$+\cos^2\frac{3\pi}{12}+\cos^2\frac{5\pi}{12}$

$=(\sin^2\frac{\pi}{12}+\cos^2\frac{\pi}{12})+(\sin^2\frac{3\pi}{12}+\cos^2\frac{3\pi}{12})$

$+(\sin^2\frac{5\pi}{12}+\cos^2\frac{5\pi}{12})$

$=1+1+1=3$ (Ans.)

(b) $\cot\theta=\frac{3}{4}$ এবং $\cos\theta$ ঋণাত্মক হলে, A এর মান নির্ণয় কর।

প্রমাণ: যেহেতু $\cot\theta=\frac{3}{4}\Rightarrow\tan\theta=\frac{4}{3}$ এবং $\cos\theta$ ঋণাত্মক

$\therefore \sec\theta=-\sqrt{1+\tan^2\theta}=-\sqrt{1+\frac{16}{9}}$

$=-\sqrt{\frac{25}{9}}=-\frac{5}{3}$

$\therefore \cos\theta=-\frac{3}{5}$ এবং

$\sin\theta=\tan\theta\cos\theta=\frac{4}{3}\times(-\frac{3}{5})=-\frac{4}{5}$

$\therefore \operatorname{cosec}\theta=-\frac{5}{4}$

এখন, $\frac{\cot(-\theta)+\operatorname{cosec}\theta}{\cos\theta+\sin(-\theta)}=\frac{-\cot\theta+\operatorname{cosec}\theta}{\cos\theta-\sin\theta}$

$=\frac{-\frac{3}{4}+(-\frac{5}{4})}{-\frac{3}{5}-\frac{-4}{5}}=\frac{-3-5}{4}\times\frac{5}{-3+4}$

$=-\frac{40}{4}=-10$ (Ans.)

(c) $4n\theta=\pi$ হলে দেখাও যে, $f(\theta)f(2\theta)f(3\theta)\cdots\cdots\cdots\cdots f\{(2n-1)\theta\}=1$

প্রমাণ: $f(\theta)f(2\theta)f(3\theta)\cdots\cdots f\{(2n-1)\theta\}$

$=\tan\theta.\tan2\theta.\tan3\theta.\cdots\tan(2n-1)\theta$

এখানে, পদসংখ্যা $=2n-1$, যা বিজোড় সংখ্যা।

$\frac{2n-1+1}{2}$ অর্থাৎ n তম পদ মধ্যপদ।

$\therefore$ মধ্যপদ $=\tan n\theta=\tan\frac{\pi}{4}=1$ [$\because 4n\theta=\pi$]

$\tan\theta.\tan(2n-1)\theta=\tan\theta.\tan(2n\theta-\theta)$

$=\tan\theta.\tan(\frac{\pi}{2}-\theta)$ [$\because 4n\theta=\pi$]

$=\tan\theta.\cot\theta=1$

$\tan2\theta.\tan(2n-2)\theta=\tan2\theta.\tan(2n\theta-2\theta)$

$=\tan2\theta.\tan(\frac{\pi}{2}-2\theta)$ উ: −10

$=\tan2\theta.\cot2\theta=1$

অনুরূপভাবে, $\tan3\theta.\tan(2n-3)\theta=1$

$\tan4\theta.\tan(2n-4)\theta=1,\cdots$ ইত্যাদি।

অর্থাৎ,মধ্যপদ হতে সমদূরবর্তী পদ দুইটির গুণফল = 1

$\therefore f(\theta)f(2\theta)f(3\theta)\cdots\cdots f\{(2n-1)\theta\}=1$

প্রশ্নমালা VII B

1. মান নির্ণয় কর : (a) tan 105° (b) cosec 165°

(a) $\tan105^\circ=\tan(60^\circ+45^\circ)$

$=\frac{\tan60^0+\tan45^0}{1-\tan60^0\tan45^0}=\frac{\sqrt{3}+1}{1-\sqrt{3}.1}$

$=\frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})}=\frac{1+2\sqrt{3}+3}{1-3}$

$=\frac{2(\sqrt{3}+2)}{-2}=-(\sqrt{3}+2)$

1(b) $\operatorname{cosec}165^\circ=\operatorname{cosec}(90^\circ+75^\circ)$

$=\sec75^\circ=\frac{1}{\cos75^0}=\frac{1}{\cos(45^0+30^0)}$

$=\frac{1}{\cos45^0\cos30^0-\sin45^0\sin30^0}$

$=\frac{1}{\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}.\frac{1}{2}}=\frac{2\sqrt{2}}{\sqrt{3}-1}$

$=\frac{2\sqrt{2}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}=\frac{2(\sqrt{6}+\sqrt{3})}{3-1}$

$=\frac{2(\sqrt{6}+\sqrt{3})}{2}=\sqrt{6}+\sqrt{3}$

2. মান নির্ণয় কর ঃ

(a) $\cos 38°15' \sin 68°15' - \cos 51°45' \sin 21°45'$

$= \cos 38°15' \sin 68°15' - \cos(90° - 38°15') \sin(90° - 68°15')$

$= \cos 38°15' \sin 68°15' - \sin 38°15' \cos 68°15'$

$= \sin(68°15' - 38°15') = \sin 30° = \frac{1}{2}$

2(b) $\cos 69°22' \cos 9°22' + \cos 80°38' \cos 20°38'$

$= \cos 69°22' \cos 9°22' + \cos(90° - 9°22') \cos(90° - 69°22')$

$= \cos 69°22' \cos 9°22' + \sin 9°22' \sin 69°22'$

$= \cos(69°22' - 9°22') = \cos 60° = \frac{1}{2}$

প্রমাণ কর যে ,

3. L.H.S. $= \sin(25° + A)\cos(25° - A) + \cos(25° + A)\cos(115° - A)$

$= \sin(25° + A)\cos(25° - A) + \cos(25° + A)\cos\{90° + (25° - A)\}$

$= \sin(25° + A)\cos(25° - A) - \cos(25° + A)\sin(25° - A)$

$= \sin\{(25° + A) - (25° - A)\}$

$= \sin(25° + A - 25° + A)$

$= \sin 2A =$ R.H.S. (Proved)

4. প্রমাণ কর যে,

(a) L.H.S.$= \sin A \sin(B - C) + \sin B \sin(C - A) + \sin C \sin(A - B)$

$= \sin A(\sin B \cos C - \sin C \cos B) + \sin B(\sin C \cos A - \sin A \cos C) + \sin C(\sin A \cos B - \sin B \cos A)$

$= \sin A \sin B \cos C - \sin A \cos B \sin C + \cos A \sin B \sin C - \sin A \sin B \cos C + \sin A \cos B \sin C - \cos A \sin B \sin C$

$= 0 =$ R.H.S. (Proved)

4(b) L.H.S. $= \sin(B + C)\sin(B - C) + \sin(C + A)\sin(C - A) + \sin(A + B)\sin(A - B)$

$= \sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B$

$= 0 =$ R.H.S. (Proved)

4(c) L.H.S. $= \sin(135° - A) + \cos(135° + A)$

$= \sin\{180° - (45° + A)\} + \cos\{180° - (45° - A)\}$

$= \sin(45° + A) - \cos(45° - A)$

$= \sin(45° + A) - \cos\{90° - (45° + A)\}$

$= \sin(45° + A) - \sin(45° + A)$

$= 0 =$ R.H.S. (Proved)

