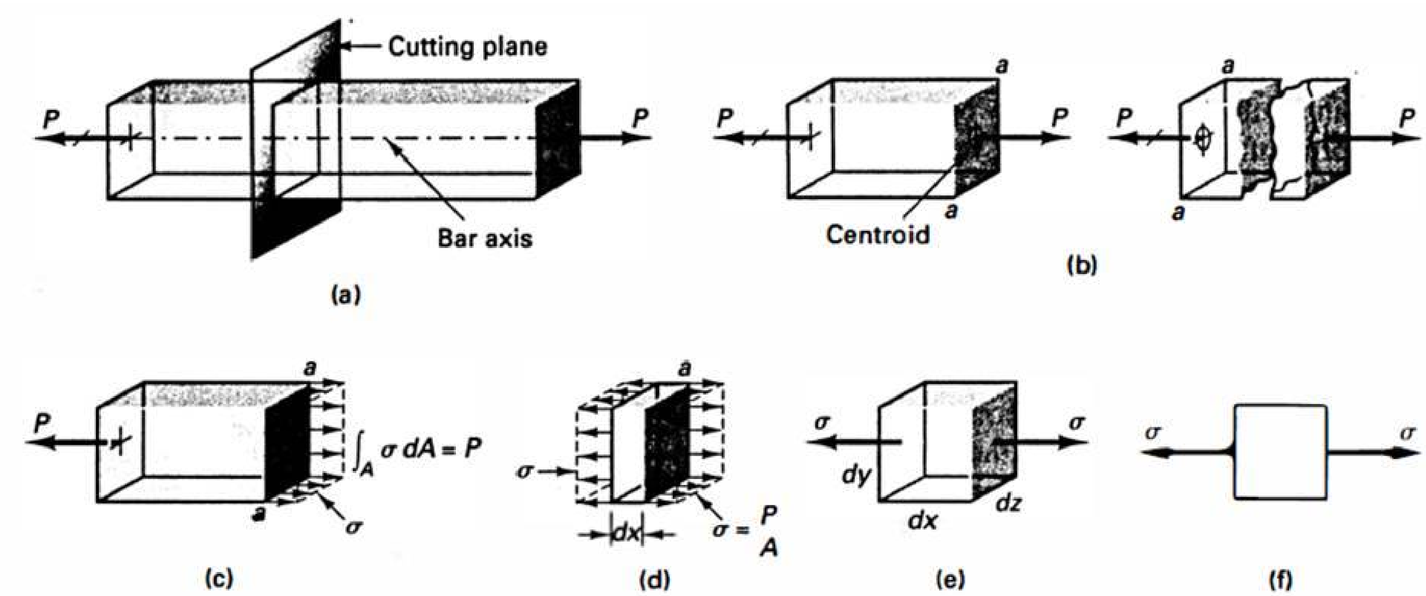
Chapter 9, POPOV

SKEW BENDING

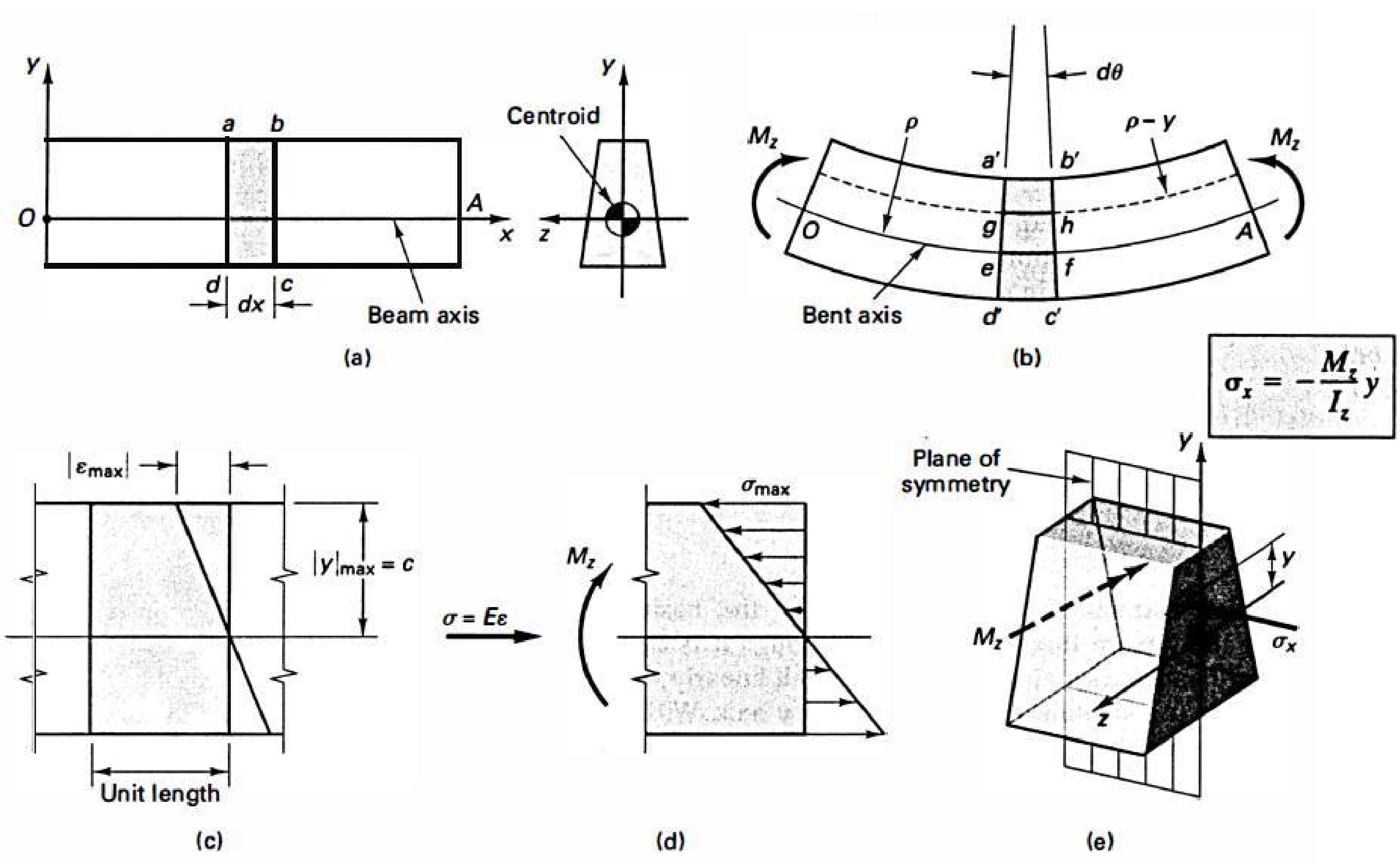
Sign Convention

1) **Axis:** Right hand screw rule to set the axis. However, convention used herein is “X” axis is the longitudinal axis of the beam. Normally “Y” axis is the vertical direction of the section. “Z” axis is from the right hand screw rule.
2) **Stress:** Tensile stress positive.
3) **Moment:** When a screw is placed along an axis and the moment is applied to it, if the screw head moves towards the positive direction of the axis, than the applied moment is positive.

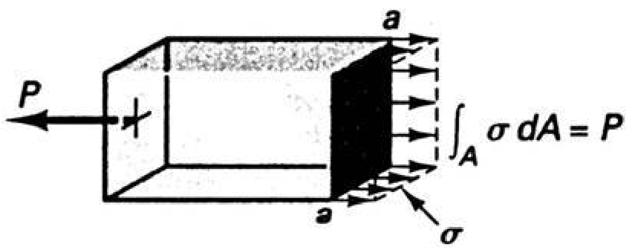
Stress for Axial Load

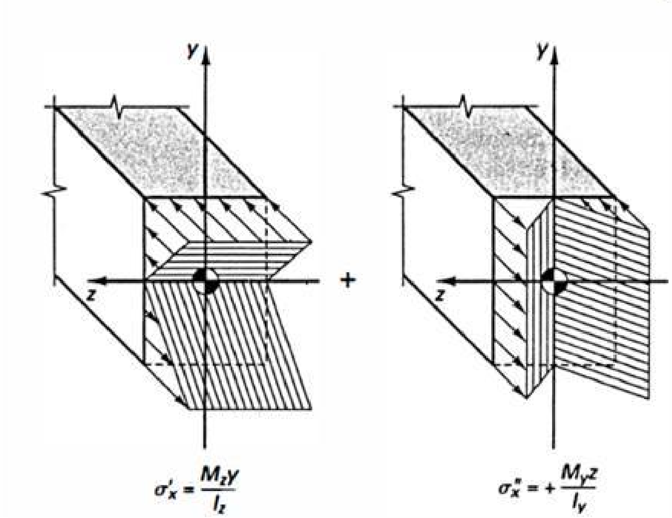


Bending Stress for Uniaxial Moment (M_z)



Bending Stress for biaxial Moment (Mz, My)

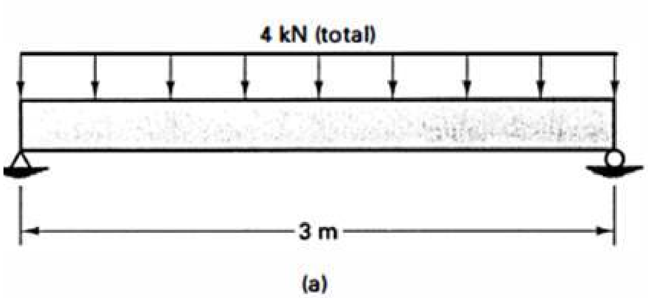




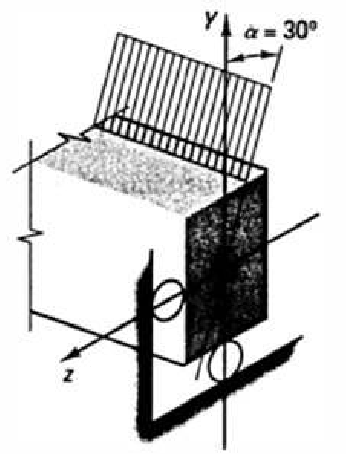
Bending Stress considering axial load and biaxial moment:

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

Example 9-1The 100-by-150-mm wooden beam shown in Fig. 9-4(a) is used to support a uniformly distributed load of 4 kN (total) on a simple span of 3 m. The applied load acts in a plane making an angle of 30° with the vertical, as shown in Fig. 9-4(b) and again in Fig. 9-4(c). Calculate the maximum bending stress at mid-span, and, for the same section, locate the neutral axis. Neglect the weight of the beam.



(a)



(b)

P_y=4Cos(30)=3.464 (kN-Total),

P_z=4Sin(30)=3 (kN-Total)

$$M_z = P_y.\frac{L}{8} = 3.644.\frac{3}{8} = 1.3\ kN.m \text{ (positive)}$$

$$M_y = P_x.\frac{L}{8} = 2.\frac{3}{8} = 0.75\ kN.m \text{ (positive)}$$

$$I_z = \frac{100x150^3}{12} = 28.1\text{x}10^6\ mm^4$$

$$I_y = \frac{150x100^3}{12} = 12.5\text{x}10^6\ mm^4$$

$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z$$

$$\sigma_x = \frac{0}{A} - \frac{1.3x10^6}{28.1x10^6}y + \frac{0.75x10^6}{12.5x10^6}z$$

A (-75,50), B (-75,-50), C (+75,-50) & D (+75,+50)

$$\sigma_{xA} = -\frac{1.3x10^6}{28.1x10^6}(-75) + \frac{0.75x10^6}{12.5x10^6}(50) = 3.47 + 3.00$$
$= 6.47$ MPa

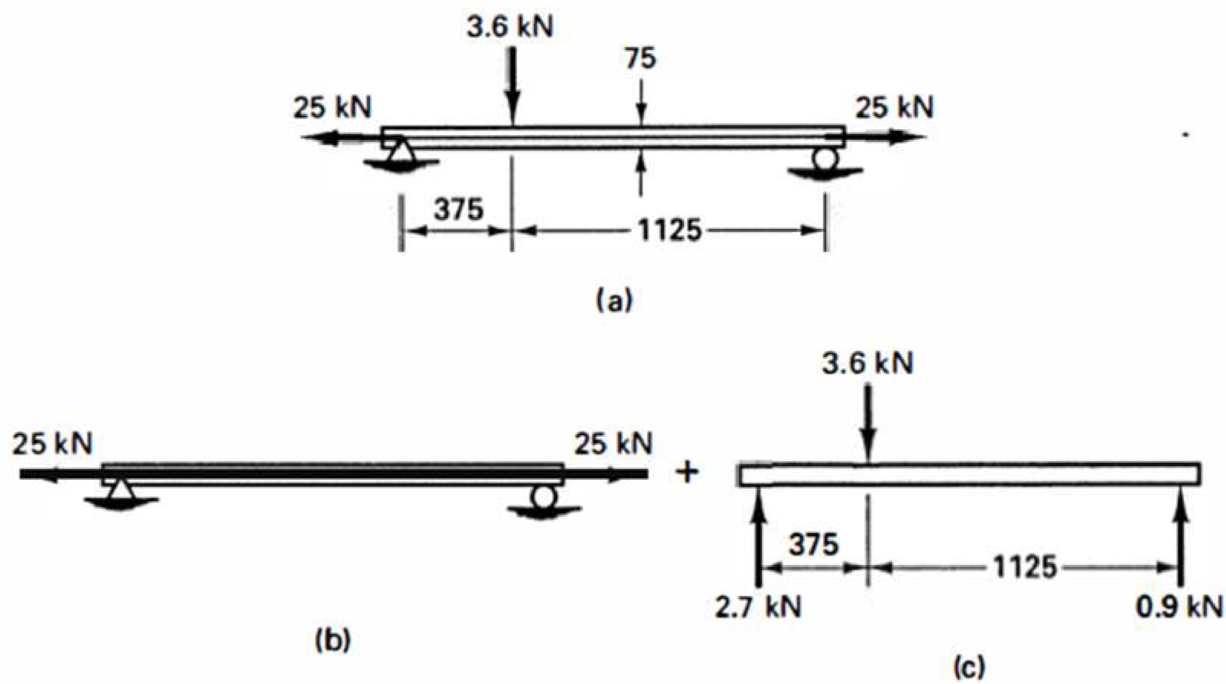
$$\sigma_{xB} = -\frac{1.3x10^6}{28.1x10^6}(-75) + \frac{0.75x10^6}{12.5x10^6}(-50) = 3.47 - 3.00$$
$= 0.47$ MPa

$$\sigma_{xC} = -\frac{1.3x10^6}{28.1x10^6}(+75) + \frac{0.75x10^6}{12.5x10^6}(-50) = -3.47 - 3.00$$
$= -6.47$ MPa

$$\sigma_{xD} = -\frac{1.3x10^6}{28.1x10^6}(+75) + \frac{0.75x10^6}{12.5x10^6}(50) = -3.47 + 3.00$$
$= -0.47$ MPa

Example 9-2

A 50-by-75-mm, 1.5-m-long elastic bar of negligible weight is loaded as shown in mm in Fig. 9-7(a). Determine the maximum tensile and compressive stresses acting normal to the section through the beam.



$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z$$

$P = 25x10^3\ N,$

$M_z = \dfrac{3.6x10^3x375x1125}{(375+1125)} = 1.025x10^6\ N - mm$

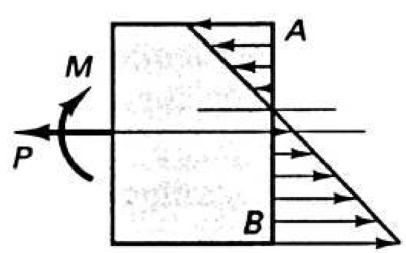
$M_y = 0$

$I_z = \dfrac{50x75^3}{12} = 1.76\text{x}10^6\ mm^4$

$\sigma_x = \dfrac{25x10^3}{50x75} - \dfrac{1.025x10^6}{1.76\text{x}10^6}y = 6.67 - 0.58y$

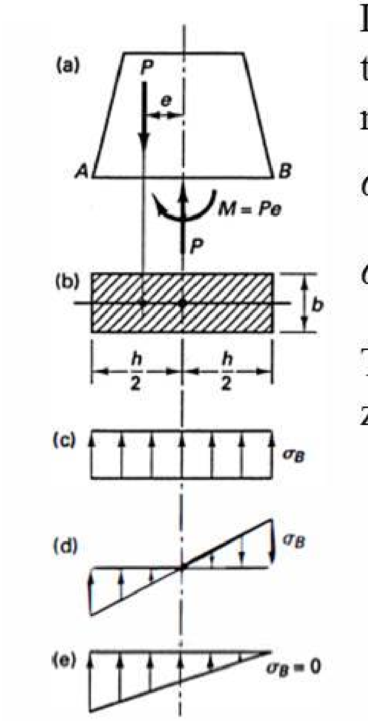
$\sigma_A = -15.08\ MPa$

$\sigma_B = 28.42\ MPa$



Example 9-4

Consider a tapered block having a rectangular cross section at the base, as shown in Figs. 9-9(a) and (b). Determine the maximum eccentricity "e" such that the stress at B caused by the applied force P is zero.



In order to maintain applied force P in equilibrium, there must be an axial compressive force P and a moment "Pe" at the base having the senses shown.

$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z$$

$$\sigma_x = \frac{-P}{bh} - 0 + \frac{-Pe}{\frac{bh^3}{12}}z$$

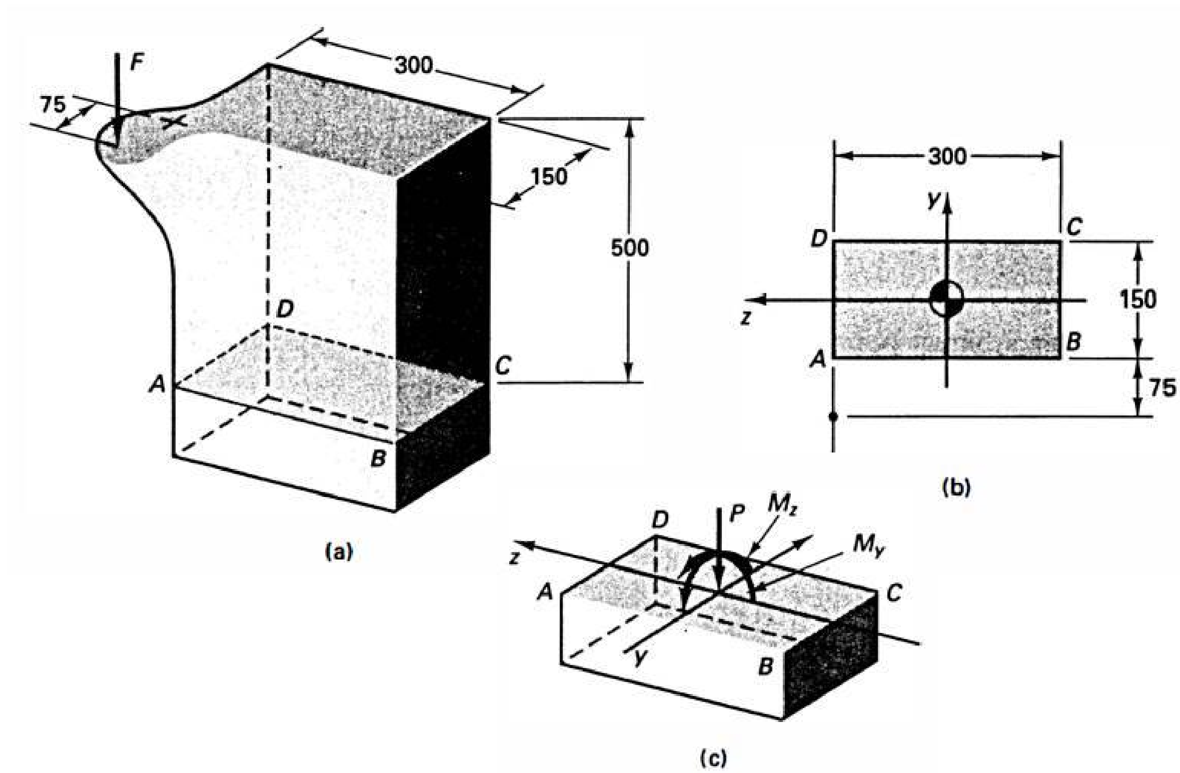
To satisfy the condition for having stress at B equal to zero, it follows that

$$0 = \frac{-P}{bh} + \frac{-Pe}{\frac{bh^3}{12}}\left(-\frac{h}{2}\right)$$

$$e = \frac{h}{6}$$

Example 9-5

Find the stress distribution at section ABCD for the block shown in mm in Fig. 9-11(a) if F = 64 kN. At the same section, locate the line of zero stress. Neglect the weight of the block.



$P = -64x10^3\ N,$

$M_z = -64x10^3x(75 + 75) = -9.6x10^6\ N - mm$

$M_y = -64x10^3x150 = -9.6x10^6\ N - mm$

$A = 150x300 = 45\text{x}10^3\ mm^2$

$$I_z = \frac{300x150^3}{12} = 84.375\text{x}10^6\ mm^4$$

$$I_y = \frac{150x300^3}{12} = 337.5\text{x}10^6\ mm^4$$

$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z$$

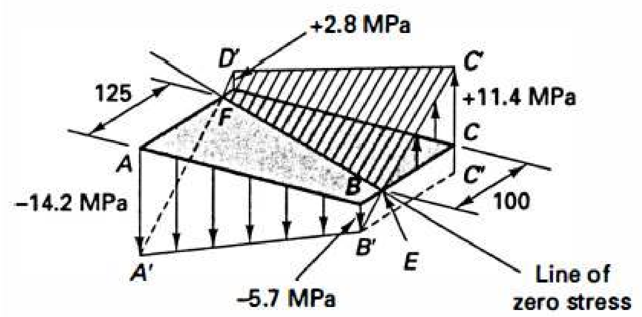
$$\sigma_x = \frac{-64x10^3}{45\text{x}10^3} - \frac{-9.6x10^6}{84.375\text{x}10^6}y + \frac{-9.6x10^6}{337.5\text{x}10^6}z$$

$$\sigma_x = -1.422 + 0.1138y - 0.0284z$$

A (-75, 150), B (-75,-150), C (+75,-150) & D (+75,+150)

$$\sigma_{xA} = -1.422 + 0.1138(-75) - 0.0284(150)\ MPa$$
$$\sigma_{xA} = -1.422 - 8.533 - 4.267 = -14.22\ MPa$$

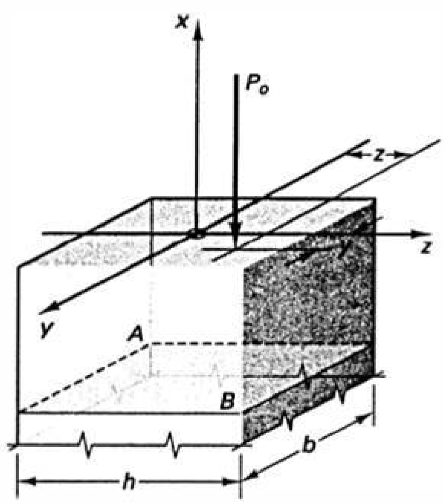
$$\sigma_{xB} = -1.422 - 8.533 + 4.267 = -5.69\ MPa$$
$$\sigma_{xC} = -1.422 + 8.533 + 4.267 = +11.38\ MPa$$
$$\sigma_{xD} = -1.422 + 8.533 - 4.267 = +2.84\ MPa$$



(d)

Example 9-6

Find the zone over which the vertical downward force P_0 may be applied to the rectangular weightless block shown in Fig. 9-12(a) without causing any tensile stresses at the section A-B.



(a)

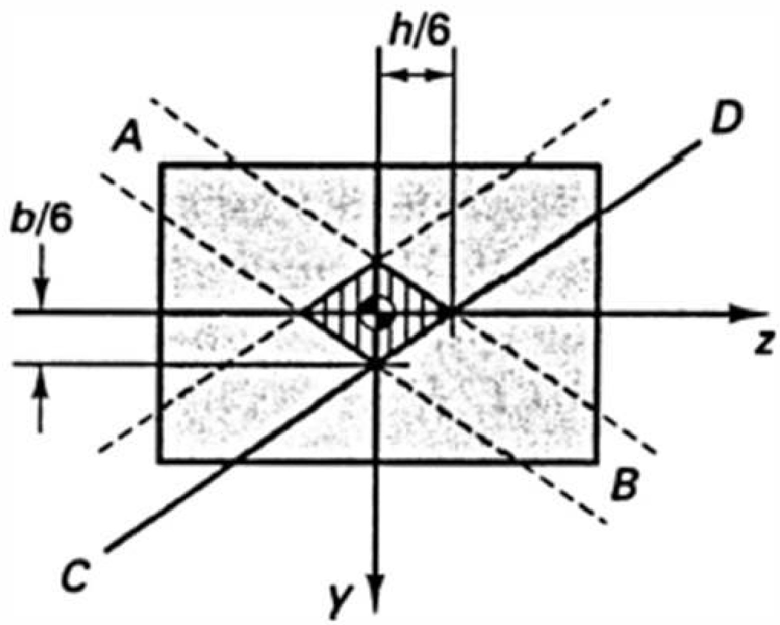
SOLUTION

The force $P = -P_0$ is placed at an arbitrary point in the first quadrant of the y-z coordinate system shown. Then the same reasoning used in the preceding example shows that with this position of the force, the greatest tendency for a tensile stress exists at A. With $P = -P_0$, $M_z = +P_0.y$, and $M_y = -P_0.z$, and setting the stress at A equal to zero fulfills the limiting condition of the problem

$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z \qquad 0 = \frac{-P_0}{bh} - \frac{P_0.y}{\frac{hb^3}{12}}\left(\frac{-b}{2}\right) + \frac{-P_0.z}{\frac{bh^3}{12}}\left(\frac{-h}{2}\right) \qquad \frac{z}{\frac{h}{6}} + \frac{y}{\frac{b}{6}} = 1$$

$$\frac{z}{\frac{h}{6}} + \frac{y}{\frac{b}{6}} = 1$$

$$\pm\frac{z}{\frac{h}{6}} \pm \frac{y}{\frac{b}{6}} = 1$$



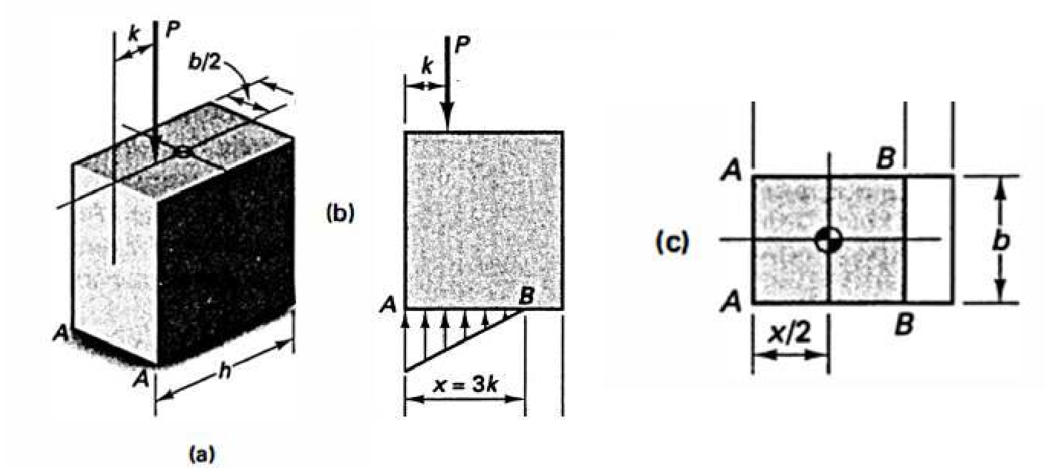
(b)

Example 9-7

Consider a "weightless" rigid block resting on a linearly elastic foundation not capable of transmitting any tensile stresses, as shown in Fig. 9-13(a). Determine the stress distribution in the foundation when applied force P is so placed that a part of the block lifts off.

SOLUTION

Assume that only a portion AB of the foundation of length x and width b is effective in resisting applied force P. This corresponds to the shaded area in Fig. 9-13(c). The stress along line B-B is zero by definition.



Hence, the following equation for the stress at B may be written:

$$\sigma_{xB} = \frac{P}{A} - \frac{M_z}{I_z} y + \frac{M_y}{I_y} z$$

$P = -P, Z_B = -\frac{x}{2}$

$M_y = -P\left(\frac{x}{2} - k\right)$

$M_z = 0$

$I_y = \frac{b.x^3}{12}$

$$\sigma_{xB} = \frac{-P}{bx} + \frac{-P\left(\frac{x}{2} - k\right)}{\frac{b.x^3}{12}}\left(-\frac{x}{2}\right) = 0$$

$x = 3k$

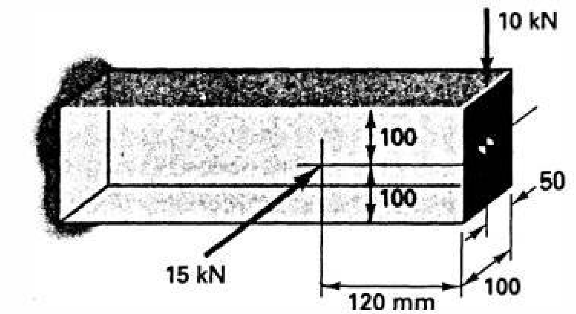
$\sigma_{xA} = \frac{-2P}{3bk}$

TO DO LIST FROM POPOV

Solve problem numbers given below from POPOV in addition to other text books.

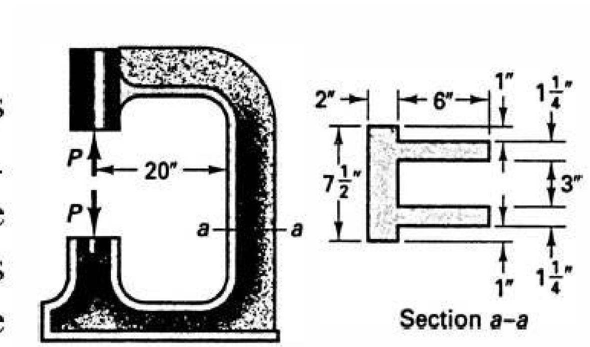
Problems from Chapter 9 (2^{nd} ed):
2,3,4,5,9,11,12,13,14,15,17,19,23,25

Problem 9-9. Determine the bending stresses at the corners in the cantilever loaded, as shown in the figure, at a section **600 mm from the free end**. Also locate the neutral axis.



$$M_z = -600x10 = -6000\ kN - mm$$
$$M_y = 15x480 = +7200\ kN - mm$$
$$I_z = \frac{100x200^3}{12} = 66.7\text{x}10^6\ mm^4$$
$$I_y = \frac{200x100^3}{12} = 16.7\text{x}10^6\ mm^4$$
$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z$$
$$\sigma_x = 0 - \frac{-6000x10^3}{66.7\text{x}10^6}y + \frac{7200x10^3}{16.7\text{x}10^6}z$$
$$\sigma_x = 0.09y + 0.431z$$
$$\sigma_{xA} = 30.6\ MPa$$
$$\sigma_{xB} = 12.6\ MPa$$
$$\sigma_{xC} = -30.6\ MPa$$
$$\sigma_{xD} = -12.6\ MPa$$

Problem 9-12. A frame for a punch press has the proportions shown in the figure. What force P can be applied to this frame controlled by the stresses in the sections such as a-a if the allowable stresses are 4000 psi in tension and 12,000 psi in compression?



$$A = 30\ in^2, \qquad \bar{y} = 3", \qquad I_z = 170\ in^4$$

$$M_z = (20 + 3)P = 23P$$

$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z$$

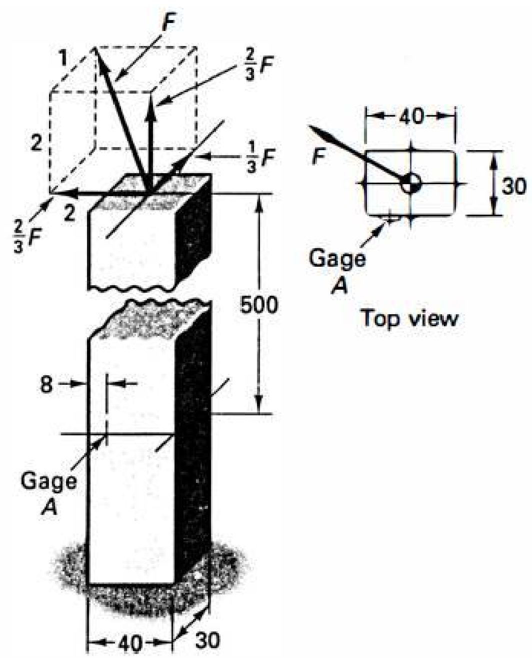
$$\sigma_x = \frac{P}{30} - \frac{23P}{170}y, \ y = -3" \ \& + 5",$$

$$\sigma_{x(y=-3)} = \frac{112P}{225} \ll 12000, P = 9100\ lb$$

$$\sigma_{x(y=5)} = \frac{164P}{225} \ll 4000, P = 18600\ lb$$

$$Ans\!: P_{all} = 9100\ lb$$

Problem 9-19. An inclined tensile force **F** is applied to an aluminum alloy bar such that its line of action goes through the centroid of the bar, as shown in mm in the figure. (The detail of the attachment is not shown.) What is the magnitude of force **F** if it causes a longitudinal strain of $\varepsilon = +20x10^{-6}$ in the gage at A? Assume that the bar behaves as a linearly clastic material and let E = 70 GPa.



$\varepsilon = +20x10^{-6}, \sigma = 20x10^{-6}x70x10^{3} = 1.4\ MPa$

$$I_z = \frac{40x30^3}{12} = 90000\ in^4,$$

$$I_y = \frac{30x40^3}{12} = 160000\ in^4$$

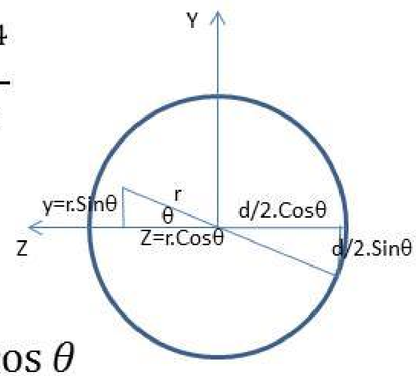
$$P = +\frac{2}{3}F,$$

$$M_z = \frac{1}{3}F.500,\ M_y = -\frac{2}{3}F.500$$

$$1.4 = \frac{\frac{2}{3}F}{1200} - \frac{\frac{1}{3}Fx500}{90000}(-15) + \frac{-\frac{2}{3}Fx500}{160000}(12)$$

$Ans: F = 26.25\ lb$

Problem 9-25. Determine the kern for a member having a solid circular cross section.



$$z = r\cos\theta\,, \qquad y = r\sin\theta\,, \qquad I_y = I_z = \frac{\pi d^4}{64}$$

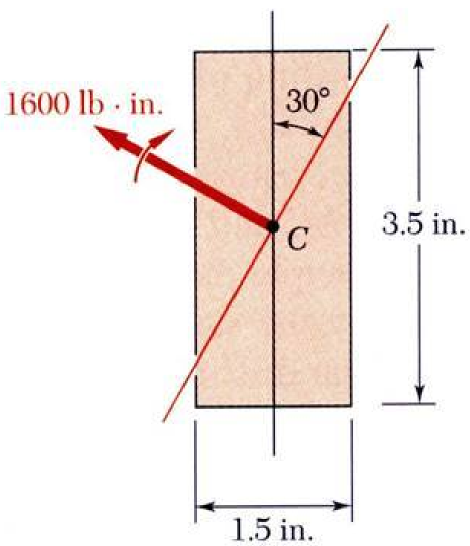
$$M_z = P.r.\sin\theta\,, M_y = -P.r.\cos\theta$$

$$\sigma_x = \frac{P}{A} - \frac{M_z}{I_z}y + \frac{M_y}{I_y}z$$

$$0 = \frac{P}{\frac{\pi d^2}{4}} - \frac{P.r.\sin\theta\, d}{\frac{\pi d^4}{64}}\frac{d}{2}\sin\theta + \frac{-P.r.\ \cos\theta\, d}{\frac{\pi d^4}{64}}\frac{d}{2}\cos\theta$$

$$r = \frac{d}{8}$$

A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.



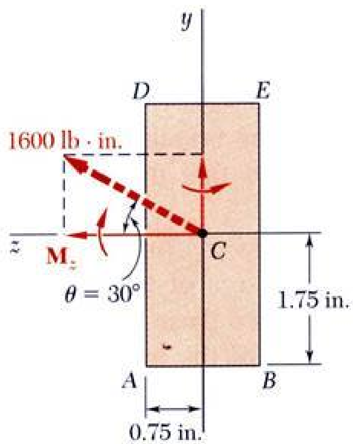
SOLUTION:

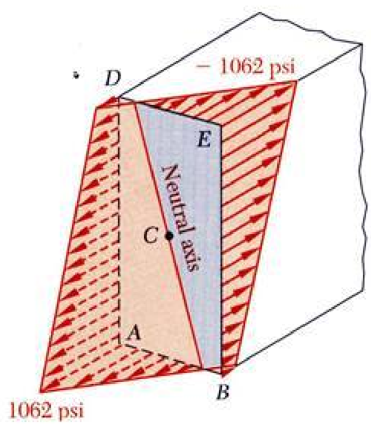
- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

- Combine the stresses from the component stress distributions.

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$





- Resolve the couple vector into components and calculate the corresponding maximum stresses.

$$M_z = (1600\,\text{lb}\cdot\text{in})\cos 30 = 1386\,\text{lb}\cdot\text{in}$$
$$M_y = (1600\,\text{lb}\cdot\text{in})\sin 30 = 800\,\text{lb}\cdot\text{in}$$
$$I_z = \tfrac{1}{12}(1.5\,\text{in})(3.5\,\text{in})^3 = 5.359\,\text{in}^4$$
$$I_y = \tfrac{1}{12}(3.5\,\text{in})(1.5\,\text{in})^3 = 0.9844\,\text{in}^4$$

The largest tensile stress due to M_z occurs along AB

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386\,\text{lb}\cdot\text{in})(1.75\,\text{in})}{5.359\,\text{in}^4} = 452.6\,\text{psi}$$

The largest tensile stress due to M_z occurs along AD

$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800\,\text{lb}\cdot\text{in})(0.75\,\text{in})}{0.9844\,\text{in}^4} = 609.5\,\text{psi}$$

- The largest tensile stress due to the combined loading occurs at A.

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 452.6 + 609.5$$

$$\boxed{\sigma_{\max} = 1062\,\text{psi}}$$

Chapter 11, POPOV

Stress and Strain Transformation

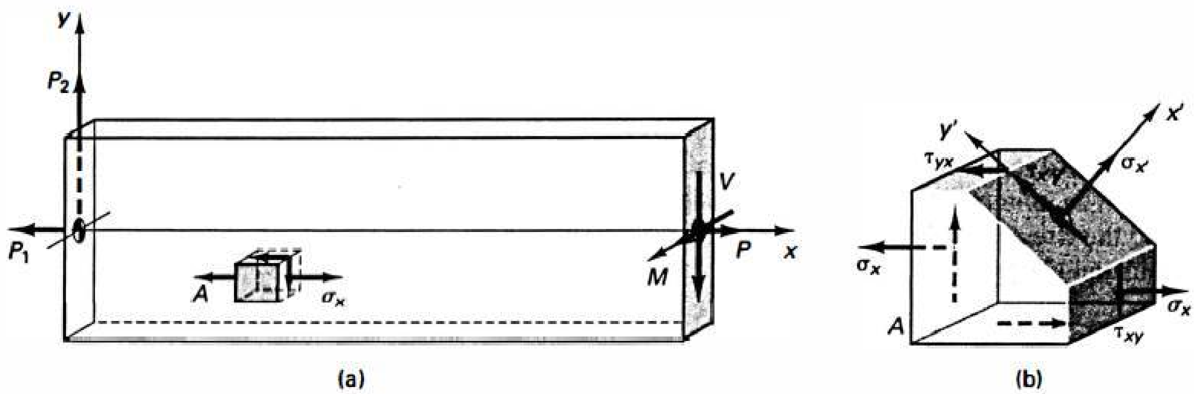


Fig. 11-1 State of stress at a point on different planes.

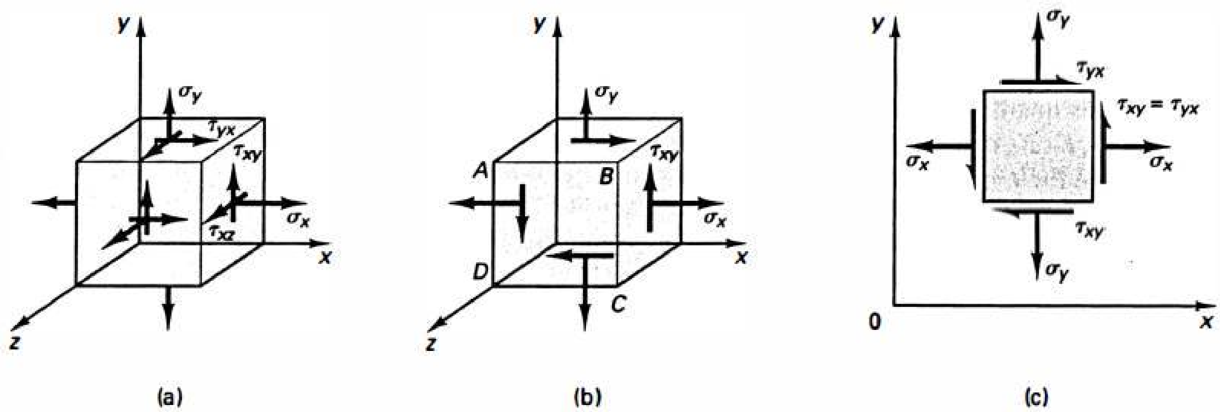


Fig. 11-2 Representations of stresses acting on an element.

Example 11·1 Let the state of stress for an element of unit thickness be as shown in Fig. 11-3(a). An alternative representation of the state of stress at the same point may be given on an infinitesimal wedge with an angle of $\alpha = 22.5^0$, as in Fig. 11-3(b). Find the stresses that must act on plane AB of the wedge to keep the element in equilibrium.

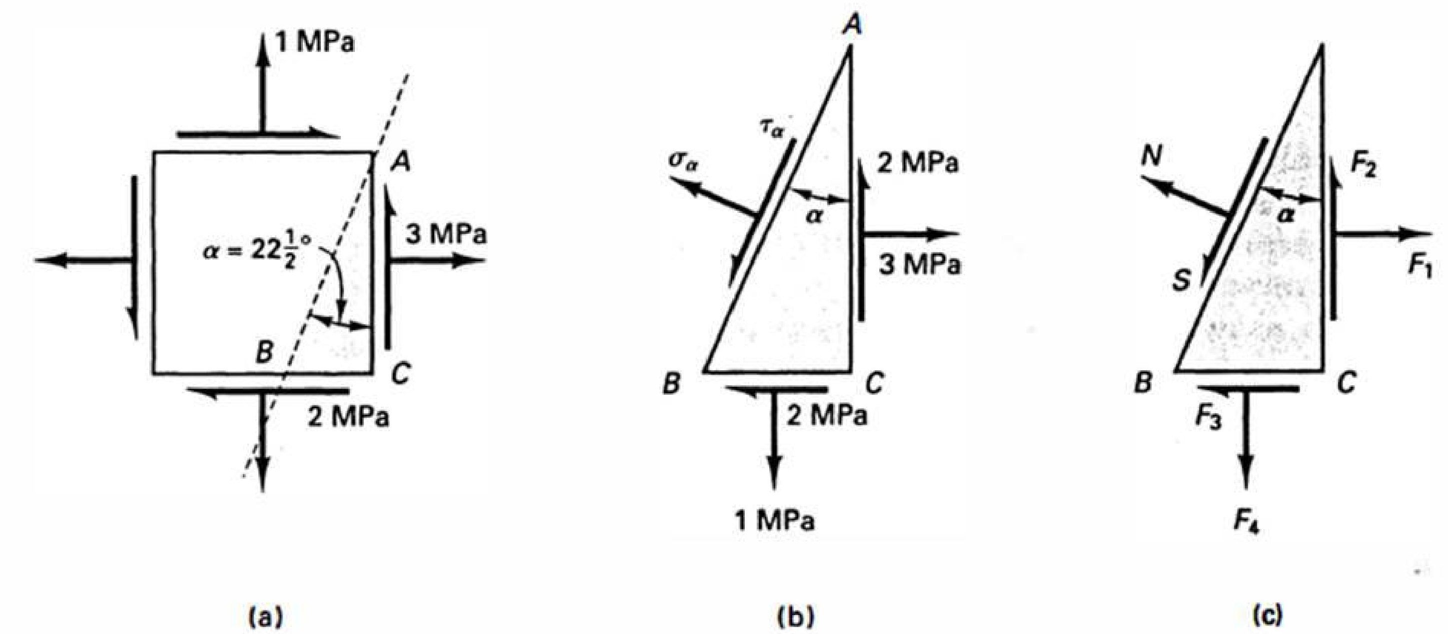


Fig. 11-3

SOLUTION

Wedge ABC is part of the element in Fig. 11-3(a): therefore, the stresses on faces AC and BC are known. The unknown normal and shear stresses acting on face AB are designated in the figure by σ_α and τ_α respectively. Their sense is assumed arbitrarily.

To determine a σ_α and τ_α for convenience only, let the area of the face defined by line AB be unity such as 1 mm^2. Then the area corresponding to line AC is equal to 1x cos α = 0.924 mm^2 and that to BC is equal to 1x sin α = 0.383 mm^2 .

Forces F1, F2, F.1, and F4, Fig. ll-3(c), can be obtained by multiplying the stresses by their respective areas. The unknown equilibrant forces N and S act, respectively, normal and tangential to plane AB. Then applying the equations of static equilibrium to the forces acting on the wedge given forces N and S.

$F_1 = 3x \cos \propto = 3x0.924 = 2.78\ N$ $\qquad$ $F_2 = 2x \cos \propto = 2x0.924 = 1.85\ N$

$F_3 = 2x \sin \propto = 2x0.383 = 0.766\ N$ $\qquad$ $F_2 = 1x \sin \propto = 2x0.383 = 0.383\ N$

$$\sum F_N = 0, \qquad N = F_1 \cos \propto -F_2 \sin \propto -F_3 \cos \propto +F_4 \sin \propto$$

$$= 2.78x0.924 - 1.85x0.383 - 0.766x0.924 + 0.383x0.383$$
$$= 1.29\ N$$

$$\sum F_S = 0, \qquad S = F_1 \sin \propto +F_2 \cos \propto -F_3 \sin \propto -F_4\ cos \propto$$

$$= 2.78x0.383 + 1.85x0.924 - 0.766x0.383 - 0.383x0.924$$
$$= 2.12\ N$$

$\sigma_\alpha = 1.29\ MPa$ and $\tau_\alpha = 2.12\ MPa$

11-3. Transformation of Stresses in Two-Dimensional Problems

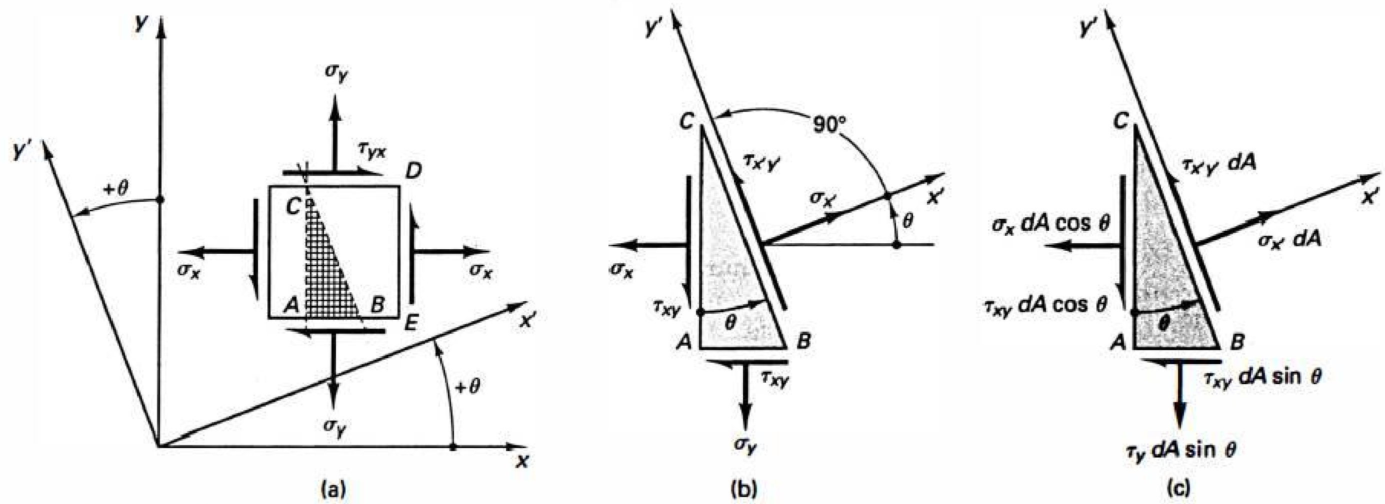


Fig. 11-4 Derivation of stress transformation on an inclined plane.

Algebraic equations are developed using an element of unit thickness in Fig. 11-4(a) in a state of two-dimensional stress initially referred to the positive, and are negative if compressive. Positive shear stress is defined as acting upward in the positive direction of the y axis on the right (positive) face DE of the element. Here the stress transformation is sought from the xy coordinate axes to the x'y' axes. The angle θ, which locates the x' axis, is positive when measured from the x axis toward they axis in a counterclockwise direction.

The outward normal to the section forms an angle θ with the x axis. If an area of the wedge isolated by this section is dA, the areas associated with the faces AC and AB are dA.cosθ and dA.sin θ, respectively. By multiplying the stresses by their respective areas, a diagram with the forces acting on the wedge is constructed. Fig. 11-4(c). Then, by applying the equations of static equilibrium to the forces acting on the wedge, stresses $\sigma_{x'}$ and $\tau_{x'y'}$ are obtained:

$$\sum F_{x'} = 0 \quad \sigma_{x'} dA = \sigma_x dA \cos\theta \cos\theta + \sigma_y dA \sin\theta \sin\theta$$

$$+ \tau_{xy} dA \cos\theta \sin\theta + \tau_{xy} dA \sin\theta \cos\theta$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$= \sigma_x \frac{1 + \cos 2\theta}{2} + \sigma_y \frac{1 - \cos 2\theta}{2} + \tau_{xy} \sin 2\theta$$

$$\boxed{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta} \qquad (11\text{-}1)$$

Similarly, from $\sum F_{y'} = 0$,

$$\boxed{\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta} \qquad (11\text{-}2)$$

Replacing θ in Eq. 11-1 by θ + 90° gives the normal stress in the direction of the y' axis. This stress can be designated as $\sigma_{y'}$; see Fig. 1-3(b). Hence, on noting that cos(2θ+180°) = -cos2θ and sin(2θ+180°) = -sin2θ, one has:

$$\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (11\text{-}3)$$

By adding Eqs. 11-1 and 11-3;

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y \qquad (11\text{-}4)$$

meaning that the sum of the normal stresses on any two mutually planes remains the same (i.e., invariant), regardless of the angle θ.

11-4. Principal Stresses in Two-Dimensional Problems

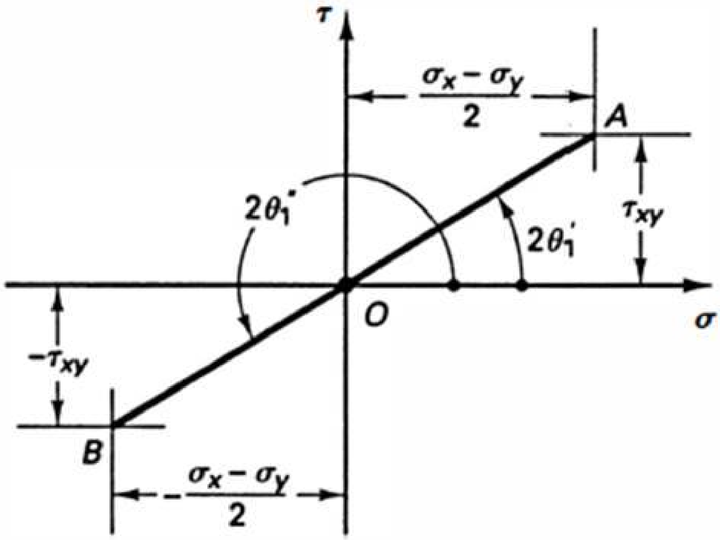
To find the plane for a maximum or a minimum normal stress. Eq. 11-1 is differentiated with respect to 6 and the derivative set equal to zero; that is:

$$\frac{d\sigma_{x'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2} 2 \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

Hence,

$$\boxed{\tan 2\theta_1 = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}} \qquad (11\text{-}6)$$

Equation 11-6 has two roots, since the value of the tangent of an angle in the diametrically opposite quadrant is the same, as may be seen from Fig. 11-5. These roots are 180° apart, and, as Eq. 11-6 is for a double angle, the roots of θ are 90° apart. One of these roots locates a plane on which the maximum normal stress acts; the other locates the corresponding plane for the minimum normal stress. To distinguish between these two roots, a prime and double prime notation is used.



$$\overline{OA} = \overline{OB} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sin 2\theta_1' = -\sin 2\theta_1'' = \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\cos 2\theta_1' = -\cos 2\theta_1'' = \frac{\frac{1}{2}(\sigma_x - \sigma_y)}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

Fig. 11-5 Angle functions for principal stresses.

