$=4(\frac{\sin^2 4\theta+1-2\sin^2 4\theta}{\sin 4\theta\cos 4\theta})$

$=4(\frac{1-\sin^2 4\theta}{\sin 4\theta\cos 4\theta})=4(\frac{\cos^2 4\theta}{\sin 4\theta\cos 4\theta})$

$=4\cot 4\theta$

অনুরূপভাবে প্রমাণ করা যায় ,

$2\tan 2\theta+4\cot 4\theta=2\cot 2\theta$ এবং

$\tan\theta+2\cot 2\theta=\cot\theta$

L.H.S.= $\tan\theta+2\tan 2\theta+4\tan 4\theta+8\cot 8\theta$

$=\tan\theta+2\tan 2\theta+4\cot 4\theta$

$=\tan\theta+2\cot 2\theta=\cot\theta=$ R.H.S. (Proved)

(c) দৃশ্যকল্প -২ হতে দেখাও যে, $\sqrt{3}\frac{AC}{AB}-\frac{AC}{BC}=4$

[ঢা.'১০;চ.'১৪,'১৫]

দৃশ্যকল্প -২ হতে পাই, $\frac{AC}{AB}=\text{cosec }20^0$ ও

$\frac{AC}{BC}=\sec 20^0$.

$\sqrt{3}\frac{AC}{AB}-\frac{AC}{BC}=\sqrt{3}\,\text{cosec}20^0-\sec 20^0$

$=\frac{\sqrt{3}}{\sin 20^0}-\frac{1}{\cos 20^0}$

$=\frac{\sqrt{3}\cos 20^0-\sin 20^0}{\sin 20^0\cos 20^0}$

$=\frac{\frac{\sqrt{3}}{2}\cos 20^0-\frac{1}{2}\sin 20^0}{\frac{1}{2}\sin 20^0\cos 20^0}$

$=\frac{\cos 30^0\cos 20^0-\sin 30^0\sin 10^0}{\frac{1}{4}\sin 40^0}$

$=\frac{4\cos(30^0+20^0)}{\sin(90^0-50^0)}=\frac{4\cos 50^0}{\cos 50^0}=4$

$\therefore\ \sqrt{3}\frac{AC}{AB}-\frac{AC}{BC}=4$

প্রশ্নমালা –VII E

প্রমাণ কর যে,

1. (a) $\frac{1-\sin x}{1+\sin x}=\tan^2(\frac{\pi}{4}-\frac{x}{2})$

L.H.S.= $\frac{1-\sin x}{1+\sin x}$

$=\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$

$=\frac{(\cos\frac{x}{2}-\sin\frac{x}{2})^2}{(\cos\frac{x}{2}+\sin\frac{x}{2})^2}=\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)^2$

$=\left(\frac{\tan\frac{\pi}{4}-\tan\frac{x}{2}}{1+\tan\frac{\pi}{4}\tan\frac{x}{2}}\right)^2=\tan^2(\frac{\pi}{4}-\frac{x}{2})=$ R.H.S

1. (b) $\cos^2\frac{\alpha}{2}+\cos^2(\frac{\alpha}{2}-60°)+\cos^2(\frac{\alpha}{2}+60°)=\frac{3}{2}$

L.H.S. $=\cos^2\frac{\alpha}{2}+\cos^2(\frac{\alpha}{2}-60°)+\cos^2(\frac{\alpha}{2}+60°)$

$=\frac{1}{2}\{1+\cos 2.\frac{\alpha}{2}+1+\cos 2.(\frac{\alpha}{2}-60°)+1+\cos 2(\frac{\alpha}{2}+60°)\}$

$=\frac{1}{2}\{3+\cos\alpha+\cos(\alpha-120°)+\cos(\alpha+120°)\}$

$=\frac{1}{2}\{3+\cos\alpha+2\cos\alpha\cos 120°\}$

$=\frac{1}{2}\{3+\cos\alpha+2\cos\alpha.(-\frac{1}{2})\}$

$=\frac{1}{2}\{3+\cos\alpha-\cos\alpha\}=\frac{3}{2}$

1(c) $\sin^2(\frac{\alpha}{2}-36°)+\sin^2(\frac{\alpha}{2}+36°)$

$=\frac{1}{4}\{4-(\sqrt{5}-1)\cos\alpha\}$

L.H.S.$= \sin^2(\frac{\alpha}{2} - 36°) + \sin^2(\frac{\alpha}{2} + 36°)$

$= \frac{1}{2}\{1 - \cos2(\frac{\alpha}{2} - 36°) + 1 - \cos2(\frac{\alpha}{2} + 36°)\}$