5. প্রমাণ কর যে,

(a) L.H.S.$= \dfrac{\cos 25^0 - \sin 25^0}{\cos 25^0 + \sin 25^0}$

$= \dfrac{\cos 25^0 (1 - \frac{\sin 25^0}{\cos 25^0})}{\cos 25^0 (1 + \frac{\sin 25^0}{\cos 25^0})} = \dfrac{1 - \tan 25^0}{1 - \tan 25^0}$

$= \dfrac{\tan 45^0 - \tan 25^0}{1 + \tan 45^0 \tan 25^0} = \tan(45° - 25°)$

$= \tan 20° =$ R.H.S. (proved)

5(b) L.H.S.$= \dfrac{\sin 75^0 + \sin 15^0}{\sin 75^0 - \sin 15^0}$

$= \dfrac{\sin(90^0 - 15^0) + \sin 15^0}{\sin(90^0 - 15^0) - \sin 15^0}$

$= \dfrac{\cos 15^0 + \sin 15^0}{\cos 15^0 - \sin 15^0} = \dfrac{\cos 15^0 (1 + \frac{\sin 15^0}{\cos 15^0})}{\cos 15^0 (1 - \frac{\sin 15^0}{\cos 15^0})}$

$= \dfrac{1 + \tan 15^0}{1 - \tan 15^0} = \dfrac{\tan 45^0 + \tan 15^0}{1 - \tan 45^0 \tan 15^0}$

$= \tan(45° + 15°) = \tan 60° = \sqrt{3}$

6. প্রমাণ কর যে,

(a) $\tan\frac{\pi}{4} = \tan(\frac{\pi}{20} + \frac{\pi}{5})$

$\Rightarrow 1 = \dfrac{\tan\frac{\pi}{20} + \tan\frac{\pi}{5}}{1 - \tan\frac{\pi}{20}\tan\frac{\pi}{5}}$

$\Rightarrow \tan\frac{\pi}{20} + \tan\frac{\pi}{5} = 1 - \tan\frac{\pi}{20}\tan\frac{\pi}{5}$

$\therefore \tan\frac{\pi}{20} + \tan\frac{\pi}{5} + \tan\frac{\pi}{20}\tan\frac{\pi}{5} = 1$

6(b) $\tan(A - B) = -\tan(B - A)$

$= -\tan\{(B - C) + (C - A)\}$

$= -\dfrac{\tan(B - C) + \tan(C - A)}{1 - \tan(B - C)\tan(C - A)}$

$\Rightarrow \tan(A - B) - \tan(A - B)\tan(B - C)\tan(C - A) = -\tan(B - C) - \tan(C - A)$

$\therefore \tan(B - C) + \tan(C - A) + \tan(A - B) = \tan(B - C)\tan(C - A)\tan(A - B)$

7(a) **L.H.S.** $= 2\sin(\theta + \frac{\pi}{4})\sin(\theta - \frac{\pi}{4})$

$= 2(\sin\theta\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\theta)$

$(\sin\theta\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\theta)$

$= 2(\sin\theta.\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cos\theta)$

$(\sin\theta.\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cos\theta)$

$= 2.\frac{1}{2}(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$

$= \sin^2\theta - \cos^2\theta =$ R.H.S. (Proved)

বিকল্প পদ্ধতি: **L.H.S.**$= 2\sin(\theta + \frac{\pi}{4})\sin(\theta - \frac{\pi}{4})$

$= 2(\sin^2\theta - \sin^2\frac{\pi}{4})$

$[\because \sin(A+B)\sin(A - B) = \sin^2 A - \sin^2 B]$

$= 2(\sin^2\theta - \frac{1}{2}) = 2\sin^2\theta - 1$

$= 2\sin^2\theta - (\sin^2\theta + \cos^2\theta)$

$= \sin^2\theta - \cos^2\theta =$ R.H.S. (Proved)

7(b) **L.H.S.**$= \tan(A + B)\tan(A - B)$

$= \dfrac{\sin(A + B)\sin(A - B)}{\cos(A + B)\cos(A - B)}$

$= \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B} =$ R.H.S.

7.(c) L.H.S.$= \dfrac{\tan(\frac{\pi}{4} + \theta) - \tan(\frac{\pi}{4} - \theta)}{\tan(\frac{\pi}{4} + \theta) + \tan(\frac{\pi}{4} - \theta)}$

$= \{\dfrac{\sin(\frac{\pi}{4} + \theta)}{\cos(\frac{\pi}{4} + \theta)} - \dfrac{\sin(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} - \theta)}\} \div$

$\{\dfrac{\sin(\frac{\pi}{4} + \theta)}{\cos(\frac{\pi}{4} + \theta)} + \dfrac{\sin(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} - \theta)}\}$

$= \dfrac{\sin(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta) - \cos(\frac{\pi}{4} + \theta)\sin(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta)} \times$

$\dfrac{\sin(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta) + \cos(\frac{\pi}{4} + \theta)\sin(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta)}$

$= \dfrac{\sin(\frac{\pi}{4} + \theta - \frac{\pi}{4} + \theta)}{\sin(\frac{\pi}{4} + \theta + \frac{\pi}{4} - \theta)} = \dfrac{\sin 2\theta}{\sin\frac{\pi}{2}}$

$= \sin 2\theta =$ R.H.S. (Proved)

8. (a) $a\cos(x + \alpha) = b\cos(x - \alpha)$ **হলে দেখাও যে,** $(a + b)\tan x = (a - b)\cot\alpha$ **[ঢা.'০৫]**

প্রমাণ ঃ দেওয়া আছে, $a\cos(x + \alpha) = b\cos(x - \alpha)$

$\Rightarrow a(\cos x\cos\alpha - \sin x\sin\alpha) = b(\cos x\cos\alpha + \sin x\sin\alpha)$

$\Rightarrow (a - b)\cos x\cos\alpha = (a + b)\sin x\sin\alpha$

$\Rightarrow (a + b)\dfrac{\sin x}{\cos x} = (a - b)\dfrac{\cos\alpha}{\sin\alpha}$

$\therefore (a + b)\tan x = (a - b)\cot\alpha$

8(b) $a\sin(x+\theta) = b\sin(x-\theta)$ **হলে দেখাও যে,** $(a+b)\tan\theta + (a-b)\tan x = 0$

প্রমাণ ঃ দেওয়া আছে, $a\sin(x+\theta) = b\sin(x-\theta)$

$\Rightarrow a(\sin x\cos\theta + \sin\theta\cos x)$

$= b(\sin x\cos\theta - \sin\theta\cos x)$

$\Rightarrow (a-b)\sin x\cos\theta = -(a+b)\sin\theta\cos x$

$\Rightarrow (a-b)\dfrac{\sin x}{\cos x} = -(a+b)\dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow (a-b)\tan x = -(a+b)\tan\theta$

$\therefore (a+b)\tan\theta + (a-b)\tan x = 0$

8.(c) θ **কোণকে** α **এবং** β **এই দুই অংশে এমন ভাবে বিভক্ত করা হল যেন,** $\tan\alpha : \tan\beta = x : y$ **হয়।**