On planes on which maximum or minimum nor mal stresses occur, there are no shear stresses. These planes are called the principal planes of stress, and the stresses acting on these planes-the maximum and minimum normal stresses-are called the principal stresses.

The magnitudes of the principal stresses can be obtained by substituting the values of the sine and cosine functions corresponding to the double angle given by Eq. 11-6 into Eq. 11-1. Then the results are simplified, and the expression for the maximum normal stress (denoted by σ_1) and the minimum normal stress (denoted by σ_2) becomes

$$(\sigma_{x'})_{\substack{\max \\ \min}} = \sigma_{1 \text{ or } 2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (11\text{-}7)$$

where the positive sign in front of the square root must be used to obtain σ_1 and the negative sign to obtain σ_2. The planes on which these stresses act can be determined by using Eq. 11-6. A particular root of Eq. 11-6 substituted into Eq. 11-1 will check the result found from Eq. 11-4 and at the same time will locate the plane on which this principal stress acts.

11.5 Maximum Shear Stresses in Two-Dimensional Problems

If σ_x, σ_y and τ_{xy} are known for an element, the shear stress on any plane defined by an angle θ is given by Eq. 11-2, and a study similar to the one made before for the normal stresses may be made for the shear stress. Thus, similarly, to locate the planes on which the maximum or the minimum shear stresses act, Eq. 11-2 must be differentiated with respect to θ and the derivative set equal to zero. When this is carried out and the results are simplified,

$$\tan 2\theta_2 = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}} \qquad (11\text{-}8)$$

where the subscript 2 is attached to θ to designate the plane on which the shear stress is a maximum or a minimum. Like Eq. 11-6, Eq. 11-8 has two roots, which again may be distinguished by attaching to θ_2 a prime or a double prime notation. The two planes defined by this equation are mutually perpendicular. Moreover, the value of $\tan 2\theta_2$ given by Eq. 11-8 is a negative reciprocal of the value of $\tan 2\theta_1$ in Eq.11-6.

$$\boxed{\tan 2\theta_2 = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}} \qquad (11\text{-}8)$$

$$\boxed{\tan 2\theta_1 = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}} \qquad (11\text{-}6)$$

Hence, the roots for the double angles of Eq. 11-8 are 90° away from the corresponding roots of Eq. 11-6. This means that the angles that locate the planes of maximum or minimum shear stress form angles of 45° with the planes of the principal stresses. A substitution into Eq. 11-2 of the sine and cosine functions corresponding to the double angle given by Eq.11-8 and determined in a manner analogous to that in Fig. 11-5 gives the maximum and the minimum values of the shear stresses. These, after simplifications, are:

$$\boxed{\tau_{\substack{max \\ min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \qquad (11\text{-}9)$$

Thus, the maximum shear stress differs from the minimum shear stress only in sign. Moreover, since the two roots given by Eq. 11-9 locate planes 90° apart, this result also means that the numerical values of the shear stresses on the mutually perpendicular planes are the same. From the physical point of view, these signs have no meaning, and for this reason, the largest shear stress regardless of sign will often be called the maximum shear stress.

The definite sense of the shear stress can always be determined by direct substitution of the particular root of θ_2 into Eq. 11-2. A positive shear stress indicates that it acts in the direction assumed in Fig. 11-4(b), and vice versa. Unlike the principal stresses for which no shear stresses occur on the principal planes, the maximum shear stresses act on planes that are usually not free of normal stresses. Substitution of θ_2 from Eq. 11-8 into Eq. 11-1 shows that the normal stresses that act on the planes of the maximum shear stresses are

$$\boxed{\sigma' = \frac{\sigma_x + \sigma_y}{2}} \tag{11-10}$$

Therefore, a normal stress acts simultaneously with the maximum shear stress unless $\sigma_x + \sigma_y$ vanishes. If σ_x and σ_y in Eq. 11-9 are the principal stress, τ_{xy} is zero and Eq. 11-9 simplifies to $\boxed{\tau_{\substack{max \\ min}} = \pm \frac{\sigma_1 - \sigma_2}{2}}$ (11-11)

$$\boxed{(\sigma_{x'})_{\substack{max \\ min}} = \sigma_{1\,or\,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \qquad (11\text{-}7)$$

Equation 11-7 clearly shows that in the absence of normal stresses, the principal stresses are numerically equal to the shear stress. The sense of the normal stresses follows from Eq. 11-6. The shear stresses act toward the diagonal DF in the direction of the principal tensile stresses; see Fig.11-6.

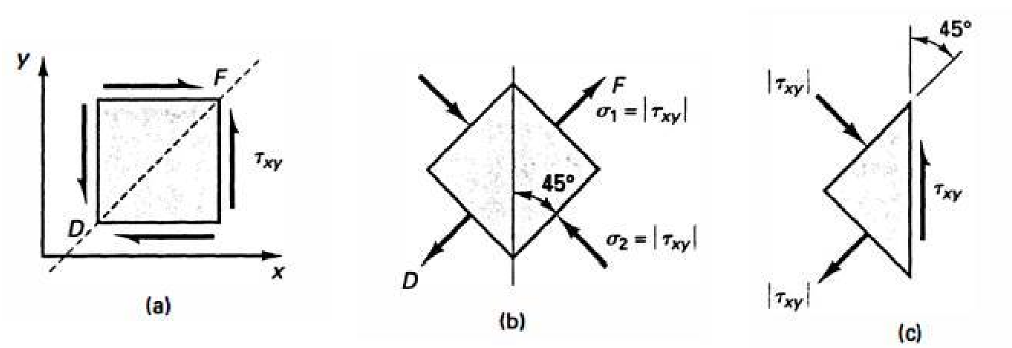


Fig.11-6 Equivalent representations for pure shear stress.

Example 11-2

For the state of stress in Example 11-1, reproduced in Fig. 11-7(a), (a) rework the previous problem for $\theta = -22\frac{1}{2}°$, using the general equations for the transformation of stress; (b) find the principal stresses and show their sense on a properly oriented element; and (c) find the maximum shear stresses with the associated normal stresses and show the results on a properly oriented element.

SOLUTION

(a) By directly applying Eqs. 11-1 and 11-2 for $\theta = -22\frac{1}{2}°$, with $\sigma_x = +3$ MPa, $\sigma_y = +1$ MPa, and $\tau_{xy} = +2$ MPa, one has

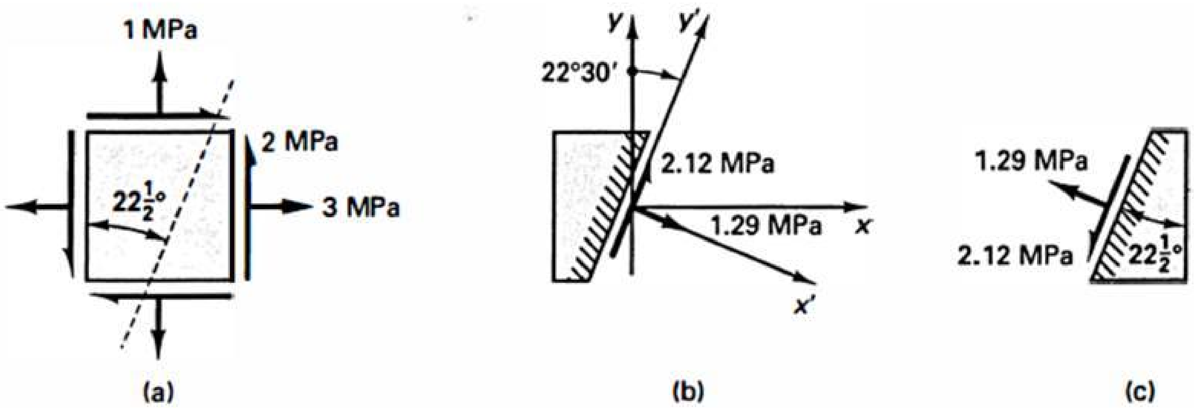
$$\sigma_{x'} = \frac{3+1}{2} + \frac{3-1}{2}\cos(-45°) + 2\sin(-45°)$$

$$= 2 + 1 \times 0.707 - 2 \times 0.707 = +1.29 \text{ MPa}$$

$$\tau_{x'y'} = -\frac{3-1}{2}\sin(-45°) + 2\cos(-45°)$$

$$= +1 \times 0.707 + 2 \times 0.707 = +2.12 \text{ MPa}$$

The positive sign of $\sigma_{x'}$ indicates tension; whereas the positive sign of $\tau_{x'y'}$ indicates that the shear stress acts in the $+y'$ direction, as shown in Fig. 11-4(b). These results are shown in Fig. 11-7(b) as well as in Fig. 11-7(c).



(b) The principal stresses are obtained by means of Eq. 11-7. The planes on which the principal stresses act are found by using Eq. 11-6.

$$\sigma_{1,2} = \frac{3+1}{2} \pm \sqrt{\left(\frac{3-1}{2}\right)^2 + 2^2} = 2 \pm 2.24$$

$$\sigma_1 = 4.24 \text{ MPa (tension)} \qquad \sigma_2 = 0.24 \text{ MPa (compression)}$$

$$\tan 2\theta_1 = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{2}{(3-1)/2} = 2$$

$$2\theta_1 = 63°26' \quad \text{or} \quad 63°26' + 180° = 243°26'$$

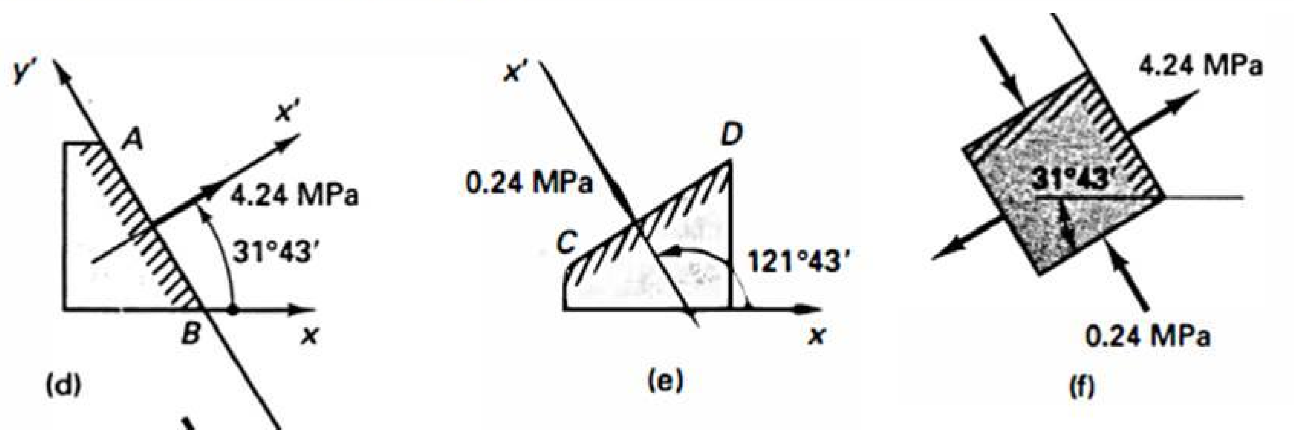
Hence,

$$\theta_1' = 31°43' \quad \text{and} \quad \theta_1'' = 121°43'$$

This locates the two principal planes, AB and CD, Figs. 11-7(d) and (e), on which σ_1 and σ_2 act. On which one of these planes the principal stresses act is unknown. So Eq. 11-1 is solved by using, for example $\theta_1' = 31°43'$. The stress found by this calculation is the stress that acts on plane AB. Then, since $2\theta_1' = 63°26'$,

$$\sigma_{x'} = \frac{3+1}{2} + \frac{3-1}{2}\cos 63°26' + 2\sin 63°26' = +4.24 \text{ MPa} = \sigma_1$$

This result, besides giving a check on the previous calculations, shows that the maximum principal stress acts on plane AB. The complete state of stress at the given point in terms of the principal stresses is shown in Fig. 11-7(f). Note that the results satisfy Eq. 11-4.



(c) The maximum shear stress is found by using Eq. 11-9. The planes on which these stresses act are defined by Eq. 11-8. The sense of the shear stresses is determined by substituting one of the roots of Eq. 11-8 into Eq. 11-2. Normal stresses associated with the maximum shear stress are determined by using Eq. 11-10.

$$\tau_{max} = \sqrt{[(3-1)/2]^2 + 2^2} = \sqrt{5} = 2.24 \text{ MPa}$$

$$\tan 2\theta_2 = -\frac{(3-1)/2}{2} = -0.500$$

$$2\theta_2 = 153°26' \qquad \text{or} \qquad 153°26' + 180° = 333°26'$$

Hence,

$$\theta_2' = 76°43' \qquad \text{and} \qquad \theta_2'' = 166°43'$$

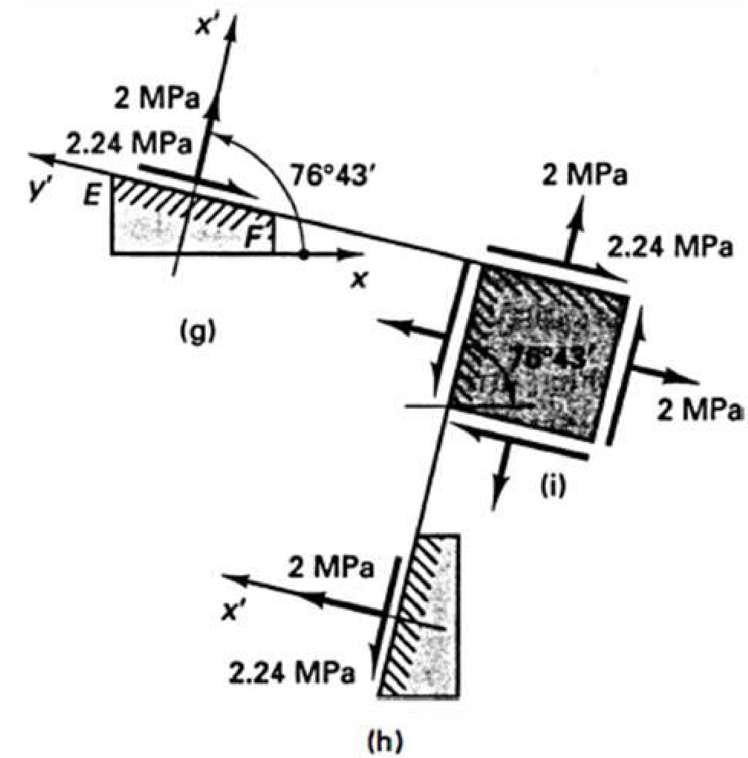
These planes are shown in Figs. 11-7(g) and (h). Then, by using $2\theta_2' = 153°26'$ in Eq. 11-2,

$$\tau_{x'y'} = -\frac{3-1}{2}\sin 153°26' + 2\cos 153°26' = -2.24 \text{ MPa}$$

which means that the shear along plane *EF* has an opposite sense to that of the y' axis. From Eq. 11-11,

$$\sigma' = \frac{3+1}{2} = 2 \text{ MPa}$$

The complete results are shown in Fig. 11-7(i). Note again that Eq. 11-4 is satisfied.



x'
2 MPa
2.24 MPa
76°43′
y'
E
F
x
(g)
2 MPa
2.24 MPa
76°43′
2 MPa
(i)
2 MPa
x'
2.24 MPa
(h)

11-6. Mohr's Circle of Stress for Two-Dimensional Problems

A careful study of Eqs. 11-1 and 11-2 shows that they represent a circle written in parametric form. That they do represent a circle is made clearer by first rewriting them as

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad (11\text{-}12)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \qquad (11\text{-}13)$$

Then by squaring both these equations, adding, and simplifying,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad (11\text{-}14)$$

In a given problem, σ_x, σ_y, and τ_{xy} are the three known constants, and $\sigma_{x'}$ and $\tau_{x'y'}$ are the variables. Hence, Eq. 11-14 may be written in more compact form as

$$(\sigma_{x'} - a)^2 + \tau_{x'y'}^2 = b^2 \qquad (11\text{-}15)$$

where $a = (\sigma_x + \sigma_y)/2$ and $b^2 = [(\sigma_x - \sigma_y)/2]^2 + \tau_{xy}^2$ are constants.

This equation is the familiar expression of analytical geometry, $(x - a)^2 + y^2 = b^2$, for a circle of radius b with its center at $(+a,0)$. Hence, if a circle satisfying this equation is plotted, the simultaneous values of a point (x, y) on this circle correspond to $\sigma_{x'}$ and $\tau_{x'y'}$ for a particular orientation of an inclined plane. The ordinate of a point on the circle is the shear stress $\tau_{x'y'}$; the abscissa is the normal stress $\sigma_{x'}$. The circle so constructed is called a *circle of stress* or *Mohr's circles of stress*.[3]

By using the previous interpretation, a Mohr's circle for the stresses given in Fig. 11-8(a) is plotted in Fig. 11-8(c) with σ and τ as the coordinate axes. The center C is at $(a,0)$, and the circle radius $R = b$. Hence,

$$a = OC = \frac{\sigma_x + \sigma_y}{2} \tag{11-16}$$

and

$$b = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{11-17}$$

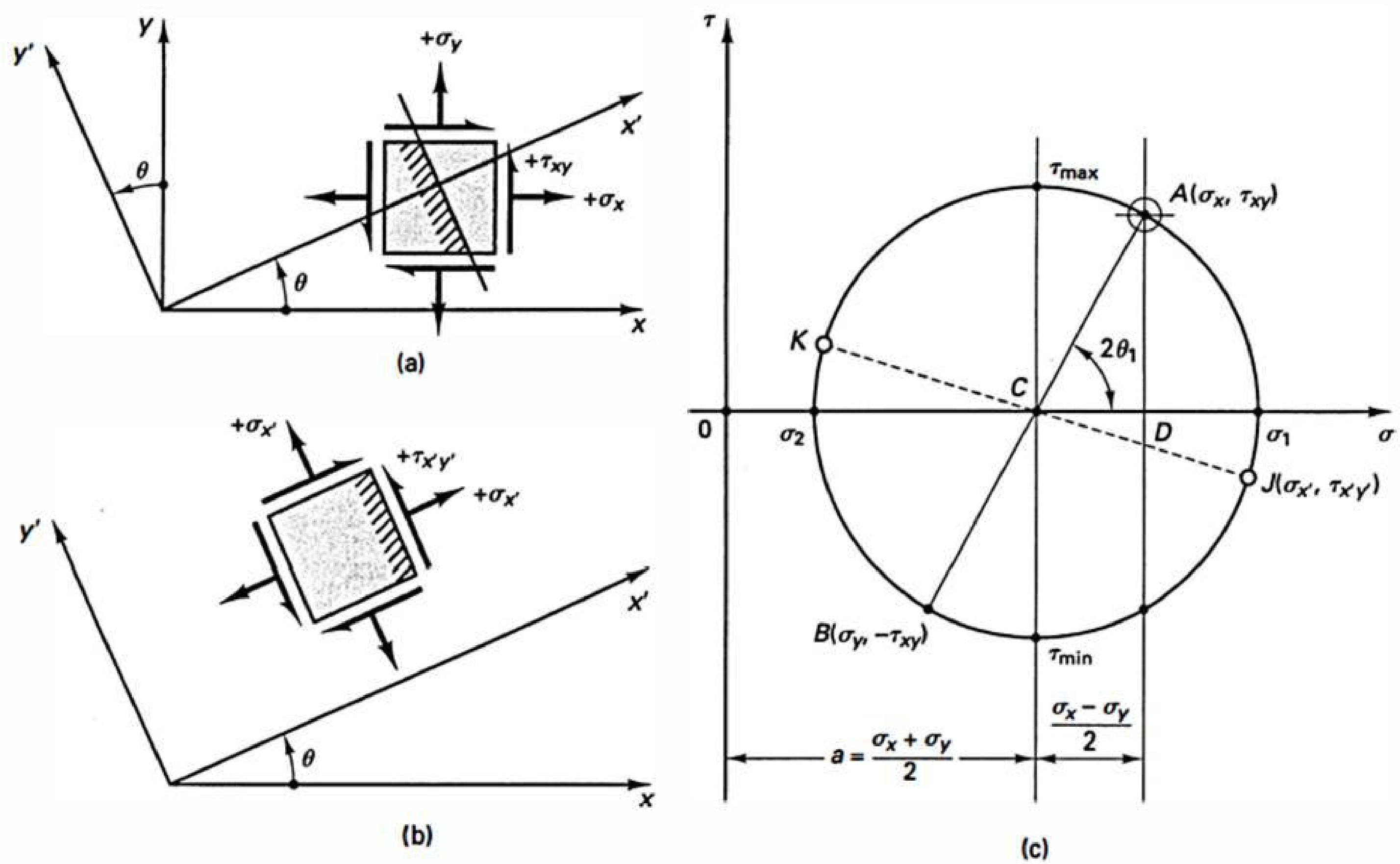


Fig. 11-8 Mohr's circle of stress.

The coordinates for point A on the circle correspond to the stresses in Fig. 11-8(a) on the right face of the element. For this face of the element, $\theta = 0°$ (i.e., the xy and the $x'y'$ axes coincide), $\sigma_{x'} = \sigma_x$, and $\tau_{x'y'} = \tau_{xy}$. The positive directions for these stresses coincide with the positive directions of the axes. Since $AD/CD = \tau_{xy}/[(\sigma_x - \sigma_y)/2]$, according to Eq. 11-6, the angle ACD is equal to $2\theta_1$. The coordinates for the conjugate point B correspond to the stresses in Fig. 11-8(a) on the upper face of the element. This follows from Eqs. 11-1 and 11-2 with $\theta = 90°$ or, for σ_y', from Eq. 11-3 with $\theta = 0°$.

The same reasoning applies to any other orientation of an element, such as shown in Fig. 11-8(b). A pair of conjugate points J and K can always be found on the circle to give the corresponding stresses, Fig. 11-8(c). An infinity of possible states of stress dependent on the angle θ are defined by the stress circle. Therefore, the following important observations regarding the state of stress at a point can be made based on the Mohr's circle:

1. The largest possible normal stress is σ_1; the smallest is σ_2. No shear stresses exist together with either one of these principal stresses.

2. The largest shear stress τ_{max} is numerically equal to the radius of the circle, also to $(\sigma_1 - \sigma_2)/2$. A normal stress equal to $(\sigma_1 + \sigma_2)/2$ acts on each of the planes of maximum shear stress.
3. If $\sigma_1 = \sigma_2$, Mohr's circle degenerates into a point, and no shear stresses at all develop in the *xy* plane.
4. If $\sigma_x + \sigma_y = 0$, the center of Mohr's circle coincides with the origin of the $\sigma\tau$ coordinates, and the state of pure shear exists.
5. The sum of the normal stresses on any two mutually perpendicular planes is invariant; that is,

$$\sigma_x + \sigma_y = \sigma_1 + \sigma_2 = \sigma_{x'} + \sigma_{y'} = \text{constant}$$

11-7. Construction of Mohr's Circles for Stress Transformation

The transformation of two-dimensional states of stress from one set of coordinates to another can always be made by direct application of statics as in Example 11-1, or, using the derived equations in Sections 11-3, 11-4, and 11-5.

Method I The basic problem consists of constructing the circle of stress for given stresses σ_x, σ_y, and τ_{xy}, such as shown in Fig. 11-9(a), and then determining the state of stress on an *arbitrary* plane *a–a*. A procedure for determining the stresses on any inclined plane requires justification on the basis of the equations derived in Section 11-3.

According to Eq. 11-16, the center C of a Mohr's circle of stress is located on the σ axis at a distance $(\sigma_x + \sigma_y)/2$ from the origin. Point A on the circle has the coordinates (σ_x, τ_{xy}) corresponding to the *stresses acting on the right-hand face of the element* in the positive direction of the coordinate axes, Fig. 11-9(a). Point A will be referred to as the *origin of planes.* This information is sufficient to draw a circle of stress, Fig. 11-9(b).

$$\theta + \alpha = 2\theta_1 \qquad < FCJ + 2\alpha = 2\theta_1$$
$$\alpha = 2\theta_1 - \theta \qquad < FCJ = 2\theta_1 - 2\alpha$$
$$< FCJ = 2\theta_1 - 2(2\theta_1 - \theta)$$
$$< FCJ = 2\theta - 2\theta_1$$

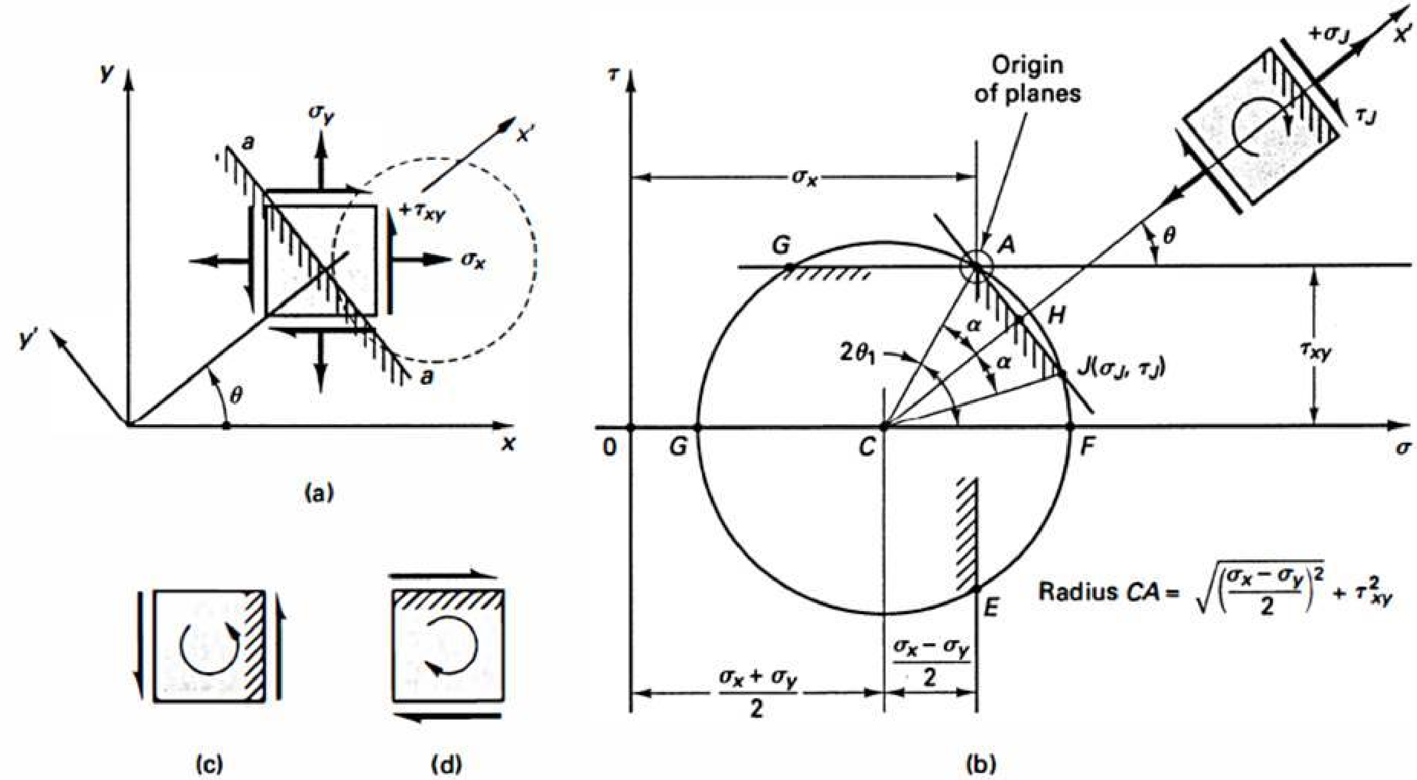


Fig. 11-9 Construction of Mohr's circle for determining stresses on an arbitrary plane.

The next step consists of drawing on the circle of stress a line through A parallel to plane a–a in the physical plane of Fig. 11-9(a). The intersection of this line at J with the stress circle gives the stresses acting on plane a–a. This requires some justification. For this purpose, the indicated geometric construction must be reviewed in detail.

According to the previous derivation shown in Fig. 11-8(c), angle ACF in Fig. 11-9(b) is equal to $2\theta_1$. Further, since line CH is drawn perpendicular to line AJ, angle ACJ is bisected, and $\alpha = 2\theta_1 - \theta$. Hence, angle JCF is $\theta - \alpha = 2\theta - 2\theta_1$, and it remains to be shown that the coordinates of point J define the stresses acting on inclined plane a–a. For this purpose, one notes from Fig. 11-9(b) that if R is the radius of a circle, $R \cos 2\theta_1 = (\sigma_x - \sigma_y)/2$ and $R \sin 2\theta_1 = \tau_{xy}$. Then, forming expressions for the normal and shear stresses at J based on the construction of the circle in Fig. 11-9(b) and making use of trigonometric identities for double angles, one has

$$\sigma_J = \frac{\sigma_x + \sigma_y}{2} + R\cos(2\theta - 2\theta_1)$$

$$= \frac{\sigma_x + \sigma_y}{2} + R\,(\cos 2\theta \cos 2\theta_1 + \sin 2\theta \sin 2\theta_1) \qquad (11\text{-}18)$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

and

$$\tau_J = R\sin(2\theta - 2\theta_1) = R\sin 2\theta \cos 2\theta_1 - R\cos 2\theta \sin 2\theta_1$$

$$= +\frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad (11\text{-}19)$$

Except for the sign of τ_J, the last expressions are identical to Eqs. 11-1 and 11-2 and, therefore, define the stresses acting on the element shown in the upper right quadrant of Fig. 11-9(b). The hatched side of this element is parallel to line *AJ* on the stress circle, which is parallel to line *a–a* in Fig. 11-9(a). However, since the sign of τ_J is opposite to that in the basic transformation, Eq. 11-2, a special rule for the direction of shear stress has to be introduced.

For this purpose, consider the initial data for the element shown in Fig. 11-9(a), where all stresses are shown with positive sense. By isolating the shear stresses acting on the vertical faces, Fig. 11-9(c), it can be seen that *these stresses alone* cause a *counterclockwise* couple. By considering lines emanating from the origin of planes A, for the first case, Fig. 11-9(c), the circle is intersected at E, whereas for the second case, Fig. 11-9(d), it is intersected at G. This can be generalized into a rule: If the point of intersection of a line emanating from the origin of planes A intersects the circle *above* the σ axis, the shear stresses on the opposite sides of an element cause a *clockwise* couple. Conversely, if the point of intersection lies *below* the σ axis, the shear stresses on the opposite sides cause a *counterclockwise* couple. According to this rule, the shear stresses at J in Fig. 11-9(b) act with a clockwise sense.

This general procedure is illustrated for two particularly important cases. For the data given in Fig. 11-10(a), the principal stresses are found in Fig. 11-10(b), and the maximum shear stresses are found in Fig. 11-10(c).

For the first case, it is known that the extreme values on the abscissa, σ_1 and σ_2, give the principal stresses. Connecting these points with the origin of planes A locates the planes on which these stresses act. Angle θ_1 can be determined by trigonometry. Either one of the two solutions is sufficient to obtain the complete solution shown on the element on the right.

The magnitudes of the maximum absolute shear stresses are known to be given by the radius of the Mohr's circle. As shown in Fig. 11-10(c), these stresses are located above and below C. Connecting these points with the origin of planes A determines the planes on which these stresses act. The corresponding elements are shown in the upper two diagrams of the elements, where the associated mean normal stresses are also indicated. Either one of these solutions with the aid of equilibrium concepts is sufficient for the complete solution shown on the bottom element in the figure.

Example 11-3

Given the state of stress shown in Fig. 11-12(a), transform it (a) into the principal stresses, and (b) into the maximum shear stresses and the associated normal stresses. Show the results for both cases on properly oriented elements. Use Method I.

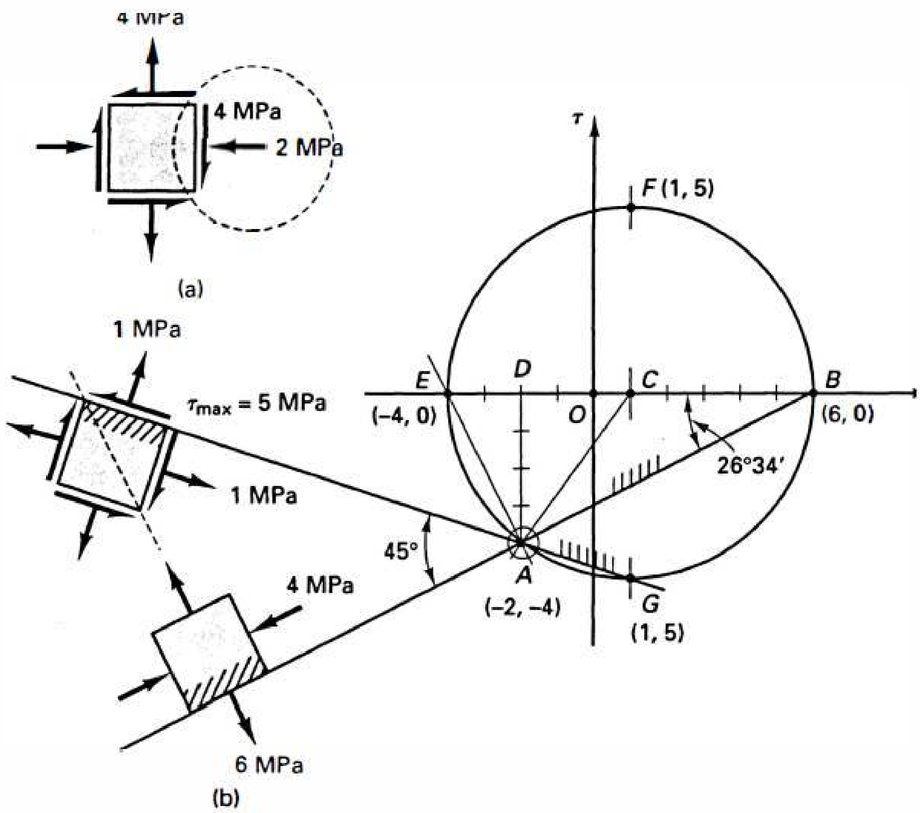


Fig. 11-12

Example 11-4

Using Mohr's circle, transform the stresses shown in Fig. 11-14(a) into stresses acting on the plane at an angle of $22\frac{1}{2}°$ with the vertical axis. Use Method I.

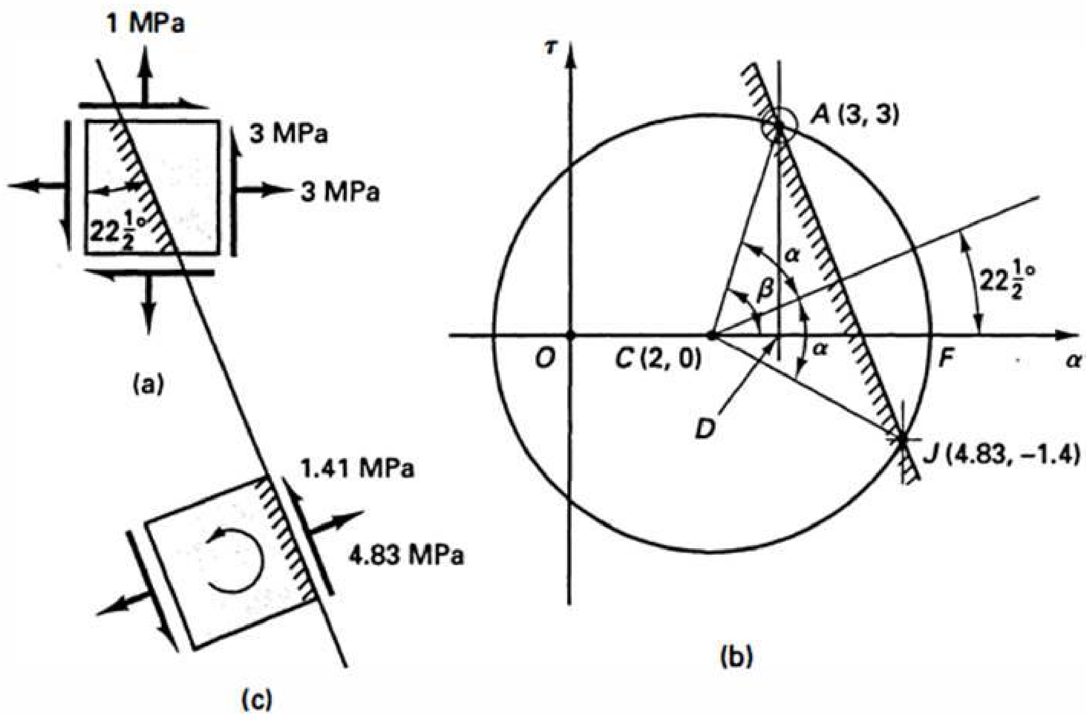


Fig. 11-14

Annex-A, Maximum shear stress angles

$$\sin 2\theta_1{}' = \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}}$$

$$2\theta_1{}' + 90^0 = 2\theta_2{}'$$
$$2\theta_1{}' = 2\theta_2{}' - 90^0$$
$$\sin(2\theta_2{}' - 90^0) = \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}}$$
$$\sin 2\theta_2{}' \cos 90^0 - \cos 2\theta_2{}' \sin 90^0 = \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}}$$

$$\cos 2\theta_2{}' = \frac{-\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}}$$
$$\sin 2\theta_2{}' = \frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}}$$

TO DO LIST FROM POPOV

Solve problem numbers given below from POPOV in addition to other text books.

Problems from Chapter 11 (2^{nd} ed): 2-6, 17-20, 22-36.

Chapter 14, POPOV

Beam Deflection by Direct Integration

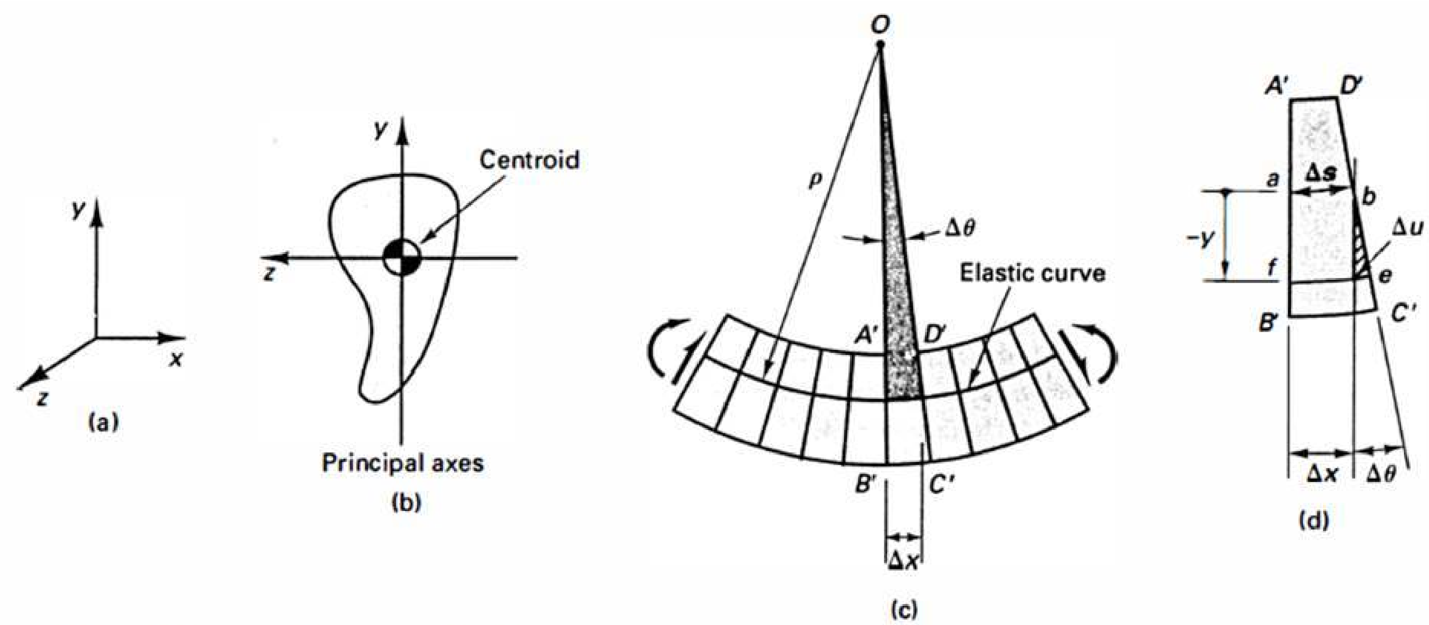
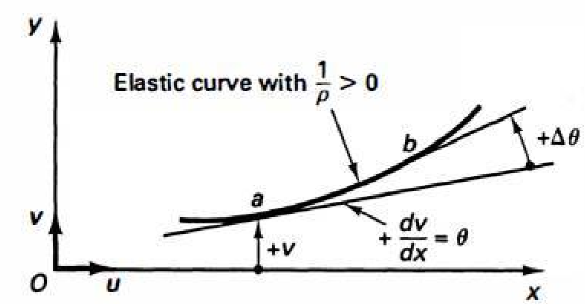
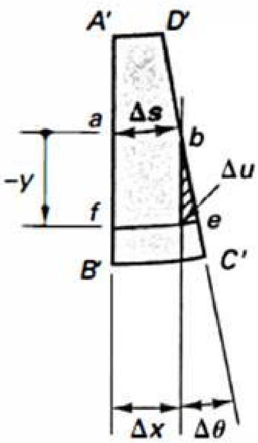


Fig. 14-1 Deformation of a beam in bending.



The center of curvature O for the elastic curve for any element can be found by extending to intersection any two adjoining sections, such as $A'B'$ and $D'C'$. In the enlarged view of element $A'B'C'D'$ in Fig. 14-1(d), it can be seen that in a bent beam, the included angle between two adjoining sections is $\Delta\theta$. If distance y from the neutral surface to the strained fibers is measured in the usual manner as being positive upward, the deformation Δu of any fiber can be expressed as



(d)

$$\Delta u = -y\,\Delta\theta \tag{14-1}$$

For negative y's, this yields elongation, which is consistent with the deformation shown in the figure.

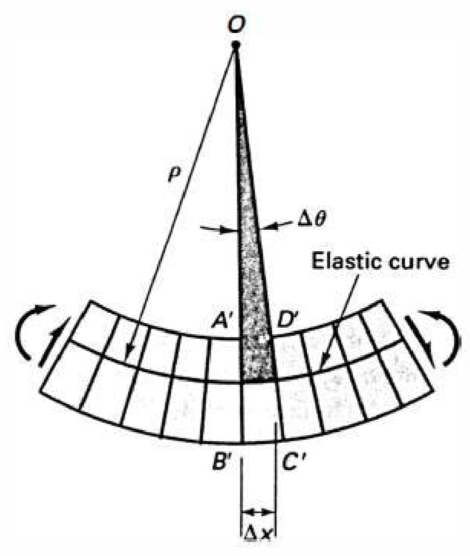
The fibers lying in the curved neutral surface of the deformed beam, characterized in Fig 14-1(d) by fiber ab, are not strained at all. Therefore, arc length Δs corresponds to the initial length of all fibers between sections $A'B'$ and $D'C'$. Bearing this in mind, upon dividing Eq. 14-1 by Δs, one can form the following relations:

$$\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = -y \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} \quad \text{or} \quad \frac{du}{ds} = -y\frac{d\theta}{ds} \tag{14-2}$$

One can recognize that du/ds is the normal strain in a beam fiber at a distance y from the neutral axis. Hence,

$$\frac{du}{ds} = \varepsilon \tag{14-3}$$

The term $d\theta/ds$ in Eq. 14-2 has a clear geometrical meaning. With the aid of Fig 14-1(c), it is seen that, since $\Delta s = \rho\Delta\theta$,



(c)

$$\lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$\frac{du}{ds} = -y\frac{d\theta}{ds} \qquad \frac{du}{ds} = \varepsilon \qquad \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$\varepsilon = -y.\frac{1}{\rho} \qquad \varepsilon = \frac{\sigma_x}{E} = -\frac{M.y}{EI}$$

$$-\frac{y}{\rho} = -\frac{M.y}{EI}$$

$$\frac{M}{EI} = \frac{1}{\rho}$$

In texts on analytic geometry, it is shown that in Cartesian coordinates, the curvature of a line is defined as

$$\frac{1}{\rho} = \frac{\dfrac{d^2v}{dx^2}}{\left[1 + \left(\dfrac{dv}{dx}\right)^2\right]^{3/2}} = \frac{v''}{[1 + (v')^2]^{3/2}} \tag{14-8}$$

where x and v are the coordinates of a point on a curve. For the problem at hand, distance x locates a point on the elastic curve of a deflected beam, and v gives the deflection of the same point from its initial position.

If Eq. 14-8 were substituted into Eq. 14-6, the exact differential equation for the elastic curve would result. In general, the solution of such an equation is very difficult to achieve. However, since the deflections tolerated in the vast majority of engineering structures are very small, slope dv/dx of the elastic curve is also very small. Therefore, the square of slope v' is a negligible quantity in comparison with unity, and Eq. 14-8 simplifies to

$$\frac{1}{\rho} \approx \frac{d^2v}{dx^2} \tag{14-9}$$

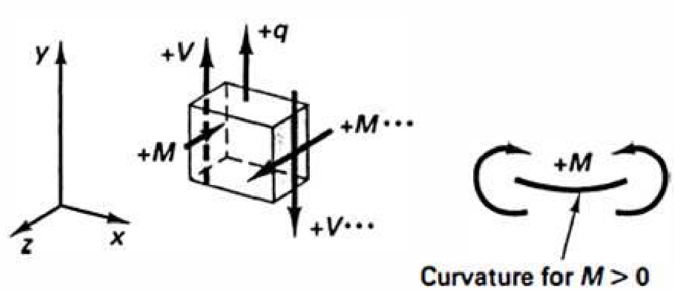
This simplification eliminates the *geometric nonlinearity* from the problem, and the governing differential equation for small deflections of elastic beams[2] using Eq. 14-6 is

$$\boxed{\frac{d^2v}{dx^2} = \frac{M}{EI}} \tag{14-10}$$

where it is understood that $M = M_z$ and $I = I_z$.

It is important to note that for the elastic curve, at the level of accuracy of Eq. 14-10, one has $ds = dx$. This follows from the fact that, as before, the square of the slope dv/dx is negligibly small compared with unity, and

$$ds = \sqrt{dx^2 + dv^2} = \sqrt{1 + (v')^2}\, dx \approx dx \tag{14-11}$$



(a)

14-5. Alternative Forms of the Governing Equation

The differential relations among the applied loads, shear, and moment, Eqs. 7-3 and 7-4, can be combined with Eq. 14-10 to yield the following useful sequence of equations:

$$v = \text{deflections of the elastic curve}$$

$$\theta = \frac{dv}{dx} = v' = \text{slope of the elastic curve}$$

$$M = EI\frac{d^2v}{dx^2} = EIv'' \tag{14-13}$$

$$V = \frac{dM}{dx} = \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = (EIv'')'$$

$$q = \frac{dV}{dx} = \frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = (EIv'')''$$

In applying these relations, the sign convention shown in Fig. 14-3 must be adhered to strictly. For beams with *constant* flexural rigidity EI, Eq. 14-13 simplifies into three alternative governing equations for determining the deflection of a loaded beam:

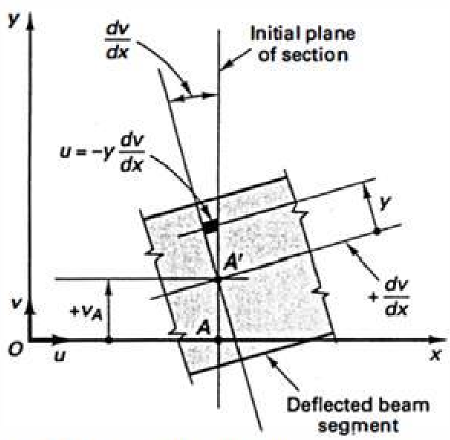


Fig. 14-4 Longitudinal displacements in a beam due to rotation of a plane section.

$$\boxed{EI\frac{d^2v}{dx^2} = M(x)} \tag{14-14a}$$

$$EI\frac{d^3v}{dx^3} = V(x) \tag{14-14b}$$

$$\boxed{EI\frac{d^4v}{dx^4} = q(x)} \tag{14-14c}$$

14-6. Boundary Conditions

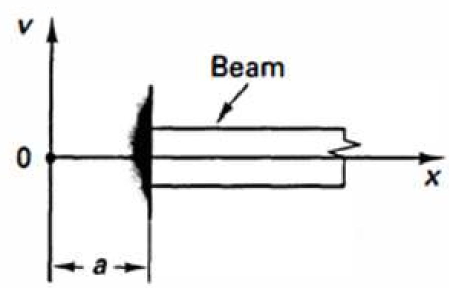
1. *Clamped or fixed support:* In this case, the displacement v and the slope dv/dx must vanish. Hence, at the end considered, where $x = a$,

$$v(a) = 0 \qquad v'(a) = 0 \tag{14-15a}$$

2. *Roller or pinned support:* At the end considered, no deflection v nor moment M can exist. Hence,

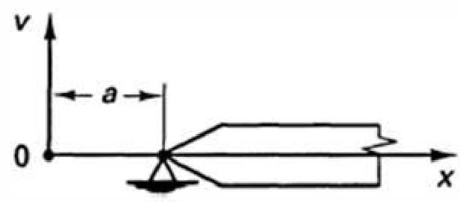
$$v(a) = 0 \qquad M(a) = EIv''(a) = 0 \tag{14-15b}$$

Here the physically evident condition for M is related to the derivative of v with respect to x from Eq. 14-14.



$\begin{cases} v(a) = 0 \\ \theta(a) = v'(a) = 0 \end{cases}$

(a) Clamped support



$\begin{cases} v(a) = 0 \\ M(a) = EIv''(a) = 0 \end{cases}$

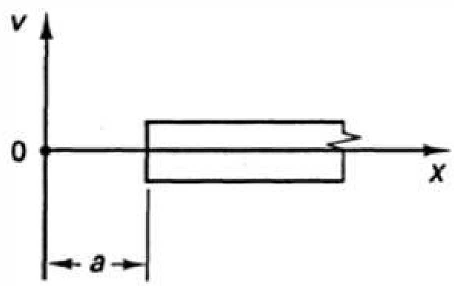
(b) Simple support

3. *Free end:* Such an end is free of moment and shear. Hence,

$$M(a) = EIv''(a) = 0 \qquad V(a) = (EIv'')'_{x=a} = 0 \tag{14-15c}$$

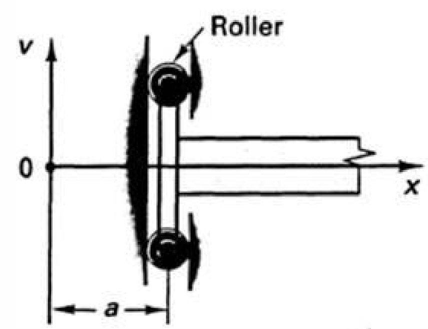
4. *Guided support:* In this case, free vertical movement is permitted, but the rotation of the end is prevented. The support is not capable of resisting any shear. Therefore,

$$v'(a) = 0 \qquad V(a) = (EIv'')'_{x=a} = 0 \tag{14-15d}$$



$\begin{cases} M(a) = EIv''(a) = 0 \\ V(a) = EIv'''(a) = 0 \end{cases}$

(c) Free end



$\begin{cases} \theta(a) = v'(a) = 0 \\ V(a) = EIv'''(a) = 0 \end{cases}$

(d) Guided support

14-7. Direct Integration Solutions

As a general example of calculating beam deflection, consider Eq. 14-14c, $EIv^{iv} = q(x)$. By successively integrating this expression four times, the formal solution for v is obtained. Thus,

$$EIv^{iv} = EI\frac{d^4v}{dx^4} = EI\frac{d}{dx}(v''') = q(x) \qquad \text{14-16a}$$

$$EIv''' = \int_0^x q\,dx + C_1$$

$$EIv'' = \int_0^x dx\int_0^x q\,dx + C_1x + C_2$$

$$EIv' = \int_0^x dx\int_0^x dx\int_0^x q\,dx + C_1x^2/2 + C_2x + C_3 \qquad (14\text{-}16b)$$

$$EIv = \int_0^x dx\int_0^x dx\int_0^x dx\int_0^x q\,dx + C_1x^3/3! + C_2x^2/2! + C_3x + C_4$$

In these equations the constants C_1, C_2, C_3, and C_4 have a special physical meaning. Since, per Eq. 14-14b, $EIv''' = V$, by substituting this relation into the second of Eqs. 14-16 and simplifying, Eq. 7-6 is reproduced; that is,

$$V = \int_0^x q\,dx + C_1 \tag{7-6}$$

By substituting this relation into Eq. 7-7 and integrating, a different form of Eq. 7-7 is obtained.