$= \frac{1}{2}[2 - \{\cos(\alpha - 72°) + \cos(\alpha + 72°)\}]$

$= \frac{1}{2}\{2 - 2\cos\alpha \cos72°\} = 1 - \cos\alpha\cos72°$

$= 1 - \cos\alpha.\cos(90° - 18°)$

$= 1 - \cos\alpha \sin18°$

$= 1 - \frac{1}{4}(\sqrt{5} - 1)\cos\alpha$

$= \frac{1}{4}\{4 - (\sqrt{5} - 1)\cos\alpha\} =$ R.H.S. (Proved)

2.(a) $2\cos\frac{\pi}{16} = 2\cos 11°15' = \sqrt{2+\sqrt{2+\sqrt{2}}}$ [কু.'০৭,'১৩; চ.'০১; রা.'০৩]

R.H.S.$= \sqrt{2+\sqrt{2+\sqrt{2}}}$

$= \sqrt{2+\sqrt{2(1+\frac{\sqrt{2}}{2})}} = \sqrt{2+\sqrt{2(1+\frac{1}{\sqrt{2}})}}$

$= \sqrt{2+\sqrt{2(1+\cos45^0)}}$

$= \sqrt{2+\sqrt{2.2\cos^2 22^0 30'}}$

$= \sqrt{2+2\cos22^0 30'} = \sqrt{2(1+\cos22^0 30')}$

$= \sqrt{2.2\cos^2 11^0 15'} = 2\cos 11°15' =$ M.H.S.

আবার, $2\cos\frac{\pi}{16} = 2\cos11°15'$

$\therefore 2\cos\frac{\pi}{16} = 2\cos11°15' = \sqrt{2+\sqrt{2+\sqrt{2}}}$

2(b) $\tan(7\frac{1}{2})^0 = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$

L.H.S. $= \tan(7\frac{1}{2})^0 = \tan 7^0 30'$

$= \frac{\sin 7^0 30'}{\cos 7^0 30'} = \frac{2\sin^2 7^0 30'}{2\sin 7^0 30' \cos 7^0 30'}$

$= \frac{1 - \cos15^0}{\sin 15^0} = \frac{1 - \cos(45^0 - 30^0)}{\sin(45^0 - 30^0)}$

$= \frac{1 - (\cos45^0\cos30^0 + \sin45^0\sin30^0)}{\sin45^0\cos30^0 - \cos45^0\sin30^0}$

$= \frac{1 - \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}}{\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}} = \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1}$

$= \frac{(2\sqrt{2} - \sqrt{3} - 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$

$= \frac{2\sqrt{6} - 3 - \sqrt{3} + 2\sqrt{2} - \sqrt{3} - 1}{3 - 1}$

$= \frac{2\sqrt{6} + 2\sqrt{2} - 4 - 2\sqrt{3}}{2}$

$= \sqrt{6} - \sqrt{3} + \sqrt{2} - 2 =$ R.H.S (Proved)

3. $\frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha} = \cot^2(\frac{\pi}{4} + \frac{\alpha}{2})$

L.H.S. $= \frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha}$

$= \frac{\frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\alpha} + \frac{\sin\alpha}{\cos\alpha}} = \frac{1 - \sin\alpha}{1 + \sin\alpha}$

$= \frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$

$= \frac{(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2})^2}{(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2})^2} = \left(\frac{\cos\frac{\alpha}{2}(\cot\frac{\alpha}{2} - 1)}{\cos\frac{\alpha}{2}(\cot\frac{\alpha}{2} + 1)}\right)^2$

$= \left(\frac{\cot\frac{\alpha}{2}\cot\frac{\pi}{2} - 1}{\cot\frac{\pi}{2} + \cot\frac{\alpha}{2}}\right)^2 = \left(\cot(\frac{\alpha}{2} + \frac{\pi}{2})\right)^2$

$= \cot^2(\frac{\alpha}{2} + \frac{\pi}{2}) =$ R.H.S. (Proved)

4. $\cos\theta = \frac{a\cos\varphi - b}{a - b\cos\varphi}$ হলে দেখাও যে,

$$\frac{\tan\frac{1}{2}\theta}{\sqrt{a+b}}=\frac{\tan\frac{1}{2}\varphi}{\sqrt{a-b}}$$