দেখাও যে, $\sin(\alpha-\beta) = \dfrac{x-y}{x+y}\sin\theta$

প্রমাণ ঃ দেওয়া আছে, $\theta = \alpha+\beta$ এবং

$\tan\alpha : \tan\beta = x : y$

$\Rightarrow \dfrac{\tan\alpha}{\tan\beta} = \dfrac{x}{y} \Rightarrow \dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{x+y}{x-y}$

$\Rightarrow \tan\alpha + \tan\beta = \dfrac{x+y}{x-y}(\tan\alpha - \tan\beta)$

$\Rightarrow \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta} = \dfrac{x+y}{x-y}\left(\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}\right)$

$\Rightarrow \dfrac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta}$

$= \dfrac{x+y}{x-y}\left(\dfrac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{\cos\alpha\cos\beta}\right)$

$\Rightarrow \sin(\alpha+\beta) = \dfrac{x+y}{x-y}\sin(\alpha-\beta)$

$\Rightarrow \sin\theta = \dfrac{x+y}{x-y}\sin(\alpha-\beta)$

$\therefore \sin(\alpha-\beta) = \dfrac{x-y}{x+y}\sin\theta$

8(d) $\tan\theta + \sec\theta = \dfrac{x}{y}$ **হলে দেখাও যে,**

$\sin\theta = \dfrac{x^2-y^2}{x^2+y^2}$

প্রমাণ : দেওয়া আছে, $\tan\theta + \sec\theta = \dfrac{x}{y}$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta} = \dfrac{x}{y} \Rightarrow \dfrac{1+\sin\theta}{\cos\theta} = \dfrac{x}{y}$

$\Rightarrow \dfrac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta} = \dfrac{x^2}{y^2}$ [উভয় পক্ষকে বর্গ করে।]

$\Rightarrow \dfrac{1+2\sin\theta+\sin^2\theta+\cos^2\theta}{1+2\sin\theta+\sin^2\theta-\cos^2\theta} = \dfrac{x^2+y^2}{x^2-y^2}$

[যোজন–বিয়োজন করে।]

$\Rightarrow \dfrac{1+2\sin\theta+(\sin^2\theta+\cos^2\theta)}{(1-\cos^2\theta)+2\sin\theta+\sin^2\theta} = \dfrac{x^2+y^2}{x^2-y^2}$

$\Rightarrow \dfrac{1+2\sin\theta+1}{\sin^2\theta+2\sin\theta+\sin^2\theta} = \dfrac{x^2+y^2}{x^2-y^2}$

$\Rightarrow \dfrac{2(1+\sin\theta)}{2\sin\theta(1+\sin\theta)} = \dfrac{x^2+y^2}{x^2-y^2}$

$\Rightarrow \dfrac{1}{\sin\theta} = \dfrac{x^2+y^2}{x^2-y^2}$

$\therefore \sin\theta = \dfrac{x^2-y^2}{x^2+y^2}$ (Showed)

8(e) $\sin(A+B) = n\sin(A-B)$ **এবং** $n \neq 1$ **হলে দেখাও যে,** $\cot A = \dfrac{n-1}{n+1}\cot B$

প্রমাণ ঃ দেওয়া আছে, $\sin(A+B) = n\sin(A-B)$

$\Rightarrow \dfrac{\sin(A+B)}{\sin(A-B)} = n$

$\Rightarrow \dfrac{\sin(A+B)+\sin(A-B)}{\sin(A+B)-\sin(A-B)} = \dfrac{n+1}{n-1}$

[যোজন–বিয়োজন করে।]

$\Rightarrow \dfrac{2\sin A\cos B}{2\sin B\cos A} = \dfrac{n+1}{n-1}$

$\Rightarrow \dfrac{\cot B}{\cot A} = \dfrac{n+1}{n-1}$

$\therefore \cot A = \dfrac{n-1}{n+1}\cot B$

9. (a) $a\sin(\theta+\alpha) = b\sin(\theta+\beta)$ হলে দেখাও যে, $\cot\theta = \dfrac{a\cos\alpha - b\cos\beta}{b\sin\beta - a\sin\alpha}$ [য.'০৫]

প্রমাণ: দেওয়া আছে, $a\sin(\theta+\alpha) = b\sin(\theta+\beta)$

$\Rightarrow a(\sin\theta\cos\alpha + \sin\alpha\cos\theta) = b(\sin\theta\cos\beta + \sin\beta\cos\theta)$

$\Rightarrow a\sin\theta\cos\alpha - b\sin\theta\cos\beta = b\sin\beta\cos\theta - a\sin\alpha\cos\theta$

$\Rightarrow (a\cos\alpha - b\cos\beta)\sin\theta = (b\sin\beta - a\sin\alpha)\cos\theta$

$\therefore \cot\theta = \dfrac{a\cos\alpha - b\cos\beta}{b\sin\beta - a\sin\alpha}$ (Showed)

9(b) $\sin\theta = k\cos(\theta-\alpha)$ হলে দেখাও যে, $\cot\theta = \dfrac{1+k\sin\alpha}{k\cos\alpha}$ [কু.'১২]

প্রমাণ: দেওয়া আছে, $\sin\theta = k\cos(\theta-\alpha)$

$\Rightarrow \sin\theta = k(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$\Rightarrow \sin\theta + k\sin\theta\sin\alpha = k\cos\theta\cos\alpha$

$\Rightarrow (1+k\sin\alpha)\sin\theta = k\cos\theta\cos\alpha$

$\Rightarrow \dfrac{1+k\sin\alpha}{k\cos\alpha} = \dfrac{\cos\theta}{\sin\theta}$

$\therefore \cot\theta = \dfrac{1+k\sin\alpha}{k\cos\alpha}$

9(c) $\cot\alpha + \cot\beta = a$, $\tan\alpha + \tan\beta = b$ এবং $\alpha+\beta = \theta$ হলে দেখাও যে, $(a-b)\tan\theta = ab$

[ঢা.'০১,'১১; য.'০১; ব.'০৮]

প্রমাণ: দেওয়া আছে,

$\cot\alpha + \cot\beta = a \cdots(1)$, $\tan\alpha + \tan\beta = b \cdots(2)$

এবং $\alpha+\beta = \theta \cdots\cdots(3)$

(1) হতে আমরা পাই, $\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta} = a$

$\Rightarrow \dfrac{\tan\beta + \tan\alpha}{\tan\alpha\tan\beta} = a$

$\Rightarrow \dfrac{b}{\tan\alpha\tan\beta} = a \Rightarrow \tan\alpha\tan\beta = \dfrac{b}{a}$

এখন, $\theta = \alpha + \beta$

$\Rightarrow \tan\theta = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$= \dfrac{b}{1-\dfrac{b}{a}} = \dfrac{ab}{a-b}$

$\therefore (a-b)\tan\theta = ab$

9(d) $\dfrac{\sin(\alpha+\theta)}{\sin\alpha} = \dfrac{2\sin(\beta+\theta)}{\sin\beta}$ হলে দেখাও যে, $\cot\alpha - \cot\theta = 2\cot\beta$ [কু.'১২]