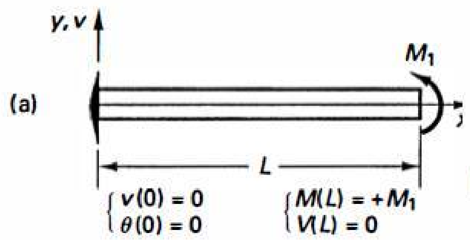
$$M = \int_0^x dx \int_0^x q\,dx + C_1 x + C_2 \tag{14-17}$$

The right side of this equation is identical to the third of Eqs. 14-16.

Example 14-2

A bending moment M_1 is applied at the free end of a cantilever of length L and of constant flexural rigidity EI, Fig. 14-8(a). Find the equation of the elastic curve.

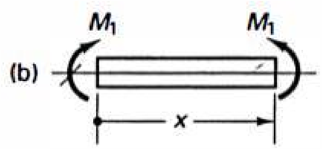
$$EI\frac{d^2v}{dx^2} = M = M_1$$



$$EI\frac{dv}{dx} = M_1x + C_3$$

But $\theta(0) = 0$; hence, at $x = 0$, one has $EIv'(0) = C_3 = 0$ and

$$EI\frac{dv}{dx} = M_1x$$



$$EIv = \frac{1}{2}M_1x^2 + C_4$$

But $v(0) = 0$; hence, $EIv(0) = C_4 = 0$ and

$$v = \frac{M_1x^2}{2EI} \tag{14-20}$$

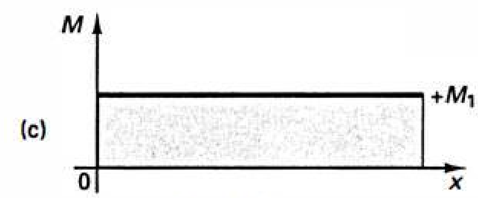
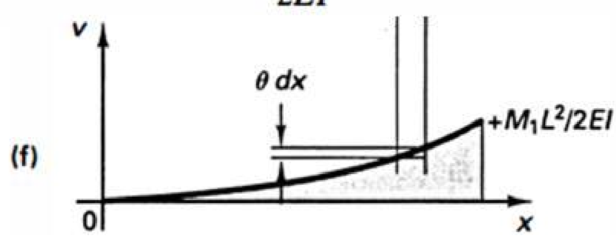
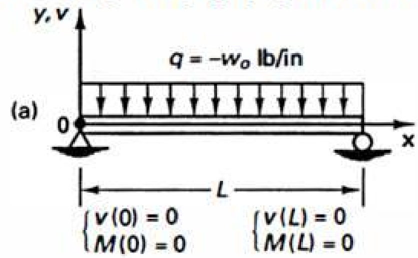


Fig. 14-8



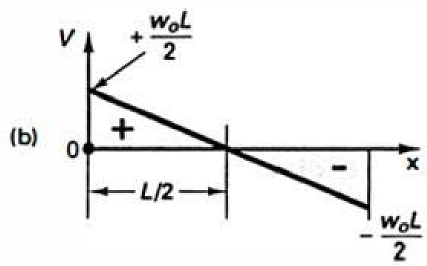
Example 14-3

A simple beam supports a uniformly distributed downward load w_o. The flexural rigidity EI is constant. Find the elastic curve by the following three methods: (a) Use the second-order differential equation to obtain the deflection of the beam. (b) Use the fourth-order equation instead of the one in part (a). (c) Illustrate a graphical solution of the problem.



$$M = \frac{w_o Lx}{2} - \frac{w_o x^2}{2}$$

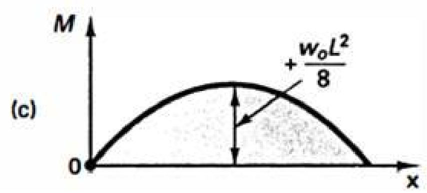
$$EI\frac{d^2v}{dx^2} = M = \frac{w_o Lx}{2} - \frac{w_o x^2}{2}$$



$$EI\frac{dv}{dx} = \frac{w_o Lx^2}{4} - \frac{w_o x^3}{6} + C_3$$

$$EIv = \frac{w_o Lx^3}{12} - \frac{w_o x^4}{24} + C_3 x + C_4$$

But $v(0) = 0$; hence, $EIv(0) = 0 = C_4$; and, since $v(L) = 0$,



$$EIv(L) = 0 = \frac{w_o L^4}{24} + C_3 L \quad \text{and} \quad C_3 = -\frac{w_o L^3}{24}$$

$$v = -\frac{w_o x}{24EI}(L^3 - 2Lx^2 + x^3) \tag{14-21}$$

Because of symmetry, the largest deflection occurs at $x = L/2$. On substituting this value of x into Eq. 14-21, one obtains

$$|v|_{max} = \frac{5w_oL^4}{384EI} \tag{14-22}$$

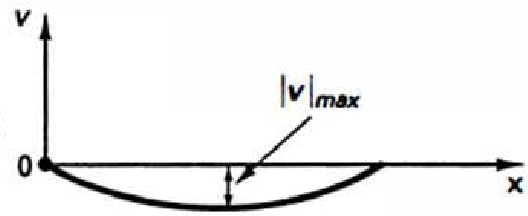
(b) Application of Eq. 14-14c to the solution of this problem is direct. The constants are found from the boundary conditions.

$$EI\frac{d^4v}{dx^4} = q(x) = -w_o$$

$$EI\frac{d^3v}{dx^3} = -w_ox + C_1$$

$$EI\frac{d^2v}{dx^2} = -\frac{w_ox^2}{2} + C_1x + C_2$$

(f)

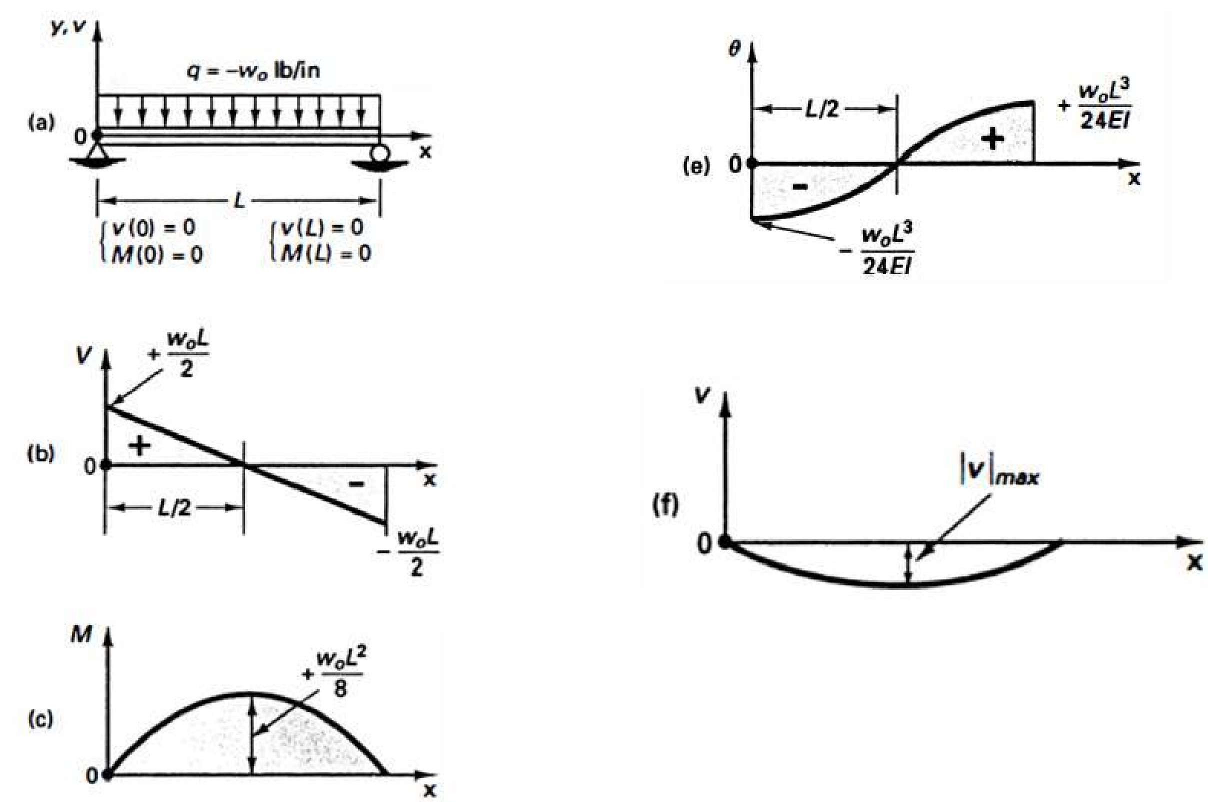


But $M(0) = 0$; hence, $EIv''(0) = 0 = C_2$; and, since $M(L) = 0$,

$$EIv''(L) = 0 = -\frac{w_oL^2}{2} + C_1L \quad \text{or} \quad C_1 = \frac{w_oL}{2}$$

hence,

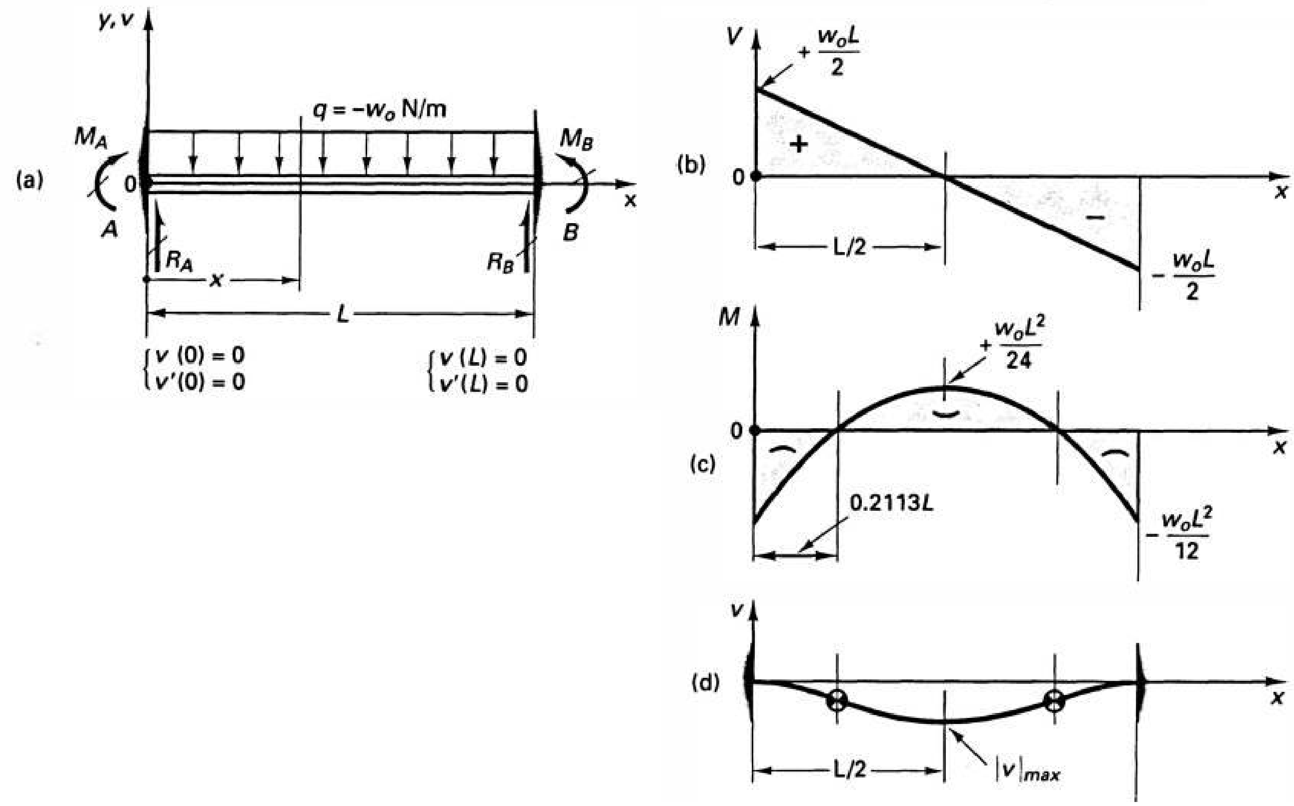
$$EI\frac{d^2v}{dx^2} = \frac{w_oLx}{2} - \frac{w_ox^2}{2}$$



(a)
y, v
q = −w_o lb/in
0
x
L
v(0) = 0
M(0) = 0
v(L) = 0
M(L) = 0
(b)
V
+ w_oL/2
+
0
L/2
−
x
− w_oL/2
(c)
M
+ w_oL²/8
0
x
(e)
θ
L/2
+ w_oL³/24EI
+
0
−
x
− w_oL³/24EI
(f)
v
|v|_max
0
x

Example 14-4

A beam fixed at both ends supports a uniformly distributed downward load w_o, Fig 14-10(a). The EI for the beam is constant. (a) Find the expression for the elastic curve using the fourth-order governing differential equation. (b) Verify the results found using the second-order differential equation.



SOLUTION

(a) As discussed in connection with Fig. 14-6(c), this beam is statically indeterminate to the second degree since horizontal reactions are assumed to be zero. The solution is obtained by four successive integrations of Eq. 14-14c in a manner shown in Eqs. 14-16. Then the constants of integration are found from the boundary conditions.

$$EI\frac{d^4v}{dx^4} = q(x) = -w_o$$

$$EI\frac{d^3v}{dx^3} = -w_o x + C_1$$

$$EI\frac{d^2v}{dx^2} = -\frac{w_o x^2}{2} + C_1 x + C_2$$

$$EI\frac{dv}{dx} = -\frac{w_o x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$EIv = -\frac{w_o x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

Four kinematic boundary conditions are available for determining the constants of integration:

$$EIv(0) = EIv_A = 0 = C_4$$

$$EIv'(0) = EIv'_A = 0 = C_3$$

$$EIv(L) = EIv_B = 0 = -\frac{w_o L^4}{24} + C_1 \frac{L^3}{6} + C_2 \frac{L^2}{2}$$

$$EIv'(L) = EIv'_B = 0 = -\frac{w_o L^3}{6} + C_1 \frac{L^2}{2} + C_2 L$$

Constants C_3 and C_4 do not enter the last two equations since they are zero. By solving the last two equations simultaneously.

$$C_1 = \frac{w_o L}{2} \qquad \text{and} \qquad C_2 = -\frac{w_o L^2}{12}$$

By substituting these constants into the equation for the elastic curve, after algebraic simplifications,

$$v = -\frac{w_o x^2}{24EI}(L - x)^2 \tag{14-23}$$

$$|v|_{\max} = \frac{w_o L^4}{384EI}$$

(b) This solution is found using Eq. 14-14a, and, although the vertical reaction at A can be determined directly from statics, it will be treated as an unknown. On this basis,

$$EI\frac{d^2v}{dx^2} = M(x) = M_A + R_A x - \frac{w_o x^2}{2}$$

Integrating twice,

$$EI\frac{dv}{dx} = M_A x + R_A\frac{x^2}{2} - \frac{w_o x^3}{6} + C_3$$

$$EIv = M_A\frac{x^2}{2} + R_A\frac{x^3}{6} - \frac{w_o x^4}{24} + C_3 x + C_4$$

Constants C_3 and C_4 *as well as* R_A and M_A are found from the four kinematic boundary conditions:

$$EIv(0) = EIv_A = 0 = C_4$$

$$EIv'(0) = EIv'^A = 0 = C_3$$

$$EIv(L) = EIv_B = 0 = M_A\frac{L^2}{2} + R_A\frac{L^3}{6} - \frac{w_o L^4}{24}$$

$$EIv'(L) = EIv'_B = 0 = M_A L + R_A\frac{L^2}{2} - \frac{w_o L^4}{6}$$

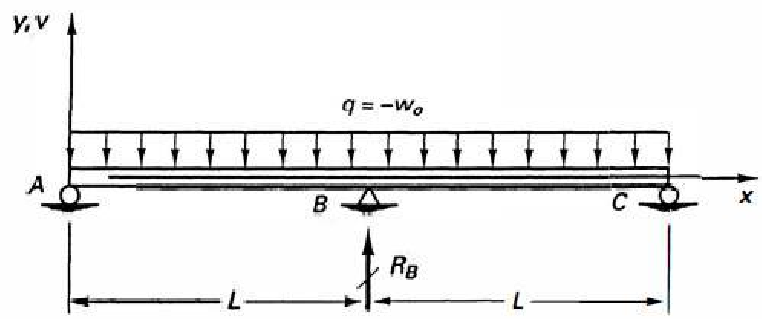
Solving the last two equations simultaneously,

$$R_A = \frac{w_o L}{2} \qquad \text{and} \qquad M_A = -\frac{w_o L^2}{12}$$

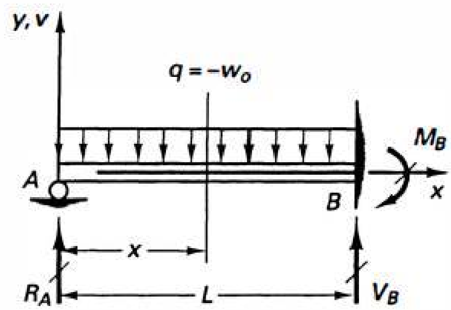
Substituting these expressions into the equation for deflection *with* $C_3 = C_4 = 0$, Eq. 14-23 is again obtained.

Example 14-5

Determine the equation of the elastic curve for the uniformly loaded continuous beam shown in Fig. 14-11(a). Use the second-order differential equation. EI is constant.



(a)



(b)

SOLUTION

Because of symmetry, the solution can be confined to determining the deflection for either span. Also, because of symmetry, it can be concluded that at the middle support, not only is the deflection zero, but since the elastic curve cannot rotate in either direction, its slope is also zero. In this manner, the problem can be reduced to the one-degree statically indeterminate problem shown in Fig. 14-11(b) with known boundary conditions.

Second-order differential-equation solution:

$$EI\frac{d^2v}{dx^2} = M(x) = R_Ax - \frac{w_ox^2}{2}$$

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} - \frac{w_ox^3}{6} + C_3$$

$$EIv = R_A\frac{x^3}{6} - \frac{w_ox^4}{24} + C_3x + C_4$$

Boundary conditions:

$$EIv(0) = EIv_A = 0 = C_4$$

$$EIv'(L) = EIv'_B = 0 = R_A\frac{L^2}{2} - \frac{w_oL^3}{6} + C_3$$

$$EIv(L) = EIv_B = 0 = R_A\frac{L^3}{6} - \frac{w_oL^4}{24} + C_3L$$

By solving the last two equations simultaneously,

$$R_A = \frac{3w_oL}{8} \qquad \text{and} \qquad C_3 = -\frac{w_oL^3}{48}$$

which, upon substitution into the equation for the elastic curve, leads to

$$v = -\frac{w_ox}{48EI}(L^3 - 3Lx^2 + 2x^3) \tag{14-25}$$

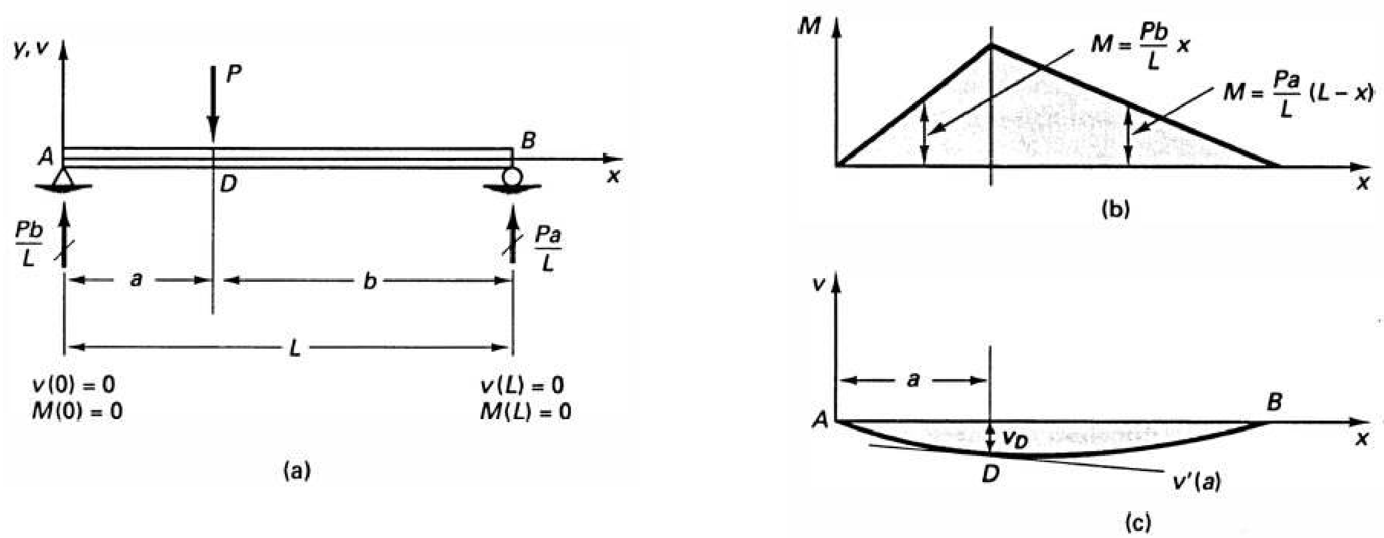
From symmetry, the reactions at A and C are equal, and, by using statics, the reaction at B is

$$R_B = \frac{5w_oL}{4} \tag{14-26}$$

This reaction is also numerically equal to $2V_B$.

Example 14-6

A simple beam supports a concentrated downward force P at a distance a from the left support, Fig. 14-12(a). The flexural rigidity EI is constant. Find the equation of the elastic curve by successive integration.



For segment AD | *For segment DB*

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{Pb}{EIL}x \qquad\qquad \frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{Pa}{EI} - \frac{Pa}{EIL}x$$

$$\frac{dv}{dx} = \frac{Pb}{EIL}\frac{x^2}{2} + A_1 \qquad\qquad \frac{dv}{dx} = \frac{Pa}{EI}x - \frac{Pa}{EIL}\frac{x^2}{2} + B_1$$

$$v = \frac{Pb}{EIL}\frac{x^3}{6} + A_1x + A_2 \qquad\qquad v = \frac{Pa}{EI}\frac{x^2}{2} - \frac{Pa}{EIL}\frac{x^3}{6} + B_1x + B_2$$

To determine the four constants $A_1, A_2, B_1,$ and B_2, two boundary and two continuity conditions must be used.

For segment AD:

$$v(0) = 0 = A_2$$

For segment DB:

$$v(L) = 0 = \frac{PaL^2}{3EI} + B_1L + B_2$$

(A) Segment AD:

$$v(0) = 0 = A_2 \Rightarrow \boxed{A_2 = 0}$$

(B) Segment DB:

$$v(L) = 0$$

$$\Rightarrow 0 = \frac{PaL^2}{3EI} + B_1 L + B_2$$

$$\Rightarrow B_2 = -B_1 L - \frac{PaL^2}{3EI} \quad \text{--(1)}$$

(C) Equating slope at D, from AD & DB

$$\frac{Pa^2 b}{2EIL} + A_1 = \frac{Pa^2}{EI} - \frac{Pa^3}{2EIL} + B_1$$

$$\Rightarrow A_1 = B_1 + \frac{Pa^2}{EI} - \frac{Pa^3}{2EIL} - \frac{Pa^2 b}{2EIL} \quad \text{--(2)}$$

(D) Equating deflection at D, from AD & DB

$$\frac{Pba^3}{6EIL} + A_1 a = \frac{Pa^3}{2EI} - \frac{Pa^4}{6EIL} + B_1 a + B_2 \quad \text{--(3)}$$

(E) Substitute B_2 & A_1 from (1) & (2) into (3)

$$\frac{Pba^3}{6EIL} + \left(\cancel{B_1 a} + \frac{Pa^3}{EI} - \frac{Pa^4}{2EI} - \frac{Pa^3 b}{2EI}\right) = \frac{Pa^3}{2EI} - \frac{Pa^4}{6EIL} + \cancel{B_1 a} - B_1 L - \frac{PaL^2}{3EI}$$

$$\Rightarrow -B_1 L = Pa\left(-\frac{a^2 b}{3EIL} + \frac{a^2}{2EI} - \frac{a^3}{3EI} + \frac{L^2}{3EI}\right)$$

$$\Rightarrow B_1 = -\frac{Pa}{6EIL}\left[-\frac{2a^2 b}{L} + 3a^2 - \frac{2a^3}{L} + 2L^2\right]$$

$$\Rightarrow B_1 = -\frac{Pa}{6EIL}\left[2L^2 + a^2 + \left\{2a^2 - \frac{2a^2 b}{L} - \frac{2a^3}{L}\right\}\right]$$

$$\Rightarrow B_1 = -\frac{Pa}{6EIL}\left[2L^2 + a^2 + \frac{2a^2}{L}(L - a - b)\right]$$

$$\Rightarrow \boxed{B_1 = -\frac{Pa}{6EIL}\left(2L^2 + a^2\right)}$$

From (1) $$B_2 = \frac{Pa}{6EI}\left(2L^2 + a^2\right) - \frac{PaL^2}{3EI}$$

$$\Rightarrow \boxed{B_2 = \frac{Pa^3}{6EI}}$$

from ②

$$A_1 = -\frac{Pa}{6EIL}\left(2L^2+a^2\right) + \frac{Pa^2}{EI} - \frac{Pa^3}{2EIL} - \frac{Pa^2 b}{2EIL}$$

$$\Rightarrow A_1 = -\frac{Pa}{6EIL}\left(2L^2+a^2\right) + \frac{Pa^2}{2EIL}\left(2L-a-b\right)$$

$$\Rightarrow A_1 = -\frac{Pa}{6EIL}\left(2L^2+a^2\right) + \frac{Pa^2}{2EIL}\cdot L$$

$$\Rightarrow A_1 = -\frac{Pa}{6EIL}\left(2L^2+a^2-3aL\right)$$

$$\Rightarrow A_1 = -\frac{Pa}{6EIL}\left(2a^2+2b^2+4ab+a^2-3a^2-3ab\right)$$

$$\Rightarrow A_1 = -\frac{Pa}{6EIL}\left(2b^2+ab\right)$$

$$\Rightarrow A_1 = -\frac{Pb}{6EIL}\left(2ab+a^2\right)$$

$$\Rightarrow \boxed{A_1 = -\frac{Pb}{6EIL}\left(L^2-b^2\right)}$$

Thus

$$\boxed{\begin{array}{ll} A_1 = -\frac{Pb}{6EIL}\left(L^2-b^2\right) & ; \quad A_2 = 0 \\ B_1 = -\frac{Pa}{6EIL}\left(2L^2+a^2\right) & ; \quad B_2 = \frac{Pa^3}{6EI} \end{array}}$$

With these constants, for example, the elastic curve for segment AD of the beam, after algebraic simplification, becomes

$$v = -\frac{Pbx}{6EIL}(L^2 - b^2 - x^2) \tag{14-27}$$

Deflection v_D at applied force P is

$$v_D = v(a) = -\frac{Pa^2b^2}{3EIL} \tag{14-28}$$

The largest deflection occurs in the longer segment of the beam. If $a > b$, the point of maximum deflection is at $x = \sqrt{a(a + 2b)/3}$, which follows from setting the expression for the slope equal to zero. The deflection at this point is

$$|v|_{\max} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,EIL} \tag{14-29}$$

Usually, the deflection at the center of the span is very nearly equal to the numerically largest deflection. Such a deflection is much simpler to determine, which recommends its use. If force P is applied at the middle of the span, when $a = b = L/2$, by direct substitution into Eq. 14-28 or 14-29.

$$|v|_{\max} = \frac{PL^3}{48EI} \tag{14-30}$$

Here it is helpful to recall the definition of the *spring constant,* or *stiffness,* k given by Eq. 3-6. In the present context, for a force P placed at an arbitrary distance a from a support,

$$k = \frac{P}{v_B} = \frac{3EIL}{a^2b^2} \tag{14-31}$$

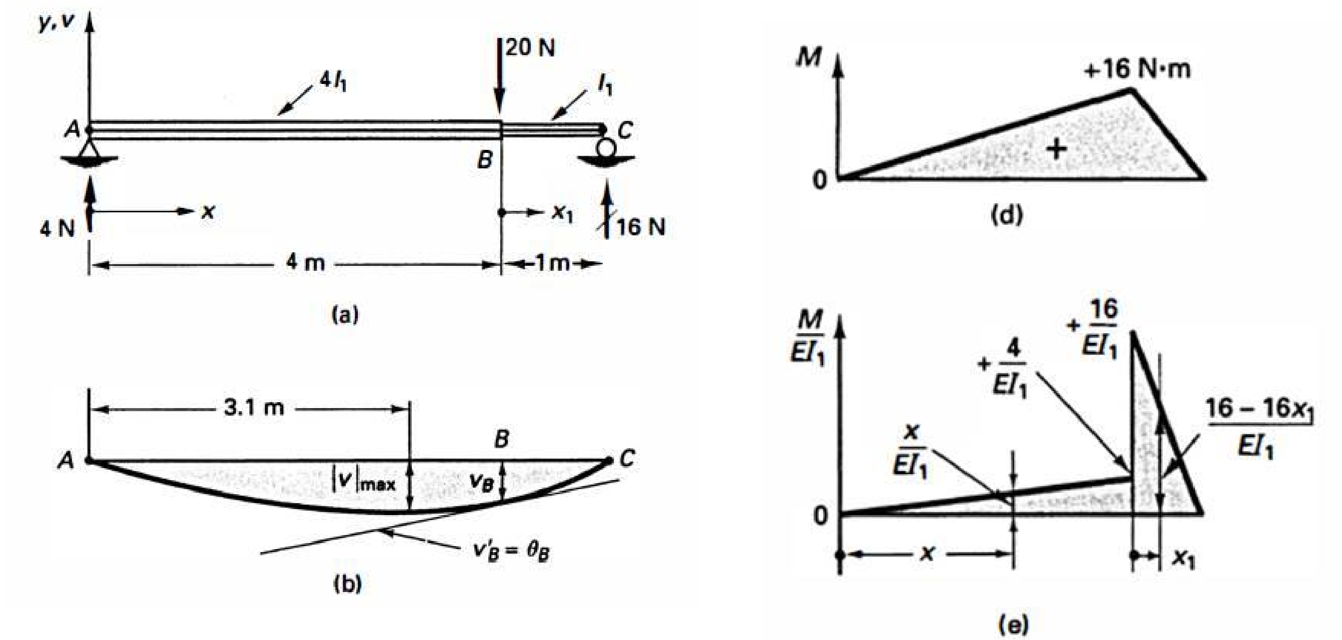
For a particular case, when $a = b = L/2$, this equation reduces to

$$k_o = \frac{48EI}{L^3} \tag{14-32}$$

This expression also follows directly from Eq. 14-30.

Example 14-7

A simply supported beam 5 m long is loaded with a 20-N downward force at a point 4 m from the left support, Fig. 14-13(a). The moment of inertia of the cross section of the beam is $4I_1$ for segment AB and I_1 for the remainder of the beam. Determine the elastic curve.



For segment AB, $0 < x < 4$:

$$M = 4x \qquad \text{and} \qquad EI = 4EI_1$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{x}{EI_1}$$

$$\theta = \frac{dv}{dx} = \frac{x^2}{2EI_1} + A_1$$

$$v = \frac{x^3}{6EI_1} + A_1x + A_2$$

At $x = 0$, $v(0) = v_A = 0$ and $\theta(0) = \theta_A$. Hence, $A_1 = \theta_A$ and $A_2 = 0$.

At the end of segment AB:

$$\theta(4) = \theta_B = \frac{8}{EI_1} + \theta_A \qquad \text{and} \qquad v(4) = v_B = \frac{32}{3EI_1} + 4\theta_A$$

For segment BC, $0 < x_1 < 1$:

$$M = 4(4 + x_1) - 20x_1 = 16 - 16x_1 \qquad \text{and} \qquad EI = EI_1$$

$$\frac{d^2v}{dx_1^2} = \frac{16}{EI_1} - \frac{16x_1}{EI_1}$$

$$\theta = \frac{dv}{dx_1} = \frac{16x_1}{EI_1} - \frac{8x_1^2}{EI_1} + A_3$$

$$v = \frac{8x_1^2}{EI_1} - \frac{8x_1^3}{3EI_1} + A_3x_1 + A_4$$

At $x_1 = 0$, $v(0) = v_B$ and $\theta(0) = \theta_B$. Hence, from the solution before, $A_4 = v_B = 32/3EI_1 + 4\theta_A$, and $A_3 = \theta_B = 8/EI_1 + \theta_A$. The expressions for θ and v in segment BC are then obtained as

$$\theta = \frac{16x_1}{EI_1} - \frac{8x_1^2}{EI_1} + \frac{8}{EI_1} + \theta_A$$

$$v = \frac{8x_1^2}{EI_1} - \frac{8x_1^3}{3EI_1} + \frac{8x_1}{EI_1} + \theta_A x_1 + \frac{32}{3EI_1} + 4\theta_A$$

Finally, the boundary condition at C is applied to determine the value of θ_A. At $x_1 = 1$, $v(1) = v_c = 0$; therefore,

$$0 = \frac{8}{EI_1} - \frac{8}{3EI_1} + \frac{8}{EI_1} + \theta_A + \frac{32}{3EI_1} + 4\theta_A \quad \text{and} \quad \theta_A = -\frac{4.8}{EI_1}$$

Substituting this value of θ_A into the respective expressions for θ and v, equations for these quantities can be obtained for either segment. For example, the equation for the slope in segment AB is $\theta = x^2/2EI_1 - 24/5EI_1$. Upon setting this quantity equal to zero, x is found to be 3.1 m. The maximum deflection occurs at this value of x, and $|v|_{max} = 9.95/EI_1$. Characteristically, the deflection at the center of the span (at $x = 2.5$ m) is nearly the same, being $9.4/EI_1$.

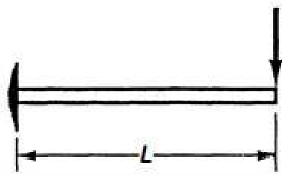


Fig. P14-9

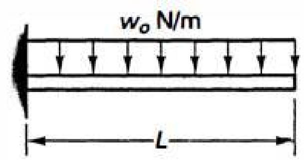


Fig. P14-10

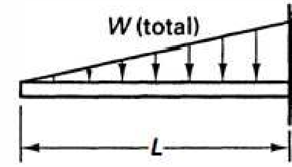


Fig. P14-11

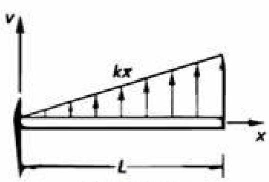


Fig. P14-12

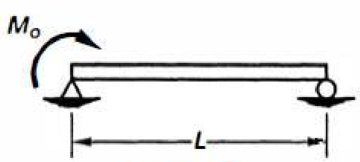


Fig. P14-15

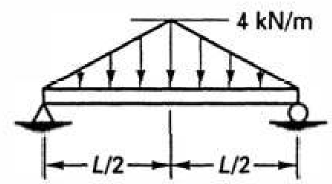


Fig. P14-16

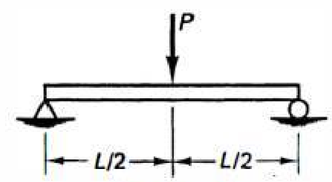


Fig. P14-17

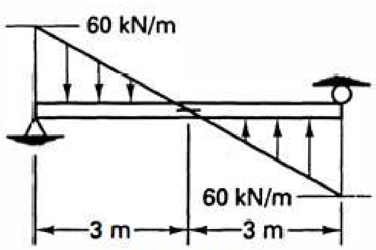


Fig. P14-18

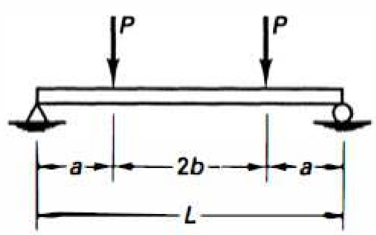


Fig. P14-19

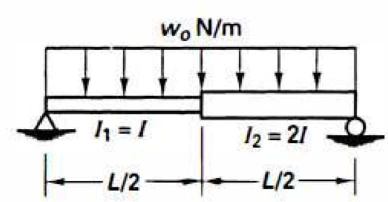


Fig. P14-20

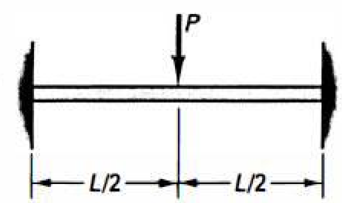


Fig. P14-21

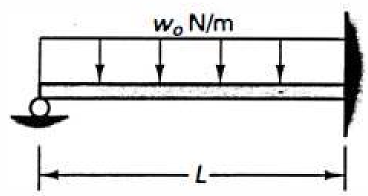


Fig. P14-23

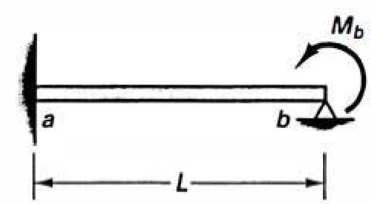


Fig. P14-24

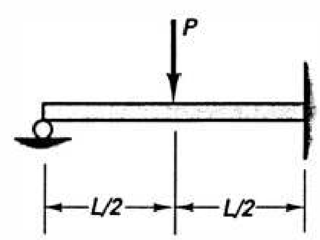


Fig. P14-29

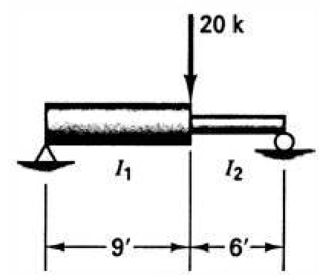
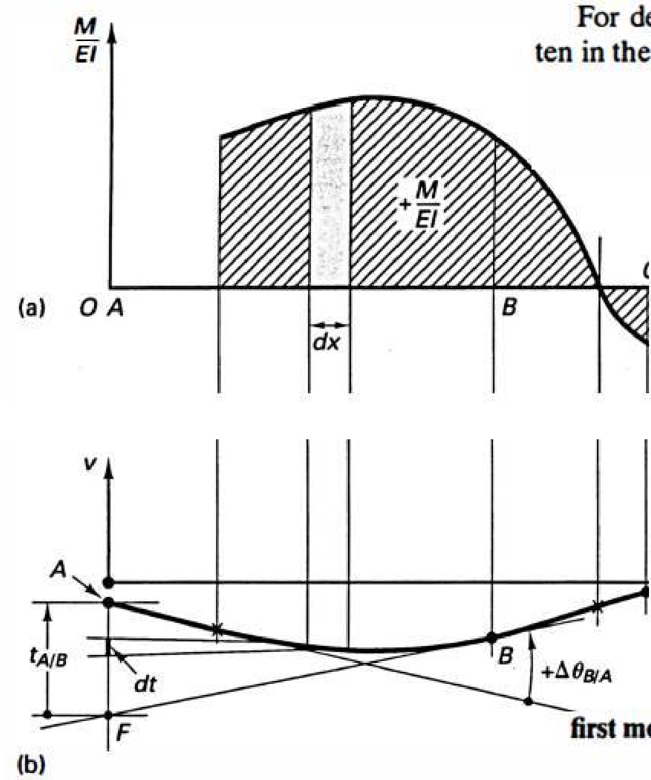


Fig. P14-37

Chapter 15, POPOV

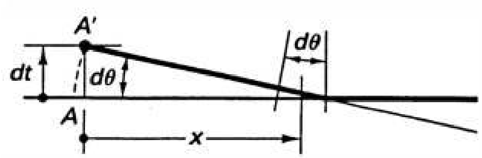
Beam Deflections by Moment-area Method



For deriving the theorems, Eq. 14-10a, $d^2v/dx^2 = M/EI$, can be rewritten in the following alternative forms:

$$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d\theta}{dx} = \frac{M}{EI} \qquad \text{or} \qquad d\theta = \frac{M}{EI}dx \tag{15-1}$$

$$dt = x\,d\theta = \frac{M}{EI}x\,dx \tag{15-2}$$



first moment-area theorem is

$$\boxed{\int_A^B d\theta = \theta_B - \theta_A = \Delta\theta_{B/A} = \int_A^B \frac{M}{EI}dx} \tag{15-3}$$

where $\Delta\theta_{B/A}$ is the *angle change between B and A*. This change in angle measured in radians between any two tangents at points A and B on the elastic curve is equal to the M/EI area bounded by the ordinates through A and B. Further, if slope θ_A of the elastic curve at A is known, slope θ_B at B is given as

$$\boxed{\theta_B = \theta_A + \Delta\theta_{B/A}} \tag{15-4}$$

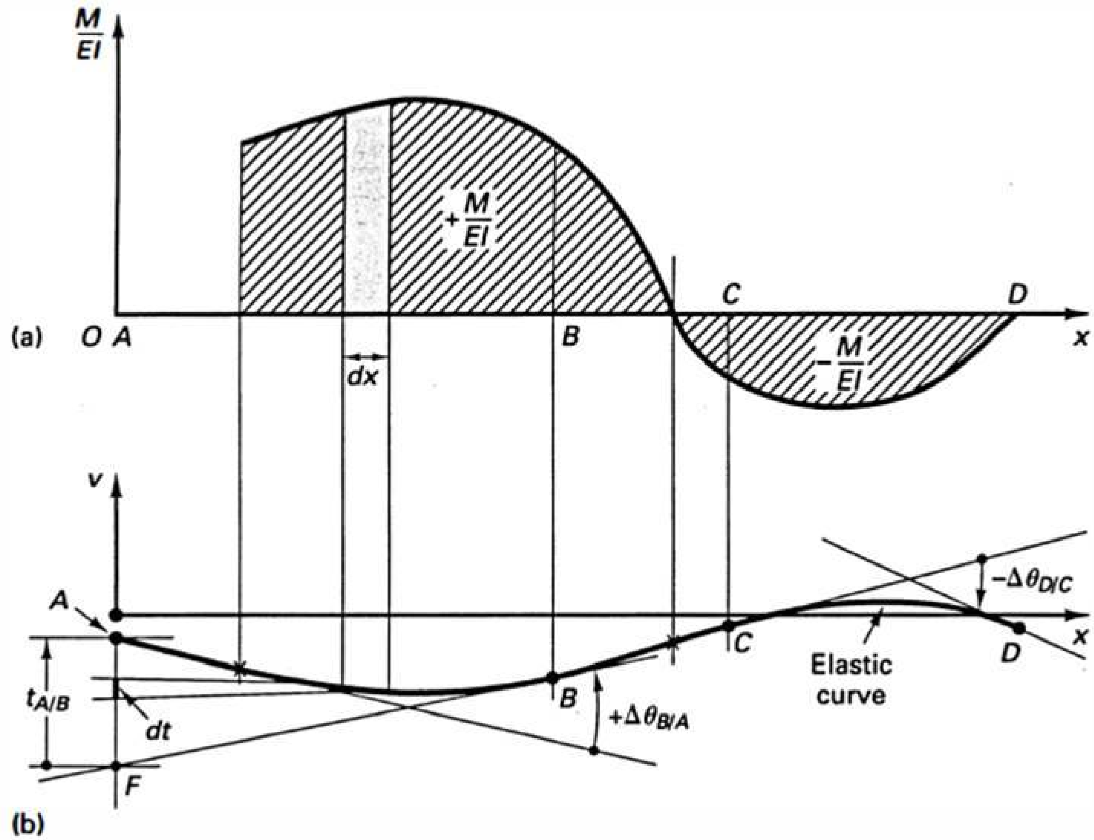


Fig. 15-2 Relationship between the *M/EI* diagram and the elastic curve.

second moment-area theorem:

$$\boxed{t_{A/B} = \int_A^B d\theta\, x = \int_A^B \frac{M}{EI} x\, dx} \tag{15-5}$$

This states that the tangential deviation of a point A on the elastic curve from a tangent through another point B also on the elastic curve is equal to the statical (or first) moment of the bounded section of the M/EI diagram around a vertical line through A. In most cases, the tangential deviation is not in itself the desired deflection of a beam.

Using the definition of the center of gravity of an area, one may, for convenience, restate Eq. 15-5 for numerical applications in a simpler form as

$$\boxed{t_{A/B} = \Phi \bar{x}} \tag{15-6}$$

where Φ is the total area of the M/EI diagram between the two points considered and $\bar{x}$ is the horizontal distance to the centroid of this area *from A*.

The previous two theorems are applicable between any two points on a *continuous* elastic curve of any beam for any loading. They apply between and beyond the reactions for overhanging and continuous beams. However, it must be emphasized that only relative rotation of the tangents and only tangential deviations are obtained directly. A further considera-tion of the geometry of the elastic curve at the supports to include the

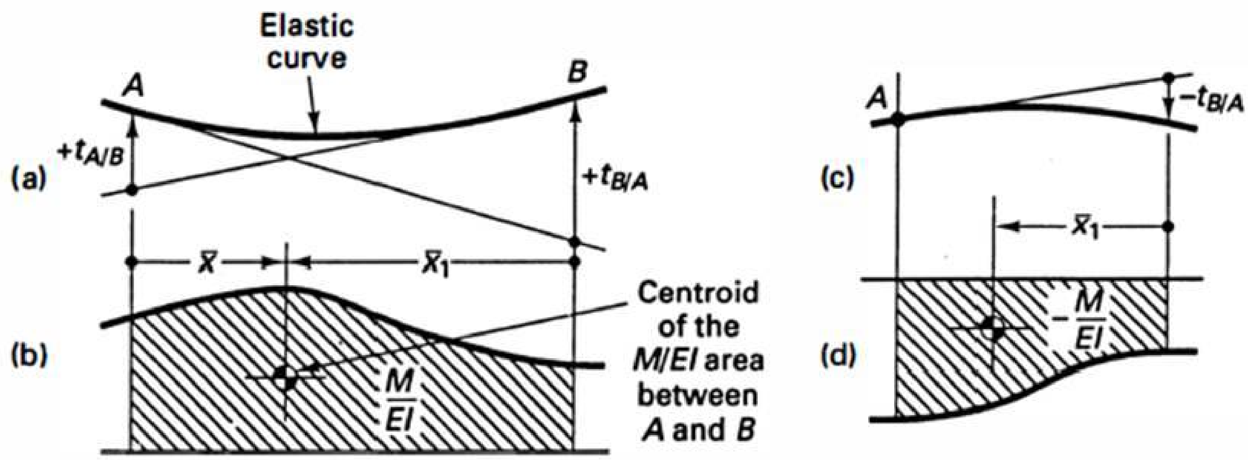
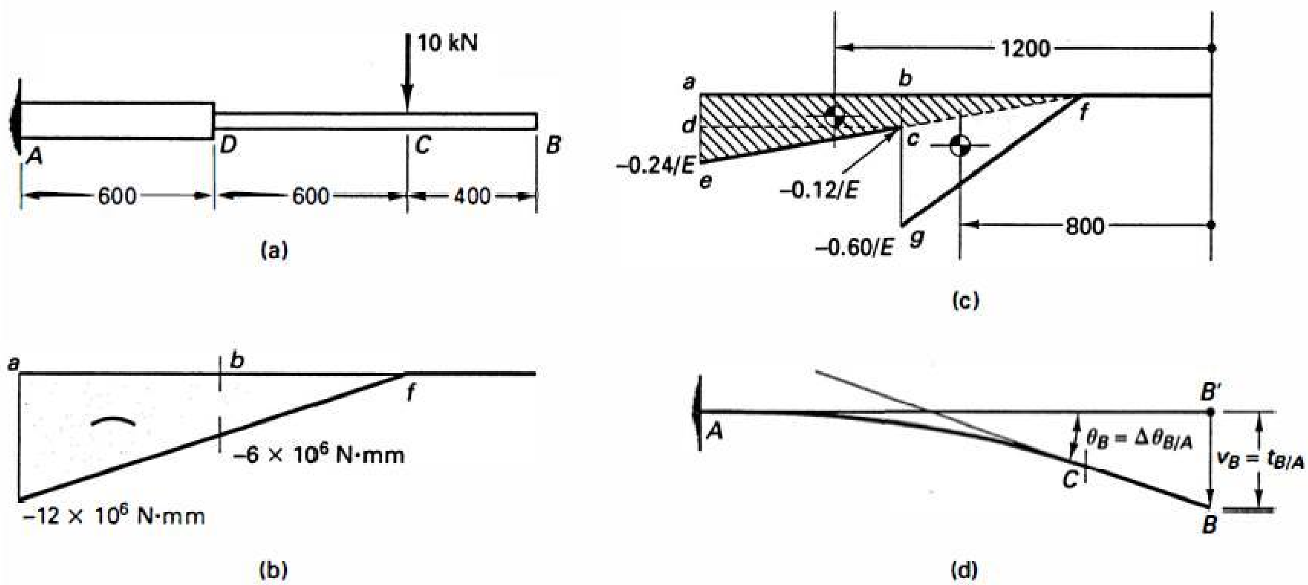


Fig. 15-3 Interpretation of signs for tangential deviation.

Example 15-1

Consider an aluminum cantilever beam 1600 mm long with a 10-kN force applied 400 mm from the free end, as shown in Fig. 15-4(a). For a distance of 600 mm from the fixed end, the beam is of greater depth than it is beyond, having $I_1 = 50 \times 10^6$ mm^4. For the remaining 1000 mm of the beam, $I_2 = 10 \times 10^6$ mm^4. Find the deflection and the angular rotation of the free end. Neglect the weight of the beam, and assume E for aluminum at 70 GPa.



The area of triangle afe:

$$\Phi_1 = -\frac{1200 \times 0.24}{2E} = -\frac{144}{E}$$

The area of triangle feg:

$$\Phi_2 = -\frac{600 \times 0.48}{2E} = -\frac{144}{E}$$

$$\theta_B = \Delta\theta_{B/A} = \int_A^B \frac{M}{EI} dx = \Phi_1 + \Phi_2 = -\frac{288}{70 \times 10^3} = -4.11 \times 10^{-3} \text{ rad}$$

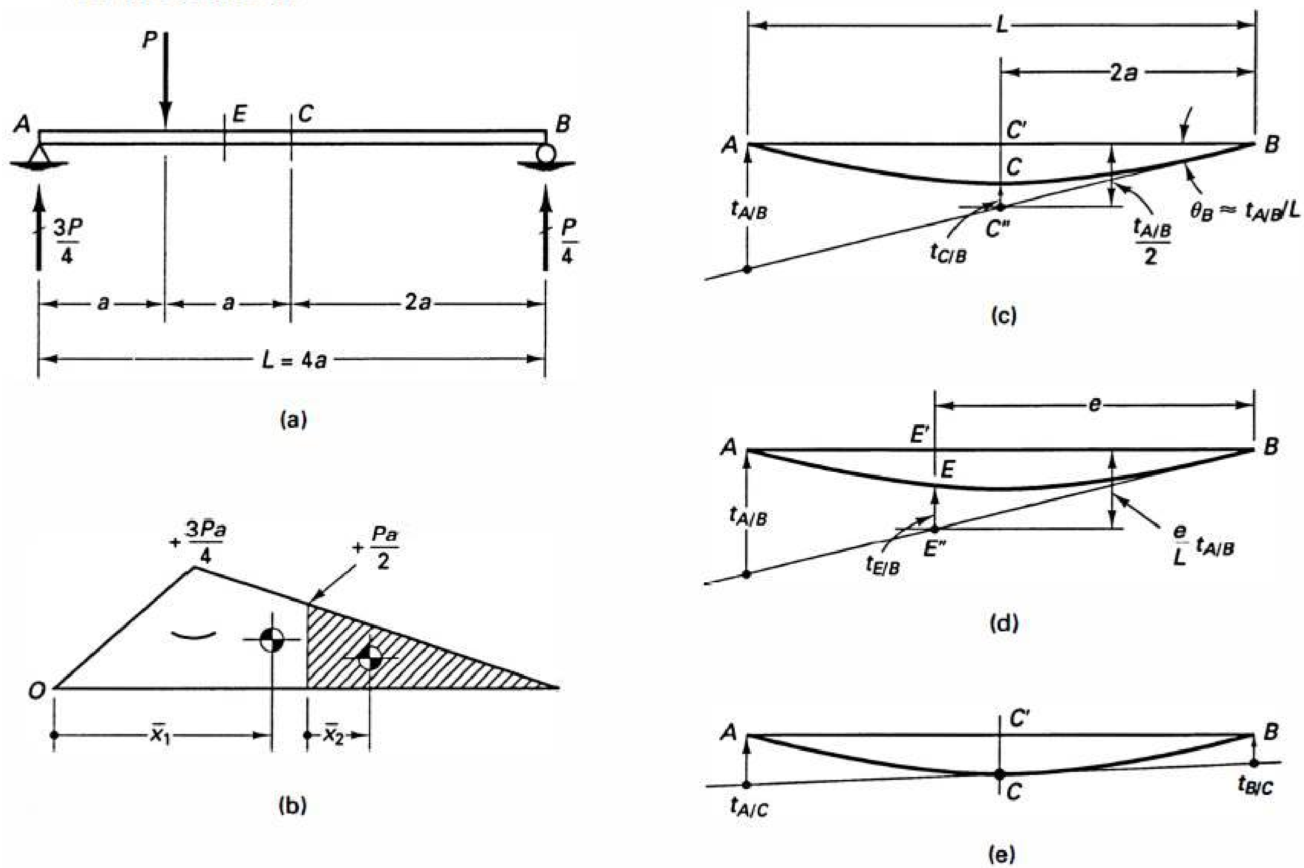
$$v_B = t_{B/A} = \Phi_1 \bar{x}_1 + \Phi_2 \bar{x}_2$$

$$= \left(-\frac{144}{E}\right) \times 1200 + \left(-\frac{144}{E}\right) \times 800 = -4.11 \text{ mm}$$

The negative sign of $\Delta\theta$ indicates clockwise rotation of the tangent at B in relation to the tangent at A. The negative sign of $t_{B/A}$ means that point B is below a tangent through A.