প্রমাণ : দেওয়া আছে , $\cos\theta=\dfrac{a\cos\varphi-b}{a-b\cos\varphi}$

$$\Rightarrow \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{a\dfrac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}-b}{a-b\dfrac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}}$$

$$\text{or, }\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{a(1-\tan^2\frac{\varphi}{2})-b(1+\tan^2\frac{\varphi}{2})}{a(1+\tan^2\frac{\varphi}{2})-b(1-\tan^2\frac{\varphi}{2})}$$

$$\text{or, }\frac{2}{-2\tan^2\frac{\theta}{2}}=$$

$$\frac{a(1-\tan^2\frac{\varphi}{2}+1+\tan^2\frac{\varphi}{2})-b(1+\tan^2\frac{\varphi}{2}+1-\tan^2\frac{\varphi}{2})}{a(1-\tan^2\frac{\varphi}{2}-1-\tan^2\frac{\varphi}{2})-b(1+\tan^2\frac{\varphi}{2}-1+\tan^2\frac{\varphi}{2})}$$

$$\Rightarrow \frac{1}{-\tan^2\frac{\theta}{2}}=\frac{2a-2b}{-2a\tan^2\frac{\varphi}{2}-2b\tan^2\frac{\varphi}{2}}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}}=\frac{a-b}{(a+b)\tan^2\frac{\varphi}{2}}$$

$$\Rightarrow \frac{\tan^2\frac{1}{2}\theta}{a+b}=\frac{\tan^2\frac{1}{2}\varphi}{a-b}$$

$$\therefore \frac{\tan\frac{1}{2}\theta}{\sqrt{a+b}}=\frac{\tan\frac{1}{2}\varphi}{\sqrt{a-b}}\quad \text{(Showed)}$$

5. (a) $\sec(\theta+\alpha)+\sec(\theta-\alpha)=2\sec\theta$ হলে দেখাও যে , $\cos\theta=\pm\sqrt{2}\cos\dfrac{\alpha}{2}$.

প্রমাণ : $\sec(\theta+\alpha)+\sec(\theta-\alpha)=2\sec\theta$

$$\Rightarrow \frac{1}{\cos(\theta+\alpha)}+\frac{1}{\cos(\theta-\alpha)}=\frac{2}{\cos\theta}$$

$$\Rightarrow \frac{\cos(\theta-\alpha)+\cos(\theta+\alpha)}{\cos(\theta+\alpha)\cos(\theta-\alpha)}=\frac{2}{\cos\theta}$$

$$\Rightarrow \frac{2\cos\theta\cos\alpha}{\cos^2\theta-\sin^2\alpha}=\frac{2}{\cos\theta}$$

$$\Rightarrow \cos^2\theta\cos\alpha=\cos^2\theta-\sin^2\alpha$$

$$\Rightarrow \cos^2\theta\,(1-\cos\alpha)=\sin^2\alpha$$

$$\Rightarrow \cos^2\theta=\frac{1-\cos^2\alpha}{1-\cos\alpha}=1+\cos\alpha$$

$$\Rightarrow \cos^2\theta=2\cos^2\frac{\alpha}{2}$$

$$\therefore \cos\theta=\pm\sqrt{2}\cos\frac{\alpha}{2}\quad \text{(Showed)}$$

5(b) $\sin A=\dfrac{1}{\sqrt{2}}$ এবং $\sin B=\dfrac{1}{\sqrt{3}}$ হলে দেখাও যে ; $\tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}=5+2\sqrt{6}$

প্রমাণ : দেওয়া আছে, $\sin A=\dfrac{1}{\sqrt{2}}$ এবং $\sin B=\dfrac{1}{\sqrt{3}}$

$$\therefore \frac{\sin A}{\sin B}=\frac{\sqrt{3}}{\sqrt{2}}$$

$$\Rightarrow \frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\ \text{[যোজন–বিয়োজন করে]}$$

$$\Rightarrow \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}=\frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3})^2-(\sqrt{2})^2}$$

$$\Rightarrow \tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right)=\frac{3+2\sqrt{3}\sqrt{2}+2}{3-2}$$

$$\therefore \tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right)=5+2\sqrt{6}$$

(Showed)

6. $A+B\neq 0$ এবং $\sin A+\sin B=2\sin(A+B)$ হলে দেখাও যে, $\tan\dfrac{A}{2}\tan\dfrac{B}{2}=\dfrac{1}{3}$ [কু.'০১]

প্রমাণ : দেওয়া আছে, $\sin A+\sin B=2\sin(A+B)$

$\Rightarrow 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$= 2\times 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)$

