compact form:

$$\boldsymbol{p} = \boldsymbol{T}.\boldsymbol{P}$$

Transformation of Coordinates

(Contd.)

in which,

$$\boldsymbol{p} = \left(p_1 \quad q_1 \quad p_2 \quad q_2 \right)^t$$

$$\boldsymbol{P} = \left(P_1 \quad Q_1 \quad P_2 \quad Q_2 \right)^t$$

and T is called ***transformation matrix****, which transforms the global nodal forces,* ***P*** *into local nodal forces,* ***p****.*

Since, displacements are also vectors like forces, a similiar transformation rule exists for them too, i.e.,

$$\mathbf{u} = \mathbf{T}\mathbf{U}$$

in which,

$$\mathbf{u} = \left(u_1 \quad v_1 \quad u_2 \quad v_2 \right)^t$$

$$\mathbf{U} = \left(U_1 \quad V_1 \quad U_2 \quad V_2 \right)^t$$

Transformation of Coordinates

(Contd.)

We can also write

$$\boldsymbol{P} = \boldsymbol{T}^{-1}\,\boldsymbol{p}$$

Since $\boldsymbol{T}$ *is an orthogonal matrix,*

$$\boldsymbol{T}^{-1} = \boldsymbol{T}^{t}$$

Stiffness Equation in Global Coordinates

$$\begin{aligned}\boldsymbol{P} &= \boldsymbol{T}^t \boldsymbol{p} \\ &= \boldsymbol{T}^t \boldsymbol{k}\,\boldsymbol{u} \\ &= \boldsymbol{T}^t \boldsymbol{k}\,\boldsymbol{T}\,\boldsymbol{U} \\ &= \boldsymbol{K}\,\boldsymbol{U}\end{aligned}$$

in which,

$$\boldsymbol{K} = \boldsymbol{T}^t \boldsymbol{k} \boldsymbol{T}$$

$$= \frac{AE}{L}\left[\begin{array}{cc|cc} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ \hline -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{array}\right]$$

Stiffness Equation in Global Coordinates (Contd.)

$$\boldsymbol{K} = \frac{AE}{L}\left[\begin{array}{cc|cc} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ \hline -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{array}\right]$$

This is stiffness matrix of the truss element as shown in the figure with reference to the global X, Y *coordinates.*

The above formulation for $\boldsymbol{K}$ *first appeared in:*

Turner MJ, Clough RW, Martin HC and Topp LJ (1956), Stiffness and deflection analysis of complex structures, *J. Aeronautical Sciences, **23**(9), 805 - 824.*

Stiffness Equation in Global Coordinates (Contd.)

A derivation of $\boldsymbol{K}$ *without involving transformation matrix,* $\boldsymbol{T}$ *is given in,*

Martin HC (1958), Truss analysis by stiffness consideration, *Trans. ASCE,* **123**, 1182 - 1194.

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