Example 15-2

Find the deflection due to the concentrated force P applied as shown in Fig. 15.5(a) at the center of a simply supported beam. The flexural rigidity EI is constant.



SOLUTION

The bending-moment diagram is in Fig. 15-5(b). Since EI is constant, the M/EI diagram need not be made, as the areas of the bending-moment diagram divided by EI give the necessary quantities for use in the moment-area theorems. The elastic curve is in Fig. 15-5(c). It is concave upward throughout its length as the bending moments are positive. This curve must pass through the points of the support at A and B.

It is apparent from the diagram of the elastic curve that the desired quantity is represented by distance CC'. Moreover, from purely geometrical or kinematic considerations, $CC' = C'C'' - C''C$, where distance $C''C$ is measured from a tangent to the elastic curve passing through the point of support B. However, since the deviation of a support point from a tangent to the elastic curve at the other support may always be computed by the second moment-area theorem, a distance such as $C'C''$ may be found by proportion from the geometry of the figure. In this case, $t_{A/B}$ follows by taking the whole M/EI area between A and B and multiplying it[2] by its $\bar{x}$ measured from a vertical through A; hence, $C'C'' = \frac{1}{2}t_{A/B}$. By another application of the second theorem, $t_{C/B}$, which is equal to $C''C$, is determined. For this case, the M/EI area is hatched in Fig. 15-5(b), and, for it, $\bar{x}$ is measured from C. Since the right reaction is P/4 and the distance $CB = 2a$, the maximum ordinate for the shaded triangle is $+\ Pa/2$.

$$v_c = C'C'' - C''C = t_{A/B}/2 - t_{C/B}$$

$$t_{A/B} = \Phi_1\bar{x}_1 = \frac{1}{EI}\left(\frac{4a}{2}\,\frac{3Pa}{4}\right)\frac{a+4a}{3} = +\frac{5Pa^3}{2EI}$$

$$t_{C/B} = \Phi_2\bar{x}_2 = \frac{1}{EI}\left(\frac{2a}{2}\,\frac{Pa}{2}\right)\frac{2a}{3} = +\frac{Pa^3}{3EI}$$

$$v_C = \frac{t_{A/B}}{2} - t_{C/B} = \frac{5Pa^3}{4EI} - \frac{Pa^3}{3EI} = \frac{11Pa^3}{12EI}$$

The positive signs of $t_{A/B}$ and $t_{C/B}$ indicate that points A and C lie above the tangent through B. As may be seen from Fig. 15-5(c), the deflection at the center of the beam is in a downward direction.

The slope of the elastic curve at C can be found from the slope of one of the ends and from Eq. 15-4. For point B on the right,

$$\theta_B = \theta_C + \Delta\theta_{B/C} \qquad \text{or} \qquad \theta_C = \theta_B - \Delta\theta_{B/C}$$

$$\theta_C = \frac{t_{A/B}}{L} - \Phi_2 = \frac{5Pa^2}{8EI} - \frac{Pa^2}{2EI} = \frac{Pa^2}{8EI} \text{ (counterclockwise)}$$

The previous procedure for finding the deflection of a point on the elastic curve is generally applicable. For example, if the deflection of point E, Fig. 15-5(d), at a distance e from B is wanted, the solution may be formulated as

$$v_E = E'E'' - E'E = (e/L)t_{A/B} - t_{E/B}$$

By locating point E at a variable distance x from one of the supports, the equation of the elastic curve can be obtained.

To simplify the arithmetical work, some care in selecting the tangent at a support must be exercised. Thus, although $v_c = t_{B/A}/2 - t_{C/A}$ (not shown in the diagram), this solution would involve the use of the unshaded portion of the bending-moment diagram to obtain $t_{C/A}$, which is more tedious.

ALTERNATIVE SOLUTION

The solution of the foregoing problem may be based on a different geometrical concept. This is illustrated in Fig. 15-5(e), where a tangent to the elastic curve is drawn at C. Then, since distances AC and CB are equal,

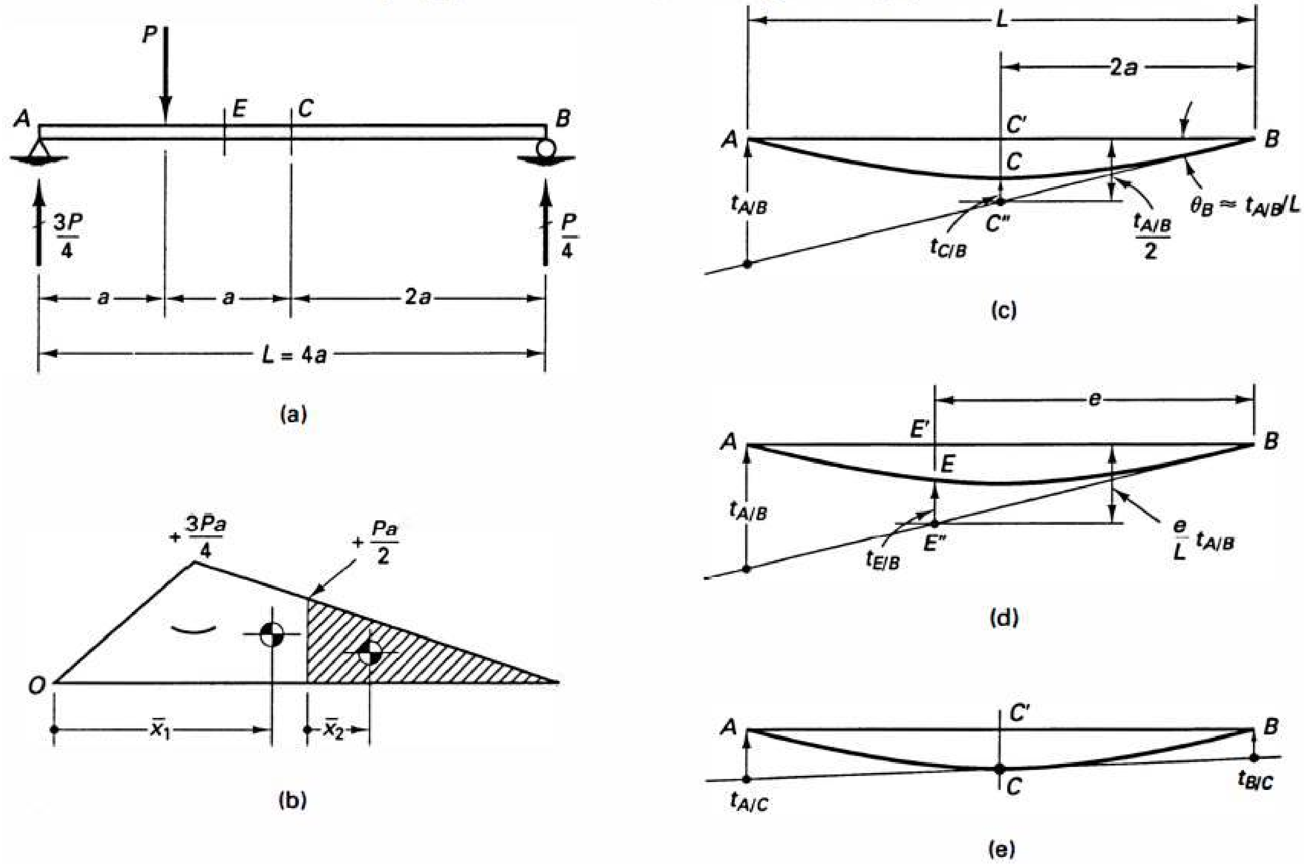
$$v_C = CC' = (t_{A/C} + t_{B/C})/2$$

That is, distance CC' is an average of $t_{A/C}$ and $t_{B/C}$. The tangential deviation $t_{A/C}$ is obtained by taking the first moment of the unshaded M/EI area in Fig. 15-5(b) about A, and $t_{B/C}$ is given by the first moment of the shaded M/EI area about B. The numerical details of this solution are left for completion by the reader. This procedure is usually longer than the first.

Note particularly that if the elastic curve is not symmetrical, the tangent at the center of the beam is *not horizontal.*

Example 15-3

For a prismatic beam loaded as in the preceding example, find the maximum deflection caused by applied force P; see Fig. 15-6(a).



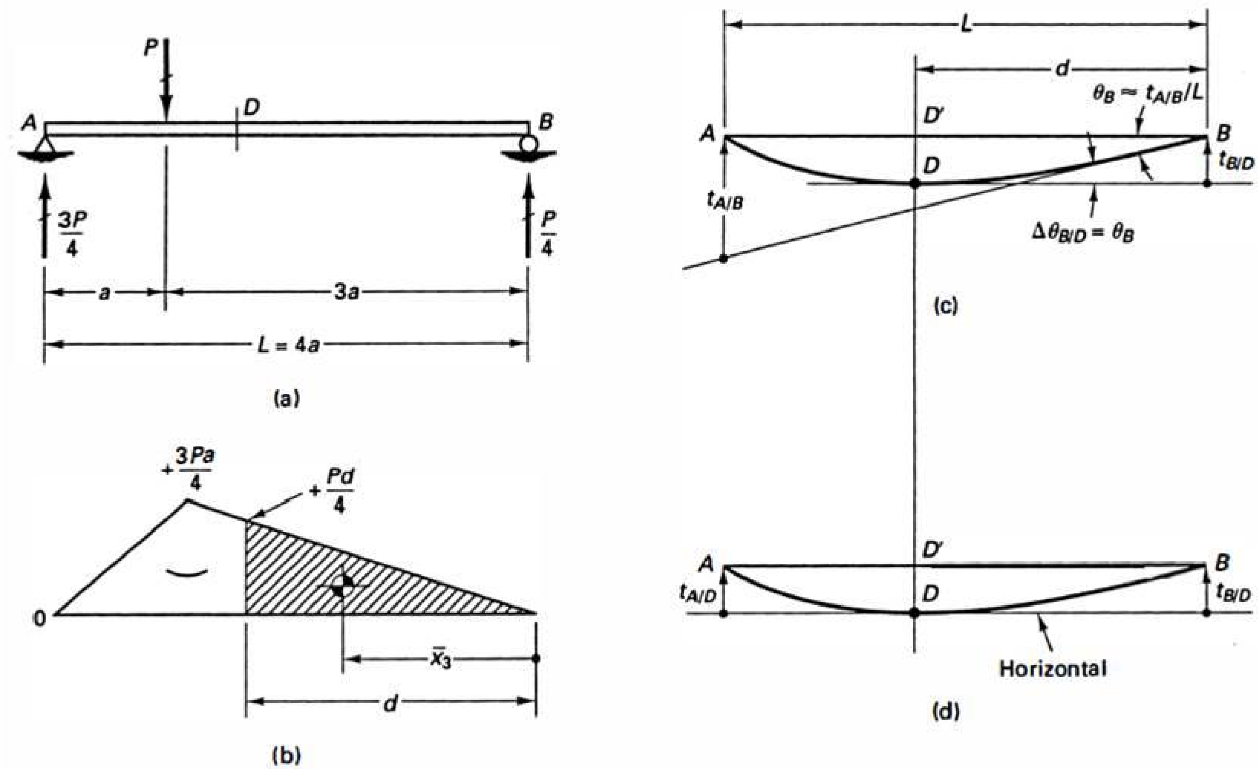
Solution

The bending-moment diagram and the elastic curve are shown in Figs. 15-6(b) and (c), respectively. The elastic curve is concave up throughout its length, and the maximum deflection occurs where the tangent to the elastic curve is horizontal. This point of tangency is designated in the figure by D and is located by the unknown horizontal distance d measured from the right support B. Then, by drawing a tangent to the elastic curve through point B at the support, one sees that $\Delta\theta_{B/D} = \theta_B$ since the line passing through the supports is horizontal. However, the slope θ_B of the elastic curve at B may be determined by obtaining $t_{A/B}$ and dividing it by the length of the span. On the other hand, by using the first moment-area theorem, $\Delta\theta_{B/D}$ may be expressed in terms of the shaded area in Fig. 15-6(b). Equating $\Delta\theta_{B/D}$ to θ_B and solving for d locates the horizontal tangent at D. Then, again from geometrical considerations, it is seen that the maximum deflection represented by DD' is equal to the tangential deviation of B from a horizontal tangent through D (i.e., $t_{B/D}$).

$$t_{A/B} = \Phi_1 \bar{x}_1 = + \frac{5Pa^3}{2EI} \qquad \text{(see Example 15-2)}$$

$$\theta_B = \frac{t_{A/B}}{L} = \frac{t_{A/B}}{4a} = \frac{5Pa^2}{8EI}$$

$$\Delta\theta_{B/D} = \frac{1}{EI}\left(\frac{d}{2}\,\frac{Pd}{4}\right) = \frac{Pd^2}{8EI} \quad \text{(area between } D \text{ and } B\text{)}$$



Since $\theta_B = \theta_D + \Delta\theta_{B/D}$ and it is required that $\theta_D = 0$,

$$\Delta\theta_{B/D} = \theta_B \quad \frac{Pd^2}{8EI} = \frac{5Pa^2}{8EI} \quad \text{hence, } d = \sqrt{5}a$$

$$v_{\max} = v_D = DD' = t_{B/D} = \Phi_3 \bar{x}_3$$

$$= \frac{1}{EI}\left(\frac{d}{2}\,\frac{Pd}{4}\right)\frac{2d}{3} = \frac{5\sqrt{5}Pa^3}{12EI} = \frac{11.2Pa^3}{12EI}$$

After distance d is found, the maximum deflection may also be obtained as $v_{\max} = t_{A/D}$, or $v_{\max} = (d/L)t_{A/B} - t_{D/B}$ (not shown). Also note that using the condition $t_{A/D} = t_{B/D}$, Fig. 15-6(d), an equation may be set up for d.

It should be apparent from this solution that it is easier to calculate the deflection at the center of the beam, which was illustrated in Example 15-2, than to determine the maximum deflections yet, by examining the end results, one sees that, numerically, the two deflections differ little: $v_{\text{center}} = 11Pa^3/12EI$ as opposed to $v_{\max} = 11.2Pa^3/12EI$. For this reason, in many practical problems of simply supported beams, where all the applied forces act in the same direction, it is often sufficiently accurate to calculate the deflection at the center instead of attempting to obtain the true maximum.

Example 15-4

In a simply supported beam, find the maximum deflection and rotation of the elastic curve at the ends caused by the application of a uniformly distributed load of w_o lb/ft; see Fig. 15-7(a). Flexural rigidity EI is constant.

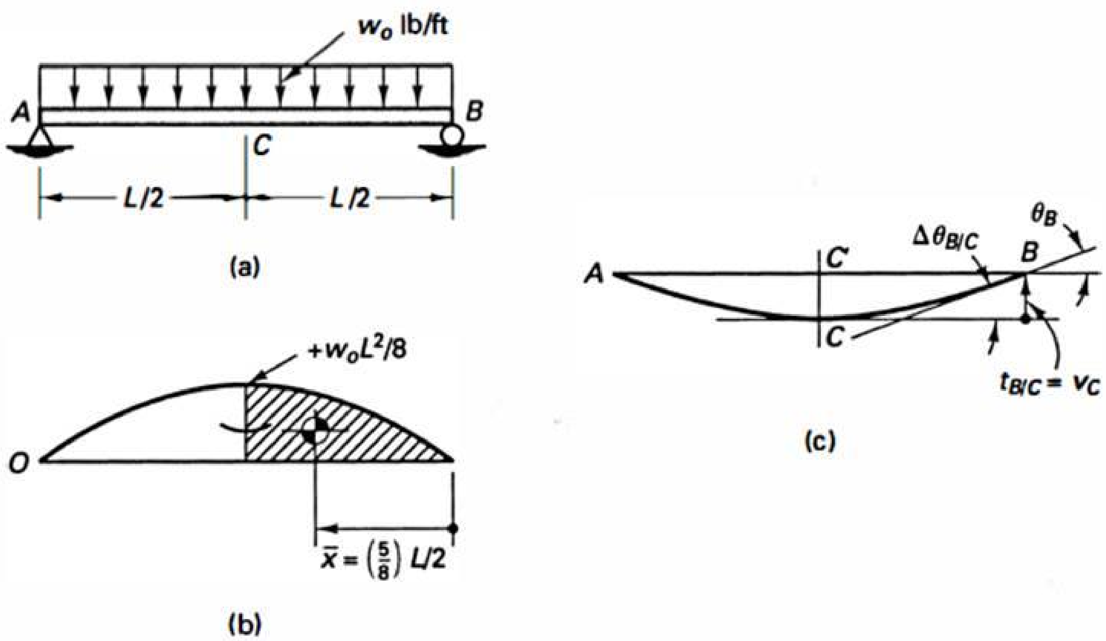


Fig. 15-7

SOLUTION

The bending-moment diagram is shown in Fig. 15-7(b). As established in Example 7-8, it is a second-degree parabola with a maximum value at the vertex of $w_o L^2/8$. The elastic curve passing through the points of supports A and B is shown in Fig. 15-7(c).

In this case, the M/EI diagram is symmetrical about a vertical line passing through the center. Therefore, the elastic curve must be symmetrical, and the tangent to this curve at the center of the beam is horizontal. From the figure, it is seen that $\Delta\theta_{B/C}$ is equal to θ_B, and the rotation of B is equal to one-half the area[3] of the whole M/EI diagram. Distance CC' is the desired deflection, and from the geometry of the figure, it is seen to be equal to $t_{B/C}$ (or $t_{A/C}$, not shown).

$$\Phi = \frac{1}{EI}\left(\frac{2}{3}\,\frac{L}{2}\,\frac{w_o L^2}{8}\right) = \frac{w_o L^3}{24EI}$$

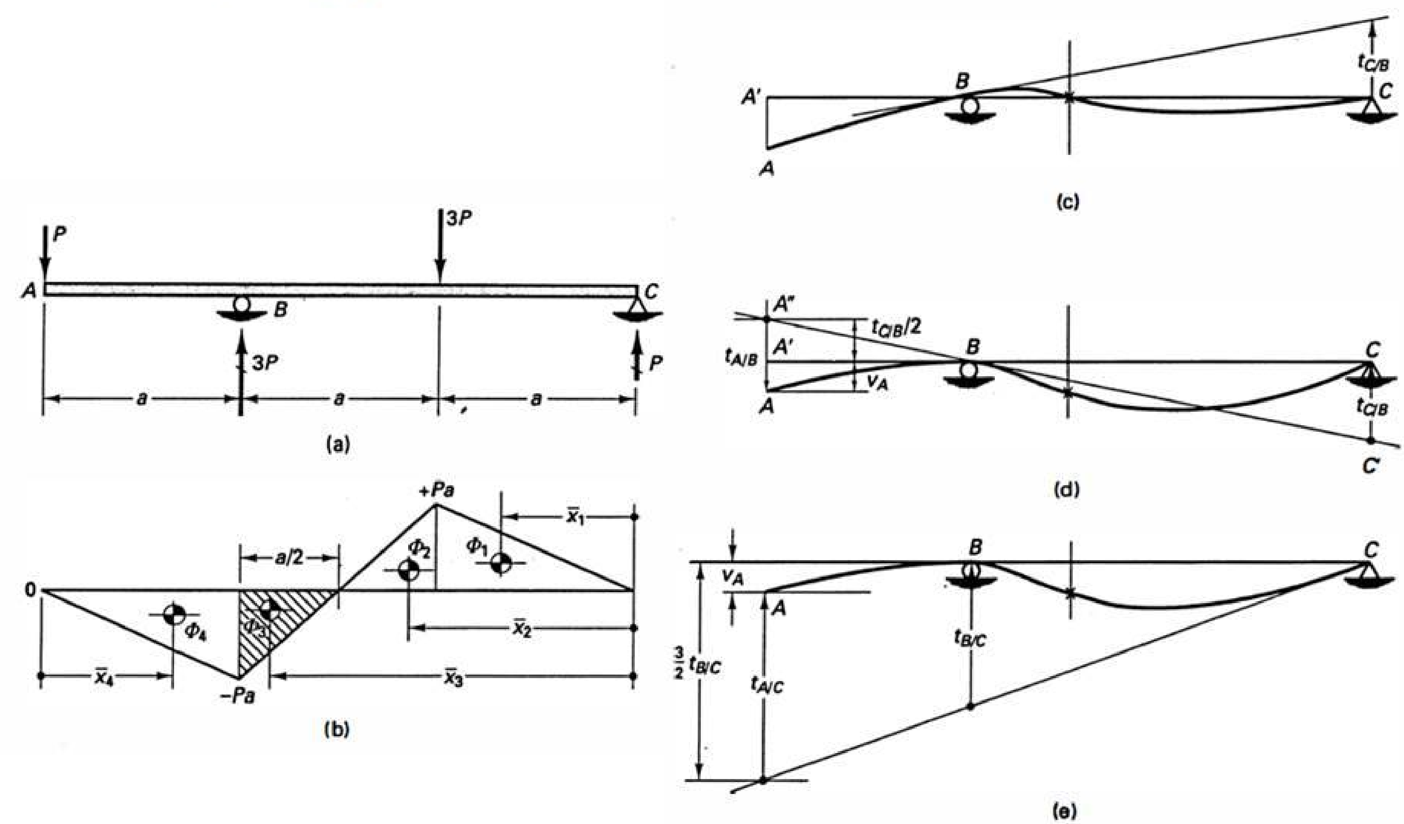
$$\theta_B = \Delta\theta_{B/C} = \Phi = +\frac{w_o L^3}{24EI}$$

$$v_C = v_{\max} = t_{B/C} = \Phi\bar{x} = \frac{w_o L^3}{24EI}\,\frac{5L}{16} = \frac{5 w_o L^4}{384EI}$$

The value of the deflection agrees with Eq. 14-22, which expresses the same quantity derived by the integration method. Since point B is above the tangent through C, the sign of v_c is positive.

Example 15-5

Find the deflection of the free end A of the beam shown in Fig. 15-8(a) caused by the applied forces. EI is constant.



SOLUTION

The bending-moment diagram for the applied forces is shown in Fig. 15-8(b). The bending moment changes sign at $a/2$ from the left support. At this point, an inflection in the elastic curve occurs. Corresponding to the positive moment, the curve is concave up, and vice versa. The elastic curve is so drawn and passes over the supports at B and C, Fig. 15-8(c). To begin, the inclination of the tangent to the elastic curve at support B is determined by finding $t_{C/B}$ as the statical moment of the areas with the proper signs of the M/EI diagram between the verticals through C and B about C.

$$
\begin{aligned}
t_{C/B} &= \Phi_1\bar{x}_1 + \Phi_2\bar{x}_2 + \Phi_3\bar{x}_3 \\
&= \frac{1}{\mathrm{EI}}\left[\frac{\mathrm{a}}{2}(+\,\mathrm{Pa})\frac{2\mathrm{a}}{3} + \frac{1}{2}\frac{\mathrm{a}}{2}(+\,\mathrm{Pa})\left(\mathrm{a} + \frac{1}{3}\frac{\mathrm{a}}{2}\right)\right. \\
&\quad \left. + \frac{1}{2}\frac{a}{2}(-Pa)\left(\frac{3a}{2} + \frac{2}{3}\frac{a}{2}\right)\right] \\
&= +\frac{Pa^3}{6EI}
\end{aligned}
$$

The positive sign of $t_{C/B}$ indicates that point C is *above the tangent at B*. Hence, a corrected diagram of the elastic curve is made, Fig. 15-8(d), where it is seen that the deflection sought is given by distance AA' and is equal to $AA'' - A'A''$. Further, since triangles $A'A''B$ and $CC'B$ are similar, distance $A'A'' = t_{C/B}/2$. On the other hand, distance AA'' is the deviation of point A from the tangent to the elastic curve at support B. Hence,

$$v_A = AA' = AA'' - A'A'' = t_{C/B}/2$$

$$t_{A/B} = \frac{1}{EI}(\Phi_4 \bar{x}_4) = \frac{1}{EI}\left[\frac{a}{2}(-Pa)\frac{2a}{3}\right] = -\frac{Pa^3}{3EI}$$

where the negative sign means that point A is below the tangent through B. This sign is not used henceforth, as the geometry of the elastic curve indicates the direction of the actual displacements. Thus, the deflection of point A *below the line passing through the supports is*

$$v_A = \frac{Pa^3}{3EI} - \frac{1}{2}\frac{Pa^3}{6EI} = \frac{Pa^3}{4EI}$$

This example illustrates the necessity of watching the signs of the quantities computed in the applications of the moment-area method, although usually less difficulty is encountered than in this example. For instance, if the deflection of end A is established by first finding the rotation of the elastic curve at C, no ambiguity in the direction of tangents occurs. This scheme of analysis is shown in Fig. 15-8(e), where $v_A = \frac{3}{2}t_{B/C} - t_{A/C}$.

Example 15-6

A simple beam supports two equal and opposite forces P at the quarter points; see Fig. 15.9(a). Find the deflection of the beam at the middle of the span. EI is constant.

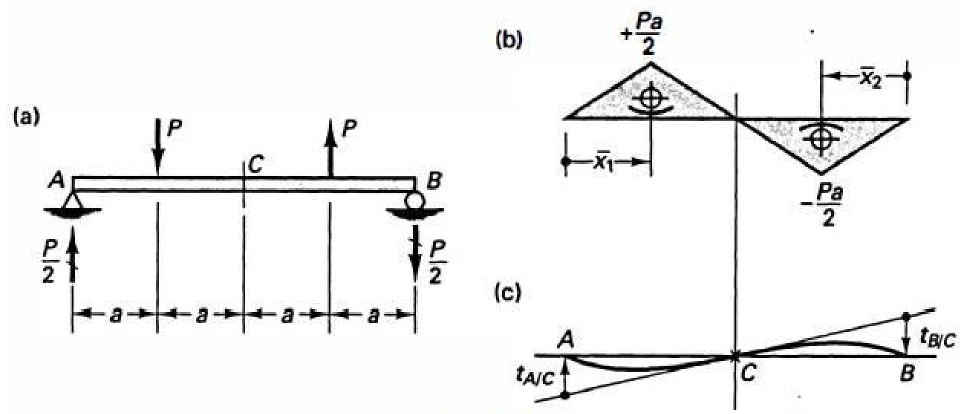


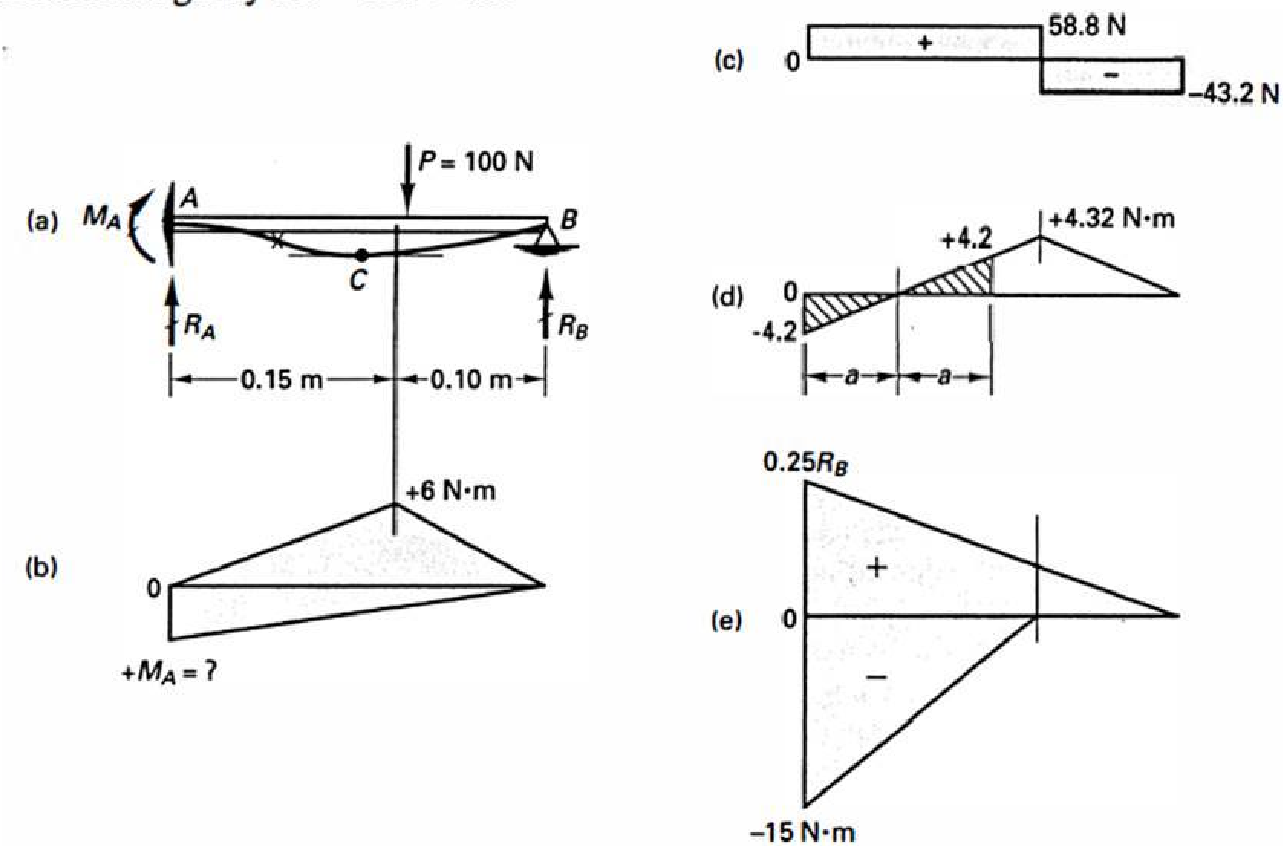
Fig. 15-9

SOLUTION

The bending-moment diagram and elastic curve with a tangent at C are shown in Figs. 15-9(b) and (c), respectively. Then, since the statical moments of the positive and negative areas of the bending-moment diagram around A and B, respectively, are numerically equal (i.e., $|t_{A/C}| = |t_{B/C}|$), the deflection of the beam at the center of the span is *zero*. The elastic curve in this case is *anti*symmetrical. Noting this, much work may be avoided in obtaining the deflections at the *center of the span*. The deflection of *any other point on the elastic curve can be found in the usual manner.*

Example 15-7

Find the maximum downward deflection of the small aluminum beam shown in Fig. 15-10(a) due to an applied force $P = 100$ N. The beam's constant flexural rigidity $EI = 60\ \text{N} \cdot \text{m}^2$.



Solution

The solution of this problem consists of two parts. First, a redundant reaction must be determined to establish the numerical values for the bending-moment diagram; then the usual moment-area procedure is applied to find the deflection.

By assuming the beam is released from the redundant end moment, a simple beam-moment diagram is constructed above the base line in Fig. 15-10(b). The moment diagram of known shape due to the unknown redundant moment M_A is shown on the same diagram below the base line. One assumes M_A to be positive, since in this manner, its correct sign is obtained automatically according to the beam sign convention. The composite diagram represents a *complete* bending-moment diagram.

The tangent at the built-in end remains horizontal after the application of force P. Hence, the geometrical condition is $t_{B/A} = 0$. An equation formulated on this basis yields a solution for M_A.[6] The equations of static equilibrium are used to compute the reactions. The final bending-moment diagram, Fig. 15-10(d), is obtained in the usual manner after the reactions are known. Thus, since $t_{B/A} = 0$,

$$\frac{1}{EI}\left[\frac{1}{2}(0.25)(6)\frac{1}{3}(0.25 + 0.10) + \frac{1}{2}(0.25)M_A\frac{2}{3}(0.25)\right] = 0$$

Hence, $M_A = -4.2$ N · m. Since, initially, M_A was assumed to be positive, and is so shown in Figs. 15-10(a) and (b), this result indicates that actually M_A has an *opposite* sense. The correct sense for M_A must be used in the equations of statics that follow and is reflected in the shear and moment diagrams constructed in Figs. 15-10(c) and (d), respectively.

$$\sum M_A = 0 \curvearrowright + \quad 100(0.15) - R_B(0.25) - 4.2 = 0 \qquad R_B = 43.2 \text{ N}$$

$$\sum M_B = 0 \curvearrowleft + \quad 100(0.10 + 4.2 - R_A(0.25) = 0 \qquad R_A = 56.8 \text{ N}$$

Check: $\sum F_y = 0 \uparrow + \qquad 43.2 + 56.8 - 100 = 0$

The maximum deflection occurs where the tangent to the elastic curve is horizontal, point C in Fig. 15-10(a). Hence, by noting that the tangent at A is also horizontal and using the first moment-area theorem, point C is located. This occurs when the hatched areas in Fig. 15-10(d) having opposite signs are equal; that is, at a distance $2a = 2(4.2/56.8) = 0.148$ m from A. The tangential deviation $t_{A/C}$ (or $t_{C/A}$) gives the deflection of point C.

$$\begin{aligned} v_{max} &= v_C = t_{A/C} \\ &= \frac{1}{EI}\left[\frac{1}{2} \times 0.074(+4.2)\left(0.074 + \frac{2}{3} \times 0.074\right)\right. \\ &\quad \left. + \frac{1}{2} \times 0.074(-4.2)\frac{1}{3} \times 0.074\right] \\ &= (15.36)10^{-3}/EI = 0.256 \text{ mm} \qquad \text{(down)} \end{aligned}$$

Example 15-8

Find the moments at the supports for a fixed-end beam loaded with a uniformly distributed load of w_o N/m; see Fig. 15-11(a).

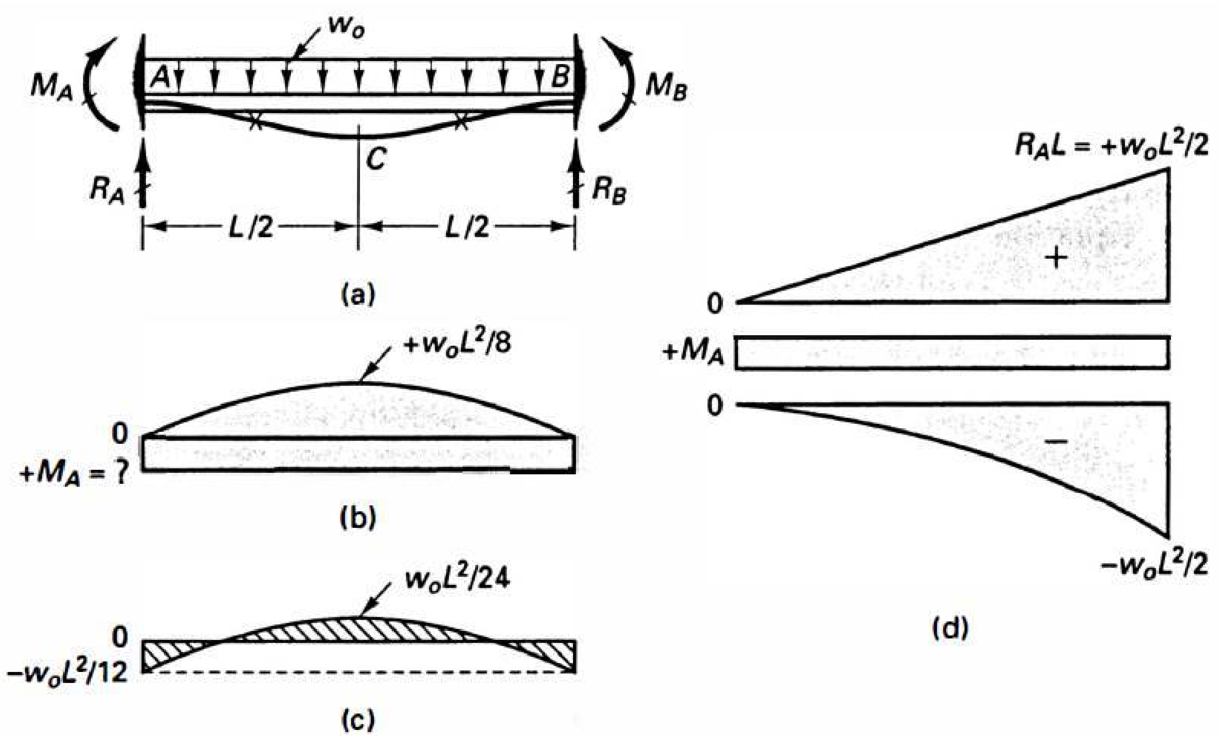


Fig. 15-11

Although this beam is statically indeterminate to the second degree, because of symmetry, a single equation based on a geometrical condition is sufficient to yield the redundant moments. From the geometry of the elastic curve, any one of the following conditions may be used: $\Delta\theta_{A/B} = 0$,[7] $t_{B/A} = 0$, or $t_{A/B} = 0$. From the first condition, $\Delta\theta_{A/B} = 0$,

$$\frac{1}{EI}\left[\frac{2}{3}L\left(+\frac{w_oL^2}{8}\right) + L(+M_A)\right] = 0$$

Then

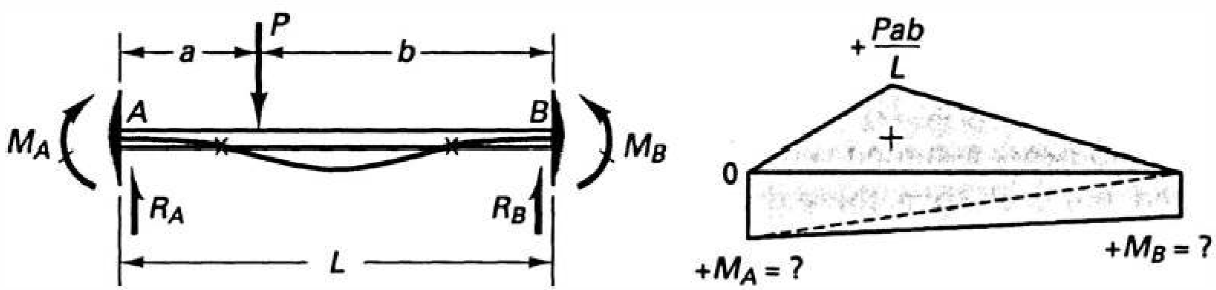
$$M_A = M_B = -\frac{w_oL^2}{12} \tag{15-8}$$

The negative sign for these moments indicates that their sense is opposite from that assumed in Figs. 15-11(a) and (b).

The composite moment diagram is shown in Fig. 15-11(c). In comparison with the maximum bending moment of a simple beam, a considerable reduction in the magnitude of the critical moments occurs.

Example 15-9

A beam fixed at both ends carries a concentrated force P, as shown in Fig. 15-12. Find the fixed-end moments. EI is constant.



(a) (b)

(a) $\Delta\theta_{A/B} = 0$, since the change in angle between the tangents at A and B is zero.
(b) $t_{B/A} = 0$, since support B does not deviate from a fixed tangent at A.
(c) Similarly, $t_{A/B} = 0$.

Any two of these conditions may be used; arithmetical simplicity of the resulting equations governs the choice. Thus, by using condition (a), which is always the simplest, and condition (b), the two equations are[8]

$$\Delta\theta_{A/B} = \frac{1}{EI}\left(\frac{1}{2}L\frac{Pab}{L} + \frac{1}{2}LM_A + \frac{1}{2}LM_B\right) = 0$$

or

$$M_A + M_B = -\frac{Pab}{L}$$

$$t_{B/A} = \frac{1}{EI}\left[\frac{1}{2}L\frac{Pab}{L}\frac{1}{3}(L+b) + \frac{1}{2}LM_A\frac{2}{3}L + \frac{1}{2}LM_B\frac{1}{3}L\right] = 0$$

or

$$2M_A + M_B = -\frac{Pab}{L^2}(L+b)$$

Solving the two reduced equations simultaneously gives

$$M_A = -\frac{Pab^2}{L^2} \qquad \text{and} \qquad M_B = -\frac{Pa^2b}{L^2}$$

These negative moments have an opposite sense from that initially assumed and shown in Figs. 15-12(a) and (b).

15-1 through 15-12. Using the moment-area method, determine the deflection and the slope of the elastic curves at points A due to the applied loads for the beams, as shown in the figures. Specify the direction of deflection and of rotation for the calculated quantities. If neither the size of a beam nor its moment of inertia are given, EI is constant. Wherever needed, let $E = 29 \times 10^3$ ksi or 200 GPa. In all cases, a well-prepared sketch of the elastic curve, showing the inflection points, should be made.

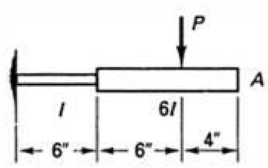


Fig. P15-1

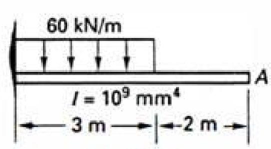


Fig. P15-2

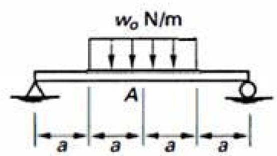


Fig. P15-3

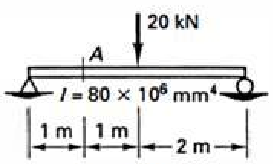


Fig. P15-4

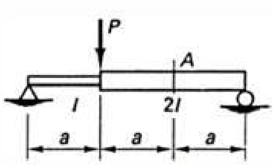


Fig. P15-5

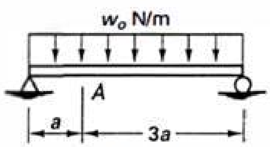


Fig. P15-6

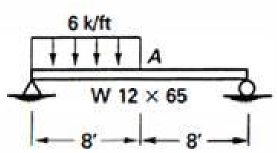


Fig. P15-7

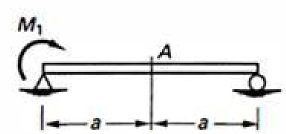


Fig. P15-8

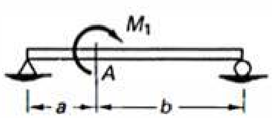


Fig. P15-9

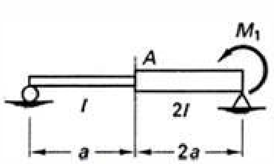


Fig. P15-10

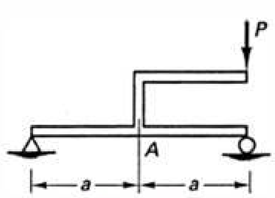


Fig. P15-11

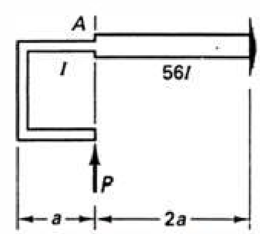


Fig. P15-12

15-13. Determine the deflection at the midspan of a simple beam, loaded as shown in the figure, by solving the two separate problems indicated and superimposing the results. Use the moment-area method. EI is constant. (*Note:* Solution of complex problems by subdividing them into a symmetrical part and an unsymmetrical part is often very advantageous because it reduces the numerical work.)

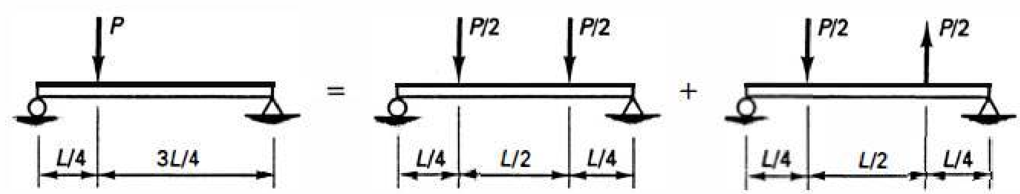


Fig. P15-13

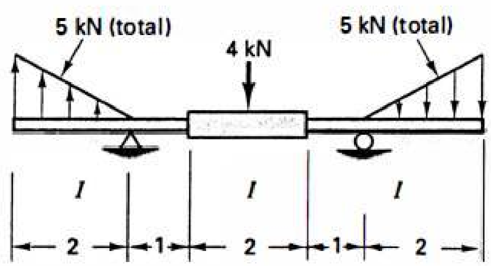


Fig. P15-14

15-22. For the beam loaded as shown in the figure, determine (a) the deflection at the center of the span, (b) the deflection at the point of inflection of the elastic curve, and (c) the maximum deflection, $EI = 1800$ lb-in^2.

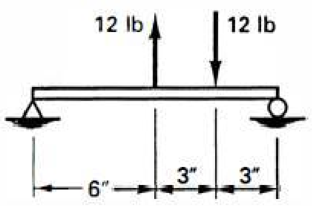


Fig. P15-22

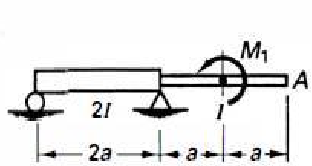


Fig. P15-23

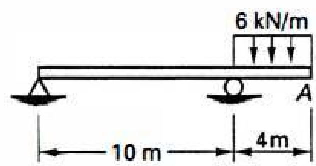


Fig. P15-24

15-25. Determine the maximum *upward* deflection for the overhang of a beam loaded as shown in the figure. E and I are constant.

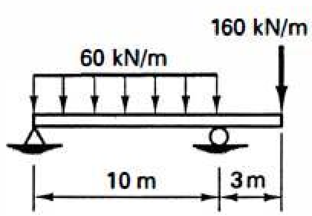


Fig. P15-25

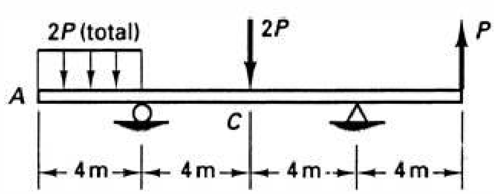


Fig. P15-26

15-28. A hinged beam system is loaded as shown in the figure. Determine the deflection and slope of the elastic curve at point A.

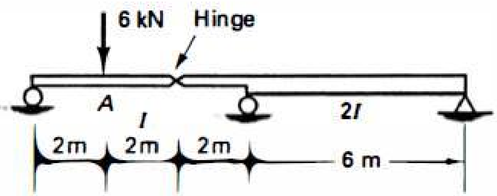


Fig. P15-28

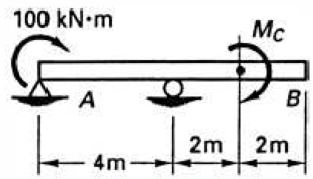


Fig. P15-29

15-33 and 15-34. For the beams loaded as shown in the figures, using the moment-area method, determine the redundant reactions and plot shear and moment diagrams. In both problems, EI is constant.

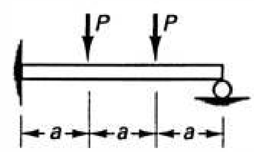


Fig. P15-33

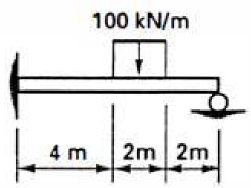


Fig. P15-34

15-35. For the beam loaded as shown in the figure, (a) determine the ratio of the moment at the fixed end to the applied moment M_A; (b) determine the rotation of the end A. EI is constant.

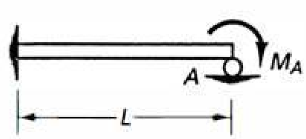


Fig. P15-35

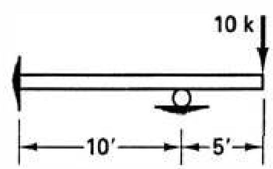


Fig. P15-36

Chapter 16, POPOV

Columns

The selection of structural and machine elements is based on three characteristics: (i) strength, (ii) stiffness, and (iii) stability.
The consideration of material strength alone is not sufficient to predict the behavior of slender members. Stability considerations are primary in some structural systems. The phenomenon of structural instability occurs in numerous situations where compressive stresses are present. Thin sheets, although fully capable of sustaining tensile loadings. are very poor in transmitting compression.

Narrow beams. unbraced laterally, can turn sidewise and collapse under an applied load. Vacuum tanks, as well as submarine hulls□ unless properly designed, can severely distort under external pressure and can assume shapes that differ drastically from their original geometry. A thin-walled tube can wrinkle like tissue paper when subjected either to axial compression or a torque; see Fig. 16-1.

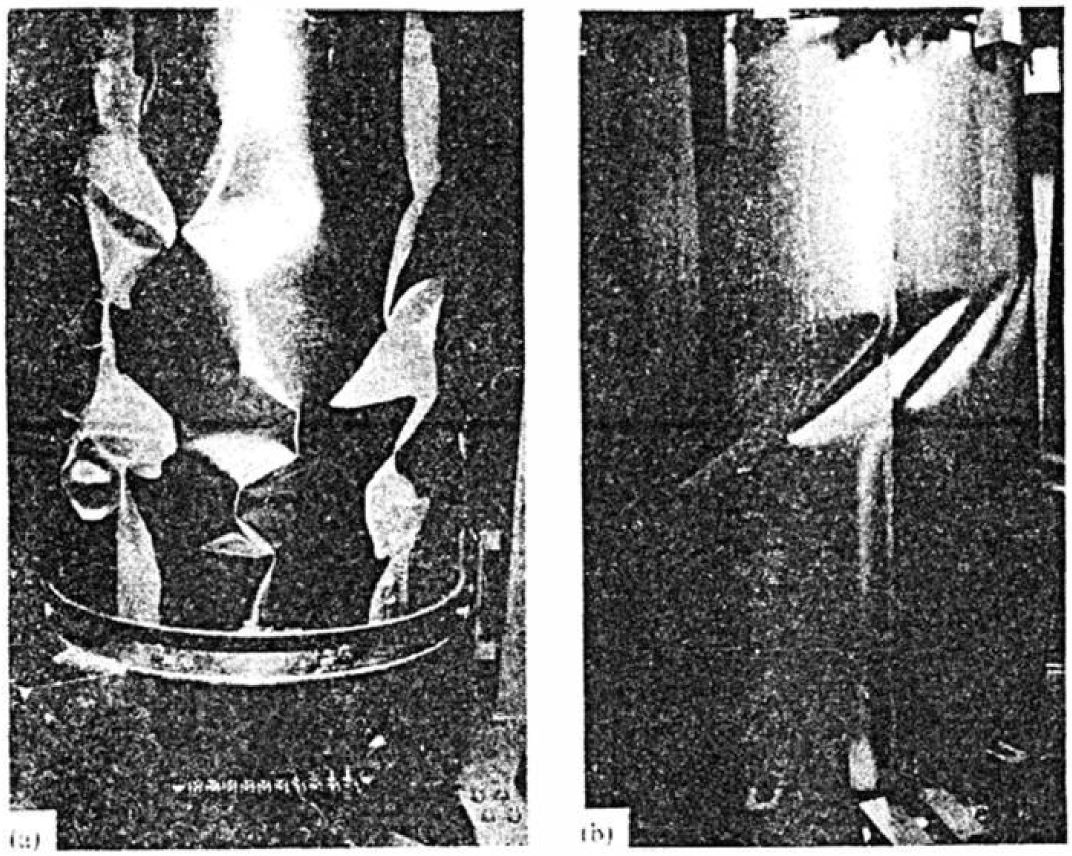


Fig. 16-1 Typical buckling patterns for thin-walled cylinders (a) in compression and (b) in torsion for a pressurized cylinder. (Courtesy L. A. Harris of North American Aviation. Inc.)

