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ROSS

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

FOURTH EDITION

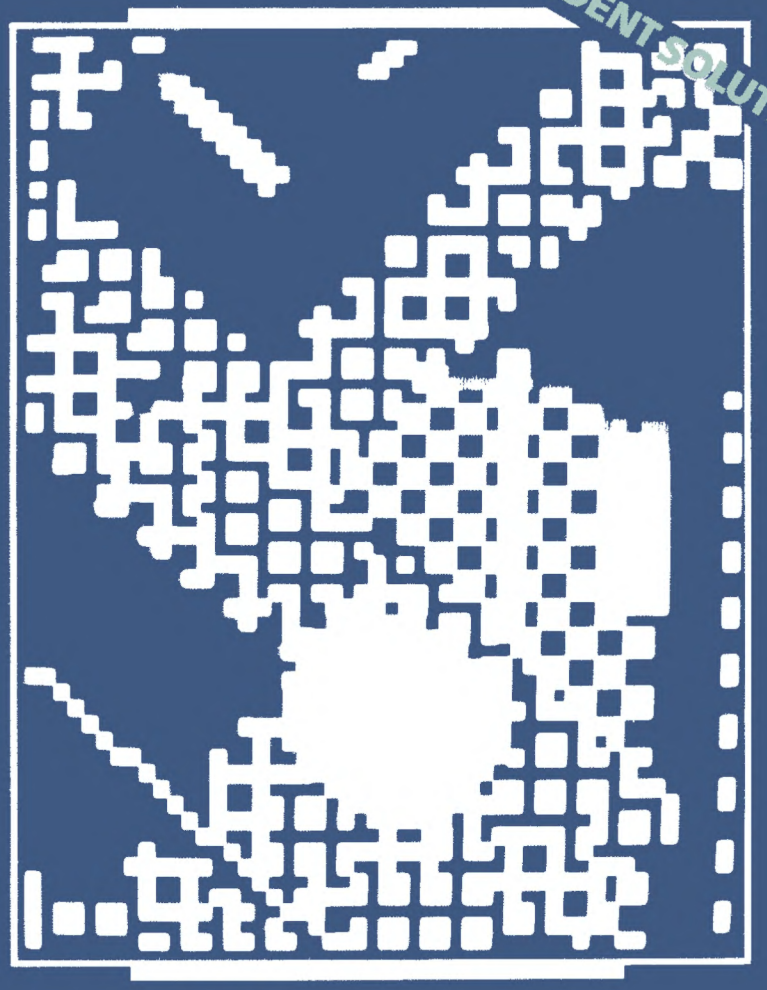
STUDENT SOLUTIONS

WILEY

SHEPLEY L. ROSS

# INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

STUDENT SOLUTIONS



FOURTH EDITION

# Student Solutions Manual

to accompany

# INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

Fourth Edition

**Shepley L. Ross**
University of New Hampshire

*With the assistance of*

**Shepley L. Ross, II**
Bates College

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# Preface

This manual is a supplement to the author's text, *Introduction to Ordinary Differential Equations*, Fourth Edition. It contains the answers to the even-numbered exercises and detailed solutions to approximately one half of both the even- and odd-numbered exercises in the regular exercise sets.

The following abbreviations have been used:

- D.E. differential equation
- G.S. general solution
- I.F. integrating factor
- I.C. initial condition
- I.V.P. initial value problem.

The author expresses his thanks to his son Shepley L. Ross, II, of Bates College for his many contributions, especially the solutions and graphs of Chapter 8. The author also thanks his colleague Ellen O'Keefe and his graduate students Rita Fairbrother and Chris McDevitt for providing solutions.

— *Shepley L. Ross*

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# Chapter 1

**Section 1.2, Page 11**

1. (d) We must show that $f(x) = (1 + x^2)^{-1}$ satisfies the D.E. $(1 + x^2)y'' + 4xy' + 2y = 0$. Differentiating $f(x)$, we find $f'(x) = -(1 + x^2)^{-2}(2x)$ and $f''(x) = (6x^2 - 2)(1 + x^2)^{-3}$. We now substitute $f(x)$ for $y$, $f'(x)$ for $y'$, and $f''(x)$ for $y''$ in the stated D.E. We obtain

$$(1 + x^2)(6x^2 - 2)(1 + x^2)^{-3} - 4x(1 + x^2)^{-2}(2x) + 2(1+x^2)^{-1} = 0.$$

This reduces to

$$(6x^2 - 2 - 8x^2)(1 + x^2)^2 + 2(1 + x^2)^{-1} = 0,$$

and hence to

$$[(-2x^2 - 2) + (2 + 2x^2)](1 + x^2)^{-2} = 0,$$

that is, $0(1 + x^2)^{-2} = 0$ or $0 = 0$. Hence the given D.E. is satisfied by $f(x) = (1 + x^2)^{-1}$.

2. (a) We must show that the relation $x^3 + 3xy^2 = 1$ defines at least one real function which is an explicit solution of the given D.E. on $0 < x < 1$. Solving the given relation for $y$, we obtain

$$y^2 = \frac{1 - x^3}{3x}, \quad y = \pm\left[\frac{1 - x^3}{3x}\right]^{1/2} \qquad (1)$$

We choose the plus sign and consider the function f defined by

$$f(x) = \left[\frac{1 - x^3}{3x}\right]^{1/2}, \quad 0 < x < 1.$$

We note that $(1 - x^3)/3x > 0$ for $0 < x < 1$, and hence $f(x)$ is indeed defined on $0 < x < 1$. Now differentiating and simplifying, we find

$$f'(x) = -\frac{1}{6}\left[\frac{1 - x^3}{3x}\right]^{-1/2}\left[\frac{2x^3 + 1}{x^2}\right].$$

Substituting $f(x)$ for $y$ and $f'(x)$ for $y'$ in the given D.E., we find

$$2x\left[\frac{1 - x^3}{3x}\right]^{1/2}\left\{-\frac{1}{6}\left[\frac{1 - x^3}{3x}\right]^{-1/2}\left[\frac{2x^3 + 1}{x^2}\right]\right\}$$

$$+ x^2 + \frac{1 - x^3}{3x} = 0.$$

This simplifies to $-\frac{2x^3 + 1}{3x} + x^2 + \frac{1 - x^3}{3x} = 0$ and thence to $\frac{0}{3x} = 0$. Thus $f(x)$ is an explicit solution of the D.E. on $0 < x < 1$, and so the given relation $x^3 + 3xy^2 = 1$ is an implicit solution on $0 < x < 1$. We note that choosing the minus sign in (1) would also have led to an explicit solution of the D.E. on $0 < x < 1$.

3. (a) We must show that $f(x) = (x^3 + c)e^{-3x}$ satisfies the D.E. $y' + 3y = 3x^2e^{-3x}$. Differentiating $f(x)$, we find $f'(x) = (x^3 + c)(-3e^{-3x}) + e^{-3x}(3x^2)$. We not substitute $f(x)$ for $y$ and $f'(x)$ for $y'$ in the stated D.E. We obtain

$$(x^3 + c)(-3e^{-3x}) + e^{-3x}(3x^2) + 3(x^3 + c)e^{-3x} = 3x^2e^{-3x}.$$

The left member reduces to $3x^2e^{-3x}$, so we have $3x^2e^{-3x} = 3x^2e^{-3x}$; and thus the D.E. is satisfied by $f(x) = (x^3 + c)e^{-3x}$.