$\Rightarrow \sin\frac{1}{2}(A+B)\{\cos\frac{1}{2}(A-B) -$

$2\cos\frac{1}{2}(A+B)\} = 0$

$A+B \neq 0$ বলে $\sin\frac{1}{2}(A+B) \neq 0$

$\therefore \cos\frac{1}{2}(A-B) - 2\cos\frac{1}{2}(A+B) = 0$

$\Rightarrow \cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2} -$

$2(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}) = 0$

$\Rightarrow 3\sin\frac{A}{2}\sin\frac{B}{2} = \cos\frac{A}{2}\cos\frac{B}{2}$

$\Rightarrow \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{1}{3}$

$\therefore \tan\frac{A}{2}\tan\frac{B}{2} = \frac{1}{3}$ (Showed)

7 $a\cos\theta + b\sin\theta = c$ সমীকরণটি θ এর দুইটি ভিন্ন মান α, β দ্বারা সিদ্ধ হলে দেখাও যে,

$$\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$$

সমাধান : $\cos\theta + b\sin\theta = c$ সমীকরণটি θ এর দুইটি ভিন্ন মান α ও β দ্বারা সিদ্ধ বলে,

$a\cos\alpha + b\sin\alpha = c$

এবং $a\cos\beta + b\sin\beta = c$

$\therefore a\cos\alpha + b\sin\alpha = a\cos\beta + b\sin\beta$

$\Rightarrow a(\cos\alpha - \cos\beta) = b(\sin\beta - \sin\alpha)$

$\Rightarrow a.2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta-\alpha)$

$= b.2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta-\alpha)$

$\alpha \neq \beta$ বলে, $\sin\frac{1}{2}(\beta-\alpha) \neq 0$

$\therefore a\sin\frac{1}{2}(\alpha+\beta) = b\cos\frac{1}{2}(\alpha+\beta)$

$\Rightarrow \tan\frac{1}{2}(\alpha+\beta) = \frac{b}{a}$

এখন, L.H.S. $= \sin(\alpha+\beta) = \sin 2.\frac{1}{2}(\alpha+\beta)$

$= \frac{2\tan\frac{1}{2}(\alpha+\beta)}{1+\tan^2\frac{1}{2}(\alpha+\beta)} = \frac{2\frac{b}{a}}{1+(\frac{b}{a})^2}$

$= \frac{2b}{a}\times\frac{a^2}{a^2+b^2} = \frac{2ab}{a^2+b^2} =$ R.H.S.

8. প্রমাণ কর যে, $\tan 6^0 \tan 42^0 \tan 66^0 \tan 78^0 = 1$

প্রমাণ :

L.H.S. $= \tan 6^0 \tan 42^0 \tan 66^0 \tan 78^0$

$= \tan 6^0 \tan 66^0 \tan 42^0 \tan 78^0$

$= \tan(36^0 - 30^0)\tan(36^0 + 30^0)$

$\tan(60^0 - 18^0)\tan(60^0 + 18^0)$

$= \frac{\tan^2 36^0 - \tan^2 30^0}{1 - \tan^2 36^0 \tan^2 30^0}\frac{\tan^2 60^0 - \tan^2 18^0}{1 - \tan^2 60^0 \tan^2 18^0}$

$= \frac{5-2\sqrt{5}-\frac{1}{3}}{1-(5-2\sqrt{5})\frac{1}{3}}\frac{3-\frac{1}{5}(5-2\sqrt{5})}{1-3.\frac{1}{5}(5-2\sqrt{5})}$

$= \frac{15-6\sqrt{5}-1}{3-5+2\sqrt{5}}\frac{15-5+2\sqrt{5}}{5-15+6\sqrt{5}}$

$= \frac{14-6\sqrt{5}}{2\sqrt{5}-2}\frac{10+2\sqrt{5}}{6\sqrt{5}-10}$

$= \frac{2(7-3\sqrt{5}).2(5+\sqrt{5})}{2(\sqrt{5}-1).2(3\sqrt{5}-5)} = \frac{(7-3\sqrt{5})(5+\sqrt{5})}{(\sqrt{5}-1)(3\sqrt{5}-5)}$

$= \frac{35-15\sqrt{5}+7\sqrt{5}-15}{15-5\sqrt{5}-3\sqrt{5}+5} = \frac{20-8\sqrt{5}}{20-8\sqrt{5}}$

$= 1 =$ R.H.S. (Proved)

সম্ভাব্য ধাপসহ প্রশ্ন:

$$\Rightarrow 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$
$$= 2\times 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)$$
$$\Rightarrow \sin\frac{1}{2}(A+B)\{\cos\frac{1}{2}(A-B) - 2\cos\frac{1}{2}(A+B)\} = 0$$