16-2. Examples of Instability

In numerous engineering applications, compression members have tubular cross sections. If the wall thickness is thin, the plate like elements of such members can buckle locally. An example of this behavior is illustrated in Fig. 16-2(a) for a square thin-walled tube. At a sufficiently large axial load, the side walls tend to subdivide into a sequence of alternating inward and outward buckles. As a consequence, the plates carry a smaller axial stress in the regions of large amount of buckling displacement away from corners; see Fig. 16-2(b). For such cases, it is customary to approximate the complex stress distribution by a constant allowable stress acting over an effective width w next to the corners or stiffeners.

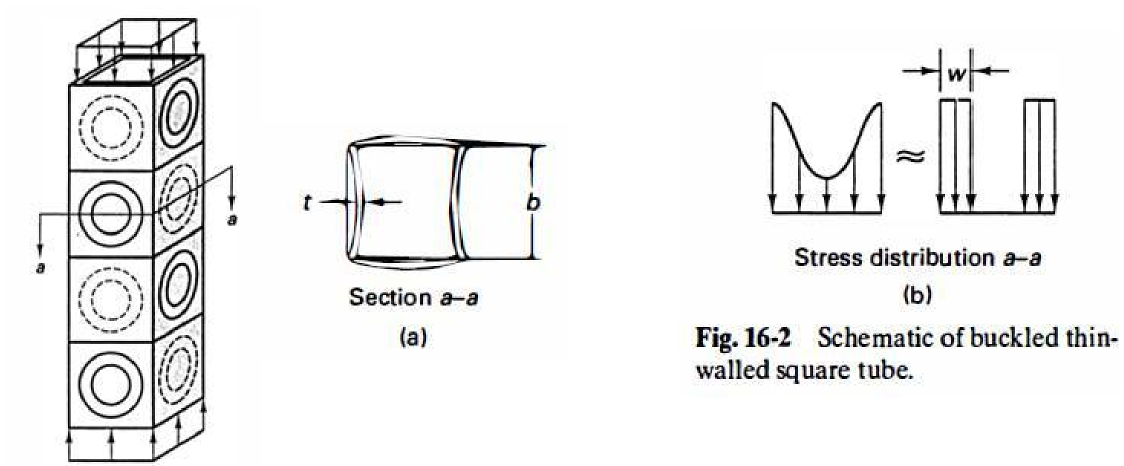


Fig. 16-2 Schematic of buckled thin-walled square tube.

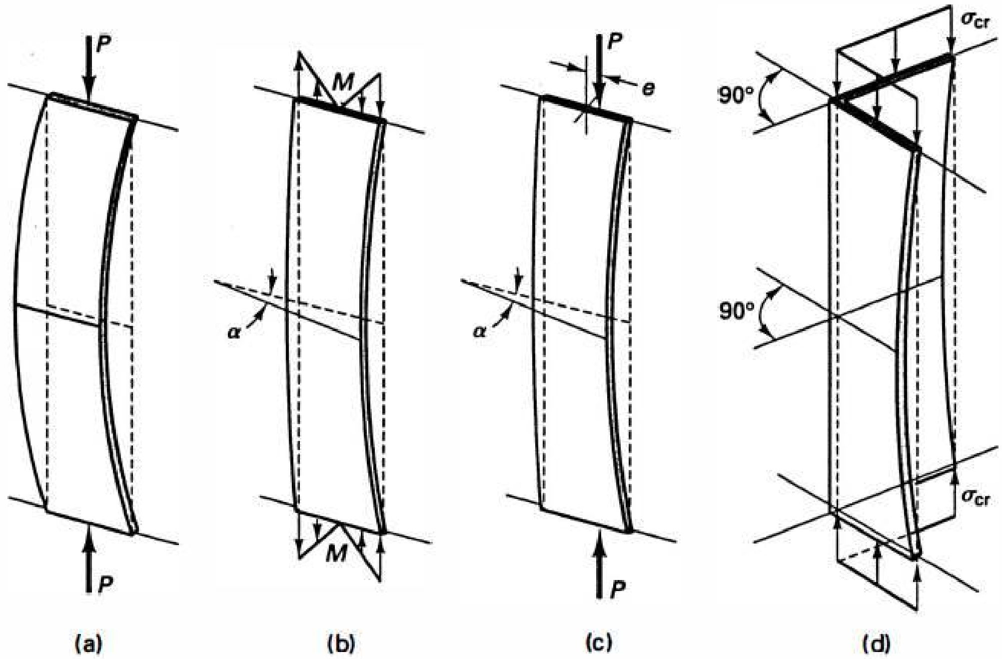
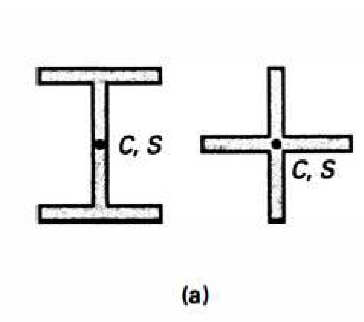
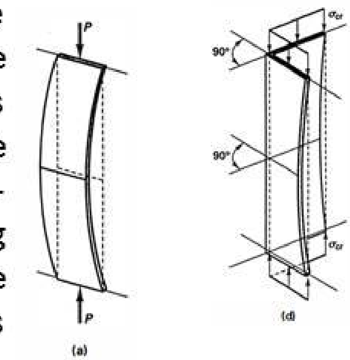


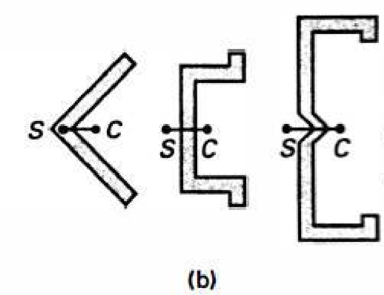
Fig. 16-3 Column buckling modes: (a) pure flexural, (b) and (c) torsional-flexural, and (d) pure torsional.

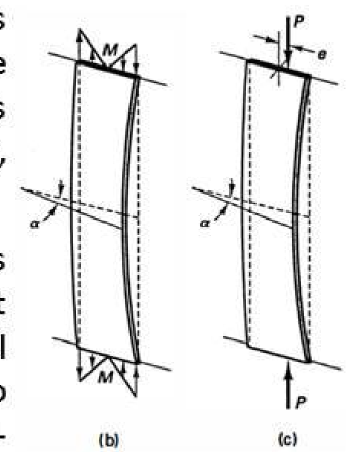
A plank of limited flexural but adequate torsional stiffness subjected to an axial compressive force is shown to buckle in a bending mode; see Fig. 16-3(a). If the same plank is subjected to end moments, Fig. 16-3(b), in addition to a flexural buckling mode, the cross sections also have a tendency to twist. This is a torsion bending mode of buckling, and the same kind of buckling may occur for the eccentric force P, as shown in Fig.16-3(c). Lastly, a pure torsional buckling mode is illustrated in Fig. 16-3(d). This occurs when the torsional stiffness of a member is small.



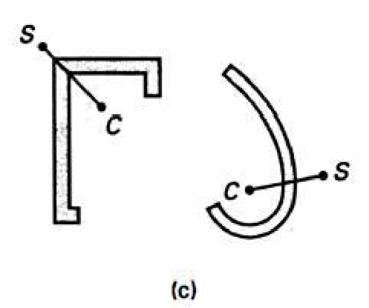


Two sections having biaxial symmetry, where centroids C and shear centers S coincide, are shown in Fig.16-4(a). Compression members having such cross sections buckle either in pure flexure, Fig. 16-3(a), or twist around S, Fig. 16-3(d). However, the torsional mode of buckling generally does not control the design, since the usual rolled or extruded metal cross sections are relatively thick.





Flexural buckling would occur for the sections in Fig. 16-4(b) if the smallest flexural stiffness around the major principal axis is less than the torsional stiffness. Otherwise, simultaneous flexural and torsional buckling would develop, with the member twisting around S.



For the sections in Fig.16-4(c), buckling always occurs in the torsional mode. In the subsequent derivations, it will be assumed that the wall thicknesses of members are sufficiently large to exclude the possibility of torsional or torsional-flexural buckling. Compression members having cross sections of the type shown in Fig.16-4(c) are not considered.

16-4. Euler Load for Columns with Pinned Ends

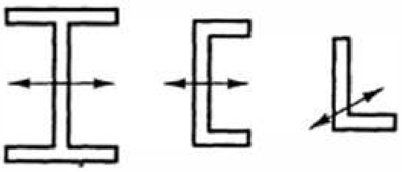


Fig. 16-12 Flexural column buckling occurs in plane of major axis.

The significant flexural rigidity EI of a column depends on the minimum I, and at the critical load a column buckles either to one side or the other in the plane of the major axis.

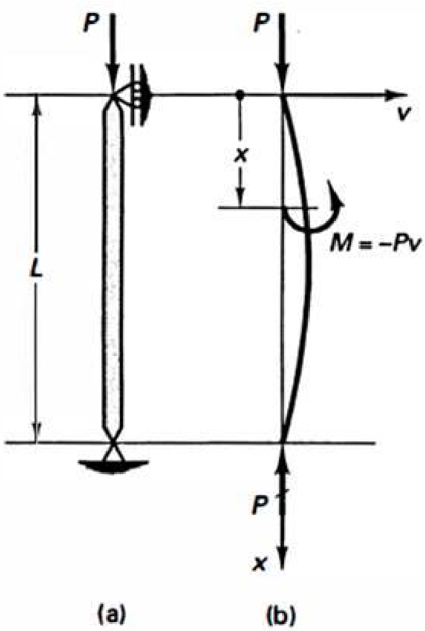


Fig. 16-13 Column pinned at both ends.

Consider the ideal perfectly straight column with pinned supports at both ends; see Fig. 16-13(a). The *least* force at which a buckled mode is possible is the *critical* or *Euler buckling load.*

In order to determine the critical load for this column, the compressed column is *displaced* as shown in Fig. 16-13(b). In this position, the bending moment according to the beam sign convention[12] is $-Pv$. By substituting this value of moment into Eq. 14-10, the differential equation for the elastic curve for the initially straight column becomes

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}v \qquad (16\text{-}5)$$

Letting $\lambda^2 = P/EI$ and transposing gives

$$\boxed{\frac{d^2v}{dx^2} + \lambda^2 v = 0} \qquad (16\text{-}6)$$

This is an equation of the same form as the one for simple harmonic motion, and its solution is

$$v = A \sin \lambda x + B \cos \lambda x \qquad (16\text{-}7)$$

where A and B are arbitrary constants that must be determined from the boundary conditions. These conditions are

$$v(0) = 0 \qquad \text{and} \qquad v(L) = 0$$

Hence,

$$v(0) = 0 = A \sin 0 + B \cos 0 \qquad \text{or} \qquad B = 0$$

and

$$v(L) = 0 = A \sin \lambda L \qquad (16\text{-}8)$$

This equation can be satisfied by taking $A = 0$. However, with AB each equal to zero, as can be seen from Eq. 16-7, is a solution for a straight column and is usually referred to as a trivial solution. An alternative solution is obtained by requiring the sine term in Eq. 16-8 to vanish. This occurs when λn equals $n\pi$, where n is an integer. Therefore, since λ was defined as $\sqrt{P/EI}$, the nth critical force P_n that makes the deflected shape of the column possible follows from setting $\sqrt{P/EI}\, L = n\pi$. Hence,

$$P_n = \frac{n^2\pi^2 EI}{L^2} \qquad (16\text{-}9)$$

These P_n's are the *eigenvalues* for this problem. However, since in stability problems only the *least value* of P_n is of importance, n must be taken as unity, and the *critical* or *Euler load*[13] P_{cr} for an initially *perfectly straight elastic* column with pinned ends becomes

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \tag{16-10}$$

where E is the elastic modulus of the material, I is the *least* moment of inertia of the constant cross-sectional area of a column, and L is its length. This case of a column pinned at both ends is often referred to as the *fundamental case.*

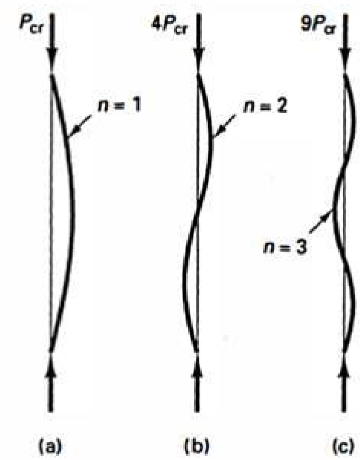


Fig. 16-14 First three buckling modes for a column pinned at both ends.

16-5. Euler Loads for Columns with Different End Restraints

The same procedure as that discussed before can be used to determine the critical axial loads for columns with different boundary conditions. The solutions of these problems are very sensitive to the end restraints. Consider, for example, a column with one end fixed and the other pinned, as shown in Fig. 16-15, where the buckled column is drawn in a *deflected* position. Here the effect of unknown end moment M_o and the reactions must be considered in setting up the differential equation for the elastic curve at the critical load:

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{-Pv + M_o(1 - x/L)}{EI} \tag{16-12}$$

Letting $\lambda^2 = P/EI$ as before, and transposing, gives

$$\frac{d^2v}{dx^2} + \lambda^2 v = \frac{\lambda^2 M_o}{P}\left(1 - \frac{x}{L}\right) \tag{16-13}$$

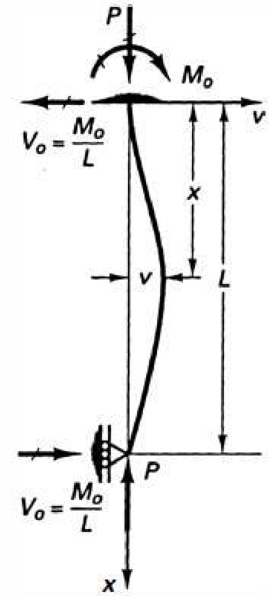


Fig. 16-15 Column fixed at one end and pinned at the other.

The *homogeneous solution* of this differential equation (i.e., when the right side is zero) is the same as that given by Eq. 16-7. The *particular solution*, due to the nonzero right side, is given by dividing the term on that side by λ^2. The complete solution then becomes

$$v = A \sin \lambda x + B \cos \lambda x + (M_o/P)(1 - x/L) \tag{16-14}$$

where A and B are arbitrary constants and M_o is the unknown moment at the fixed end. The three kinematic boundary conditions are

$$v(0) = 0 \qquad v(L) = 0 \qquad \text{and } v'(0) = 0$$

Hence,

$$v(0) = 0 = B + M_o/P$$

$$v(L) = 0 = A \sin \lambda L + B \cos \lambda L$$

and

$$v'(0) = 0 = A\lambda - M_o/PL$$

Solving these equations simultaneously, one obtains the following transcendental equation:

$$\lambda L = \tan \lambda L \tag{16-15}$$

which must be satisfied for a nontrivial equilibrium shape of the column at the critical load. The smallest root of Eq. 16-15 is

$$\lambda L = 4.493$$

from which the corresponding least eigenvalue or critical load for a *column fixed at one end and pinned at the other* is

$$P_{cr} = \frac{20.19EI}{L^2} = \frac{2.05\pi^2 EI}{L^2} \tag{16-16}$$

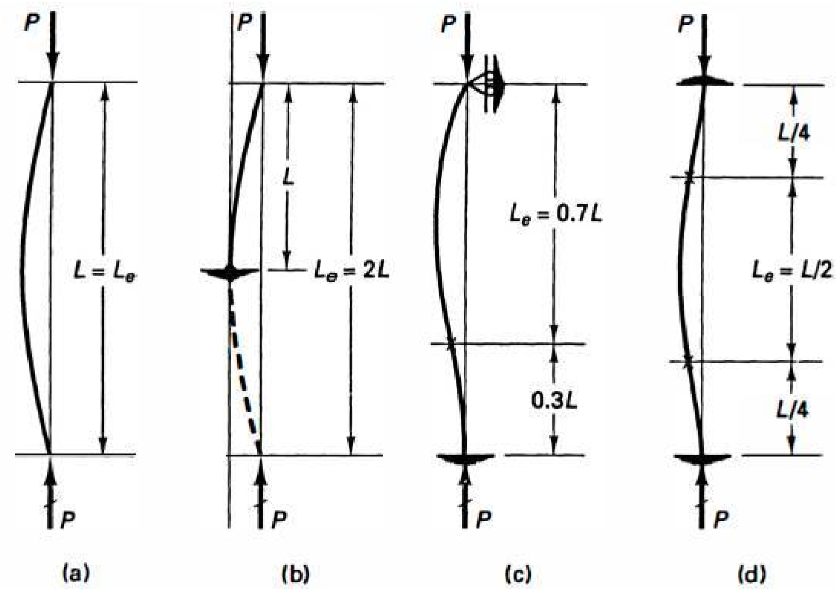


Fig. 16-16 Effective lengths of columns with different restraints.

All the previous formulas can be made to resemble the fundamental case, provided that the *effective column length* is used instead of the actual column length. This length turns out to be the distance between the inflection points on the elastic curves. The effective column length L_e for the fundamental case is L, but for the cases discussed it is $0.7L$, $0.5L$, and $2L$, respectively. For a general case, $L_e = KL$, where K is the effective length factor, which depends on the end restraints. Hence, a more general form of the Euler formula, incorporating the concept of the *effective column length* L_e, can be written as

$$\boxed{P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 EI}{L_e^2}} \qquad (16\text{-}18)$$

16-6. Limitations of the Euler Formulas

The elastic modulus E was used in the derivation of the Euler formulas for columns; therefore, all the reasoning presented earlier is applicable *while the material behavior remains linearly elastic.* To bring out this significant limitation, Eq. 16-10 is rewritten in a different form. By definition, $I = Ar^2$, where A is the cross-sectional area and r is its *radius of gyration.* Substitution of this relation into Eq. 16-10 gives

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EAr^2}{L_e^2}$$

or

$$\boxed{\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L_e/r)^2}} \tag{16-19a}$$

where the *critical stress* σ_{cr} for a column is defined as P_{cr}/A (i.e., as an *average* stress over the cross-sectional area A of a column at the critical load P_{cr}). The length of the column is L_e, and r is the *least* radius of gyration of the cross-sectional area, since the original Euler formula is in terms of the minimum I. By using the effective length L_e, the expression becomes general. The ratio L_e/r of the column length to the *least* radius of gyration is called the column *slenderness ratio. No factor of safety is included in the last equation.*

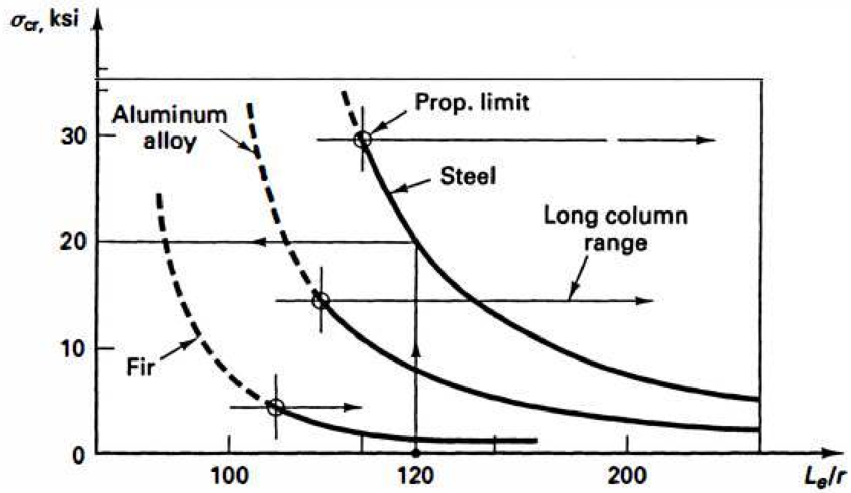


Fig. 16-17 Variation of critical column stress with slenderness ratio for three different materials.

A graphical interpretation of Eq. 16-19a is shown in Fig. 16-17, where the critical column stress is plotted versus the slenderness ratio for three different materials. For each material, E is constant, and the resulting curve is a hyperbola. However, since Eq. 16-19a is based on the elastic behavior of a material, σ_{cr} determined by this equation cannot exceed the proportional limit of a material. Therefore, the hyperbolas shown in Fig. 16-17 are drawn dashed beyond the individual material's proportional limit, and these portions of the curves *cannot be used*. The necessary modifications of Eq. 16-19a to include inelastic material response will be discussed in the next section.

Example 16-2

Find the shortest length L for a steel column with pinned ends having a cross-sectional area of 60 by 100 mm, for which the elastic Euler formula applies. Let $E = 200$ GPa and assume the proportional limit to be 250 MPa.

SOLUTION

The minimum moment of inertia of the cross-sectional area $I_{\min} = 100 \times 60^3/12 = 1.8 \times 10^6\ \text{mm}^4$. Hence, the *least radius of gyration* r is given as

$$\boxed{r = r_{\min} = \sqrt{\frac{I_{\min}}{A}}} \tag{16-19b}$$

and

$$r_{\min} = \sqrt{\frac{1.8 \times 10^6}{60 \times 100}} = \sqrt{3} \times 10\ \text{mm}$$

Then, using Eq. 16-19a and noting that for a column with pinned ends $L_e = L$, $\sigma_{cr} = \pi^2 E/(L/r)^2$. Solving for the slenderness ratio L/r at the proportional limit,

$$\left(\frac{L}{r}\right)^2 = \frac{\pi^2 E}{\sigma_{cr}} = \frac{\pi^2 \times 200 \times 10^3}{250} = 800\pi^2$$

or

$$\frac{L}{r} = 88.9 \qquad \text{and} \qquad L = 88.9\sqrt{3} \times 10 = 1540\ \text{mm}$$

16-7. Generalized Euler Buckling-load Formulas

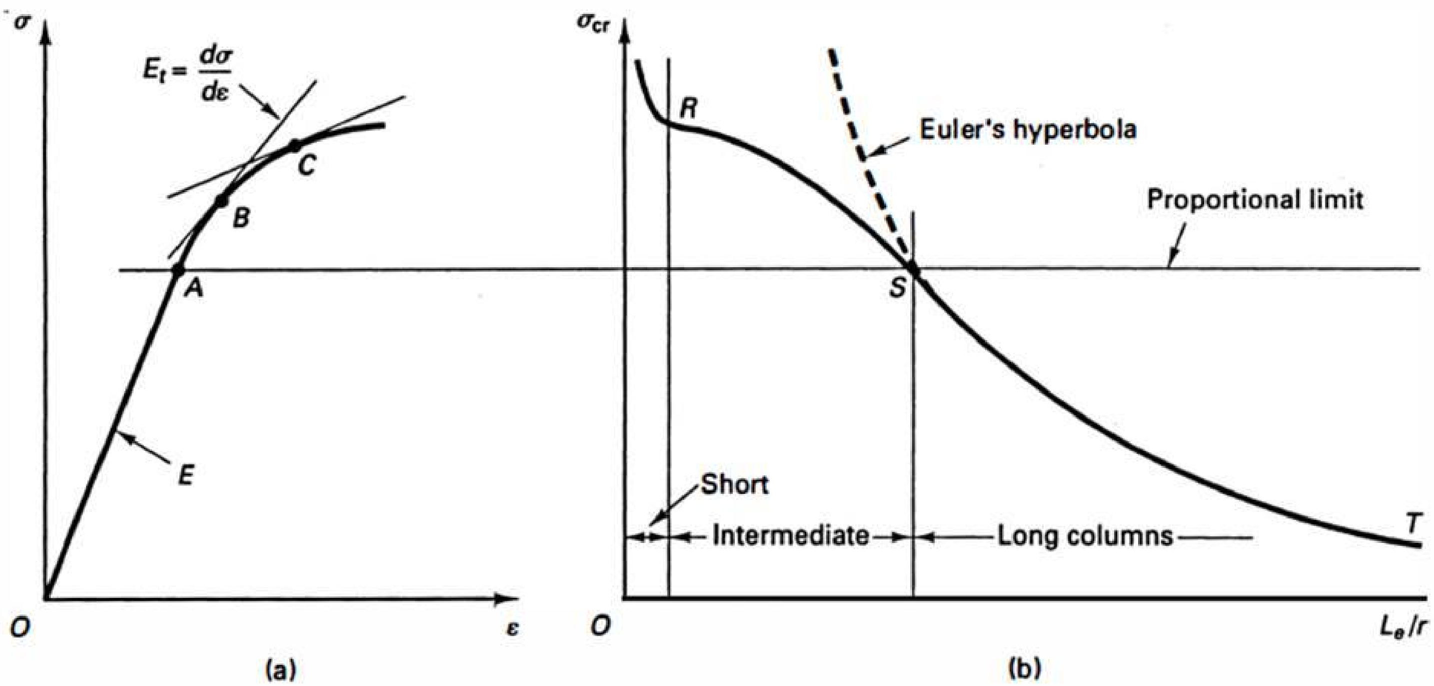


Fig. 16-18 (a) Compression stress-strain diagram, and (b) critical stress in columns versus slenderness ratio.

A typical compression stress-strain diagram for a specimen that is prevented from buckling is shown in Fig. 16-18(a). In the stress range from O to A, the material behaves elastically. If the stress in a column at buckling does not exceed this range, the column buckles elastically. The hyperbola expressed by Eq. 16-19a with an elastic E, is applicable in such a case. This portion of the curve is shown as ST in Fig. 16-18(b). It is important to recall that this curve does not represent the behavior of one column, but rather the behavior of an infinite number of ideal columns of different lengths. The hyperbola beyond the useful range is shown in the figure by dashed lines.

A column with an L_e/r ratio corresponding to point S in Fig. 16-18(b) is the shortest column of a given material and size that will buckle elastically. A shorter column, having a still smaller L_e/r ratio, will not buckle at the proportional limit of the material. On the compression stress-strain diagram, Fig. 16-18(a), this means that the stress level in the column has passed point A and has reached some point B perhaps. At this higher stress level, it may be said that a column of different material has been created, since the stiffness of the material is no longer represented by the elastic modulus. At this point, the material stiffness is given instantaneously by the tangent to the stress-strain curve [i.e., by the *tangent modulus* E_t; see Fig. 16-18(a)]. The column remains stable if its new flexural rigidity E_tI at B is sufficiently large, and it can carry a higher load. As the load is increased, the stress level rises, whereas the tangent modulus decreases. A column of ever "less stiff material" is acting under an increasing load. Substitution of the tangent modulus E_t for the elastic modulus E is then the only modification necessary to make the elastic buckling formulas applicable in the inelastic range. Hence, the *generalized Euler buckling-load formula,* or the *tangent modulus formula,* becomes

$$\boxed{\sigma_{cr} = \frac{\pi^2 E_t}{(L_e/r)^2}} \qquad (16\text{-}20)$$

Since stresses corresponding to the tangent moduli can be obtained from the compression stress-strain diagram, the L_e/r ratio at which a column will buckle with these values can be obtained from Eq. 16-20. A plot representing this behavior for low and intermediate ratios of L_e/r is shown in Fig. 16-18(b) by the curve from R to S. Tests on individual columns verify this curve with remarkable accuracy.

The tangent modulus formula gives the carrying capacity of a column at the *instant it tends to buckle.* As a column deforms further, the stiffness of the fibers on the concave side continues to exhibit approximately the tangent modulus E_t. The fibers on the convex side, however, on being relieved of some stress, rebound with the original elastic modulus E, as shown in Fig. 16-19 at point C. Inasmuch as two moduli, E_t and E, are used in developing this theory,[15] it is referred to as either the *double-modulus* or the *reduced-modulus theory* of column buckling. For the same column slenderness ratio, this theory always gives a slightly higher column buckling capacity than the tangent-modulus theory.

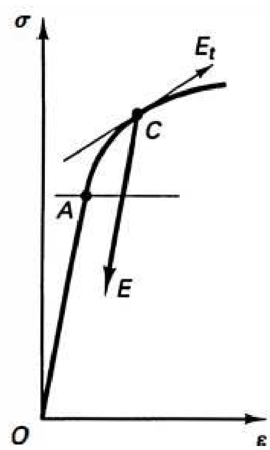


Fig. 16-19 Stress-strain behavior in buckled column.

As mentioned earlier, columns that buckle elastically are generally referred to as long columns. Columns having small $\frac{L_e}{r}$ ratios exhibiting no buckling phenomena are called short columns. The remaining columns are of intermediate length. At small $\frac{L_e}{r}$ ratios, ductile materials "squash out" and can carry very large loads.

Since length L_e in Eq. 16-20 is treated as the effective length of a column, different end conditions can be analyzed. Following this procedure for comparative purposes, plots of critical stress σ_{cr}, versus the slenderness ratio $\frac{L_e}{r}$ for fixed-ended columns and pin-ended ones are shown in Fig.16-21. It is important to note that the carrying capacity for these two cases per Eqs.16-10 and 16-17 is in a ratio of 4 to 1 only for columns having the slenderness ratio $\left(\frac{L_e}{r}\right)_1$ or greater. For smaller $\frac{L_e}{r}$ ratios, progressively less benefit is derived from restraining the ends. At small $\frac{L_e}{r}$ ratios, the curves merge. It makes little difference whether a "short block" is pinned or fixed at the ends, as strength, rather than buckling determines the behavior.

$$\boxed{\sigma_{cr} = \frac{\pi^2 E_t}{(L_e/r)^2}} \qquad (16\text{-}20)$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \qquad (16\text{-}17a)$$

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \qquad (16\text{-}10)$$

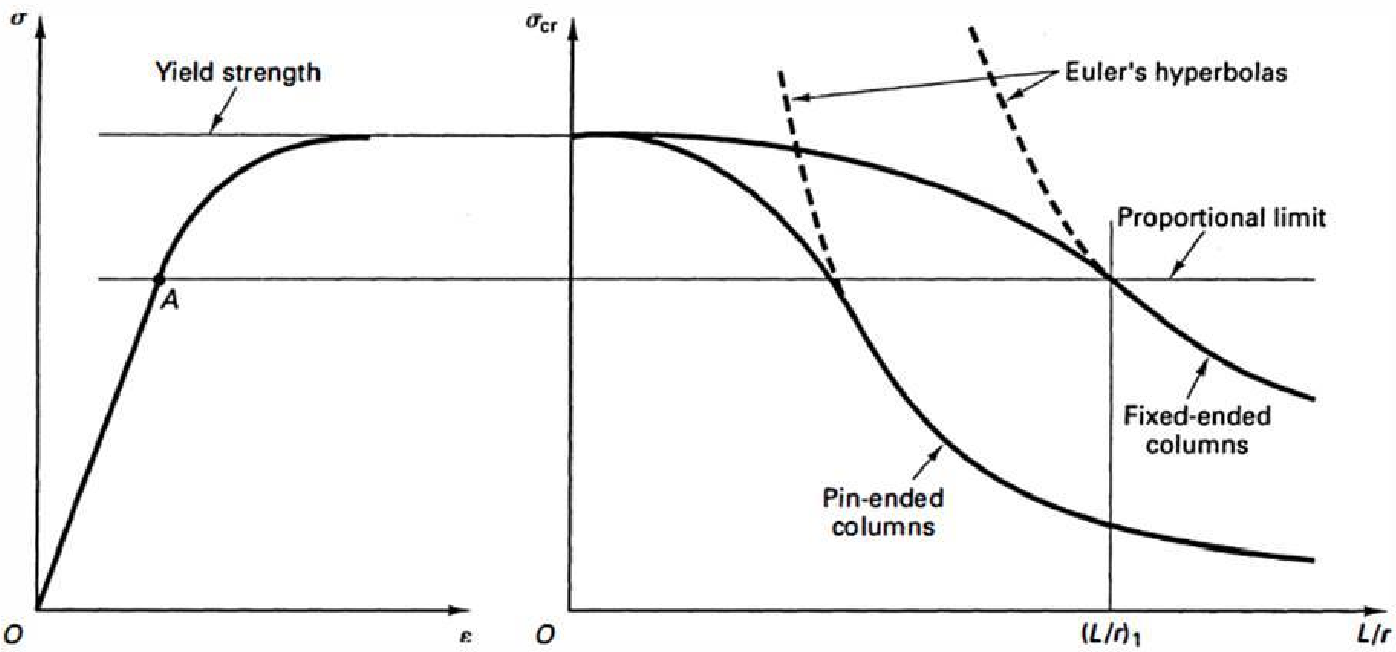
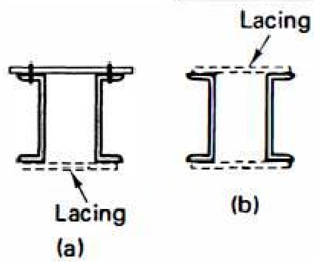


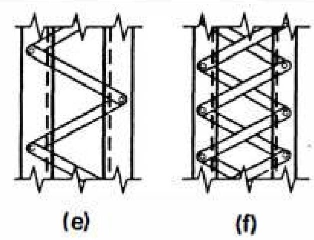
Fig. 16-21 Comparison of the behavior of columns with different end conditions.

Part B DESIGN OF COLUMNS

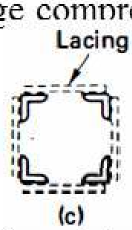
16-11. General Considerations



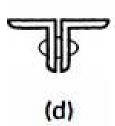
Cross sections for typical bridge compression members



The main longitudinal shapes in the other members are separated by plates or are laced (latticed) together by light bars



Derrick boom or a radio tower



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For other than short columns and blocks, the buckling theory for columns shows that their cross-sectional areas should have the largest possible least radius of gyration r. Such a provision for columns assures the smallest possible slenderness ratio, $\frac{L_e}{r}$ permitting the use of higher stresses. However, limitations must be placed on the minimum thickness of the material to prevent local plate buckling.

Since tubular members have a large radius of gyration in relation to the amount of material in a cross section, they are excellent for use as columns. Wide flange sections (sometimes referred to as H sections) are also very suitable for use as columns and are superior to I sections, which have narrow flanges, resulting in larger ratios of $\frac{L_e}{r}$. In order to obtain a large radius of gyration, columns are often built up from rolled or extruded shapes, and the individual pieces are spread out to obtain the desired effect. Local instability must be carefully guarded against to prevent failures in lacing bars, as shown in Fig.16-29.

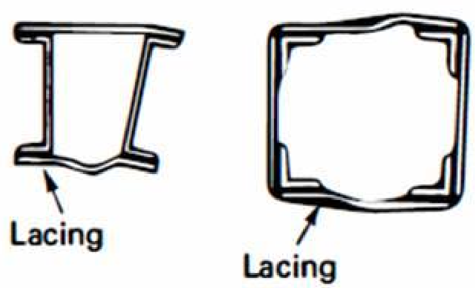


Fig. 16-29 Lattice instability.

Unavoidable imperfections must be recognized in the practical design of columns. Therefore, specifications usually stipulate not only the quality of material, but also fabrication tolerances for permissible out-of-straightness. The residual stresses caused by the manufacturing process must also be considered. For example, steel wide-flange sections, because of uneven cooling during a hot rolling operation, develop residual stress patterns of the type shown in Fig. 16-30. The maximum residual compressive stresses may be on the order of $0.3\sigma_{yp}$ in such members.

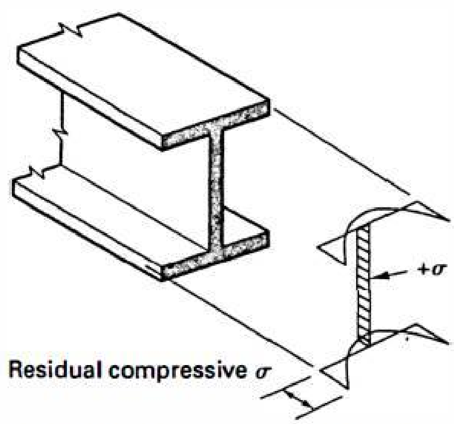


Fig. 16-30 Schematic residual stress pattern.

For the lower range of column length, usually a parabola (and, in a few instances, an inclined straight line) is specified. In this manner, the basic compressive strength of the material, residual stresses, and fabrication tolerances are accounted for. For slender (long) columns, the Euler elastic buckling load provides the basis for the critical stress. In this range of column lengths, the residual stresses play a relatively minor role. The dominant parameters are the material stiffness, E, and geometric fabrication imperfections. Often the two specified complementary equations have a common tangent at a selected value of L/r. Such a condition cannot be fulfilled if a straight line is used instead of a parabola. In a few specifications, the more conservative approach of using the elastic formula and an allowable stress is made by assuming an accidental eccentricity based on manufacturing tolerances.

For some materials, a sequence of three different equations is specified for the design of columns, Fig. 16-31(b). One of these equations for short columns defines the basic compressive strength of a material. Another equation, specifically applicable for the long column range, is based on the Euler buckling load. An empirical relation, such as an inclined straight line shown in the figure, or a parabola, is specified for columns of intermediate lengths. Such a type of formula is generally given for aluminum alloys and wood.

In applying the design formulas, it is important to observe the-following items:

1. The material and fabrication tolerances for which the formula is written.
2. Whether the formula gives the working load (or stress) or whether it estimates the ultimate carrying capacity of a member. If the formula is of the latter type, a safety factor must be introduced.
3. The range of applicability of the formula. Some empirical formulas can lead to unsafe design if used beyond the specified range.

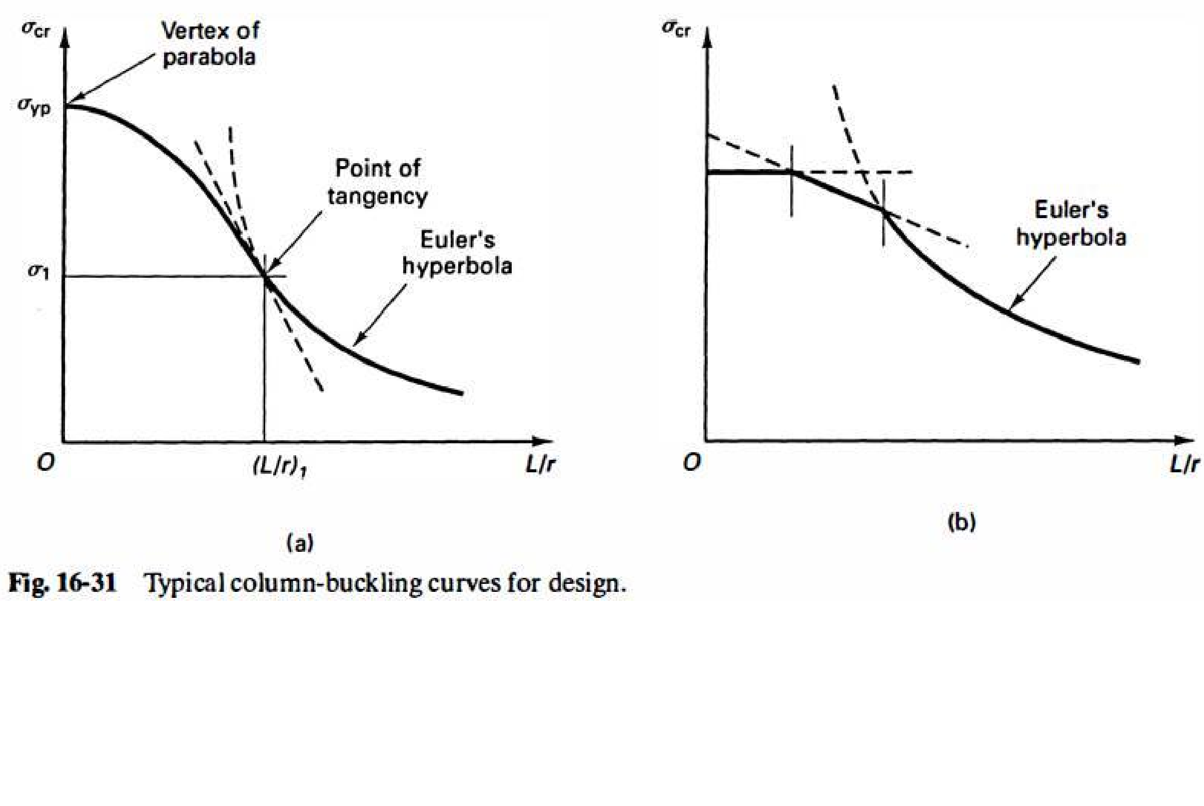


Fig. 16-31 Typical column-buckling curves for design.

AISC ASD Formulas for Columns. The AISC formula for *allowable* stress, σ_{allow}, for *slender* columns is based on the Euler elastic buckling load with a safety factor of $23/12 = 1.92$. Slender columns are defined as having the slenderness ratio $(L_e/r)_1 = C_c = \sqrt{2\pi^2 E/\sigma_{yp}}$ or greater. Constant C_c corresponds to the critical stress σ_{cr} at the Euler load equal to one-half the steel yield stress σ_{yp}.

The formula for long columns when $(L_e/r) > C_c$ is

$$\sigma_{allow} = \frac{12\pi^2 E}{23(L_e/r)^2} \tag{16-56}$$

where L_e is the effective column length and r is the least radius of gyration for the cross-sectional area. No columns are permitted to exceed an L_e/r of 200.

For an L_e/r ratio less than C_c, AISC specifies a parabolic formula:

$$\sigma_{allow} = \frac{[1 - (L_e/r)^2/2C_c^2]\sigma_{yp}}{\text{F.S.}} \tag{16-57}$$

where F.S., the factor of safety, is defined as

$$\text{F.S.} = \frac{5}{3} + \frac{3(L_e/r)}{8C_c} - \frac{(L_e/r)^3}{8C_c^3}$$

Since, in practical applications, the ideal restraint of the column ends, assumed in Section 16-5, cannot always be relied upon, conservatively, AISC specifies modification of the effective lengths as follows:

For columns built in at both ends: $L_e = 0.65L$
For columns built in at one end and pinned at the other: $L_e = 0.80L$
For columns built in at one end and free to translate and rotate at the other: $L_e = 2.10L$

No modification need be made for columns pinned at both ends, where $L_e = L$. For other end restraints, see AISC Specifications.

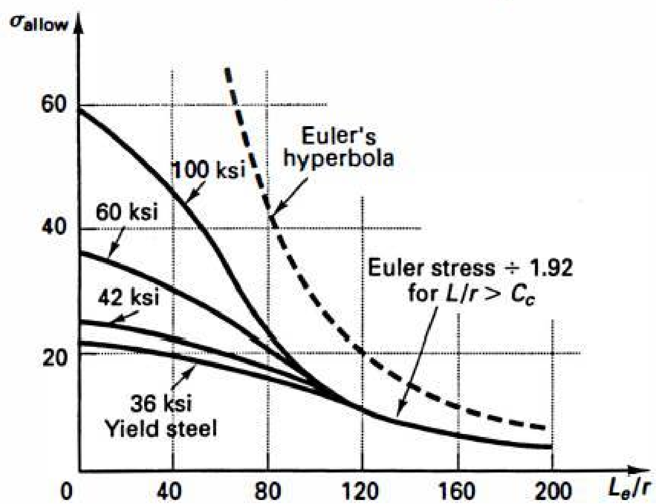


Fig. 16-32 Allowable stress for concentrically loaded columns per AISC specifications.

AISC LRFD Formulas for Columns. Here, again, there are two equations governing column strength, one for elastic and the other for inelastic buckling. The boundary between the inelastic and elastic instability is at $\lambda_c = 1.5$, where the *column slenderness parameter* λ_c is defined as

$$\lambda_c = \frac{L_e}{r\pi}\sqrt{\frac{\sigma_{yp}}{E}} \tag{16-58}$$

This expression results from normalizing the slenderness ratio L_e/r with respect to the slenderness ratio for the Euler elastic critical stress, assuming $\sigma_{cr} = \sigma_{yp}$.

For $\lambda_c > 1.5$, the *critical* buckling stress σ_{cr} is based on the Euler load and is given as

$$\sigma_{cr} = \left[\frac{0.877}{\lambda_c^2}\right]\sigma_{yp} \tag{16-59}$$

where the factor 0.877 is introduced to account for the initial out-of-straightness of the column, see Fig. 16-11(c), and the effects of residual stresses.

For $\lambda_c = 1.5$, an empirical relationship based on extensive experimental and probabilistic studies is given as

$$\sigma_{cr} = (0.658^{\lambda_c^2})\sigma_{yp} \tag{16-60}$$

This equation includes the effects of residual stresses and initial out-of-straightness.

Both of the previous formulas give the nominal axial strength (capacity) of columns and must be used in conjunction with factored loads and a resistance factor ϕ_c of 0.85. The effective slenderness ratios L_e/r are determined as for the ASD.

Example 16-6

(a) Determine the allowable axial loads for two 15-ft W 14 × 159 steel columns using AISC ASD formulas when one of the columns has pinned ends and the other has one end fixed and the other pinned. (b) Repeat the solution for two 40-ft W 14 × 159 columns. For the given section, $A = 46.7$ in^2 and $r_{min} = 4.00$ in. Assume A36 steel having $\sigma_{yp} = 36$ ksi.

SOLUTION

For both cases, it is necessary to calculate C_c to determine whether Eq. 16-56 or 16-57 is applicable:

$$C_c = \sqrt{2\pi^2 E/\sigma_{yp}} = \sqrt{2\pi^2 \times 29 \times 10^3/36} = 126.1$$

(a) For the W 14 × 159 shape, the *minimum* $r = 4.00$ in. Hence, for the 15-ft column with pinned ends, $L_e/r = 15 \times 12/4 = 45 < C_c$, and Eq. 16-57 applies. Hence,

$$\sigma_{allow} = \frac{[1 - 45^2/(2 \times 126.1^2)]36}{5/3 + 3 \times 45/(8 \times 126.1) - 45^3/(8 \times 126.1^3)} = 18.78 \text{ ksi}$$

and

$$P_{allow} = A\sigma_{allow} = 46.7 \times 18.78 = 877 \text{ kips}$$

For the column with one end fixed and the other pinned, according to the AISC, the effective length $L_e = 0.8L = 12$ ft. Hence, $L_e/r = 12 \times 12/4 = 36$, and again applying Eq. 16-57, $\sigma_{allow} = 19.50$ ksi and $P_{allow} = A\sigma_{allow} = 46.7 \times 19.50 = 911$ kips.

Here the allowable axial force is increased by 3.9% by fixing one of the column ends.

(b) For a 40-ft column with pinned ends, $L_e/r = 40 \times 12/4 = 120 < C_c$. Hence, using Eq. 16-57 again, it can be determined that $\sigma_{allow} = 10.28$ ksi and $P_{allow} = A\sigma_{allow} = 46.7 \times 10.28 = 480$ kips. Similarly, since for a column fixed at one end and pinned at the other, $L_e/r = 0.8 \times 120 = 96$, Eq. 16-57 gives $\sigma_{allow} = 13.48$ ksi and $P_{allow} = A\sigma_{allow} = 46.7 \times 13.48 = 630$ kips.

For this case, the allowable axial force is increased 31.2% by fixing one of the column ends. This contrasts with the 3.9% found earlier for the shorter columns. This finding is in complete agreement with the generalized Euler theory for columns, Section 16-7. As can be noted from Fig. 16-21, by restraining the ends of *long* columns, a large increase in their strength is obtained at large values of L_e/r. Restraining the ends of short columns results only in a modest increase in their strength.

Example 16-7

Using the AISC ASD column formulas, select a 15-ft-long pin-ended column to carry a concentric load of 200 kips. The structural steel is to be A572, having $\sigma_{yp} = 50$ ksi.

SOLUTION

The required size of the column can be found directly from the tables in the *AISC Steel Construction Manual*. However, this example provides an opportunity to demonstrate the trial-and-error procedure that is so often necessary in design, and the solution presented follows from using this method.

First try: Let $L/r = 0$ (a poor assumption for a column 15 ft long). Then, from Eq. 16-57, since F.S. $= 5/3$, $\sigma_{allow} = 50/\text{F.S.} = 30$ ksi and $A = P/\sigma_{allow} = 200/30 = 6.67\ \text{in}^2$. From Table 4A in the Appendix, this requires a W 8×24 section, whose $r_{min} = 1.61$ in. Hence, $L/r = 15(12)/1.61 = 112$. With this L/r, the allowable stress is found using Eq. 16-56 or Eq. 16-57, whichever is applicable depending on C_c:

$$C_c = \sqrt{2\pi^2 E/\sigma_{yp}} = \sqrt{2\pi^2 \times 29 \times 10^3/50} = 107 < L/r = 112$$

Hence, using Eq. 16-56,

$$\sigma_{\text{allow}} = \frac{12\pi^2 \times 29 \times 10^3}{23 \times 112^2} = 11.9 \text{ ksi}$$

This is much smaller than the initially assumed stress of 30 ksi, and another section must be selected.

Second try: Let $\sigma_{\text{allow}} = 11.9$ ksi, as found before. Then $A = 200/11.9 = 16.8 \text{ in}^2$, requiring a W 8×58 section having $r_{\text{min}} = 2.10$ in. Now $L/r = 15(12)/2.10 = 85.7$, which is less than C_c found before. Therefore, Eq. 16-57 applies, and

$$\text{F.S.} = 5/3 + 3(85.7)/(8 \times 107) - (85.7)^3/(8 \times 107^3) = 1.90$$

and

$$\sigma_{\text{allow}} = [1 - (85.7)^2/(2 \times 107^2)]50/1.90 = 17.9 \text{ ksi}$$

This stress requires $A = 200/17.9 = 11.2 \text{ in}^2$, which is met by a W 8×40 section with $r_{\text{min}} = 2.04$ in. A calculation of the capacity for this section shows that the allowable axial load for it is 204 kips, which meets the requirements of the problem.

Example 16-8

Determine the design compressive strength P_u for a 15-ft W 14 $\times$ 159 steel column pinned at both ends based on the AISC LRFD provisions. For this section, $A = 46.7\ \text{in}^2$ and $r_{\text{min}} = 4.00$ in. Assume A36 steel having $\sigma_{\text{allow}} = 36$ ksi.

SOLUTION

The column slenderness parameter as defined by Eq. 16-58 is

$$\lambda_c = \frac{15 \times 12}{4\pi}\sqrt{\frac{36}{29 \times 10^3}} = 0.5047$$

Since λ_c is less than 1.5, Eq. 16-60 applies for determining the critical stress and

$$\sigma_{\text{cr}} = (0.658^{0.5047^2})36 = 32.36\ \text{ksi}$$

Hence, for this column, the *nominal* compressive strength

$$P_n = A\sigma_{\text{cr}} = 46.7 \times 32.36 = 1510\ \text{kips}$$

and since the resistance factor $\phi_c = 0.85$, the column-design compressive strength

$$P_u = \phi_c P_n = 0.85 \times 1510 = 1289\ \text{kips}$$

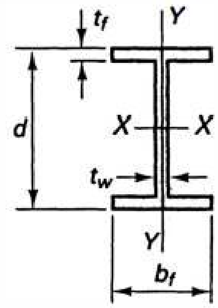


TABLE 4A. AMERICAN STANDARD STEEL W SHAPES DIMENSIONS AND PROPERTIES U.S. CUSTOMARY UNITS (ABRIDGED LIST)

				Flange		Axis X–X		Axis Y–Y	
Designation*	Area A	Depth d	Web Thickness t_w	Width b_f	Thickness t_f	I_x	r_x	I_y	r_y
in × lb/ft	in²	in	in	in	in	in⁴	in	in⁴	in
W36 × 245	72.1	36.08	0.800	16.510	1.350	16100	15.0	1010	3.75
230	67.6	35.90	0.760	16.470	1.260	15000	14.9	940	3.73
150	44.2	35.85	0.625	11.975	0.940	9040	14.3	270	2.47
135	39.7	35.55	0.600	11.950	0.790	7800	14.0	225	2.38
W33 × 201	59.1	33.68	0.715	15.745	1.150	11500	14.0	749	3.56
130	38.3	33.09	0.580	11.510	0.855	6710	13.2	218	2.39
118	34.7	32.86	0.550	11.480	0.740	5900	13.0	187	2.32
W30 × 191	56.1	30.68	0.710	15.040	1.185	9170	12.8	673	3.46
173	50.8	30.44	0.655	14.985	1.065	8200	12.7	598	3.43
W27 × 161	47.4	27.59	0.660	14.020	1.080	6280	11.5	497	3.24
146	42.9	27.38	0.605	13.965	0.975	5630	11.4	443	3.21
94	27.7	26.92	0.490	9.990	0.745	3270	10.9	124	2.12
84	24.8	26.71	0.460	9.960	0.640	2850	10.7	106	2.07
W18 × 60	17.6	18.24	0.415	7.555	0.695	984	7.47	50.1	1.69
50	14.7	17.99	0.355	7.495	0.570	800	7.38	40.1	1.65
46	13.5	18.06	0.360	6.060	0.605	712	7.25	22.5	1.29
35	10.3	17.70	0.300	6.000	0.425	510	7.04	15.3	1.22

				Flange		Axis X–X		Axis Y–Y	
Designation*	Area A	Depth d	Web Thickness t_w	Width b_f	Thickness t_f	I_x	r_x	I_y	r_y
in × lb/ft	in^2	in	in	in	in	in^4	in	in^4	in
W16 × 26	7.68	15.69	0.250	5.500	0.345	301	6.26	9.59	1.12
W14 × 193	56.8	15.48	0.890	15.710	1.440	2400	6.50	931	4.05
159	46.7	14.98	0.745	15.565	1.190	1900	6.38	748	4.00
99	29.1	14.16	0.485	14.565	0.780	1110	6.17	402	3.71
90	26.5	14.02	0.440	14.520	0.710	999	6.14	362	3.70
W12 × 72	21.1	12.25	0.430	12.040	0.670	597	5.31	195	3.04
65	19.1	12.12	0.390	12.000	0.605	533	5.28	174	3.02
50	14.7	12.19	0.370	8.080	0.640	394	5.18	56.3	1.96
45	13.2	12.06	0.335	8.045	0.575	350	5.15	50.0	1.94
40	11.8	11.94	0.295	8.005	0.515	310	5.13	44.1	1.93
W10 × 112	32.9	11.36	0.755	10.415	1.250	716	4.66	236	2.68
60	17.6	10.22	0.420	10.080	0.680	341	4.39	116	2.57
49	14.4	9.98	0.340	10.000	0.560	272	4.35	93.4	2.54
45	13.3	10.10	0.350	8.020	0.620	248	4.33	53.4	2.01
39	11.5	9.92	0.315	7.985	0.530	209	4.27	45.0	1.98
33	9.71	9.73	0.290	7.960	0.435	170	4.19	36.6	1.94
W8 × 67	19.7	9.00	0.570	8.280	0.935	272	3.72	88.6	2.12
58	17.1	8.75	0.510	8.220	0.810	228	3.65	75.1	2.10
40	11.7	8.25	0.360	8.070	0.560	146	3.53	49.1	2.04
31	9.13	8.00	0.285	7.995	0.435	110	3.47	37.1	2.02
28	8.25	8.06	0.285	6.535	0.465	98.0	3.45	21.7	1.62
24	7.08	7.93	0.245	6.495	0.400	82.8	3.42	18.3	1.61
21	6.16	8.28	0.250	5.270	0.400	75.3	3.49	9.77	1.26
18	5.26	8.14	0.230	5.250	0.330	61.9	3.43	7.97	1.23

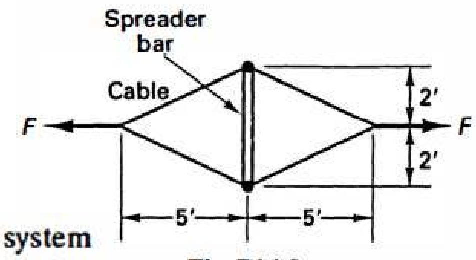


Fig. P16-8

16-8. A 1-in round steel bar 4 ft long acts as a spreader bar in the arrangement shown in the figure. If cables and connections are properly designed, what pull F can be applied to the assembly? Use Euler's formula and assume a factor of safety of 3. $E = 29 \times 10^6$ psi.