4. (b) We must show that $g(x) = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x}$ satisfies the D.E. $y''' - 2y'' - 4y' + 8y = 0$. Differentiating $g(x)$, we find $g'(x) = 2c_1e^{2x} + 2c_2xe^{2x} + c_2e^{2x} - 2c_3e^{-2x}$, $g''(x) = 4c_1e^{2x} + 4c_2xe^{2x} + 4c_2e^{2x} + 4c_3e^{-2x}$, and $g'''(x) = 8c_1e^{2x} + 8c_2xe^{2x} + 12c_2e^{2x} - 8c_3e^{2x}$. We now substitute $g(x)$ for $y$, $g'(x)$ for $y'$. $g''(x)$ for $y''$, and $g'''(x)$ for $y'''$ in the stated D.E. We obtain

$$\begin{aligned} &8c_1e^{2x} + 8c_2xe^{2x} + 12c_2e^{2x} - 8c_3e^{-2x} \\ &\quad - 2(4c_1e^{2x} + 4c_2xe^{2x} + 4c_2e^{2x} + 4c_3e^{-2x}) \\ &\quad - 4(2c_1e^{2x} + 2c_2xe^{2x} + c_2e^{2x} - 2c_3e^{-2x}) \\ &\quad + 8(c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x}) = 0. \end{aligned}$$

This reduces to

$$\begin{aligned} &8c_1e^{2x} + 8c_2xe^{2x} + 12c_2e^{2x} - 8c_3e^{-2x} \\ &\quad - 8c_1e^{2x} - 8c_2xe^{2x} - 8c_2e^{2x} - 8c_3e^{-2x} \\ &\quad - 8c_1e^{2x} - 8c_2xe^{2x} - 4c_2e^{2x} + 8c_3e^{-2x} \\ &\quad + 8c_1e^{2x} + 8c_2xe^{2x} + 8c_3e^{-2x} = 0 \end{aligned}$$

and hence to

$$\begin{aligned} &(8c_1 + 12c_2 - 8c_1 - 8c_2 - 8c_1 - 4c_2 + 8c_1)e^{2x} \\ &\quad + (8c_2 - 8c_2 - 8c_2 + 8c_2)\, xe^{2x} \\ &\quad + (-8c_3 - 8c_3 + 8c_3 + 8c_3)e^{-2x} = 0, \end{aligned}$$

that is, $0 \cdot e^{2x} + 0 \cdot xe^{2x} + 0 \cdot e^{-2x} = 0$, or $0 = 0$. Hence the given D.E. is satisfied by $g(x) = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x}$.

5. (a) To determine the values of m for which $f(x) = e^{mx}$ satisfies the given D.E., we differentiate f(x) the required number of times and substitute into the D.E. We have $f(x) = e^{mx}$, $f'(x) = me^{mx}$, $f''(x) = m^2e^{mx}$, $f'''(x) = m^3e^{mx}$. Then substituting f(x) for y, f′(x) for y′, f″(x) for y″, and f‴(x) for y‴, we find $m^3e^{mx} - 3m^2e^{mx} - 4me^{mx} + 12e^{mx} = 0$ or $e^{mx}(m^3 - 3m^2 - 4m + 12) = 0$. Then since $e^{mx} \neq 0$ for all m and x, we must have $m^3 - 3m^2 - 4m + 12 = 0$. That is, if $f(x) = e^{mx}$ is a solution of the given D.E., then the constant m must satisfy the cubic equation $m^3 - 3m^2 - 4m + 12 = 0$. By inspection we observe that m = 2 is a root, and so

$(m - 2)$ is a factor of the left member. We now use synthetic division to find the other factor. We have

| | | | | |
|---|---|---|---|---|
| 2 | 1 | -3 | -4 | 12 |
| | | 2 | -2 | -12 |
| | 1 | -1 | -6 | 0 |

From this we see that the reduced quadratic factor is $m^2 - m - 6$. Hence the cubic equation may be written $(m - 2)(m^2 - m - 6) = 0$ and so $(m - 2)(m - 3)(m + 2) = 0$. Thus we see that its roots are $m = 2, 3, -2$. These then are the values of m for which $f(x) = e^{mx}$ is a solution of the given D.E.

6. (b) From $f(x) = 3e^{2x} - 2xe^{2x} - \cos 2x$, we find $f'(x) = 4e^{2x} - 4xe^{2x} + 2 \sin 2x$, $f''(x) = 4e^{2x} - 8xe^{2x} + 4 \cos 2x$. We substitute $f(x)$ for y, $f'(x)$ for $y'$, $f''(x)$ for $y''$ in the given D.E., obtaining

$$(4e^{2x} - 8xe^{2x} + 4 \cos 2x) - 4(4e^{2x} - 4xe^{2x} + 2 \sin 2x) + 4(3e^{2x} - 2xe^{2x} - \cos 2x) = -8 \sin 2x .$$

Collecting like terms in the left member, we find

$$(4 - 16 + 12)e^{2x} + (-8 + 16 - 8)xe^{2x} + (4 - 4) \cos 2x - 8 \sin 2x = -8 \sin 2x$$

or $$-8 \sin 2x = -8 \sin 2x .$$

Thus $f(x)$ satisfies the D.E. Now note that $f(0) = 3e^0 - 2(0)e^0 - \cos 0 = 3 - 0 - 1 = 2$ and $f'(0) = 4e^0 - 4(0)e^0 + 2 \sin 0 = 4 - 0 + 0 = 4$. Hence $f(x)$ also satisfies the stated conditions.

7. (a) We must show that the D.E. $|y'| + |y| + 1 = 0$ has no (real) solutions. Assume, to the contrary, that this D.E. has a real differentiable function f as a solution on some interval I. Since f is a solution on I, we must have $|f'(x)| + |f(x)| + 1 = 0$ for all $x \in I$. But for a real function f, both $|f'(x)| \geq 0$ and $|f(x)| \geq 0$, so $|f'(x)| + |f(x)| + 1 \geq 1$ for all $x \in I$. Thus we obtain two contradictory statements about $|f'(x)| + |f(x)| + 1$, and so our original assumption that the D.E. has a real solution is incorrect. So it has no real solutions.

**Section 1.3, Page 21**

2. (a) We apply the I.C. $y(0) = 2$ to the given family of solutions. That is, we let $x = 0$, $y = 2$ in $y = (x^2 + c)e^{-x}$. We obtain $2 = (0 + c)e^0$ and hence $c = 2$. We thus obtain the particular solution $y = (x^2 + 2)e^{-x}$ satisfying the stated I.V.P.

3. (a) We first apply the I.C. $y(0) = 5$ to the given family of solutions. That is, we let $x = 0$, $y = 5$ in $y = c_1e^{4x} + c_2e^{-3x}$. We obtain $c_1 + c_2 = 5$. We next differentiate the given family to obtain $y' = 4c_1e^{4x} - 3c_2e^{-3x}$. We apply the I.C. $y'(0) = 6$ to this derived

relation. That is, we let $x = 0$, $y' = 6$ in $y' = 4c_1e^{4x} - 3c_2e^{-3x}$. We obtain $4c_1 - 3c_2 = 6$. The two equations

$$\begin{cases} c_1 + c_2 = 5, \\ 4c_1 - 3c_2 = 6 \end{cases}$$ determine $c_1$ and $c_2$ uniquely.

Solving this system, we find $c_1 = 3$, $c_2 = 2$. Substituting these values back into $y = c_1e^{4x} + c_2e^{-3x}$ we obtain the particular solution $y = 3e^{4x} + 2e^{-3x}$ satisfying the stated I.V.P.

4. (a) We first apply the B.C. $y(0) = 0$ to the given family of solutions. That is, we let $x = 0$, $y = 0$, $y = c_1 \sin x + c_2 \cos x$. We obtain $c_2 = 0$. We next apply the B.C. $y(\pi/2) = 1$ to the given family of solutions. That is, we let $x = \pi/2$, $y = 1$ in $y = c_1 \sin x + c_2 \cos x$. We obtain $c_1 = 1$. Substituting the values $c_1 = 1$, $c_2 = 0$ back into $y = c_1 \sin x + c_2 \cos x$, we obtain the particular solution $y = \sin x$ satisfying the stated boundary-value problem.