প্রমাণ: দেওয়া আছে, $\dfrac{\sin(\alpha+\theta)}{\sin\alpha} = \dfrac{2\sin(\beta+\theta)}{\sin\beta}$

$\Rightarrow \sin\beta.\sin(\alpha+\theta) = 2\sin\alpha.\sin(\beta+\theta)$

$\Rightarrow (\sin\alpha\cos\theta + \cos\alpha\sin\theta)\sin\beta = 2\sin\alpha(\sin\beta\cos\theta + \sin\theta\cos\beta)$

$\Rightarrow \sin\alpha\cos\theta\sin\beta + \cos\alpha\sin\theta\sin\beta = 2\sin\alpha\sin\beta\cos\theta + 2\sin\alpha\sin\theta\cos\beta$

$\Rightarrow \cos\alpha\sin\theta\sin\beta - \sin\alpha\sin\beta\cos\theta = 2\sin\alpha\sin\theta\cos\beta$

ধরি, $\sin\theta\sin\alpha\sin\beta \neq 0$ এবং উভয় পক্ষকে $\sin\theta\sin\alpha\sin\beta$ দ্বারা ভাগ করে আমরা পাই,

$\dfrac{\cos\alpha}{\sin\alpha} - \dfrac{\cos\theta}{\sin\theta} = 2\dfrac{\cos\beta}{\sin\beta}$

$\therefore \cot\alpha - \cot\theta = 2\cot\beta$

বিকল্প পদ্ধতি: দেওয়া আছে,

$\dfrac{\sin(\alpha+\theta)}{\sin\alpha} = \dfrac{2\sin(\beta+\theta)}{\sin\beta}$

$\Rightarrow \dfrac{\sin\alpha\cos\theta + \cos\alpha\sin\theta}{\sin\alpha\sin\theta} = \dfrac{2(\sin\beta\cos\theta + \cos\beta\sin\theta)}{\sin\beta\sin\theta}$

$\Rightarrow \dfrac{\cos\theta}{\sin\theta} + \dfrac{\cos\alpha}{\sin\alpha} = 2\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{\cos\beta}{\sin\beta}\right)$

$\Rightarrow \cot\theta + \cot\alpha = 2(\cot\theta + \cot\beta)$

$\therefore \cot\alpha - \cot\theta = 2\cot\beta$

10. (a) $A+B+C = \pi$ এবং $\cos A = \cos B\cos C$ হলে দেখাও যে, $\tan B\tan C = 2$

[য.'০৩,'০৯]

প্রমাণ: দেওয়া আছে,

$A+B+C = \pi$ এবং $\cos A = \cos B\cos C$

$\therefore B + C = \pi - A$

$\Rightarrow \cos(B + C) = \cos(\pi - A)$

$\Rightarrow \cos B \cos C - \sin B \sin C = -\cos A$

$\Rightarrow \cos B \cos C - \sin B \sin C = -\cos B \cos C$

$[\because \cos A = \cos B \cos C]$

$\Rightarrow 2\cos B \cos C = \sin B \sin C$

$\Rightarrow \dfrac{\sin B \sin C}{\cos B \cos C} = 2$

$\therefore \tan B \tan C = 2$ (Showed)

10.(b) $A + B = \dfrac{\pi}{4}$ হলে দেখাও যে,

$(1 + \tan A)(1 + \tan B) = 2$

প্রমাণ ঃ দেওয়া আছে, $A + B = \dfrac{\pi}{4}$

$\Rightarrow \tan(A + B) = \tan\dfrac{\pi}{4} \Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \tan A + \tan B + \tan A \tan B + 1 = 2$

$\Rightarrow 1(1 + \tan A) + \tan B(1 + \tan A) = 2$

$\therefore (1 + \tan A)(1 + \tan B) = 2$ (Showed)

11.(a) $\sin\alpha \sin\beta - \cos\alpha \cos\beta + 1 = 0$ হলে প্রমাণ কর যে, $1 + \cot\alpha \tan\beta = 0$ [য.'০৭]

প্রমাণ ঃ দেওয়া আছে,

$\sin\alpha \sin\beta - \cos\alpha \cos\beta + 1 = 0$

$\Rightarrow \cos\alpha \cos\beta - \sin\alpha \sin\beta = 1$

$\Rightarrow \cos(\alpha + \beta) = 1 \Rightarrow \cos(\alpha + \beta) = \cos 0$

$\therefore \alpha + \beta = 0 \Rightarrow \beta = -\alpha$

এখন, L.H.S. $= 1 + \cot\alpha \tan(-\alpha)$

$= 1 + \dfrac{1}{\tan\alpha}(-\tan\alpha) = 1 - 1 = 0 =$ R.H.S.

11.(b) $\tan\beta = \dfrac{2\sin\alpha \sin\gamma}{\sin(\alpha + \gamma)}$ হলে দেখাও যে,

$\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\gamma} = \dfrac{2}{\tan\beta}$.

প্রমাণ ঃ দেওয়া আছে, $\tan\beta = \dfrac{2\sin\alpha \sin\gamma}{\sin(\alpha + \gamma)}$

$\Rightarrow \dfrac{\sin\beta}{\cos\beta} = \dfrac{2\sin\alpha \sin\gamma}{\sin(\alpha + \gamma)}$

$\Rightarrow \sin\beta(\sin\alpha\cos\gamma + \sin\gamma\cos\alpha)$

$= 2\sin\alpha \cos\beta \sin\gamma$

$\Rightarrow \sin\alpha\sin\beta \cos\gamma + \cos\alpha \sin\beta \sin\gamma$

$= 2\sin\alpha \cos\beta \sin\gamma$

ধরি, $\sin\alpha \sin\beta \sin\gamma \neq 0$ এবং উভয় পক্ষকে $\sin\alpha \sin\beta \sin\gamma$ দ্বারা ভাগ করে আমরা পাই,

$\dfrac{\cos\gamma}{\sin\gamma} + \dfrac{\cos\alpha}{\sin\alpha} = 2\dfrac{\cos\beta}{\sin\beta}$

$\Rightarrow \cot\gamma + \cot\alpha = 2\cot\beta$

$\therefore \dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\gamma} = \dfrac{2}{\tan\beta}$ (Showed)

11(c) $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$ হলে দেখাও যে,

$\tan(\alpha - \beta) = (1 - n)\tan\alpha$

প্রমাণ ঃ $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$ ....(1)

এখন, $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \dfrac{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}}{1 + \dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}}$

$= \dfrac{\dfrac{\sin\alpha}{\cos\alpha}\left(1 - \dfrac{n\cos^2\alpha}{1 - n\sin^2\alpha}\right)}{1 + \dfrac{n\sin^2\alpha}{1 - n\sin^2\alpha}}$

$= \tan\alpha\left(\dfrac{1 - n\sin^2\alpha - n\cos^2\alpha}{1 - n\sin^2\alpha}\right) \times \dfrac{1 - n\sin^2\alpha}{1 - n\sin^2\alpha + n\sin^2\alpha}$

$= \tan\alpha \dfrac{1 - n(\sin^2\alpha + \cos^2\alpha)}{1}$

$\therefore \tan(\alpha - \beta) = (1 - n)\tan\alpha$ (Showed)

12(a) $\tan\alpha - \tan\beta = x$ এবং $\cot\beta - \cot\alpha = y$ হলে দেখাও যে, $\cot(\alpha - \beta) = \dfrac{1}{x} + \dfrac{1}{y}$.