16-9. A solid steel bar having a 21 mm radius acts as a spreader bar in the system shown in the figure. Based on the Euler formula with a factor of safety of 1.7, what is the capacity of the system based on the spreader bar strength? Let $E = 200$ GPa.

16-10. A boom is made from an aluminum pipe of 60 mm outside diameter and having a 4-mm wall thickness, and is part of an arrangement for lifting weights, as shown in the figure. Determine the magnitude of the force F that could be applied to this planar system as controlled by the capacity of the boom. Assume a factor of safety of 2 for the Euler buckling load. $E = 75$ GPa. All dimensions are shown in mm.

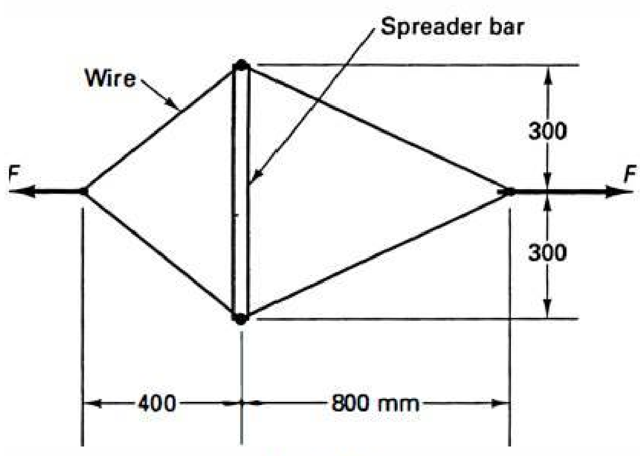


Fig. P16-9

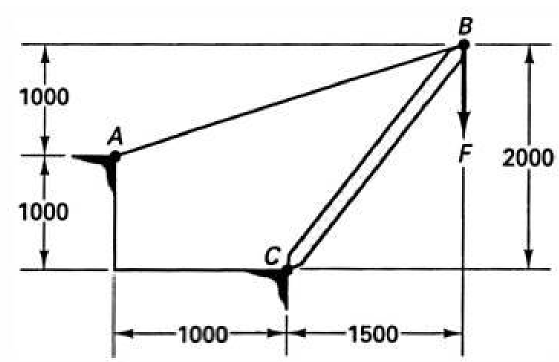


Fig. P16-10

16-11. The mast of a derrick is made of a standard rectangular 4×2-in steel tubing weighing 6.86 lb/ft. ($A = 2.02\ \text{in}^2$, $I_x = 1.29\ \text{in}^4$, and $I_z = 3.87\ \text{in}^4$.) If this derrick is assembled as indicated in the figure, what vertical force F, governed by the size of the mast, can be applied at A? Assume that all joints are pin connected and that the connection details are so made that the mast is loaded concentrically. The top of the mast is braced to prevent sidewise displacement. Use Euler's formula with a factor of safety of 3.3. $E = 29 \times 10^6$ psi.

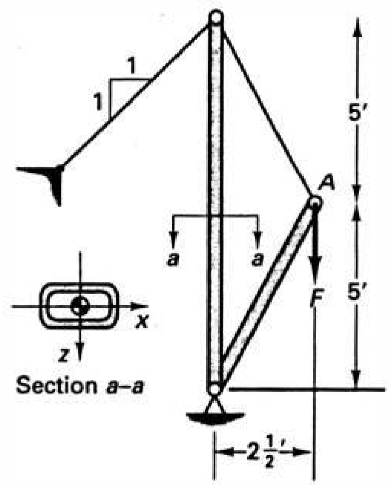


Fig. P16-11

16-12. What force F can be applied to the system shown in the figure, governed by the 25×16-mm aluminum-alloy bar AB? The factor of safety on the Euler buckling load is to be 2.5. Assume the ends are pinned. $E = 70$ GPa.

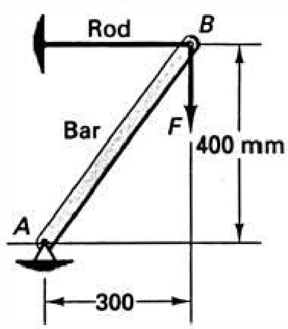


Fig. P16-12

16-15. What size standard steel pipe should be used for the horizontal member of the jib crane shown in the figure for supporting the maximum force of 20 kN, which includes an impact factor? Use the Euler buckling formula for columns with pinned ends and a factor of safety of 2.5. Neglect the weight of construction. $E = 200$ GPa.

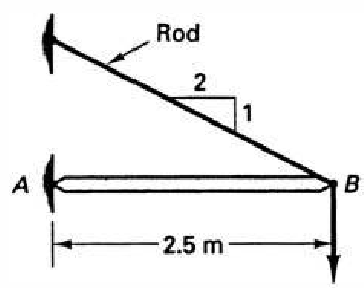


Fig. P16-15

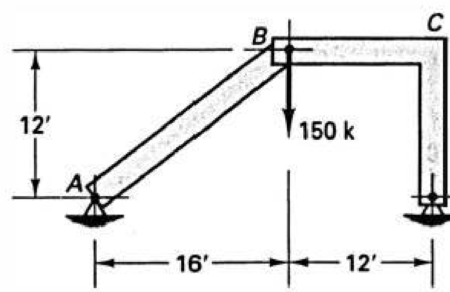


Fig. P16-16

16-16. Select a W steel section for member AB for the system shown in the figure to resist a vertical force of 150 k. The system is laterally braced at B and C. Neglect the weight of the members. Assume pinned ends and a factor of safety of 2. $E = 29 \times 10^3$ ksi.

16-17. Select standard steel pipes for members AC and AD shown in the figure to support a vertical load $F = 4.75$ k with a factor of safety of 2.5 on the Euler buckling load. Neglect the weight of the members. $E = 29 \times 10^3$ ksi.

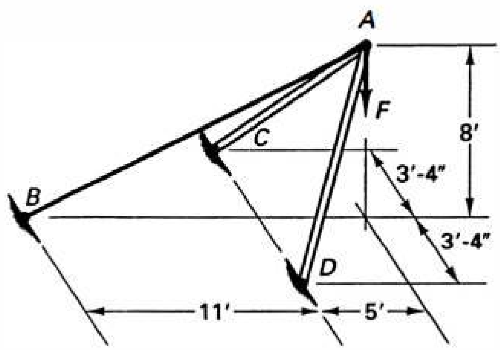


Fig. P16-17

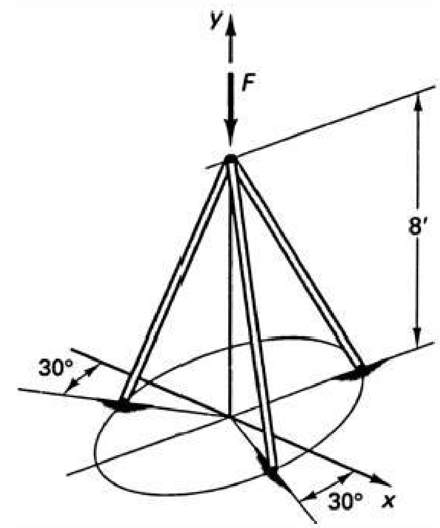


Fig. P16-18

16-18. A tripod is to be made up from 3 × 3-in steel angles, each 10 ft long, to support a vertical load $F = 8$ k at the center, as shown in the figure. Using the Euler buckling formula with a factor of safety of 3 to account for impact, determine the required thickness of the angles. Neglect the weight of the angles; assume that they are loaded concentrically and that the ends are pinned. $E = 30 \times 10^6$ psi.

Specification for Structural Steel Buildings

March 9, 2005

ANSI/AISC 360-05
An American National Standard

3. Design for Strength Using Load and Resistance Factor Design (LRFD)

Design according to the provisions for *Load and Resistance Factor Design* (LRFD) satisfies the requirements of this Specification when the *design strength* of each *structural component* equals or exceeds the *required strength* determined on the basis of the *LRFD load combinations*. All provisions of this Specification, except for those in Section B3.4, shall apply.

Design shall be performed in accordance with Equation B3-1:

$$\boxed{R_u \leq \phi R_n} \qquad \text{(B3-1)}$$

where

R_u = required strength (LRFD)
R_n = *nominal strength*, specified in Chapters B through K
ϕ = *resistance factor*, specified in Chapters B through K
ϕR_n = design strength

4. Design for Strength Using Allowable Strength Design (ASD)

Design according to the provisions for *Allowable Strength Design* (ASD) satisfies the requirements of this Specification when the *allowable strength* of each *structural component* equals or exceeds the *required strength* determined on the basis of the *ASD load combinations*. All provisions of this Specification, except those of Section B3.3, shall apply.

Design shall be performed in accordance with Equation B3-2:

$$\boxed{R_a \leq R_n / \Omega} \qquad \text{(B3-2)}$$

where

R_a = required strength (ASD)
R_n = *nominal strength*, specified in Chapters B through K
Ω = safety factor, specified in Chapters B through K
R_n/Ω = allowable strength

CHAPTER E

DESIGN OF MEMBERS FOR COMPRESSION

E1. GENERAL PROVISIONS

The *design compressive strength*, $\phi_c P_n$, and the *allowable compressive strength*, P_n/Ω_c, are determined as follows:

The *nominal compressive strength*, P_n, shall be the lowest value obtained according to the *limit states* of *flexural buckling*, *torsional buckling* and *flexural-torsional buckling*.

(a) For doubly symmetric and singly symmetric members the limit state of flexural buckling is applicable.

(b) For singly symmetric and unsymmetric members, and certain doubly symmetric members, such as cruciform or built-up *columns*, the limit states of torsional or flexural-torsional buckling are also applicable.

$$\boxed{\phi_c = 0.90 \text{ (LRFD)} \quad \Omega_c = 1.67 \text{ (ASD)}}$$

E2. SLENDERNESS LIMITATIONS AND EFFECTIVE LENGTH

The effective length factor, K, for calculation of column slenderness, KL/r, shall be determined in accordance with Chapter C,

where

L = laterally unbraced length of the member, in. (mm)
r = governing radius of gyration, in. (mm)
K = the *effective length factor* determined in accordance with Section C2

User Note: For members designed on the basis of compression, the slenderness ratio KL/r preferably should not exceed 200.

E3. COMPRESSIVE STRENGTH FOR FLEXURAL BUCKLING OF MEMBERS WITHOUT SLENDER ELEMENTS

This section applies to compression members with *compact* and *noncompact sections*, as defined in Section B4, for uniformly compressed elements.

> **User Note:** When the torsional unbraced length is larger than the lateral unbraced length, Section E4 may control the design of wide flange and similarly shaped columns.

The *nominal compressive strength*, P_n, shall be determined based on the *limit state* of *flexural buckling*.

$$\boxed{P_n = F_{cr} A_g} \tag{E3-1}$$

The *flexural buckling stress*, F_{cr}, is determined as follows:

(a) When $\frac{KL}{r} \leq 4.71\sqrt{\frac{E}{F_y}}$ (or $F_e \geq 0.44F_y$)

$$\boxed{F_{cr} = \left[0.658^{\frac{F_y}{F_e}}\right] F_y} \qquad \text{(E3-2)}$$

(b) When $\frac{KL}{r} > 4.71\sqrt{\frac{E}{F_y}}$ (or $F_e < 0.44F_y$)

$$\boxed{F_{cr} = 0.877F_e} \qquad \text{(E3-3)}$$

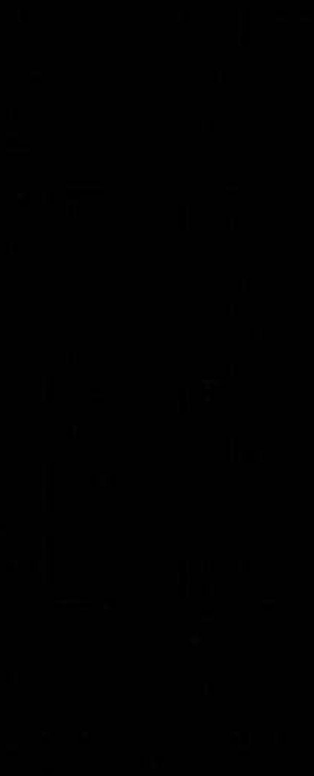
where

F_e = elastic critical buckling stress determined according to Equation E3-4, Section E4, or the provisions of Section C2, as applicable, ksi (MPa)

$$\boxed{F_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2}} \qquad \text{(E3-4)}$$

User Note: The two equations for calculating the limits and applicability of Sections E3(a) and E3(b), one based on KL/r and one based on F_e, provide the same result.

EXAMPLE:
Determine the allowable compressive load carrying capacity of the column shown in Fig. It consists of W10x45 section having A992 (F_y = 50 ksi) steel. There are hinge support at top and bottom that allows rotation in any direction. Also the column has weak direction support (braced) at mid-height so that lateral deflection is prevented in x direction. Use ASD approach.



SOLUTION:
For W10x45 section, from AISC Manual Chart we have
A = 13.3 in^2, r_x=4.32 in., r_y=2.01 in.
$x \rightarrow$ strong axis
$y \rightarrow$ weak axis
Column length, L = (13x2)x12 = 312 in.

Possibility of buckling in both x and y directions to be checked.

Buckling in y direction causes bending about x axis or strong axis. For strong axis buckling, the buckling shape is like a half sine wave over full column length. Thus for strong (or x axis) axis buckling, K_x = 1.0

$\therefore K_x L/r_x = 1.0\times312/4.32 = 72.22$

$\therefore F_{ex} = \pi^2 E/(K_x L/r_x)^2 = 3.14^2\times29000/(72.2)^2 = 54.82$ ksi. (> F_y, note)

And $4.71\sqrt{(E/F_y)} = 4.71\sqrt{(29000/50)} = 113.43 \therefore K_x L/r_x < 4.71\sqrt{(E/F_y)}$

$$F_{cr} = \left[0.658^{\frac{F_y}{F_e}}\right] F_y = [0.658^{(50/54.82)}]50 = 34.13 \text{ ksi}$$

Nominal strength for x-axis buckling $P_{nx} = F_{cr} A_g = 34.13\times13.3 = 454$ kip

Buckling in *x* direction causes bending about *y* axis or weak axis. For weak axis buckling, the buckling shape is like a full sine wave over full column length. Thus for weak (or *y* axis) axis buckling, $K_y = 0.5$

$\therefore K_y L/r_y = 0.5\times312/2.01 = 77.61$

$\therefore F_{ey} = \pi^2 E/(K_y L/r_y)^2 = 3.14^2\times29000/(77.61)^2 = 47.47$ ksi.

$$F_{cr} = \left[0.658^{\frac{F_y}{F_e}}\right] F_y = [0.658^{(50/47.47)}]50 = 32.17 \text{ ksi}$$

Nominal strength for y axis buckling $P_{ny} = F_{cr} A_g = 32.17\times13.3 = 427.9$ kip

$\therefore P_n$ = smaller of P_{nx} and P_{ny} = 427.9 kip

$\therefore$ Allowable strength $P = P_n/\Omega = 427.9/1.67 = 256.2$ kip

EXAMPLE 6.10.1 in ASD:

Select the lightest *W* section of A992 (F_y = 50 ksi) steel to serve as a pinned-end main member column 16 ft long to carry an axial compression load of 115 kips dead load and 125 kips live load in a braced structure, as shown in Fig. Use ASD approach.

SOLUTION:

P= 115+125= 240 kip, L = 16' = 192"

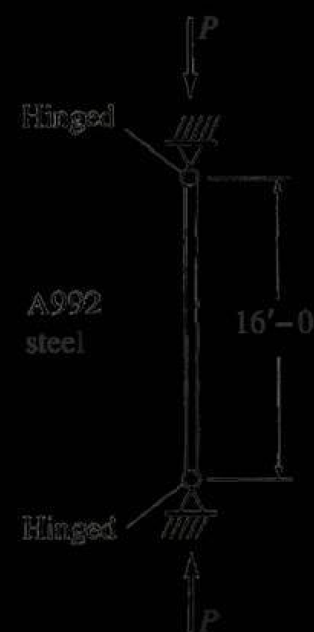
Both ends hinged, therefore K = 1.0

Nominal strength $P_n = F_{cr} A_g$

1. $F_{cr} = \left[0.658^{\frac{F_y}{F_e}}\right] F_y$ For $\dfrac{KL}{r} \leq 4.71\sqrt{\dfrac{E}{F_y}}$ or $F_e \geq 0.44F_y$ **(6.7.7)**

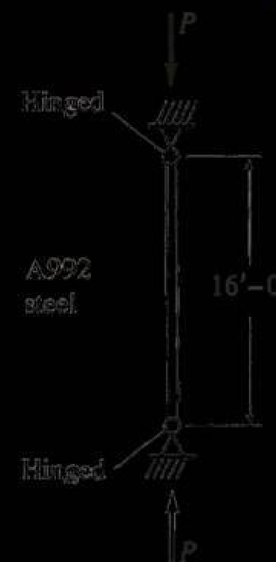
2. $F_{cr} = 0.877F_e$ For $\dfrac{KL}{r} > 4.71\sqrt{\dfrac{E}{F_y}}$ or $F_e < 0.44F_y$ **(6.7.8)**

$$F_e = F_{cr} = \frac{\pi^2 E}{\left(\dfrac{KL}{r}\right)^2}$$



$4.71\sqrt{(E/F_y)} = 4.71\sqrt{(29000/50)} = 113.4$

EXAMPLE 6.10.1 in ASD: Solution Contd....



TRIAL-1

Assume $KL/r = 90$, $\therefore$ $r = KL/90 = 192/90 = 2.133$ in.
$F_e = \pi^2 E/(KL/r)^2 = 3.14^2 \times 29000/(90)^2 = 35.33$ ksi.

$F_{cr} = \left[0.658^{\frac{F_y}{F_e}}\right] F_y = [0.658^{(50/35.33)}]50 = 27.65$ ksi For $\frac{KL}{r} \le 4.71\sqrt{\frac{E}{F_y}}$

Nominal strength $P_n = \Omega P = 1.67 \times 240 = 400.8$ kip
But $P_n = F_{cr} A_g$
$\therefore A_g = P_n/F_{cr} = 400.8/27.63 = 14.5$ in^2.

Now go to W section charts of AISC Manual and find a section having $r \ge 2.133$ and $A_g \ge 14.5$.
From AISC Manual Chart on Pages 1-24 and 1-25,
Select W12x53 with A=15.6 in^2 and r = 2.48 in

TRIAL-2

Assume $KL/r = 80$, $\therefore$ $r = KL/80 = 192/80 = 2.4$ in.
$F_e = \pi^2 E/(KL/r)^2 = 3.14^2 \times 29000/(80)^2 = 44.72$ ksi.

$F_{cr} = \left[0.658^{\frac{F_y}{F_e}}\right] F_y = [0.658^{(50/44.72)}]50 = 31.31$ ksi

But $P_n = F_{cr} A_g$ $\therefore A_g = P_n/F_{cr} = 400.8/31.31 = 12.8$ in^2.

From AISC Manual Chart on Pages 1-24 and 1-25,
Select W10x49 with A=14.4 in^2 and r = 2.54

EXAMPLE 6.10.1 in ASD: Solution Contd….

<u>TRIAL-3</u>

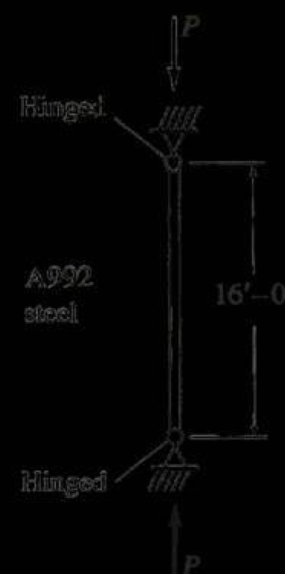
Assume $KL/r = 70$, $\therefore r = KL/70 = 192/70 =$ 2.743 in.
$F_e = \pi^2E/(KL/r)^2 = 3.14^2 \times 29000/(70)^2 = 58.35$ ksi. (> F_y Note)

$$F_{cr} = \left[0.658^{\frac{F_y}{F_e}}\right]F_y = [0.658^{(50/58.35)}]50 = 34.93 \text{ ksi}$$

But $P_n = F_{cr}A_g$ $\therefore A_g = P_n/F_{cr} = 400.8/34.93 =$ 11.45 in^2.

Now go to W section charts of AISC Manual and find a section having $r \geq 2.743$ and $A_g \geq 11.45$

From AISC Manual Chart on Pages 1-22 and 1-23,
Select W12x65 with A=19.1 in^2 and r = 3.02



Based on above three trials, the finally chosen section is W10x49

Structural Connections

Chapter 14

Mechanics of Materials

E P Popov (2nd ed)

Types of Connectors

Components which make up the complete structure are fastened together by means of:

1. RIVETS (older version)
2. BOLTS (newer version)
3. WELDS

Round Head Rivet

Countersunk Rivet

Modern Riveting Gun

Typical High Strength Bolt

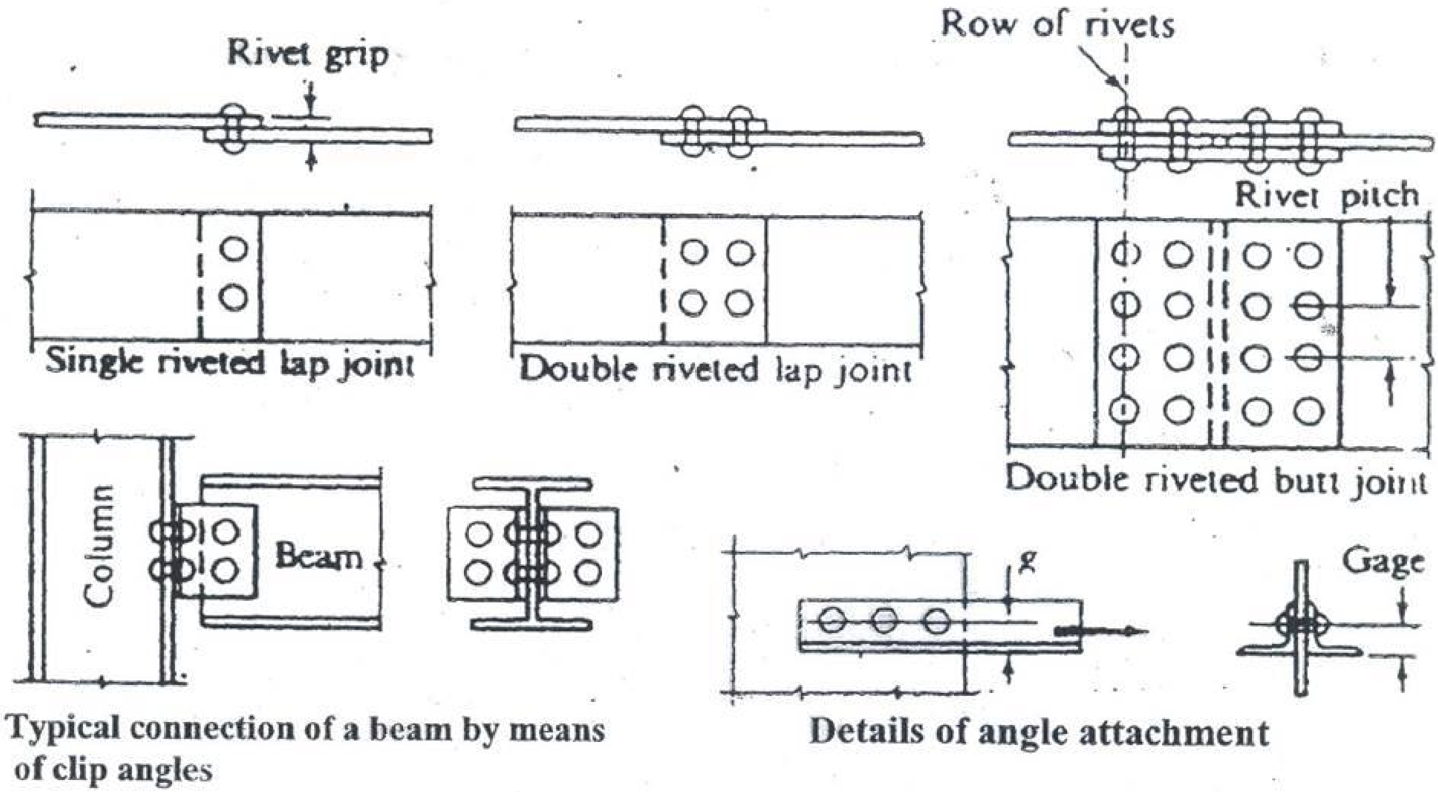
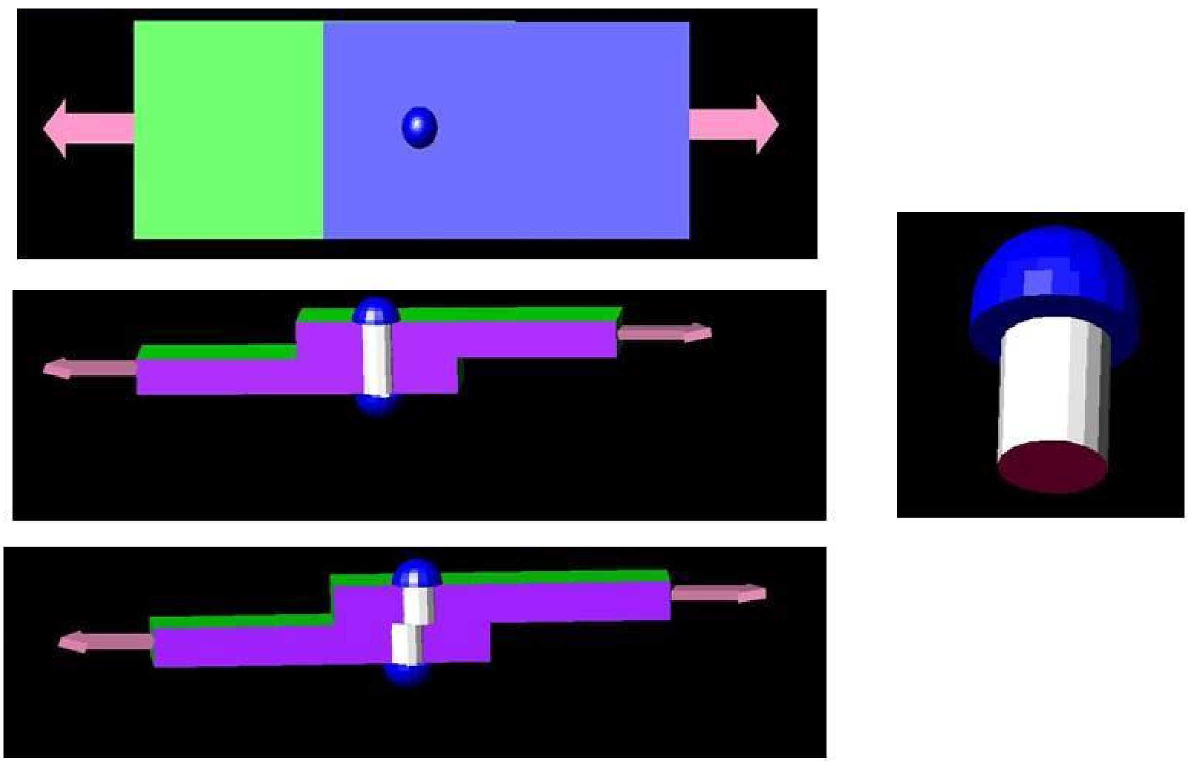


Fig. 14-2. Typical arrangements of riveted connections

Type of connection Failures

1. Shearing Failure of Rivet/Bolts.
2. Bearing Failure of plate.
3. Tearing failure of plate.

1. Shearing failure of bolts



Failure in Shear

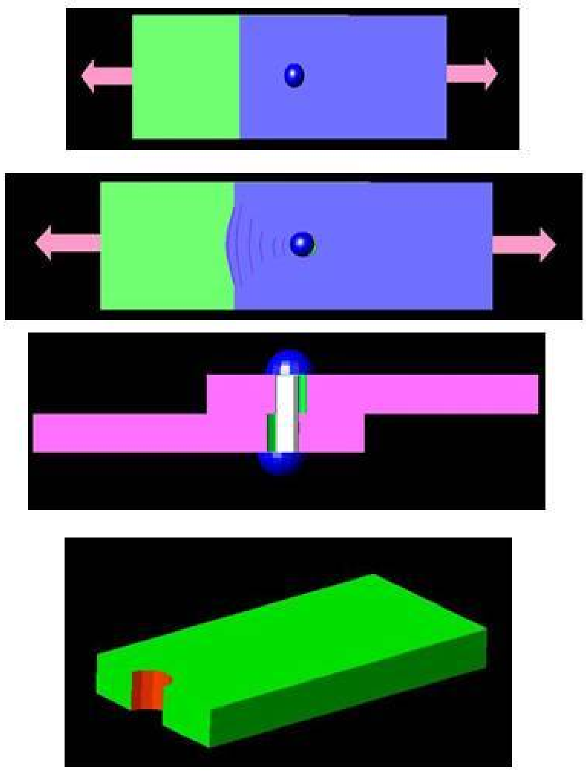

(a) (b)

Fig. 14-5. (a) Single and (b) double shear of a rivet

In a riveted joint, the rivets themselves may fail in shear. This type of failure is **shown in Figs. 14-5(a) and (b).** In analyzing this possible manner of failure, one must always note whether a **rivet acts in single or double shear.**

In the latter case, ***two* cross-sectional areas** of the same rivet resist the **applied force**. In practical calculations, the shearing **stress** is assumed to be ***uniformly distributed*** *over the* ***cross section*** *of* ***a rivet***. This assumption is justified for **ductile materials** after the **elastic limit** has been **exceeded**.

2. Bearing Failure of Plate



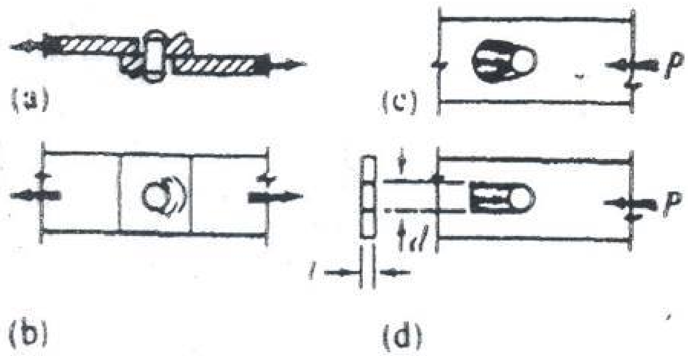


Fig 14-6. A bearing failure of a riveted joint and bearing stress distributions, (c) probably, (d) assumed

Failure in Bearing

A riveted joint-may fail if a rivet crushes the material of the plate against which it bears. Figs. **14-6(a)** and **(b)**, or if **the rivet itself is deformed** by the plate acting on it. The stress distribution is very **complicated in this type of failure** and is somewhat like that shown in **Fig. 14-6(c)**. In practice, this stress distribution .is approximated on the basis of an ***average: bearing: stress acting over the. projected area of the rivets shank onto a plate,* i.e., on area *td* in**

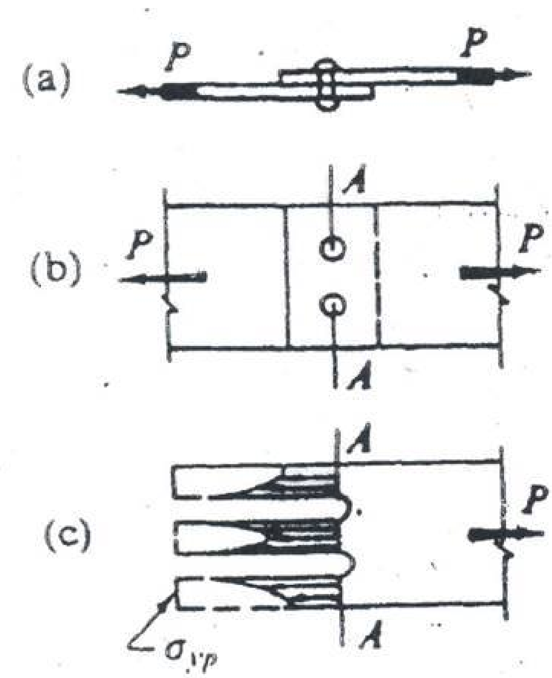


Fig. 14-4. Tensile stresses in a plate of a riveted joint

Failure in Tension

- A riveted joint transmitting a tensile force may fail in a plate weakened by the rivet holes. For example, in a single-riveted lap joint, Fig. 14-4(a), the *net* area in either plate across the section *A-A* in Fig. 14-4(b) is the least area, and a tear would occur there. At working loads, before failure occurs, large stress concentrations exist at the rivet holes in a plate. Fig. 14-4(c), since the holes interrupt the continuity of the plate. However, a nearly uniform stress distribution will prevail at the yield point for ductile materials (Art. 2-11). Since riveting is employed only for ductile materials, it is customary to base the capacity of a joint in tension on the assumption of a *uniform stressdistribution across the net section of a plate.*.

4. Other Failure Modes

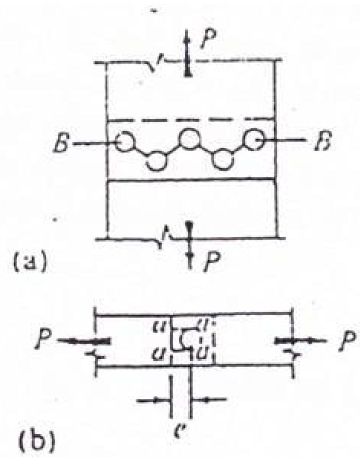
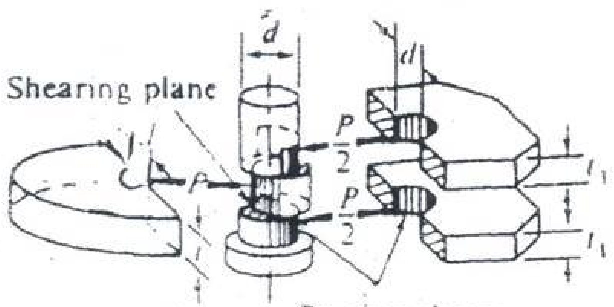


Fig. 14-7. Possible methods of failure of a riveted joint, (a) zig-zag tear, and (b) "tear out" due to insufficient edge (end) distance

Fig. 14-8. Bending of the plates in lap joints, commonly neglected

Some Further Remarks

In this text the capacity of a riveted or ordinary bolted joint will be based only on the probable tear, shear, and bearing capacities. This assumed action



of a riveted joint is illustrated in Fig. 14-9.* The frictional resistance between the plates is neglected. The *smallest* of the three resistances is the strength of a joint. The ratio of this strength divided by the strength of a solid plate or member, expressed in percent, is called the *efficiency of a joint,* i.e.,

$$\textit{efficiency} = \frac{\textit{strength of the joint}}{\textit{strength of the solid member}} \times 100 \tag{14-1}$$

EXAMPLE 14--I

Find the capacity of the tension member ***AB*** of the Fink truss shown in Fig. 14-10(a) if it is made from **two 76 x 51 x 7.9 mm angles** attached to a **10mm thick gusset plate by four 19mm** A502-1 rivets in **21 mm diameter holes**. The allowable stresses (AISC) are **150MPa** in tension, **100** MPa in shear, and **335** MPa in bearing on the angles as well as the gusset.

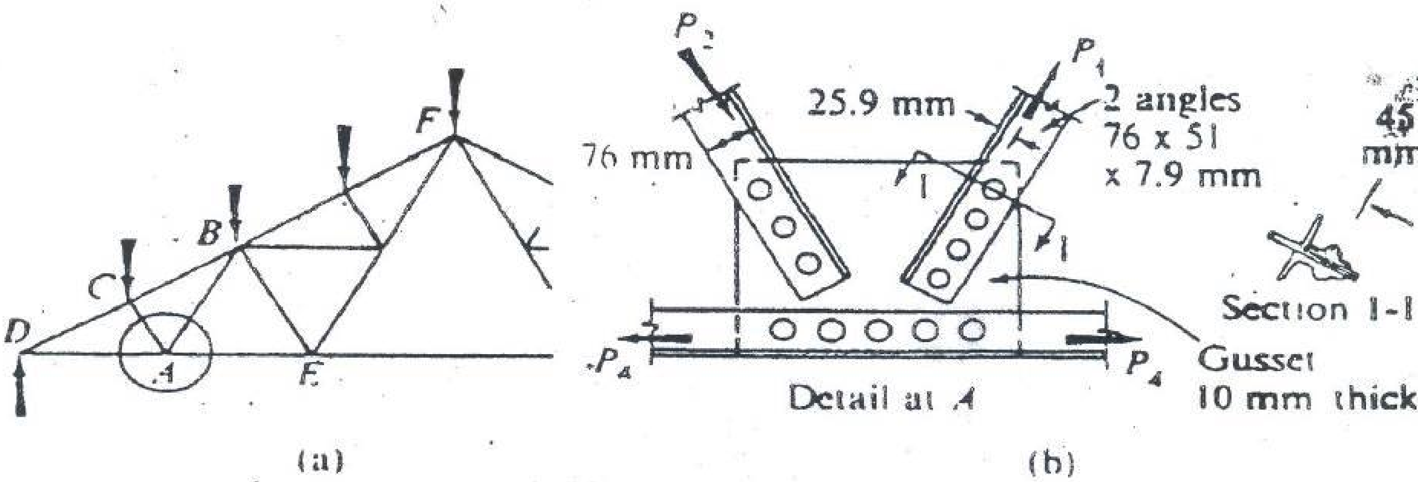


Fig. 14-10

Tear at Section 1-1:

$$A_{net} = 2[942 - \underbrace{7.9(19 + 3)}_{121}] = 1536 \text{ mm}^2$$

$$P_{allow} = A_{net}F_t = (1536)(150)10^{-3} = 230 \text{ kN}$$

Bearing on angles:

$$A = 8td = 8(7.9)(19) = 1200 \text{ mm}^2$$

$$P_{allow} = AF_p = (1200)(335)10^{-3} = 402 \text{ kN}$$

Bearing on gusset:

$$A = 4t_1d = 4(10)(19) = 760 \text{ mm}^2$$

$$P_{allow} = AF_p = (760)(335)10^{-3} = 255 \text{ kN}$$

Shear of rivets:

$$A_{one\ rivet} = \tfrac{1}{4}\pi(19)^2 = 283.5 \text{ mm}^2$$

$$P_{allow} = 8A_{one\ rivet}F_v = 8(283.5)(100)10^{-3} = 227 \text{ kN} \quad \text{(governs)}$$

EXAMPLE 14-3

Find the capacity of the standard AISC connection for the W 310 x 54 beam shown in Fig. 14-11. The connection consists of two 102 x 89 x 9.5 mm angles, each 215 mm long; 22 mm A502-1 rivets spaced 75 mm apart are used in 24mm holes. Use the AISC allowable stresses given in Example 14-1.

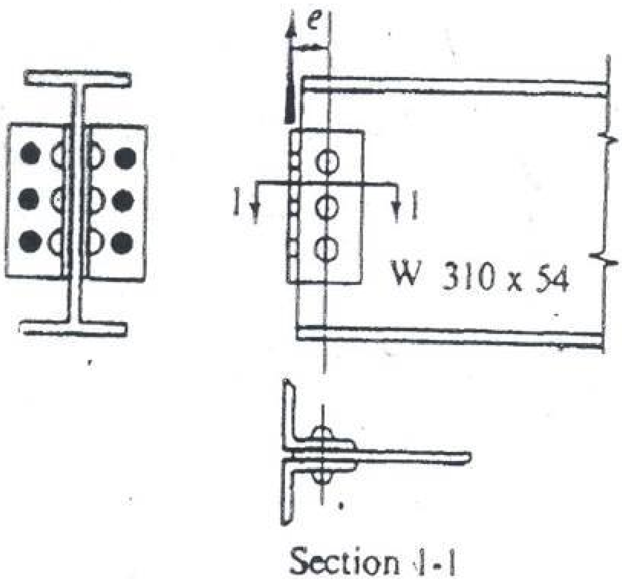


FIG 14-11

SOLUTION

A tension tear cannot occur in this connection, so only shearing and bearing capacities need be investigated. *Bearing on the web of W 310 x 54. beam:* (Thickness of the web is found from Table 4 of the Appendix.)

Bearing of Rivets on Beam: $P_b = 3(7.75 X 22)(335)10^{-3} = 171$ kN (governs)

Shear of rivets: There **are six** cross-sectional areas.

$$A_{\text{one rivet}} = 380 \text{ mm}^2$$

$$P_s = (6)(380)(100)10^{-3} = 228 \text{ kN}$$

Bearing of Rivets on Angles:

$$P_b = 6(\underset{d}{22})(\underset{t}{9.5})(335)10^{-3} = 420 \text{ kN}$$

Hence the capacity **of this** connection is **171 kilonewtons**.

EXAMPLE 14-5

Find the allowable tensile force that the multiple-riveted structural joint shown in Figs. 14-12(a) and (b) can transmit. Also find the efficiency of this joint. All rivets are nominally 22 mm in 25 mm holes. Use AISC allowable stresses given in example 14-1.

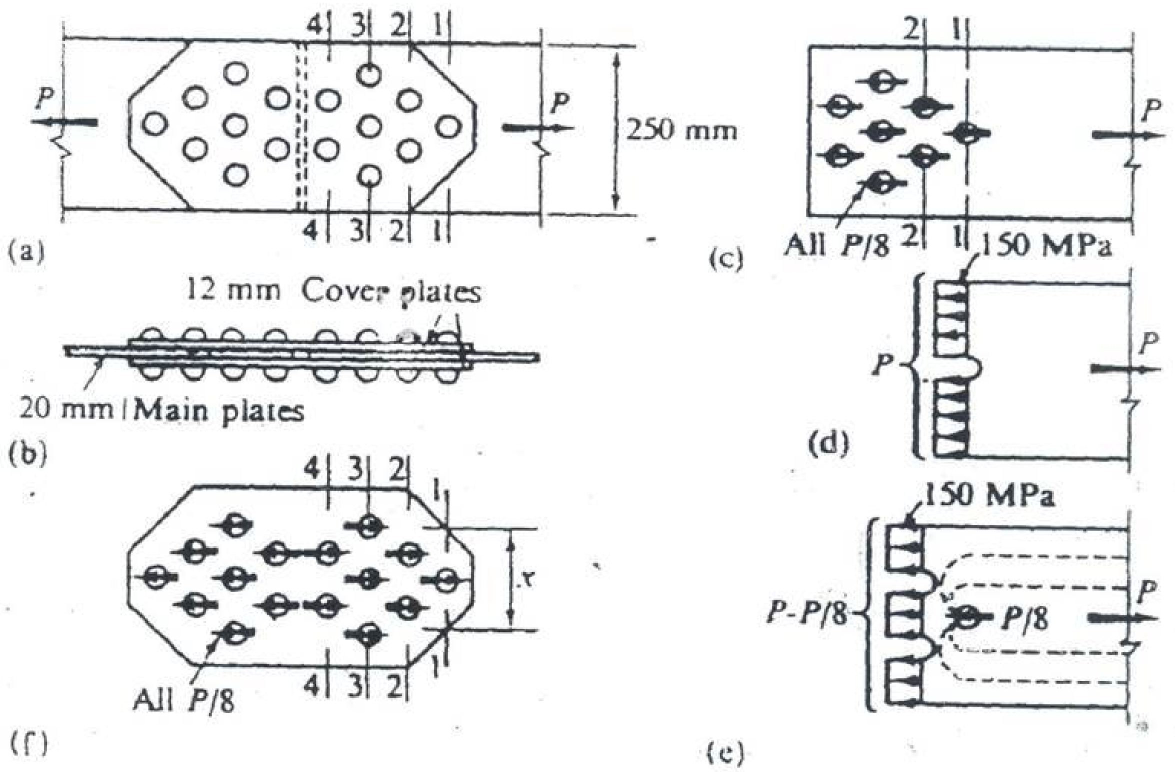


Fig. 14-12

Capacity P_s *of the joint in shear:* 16 cross-sectional areas of rivets are available. $F_v = 100$ MPa.

$A_{\text{one rivet}} = \pi/4\ (22)^2 = 380\ \text{mm}^2$

$P_s = 16(380)(100)\ 10^{-3} = 608$ kN (governs)

Capacity P_B ***of the joint in bearing on the main plate:*** 8 bearing surfaces in double shear are available, $F_p = 335$ MPa.

$P_B = 8(22)(20)(335)10^{-3} = 1180$ kN

Capacity P_b ***of the joint in bearing on the cover plates:*** 16 bearing surfaces in single shear. $F_p = 335$ MPa.

$P_b = 16(22)(12)(335)10^{-3} = 1420$ kN

Capacity of the main plate in tension:

F_t = 150 MPa

Without holes:

$$\mathbf{P}_t = (20)(250)(150)10^{-3} = \mathbf{750}\ \text{kN}$$

Section 1-1, frec body. Fig. 14-12(d)

$$\mathbf{P}_{1\text{-}1} = (20)[250 - (22+3)](150)10^{-3} = \mathbf{675}\ \text{kN}$$

Section 2-2, free body. Fig. 14-12(c),

$$\mathbf{P}_{2\text{-}2} = 1/8\ P_{2\text{-}2} + (20)[250 - 2(22+3)](150)10^{-3} \quad \text{or} \quad 7/8\ P_{2\text{-}2} = 600\ \text{kN}$$

hence $\mathbf{P_{2\text{-}2} = 686}$ kN.

This result means that, if a tensile force $P_{2\text{-}2}$ --- 686 kN were applied to the joint, only seven-eighths of this force need be resisted by section 2-2 at a 150 MPa stress, since one-eighth of this force is resisted by the *outer rivet*. Similarly,

$$\frac{5}{8}\ P_{3\text{-}3} = (20)(250 - 75)(150)10^{-3} \qquad \text{or } P_{3\text{-}3} = 840\ \text{kN}$$

and $\frac{2}{8} P_{4\text{-}4} = (20)(250 - 50)(150)10^{-3}$ or $P_{4\text{-}4} = 2400$ kN

Capacity of the **two cover plates in tension:** F_t = 150 MPa. *Across section 4-4, Fig. 14-12(f) the whole force P* is transmitted, hence,

$$\mathbf{P_{4\text{-}4}} = 2(12)(250 - 2(22+3)](150)10^{-3} = 720 \text{ kN}$$

Then, by *considering* free-body diagrams of the cover plates to one side of a section, as was done for the main plate,

$$6/8\ \mathbf{P_{3\text{-}3}} = (24X250 - 75X150)10^{-3} \quad \text{or } \mathbf{P_{3\text{-}3}} = 840 \text{ kN}$$

and $$3/8\ \mathbf{P_{2\text{-}2}} = (24X250 - 50X150)10^{-3} \quad \text{or } \mathbf{P_{2\text{-}2}} = 1920 \text{ kN}$$

Similarly, section 1-1 is not critical. Therefore the width of the *cover plates* at section 1-1 may be reduced to a width *x to* provide, at an allowable stress, a net area for, only one-eighth of the applied force.

The capacity of this joint is limited by the allowable shearing stress in the rivets. Hence the capacity of this joint is 608 kN. The efficiency of this joint is (608/750)100 = 81.1%, which **is** good.

14-4. ECCENTRIC RIVETED AND BOLTED CONNECTIONS

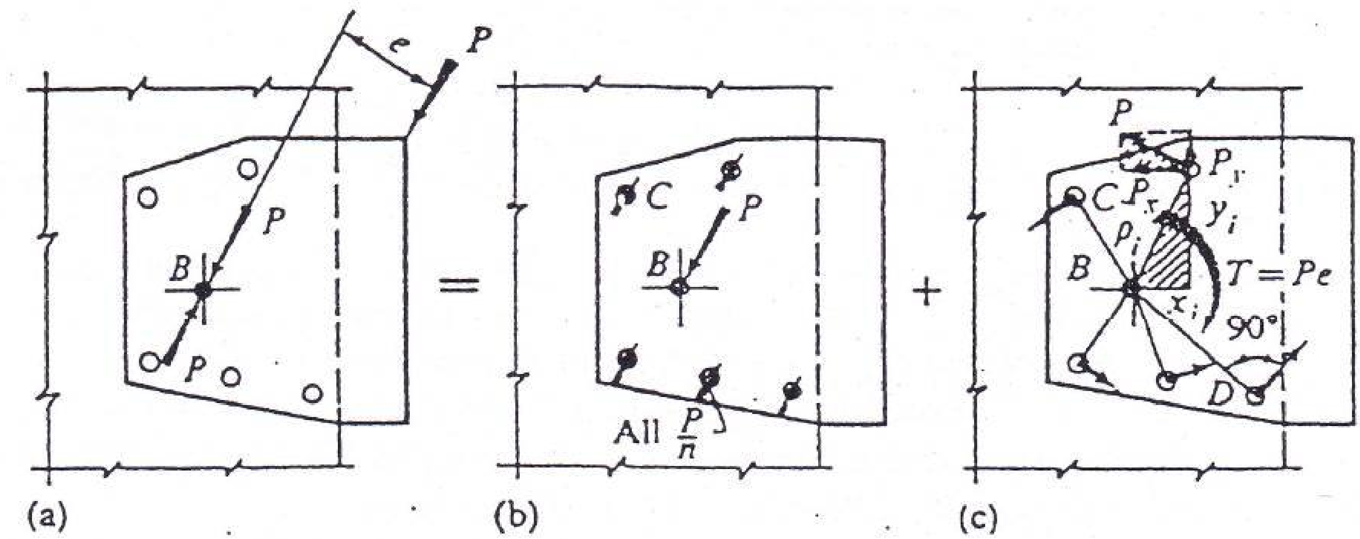


Fig. 14-14. Resolution of a problem of an eccentrically loaded riveted connection into two problems.