   (c) We first apply the B.C. $y(0) = 0$ to the given family of solutions. That is, we let $x = 0$, $y = 0$ in $y = c_1 \sin x + c_2 \cos x$. We obtain $c_2 = 0$. We next apply the B.C. $y(\pi) = 1$ to the given family of

solutions. That is, we let $x = \pi$, $y = 1$ in $y = c_1 \sin x + c_2 \cos x$. We obtain $-c_2 = 1$, so $c_2 = -1$. At this point we have both $c_2 = 0$ and, at the same time, $c_2 = -1$. This is impossible! So the given boundary-value problem has no solution.

5. We are given that every solution of the stated D.E. may be written in the form

$$y = c_1 x + c_2 x^2 + c_3 x^3 \tag{1}$$

for some choice of the constants $c_1$, $c_2$, $c_3$. We must determine these constants so that (1) will satisfy the three stated conditions. We differentiate (1) twice to obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 \tag{2}$$

and

$$y'' = 2c_2 x + 6c_3 x \tag{3}$$

We now apply the condition $y(2) = 0$ to (1), letting $x = 2$, $y = 0$ in (1). We obtain $2c_1 + 4c_2 + 8c_3 = 0$. Similarly, we apply the condition $y'(2) = 2$ to (2), thereby obtaining $c_1 + 4c_2 + 12c_3 = 2$. Finally, we apply the condition $y''(2) = 6$ to (3), obtaining $2c_2 + 12c_3 = 6$. Thus we have the three equations

$$\begin{cases} 2c_1 + 4c_2 + 8c_3 = 0, \\ c_1 + 4c_2 + 12c_3 = 2, \\ 2c_2 + 12c_3 = 6, \end{cases}$$

in the three unknowns. These can be solved in various ways. One easy way is to eliminate $c_1$ from the first two equations, obtaining the equivalent of $c_2 + 4c_3 = 1$. Combining this last with $c_2 + 6c_3 = 3$, which is equivalent to the third equation of the system, we readily find $c_2 = -3$, $c_3 = 1$. Then from the second equation one finds $c_1 = 2$. Thus $c_1 = 2$, $c_2 = -3$, $c_3 = 1$. Substituting these values back into (1), we find the solution of the stated I.V.P. is

$$y = 2x - 3x^2 + x^3.$$

(a) Let us apply Theorem 1.1. We first check the hypothesis. Here $f(x,y) = x^2 \sin y$ and $\frac{\partial f(x,y)}{\partial y} = x^2 \cos y$. Both f and $\frac{\partial f}{\partial y}$ are continuous in every domain D in the xy plane. The initial condition $y(1) = -2$ means that $x_0 = 1$ and $y_0 = -2$, and the point $(1, -2)$ certainly lies in some such domain D. Thus all hypothesis are satisfied and the conclusion holds. That is, there is a unique solution $\phi$ of the D.E. $y' = x^2 \sin y$, defined on some interval $|x - 1| \leq h$ about $x_0 = 1$, which satisfies the initial condition, that is, which is such that $\phi(1) = -2$.

# Chapter 2

**Section 2.1, Page 36**

In these solutions we denote derivatives by primes and partial derivatives by subscripts. The solutions of Exercises 3, 5, 6, 7, and 9 follow the pattern of Example 2.5 on page 31.

3. Here $M(x,y) = 2xy + 1$, $N(x,y) = x^2 + 4y$. From these we find $M_y(x,y) = 2x = N_x(x,y)$, so the D.E. is exact. We seek $F(x,y)$ such that

$$F_x(x,y) = M(x,y) = 2xy + 1 \text{ and } F_y(x,y) = N(x,y) = x^2 + 4y.$$

From the first of these, we find

$$F(x,y) = \int M(x,y)\,\partial x + \phi(y) = \int (2xy + 1)\,\partial x + \phi(y)$$
$$= x^2y + x + \phi(y). \text{ From this,}$$
$$F(x,y) = x^2 + \phi'(y).$$

But we must have $F_y(x,y) = N(x,y) = x^2 + 4y$. Therefore

$$x^2 + \phi'(y) = x^2 + 4y$$

or $\frac{d\phi(y)}{dy} = 4y$. Then $\phi(y) = 2y^2 + c_0$. Thus $F(x,y) = x^2y + x + 2y^2 + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is $x^2y + x + 2y^2 = c$, where $c = c_1 - c_0$.

Alternatively, by the method of grouping, we first write the D.E. in the form $2xy\,dx + x^2\,dy + dx + 4y\,dy = 0$. We recognize this as $d(x^2y) + d(x) + d(2y^2) = d(c)$ or $d(x^2y + x + 2y^2) = d(c)$. Hence we obtain the solution $x^2y + x + 2y^2 = c$.

4. Here $M(x,y) = 3x^2y + 2$, $N(x,y) = -(x^3 + y)$. From these we find $M_y(x,y) = 3x^2 \neq -3x^2 = N_x(x,y)$. Since $M_y(x,y) \neq N_x(x,y)$, the D.E. is not exact.

5. Here $M(x,y) = 6xy + 2y^2 - 5$, $N(x,y) = 3x^2 + 4xy - 6$. From these we find $M_y(x,y) = 6x + 4y = N_x(x,y)$, so the D.E. is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 6xy + 2y^2 - 5$ and $F_y(x,y) = N(x,y) = 3x^2 + 4xy - 6$. From the first of these, we find $F(x,y) = \int M(x,y)\,\partial x + \phi(y)$ $\int (6xy + 2y^2 - 5)\,\partial x + \phi(y) = 3x^2y + 2y^2x - 5x + \phi(y)$. From this, $F_y(x,y) = 3x^2 + 4xy + \phi(y)$. But we must have $F_y(x,y) = N(x,y) = 3x^2 + 4xy - 6$. Therefore

$$3x^2 + 4xy + \phi'(y) = 3x^2 + 4xy - 6,$$

or $\phi'(y) = -6$. Then $\phi(y) = -6y + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is $3x^2y + 2y^2x - 5x - 6y = c$, where $c = c_1 - c_0$.

Alternatively, by the method of grouping, we first write the D.E. in the form $(6xy\,dx + 3x^2dy) + (2y^2dx + 4xy\,dy) - 5dx - 6dy = 0$. We recognize this as $d(3x^2y) + d(2y^2x) - d(5x) - d(6y) = d(c)$ or $d(3x^2y + 2y^2x - 5x - 6y) = d(c)$. Hence we obtain the solution $3x^2y + 2y^2x - 5x - 6y = c$.

6. Here $M(r,\theta) = (\theta^2 + 1)\cos r$, $N(r,\theta) = 2\theta \sin r$. From these we find $M_\theta(r,\theta) = 2\theta \cos r = N_r(r,\theta)$, so the D.E. is exact. We seek $F(r,\theta)$ such that $F_r(r,\theta) = M(r,\theta) = (\theta^2 + 1)\cos r$ and $F_\theta(r,\theta) = N(r,\theta) = 2\theta \sin r$. From the first of these, we find $F(r,\theta) = \int M(r,\theta)\,\partial r + \phi(\theta) = \int (\theta^2 + 1)\cos r\,\partial r + \phi(\theta) = (\theta^2 + 1)\sin r + \phi(\theta)$. From this, $F_\theta(r,\theta) = 2\theta \sin r + \phi'(\theta)$. But we must have $F_\theta(r,\theta) = N(r,\theta) = 2\theta \sin r$. Therefore

$$2\theta \sin r + \phi'(\theta) = 2\theta \sin r$$

or $\phi'(\theta) = 0$. Then $\phi'(\theta) = c_0$. Thus $F(r,\theta) = (\theta^2 + 1)\sin r + c_0$. The one-parameter family of solutions $F(r,\theta) = c_1$ is $(\theta^2 + 1)\sin r = c_1$ where $c = c_1 - c_0$.

Alternately, by the method of grouping, we first write the D.E. in the form $(\theta^2 \cos r\,dr + 2\theta \sin r\,d\theta) + \cos r\,dr = 0$. We recognize this as $d(\theta^2 \sin r) + d(\sin r) = d(c)$ or $d(\theta^2 \sin r + \sin r) = d(c)$. Hence we obtain the solution $\theta^2 \sin r + \sin r = c$, or $(\theta^2 + 1)\sin r = c$.

