

Lecture Note – 3

Example 3.2.1 At a climate station, air pressure is measured as 100 kPa, air temperature as 20°C, and the wet-bulb, or dew-point, temperature as 16°C. Calculate

the corresponding vapor pressure, relative humidity, specific humidity, and air density.

Solution. The saturated vapor pressure at $T = 20^\circ\text{C}$ is given by Eq. (3.2.9)

$$\begin{aligned}e_s &= 611 \exp\left(\frac{17.27T}{237.3 + T}\right) \\ &= 611 \exp\left(\frac{17.27 \times 20}{237.3 + 20}\right) \\ &= 2339 \text{ Pa}\end{aligned}$$

and the actual vapor pressure e is calculated by the same method substituting the dew-point temperature $T_d = 16^\circ\text{C}$:

$$\begin{aligned}e &= 611 \exp\left(\frac{17.27T_d}{237.3 + T_d}\right) \\ &= 611 \exp\left(\frac{17.27 \times 16}{237.3 + 16}\right) \\ &= 1819 \text{ Pa}\end{aligned}$$

The relative humidity from (3.2.11) is

$$\begin{aligned}R_h &= \frac{e}{e_s} \\ &= \frac{1819}{2339} \\ &= 0.78 \\ &= 78\%\end{aligned}$$

and the specific humidity is given by (3.2.6) with $p = 100 \text{ kPa} = 100 \times 10^3 \text{ Pa}$:

$$\begin{aligned} q_v &= 0.622 \frac{e}{p} \\ &= 0.622 \left(\frac{1819}{100 \times 10^3} \right) \\ &= 0.0113 \text{ kg water/kg moist air} \end{aligned}$$

The air density is calculated from the ideal gas law (3.2.7). The gas constant R_a is given by (3.2.8) with $q_v = 0.0113 \text{ kg/kg}$ as $R_a = 287(1 + 0.608q_v) = 287(1 + 0.608 \times 0.0113) = 289 \text{ J/kg}\cdot\text{K}$, and $T = 20^\circ\text{C} = (20 + 273) \text{ K} = 293 \text{ K}$, so that

$$\begin{aligned} \rho_a &= \frac{p}{R_a T} \\ &= \frac{100 \times 10^3}{289 \times 293} \\ &= 1.18 \text{ kg/m}^3 \end{aligned}$$

Water Vapor in a Static Atmospheric Column

Two laws govern the properties of water vapor in a static column, the ideal gas law

$$p = \rho_a R_a T \quad (3.2.12)$$

and the *hydrostatic pressure law*

$$\frac{dp}{dz} = -\rho_a g \quad (3.2.13)$$

The variation of air temperature with altitude is described by

$$\frac{dT}{dz} = -\alpha \quad (3.2.14)$$

where α is the lapse rate. As shown in Fig. 3.2.2, a linear temperature variation combined with the two physical laws yields a nonlinear variation of pressure with altitude. Density and specific humidity also vary nonlinearly with altitude. From (3.2.12), $\rho_a = p/R_a T$, and substituting this into (3.2.13) yields

$$\frac{dp}{dz} = \frac{-pg}{R_a T}$$

or

$$\frac{dp}{p} = \left(\frac{-g}{R_a T} \right) dz$$

Substituting $dz = -dT/\alpha$ from (3.2.14):

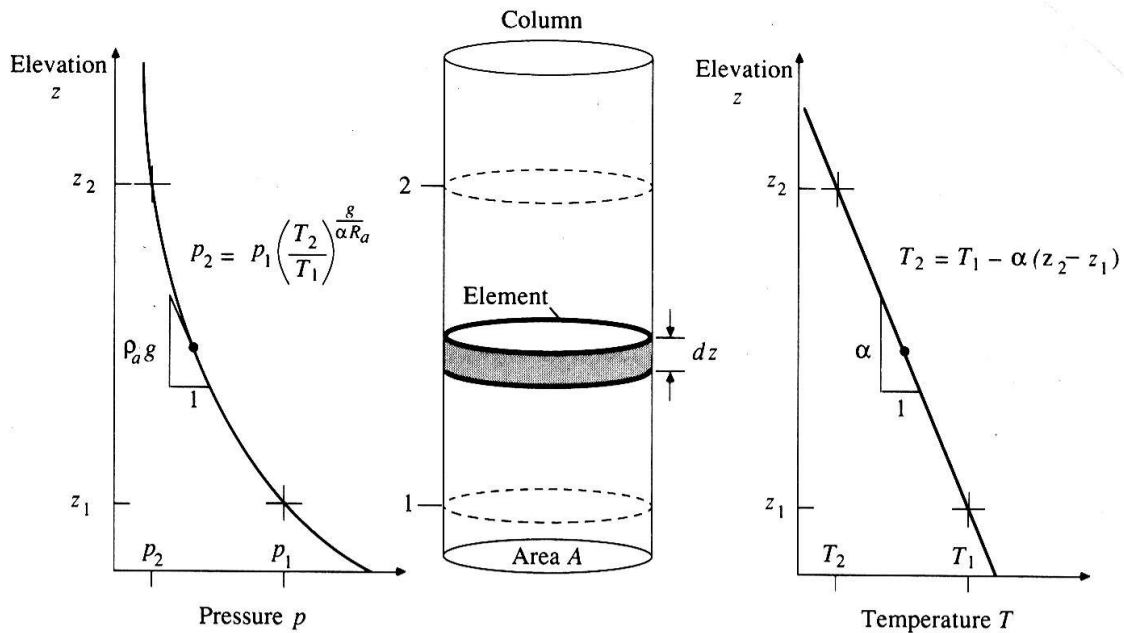


FIGURE 3.2.2
Pressure and temperature variation in an atmospheric column.