The new problem can be conveniently resolved into the two problems shown in Figs. 14-14(b) and (c). The first problem. Fig. 14-14(b), was discussed earlier, and the average *direct* shearing stress f_d in *all* rivets is

$$f_d = \frac{P}{\sum A_i}$$

(14-2) where $\sum A_i$, is the sum of all the cross-sectional areas of the rivets. If all *n* rivets are of equal size, $\sum A_i == nA$, where *A* is the cross-sectional area of one rivet, and.

$$f_d = \frac{P}{nA} \quad \text{or} \quad P_d = Af_d = \frac{P}{n} \qquad (14\text{-}2a)$$

where P_d is the direct force in each rivet. For equilibrium, f_d and P_d act in a direction opposite to the force *P* applied at *B*.

the torsion formula $\tau = T\rho/I_p$, may be adopted for its solution. *[I_p=J (polar moment of Inertia)]*

$$J = I_p \approx \Sigma \rho_i^2 A_i$$

$$\text{Since } \rho_i^2 = x_i^2 + y_i^2, \text{ so, } J = I_p \approx \Sigma (x_i^2 + y_i^2) A_i$$

where x_i and y_i are the co-ordinates of a particular rivet's centre from the centroid of all the rivet areas.

By using the approximation for I_p *(J)* established above, the *torsional* shearing stress f_t, on any one rivet at a distance ρ_i, from the centroid of all **rivet** areas becomes

$$(f_t)_i = \frac{T\rho_i}{I_p} \approx \frac{T\rho_i}{\sum \rho_i^2 A_i} = \frac{T\rho_i}{\sum (x_i^2 + y_i^2) A_i} \qquad (14\text{-}3)$$

whereas, if all rivets are of equal size,

$$(f_t)_i = \frac{T\rho_i}{A\sum (x_i^2 + y_i^2)} \qquad \text{or} \qquad (P_t)_i = A(f_t)_i = \frac{T\rho_i}{\sum (x_i^2 + y_i^2)} \qquad (14\text{-}3a)$$

the centroid of all rivet areas. *Either* $(f_t)_i$. *or* $(P_t)_i$, *acts perpendicular to the direction of* ρ_i. Further, by noting the similarity of the triangle with sides x_i, y_i, and ρ_i, to the triangle of force at a rivet and its components, shown shaded in Fig. 14-14(c), it follows that the *components* of the torsional stresses in the *x*- and y-directions, respectively, are

$$(f_{tx})_i = \frac{Ty_i}{\sum (x_i^2 + y_i^2) A_i} \qquad \text{and} \qquad (f_{ty})_i = \frac{Tx_i}{\sum (x_i^2 + y_i^2) A_i} \qquad (14\text{-}3b)$$

EXAMPLE 14-6

Find the maximum shearing stress caused by an inclined force $P = 50$ kN in the rivets of the connection shown in Fig. 14-15(a). The rivets are of 25 mm diameter $(A = 491\ \text{mm}^2)$. All dimensions are in mm.

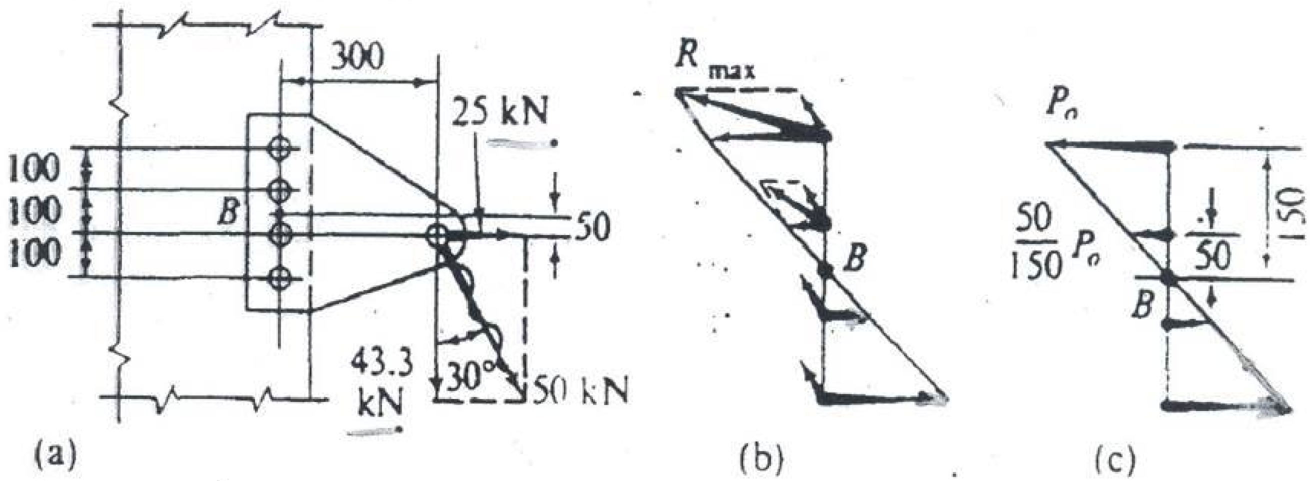


Fig. 14-15

$$P_x = 50 \sin 30^\circ = 25 \text{ kN} \rightarrow, \; P_y = 50 \cos 30^\circ = 43.3 \text{ kN}\downarrow$$

$$T = (43.3)(300) - (25)(50) = 11\,740 \text{ kN}\cdot\text{mm}$$

$$f_{dy} = \frac{P_y}{nA} = \frac{43\,300}{4(491)} = 22.0 \text{ MPa, resisting} \uparrow$$

$$f_{dx} = \frac{P_x}{nA} = \frac{25\,000}{4(491)} = 12.7 \text{ MPa, resisting} \leftarrow$$

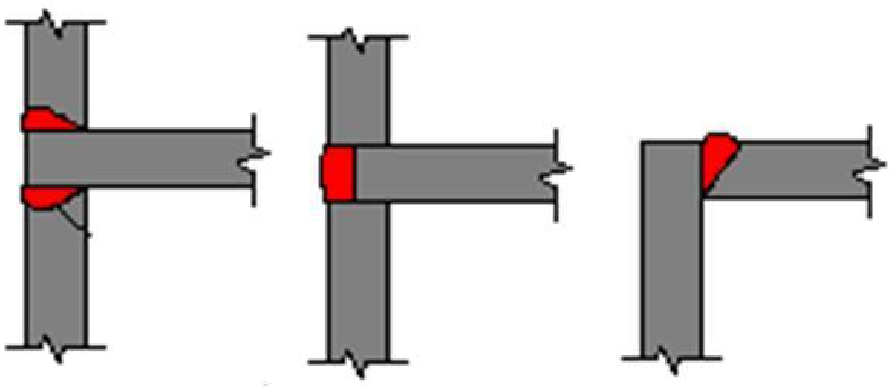
$$f_{tx} = f_t = \frac{Tc}{A \sum (x^2 + y^2)} = \frac{(11\,740 \times 10^3)(150)}{(491)(150^2 + 50^2)2} = 71.7 \text{ MPa} \leftarrow$$

$$f_{max} = \sqrt{(12.7 + 71.7)^2 + 22.0^2} = 87.2 \text{ MPa}$$

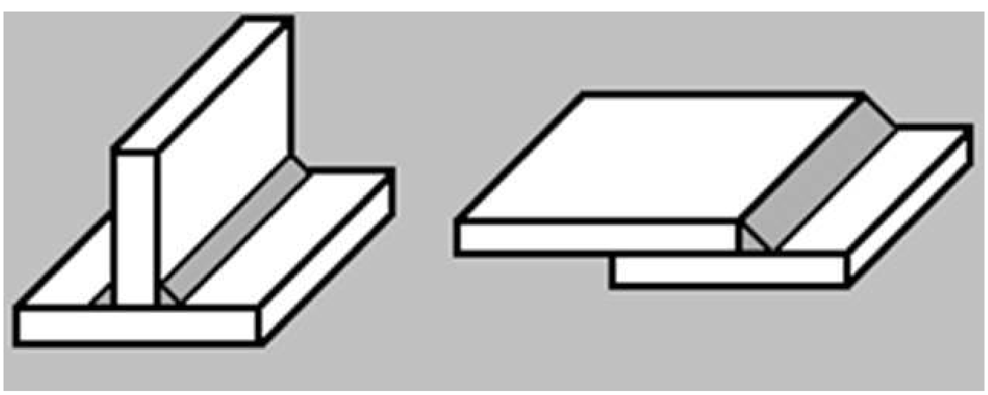
Types of Welds

Welds are classified according to their shape and method of deposition into:

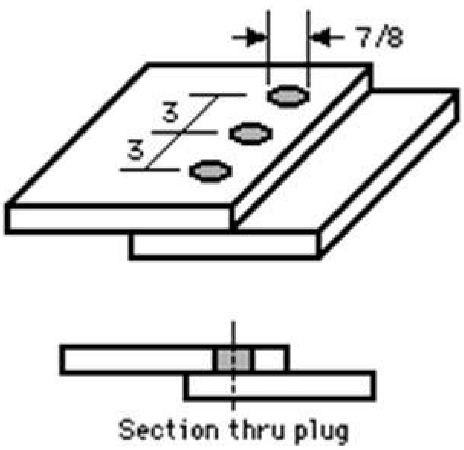
1. Groove Weld
2. Fillet Weld
3. Plug Weld
4. Slot Weld



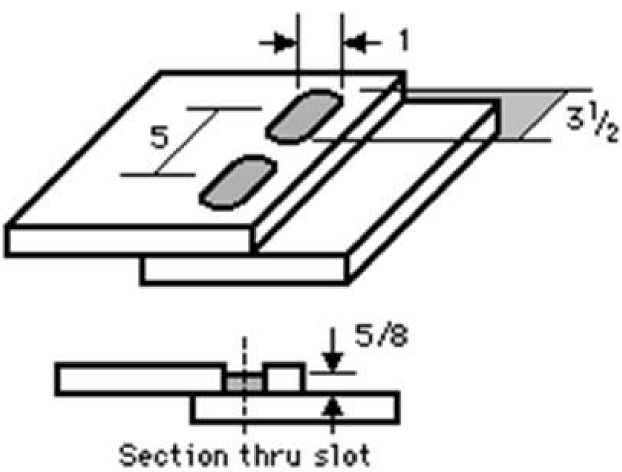
1. Groove Weld is made in opening between two parts being joined.



2. Fillet Weld triangular in shape, joins surfaces which are at an angle with one another.



3. Plug Weld is made by depositing weld metal in a circular hole in one of two lapped places.

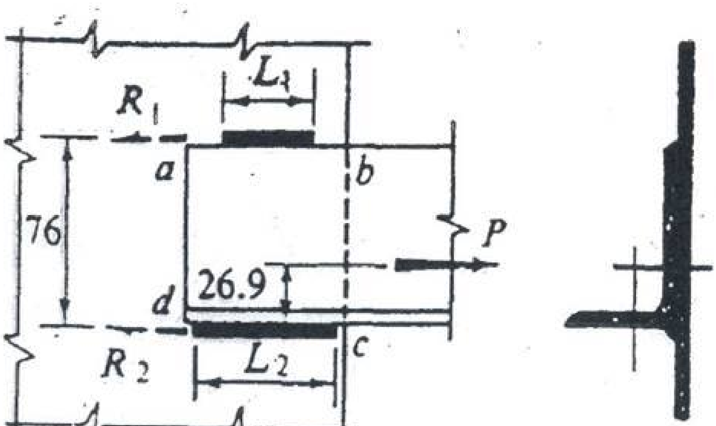


4. Slot Weld similar to plug but the hole is elongated.

The strength of a fillet weld, ***regardless of the direction of the applied force**** is based on the **cross-sectional area at the *throat* multiplied by the allowable *shearing stress*** for the weld metal. The **AWS allowable shear stress** is **0.3 times the electrode tensile strength**. For example, **E70 electrodes** (i.e., tensile strength of **483** MPa) used as weld metal have an allowable shear stress of 0.3 x 483 = **145** MPa. **The allowable force *q* per mm of the weld is then given as**

$$q = (145)(0.707)\, W = 102\, W \quad [\text{N/mm}] \qquad (14\text{-}4)$$

$$q = \frac{0.3 f_y}{\sqrt{2}} W \ /\text{unit length}$$



(a) (b)

EXAMPLE 14-7

Determine the proper lengths of welds for the connection of a 76 x 51 x 11.1 mm steel angle to a steel plate, Fig. 14-17. The connection is to develop the full strength in the angle uniformly stressed to 140 MPa. Use 10 mm fillet welds, whose strength per AWS specification is 1020 Newton per linear millimeter.

Hence, using the specified value for the strength of the 10 mm weld

$$A_{angle} = 1290 \text{ mm}^2 \qquad \sigma_{allow} = 140 \text{ MPa}$$

$$P = A\sigma_{allow} = 1290(140) = 181 \times 10^3 \text{ N}$$

$$\sum M_d = 0 \circlearrowleft +, \quad R_1(76) - 181(26.9) \qquad R_1 = 64.1 \text{ kN}$$

$$\sum M_a = 0 \circlearrowleft +, \quad R_2(76) - 181(76 - 26.9) = 0, \quad R_2 = 116.9 \text{ kN}$$

Check: $R_1 + R_2 = 64.1 + 116.9 = 181 \text{ kN} = P$

note that, = (64.1)/(1.02) = 62.8 mm and L_2 = (116.9)/(1.02) = 114.6mm.

14-6. ECCENTRIC WELDED CONNECTIONS

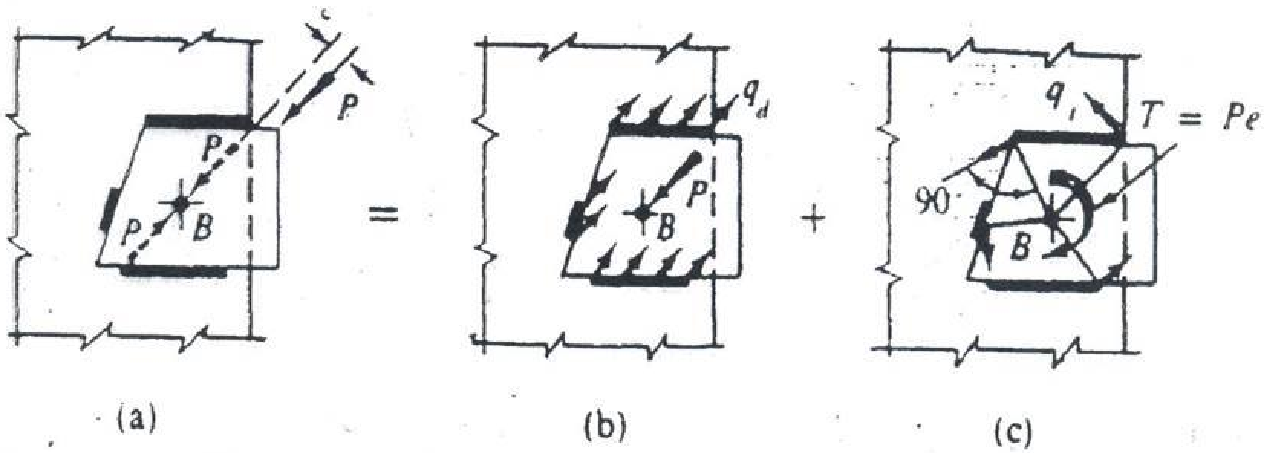


Fig. 14-18. Resolution of a problem of an eccentncally loaded welded connection into two sèparate problems

Fig. 14-18(b), is a concentrically loaded connection where the applied force ***P*** acts through the centroid ***B*** of all welds. For this case, the *direct* force q_d per mm length of weld acting in the direction opposite to *P* is

$$q_d = \frac{P}{\sum L_i} \quad \text{[N/mm]} \qquad (14\text{-}5)$$

where, $\sum L_i$ is the ***total length*** of all welds. If the force *P* is resolved into the horizontal and vertical components P_x, and P_y respectively, the components of the direct force per mm of weld are

$$q_{dx} = \frac{P_x}{\sum L_i} \quad \text{and} \quad q_{dy} = \frac{P_y}{\sum L_i} \qquad (14\text{-}5a)$$

The second problem, Fig. 14-18(c), is analogous to the torsion problem, if the plate is assumed to be *rigid* and to twist around the point *B*. Then, by further assuming elastic action of the welds, the torsion formula, with a modified value *of* I_p may be applied.

$$I_p = \sum I_{pi} = \sum\left(\frac{L_i^3}{12} + L_i\bar{x}_i^2 + L_i\bar{y}_i^2\right)$$

Using the above equivalent value of I_p, in the torsion formula, the *torsional* force q_t, per mm of weld is

$$q_t = \frac{T\rho}{\sum\left(\frac{L_i^3}{12} + L_i\bar{x}_i^2 + L_i\bar{y}_i^2\right)} \qquad \text{[N/mm]} \qquad (14\text{-}6)$$

where ρ is the distance from the centroid of all welds to *a particular' point on any weld*. The torsional force q_t, acts perpendicular to the radius vector ρ. The horizontal and vertical *components* of q_t, in a manner analogous to that employed in the analysis of riveted connections, may be shown to be, respectively,

$$q_{tx} = \frac{Ty}{\sum\left(\frac{L_i^3}{12} + L_i\bar{x}_i^2 + L_i\bar{y}_i^2\right)}$$

$$q_{ty} = \frac{Tx}{\sum\left(\frac{L_i^3}{12} + L_i\bar{x}_i^2 + L_i\bar{y}_i^2\right)} \qquad (14\text{-}6a)$$

EXAMPLE 14-8

Find the size of the two welds required to attach a plate to a machine as shown in Fig. 14-20(a) if the plate carries an inclined force $P = 50$ kN. Use the stresses allowed by the AWS. All dimensions shown are in mm.

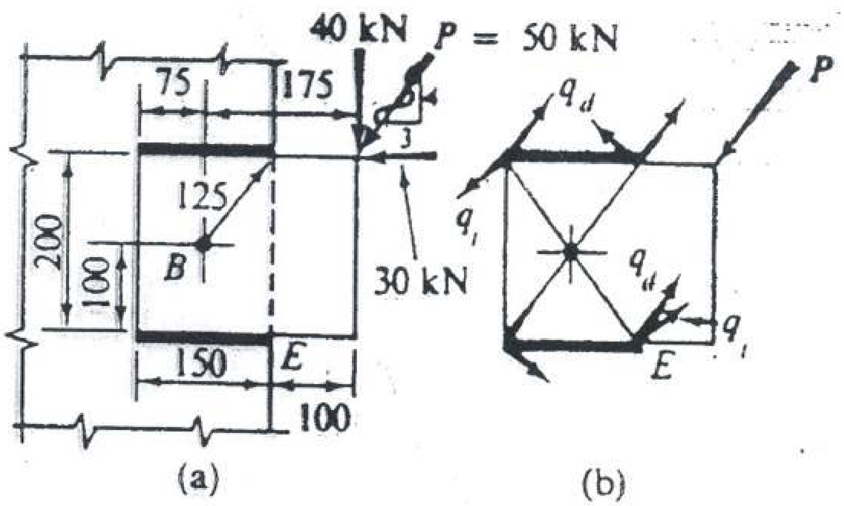


Fig. 14-20

$$P_x = \tfrac{3}{5}(50) = 30 \text{ kN} \qquad P_y = \tfrac{4}{5}(50) = 40 \text{ kN}$$

$$T = 40(175) - 30(100) = 4000 \text{ kN}\cdot\text{mm}$$

$$I_p = 2\left(\frac{L^3}{12} + L\bar{y}^2\right) = 2\left[\frac{150^3}{12} + 150(100)^2\right] = 3.56 \times 10^6 \text{ mm}^3$$

$$(q_{dx})_E = \frac{P_x}{\sum L_i} = \frac{30\,000}{150 + 150} = 100 \text{ N/mm} \rightarrow$$

$$(q_{dy})_E = \frac{P_y}{\sum L_i} = \frac{40\,000}{300} = 133 \text{ N/mm}\uparrow$$

$$(q_{tx})_E = = \frac{Ty_E}{I_p} = \frac{(4000)(10^3)(100)}{(3.56)(10^6)} = 112 \text{ N/mm} \rightarrow$$

$$(q_{ty})_E = \frac{Tx}{I_p} = \frac{(4000)(10^3)(75)}{(3.56)(10^6)} = 84 \text{ N/mm}\uparrow$$

$$q_{max} = q_E = \sqrt{(100 + 112)^2 + (133 + 84)^2} = 303 \text{ N/mm}$$

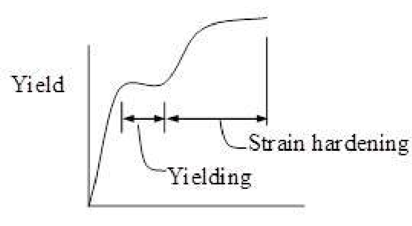
Finally, since by Eq. 14-4 the allowable force per millimcter of weld, regardless of the direction of the applied force, is 101.8 *W*, where *W* is the width of the leg,

$q = 101.8\ W = 303$ N/mm or $W = 2.98$ mm.

THEORIES OF FAILURE

Failure refers to any action leading to an inability on the part of the structure or machine to function in the manner intended. Thus permanent deformation, fracture of even excessive linear elastic deflection may be regarded as ***Modes of Failure***.

One of the most important factors in regard to influencing the threshold of failure is the rate at which the load is applied. Loading at high rate may load to a variety of adverse phenomena associated with impact, acceleration and vibration, with the concomitant high levels of stress and strain so well as rapid stress reversal.



Creep: Under certain circumstances, deformation may continue with time while the load remains constant. This deformation beyond that experienced as the material is initially loaded is termed ***CREEP***. Creep strength refers to the maximum employable strength of material at a prescribed elevated temperature.

Fracture: Separation of a material under stress into two or more parts (i.e. formation of new surface) is referred as fracture. Griffith equated strain energy associated with material failure to that required for the formation of new surface and concluded that with respect to its capacity to cause failure, tensile stress represents a more important inference than does compressive.

Fatigue: Structural members subjected to repeated, fluctuating or alternating stresses that are below the Ultimate strength or Yield Strength may show diminished strength and ductility. This response is termed as ***Fatigue***.

When the state or stress is not uniaxial, the theories of failure attempts to predict yielding by limiting certain parameters (such as strain, stress, energy) to those in a simple tension test. This is necessary as tests can't be performed for all state of stresses.

Yielding Theories of Failure

- ❖ Maximum principal stress theory (Rankin's theory)
- ❖ Maximum shear stress theory (Coulomb's theory)
- ❖ Maximum principal strain theory (Saint Venant's theory)
- ❖ Maximum energy of distortion theory (Von Mises theory)
- ❖ Maximum octahedral shear stress theory (Von Mises theory)
- ❖ Internal friction theory
- ❖ Mohr's theory

MAXIMUM PRINCIPAL STRESS THEORY (Rankin's theory)

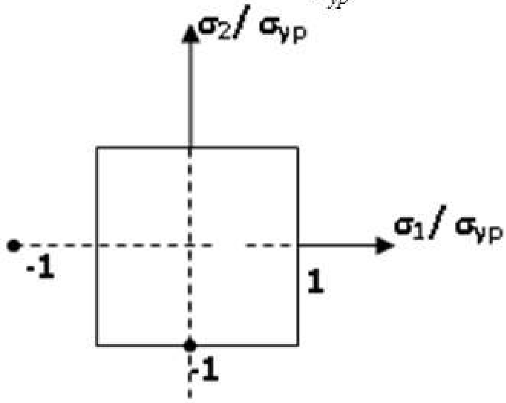
According to this theory (Rankin's theory), a material fails by yielding when the maximum principal stress exceeds the tensile yield strength or minimum principal stress exceeds the compressive yield strength. Thus at the onset of yielding: $|\sigma_1| = \sigma'_{yp}$ *or* $|\sigma_2| = \sigma''_{yp}$

For materials possessing the same yield stress in tension and compression

$$\sigma'_{yp} = \sigma''_{yp} = \sigma_{yp}.$$

and in case of plane stress ; $\sigma_3 = 0$

Thus: $\dfrac{\sigma_1}{\sigma_{yp}} = \pm 1$ *or* $\dfrac{\sigma_2}{\sigma_{yp}} = \pm 1$



Limitations:

(1) Material may have different tensile and compressive strength.

(2) Failure in ductile material is fundamentally shearing phenomenon, and failure criteria of such materials should be more reliably obtained from shearing rather than tensile stress.

MAXIMUM SHEARING STRESS THEORY (Coulomb's theory)

This theory was originally proposed by ***Coulomb*** presently known as ***Tresca*** theory. According to this theory, in a ductile material slipping occurs during yielding along critically oriented planes. It is assumed that yielding of the material depends only on the maximum shearing stress which is attained within an element. Therefore, whenever a certain critical value τ_{cr} is reached for shear stress, yielding starts.

Since:

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- If $\sigma_x = \pm\sigma_1 \neq 0$ *and* $\sigma_y = \tau_{xy} = 0$

$$\tau_{\max} = \tau_{cr} = \pm\frac{\sigma_1}{2} = \frac{\sigma_{yp}}{2}$$ [Failure criteria in Uniaxial stress]

- For biaxial state of stress the critical value of τ_{cr} is to be used from above. Two causes are to be considered

  (i) σ_1, σ_2 are of same sign; $\sigma_3 = 0$

  (ii) σ_1, σ_2 are of opposite sign; $\sigma_3 = 0$

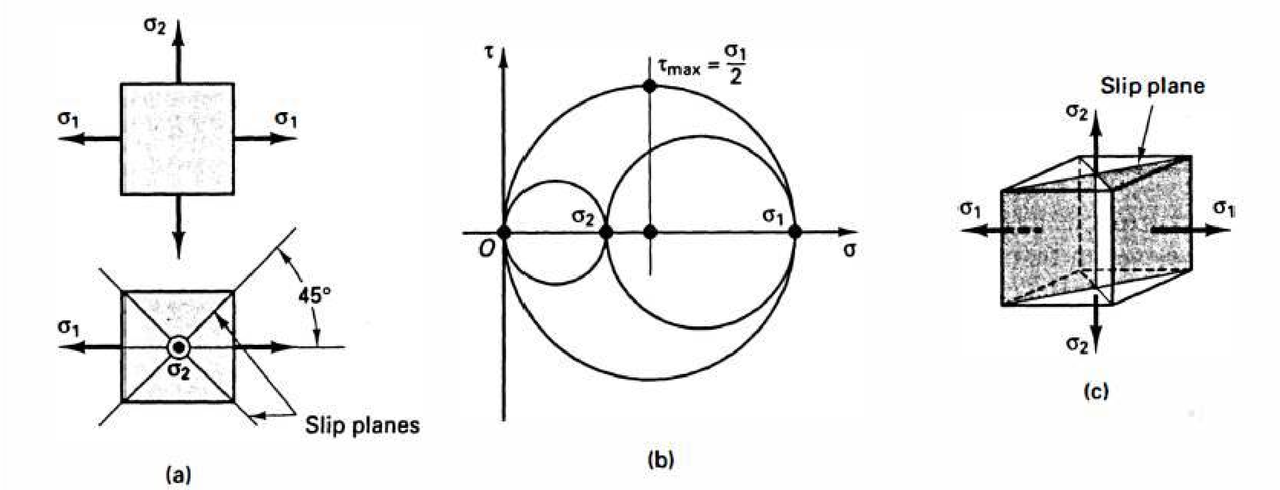
▫For case (i) it can be seen that (using the failure criteria established from uniaxial case):

$$\frac{|\sigma_1|}{2} = \tau_{\max} = \frac{|\sigma_{yp}|}{2} \quad and \quad \frac{|\sigma_2|}{2} = \tau_{\max} = \frac{|\sigma_{yp}|}{2}$$

$$|\sigma_1| \leq \sigma_{yp} \;\; and \;\; |\sigma_2| \leq \sigma_{yp}$$

$$\Rightarrow \frac{\sigma_1}{\sigma_{yp}} = \pm\, 1 \qquad\qquad \Rightarrow \frac{\sigma_2}{\sigma_{yp}} = \pm\, 1$$

[when σ_1, σ_2 are of same sign]



(a) (b) (c)

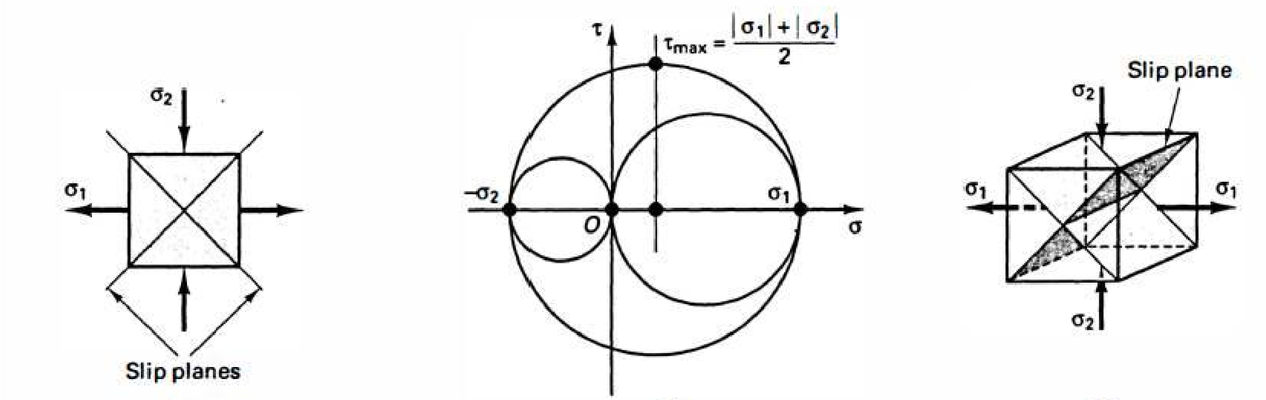
▫For case (ii) it is observed that:

$$\frac{|\sigma_1| + |\sigma_2|}{2} = \tau_{max}$$

$$ie \quad \pm \frac{\sigma_1 - \sigma_2}{2} \leq \frac{\sigma_{yp}}{2}$$

$$\Rightarrow \frac{\sigma_1}{\sigma_{yp}} - \frac{\sigma_2}{\sigma_{yp}} = \pm 1$$

[when σ_1, σ_2 are of opposite sign]



(d) (e) (f)

Above two equations are drawn and known as ***TRESCA*** yield condition. If the points are within the hexagon, material has not yielded. If the points fall on the hexagon then material is yielding. No point can lay out side the hexagon.

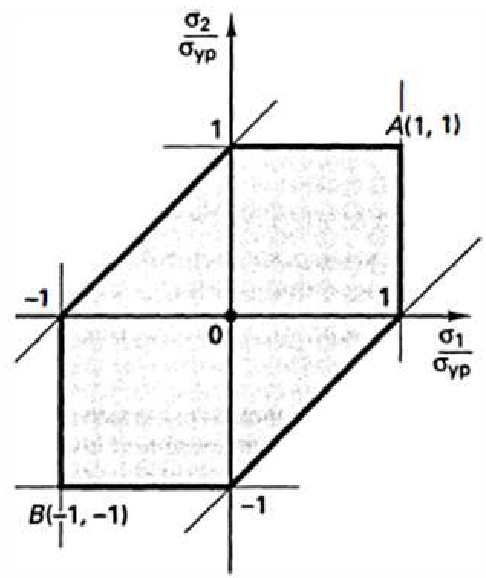


Fig. 12-4 Yield criterion based on maximum shear stress.

MAXIMUM PRINCIPAL STRAIN THEORY (Saint Venant's theory)

According to ***St. Venants*** theory a material fails by yielding when the maximum principal strain exceeds the tensile yield strain ε'_{yp} or the minimum principal strain exceeds the compressive yield strain ε''_{yp}.

From Hook's law:

$$\left|\sigma_1 - \upsilon(\sigma_2 + \sigma_3)\right| = \sigma'_{yp} \qquad \begin{pmatrix} \varepsilon_1 E = \sigma'_{yp} \\ \varepsilon_2 E = \sigma''_{yp} \end{pmatrix}$$

$$\left|\sigma_2 - \upsilon(\sigma_1 + \sigma_3)\right| = \sigma''_{yp}$$

For plane stress $\sigma_3 = 0$

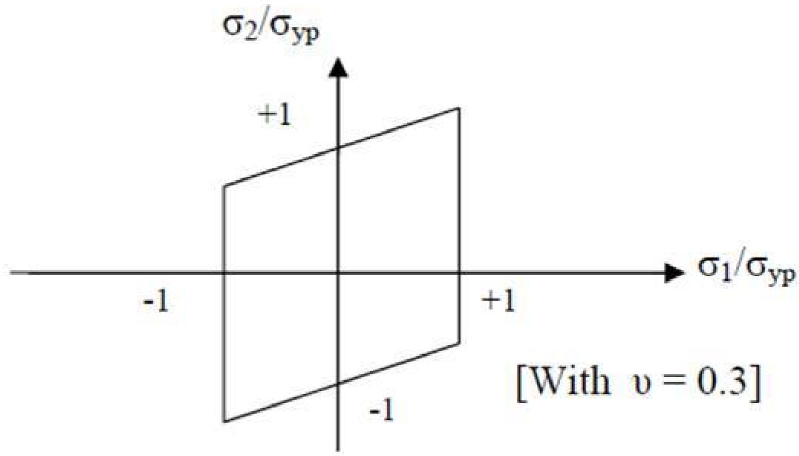
$$\therefore \ \left|\sigma_1 - \upsilon\sigma_2\right| = \sigma'_{yp}$$

$$\left|\sigma_2 - \upsilon\sigma_1\right| = \sigma''_{yp}$$

For $\sigma'_{yp} = \sigma''_{yp} = \sigma_{yp}$

$$\frac{\sigma_1}{\sigma_{yp}} - \upsilon \frac{\sigma_2}{\sigma_{yp}} = \pm 1$$

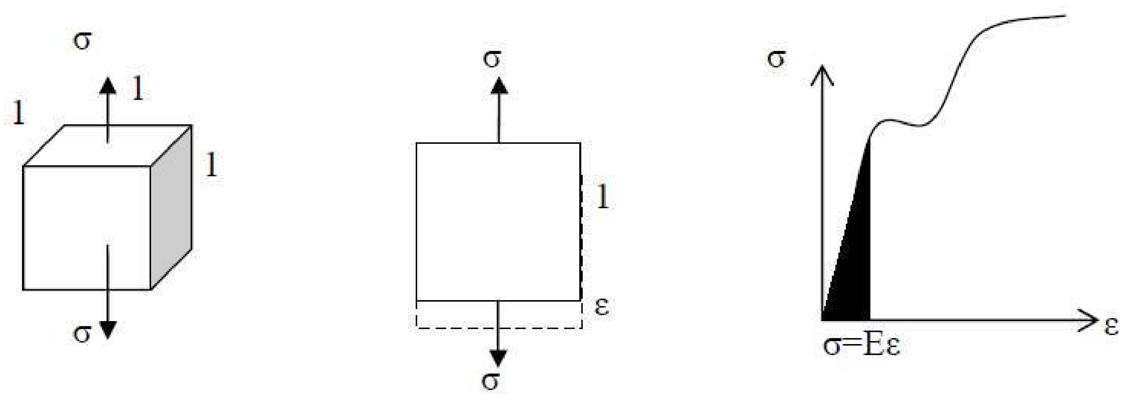
$$\frac{\sigma_2}{\sigma_{yp}} - \upsilon \frac{\sigma_1}{\sigma_{yp}} = \pm 1$$



[With υ = 0.3]

TOTAL STRAIN ENERGY

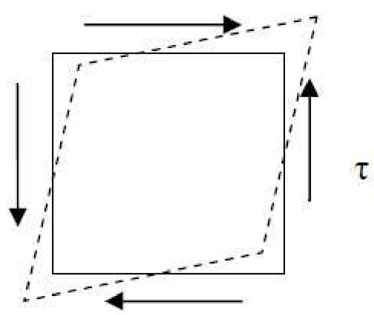
Simple Tension

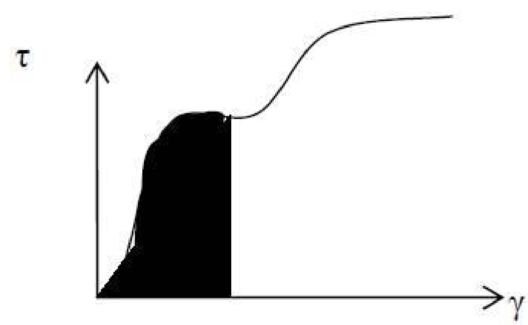


Work done $= \frac{1}{2}\ \sigma\,\varepsilon = \frac{1}{2}\frac{\sigma^2}{E}$

Work done in stressing the material up to proportional limit σ_e is $w = \frac{1}{2}\frac{\sigma_e^2}{E}$

Pure Shear





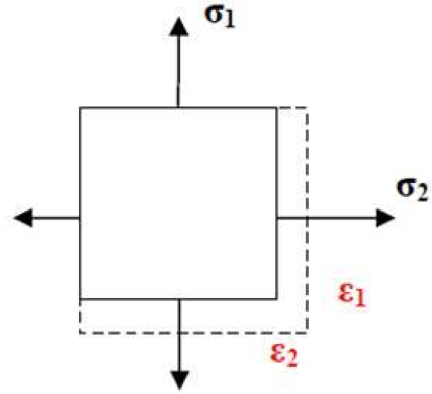
$$w = \frac{1}{2}\ \tau_r\ = \frac{1}{2}\frac{\tau_e^2}{G}$$

One Normal Stress with Sharing Stress

$$w = \frac{1}{2}\ \frac{\sigma^2}{E} + \ \frac{1}{2}\ \ \frac{\tau^2}{G}$$

Two Principle Stress

$$w = \frac{1}{2}\sigma_1 \varepsilon_1 + \frac{1}{2}\sigma_2 \varepsilon_2$$



$$\therefore \quad w = \frac{1}{2}\sigma_1\left(\frac{\sigma_1}{E} - \frac{\upsilon\sigma_2}{E}\right) + \frac{1}{2}\sigma_2\left(\frac{\sigma_2}{E} - \frac{\upsilon\sigma_1}{E}\right)$$

$$= \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\upsilon\sigma_1\sigma_2\right)/E$$

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\upsilon\sigma_2}{E}$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\upsilon\sigma_1}{E}$$

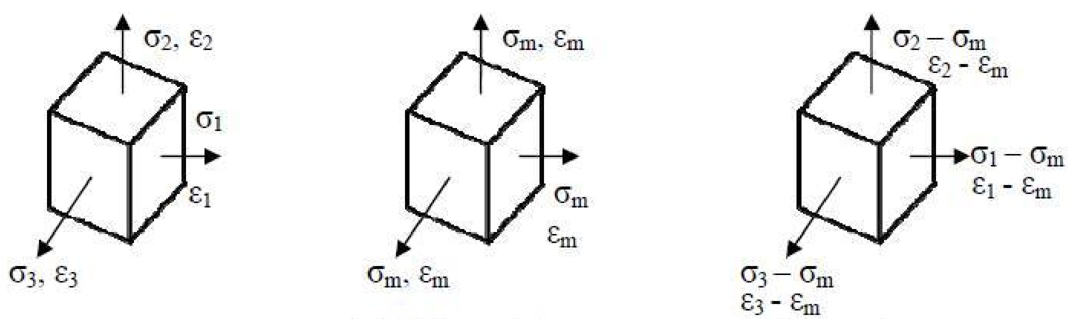
Similarly for Three Principle Stress

$$w = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\upsilon}{E}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)$$

Components of total strain energy

(energy required to change volume + energy required to change of shape or energy of distortion) [$w = w_v + w_d$]

- Mean stress - causes volume change
- Deviatoric stress - causes change of shape or causes distortion



= (a) Volume Change + (b) Distortion

From basic element: $\sigma_m = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ $\quad \varepsilon_m = \dfrac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} = \dfrac{\varepsilon_v}{3}$

(a) $\varepsilon_m + \varepsilon_m + \varepsilon_m = \varepsilon_v$ = change of volume per unit volume

(b) $(\varepsilon_1 - \varepsilon_m) + (\varepsilon_2 - \varepsilon_m) + (\varepsilon_3 - \varepsilon_m) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 3\varepsilon_m = 0$

[i.e. no volume change only distortion]

Total strain energy = strain energy in (a) + strain energy in (b) i.e. $w = w_v + w_d$ thus $w_d = w - w_v$

From (a):

$$w_v = \frac{1}{2}\sigma_m \varepsilon_m + \frac{1}{2}\sigma_m \varepsilon_m + \frac{1}{2}\sigma_m \varepsilon_m$$

$$w_v = \frac{1}{2}\sigma_m (3\varepsilon_m) = \frac{1}{2}\sigma_m \varepsilon_v$$

Bulk modulus, K (Modulus of volume expansion)

$$\text{Bulk modulus} \quad K = \frac{\textit{Mean stress}}{\textit{Volumetric strain}}$$

$$\text{Mean stress} \quad = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$\text{Volumetric strain} \quad = \frac{\textit{Change in Vol}^m}{\textit{Original Vol}^m} = \varepsilon_v$$

Let original volume = 1

$$\therefore \quad \textit{New volume} = 1 + 1.\varepsilon_v = 1 + \varepsilon_v$$

$$= (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) \approx 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

Neglecting 2nd & 3rd order strain products we get

$$\therefore \quad \varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\therefore \quad \varepsilon_v = \frac{1}{E}[\sigma_1 - \upsilon(\sigma_2 + \sigma_3) + \sigma_2 - \upsilon(\sigma_3 + \sigma_1) + \sigma_3 - \upsilon(\sigma_1 + \sigma_2)]$$

$$\varepsilon_v = \frac{1}{E}[(\sigma_1 + \sigma_2 + \sigma_3) - 2\upsilon\,(\sigma_1 + \sigma_2 + \sigma_3)]$$

$$\varepsilon_v = \frac{1 - 2\upsilon}{E}\,(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\therefore \quad K = \frac{\sigma_m}{\varepsilon_v} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)/3}{\frac{1-2}{E}(\sigma_1 + \sigma_2 + \sigma_3)} \qquad \boxed{K = \frac{E}{3(1-2\upsilon)}}$$

$$\therefore \quad \varepsilon_v = \sigma_m \frac{3(1-2\upsilon)}{E}$$

$$\therefore \quad w_v = \frac{1}{2}\sigma_m^2 \, \frac{3(1-2\upsilon)}{E}$$

$$but \Rightarrow \quad w = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\upsilon}{E}\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)$$

also $w_d = w - w_v$

$$\therefore w_d = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\upsilon}{E}\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) - \frac{1}{2}\sigma_m^2\,\frac{3(1-2\upsilon)}{E}$$

$$w_d = \frac{1}{2E}\Big[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon\sigma_1\sigma_2 - 2\upsilon\sigma_1\sigma_3 - 2\upsilon\sigma_2\sigma_3$$

$$-\frac{3}{9}(1-2\upsilon)\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_1\sigma_3 + 2\sigma_2\sigma_3\right)\Big]$$

$$w_d = \frac{1+\upsilon}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

[Energy of distortion when all the three principal stress are present]

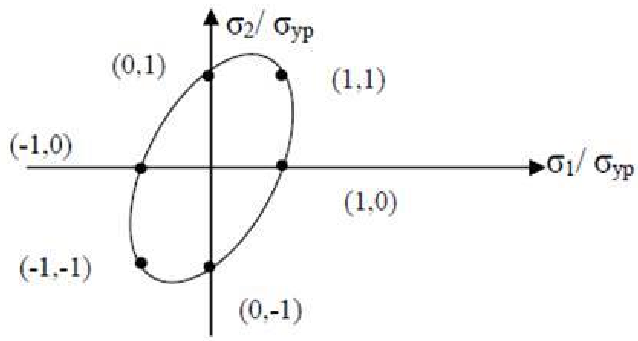
For uniaxial case $\sigma_1 \neq 0$; $\sigma_2 = \sigma_3 = 0$

$$\therefore w_d = \frac{1+\upsilon}{6E}\left(\sigma^2{}_1 + \sigma^2{}_1\right)$$

$$\Rightarrow w_d = \frac{1+\upsilon}{3E}\sigma^2{}_1$$

$$= \frac{1+\upsilon}{3E}\,.\,\sigma^2{}_{yp}$$

MAXIMUM ENERGY OF DISTORTION THEORY (Von Mises theory)



$$w_d = \frac{1+\upsilon}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

Equating this with the maximum distortion energy in simple tension i.e., uni-axial case of loading

$$w_d = \frac{1+\upsilon}{3E} \quad \sigma^2{}_{yp}$$

We get

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{yp}^2$$

For plane stress condition $\sigma_3 = 0$

$$\therefore \left(\frac{\sigma_1}{\sigma_{yp}}\right)^2 - \left(\frac{\sigma_1}{\sigma_{yp}} \cdot \frac{\sigma_2}{\sigma_{yp}}\right) + \left(\frac{\sigma_2}{\sigma_{yp}}\right)^2 = 1$$

The equation of the ellipses defines the yield criteria. Points on the ellipse indicate that the material is yielding.

Comparison of Yield and Fracture Criteria

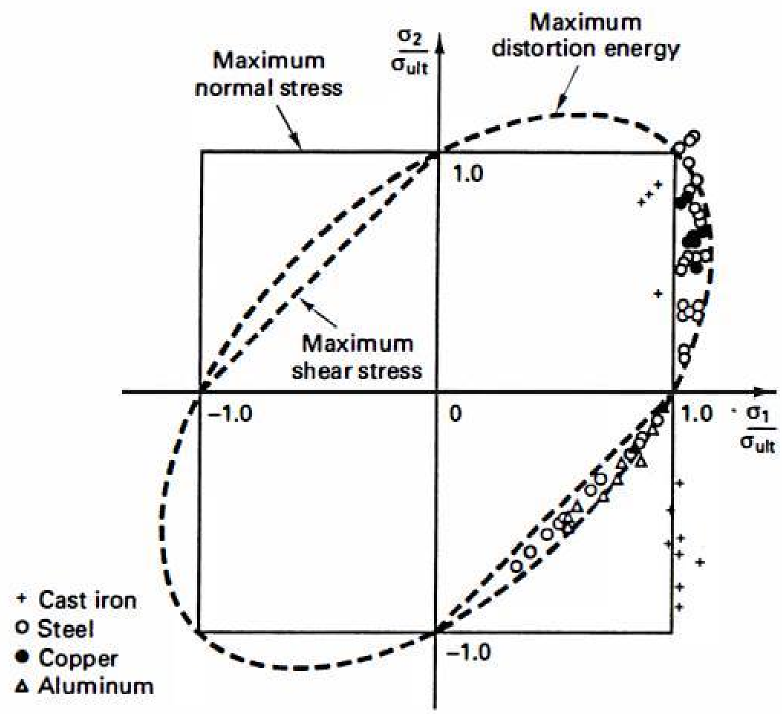
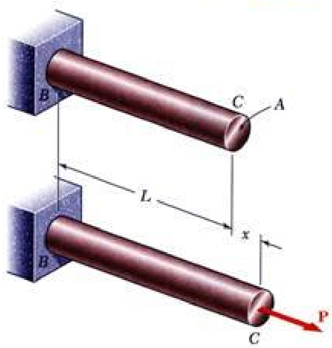
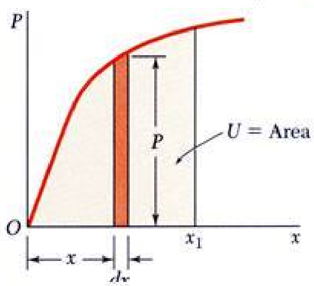
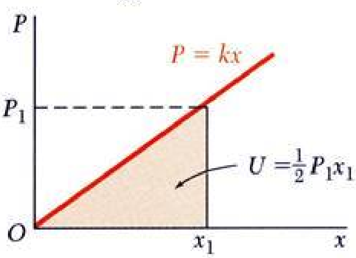


Fig. 12-10 Comparison of yield and fracture criteria with test data.

Strain Energy







- A uniform rod is subjected to a slowly increasing load

- The *elementary work* done by the load P as the rod elongates by a small *dx* is

$$dU = P\,dx = \textit{elementary work}$$

which is equal to the area of width *dx* under the load-deformation diagram.

- The *total work* done by the load for a deformation x_1,

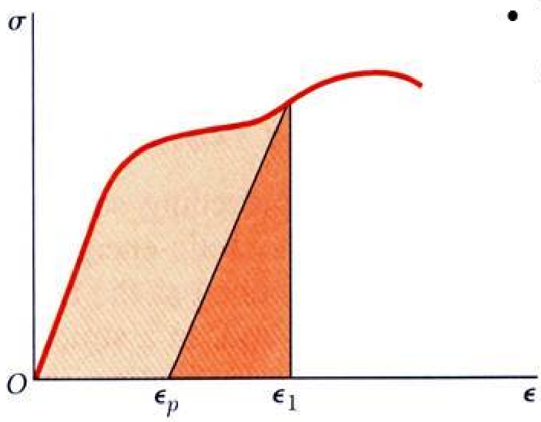
$$U = \int_0^{x_1} P\,dx = \textit{total work} = \textit{strain energy}$$

which results in an increase of *strain energy* in the rod.

- In the case of a linear elastic deformation,

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}P_1x_1$$

Strain Energy Density



- To eliminate the effects of size, evaluate the strain-energy per unit volume,

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L}$$

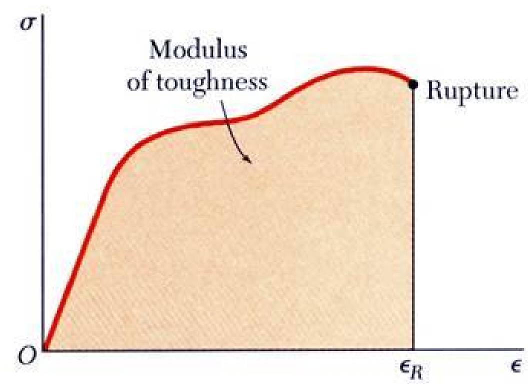
$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon = \textit{strain energy density}$$

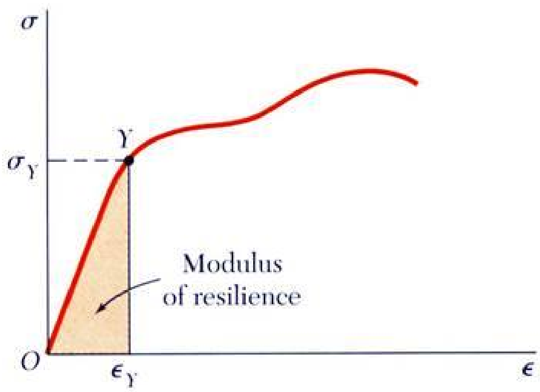
- The total strain energy density resulting from the deformation is equal to the area under the curve to ε_1.

- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.

- Remainder of the energy spent in deforming the material is dissipated as heat.