7. Here $M(x,y) = y \sec^2 x + \sec x \tan x$, $N(x,y) = \tan x + 2y$. From these we find $M_y(x,y) = \sec^2 x = N_x(x,y)$, so the D.E. is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = y \sec^2 x + \sec x \tan x$ and $F_y(x,y) = N(x,y) = \tan x + 2y$. From the first of these, we find $F(x,y) = \int M(x,y)\,\partial x + \phi(y) = \int (y \sec^2 x + \sec x \tan x)\,\partial x + \phi(y) = y \tan x + \sec x + \phi(y)$. From this, $F_y(x,y) = \tan x + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = \tan x + 2y$. Therefore, $\tan x + \phi'(y) = \tan x + 2y$ or $\phi'(y) = 2y$. Then $\phi(y) = y^2 + c_0$. Thus $F(x,y) = y \tan x + \sec x + y^2 + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is $y \tan x + \sec x + y^2 = c$, where $c = c_1 - c_0$.

Alternatively, by the method of grouping, we first write D.E. in the form $(y \sec^2 x\,dx + \tan x\,dy) + \sec x \tan x\,dx + 2y\,dy = 0$. We recognize this as $d(y \tan x) + d(\sec x) + d(y^2) = d(c)$, and hence obtain the solution $y \tan x + \sec x + y^2 = c$.

8. Here $M(x,y) = \frac{x}{y^2} + x$. $N(x,y) = \frac{x^2}{y^3} + y$. From these we find $M_y(x,y) = \frac{-2x}{y^3} \neq \frac{2x}{y^3} = N_x(x,y)$. Since $M_y(x,y) \neq N_x(x,y)$, the D.E. is not exact.

9. Here $M(s,t) = \frac{2s - 1}{t}$, $N(s,t) = \frac{s - s^2}{t^2}$. From these we find $M_t(s,t) = \frac{1 - 2s}{t^2} = N_s(s,t)$, so the D.E. is exact. We seek $F(s,t)$ such that $F_s(s,t) = M(s,t) = \frac{2s - 1}{t}$ and $F_t(s,t) = N(s,t) = \frac{s - s^2}{t^2}$. From the first of these, we find $F(s,t) = \int M(s,t)\,\partial s + \phi(t) = \int \frac{2s - 1}{t}\,\partial s + \phi(t) = \frac{s^2 - s}{t} + \phi(t)$. From this, $F_t(s,t) = \frac{s - s^2}{t^2} + \phi'(t)$. Therefore, $\frac{s - s^2}{t^2} + \phi'(t) = \frac{s - s^2}{t^2}$, or $\phi'(t) = 0$. Then $\phi(t) = c_0$. Thus $F(s,t) = \frac{s - s^2}{t} + c_0$. The one-parameter family of solutions $F(s,t) = c_1$ is $\frac{s^2 - s}{t} + c_0 = c_1$ or $s^2 - s = ct$, where $c = c_1 - c_0$.

The solutions of Exercises 12, 13, 14, and 16 follow the pattern of Example 2.6 on page 32.

12. Here $M(x,y) = 3x^2y^2 - y^3 + 2x$, $N(x,y) = 2x^3y - 3xy^2 + 1$. From these we find $M_y(x,y) = 6x^2y - 3y^2 = N_x(x,y)$, so D.E. is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 3x^2y^2 - y^3 + 2x$ and $F_y(x,y) = N(x,y) = 2x^3y - 3xy^2 + 1$. From the first of these, we have

$$F(x,y) = \int M(x,y)\,\partial x + \phi(y) = \int (3x^2y^2 - y^3 + 2x)\,\partial x + \phi(y) = x^3y^2 - xy^3 + x^2 + \phi(y)$$

From this, $F_y(x,y) = 2x^3y - 3xy^2 + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = 2x^3y - 3xy^2 + 1$. Therefore, $2x^3y - 3xy^2 + \phi'(y) = 2x^3y - 3xy^2 + 1$, or $\phi'(y) = 1$. Then $\phi(y) = y + c_0$. Thus $F(x,y) = x^3y^2 - xy^3 + x^2 + y + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is

$$x^3y^2 - xy^3 + x^2 + y = c, \qquad (*)$$

where $c = c_1 - c_0$.

Applying the I.C. $y(-2) = 1$, we let $x = -2$, $y = 1$ in (*), obtaining $-8 + 2 + 4 + 1 = c$, from which $c = -1$. Thus the particular solution of the stated I.V. problem is $x^3y^2 - xy^3 + x^2 + y = -1$ or

$$x^3y^2 - xy^3 + x^2 + y + 1 = 0. \qquad (**)$$

Alternately, the one-parameter family of solutions (*) could also be found by the method of grouping. We write the D.E. in the form $(3x^2y^2dx + 2x^3y\,dy) - (y^3dx + 3xy^2dy) + 2x\,dx + dy = 0$. We recognize this as $d(x^3y^2) - d(xy^3) + d(x^2) + d(y) = d(c)$, and so again obtain the one-parameter family of solutions

$$x^3y^2 - xy^3 + x^2 + y = c. \qquad (*)$$

The I.C. again yields the particular solution (**).

13. Here $M(x,y) = 2y \sin x \cos x + y^2 \sin x$, $N(x,y) = \sin^2 x - 2y \cos x$. From these we find $M_y(x,y) = 2 \sin x \cos x + 2y \sin x = N_x(x,y)$, so D.E. is exact. We first seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 2y \sin x \cos x + y^2 \sin x$ and $F_y(x,y) = N(x,y) = \sin^2 x - 2y \cos x$. From the first of these, we have $F(x,y) = \int M(x,y)\,\partial x + \phi(y) = \int (2y \sin x \cos x + y^2 \sin x)\,\partial x + \phi(y) = y \sin^2 x - y^2 \cos x + \phi(y)$. From this, $F_y(x,y) = \sin^2 x - 2y \cos x + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = \sin^2 x - 2y \cos x$. Therefore, $\sin^2 x - 2y \cos x + \phi'(y) = \sin^2 x - 2y \cos x$ or $\phi'(y) = 0$. Then $\phi(y) = c_0$. Thus $F(x,y) = y \sin^2 x - y^2 \cos x + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is

$$y \sin^2 x - y^2 \cos x = c, \qquad (*)$$

where $c = c_1 - c_0$.

Applying the I.C. $y(0) = 3$, we let $x = 0$, $y = 3$ in (*), obtaining $3 \sin^2 0 - 9 \cos 0 = c$, from which $c = -9$. Thus the particular solution of the stated I.V. problem is $y \sin^2 x - y^2 \cos x = -9$ or

$$y^2 \cos x - y \sin^2 x = 9. \qquad (**)$$

Alternatively, the one-parameter family of solutions (*) could also be found by the method of grouping. To do

so, we first write the D.E. in the form $(2y \sin x \cos x\,dx + \sin^2 x\,dy) + (y^2 \sin x\,dx - 2y \cos x\,dy) = 0$. We recognize this as $d(y \sin^2 x) + d(-y^2 \cos x) = d(c)$ and hence again obtain the solution (*) in the form $y \sin^2 x - y^2 \cos x = c$. The I.C. again yields the particular solution (**).

14. Here $M(x,y) = ye^x + 2e^x + y^2$, $N(x,y) = e^x + 2xy$. From these we find $M_y(x,y) = e^x + 2y = N_x(x,y)$, so the D.E. is exact. We first seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = ye^x + 2e^x + y^2$ and $F_y(x,y) = N(x,y) = e^x + 2xy$. From the first of these, we have $F(x,y) = \int M(x,y)\,\partial x + \phi(y) = \int (ye^x + 2e^x + y^2)\,\partial x + \phi(y) = ye^x + 2e^x + xy^2 + \phi(y)$. From this, $F_y(x,y) = e^x + 2xy + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = e^x + 2xy$. Therefore, $e^x + 2xy + \phi'(y) = e^x + 2xy$ or $\phi'(y) = 0$. Then $\phi(y) = c_0$. Thus $F(x,y) = ye^x + 2e^x + xy^2 + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is

$$ye^x + 2e^x + xy^2 = c, \qquad (*)$$

where $c = c_1 - c_0$.