প্রমাণ ঃ দেওয়া আছে, $\tan\alpha - \tan\beta = x$ এবং $\cot\beta - \cot\alpha = y$

এখন, $\frac{1}{x}+\frac{1}{y}=\frac{1}{\tan\alpha-\tan\beta}+\frac{1}{\cot\beta-\cot\alpha}$

$$=\frac{1}{\frac{1}{\cot\alpha}-\frac{1}{\cot\beta}}+\frac{1}{\cot\beta-\cot\alpha}$$

$$=\frac{\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}+\frac{1}{\cot\beta-\cot\alpha}$$

$$=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=\cot(\alpha-\beta)$$

$\therefore\ \cot(\alpha-\beta)=\frac{1}{x}+\frac{1}{y}$ (Showed)

(b) $\tan\theta=\frac{x\sin\varphi}{1-x\cos\varphi}$ এবং $\tan\varphi=\frac{y\sin\theta}{1-y\cos\theta}$ হলে দেখাও যে, $\frac{\sin\theta}{\sin\varphi}=\frac{x}{y}$.

প্রমাণ ঃ দেওয়া আছে , $\tan\theta=\frac{x\sin\varphi}{1-x\cos\varphi}$

$\Rightarrow \frac{\sin\theta}{\cos\theta}=\frac{x\sin\varphi}{1-x\cos\varphi}$

$\Rightarrow x\cos\theta\sin\varphi=\sin\theta-x\sin\theta\cos\varphi$

$\Rightarrow x(\cos\theta\sin\varphi+\sin\theta\cos\varphi)=\sin\theta$

$\Rightarrow x\sin(\theta+\varphi)=\sin\theta \Rightarrow x=\frac{\sin\theta}{\sin(\theta+\varphi)}$

এবং $\tan\varphi=\frac{y\sin\theta}{1-y\cos\theta}\Rightarrow\frac{\sin\varphi}{\cos\varphi}=\frac{y\sin\theta}{1-y\cos\theta}$

$\Rightarrow y(\sin\theta\cos\varphi+\sin\varphi\cos\theta)=\sin\varphi$

$\Rightarrow y=\frac{\sin\varphi}{\sin(\theta+\varphi)}$

এখন, $\frac{x}{y}=\frac{\sin\theta}{\sin(\theta+\varphi)}\times\frac{\sin(\theta+\varphi)}{\sin\varphi}=\frac{\sin\theta}{\sin\varphi}$

$\therefore\ \frac{\sin\theta}{\sin\varphi}=\frac{x}{y}$ (Showed)

13.(a) $\sin x+\sin y=a$ এবং $\cos x+\cos y=b$ হলে প্রমাণ কর যে, $\sin\frac{1}{2}(x-y)=\pm\frac{1}{2}\sqrt{4-a^2-b^2}$

প্রমাণ ঃ দেওয়া আছে , $\sin x+\sin y=a$

$\Rightarrow \sin^2x+\sin^2y+2\sin x\sin y=a^2\cdots(1)$

এবং $\cos x+\cos y=b$

$\Rightarrow \cos^2x+\cos^2y+2\cos x\cos y=b^2\cdots(2)$

(1) ও (2) যোগ করে পাই,

$(\sin^2x+\cos^2x)+(\sin^2y+\cos^2y)+$
$2(\cos x\cos y+\sin x\sin y)=a^2+b^2$

$\Rightarrow 1+1+2\cos(x-y)=a^2+b^2$

$\Rightarrow 2\{1+\cos(x-y)\}=a^2+b^2$

$\Rightarrow 2\{2\cos^2\frac{1}{2}(x-y)\}=a^2+b^2$

$\Rightarrow 4\{1-\sin^2\frac{1}{2}(x-y)\}=a^2+b^2$

$\Rightarrow 4\sin^2\frac{1}{2}(x-y)=4-a^2+b^2$

$\Rightarrow \sin^2\frac{1}{2}(x-y)=\frac{1}{4}(4-a^2-b^2)$

$\therefore\ \sin\frac{1}{2}(x-y)=\pm\frac{1}{2}\sqrt{4-a^2-b^2}$

13(b) $\cos(\alpha-\beta)\cos\gamma=\cos(\alpha-\gamma+\beta)$ **হলে দেখাও যে,** $\cot\alpha$ **,** $\cot\gamma$ **এবং** $\cot\beta$ **সমান্তর প্রগমন ভুক্ত।**

প্রমাণ ঃ $\cos(\alpha-\beta)\cos\gamma=\cos(\alpha-\gamma+\beta)$

$\Rightarrow \cos(\alpha-\beta)\cos\gamma-\cos\{(\alpha+\beta)-\gamma\}=0$

$\Rightarrow \cos(\alpha-\beta)\cos\gamma-\{\cos(\alpha+\beta)\cos\gamma+\sin(\alpha+\beta)\sin\gamma\}=0$

$\Rightarrow \{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\cos\gamma-\sin(\alpha+\beta)\sin\gamma\}=0$

$\Rightarrow 2\sin\alpha\sin\beta\cos\gamma-(\sin\alpha\cos\beta+\sin\beta\cos\alpha)\sin\gamma=0$

$\Rightarrow 2\sin\alpha\sin\beta\cos\gamma-\sin\alpha\cos\beta\sin\gamma-\sin\beta\cos\alpha\ \sin\gamma=0$

$\Rightarrow 2\cot\gamma-\cos\beta-\cot\alpha=0$

[উভয় পক্ষকে $\sin\alpha\sin\beta\sin\gamma$ দ্বারা ভাগ করে]

$\Rightarrow \cot\gamma-\cos\beta=\cot\alpha-\cot\gamma$

$\Rightarrow \cot\alpha-\cot\gamma=\cot\gamma-\cos\beta$

$\therefore$ $\cot\alpha$, $\cot\gamma$ একং $\cot\beta$ সমান্তর প্রগমন ভুক্ত।

সম্ভাব্য ধাপসহ সমস্যা :

14. $\cot 165°$ এর মান নির্ণয় কর।

সমাধান: $\cot 165° = \cot(90° + 75°)$

$= -\tan 75° = -\tan(30° + 45°)$ (১)

$= -\dfrac{\tan 30^0 + \tan 45^0}{1 - \tan 30^0 \tan 45^0}$ (১)

$= -\dfrac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}.1} = -\dfrac{1+\sqrt{3}}{\sqrt{3}-1} = -\dfrac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= -\dfrac{3 + 2\sqrt{3} + 1}{3-1} = -\dfrac{2(\sqrt{3}+2)}{2} = -(\sqrt{3}+2)$ (১)

15. মান নির্ণয় কর :

(a) $\sin 76°40' \cos 16°40' - \cos 73°20' \sin 13°20'$

$= \sin 76°40' \cos 16°40' - \cos(90° - 16°40') \sin(90° - 76°40')$

$= \sin 76°40' \cos 16°40' - \sin 16°40' \cos 76°40'$ (১)