$A+B\neq 0$ বলে $\sin\frac{1}{2}(A+B)\neq 0$

$$\therefore \cos\frac{1}{2}(A-B) - 2\cos\frac{1}{2}(A+B) = 0$$
$$\Rightarrow \cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2} - 2(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}) = 0$$
$$\Rightarrow 3\sin\frac{A}{2}\sin\frac{B}{2} = \cos\frac{A}{2}\cos\frac{B}{2}$$
$$\Rightarrow \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{1}{3}$$
$$\therefore \tan\frac{A}{2}\tan\frac{B}{2} = \frac{1}{3} \text{ (Showed)}$$

7 $a\cos\theta + b\sin\theta = c$ সমীকরণটি θ এর দুইটি ভিন্ন মান α, β দ্বারা সিদ্ধ হলে দেখাও যে,

$$\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$$

সমাধান ঃ $\cos\theta + b\sin\theta = c$ সমীকরণটি θ এর দুইটি ভিন্ন মান α ও β দ্বারা সিদ্ধ বলে,

$a\cos\alpha + b\sin\alpha = c$

এবং $a\cos\beta + b\sin\beta = c$

$\therefore a\cos\alpha + b\sin\alpha = a\cos\beta + b\sin\beta$

$\Rightarrow a(\cos\alpha - \cos\beta) = b(\sin\beta - \sin\alpha)$

$$\Rightarrow a.2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta-\alpha) = b.2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta-\alpha)$$

$\alpha\neq\beta$ বলে, $\sin\frac{1}{2}(\beta-\alpha)\neq 0$

$$\therefore a\sin\frac{1}{2}(\alpha+\beta) = b\cos\frac{1}{2}(\alpha+\beta)$$
$$\Rightarrow \tan\frac{1}{2}(\alpha+\beta) = \frac{b}{a}$$

এখন, L.H.S. $= \sin(\alpha+\beta) = \sin 2.\frac{1}{2}(\alpha+\beta)$

$$= \frac{2\tan\frac{1}{2}(\alpha+\beta)}{1+\tan^2\frac{1}{2}(\alpha+\beta)} = \frac{2\frac{b}{a}}{1+(\frac{b}{a})^2}$$
$$= \frac{2b}{a}\times\frac{a^2}{a^2+b^2} = \frac{2ab}{a^2+b^2} = \text{R.H.S.}$$

8. প্রমাণ কর যে, $\tan 6^0\tan 42^0\tan 66^0\tan 78^0 = 1$

প্রমাণ ঃ

L.H.S. $= \tan 6^0\tan 42^0\tan 66^0\tan 78^0$

$= \tan 6^0\tan 66^0\tan 42^0\tan 78^0$

$= \tan(36^0-30^0)\tan(36^0+30^0)\tan(60^0-18^0)\tan(60^0+18^0)$

$$= \frac{\tan^2 36^0 - \tan^2 30^0}{1-\tan^2 36^0\tan^2 30^0}\frac{\tan^2 60^0 - \tan^2 18^0}{1-\tan^2 60^0\tan^2 18^0}$$
$$= \frac{5-2\sqrt5-\frac{1}{3}}{1-(5-2\sqrt5)\frac{1}{3}}\frac{3-\frac{1}{5}(5-2\sqrt5)}{1-3.\frac{1}{5}(5-2\sqrt5)}$$
$$= \frac{15-6\sqrt5-1}{3-5+2\sqrt5}\frac{15-5+2\sqrt5}{5-15+6\sqrt5}$$
$$= \frac{14-6\sqrt5}{2\sqrt5-2}\frac{10+2\sqrt5}{6\sqrt5-10}$$
$$= \frac{2(7-3\sqrt5).2(5+\sqrt5)}{2(\sqrt5-1).2(3\sqrt5-5)} = \frac{(7-3\sqrt5)(5+\sqrt5)}{(\sqrt5-1)(3\sqrt5-5)}$$
$$= \frac{35-15\sqrt5+7\sqrt5-15}{15-5\sqrt5-3\sqrt5+5} = \frac{20-8\sqrt5}{20-8\sqrt5}$$

$= 1 =$ R.H.S. (Proved)

সম্ভাব্য ধাপসহ প্রশ্ন:

প্রমাণ কর যে,

9. $\cos^2\left(\frac{\alpha}{2}-18°\right)+\cos^2\left(\frac{\alpha}{2}+18°\right)$

$=\frac{1}{4}\{4+(\sqrt{5}+1)\cos\alpha\}$

$L.H.S.=\cos^2\left(\frac{\alpha}{2}-18°\right)+\cos^2\left(\frac{\alpha}{2}+18°\right)$