Applying the I.C. $y(0) = 6$, we let $x = 0$, $y = 6$ in (*), obtaining $6e^0 + 2e^0 + 0(6^2) = c$, from which $c = 8$. Thus the particular solution of the stated I.V. problem is $ye^x + 2e^x + xy^2 = 8$ or

$$e^x y + xy^2 + 2e^x = 8. \qquad (**)$$

Alternatively, the one-parameter family of solutions (*) could also be found by the method of grouping. To do so, we first write the D.E. in the form $(ye^x dx + e^x dy) + (y^2 dx + 2xy\,dy) + 2e^x dx = 0$. We recognize this as $d(e^x y) + d(xy^2) + d(2e^x) = d(c)$ and hence once again obtain the solution (*) in the form $e^x y + xy^2 + 2e^x = c$. The I.C. again yields the particular solution (**).

16. Here $M(x,y) = x^{-2/3}y^{-1/3} + 8x^{1/3}y^{1/3}$, $N(x,y) = 2x^{4/3}y^{-2/3} - x^{1/3}y^{-4/3}$. From these we find $M_y(x,y) = -\frac{1}{3}x^{-2/3}y^{-4/3} + \frac{8}{3}x^{1/3}y^{-2/3} = N_x(x,y)$ so the D.E. is exact. We first seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = x^{-2/3}y^{-1/3} + 8x^{1/3}y^{1/3}$ and $F_y(x,y) = N(x,y) = 2x^{4/3}y^{-2/3} - x^{1/3}y^{-4/3}$. From the first of these, we have $F(x,y) = \int M(x,y)\partial x + \phi(y) = \int (x^{-2/3}y^{-1/3} + 8x^{1/3}y^{1/3})\partial x + \phi(y) = 3x^{1/3}y^{-1/3} + 6x^{4/3}y^{1/3} + \phi(y)$. From this, $F_y(x,y) = -x^{1/3}y^{-4/3} + 2x^{4/3}y^{-2/3} + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = 2x^{4/3}y^{-2/3} - x^{1/3}y^{-4/3}$. Thus $-x^{1/3}y^{-4/3} + 2x^{4/3}y^{-2/3} + \phi'(y) = 2x^{4/3}y^{-2/3} - x^{1/3}y^{-4/3}$ or $\phi'(y) = 0$. Then $\phi(y) = c_0$. Thus $F(x,y) = 3x^{1/3}y^{-1/3} + 6x^{4/3}y^{1/3}$. The one-parameter family of solutions $F(x,y) = c_1$ is $3x^{1/3}y^{-1/3} + 6x^{4/3}y^{1/3} = c_2$, where $c_2 = c_1 - c_0$. We can simplify this slightly by dividing through by 3 and replacing $c_2/3$ by c, thus obtaining

$$x^{1/3}y^{-1/3} + 2x^{4/3}y^{1/3} = c. \qquad (*)$$

Applying the I.C. $y(1) = 8$, we let $x = 1$, $y = 8$ in (*) to obtain $\frac{1}{2} + 4 = c$, from which $c = 9/2$. Thus the particular solution of the stated I.V. problem is $x^{1/3}y^{-1/3} + 2x^{4/3}y^{1/3} = 9/2$ or

$$2x^{1/3}y^{-1/3} + 4x^{4/3}y^{1/3} = 9. \qquad (**)$$

Alternatively, the one-parameter family of solutions (*) could also be found by the method of grouping. To do so, we first write the D.E. in the form $(x^{-2/3}y^{-1/3}dx - x^{1/3}y^{-4/3}dy) + (8x^{1/3}y^{1/3}dx + 2x^{4/3}y^{-2/3}dy) = 0$. We recognize this as $d(3x^{1/3}y^{-1/3}) + d(6x^{4/3}y^{1/3}) = d(c_2)$ and hence obtain the solution $3x^{1/3}y^{-1/3} + 6x^{4/3}y^{1/3} = c_2$. Once again, this quickly reduces to (*), and the I.C. again yields the particular solutions (**).

18. (a) Here $M(x,y) = Ax^2y + 2y^2$, $N(x,y) = x^3 + 4xy$. From these we find $M_y(x,y) = Ax^2 + 4y$ and $N_x(x,y) = 3x^2 + 4y$. The given D.E. is exact if and only if $M_y(x,y) = N_x(x,y)$, i.e., if and only if $Ax^2 + 4y = 3x^2 + 4y$. So the given D.E. is exact if and only if $A = 3$. Substituting 3 for A in the given D.E. yields the exact D.E. $(3x^2y + 2y^2)dx + (x^3 + 4xy)dy = 0$. We now proceed to solve this D.E.

Here $M(x,y) = 3x^2y + 2y^2$, $N(x,y) = x^3 + 4xy$. From these we find $M_y(x,y) = 3x^2 + 4y = N_x(x,y)$, so

the D.E. is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 3x^2y + 2y^2$ and $F_y(x,y) = N(x,y) = x^3 + 4xy$. From the first of these, we find $F(x,y) = \int M(x,y)\,\partial x + \phi(y) = \int (3x^2y + 2y^2)\,\partial x + \phi(y) = x^3y + 2xy^2 + \phi(y)$. From this, $F_y(x,y) = x^3 + 4xy + \phi'(y)$. Therefore $x^3 + 4xy + \phi'(y) = x^3 + 4xy$, or $\phi'(y) = 0$. Then $\phi(y) = c_0$. Thus $F(x,y) = x^3y + 2xy^2 + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is $x^3y + 2xy^2 = c$, where $c = c_1 - c_0$.

Alternatively, by the method of grouping, we first write the D.E. in the form $(3x^2y\,dx + x^3dy) + (2y^2dx + 4xydy) = 0$. We recognize this as $d(x^3y) + d(2xy^2) = d(c)$ or $d(x^3y + 2xy^2) = d(c)$. Hence we obtain the solution $x^3y + 2xy^2 = c$.

20. (b) Here $N(x,y) = 2ye^x + y^2e^{3x}$, so $N_x(x,y) = 2ye^x + 3y^2e^{3x}$. For the stated D.E. to be exact, we need $M_y(x,y) = N_x(x,y) = 2ye^x + 3y^2e^{3x}$. From this, we find $M(x,y) = \int M_y(x,y)\,\partial y + \phi(x) = \int (2ye^x + 3y^2e^{3x})\,\partial y + \phi(x) = y^2e^x + y^3e^{3x} + \phi(x)$. Hence the most general function $M(x,y)$ such that the stated D.E. is exact is $M(x,y) = y^2e^x + y^3e^{3x} + \phi(x)$, where $\phi(x)$ is an arbitrary function of $x$.

21. Here $M(x,y) = 4x + 3y^2$, $N(x,y) = 2xy$.

(a) Since $M_y(x,y) = 6y \neq 2y = N_x(x,y)$, the D.E. is not exact.

(b) We multiply the given equation through by $x^n$ to obtain $(4x^{n+1} + 3x^n y^2)dx + 2x^{n+1} y\, dy = 0$. For this equation, we have $M(x,y) = 4x^{n+1} + 3x^n y^2$, $N(x,y) = 2x^{n+1} y$. For this equation to be exact, we must have $M_y(x,y) = 6x^n y = 2(n + 1)x^n y = N_x(x,y)$, and hence $6 = 2(n + 1)$, from which $n = 2$. Thus an I.F. of the form $x^n$ is $x^2$.