$= \sin(76°40' - 16°40') = \sin 60° = \dfrac{\sqrt{3}}{2}$ (১)

(b) $\cos 17°40' \sin 77°40' + \cos 107°40' \sin 12°20'$

$= \cos 17°40' \sin 77°40' + \cos(90° + 17°40') \sin(90° - 77°40')$

$= \cos 17°40' \sin 77°40' - \sin 17°40' \cos 77°40'$ (১)

$= \sin(77°40' - 17°40') = \sin 60° = \dfrac{\sqrt{3}}{2}$ (১)

(c) $\dfrac{\tan 68^0 35' - \cot 66^0 25'}{1 + \tan 68^0 35' \cot 66^0 25'}$

$= \dfrac{\tan 68^0 35' - \cot(90^0 - 23^0 35')}{1 + \tan 68^0 35' \cot(90^0 - 23^0 35')}$

$= \dfrac{\tan 68^0 35' - \tan 23^0 35'}{1 + \tan 68^0 35' \tan 23^0 35'}$ (১)

$= \tan(68°35' - 23^0 35')$

$= \tan 45° = 1$ (Ans.) (১)

প্রমাণ কর যে,

16. $\cos(A - B)\cos(A - C) + \sin(A - B)\sin(A - C) = \cos(B - C)$

L.H.S. $= \cos(A - B)\cos(A - C) + \sin(A - B)\sin(A - C)$

$= \cos\{(A - B) - (A - C)\}$ (১)

$= \cos(A - B - A + C) = \cos(-B + C)$

$= \cos(B - C) =$ R.H.S. (Proved) (১)

17. $\dfrac{\cot(3A - B)\cot B - 1}{-\cot B - \cot(3A - B)} = -\cot 3A$

L.H.S. $= \dfrac{\cot(3A - B)\cot B - 1}{-\cot B - \cot(3A - B)}$

$= \dfrac{\cot(3A - B)\cot B - 1}{-\{\cot B + \cot(3A - B)\}}$

$= -\dfrac{\cot(3A - B)\cot B - 1}{\cot B + \cot(3A - B)}$

$= -\cot(3A - B + B)$ (১)

$= -\cot 3A$

$=$ R.H.S. (Proved) (১)

18. $\cos A + \cos\left(\dfrac{2\pi}{3} - A\right) + \cos\left(\dfrac{2\pi}{3} + A\right) = 0$

L.H.S. $= \cos A + \cos\left(\dfrac{2\pi}{3} - A\right) + \cos\left(\dfrac{2\pi}{3} + A\right)$

$= \cos A + 2\cos\dfrac{2\pi}{3}\cos A$ (১)

$= \cos A + 2.\left(-\dfrac{1}{2}\right)\cos A$

$= \cos A - \cos A = 0 =$ R.H.S. (Proved) (১)

19. $\dfrac{\cos 15^0 + \sin 15^0}{\cos 15^0 - \sin 15^0} = \sqrt{3}$

L.H.S. $= \dfrac{\cos 15^0 + \sin 15^0}{\cos 15^0 - \sin 15^0}$

$$=\frac{\cos 15^0(1+\frac{\sin 15^0}{\cos 15^0})}{\cos 15^0(1-\frac{\sin 15^0}{\cos 15^0})}=\frac{1+\tan 15^0}{1-\tan 15^0}$$

$$=\frac{\tan 45^0+\tan 15^0}{1-\tan 45^0 \tan 15^0}=\tan(45°+15°) \quad (১)$$

$=\tan 60° = \sqrt{3}$ = R.H.S. (Proved) (১)

20. $\frac{\sin 75^0-\sin 15^0}{\sin 75^0+\sin 15^0}=\frac{1}{\sqrt{3}}$

$$\text{L.H.S.}=\frac{\sin 75^0-\sin 15^0}{\sin 75^0+\sin 15^0}$$

$$=\frac{\sin(90^0-15^0)-\sin 15^0}{\sin(90^0-15^0)+\sin 15^0}$$

$$=\frac{\cos 15^0-\sin 15^0}{\cos 15^0+\sin 15^0} \quad (১)$$

$$=\frac{\cos 15^0(1-\frac{\sin 15^0}{\cos 15^0})}{\cos 15^0(1+\frac{\sin 15^0}{\cos 15^0})}$$

$$=\frac{1-\tan 15^0}{1+\tan 15^0}=\frac{\tan 45^0-\tan 15^0}{1+\tan 45^0 \tan 15^0}$$

$= \tan(45°-15°)$ (১)

$= \tan 30° = \frac{1}{\sqrt{3}}$ = R.H.S. (proved) (১)

21. (a) $\tan 5A \tan 3A \tan 2A = \tan 5A - \tan 3A - \tan 2A$

প্রমাণ: $\tan 5A = \tan(3A+2A)$

$$\Rightarrow \tan 5A = \frac{\tan 3A+\tan 2A}{1-\tan 3A \tan 2A} \quad (১)$$

$\Rightarrow \tan 3A + \tan 2A = \tan 5A - \tan 5A \tan 3A \tan 2A$

$\therefore \tan 5A \tan 3A \tan 2A = \tan 5A - \tan 3A - \tan 2A$ (১)

(b) $\tan 32° + \tan 13° + \tan 32° \tan 13° = 1$

প্রমাণ: $\tan 45° = \tan(32°+13°)$

$$\Rightarrow 1 = \frac{\tan 32^0+\tan 13^0}{1-\tan 32^0 \tan 13^0} \quad (১)$$

$\Rightarrow \tan 32° + \tan 13° = 1 - \tan 32° \tan 13°$

$\therefore \tan 32° + \tan 13° + \tan 32° \tan 13° = 1$ (১)

(c) $\tan 50° = \tan 40° + 2\tan 10°$

প্রমাণ: $\tan 50° = \tan(40^0+10^0)$

$$\Rightarrow \tan 50° = \frac{\tan 40^0+\tan 10^0}{1-\tan 40^0 \tan 10^0} \quad (১)$$

$\Rightarrow \tan 50° - \tan 50° \tan 40° \tan 10° = \tan 40° + \tan 10°$

$\Rightarrow \tan 50° - \tan(90°-40°) \tan 40° \tan 10° = \tan 40° + \tan 10°$

$\Rightarrow \tan 50° - \cot 40° \tan 40° \tan 10° = \tan 40° + \tan 10°$ (১)

$\Rightarrow \tan 50° - \tan 10° = \tan 40° + \tan 10°$

$\therefore \tan 50° = \tan 40° + 2\tan 10°$ (১)

22. (a) $\tan(45°+A)\tan(45°-A) = 1$

প্রমাণ: L.H.S. $= \tan(45°+A)\tan(45°-A)$

$= \tan(45°+A)\tan\{90°-(45°+A)\}$

$= \tan(45°+A).\cot(45°+A)$ (১)

$= 1 =$ R.H.S. (Proved) (১)

(b) $\cos^2(A-B) - \sin^2(A+B) = \cos 2A \cos 2B.$

প্রমাণ: L.H.S.$= \cos^2(A-B) - \sin^2(A+B)$

$= \cos\{(A-B)+(A+B)\}\cos\{(A-B)-(A+B)\}$ (১)

$= \cos(A-B+A+B)\cos(A-B-A-B)$

$= \cos 2A \cos(-2B) = \cos 2A \cos 2B =$ R.H.S. (১)

23. (a) $\sin\alpha = k\sin(\alpha+\beta)$ **হলে দেখাও যে,** $\tan(\alpha+\beta) = \frac{\sin\beta}{\cos\beta - k}$.