$=\frac{1}{2}\{1+\cos 2\left(\frac{\alpha}{2}-18°\right)+1+\cos 2\left(\frac{\alpha}{2}+18°\right)\}$ (১)

$=\frac{1}{2}\{2+\cos(\alpha-36°)+\cos(\alpha+36°)\}$

$=\frac{1}{2}(2+2\cos\alpha\cos 36°)$

$=\{1+\frac{1}{4}(\sqrt{5}+1)\cos\alpha\}$

$=\frac{1}{4}\{4+(\sqrt{5}+1)\cos\alpha\}=R.H.S.$ (১)

10.(a) $\sin(292.5)^0=-\frac{1}{2}\sqrt{2+\sqrt{2}}$

$L.H.S.=\sin(292.5)^0$

$=\sin\{270^0+(22.5)^0\}=-\cos(22.5)^0$ (১)

$=-\sqrt{\cos^2(22.5)^0}=-\sqrt{\frac{1}{2}(1+\cos 45^0)}$ (১)

$=-\sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{2}})}=-\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

$=-\sqrt{\frac{2+\sqrt{2}}{4}}=-\frac{1}{2}\sqrt{2+\sqrt{2}}=R.H.S.$ (১)

10.(b) $\cot(142.5)^0=\sqrt{2}+\sqrt{3}-2-\sqrt{6}$

$L.H.S.=\cot(142.5)^0=\cot 142^0 30'$

$=\cot(180^0-37^0 30')=-\cot 37^0 30'$ (১)

$=-\frac{\cos 37^0 30'}{\sin 37^0 30'}=-\frac{2\cos^2 37^0 30'}{2\sin 37^0 30'\cos 37^0 30'}$

$=-\frac{1+\cos 75^0}{\sin 75^0}=-\frac{1+\cos(45^0+30^0)}{\sin(45^0+30^0)}$ (১)

$=-\frac{1+\cos 45^0\cos 30^0-\sin 45^0\sin 30^0}{\sin 45^0\cos 30^0+\cos 45^0\sin 30^0}$

$=-\frac{1+\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\frac{1}{2}}{\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\frac{1}{2}}=-\frac{2\sqrt{2}+\sqrt{3}-1}{\sqrt{3}-1}$

$=-\frac{(2\sqrt{2}+\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$ (১)

$=-\frac{2\sqrt{6}+3-\sqrt{3}-2\sqrt{2}-\sqrt{3}+1}{3-1}$

$=-\frac{2\sqrt{6}+4-2\sqrt{3}-2\sqrt{2}}{2}$

$=-(\sqrt{6}+2-\sqrt{3}-\sqrt{2})=\sqrt{3}+\sqrt{2}-2-\sqrt{6}$ (১)

10(c) $\tan(82.5)^0=\sqrt{6}+\sqrt{3}+\sqrt{2}+2$

$L.H.S.=\tan(82.5)^0=\tan 82^0 30'$

$=\tan(90^0-7^0 30')=\cot 7^0 30'$ (১)

$=\frac{\cos 7^0 30'}{\sin 7^0 30'}=\frac{2\cos^2 7^0 30'}{2\sin 7^0 30'\cos 7^0 30'}$

$=\frac{1+\cos 15^0}{\sin 15^0}=\frac{1+\cos(45^0-30^0)}{\sin(45^0-30^0)}$ (১)

$=\frac{1+\cos 45^0\cos 30^0+\sin 45^0\sin 30^0}{\cos 45^0\cos 30^0-\sin 45^0\cos 30^0}$

$=\frac{1+\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\frac{1}{2}}{\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\frac{1}{2}}=\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$

$=\frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$ (১)

$=\frac{2\sqrt{6}+3+\sqrt{3}+2\sqrt{2}+\sqrt{3}+1}{3-1}$

$=\frac{2\sqrt{6}+4+2\sqrt{3}+2\sqrt{2}}{2}$

$=\sqrt{6}+2+\sqrt{3}+\sqrt{2}=\sqrt{6}+\sqrt{3}+2+\sqrt{2}$ (১)

11. $a\sin\theta+b\sin\varphi=c=a\cos\theta+b\cos\varphi$ হলে দেখাও যে,

$$\cos\frac{1}{2}(\theta-\varphi)=\pm\sqrt{\frac{2c^2-(a-b)^2}{4ab}}$$

প্রমাণ ঃ দেওয়া আছে , $a \sin\theta + b \sin\varphi = c$

$\Rightarrow a^2 \sin^2\theta + b^2 \sin^2\varphi + 2ab\sin\theta \sin\varphi = c^2 \quad \cdots(1)$