(c) We multiply the given equation through by the I.F. $x^2$, obtaining $(4x^3 + 3x^2y^2)dx + 2x^3 y\, dy = 0$. Here $M(x,y) = 4x^3 + 3x^2y^2$, $N(x,y) = 2x^3 y$. Since $M_y(x,y) = 6x^2 y = N_x(x,y)$, this D.E. is indeed exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 4x^3 + 3x^2y^2$ and $F_y(x,y) = N(x,y) = 2x^3 y$. From the first of these, $F(x,y) = \int M(x,y)\partial x + \phi(y) = \int (4x^3 + 3x^2y^2)\partial x + \phi(y) = x^4 + x^3y^2 + \phi(y)$. From this $F_y(x,y) = 2x^3 y + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = 2x^3 y$, so $\phi'(y) = 0$. Then $\phi(y) = c_0$. Thus $F(x,y) = x^4 + x^3y^2 + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is $x^4 + x^3y^2 = c$, where $c = c_1 - c_0$. Alternatively, by the **method of grouping**, we first write the D.E. $(4x^3 + 3x^2y^2)dx + 2x^3 y\, dy = 0$ in the form $4x^3 dx + (3x^2y^2 dx + 2x^3 y\, dy) = 0$. We recognize this as $d(x^4) + d(x^3y^2) = d(c)$, and hence we obtain the solutions $x^4 + x^3y^2 = c$.

23. (a) Here $M(x,y) = y + x\,f(x^2 + y^2)$ and $N(x,y) = y\,f(x^2 + y^2) - x$. Since $M_y(x,y) = 1 + 2xy\,f'(x^2 + y^2) \neq 2xy\,f'(x^2 + y^2) - 1 = N_x(x,y)$, the given D.E. is not exact.

(b) We multiply the given D.E. through by $1/(x^2 + y^2)$ to obtain

$$\left[\frac{y}{x^2 + y^2} + \frac{xf(x^2 + y^2)}{x^2 + y^2}\right]dx + \left[\frac{yf(x^2 + y^2)}{x^2 + y^2} - \frac{x}{x^2 + y^2}\right]dy = 0.$$

For this equation, we have

$$M(x,y) = \frac{y}{x^2 + y^2} + \frac{xf(x^2 + y^2)}{x^2 + y^2} \text{ and}$$

$$N(x,y) = \frac{yf(x^2 + y^2)}{x^2 + y^2} - \frac{x}{x^2 + y^2}$$

From these we find

$$M_y(x,y) = \frac{2xy(x^2 + y^2)f'(x^2 + y^2) - 2xyf(x^2 + y^2) + x^2 - y^2}{(x^2 + y^2)^2}$$

$$= N_x(x,y)$$

So the D.E. of part (b) is exact, and hence $1/(x^2 + y^2)$ is an I.F. of the given D.E.

24. Applying Exercise 23(a) with $f(x^2 + y^2) = (x^2 + y^2)^2$, we see that the given D.E. is not exact. By Exercise 23(b), we know that $1/(x^2 + y^2)$ is an I.F. of the given D.E. Hence we multiply the given D.E. through by $1/(x^2 + y^2)$ to obtain the equivalent D.E.

$$\left[\frac{y}{x^2 + y^2} + x(x^2 + y^2)\right]dx + \left[y(x^2 + y^2) - \frac{x}{x^2 + y^2}\right]dy = 0,$$

which is therefore exact. Here $M(x,y) = y/(x^2 + y^2) + x(x^2 + y^2)$, $N(x,y) = y(x^2 + y^2) - x/(x^2 + y^2)$, $M_y(x,y) = (x^2 - y^2)/(x^2 + y^2)^2 + 2xy = N_x(x,y)$. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = y/(x^2 + y^2) + x(x^2 + y^2)$ and $F_y(x,y) = N(x,y) = y(x^2 + y^2) - x/(x^2 + y^2)$. From the first of these, we find $F(x,y) = \int M(x,y)\partial x + \phi(y) = \int [y/(x^2 + y^2) + x(x^2 + y^2)]\partial x + \phi(y) = \arctan(x/y) + x^4/4 + x^2y^2/2 + \phi(y)$. From this, $F_y(x,y) = -x/(x^2 + y^2) + x^2y + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = x^2y + y^3 - x/(x^2 + y^2)$. Therefore, $\phi'(y) = y^3$ or $\frac{d\phi(y)}{dy} = y^3$. Then, $\phi(y) = y^4/4 + c_0$. Thus $F(x,y) = \arctan(x/y) + x^4/4 + x^2y^2/2 + y^4/4 + c_0$, or more simply, $F(x,y) = \arctan(x/y) + (x^2 + y^2)^2/4 + c_0$. The one-parameter family of solutions $F(x,y) = c$ is $\arctan(x/y) + (x^2 + y^2)^2/4 = c$, where $c = c_1 - c_0$.

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The equations in Exercises 1-7 and 15-17 are separable, and those in Exercises 8-14 and 18-20 are homogeneous.

2. Since $xy + 2x + y + 2 = x(y + 2) + 1(y + 2) = (x + 1)(y + 2)$, the D.E. can be rewritten as $(x + 1)(y + 2)dx + (x^2 + 2x)dy = 0$. The D.E. is separable. We first separate variables to obtain $\frac{(x + 1)dx}{x^2 + 2x} + \frac{dy}{y + 2} = 0$. Next we integrate:

$$\int \frac{(x + 1)dx}{x^2 + 2x} = \frac{1}{2} \ln|x^2 + 2x| \text{ and } \int \frac{dy}{y + 2} = \ln|y + 2|.$$

Hence we find the one-parameter family of solutions in the form $\frac{1}{2} \ln|x^2 + 2x| + \ln|y + 2| = \ln c_1$ (where we write $\ln c_1$ for the arbitrary constant, since each term on the left is an ln term). We multiply by 2 and simplify to obtain $\ln|x^2 + 2x| + \ln(y + 2)^2 = \ln c_1^2$ or $\ln\left[|x^2 + 2x|(y + 2)^2\right] = \ln c_1^2$. From this, we have $|x^2 + 2x|(y + 2)^2 = c$. If $x \geq 0$ (or $x \leq -2$), this may be expressed somewhat more simply as $(x^2 + 2x)(y + 2)^2 = c$ or $x(x + 2)(y + 2)^2 = c$.

3. The D.E. is separable. We first separate variables to obtain $2r\,dr/(r^4 + 1) + ds/(s^2 + 1) = 0$. Next we integrate. By a well-known formula, $\int ds/(s^2 + 1) = \arctan s$. We next apply the same formula with $s = r^2$, $ds = 2r\,dr$. Thus we obtain $\int 2r\,dr/(r^4 + 1) = \arctan r^2$.

Hence we find the one-parameter family of solutions in the form

$$\arctan r^2 + \arctan s = \arctan c, \qquad (*)$$

(where we write arc tan c for the arbitrary constant, since each term on the left is an arc tan). We could leave the solutions in this form, but they are unweildy. We take the tangent of each side of (*), applying the formula $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ with $A = \arctan r^2$ and $B = \arctan s$, to the left member. We obtain

$$\frac{\tan(\arctan r^2) + \tan(\arctan s)}{1-\tan(\arctan r^2)\tan(\arctan s)} = \tan(\arctan c),$$

which reduces to

$$\frac{r^2 + s}{1 - r^2 s} = c \text{ or } r^2 + s = c(1 - r^2 s).$$

6. The D.E. is separable. We first separate variables to obtain $\dfrac{\cos u\, du}{\sin u + 1} + \dfrac{e^v dv}{e^v + 1} = 0$. Next we integrate:

$$\int \frac{\cos u\, du}{\sin u + 1} = \ln(\sin u + 1) \text{ and } \int \frac{e^v dv}{e^v + 1} = \ln(e^v + 1).$$