$$\frac{dp}{p} = \left(\frac{g}{\alpha R_a} \right) \frac{dT}{T}$$

and integrating both sides between two levels 1 and 2 in the atmosphere gives

$$\ln \left(\frac{p_2}{p_1} \right) = \left(\frac{g}{\alpha R_a} \right) \ln \left(\frac{T_2}{T_1} \right)$$

or

$$p_2 = p_1 \left(\frac{T_2}{T_1} \right)^{g/\alpha R_a} \quad (3.2.15)$$

From (3.2.14) the temperature variation between altitudes z_1 and z_2 is

$$T_2 = T_1 - \alpha(z_2 - z_1) \quad (3.2.16)$$

Precipitable Water

(The amount of moisture in an atmospheric column is called its *precipitable water*.) Consider an element of height dz in a column of horizontal cross-sectional area A (Fig. 3.2.2). The mass of air in the element is $\rho_a A dz$ and the mass of water contained in the air is $q_v \rho_a A dz$. The total mass of precipitable water in the column between elevations z_1 and z_2 is

$$m_p = \int_{z_1}^{z_2} q_v \rho_a A dz \quad (3.2.17)$$

The integral (3.2.17) is calculated using intervals of height Δz , each with an incremental mass of precipitable water

$$\Delta m_p = \bar{q}_v \bar{\rho}_a A \Delta z \quad (3.2.18)$$

where \bar{q}_v and $\bar{\rho}_a$ are the average values of specific humidity and air density over the interval. The mass increments are summed over the column to give the total precipitable water.

Example 3.2.2. Calculate the precipitable water in a saturated air column 10 km high above 1 m² of ground surface. The surface pressure is 101.3 kPa, the surface air temperature is 30°C, and the lapse rate is 6.5°C/km.

Solution. The results of the calculation are summarized in Table 3.2.2. The increment in elevation is taken as $\Delta z = 2 \text{ km} = 2000 \text{ m}$. For the first increment, at $z_1 = 0 \text{ m}$, $T_1 = 30^\circ\text{C} = (30 + 273) \text{ K} = 303 \text{ K}$; at $z_2 = 2000 \text{ m}$, by Eq. (3.2.16) using $\alpha = 6.5^\circ\text{C/km} = 0.0065^\circ\text{C/m}$,

$$\begin{aligned} T_2 &= T_1 - \alpha(z_2 - z_1) \\ &= 30 - 0.0065(2000 - 0) \\ &= 17^\circ\text{C} \end{aligned}$$

$$\begin{aligned} &= (17 + 273) \text{ K} \\ &= 290 \text{ K} \end{aligned}$$

as shown in column 3 of the table. The gas constant R_a can be taken as 287 J/kg·K in this example because its variation with specific humidity is small [see Eq. (3.2.8)]. The air pressure at 2000 m is then given by (3.2.15) with $g/\alpha R_a = 9.81/(0.0065 \times 287) = 5.26$, as

$$\begin{aligned} p_2 &= p_1 \left(\frac{T_2}{T_1} \right)^{g/\alpha R_a} \\ &= 101.3 \left(\frac{290}{303} \right)^{5.26} \\ &= 80.4 \text{ kPa} \end{aligned}$$

as shown in column 4.

The air density at the ground is calculated from (3.2.12):

$$\begin{aligned} \rho_a &= \frac{p}{R_a T} \\ &= \frac{101.3 \times 10^3}{(287 \times 303)} \\ &= 1.16 \text{ kg/m}^3 \end{aligned}$$

and a similar calculation yields the air density of 0.97 kg/m³ at 2000 m. The average density over the 2 km increment is therefore $\bar{\rho}_a = (1.16 + 0.97)/2 = 1.07 \text{ kg/m}^3$ (see columns 5 and 9).

The saturated vapor pressure at the ground is determined using (3.2.9):

$$\begin{aligned} e &= 611 \exp \left(\frac{17.27T}{237.3 + T} \right) \\ &= 611 \exp \left(\frac{17.27 \times 30}{237.3 + 30} \right) \\ &= 4244 \text{ Pa} \\ &= 4.24 \text{ kPa} \end{aligned}$$

The corresponding value at 2000 m where $T = 17^\circ\text{C}$, is $e = 1.94 \text{ kPa}$ (column 6). The specific humidity at the ground surface is calculated by Eq. (3.2.6):

$$\begin{aligned} q_v &= 0.622 \frac{e}{p} \\ &= 0.622 \times \frac{4.24}{101.3} \\ &= 0.026 \text{ kg/kg} \end{aligned}$$

At 2000 m $q_v = 0.015 \text{ kg/kg}$. The average value of specific humidity over the 2-km increment is therefore $\bar{q}_v = (0.026 + 0.015)/2 = 0.0205 \text{ kg/kg}$ (column 8). Substituting into (3.2.18), the mass of precipitable water in the first 2-km increment is

$$\begin{aligned}\Delta m_p &= \bar{q}_v \bar{\rho}_a A \Delta z \\ &= 0.0205 \times 1.07 \times 1 \times 2000 \\ &= 43.7 \text{ kg}\end{aligned}$$

By adding the incremental masses, the total mass of precipitable water in the column is found to be $m_p = 77 \text{ kg}$ (column 10). The equivalent depth of liquid water is $m_p / \rho_w A = 77 / (1000 \times 1) = 0.077 \text{ m} = 77 \text{ mm}$.