এবং $a \cos\theta + b \cos\varphi = c$

$\Rightarrow a^2 \cos^2\theta + b^2 \cos^2\varphi + 2ab\cos\theta \cos\varphi = c^2 \quad \cdots(2)$

(1) ও (2) যোগ করে পাই ,

$a^2 + b^2 + 2ab(\sin\theta \sin\varphi + \cos\theta \cos\varphi) = 2c^2$

$\Rightarrow 2ab \cos(\theta - \varphi) = 2c^2 - a^2 - b^2$ (১)

$\Rightarrow 2ab\{2\cos^2 \frac{1}{2}(\theta - \varphi) - 1\} = 2c^2 - a^2 - b^2$

$\Rightarrow 4ab \cos^2 \frac{1}{2}(\theta - \varphi) = 2c^2 - a^2 - b^2 + 2ab$

$= 2c^2 - (a-b)^2$

$\Rightarrow \cos^2 \frac{1}{2}(\theta - \varphi) = \frac{2c^2 - (a-b)^2}{4ab}$

$\therefore \cos \frac{1}{2}(\theta - \varphi) = \pm\sqrt{\frac{2c^2 - (a-b)^2}{4ab}}$ (১)

12. দেখাও যে, $\sin x = 2^n \cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.\cdots\cdots\cos\frac{x}{2^n}.\sin\frac{x}{2^n}$

প্রমাণ : $\sin x = \sin 2.\frac{x}{2} = 2\sin\frac{x}{2}.\cos\frac{x}{2}$ (১)

$= 2\cos\frac{x}{2}.\sin 2.\frac{x}{2^2} = 2\cos\frac{x}{2}.2\sin\frac{x}{2^2}.\cos\frac{x}{2^2}$

$= (2\cos\frac{x}{2}).(2\cos\frac{x}{2^2}).\sin\frac{x}{2^2}$

$= (2\cos\frac{x}{2}).(2\cos\frac{x}{2^2}).(2\cos\frac{x}{2^3}).\sin\frac{x}{2^3}$

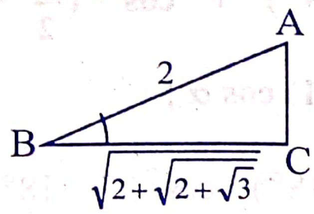
$= (2\cos\frac{x}{2}).(2\cos\frac{x}{2^2}).(2\cos\frac{x}{2^3}).\cdots\cdots\cdots$

$\cdots (2\cos\frac{x}{2^{n-1}})(2\cos\frac{x}{2^n}).\sin\frac{x}{2^n}$

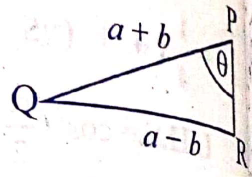
$\therefore \sin x = 2^n \cos\frac{x}{2}.\cos\frac{x}{2^2}\cdots\cdots\cos\frac{x}{2^n}.\sin\frac{x}{2^n}$ (১)

সৃজনশীল প্রশ্ন:

13.



দৃশ্যকল্প-১:



দৃশ্যকল্প -২:

(a) $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}$ হলে প্রমাণ কর যে, $\cos\varphi = \frac{\cos\theta - e}{1 - e\cos\theta}$. [ঢা.'১৪,'১৫; চ.'০৮; সি.'০৮,'১২,'১৫; রা.'০৯,'১৫; মা.'১৩; কু.,ব.'১৫]

প্রমাণ : দেওয়া আছে , $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}$

$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1-e}{1+e}\tan^2\frac{\varphi}{2}$

$\Rightarrow \frac{1}{\tan^2\frac{\varphi}{2}} = \frac{1-e}{1+e}\frac{1}{\tan^2\frac{\theta}{2}} = \frac{(1-e)\cos^2\frac{\theta}{2}}{(1+e)\sin^2\frac{\theta}{2}}$

$\Rightarrow \frac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}} = \frac{(1-e)\cos^2\frac{\theta}{2} - (1+e)\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2} + (1+e)\sin^2\frac{\theta}{2}}$

$= \frac{(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}) - e(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2})}{(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}) - e(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2})}$