Hence we find the one-parameter family of solutions in the form $\ln(\sin u + 1) + \ln(e^v + 1) = \ln c$ (where we write $\ln c$ for the arbitrary constant, since each term on the

left is a ln term). We simplify to obtain $\ln[(\sin u + 1)(e^v + 1)] = \ln c$. From this, we have

$$(\sin u + 1)(e^v + 1) = c.$$

7. This equation is separable. We first separate variables to obtain $(x + 4)\,dx/(x^2 + 3x + 2) + y\,dy/(y^2 + 1) = 0$. Next we integrate: $\int \frac{y\,dy}{y^2 + 1} = \frac{1}{2}\ln(y^2 + 1)$. To integrate the dx term, we use partial fractions. We set $\frac{x + 4}{x^2 + 3x + 2} = \frac{x + 4}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}$, $x + 4 = A(x + 2) + B(x + 1)$. Then $x = -1$ gives $A = 3$ and $x = -2$ gives $B = -2$. Thus we find $\int \frac{(x + 4)\,dx}{x^2 + 3x + 2} = 3\int \frac{dx}{x + 1} - 2\int \frac{dx}{x + 2} = 3\ln|x + 1| - 2\ln|x + 2| = \ln\frac{|x + 1|^3}{(x + 2)^2}$. Hence we obtain solutions in the form $\ln\frac{|x + 1|^3}{(x + 2)^2} + \frac{1}{2}\ln(y^2 + 1) = \ln c_1$ (where we write $\ln c_1$ for the arbitrary constant, since each term on the left is a ln term). We multiply by 2 and simplify to obtain $\ln\frac{(x + 1)^6}{(x + 2)^4} + \ln(y^2 + 1) + \ln c_1^2$ or $\ln\frac{(x + 1)^6(y^2 + 1)}{(x + 2)^4} = \ln c_1^2$. From this, we have $(x + 1)^6(y^2 + 1) = c(x + 2)^4$.

9. We first write the D.E. in the form $\frac{dy}{dx} = \frac{2xy + 3y^2}{2xy + x^2}$ and thence $\frac{dy}{dx} = \frac{2(y/x) + 3(y/x)^2}{2(y/x) + 1}$. In this form we recognize

the D.E. is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = \frac{y}{x}$. We make these substitutions to obtain $v + x\frac{dv}{dx} = \frac{2v + 3v^2}{2v + 1}$ or $x\frac{dv}{dx} = \frac{2v + 3v^2}{2v + 1} - v = \frac{2v + 3v^2 - 2v^2 - v}{2v + 1}$ or $x\frac{dv}{dx} = \frac{v^2 + v}{2v + 1}$. We now separate variables to obtain $\frac{(2v + 1)\,dv}{v^2 + v} = \frac{dx}{x}$. We integrate to obtain $\ln|v^2 + v| = \ln|x| + \ln|c|$, or $\ln|v^2 + v| = \ln|cx|$. From this, we have $|v^2 + v| = |cx|$. We now resubstitute $v = \frac{y}{x}$ to obtain $\left|\frac{y^2}{x^2} + \frac{y}{x}\right| = |cx|$. We simplify to obtain $|y^2 + xy| = |cx|x^2$, from which we find $y^2 + xy = cx^3$.

10. We first write the D.E. in the form $\frac{du}{dv} = \frac{uv^2 - u^3}{v^3}$ from which we at once obtain $\frac{du}{dv} = \frac{u}{v} - \left(\frac{u}{v}\right)^3$. In this form we recognize that the D.E. is homogeneous in u and v. If a D.E. $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ is homogeneous in x and y, we introduce a new variable v by letting $y = vx$. Here u and v are the original variables, so we introduce a new variable w by letting $u = wv$. Then $\frac{du}{dv} = w + v\frac{dw}{dv}$ and $w = \frac{u}{v}$. Substituting into the D.E. $\frac{du}{dv} = \frac{u}{v} - \left(\frac{u}{v}\right)^3$, we obtain $w + v\frac{dw}{dv} = w - w^3$. Hence $v\frac{dw}{dv} = -w^3$ and separating variables, we have $-w^{-3}dw = \frac{dv}{v}$. Integrating we obtain $-\frac{w^{-2}}{-2} = \ln|v| + c_1$ or $\frac{1}{w^2} = 2\ln|v| + 2c_1$. Resubstituting $w = \frac{u}{v}$, we have $\frac{v^2}{u^2} = \ln v^2 + c$ or $v^2 = u^2(\ln v^2 + c)$.

11. We first write the D.E. in the form $\frac{dy}{dx} = \tan\left(\frac{y}{x}\right) + \frac{y}{x}$. In this form we recognize that the D.E. is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = \frac{y}{x}$. We make these substitutions in the D.E. to obtain $v + x\frac{dv}{dx} = \tan v + v$ or $x\frac{dv}{dx} = \tan v$. We now separate variables to obtain $\cot v\, dv = \frac{dx}{x}$. We integrate to obtain $\ln|\sin v| = \ln|x| + \ln|c|$, or $\ln|\sin v| = \ln|cx|$. From this we have $|\sin v| = |cx|$. We now resubstitute $v = \frac{y}{x}$ to obtain $\left|\sin \frac{y}{x}\right| = |cx|$. For suitable x and y, this may be expressed somewhat more simply as $\sin \frac{y}{x} = cx$.

13. We first write the D.E. in the form $\frac{dy}{dx} = \frac{x^3 + y^2\sqrt{x^2 + y^2}}{xy\sqrt{x^2 + y^2}}$. We then divide numerator and denominator by $x^3$ to obtain $\frac{dy}{dx} = \frac{1 + \frac{y^2}{x^3}\sqrt{x^2 + y^2}}{\frac{y}{x^2}\sqrt{x^2 + y^2}} = \frac{1 + \left[\frac{y}{x}\right]^2\left[\frac{1}{x}\right]\sqrt{x^2 + y^2}}{\left[\frac{y}{x}\right]\left[\frac{1}{x}\right]\sqrt{x^2 + y^2}}$. Assuming $x > 0$ so that $x = \sqrt{x^2}$, this becomes

$$\frac{dy}{dx} = \frac{1 + \left[\frac{y}{x}\right]^2\left[\frac{1}{\sqrt{x^2}}\right]\left(\sqrt{x^2 + y^2}\right)}{\left[\frac{y}{x}\right]\left[\frac{1}{\sqrt{x^2}}\right]\sqrt{x^2 + y^2}} = \frac{1 + \left[\frac{y}{x}\right]^2\sqrt{1 + y^2/x^2}}{\left[\frac{y}{x}\right]\sqrt{1 + y^2/x^2}} \text{ or}$$

$$\frac{dy}{dx} = \frac{1 + \left[\frac{y}{x}\right]^2\sqrt{1 + (y/x)^2}}{\left[\frac{y}{x}\right]\sqrt{1 + (y/x^2)}}.$$ In this form we recognize the D.E. is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

and $v = \frac{y}{x}$. We make these substitutions to obtain

$$v + x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}} \quad \text{or} \quad x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}} - v$$

$$= \frac{1 + v^2\sqrt{1 + v^2} - v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}} \quad \text{or} \quad x\frac{dv}{dx} = \frac{1}{v\sqrt{1 + v^2}}.$$ We

now separate variables to obtain $v\sqrt{1 + v^2}\, dv = \frac{dx}{x}$. We integrate to obtain $1/3\ (1 + v^2)^{3/2} = \ln|x| + \ln|c_0|$ (where we have chosen to write $\ln|c_0|$ for the arbitrary constant). We multiply by 3 and simplify to obtain $(1 + v^2)^{3/2} = 3\ln|c_0 x|$ or $(1 + v^2)^{3/2} = \ln|c_0|^3|x|^3$. Since we have assumed $x > 0$, this may be expressed somewhat more simply as $(1 + v^2)^{3/2} = \ln cx^3$, where $c = |c_0|^3$. We now resubstitute $v = \frac{y}{x}$ to obtain $\left(1 + \left(\frac{y}{x}\right)^2\right)^{3/2} = \ln c x^3$. We simplify to obtain $\left(\frac{x^2 + y^2}{x^2}\right)^{3/2} = \ln c x^3$ or $\frac{\left[x^2 + y^2\right]^{3/2}}{x^3} = \ln c x^3$, from which we find $\left(x^2 + y^2\right)^{3/2} = x^3 \ln c x^3$.