প্রমাণ ঃ দেওয়া আছে, $\sin\alpha = k\sin(\alpha+\beta)$

$\Rightarrow \sin\alpha = k(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$ (১)

$\Rightarrow \sin\alpha = k\sin\alpha\cos\beta + k\sin\beta\cos\alpha$

$\Rightarrow \sin\alpha(1-k\cos\beta) = k\sin\beta\cos\alpha$

$$\Rightarrow \tan\alpha = \frac{k\sin\beta}{1-k\cos\beta} \quad (১)$$

এখন, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ (১)

$$= \frac{\dfrac{k\sin\beta}{1-k\cos\beta} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{k\sin\beta}{1-k\cos\beta}\dfrac{\sin\beta}{\cos\beta}}$$

$$= \frac{\dfrac{k\sin\beta\cos\beta + \sin\beta - k\sin\beta\cos\beta}{(1-k\cos\beta)\cos\beta}}{\dfrac{\cos\beta - k\cos^2\beta - k\sin^2\beta}{(1-k\cos\beta)\cos\beta}}$$

$$= \frac{\sin\beta}{\cos\beta - k(\cos^2\beta + \sin^2\beta)}$$

$\therefore\ \tan(\alpha+\beta) = \dfrac{\sin\beta}{\cos\beta - k}$ (Showed) (১)

(b) $\tan\alpha = \dfrac{b}{a}$ হলে দেখাও যে,

$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\alpha)$.

প্রমাণ ঃ দেওয়া আছে , $\tan\alpha = \dfrac{b}{a}$

এখন, $\sqrt{a^2+b^2}\cos(\theta-\alpha)$

$= \sqrt{a^2(1+\dfrac{b^2}{a^2})}.\cos(\theta-\alpha)$

$= a\sqrt{1+\tan^2\alpha}\cos(\theta-\alpha)$

$= a\sqrt{\sec^2\alpha}\cos(\theta-\alpha)$

$= a\sec\alpha\ \cos(\theta-\alpha)$

$= \dfrac{a}{\cos\alpha}(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$ (১)

$= a\cos\theta + a\sin\theta\tan\alpha$

$= a\cos\theta + a\sin\theta\,.\dfrac{b}{a}$

$= a\cos\theta + b\sin\theta$

$\therefore\ a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\alpha)$ (১)

বিকল্প পদ্ধতি: দেওয়া আছে, $\tan\alpha = \dfrac{b}{a} \Rightarrow \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{b}{a}$

$\Rightarrow \dfrac{\sin\alpha}{b} = \dfrac{\cos\alpha}{a} = \dfrac{\sqrt{\sin^2\alpha+\cos^2\alpha}}{\sqrt{b^2+a^2}} = \dfrac{\sqrt{1}}{\sqrt{a^2+b^2}}$ (১)

$\therefore\ b = \sqrt{a^2+b^2}\sin\alpha$, $a = \sqrt{a^2+b^2}\cos\alpha$

এখন , $a\cos\theta + b\sin\theta$

$= \sqrt{a^2+b^2}(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$

$\therefore a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\alpha)$ (১)

(showed)

24.(a) $\cos\alpha + \cos\beta = a$ এবং $\sin\alpha + \sin\beta = b$ হলে দেখাও যে, $\cos(\alpha-\beta) = \dfrac{1}{2}(a^2+b^2-2)$

প্রমাণ ঃ দেওয়া আছে , $\cos\alpha + \cos\beta = a$

$\Rightarrow \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta = a^2$ ...(1)

এবং $\sin\alpha + \sin\beta = b$

$\Rightarrow \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta = b^2$ ... (2)

(1) ও (2) যোগ করে পাই,

$(\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) +$
$2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = a^2 + b^2$

$\Rightarrow 1 + 1 + 2\cos(\alpha-\beta) = a^2 + b^2$ (১)

$\Rightarrow 2\cos(\alpha-\beta) = a^2 + b^2 - 2$

$\therefore\ \cos(\alpha-\beta) = \dfrac{1}{2}(a^2+b^2-2)$ (Showed) (১)

(b) $\tan\theta = \dfrac{a\sin x + b\sin y}{a\cos x + b\cos y}$ হলে দেখাও যে, $a\sin(\theta-x) + b\sin(\theta-y) = 0$.

প্রমাণ ঃ দেওয়া আছে , $\tan\theta = \dfrac{a\sin x + b\sin y}{a\cos x + b\cos y}$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} = \dfrac{a\sin x + b\sin y}{a\cos x + b\cos y}$ (১)

$\Rightarrow a\sin\theta\cos x + b\sin\theta\cos y =$
$a\sin x\cos\theta + b\cos\theta\sin y$

$\Rightarrow a(\sin\theta\cos x - \sin x\cos\theta) +$
$b(\sin\theta\cos y - \cos\theta\sin y) = 0$ (১)

$\therefore\ a\sin(\theta-x) + b\sin(\theta-y) = 0$

(c) $\tan\beta = \dfrac{\sin 2\alpha}{5+\cos 2\alpha}$ হলে দেখাও যে, $3\tan(\alpha-\beta) = 2\tan\alpha$.

প্রমাণ ঃ দেওয়া আছে , $\tan\beta = \dfrac{\sin 2\alpha}{5+\cos 2\alpha}$

$$\Rightarrow \tan\beta = \frac{\dfrac{2\tan\alpha}{1+\tan^2\alpha}}{5+\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}} \quad (১) + (১)$$

$$= \frac{\dfrac{2\tan\alpha}{1+\tan^2\alpha}}{\dfrac{5+5\tan^2\alpha+1-\tan^2\alpha}{1+\tan^2\alpha}} = \frac{2\tan\alpha}{6+4\tan^2\alpha}$$

$$= \frac{\tan\alpha}{3+2\tan^2\alpha}$$

এখন, $3\tan(\alpha-\beta) = 3\dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$ (১)

$$= 3\frac{\tan\alpha - \dfrac{\tan\alpha}{3+2\tan^2\alpha}}{1+\tan\alpha.\dfrac{\tan\alpha}{3+2\tan^2\alpha}}$$

$$= 3\frac{3\tan\alpha + 2\tan^3\alpha - \tan\alpha}{3+2\tan^2\alpha+\tan^2\alpha}$$

$$= 3\frac{2\tan\alpha + 2\tan^3\alpha}{3+3\tan^2\alpha}$$

$$= 3\frac{2\tan\alpha(1+\tan^2\alpha)}{3(1+\tan^2\alpha)} = 2\tan\alpha$$

$\therefore\ 3\tan(\alpha-\beta) = 2\tan\alpha$ (১)

25. (a) $\cos(\alpha+\beta)\sin(\gamma+\theta) = \cos(\alpha-\beta)\sin(\gamma-\theta)$ হলে দেখাও যে, $\tan\theta = \tan\alpha\tan\beta\tan\gamma$