Strain-Energy Density





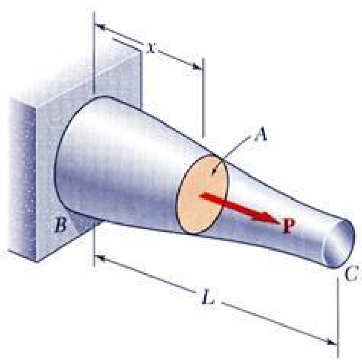
- The strain energy density resulting from setting $\varepsilon_1 = \varepsilon_R$ is the *modulus of toughness*.
- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.
- If the stress remains within the proportional limit,

$$u = \int_0^{\varepsilon_1} E\varepsilon \, d\varepsilon = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

- The strain energy density resulting from setting $\sigma_1 = \sigma_Y$ is the *modulus of resilience*.

$$u_Y = \frac{\sigma_Y^2}{2E} = \textit{modulus of resilience}$$

Elastic Strain Energy for Normal Stresses



- In an element with a nonuniform stress distribution,

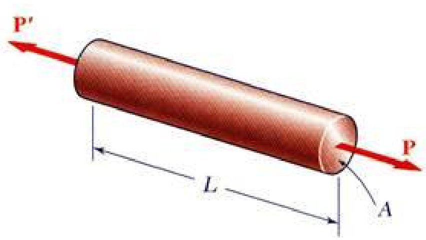
$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

- For values of $u < u_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} dV = elastic\ strain\ \ energy$$

- Under axial loading, $\sigma_x = P/A \qquad dV = A\ dx$

$$U = \int_0^L \frac{P^2}{2AE} dx$$

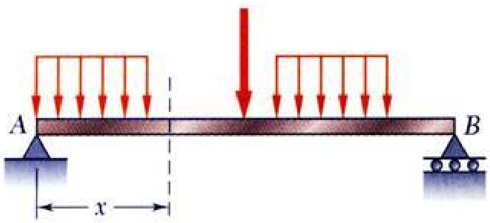


- For a rod of uniform cross-section,

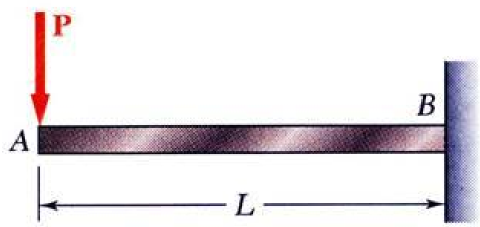
$$U = \frac{P^2 L}{2AE}; \quad W = P.\frac{\Delta}{2}; \quad U = W; \quad \frac{P^2 L}{2AE} = P.\frac{\Delta}{2}$$

$$\Delta = \frac{PL}{AE}$$

Elastic Strain Energy for Normal Stresses



$$\sigma_x = \frac{My}{I}$$



- For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting $dV = dA\,dx$,

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\,dx = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx$$

$$= \int_0^L \frac{M^2}{2EI} dx$$

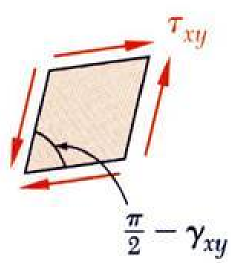
- For an end-loaded cantilever beam,

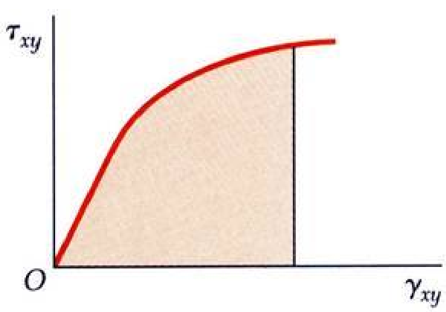
$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}; \quad W = P.\frac{\Delta}{2}$$

$$\Delta = \frac{PL^3}{3EI}$$

Strain Energy For Shearing Stresses





- For a material subjected to plane shearing stresses,

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}$$

- For values of τ_{xy} within the proportional limit,

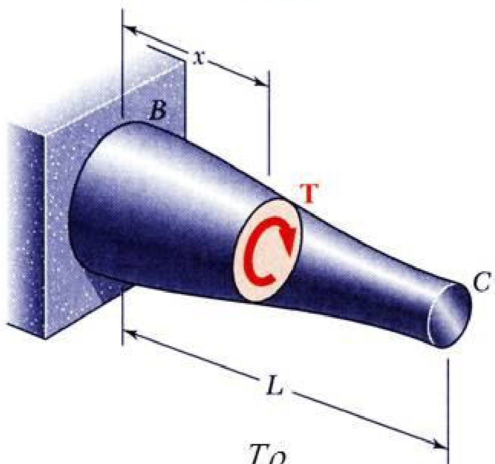
$$u = \tfrac{1}{2} G \gamma_{xy}^2 = \tfrac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

- The total strain energy is found from

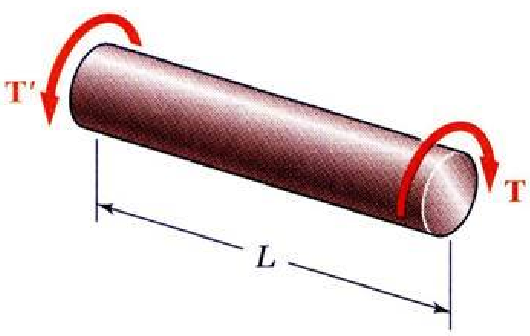
$$U = \int u\, dV$$

$$= \int \frac{\tau_{xy}^2}{2G} dV$$

Strain Energy For Shearing Stresses



$$\tau_{xy} = \frac{T\rho}{J}$$



- For a shaft subjected to a torsional load,

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GJ^2} dV$$

- Setting $dV = dA\, dx$,

$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} dA\, dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

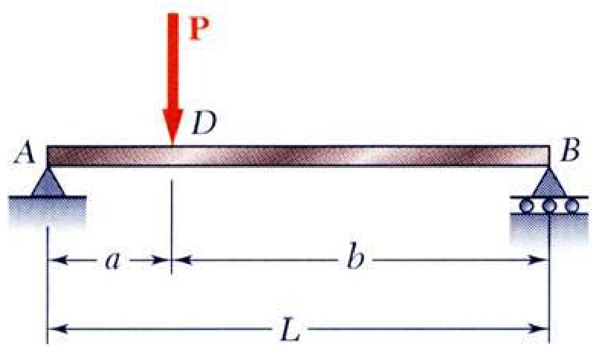
$$= \int_0^L \frac{T^2}{2GJ} dx = \frac{T^2 L}{2GJ}$$

- In the case of a uniform shaft,

$$U = \frac{T^2 L}{2GJ}; \quad W = T.\frac{\Phi}{2}$$

$$\Phi = \frac{TL}{GJ}$$

Sample Problem 11.2



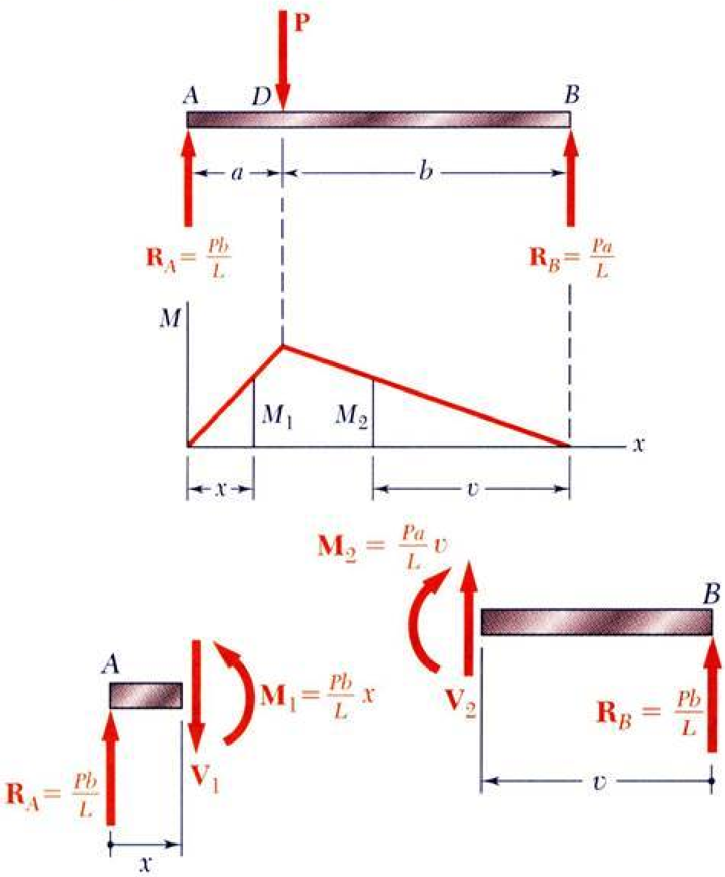
a) Taking into account only the normal stresses due to bending, determine the strain energy of the beam for the loading shown.

b) Evaluate the strain energy knowing that the beam is a W10x45, $P = 40$ kips, $L = 12$ ft, $a = 3$ ft, $b = 9$ ft, and $E = 29\text{x}10^6$ psi.

SOLUTION:

- Determine the reactions at A and B from a free-body diagram of the complete beam.
- Develop a diagram of the bending moment distribution.
- Integrate over the volume of the beam to find the strain energy.
- Apply the particular given conditions to evaluate the strain energy.

Sample Problem 11.2



SOLUTION:

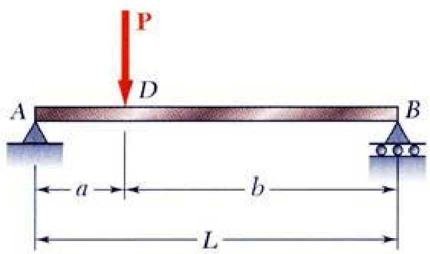
- Determine the reactions at A and B from a free-body diagram of the complete beam.

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

- Develop a diagram of the bending moment distribution.

$$M_1 = \frac{Pb}{L}x \qquad M_2 = \frac{Pa}{L}v$$

Sample Problem 11.2



Over the portion AD,

$$M_1 = \frac{Pb}{L}x$$

Over the portion BD,

$$M_2 = \frac{Pa}{L}v$$

$P = 45\,\text{kips}$ $\quad L = 144\text{ in.}$

$\text{a} = 36\text{ in.}$ $\quad b = 108\text{ in.}$

$E = 29\times10^3\,\text{ksi}$ $\quad I = 248\,\text{in}^4$

- Integrate over the volume of the beam to find the strain energy.

$$U = \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$

$$= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}v\right)^2 dv$$

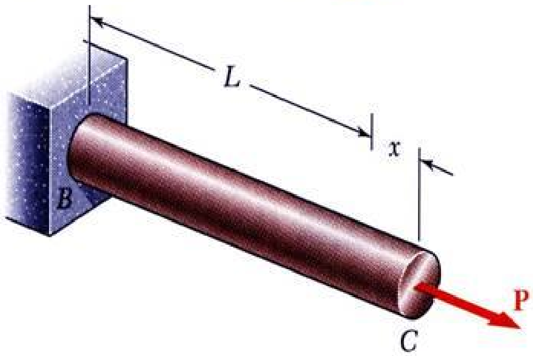
$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b) = \frac{P^2a^2b^2}{6EIL}$$

$$\boxed{U = \frac{P^2a^2b^2}{6EIL}} \quad \boxed{W = P\frac{\Delta}{2}} \quad \boxed{\Delta = \frac{Pa^2b^2}{3EIL}}$$

$$U = \frac{(40\,\text{kips})^2(36\,\text{in})^2(108\,\text{in})^2}{6\left(29\times10^3\,\text{ksi}\right)\left(248\,\text{in}^4\right)(144\,\text{in})}$$

$$\boxed{U = 3.89\,\text{in}\cdot\text{kips}}$$

Work and Energy Under a Single Load



- Previously, we found the strain energy by integrating the energy density over the volume.
  For a uniform rod,

$$U = \int u\,dV = \int \frac{\sigma^2}{2E}\,dV$$
$$= \int_0^L \frac{(P/A)^2}{2E}\,A\,dx = \frac{P^2 L}{2AE}$$

- Strain energy may also be found from the work of the single load P,

$$U = \int_0^{x_1} P\,dx$$

- For an elastic deformation,

$$U = \int_0^{x_1} P\,dx = \int_0^{x_1} kx\,dx = \tfrac{1}{2}k x_1^2 = \tfrac{1}{2}P x_1$$

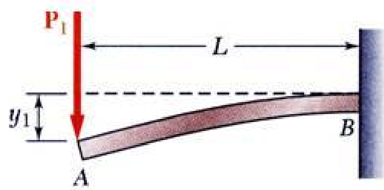
- Knowing the relationship between force and displacement,

$$x_1 = \frac{PL}{AE}$$
$$U = \tfrac{1}{2}P\left(\frac{PL}{AE}\right) = \frac{P^2 L}{2AE}$$

Work and Energy Under a Single Load

- Strain energy may be found from the work of other types of single concentrated loads.

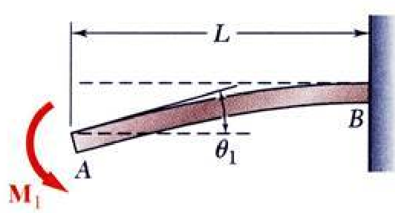
- Transverse load



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2}P_1 y_1$$

$$= \tfrac{1}{2}P_1\left(\frac{P_1 L^3}{3EI}\right) = \frac{P_1^2 L^3}{6EI}$$

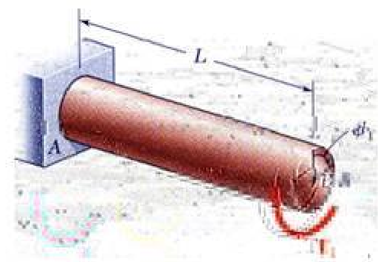
- Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2}M_1\theta_1$$

$$= \tfrac{1}{2}M_1\left(\frac{M_1 L}{EI}\right) = \frac{M_1^2 L}{2EI}$$

- Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2}T_1\phi_1$$

$$= \tfrac{1}{2}T_1\left(\frac{T_1 L}{JG}\right) = \frac{T_1^2 L}{2JG}$$

CABLES

Permanent structure

- Main load carrying elements of suspension bridges, cable car systems
- Guys of radio towers
- Electric transmission lines

Temporary structure

- Guys during erection processes

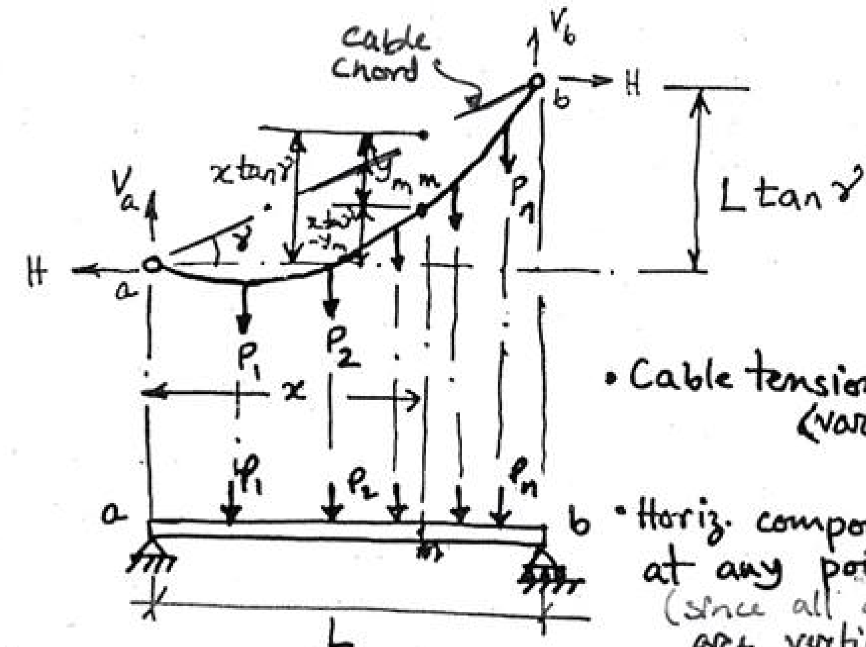
Shape of cable

- Under uniform self weight – Catenary Shape
- If sag/span ratio is small, it is reasonable to assume parabolic shape.
- If self-weight of the cable is negligible compared to concentrated loads supported, series of straight lines may be assumed.

Cable behaviour

Under load cable can only take tension and cannot take any bending.

General Cable Theorem (GCT)



- Cable tension = T (variable along Length)
- Horiz. component of T at any point = H (since all applied loads are vertical)

Let ΣM_b = Sum of M at b of all loads $P_1, P_2, P_3 \dots P_n$ (not reactions)

ΣM_m = Sum of M at m of all loads to left of m (not reactions).

Summing M at b

$$H L\tan\gamma + V_a L - \Sigma M_b = 0$$

$$V_a = \frac{\Sigma M_b}{L} - H\tan\gamma$$

Bending moment at $m = 0$

i.e Moment at m of all forces and reactions = 0

$$H(x\tan\gamma - y_m) + V_a x - \Sigma M_m = 0$$

or, $$Hx\tan\gamma - Hy_m + V_a x - \Sigma M_m = 0$$

or. $$Hy_m = Hx\cancel{\tan\gamma} + \frac{\Sigma M_b}{L}x - Hx\cancel{\tan\gamma} - \Sigma M_m$$

$$= x\,\frac{\Sigma M_b}{L} - \Sigma M_m$$

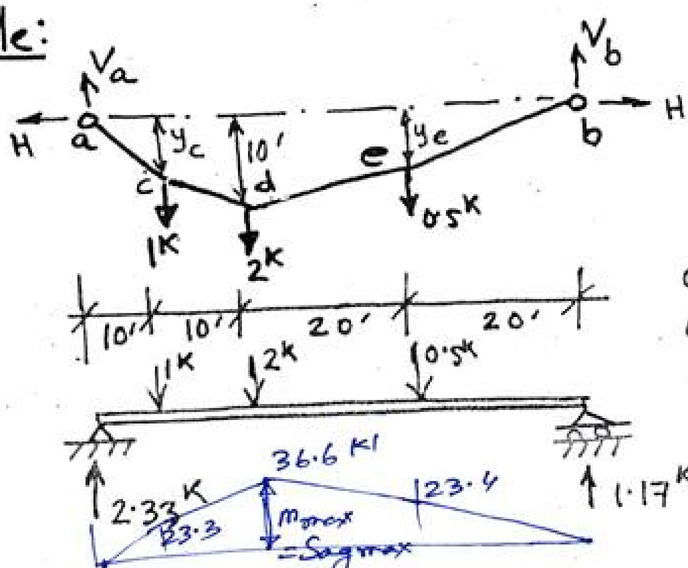
← Reaction at 'a' of a s.s. beam of same span and load.

$\overset{\leftarrow \text{Const}}{H} \cdot \overset{\leftarrow \text{max}^m \text{ at max B.M.}}{y_m}$ = bending moment at m of the corresponding s.s. beam

i.e. Horizontal component of the cable tension times the sag at any point = bending moment at the corresponding point of a simply supported beam of same span and subjected to the same loads.

— x —

Example:



Find y_c and y_e and cable tension at various locations.

At d, $y = 10'$

∴ From G.C.T

Hy = b.m at d of Single beam

or $H \times 10 = 2.33 \times 20 - 1 \times 10$

$\therefore H = 3.67^K$

H is same at all points on cable.

At c

$H \cdot y_c = 2.33 \times 10$

$y_c = \frac{23.3}{3.67} = 6.36'$

At e

$H \cdot y_e = 1.17 \times 20$

$y_e = \frac{1.17 \times 20}{3.67} = 6.36'$

Cable Tension:

$T_{ac} = 3.67 \times \frac{\sqrt{10^2 + 6.36^2}}{10} = 4.35^K \Leftarrow$

$T_{cd} = 3.67 \times \frac{\sqrt{(10-6.36)^2 + 10^2}}{10} = 3.9^K$

$T_{de} = 3.67 \times \frac{\sqrt{(10-6.36)^2 + 20^2}}{20} = 3.73^K$

$T_{eb} = 3.67 \times \frac{\sqrt{6.36^2 + 20^2}}{20}$
$= 3.85^K$

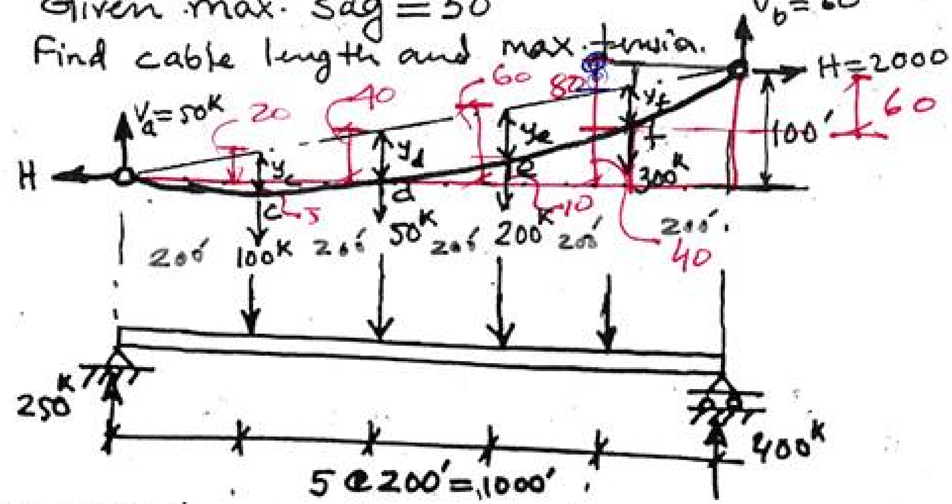
Note that max^m. tension occurs where cable slope is max^m. (portion ac.)

$T_{ac} \cos\theta = H$

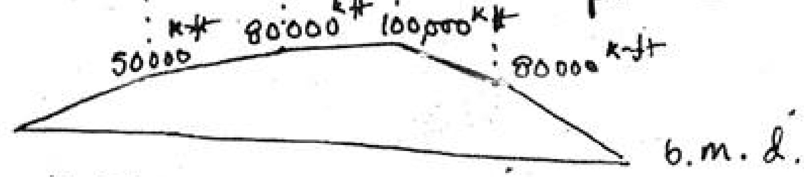
$\therefore T = \frac{H}{\cos\theta} = \frac{H}{\frac{10}{\sqrt{10^2+6.36^2}}}$

Prob. 10.1
NW

Given max. sag = 50'
Find cable length and max. tension.



Max. sag = 50 ft.
Occurs at point of max. b.m. in simple beam.



From G.C.T. $H \cdot 50 = 100{,}000$

$\therefore H = 2000^k$

$y_c = \dfrac{50000}{2000} = 25'$; $y_d = \dfrac{80{,}000}{2000} = 40'$; $y_f = 40'$

$Y_e = 50'$

Cable Length:

$= \sqrt{5^2 + 200^2} + \sqrt{5^2 + 200^2} + \sqrt{10^2 + 200^2} + \sqrt{30^2 + 200^2}$

$+ \sqrt{60^2 + 200^2}$

$= 1011.42$ ft.

Cable slope is steepest in segment fb

$\therefore T_{fb}$ is T_{max}

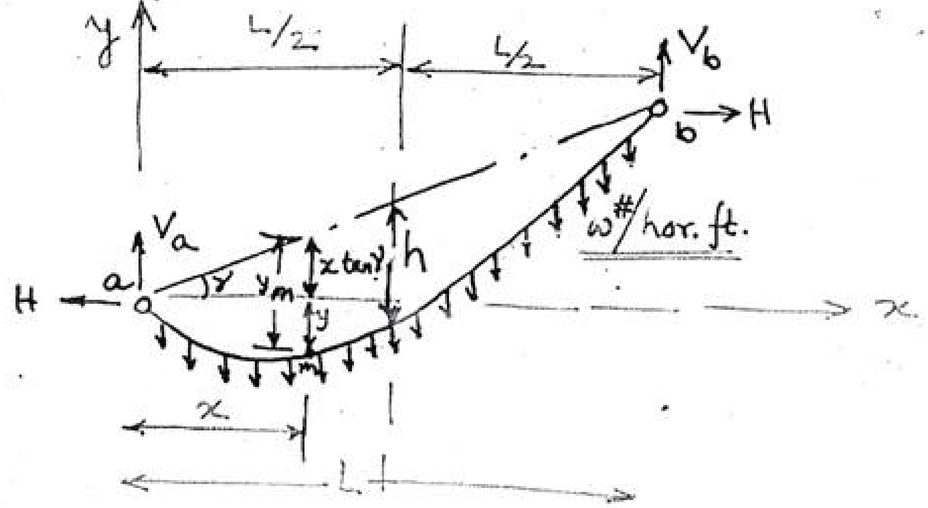
$\therefore T_{max} = 2000 \times \dfrac{\sqrt{60^2 + 200^2}}{200} = 2088^K$

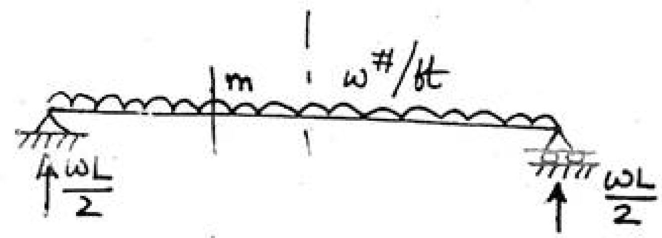
Cable under uniformly distributed Load:

Suspension bridges – load is uniformly distributed per unit length (horizontal length).

Self weight – treated approximately on the assumption of uniformly distributed load per unit length (horizontal length).

Cable Shape – Uniform Load





Applying G.C.T at m

$$H \cdot y_m = \frac{wLx}{2} - \frac{wx^2}{2} \quad \text{——(1)}$$

At mid-span $y_m = h$ called 'cable sag' measured vertically at mid-span.

At mid-span, G.C.T

$$Hh = \frac{wL^2}{8} \Rightarrow H = \frac{wL^2}{8h}$$

substitute this value of H into eq^n. (1)

$$\frac{wL^2}{8h} y_m = \frac{wLx}{2} - \frac{wx^2}{2} = \frac{wx}{2}(L-x)$$

(a) $\therefore$ $\boxed{y_m = \frac{4hx}{L^2}(L-x)}$ $\Rightarrow$ defines shape of cable w.r.t cable chord ...

In the above cable, taking x, y axes as shown

$y = +x\tan\gamma - y_m$ (negative as m is below x-axis)

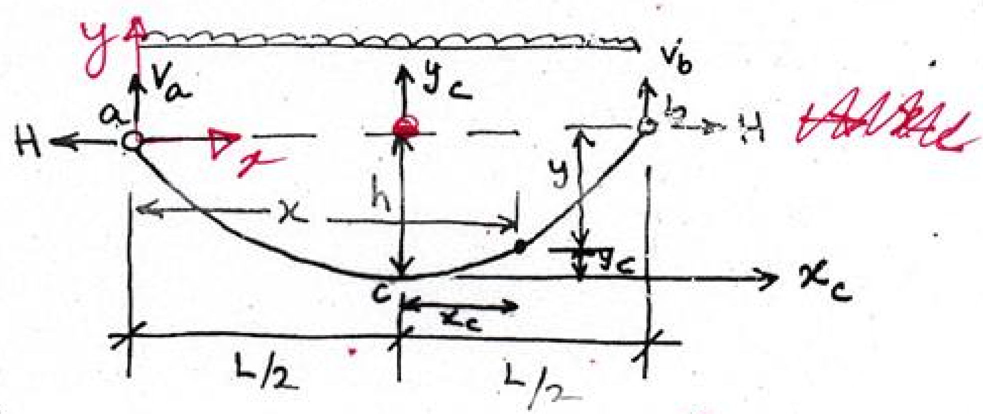
substituting for y_m

$$y = x\tan\gamma - \frac{4hx}{L^2}(L-x)$$

$$\therefore \boxed{y = \frac{4hx}{L^2}(x-L) + x\tan\gamma} \quad \text{----(b)}$$

Defines shape of cable w.r.t horizontal axis with origin at left end of cable.

<u>When the cable chord is horizontal:</u>



$\gamma = 0$

$$\therefore y = \frac{4hx}{L^2}(x-L)$$

If the origin is taken at mid-span, where sag is maximum (point c)

$$x = \frac{L}{2} + x_c \qquad y = -h + y_c$$

substituting these in the above eqn (b)

$$-h + y_c = \frac{4h}{L^2}\left(\frac{L}{2} + x_c\right)\left(\frac{L}{2} + x_c - L\right) \quad [\because \gamma = 0]$$

$$y_c = h + \frac{4h}{L^2}\left(x_c + \frac{L}{2}\right)\left(x_c - \frac{L}{2}\right)$$

$$= \frac{L^2h + 4h\left(x_c^2 - L^2/4\right)}{L^2}$$

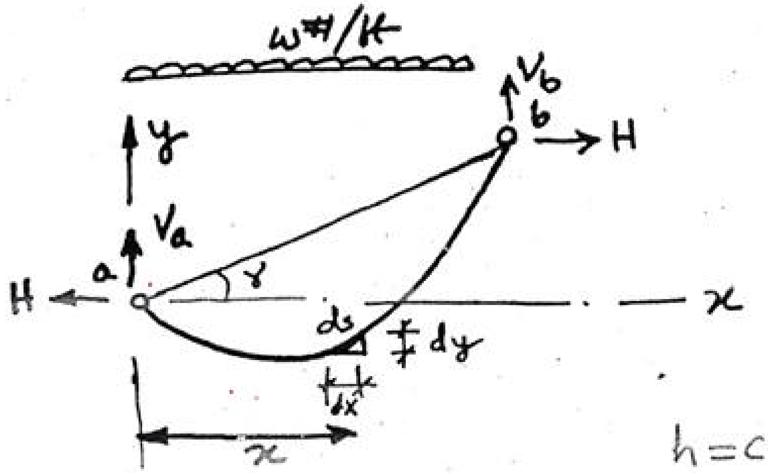
$$= \frac{\cancel{L^2h} + 4hx_c^2 - \cancel{L^2h}}{L^2}$$

$$\therefore \boxed{y_c = \frac{4h x_c^2}{L^2}} \quad \cdots (c)$$

This defines the shape of the cable with horizontal cable chord w.r.t axes (x_c, y_c) with origin at the lowest point of the cable.

—— x ——

Cable Tension — Uniform Load



h = cable sag (at mid span)

By G.C.T. $\quad H = \frac{wL^2}{8h}$

cable tension at distance x from the origin

$T_x = H \frac{ds}{dx}$ as slope of cable changes continuously

We know that cable shape w.r.t x-axis is given by

$$y = \frac{4hx}{L^2}(x-L) + x\tan\gamma$$

$$\frac{dy}{dx} = \frac{8hx}{L^2} - \frac{4h}{L} + \tan\gamma$$

$$= \frac{8\theta x}{L} - 4\theta + \tan\gamma$$

defining "sag ratio" $\theta = \frac{h}{L}$

But $ds = \sqrt{(dx)^2 + (dy)^2}$

$= \sqrt{1 + (dy/dx)^2} \cdot dx$

or $\frac{ds}{dx} = \sqrt{1 + (dy/dx)^2}$

$$T_x = H\sqrt{1+(dy/dx)^2}$$

$$= H\left[1+\left(\frac{8\theta x}{L} - 4\theta + \tan\gamma\right)^2\right]^{1/2}$$

$$= H\left(1 + \frac{64\theta^2 x^2}{L^2} + 16\theta^2 + \tan^2\gamma - \frac{64\theta^2 x}{L} - 8\theta\tan\gamma + \frac{16\theta x}{L}\tan\gamma\right)^{1/2}$$

Noting that slope $\frac{dy}{dx}$ is maximum at the end of the cable (ie at $x=0$ or $x=L$)

At $x=0$ $T_{max} = H\left(1+16\theta^2 + \tan^2\gamma - 8\theta\tan\gamma\right)^{1/2}$

At $x=L$ $T_{max} = H\left(1+16\theta^2 + \tan^2\gamma + 8\theta\tan\gamma\right)^{1/2}$

For the case of <u>horizontal cable chord</u>

$\gamma = 0$; $\therefore$ T_{max} is the same at both ends

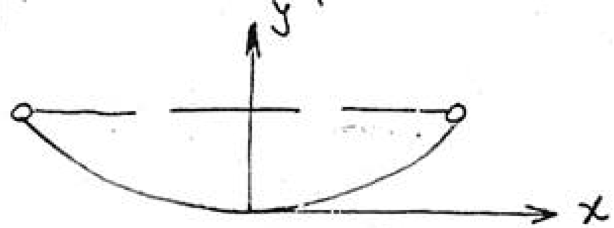
$$\boxed{T_{max} = H\left(1+16\theta^2\right)^{1/2}}$$

<u>Cable Length – Uniform Load</u>

If s is the total length of the cable

$$s = \int_0^L ds = \int_0^L \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

For the special case of a <u>horizontal cable chord</u> and origin at lowest point,



$$y = \frac{4h}{L^2}x^2 \qquad \therefore \frac{dy}{dx} = \frac{8hx}{L^2}$$

and $$s = 2\int_0^{L/2}\left[1+\frac{64h^2x^2}{L^4}\right]^{1/2} dx$$

<u>On integration:</u>

$$S = \frac{L}{2}\left(1+16\theta^2\right)^{1/2} + \frac{L}{8\theta}\ln\left[4\theta + \left(1+16\theta^2\right)^{1/2}\right]$$

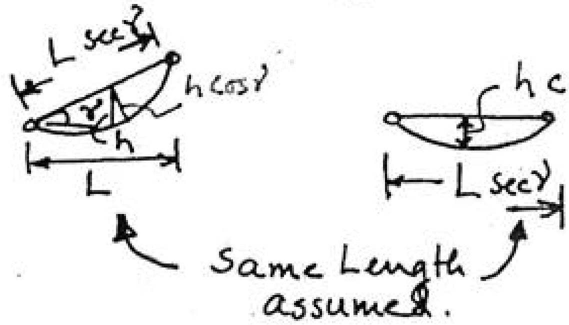
℄ cable length : hor. chord.

Cable Length – Inclined Cable Chord

For an inclined cable chord, the integration becomes cumbersome. An approximate treatment is as follows:

Assume that inclined cable has the same length as that of a horizontal cable with span equal to the length of the inclined chord

Sag = h



Sag = h cos γ

$$\theta = \frac{h}{L} \qquad \theta' = \frac{h\cos\gamma}{L\sec\gamma} = \frac{\theta}{\sec^2\gamma}$$

This value of θ' (sag ratio) may be used for obtaing cable length in inclined chord position:

$$\theta \Rightarrow \theta' \qquad L \Rightarrow L\sec\gamma$$

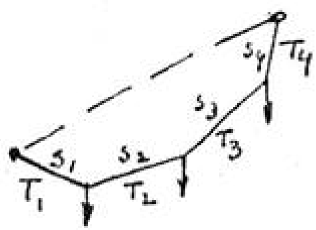
$$\therefore\ s = \frac{L\sec\gamma}{2}\left(1+16\theta'^2\right)^{1/2} + \frac{L\sec\gamma}{8\theta'}\ \mathrm{Ln}\left[4\theta' + \left(1+16\theta'^2\right)^{1/2}\right]$$

$$\boxed{\therefore\ s = \frac{L\sec\gamma}{2}\left(1+\frac{16\theta^2}{\sec^4\gamma}\right)^{1/2} + \frac{L\sec^3\gamma}{8\theta}\ \ln\left[\frac{4\theta}{\sec^2\gamma} + \left(1+\frac{16\theta^2}{\sec^4\gamma}\right)^{1/2}\right]}$$

Cable length: Inclined chord (approx. formula)

Cable Stretch – Uniform Load

stretch – Change in cable Length
For concentrated loads alone, cable segments are straight. T is constant in each segment
Total stretch is calculated as the summation of stretches of component straight segments.



$$E = \frac{\sigma}{\varepsilon} = \frac{T/A}{\Delta L/L}$$

$$\Delta L = \frac{TL}{AE}$$

$$\text{stretch} = \Delta S = \frac{T_1 S_1}{A_1 E} + \frac{T_2 S_2}{A_2 E} + \cdots$$
$$= \sum \frac{TS}{AE} = \int \frac{T_x\, ds}{AE}$$

For Parabolic Cable – (Under uniform load)

In this case procedure is more involved as T changes continuously along lengte of cable.

Approx. stretch $\Delta s = \dfrac{T_{av}\, S}{AE}$ [Assuming AE constant

where s = total length of the cable (diff. from original length but assumed o.k.)

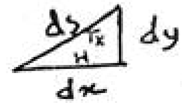
and T_{av} = hypothetical average cable tension.

T_{av} is such that if applied through out the length of cable, it will produce the same cable stretch as actually occurs.

i.e. $$\frac{T_{av}\, S}{AE} = \int_0^s \frac{T_x\, ds}{AE}$$

$$T_{av} = \frac{1}{s}\int_0^s T_x\, ds \quad (\text{AE const. assumed})$$

But $T_x = H \frac{ds}{dx}$



$$(ds)^2 = (dx)^2 + (dy)^2$$

$$\frac{(ds)^2}{dx} = \frac{(dx)^2 + (dy)^2}{dx}$$

$$= \frac{dx^2 + dy^2}{dx^2} \cdot dx$$

$$= \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} dx$$

$$\therefore T_{av} = \frac{H}{S} \int_0^S \frac{ds}{dx}\, ds$$

$$= \frac{H}{S} \int_0^L \left[1 + \left(\frac{dy}{dx}\right)^2\right] dx$$

Again,

$$y = \frac{4hx}{L^2}(x - L) + x \tan\gamma$$

$$\frac{dy}{dx} = \frac{8hx}{L^2} - \frac{4h}{L} + \tan\gamma$$

or, $$\frac{dy}{dx} = \frac{8\theta x}{L} - 4\theta + \tan\gamma$$

$$\therefore T_{av} = \frac{H}{S} \int_0^L \left[1 + \left(\frac{8\theta x}{L} - 4\theta + \tan\gamma\right)^2\right] dx$$

$$= \frac{H}{S} \int_0^L \left[1 + \frac{64\theta^2 x^2}{L^2} + 16\theta^2 + \tan^2\gamma - \frac{64\theta^2 x}{L} - 8\theta\tan\gamma + \frac{16\theta x}{L}\tan\gamma\right] dx$$

$$= \frac{H}{S}\left[x + \frac{64\theta^2 x^3}{3L^2} + 16\theta^2 x + \tan^2\gamma\, x - \frac{64\theta^2 x^2}{2L} - 8\theta\tan\gamma\, x + \frac{16\theta x^2}{2L}\tan\gamma\right]_0^L$$

$$\boxed{T_{av} = \frac{HL}{S}\left[1 + \frac{16}{3}\theta^2 + \tan^2\gamma\right]}$$

$\therefore$ <u>Cable stretch</u> $\Delta S = \frac{T_{av} S}{AE}$

$$\therefore \boxed{\Delta S = \frac{HL}{AE}\left[1 + \frac{16}{3}\theta^2 + \tan^2\gamma\right]}$$

Note S is stressed length of cable

original length $= S - \Delta S$

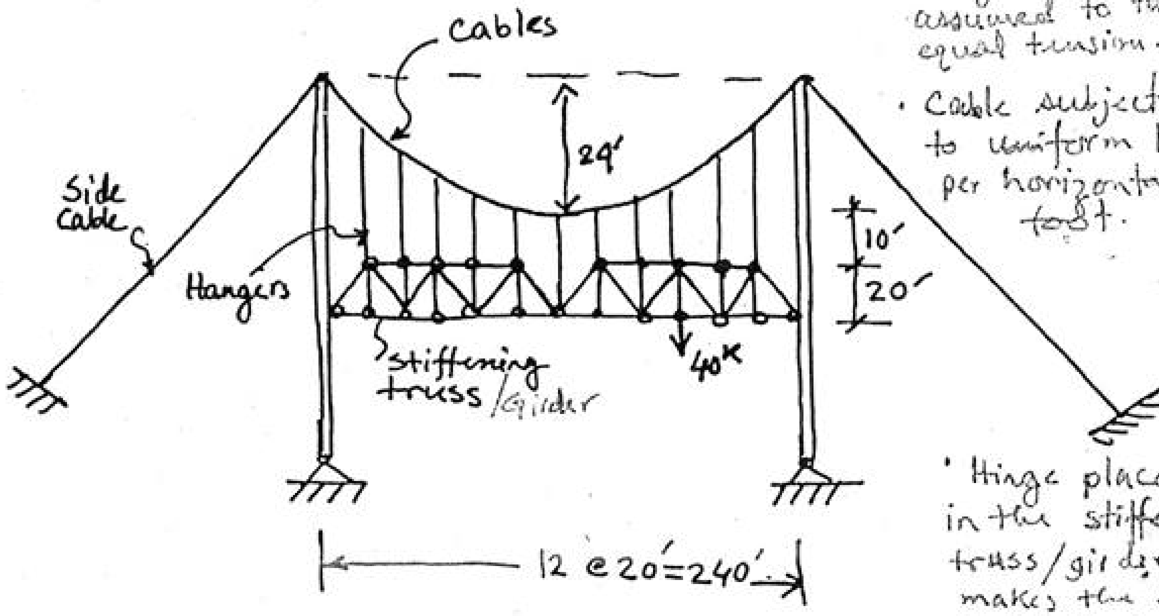
$\therefore$ ΔS is approximate on this count as well.

Statically Determinate Suspension Bridges

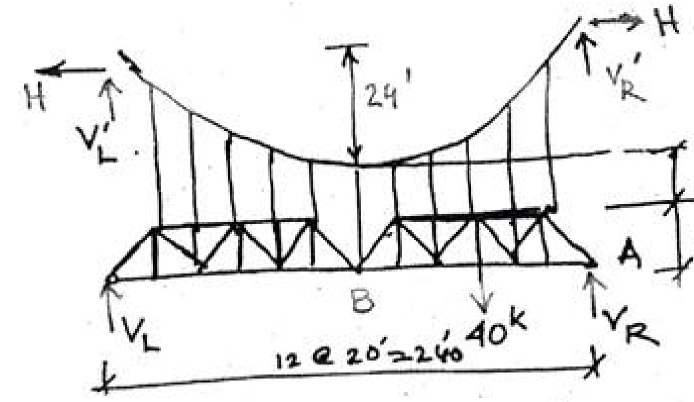
"Elastic Th." of Suspension Bridge:

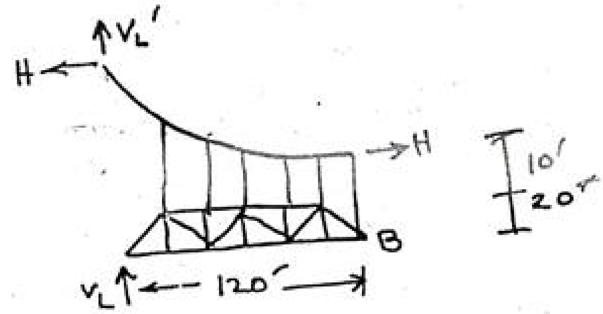
Assumptions

- Hangers are assumed to take equal tension.
- Cable subjected to uniform load per horizontal ~~foot~~.



- Hinge placed in the stiffening truss/girder makes the struc. statically deter.
- Cable remains parabolic.





$\Sigma M = 0$ @ A

$$(V_L + V_L') \times 240 - 40 \times 60 = 0$$

$$V_L + V_L' = 10^k$$

$\Sigma M = 0$ @ B

$$(V_L + V_L') \times 120 + H \times 30 - H \times 54 = 0$$

$$10 \times 120 - H \times 24 = 0$$

$$H = 50^K$$

T_{max} occurs at ends.
For a horizontal cable chord:

$$T_{max} = H\,(1 + 16\theta^2)^{1/2}$$

$$= 50\left[1 + 16\left(\frac{24}{240}\right)^2\right]^{1/2}$$

$$= 53.85^{\,kip}$$

Tension in hangers:

Let tension in each hanger $= x$

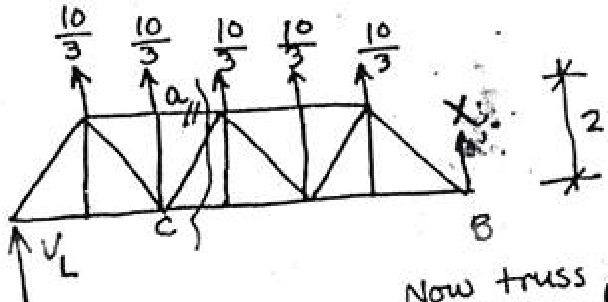
Equivalent uniform load on cable $= \dfrac{x}{\text{hanger spacing}}$

$= \dfrac{x}{20}$ K/ft.

Applying G.C.T

$$H = \frac{\omega L^2}{8h} \quad \text{where } \omega = \frac{x}{20}$$

$$\text{or, } 50 = \frac{x}{20} \cdot \frac{240^2}{8 \times 24} \qquad \therefore x = 10/3 \text{ kips}$$



$\Sigma M = 0$ @ B

$$V_L = \frac{1}{120}\left[\frac{10}{3} \times (100 + 80 + 60 + 40 + 20)\right]$$

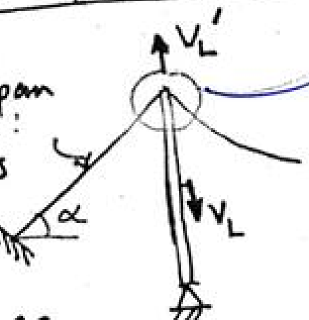
$$= \frac{25}{3} \text{ kips} \downarrow$$

Now truss member forces can be obtained in the usual manner.

Verify that: $F_a = 13.33^k (+)$

Side Span Cable:

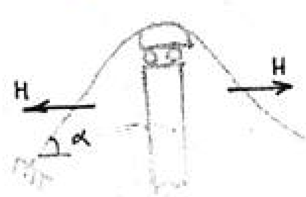
Side span cable:
- carries no suspn. load.
- acts as guy to the tower.



Guy force depends on the details at the point.

Three possible alternatives

(i) saddle on rollers.



H same in side and centre spans.

T different.

T in side span $= \dfrac{H}{\cos\alpha}$

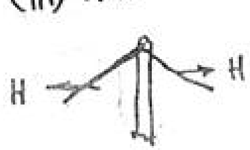
(ii) Pully.

Tension same in side span and end of centre span.

H different.

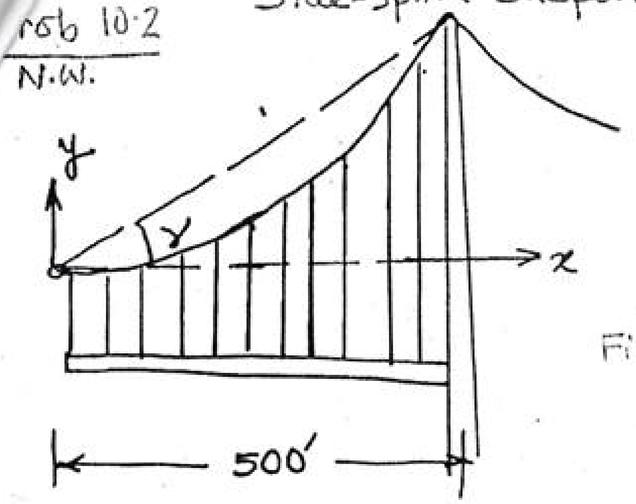
$T_{\text{in side span}} = T_{max}$.

(iii) Pins



$T = \dfrac{H}{\cos\alpha}$

Prob 10.2 Side-span suspension cable.
N.W.



$E = 27 \times 10^6$ psi
X-area of cable = 50 in^2
$\tan\gamma = 0.7$ $w = 1000$ #/hor.ft (distributed hanger loads;
sag ratio $= \frac{h}{L} = \frac{1}{40}$

Find: (a) Max. slope of cable $\left.\frac{dy}{dx}\right|_{max}$
(b) T_{max}
(c) s
(d) Unstressed length.

(a) Max. Slope $\left.\frac{dy}{dx}\right|_{max}$.

$$y = \frac{4hx}{L^2}(x-L) + x\tan\gamma$$

$$\frac{dy}{dx} = \frac{8hx}{L^2} - \frac{4h}{L} + \tan\gamma$$

$$= \frac{8 \times 1\; x}{40 \times 500} - 4 \times \frac{1}{40} + 0.7$$

$$= 0.0004x + 0.6$$

$\therefore \frac{dy}{dx}$ is max. at $x = max$
i.e. $x = 500'$

$$\therefore \left.\frac{dy}{dx}\right|_{max} = 0.0004 \times 500 + 0.6$$
$$= 0.8 \text{ (Ans)}.$$

(b) $T_{max} = ?$
Max. cable tension occurs where $\frac{dy}{dx}$ is max.

$$T_{max} = H \cdot \frac{ds}{dx}$$
$$= H \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2_{max}}$$

$\frac{ds}{dx}$ dy. $ds = \sqrt{dx^2 + dy^2}$
$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

By G.C.T $H = \frac{wL^2}{8h}$

$$= \frac{w \times L \times 40}{8}$$
$$= \frac{1 \times 500 \times 40}{8}$$
$$= 2500^K$$

$w = 1^{K/'}$
$\frac{h}{L} = \frac{1}{40}$

$$\therefore T_{max} = 2500\sqrt{1 + 0.8^2}$$
$$= 3200^K \text{ (Ans)}.$$

Cont'd

(c) compute, to nearest foot, the length of this cable.

$$s = \frac{L\sec\gamma}{2}\left(1+\frac{16\theta^2}{\sec^4\gamma}\right)^{1/2} + \frac{L\sec^3\gamma}{8\theta}\,\mathrm{Ln}\left[\frac{4\theta}{\sec^2\gamma} + \left(1+\frac{16\theta^2}{\sec^4\gamma}\right)^{1/2}\right]$$

$$= \frac{500\times1{\cdot}22}{2}\left(1+\frac{16}{40^2\times1{\cdot}22^4}\right)^{1/2} + \frac{500\times1{\cdot}22^3\times40}{8}\,\mathrm{Ln}\left[\frac{4}{40\times1{\cdot}22^2} + \left(1+\frac{16}{40^2\times1{\cdot}22^4}\right)^{1/2}\right]$$

$= 610{\cdot}787$ ft $\simeq 611$ ft.

(d) Compute, to the nearest foot, the unstressed length of this cable.

Cable stretch: $\Delta s = \frac{HL}{AE}\left[1+\frac{16}{3}\theta^2+\tan^2\gamma\right]$

$$= \frac{2500\times500}{50\times27\times10^3}\left[1+\frac{16}{3}\times\frac{1}{40^2}+0{\cdot}7^2\right]$$

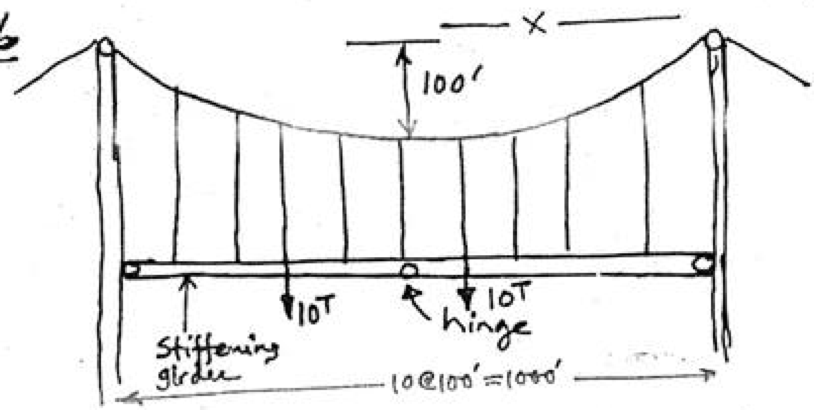
$= 1{\cdot}38$ ft.

$\therefore$ Unstressed length of cable = Stressed length, s − Cable stretch

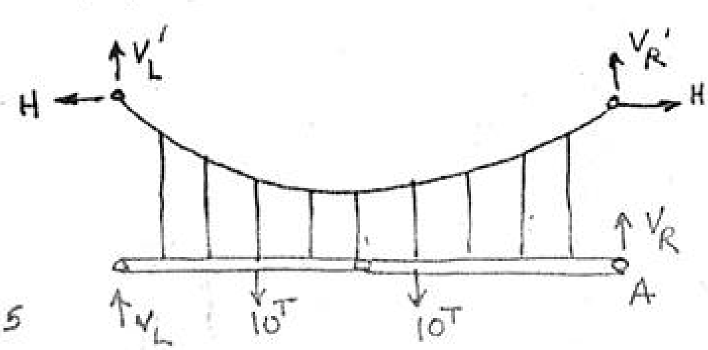
$= 610{\cdot}787 - 1{\cdot}38$

$\simeq 609$ ft. (Ans).

Prob



Draw b.m diagram of the stiffening girder.

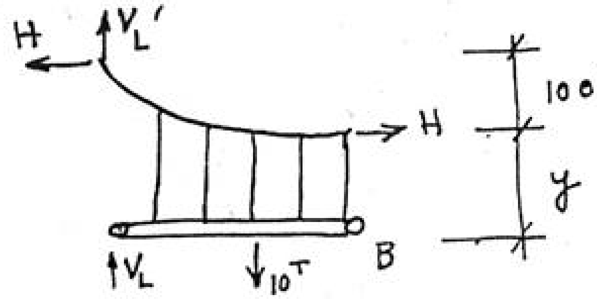


$\Sigma M = 0 @ A$

$(V_L+V_L')\times1000 - 10(700+400) = 0$

$\therefore V_L+V_L' = \frac{11,000}{1000} = 11^T$

6-15



$\Sigma M = 0 @ B$

$(V_L + V_L') \times 500 + \cancel{Hy} - H(\cancel{y} + 100) - 10 \times 200 = 0$

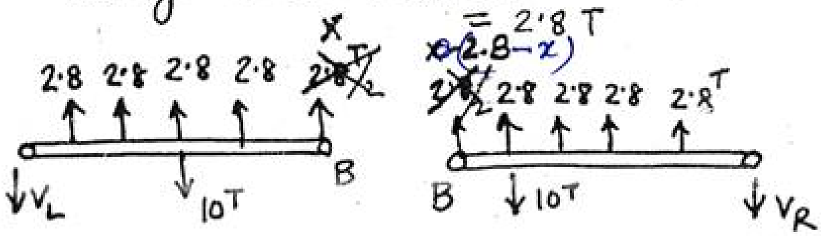
$11 \times 500 - H \times 100 - 2000 = 0$

$H = \dfrac{5500 - 2000}{100} = 35^T$

Again: $H = \dfrac{\omega L^2}{8h}$

$\therefore \omega = \dfrac{35 \times 8 \times 100}{1000^2} = 0.028\ ^T/ft.$

Hanger tension $= X = 100 \times \omega$
$= 2.8\ T$



$\Sigma M @ B = 0$

$V_L = \dfrac{1}{500}\left[2.8 \times (400 + 300 + 200 + 100) - 10 \times 200\right]$

$= 1.6^T \downarrow$

$V_R = \dfrac{1}{500} \times \left[2.8 \times (100 + 200 + 300 + 400) - 10 \times 100\right]$

$= 3.6^T \downarrow$