$\therefore \cos\varphi = \frac{\cos\theta - e}{1 - e\cos\theta}$

(b) দৃশ্যকল্প-১ হতে দেখাও যে, $\angle B = (7\frac{1}{2})^0$

প্রমাণ : দৃশ্যকল্প-১ হতে পাই, $\cos B = \frac{BC}{AB}$

$\Rightarrow \cos B = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}$

$= \frac{1}{2}\sqrt{2+\sqrt{2(1+\frac{\sqrt{3}}{2})}}$

$=\frac{1}{2}\sqrt{2+\sqrt{2(1+\cos 30^0)}}$

$=\frac{1}{2}\sqrt{2+\sqrt{2.2\cos^2 15^0)}} = \frac{1}{2}\sqrt{2+2\cos 15^0}$

$=\frac{1}{2}\sqrt{2(1+\cos 15^0)} = \frac{1}{2}\sqrt{2.2\cos^2 (7\frac{1}{2})^0}$

$=\frac{1}{2}.2\cos(7\frac{1}{2})^0 = \cos(7\frac{1}{2})^0$

$\therefore \angle B = (7\frac{1}{2})^0$; যেহেতু $\angle B$ সূক্ষ্মকোণ

(c) দৃশ্যকল্প -২ হতে প্রমাণ কর যে,

$\tan(\frac{\pi}{4}-\frac{\theta}{2}) = \sqrt{\frac{b}{a}}$.

প্রমাণ : দৃশ্যকল্প-১ হতে পাই, $\sin\theta = \frac{QR}{PQ}$

$\Rightarrow \sin\theta = \frac{a-b}{a+b}$

$\therefore$ L.H.S. $= \tan(\frac{\pi}{4}-\frac{\theta}{2}) = \frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{\cos(\frac{\pi}{4}-\frac{\theta}{2})}$

$=\frac{2\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})}{2\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}$

$=\frac{\sin 2(\frac{\pi}{4}-\frac{\theta}{2})}{1+\cos 2(\frac{\pi}{4}-\frac{\theta}{2})} = \frac{\sin(\frac{\pi}{2}-\theta)}{1+\cos(\frac{\pi}{2}-\theta)}$

$=\frac{\cos\theta}{1+\sin\theta} = \frac{\sqrt{1-\sin^2\theta}}{1+\sin\theta} = \frac{\sqrt{1-(\frac{a-b}{a+b})^2}}{1+\frac{a-b}{a+b}}$

$=\frac{\frac{\sqrt{(a+b)^2-(a-b)^2}}{a+b}}{\frac{a+b+a-b}{a+b}} = \frac{\sqrt{4ab}}{2a}$

$=\frac{2\sqrt{a}\sqrt{b}}{2a} = \sqrt{\frac{b}{a}}$ = R.H.S.

বিকল্প পদ্ধতি : দৃশ্যকল্প-১ হতে পাই, $\sin\theta = \frac{QR}{PQ}$

$\Rightarrow \sin\theta = \frac{a-b}{a+b}$

$\Rightarrow \frac{1}{\sin\theta} = \frac{a+b}{a-b} \Rightarrow \frac{1-\sin\theta}{1+\sin\theta} = \frac{a+b-a+b}{a+b+a-b}$

[বিয়োজন-যোজন করে।]

$\Rightarrow \frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\cos\frac{\theta}{2}\sin\frac{\theta}{2}} = \frac{2b}{2a}$

$\Rightarrow \frac{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})^2}{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})^2} = \frac{b}{a} \Rightarrow \frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}} = \sqrt{\frac{b}{a}}$

$\Rightarrow \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = \sqrt{\frac{b}{a}} \Rightarrow \frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}} = \sqrt{\frac{b}{a}}$

$\therefore \tan(\frac{\pi}{4}-\frac{\theta}{2}) = \sqrt{\frac{b}{a}}$

প্রশ্নমালা VII F

$A+B+C=\pi$ হলে প্রমাণ কর যে,

1. (a) $\sin A+\sin B+\sin C$

$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ [য.'০২]

প্রমাণ ঃ L.H.S. $= \sin A+\sin B+\sin C$

$=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)+2\sin\frac{C}{2}\cos\frac{C}{2}$

$=2\sin(\frac{\pi}{2}-\frac{C}{2})\cos\frac{1}{2}(A-B)+2\sin\frac{C}{2}\cos\frac{C}{2}$

$=2\cos\frac{C}{2}\cos\frac{1}{2}(A-B)+2\sin\frac{C}{2}\cos\frac{C}{2}$

$=2\cos\frac{C}{2}\{\cos\frac{1}{2}(A-B)+\sin\frac{C}{2}\}$

$=2\cos\frac{C}{2}\{\cos\frac{1}{2}(A-B)+\sin(\frac{\pi}{2}-\frac{A+B}{2})\}$

$=2\cos\frac{C}{2}\{\cos(\frac{A}{2}-\frac{B}{2})+\cos(\frac{A}{2}+\frac{B}{2})\}$