প্রমাণ ঃ দেওয়া আছে , $\cos(\alpha+\beta)\sin(\gamma+\theta) = \cos(\alpha-\beta)\sin(\gamma-\theta)$

$$\Rightarrow \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{\sin(\gamma-\theta)}{\sin(\gamma+\theta)}$$

$$\Rightarrow \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{\cos(\alpha+\beta)-\cos(\alpha-\beta))} = \frac{\sin(\gamma-\theta)+\sin(\gamma+\theta)}{\sin(\gamma-\theta)-\sin(\gamma+\theta)}$$

$$\Rightarrow \frac{2\cos\alpha\cos\beta}{-2\sin\alpha\sin\beta} = \frac{2\sin\gamma\cos\theta}{-2\sin\theta\cos\gamma} \quad (১)$$

$$\Rightarrow \frac{1}{\tan\alpha\tan\beta} = \frac{\tan\gamma}{\tan\theta}$$

$\therefore\ \tan\theta = \tan\alpha\tan\beta\tan\gamma$ (Showed) (১)

(b) $(\theta-\varphi)$ সূক্ষ্মকোণ এবং $\sin\theta + \sin\varphi = \sqrt{3}(\cos\varphi - \cos\theta)$ হলে দেখাও যে, $\sin3\theta + \sin3\varphi = 0$

প্রমাণ ঃ $\sin\theta + \sin\varphi = \sqrt{3}(\cos\varphi - \cos\theta)$

$$\Rightarrow 2\sin\frac{1}{2}(\theta+\varphi)\cos\frac{1}{2}(\theta-\varphi) = \sqrt{3}\left\{2\sin\frac{1}{2}(\theta+\varphi)\sin\frac{1}{2}(\theta-\varphi)\right\} \quad (১)$$

$$\Rightarrow \cos\frac{1}{2}(\theta-\varphi) = \sqrt{3}\sin\frac{1}{2}(\theta-\varphi)$$

$$\Rightarrow \cot\frac{1}{2}(\theta-\varphi) = \sqrt{3} = \cot30^0$$

$\therefore\ \frac{1}{2}(\theta-\varphi) = 30^0$, যেহেতু $(\theta-\varphi)$ সূক্ষ্মকোণ । (১)

$$\Rightarrow \theta-\varphi = 60^0$$

এখন, $\sin3\theta + \sin3\varphi$

$$= 2\sin\frac{3}{2}(\theta+\varphi)\cos\frac{3}{2}(\theta-\varphi) \quad (১)$$

$$= 2\sin\frac{3}{2}(\theta+\varphi)\cos\frac{3}{2}(60^0)$$

$$= 2\sin\frac{3}{2}(\theta+\varphi)\cos90^0$$

$$= 2\sin\frac{3}{2}(\theta+\varphi)\times0$$

$\therefore\ \sin3\theta + \sin3\varphi = 0$ (১)

সৃজনশীল প্রশ্ন:

26. A= $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta)$, B = $\tan70°$

(a) প্রমাণ কর যে, $\cos(\frac{\pi}{3}-\alpha)\cos(\frac{\pi}{6}-\beta) - \sin(\frac{\pi}{3}-\alpha)\sin(\frac{\pi}{6}-\beta) = \sin(\alpha+\beta)$

প্রমাণ : L.H. S. = $\cos(\frac{\pi}{3}-\alpha)\cos(\frac{\pi}{6}-\beta) - \sin(\frac{\pi}{3}-\alpha)\sin(\frac{\pi}{6}-\beta)$

$$= \cos\{(\frac{\pi}{3}-\alpha) + (\frac{\pi}{6}-\beta)\}$$

$= \cos\{(\frac{\pi}{3} + \frac{\pi}{6}) - (\alpha + \beta)\}$

$= \cos\{\frac{\pi}{2} - (\alpha + \beta)\}$

$= \sin(\alpha + \beta) =$ R.H.S. (Proved)

(b) প্রমাণ কর যে, $B = \tan 20° + 2\tan 50°$

[চ.'০৫; ঢা'১০,'১৫]

প্রমাণ : $B = \tan 70° = \tan(50° + 20°)$

$\Rightarrow \tan 70° = \frac{\tan 50^0 + \tan 20^0}{1 - \tan 50^0 \tan 20^0}$

$\Rightarrow \tan 70° - \tan 70° \tan 50° \tan 20° = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° - \tan(90° - 20°)\tan 50° \tan 20° = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° - \cot 20° \tan 50° \tan 20° = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° - \tan 50° = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° = \tan 20° + 2\tan 50°$

$\therefore B = \tan 20° + 2\tan 50°$

(c) $A = -\frac{3}{2}$ হলে দেখাও যে, $\Sigma\cos\alpha = 0$ এবং $\Sigma\sin\alpha = 0$

প্রমাণ : দেওয়া আছে, $A = -\frac{3}{2}$

$\Rightarrow \cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$

$\Rightarrow 2(\cos\beta\cos\gamma + \sin\beta\sin\gamma + \cos\gamma\cos\alpha + \sin\gamma\sin\alpha + \cos\alpha\cos\beta + \sin\alpha\sin\beta) = -3$

$\Rightarrow 2(\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha) + 2(\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha) + 1 + 1 + 1 = 0$

$\Rightarrow 2(\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha) + 2(\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha) + (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) + (\sin^2\gamma + \cos^2\gamma) = 0$

$\Rightarrow \{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2(\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha)\} + \{\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2(\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha)\} = 0$

$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\therefore \cos\alpha + \cos\beta + \cos\gamma = 0$ এবং $\sin\alpha + \sin\beta + \sin\gamma = 0$

[$\because$ দুইটি সংখ্যার বর্গের সমষ্টি শূন্য হলে সংখ্যা দুইটি পৃথক পৃথক ভাবে শূন্য হয়।]

$\therefore \sum\cos\alpha = 0$ এবং $\sum\sin\alpha = 0$

প্রশ্নমালা VII C

1. প্রমাণ কর যে,

(a) $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$

L.H.S. $= \sin 10° \sin 30° \sin 50° \sin 70°$

$= \sin 10° . \frac{1}{2} . \frac{1}{2}\{\cos(70° - 50°) - \cos(70° + 50°)\}$

$= \frac{1}{4}\sin 10°(\cos 20° - \cos 120°)$

$= \frac{1}{4}\sin 10° \cos 20° - \frac{1}{4}\cdot(-\frac{1}{2})\sin 10°$

$= \frac{1}{4}.\frac{1}{2}\{\sin(20° + 10°) - \sin(20° - 10°)\} + \frac{1}{8}\sin 10°$

$= \frac{1}{8}\sin 30° - \frac{1}{8}\sin 10° + \frac{1}{8}\sin 10°$

$= \frac{1}{8}.\frac{1}{2} = \frac{1}{16} =$ R.H.S. (Proved)

1(b) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

L.H.S. $= \cos 20° \cos 40° \cos 60° \cos 80°$

$= \frac{1}{2}\{\cos(40° + 20°) + \cos(40° - 20°)\}\frac{1}{2}.\cos 80°$

$= \frac{1}{4}\{\cos 60° + \cos 20°\}\cos(90° - 10°)$