14. This D.E. is homogeneous. Recognizing this, we could let $y = vx$ and substitute in order to separate the variables. However, the resulting separable equation is not readily tractable. This being the case, we assume $x > y > 0$ and

divide the entire equation through by $\sqrt{y}$. Then we solve for dx/dy, putting the D.E. in the form

$$\frac{dx}{dy} = \frac{\sqrt{x/y + 1} - \sqrt{x/y - 1}}{\sqrt{x/y + 1} + \sqrt{x/y - 1}} .$$

We now let $x = uy$ (see Exercise 24 concerning this). Then $\frac{dx}{dy} = u + y\frac{du}{dy}$ and $x/y = u$. Substituting in the D.E. it takes the form

$$u + y\frac{du}{dy} = \frac{\sqrt{u + 1} - \sqrt{u - 1}}{\sqrt{u + 1} + \sqrt{u - 1}} .$$

We simplify the right member by multiplying both its numerator and denominator by $\sqrt{u + 1} - \sqrt{u - 1}$ and then simplifying. As a result of this, the D.E. takes the form $u + y\frac{du}{dy} = u - \sqrt{u^2 - 1}$ which readily simplifies to the separable equation $\frac{du}{\sqrt{u^2 - 1}} = -\frac{dy}{y}$. Integrating (tables are useful!), we obtain $\ln\left|u + \sqrt{u^2 - 1}\right| = -\ln|y| + \ln|c|$ or $\ln\left|u + \sqrt{u^2 - 1}\right| = \ln\left|\frac{c}{y}\right|$. Since $x > y > 0$ and $u = x/y$, we can write this more simply as $\ln\left(u + \sqrt{u^2 - 1}\right) = \ln(c/y)$, from which we at once have $u + \sqrt{u^2 - 1} = c/y$. Now substituting $u = x/y$, we have $x/y + \sqrt{(x/y)^2 - 1} = c/y$ which readily simplifies to $x + \sqrt{x^2 - y^2} = c$.

17. The D.E. is separable. We separate variables to obtain

$\frac{(3x + 8)}{x^2 + 5x + 6} - \frac{4y\,dy}{y^2 + 4} = 0.$ tegrate the dx term, we use partial fractions. We write $\frac{3x + 8}{x^2 + 5x + 6} = \frac{3x + 8}{(x + 2)(x + 3)} = \frac{A}{x + 2} + \frac{B}{x + 3}$, and so $3x + 8 = A(x + 3) + B(x + 2)$. Then $x = -2$ gives $A = 2$; and $x = -3$ gives $B = 1$. Thus we find

$\int \frac{3x + 8}{x^2 + 5x + 6}\,dx = 2\int \frac{dx}{x + 2} + \int \frac{dx}{x + 3} = 2\ln|x + 2| + \ln|x + 3| = \ln(x + 2)^2(x + 3)$. Using this, we obtain solutions in the form $\ln(x + 2)^2(x + 3) - 2\ln(y^2 + 4) = \ln|c|$ or $\ln\frac{(x + 2)^2(x + 3)}{(y^2 + 4)^2} = \ln|c|$. Take $|x + 3| = x + 3 \geq 0$ and $|c| = c \geq 0$, and we have $(x + 2)^2(x + 3) = c(y^2 + 4)^2$. We now apply the I.C. $y(1) = 2$ to this, obtaining $36 = 64c$, $c = \frac{9}{16}$. Thus we find the particular solution

$$16(x + 2)^2(x + 3) = 9(y^2 + 4).$$

19. We first write the D.E. in the form $\frac{dy}{dx} = \frac{5y - 2x}{4x - y}$ or $\frac{dy}{dx} = \frac{5(y/x) - 2}{4 - y/x}$. We recognize that this D.E. is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = y/x$. Making these substitutions, the D.E. becomes $v + x\frac{dv}{dx} = \frac{5v - 2}{4 - v}$ or $x\frac{dv}{dx} = \frac{v^2 + v - 2}{4 - v}$. We now separate variables to obtain $\frac{(4 - v)\,dv}{v^2 + v - 2} = \frac{dx}{x}$. Next we integrate:

$\int \frac{dx}{x} = \ln|x|$. To integrate the dv term, we use partial fractions. We set $\frac{4 - v}{v^2 + v - 2} = \frac{4 - v}{(v + 2)(v - 1)} = \frac{A}{v + 2} + \frac{B}{v - 1}$, so $4 - v = A(v - 1) + B(v + 2)$. Then $v = -2$ gives $A = -2$ and $v = 1$ gives $B = 1$. Thus we find $\int \frac{(4 - v)\, dv}{v^2 + v - 2} = -2\int \frac{dv}{v + 2} + \int \frac{dv}{v - 1} = -2 \ln|v + 2| + \ln|v - 1| = \ln \frac{|v - 1|}{(v + 2)^2}$. Hence we obtain solutions in the form $\ln \frac{|v - 1|}{(v + 2)^2} = \ln|x| + \ln c$ or $\ln \frac{|v - 1|}{(v + 2)^2} = \ln c\,|x|$. From this we have $\frac{|v - 1|}{(v + 2)^2} = c|x|$ or $|v - 1| = c|x|(v + 2)^2$. We resubstitute $v = y/x$ to obtain $|y/x - 1| = c|x|(y/x + 2)^2$. We multiply both sides by $x^2$ to obtain $|yx - x^2| = c|x|(y + 2x)^2$ or $|x||y - x| = c|x|(y + 2x)^2$ from which we obtain $|y - x| = c(y + 2x)^2$. Taking $y \geq x$, this may be expressed somewhat more simply as $y - x = c(y + 2x)^2$.

Applying the I.C. $y(1) = 4$ we let $x = 1$, $y = 4$ and find $3 = c(6^2)$, from which we find $c = 1/12$. We thus obtain the particular solution $y - x = (1/12)(y + 2x)^2$, or $(2x + y)^2 = 12(y - x)$.

20. We first write the D.E. in the form

$$\frac{dy}{dx} = \frac{3x^2 + 9xy + 5y^2}{6x^2 + 4xy}$$

or

$$\frac{dy}{dx} = \frac{3 + 9(y/x) + 5(y/x)^2}{6 + 4\ (y/x)} .$$

We recognize that this D.E. is homogeneous. We let $y = vx$, and then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = \frac{y}{x}$. Making these substitutions, the D.E. becomes

$$v + x\frac{dv}{dx} = \frac{3 + 9v + 5v^2}{6 + 4v}$$

or

$$x\frac{dv}{dx} = \frac{v^2 + 3v + 3}{4v + 6}.$$

We now separate variables to obtain

$$\frac{4v + 6}{v^2 + 3v + 3}\, dv = \frac{dx}{x}.$$

Integrating, we find $2 \ln|v^2 + 3v + 3| = \ln|x| + \ln|c|$ or $\ln(v^2 + 3v + 3)^2 = \ln|cx|$. From this, $(v^2 + 3v + 3)^2 = \ln|cx|$. We resubstitute $v = y/x$ to obtain

$$\left[\frac{y^2 + 3xy + 3x^2}{x^2}\right]^2 = |cx|.$$

We simplify, taking $|x| = x > 0$, thereby obtaining

$$(y^2 + 3xy + 3x^2)^2 = cx^5.$$

Applying the I.C. $y(2) = -6$, we let $x = 2$, $y = -6$ and find $144 = 32c$ or $c = 9/2$. We thus obtain the particular solution $2(y^2 + 3xy + 3x^2)^2 = 9x^5$.

21. (b) Here $M(x,y) = Ax^2 + Bxy + Cy^2$, $N(x,y) = Dx^2 + Exy + Fy^2$. From these we find $M_y(x,y) = Bx + 2Cy$ and $N_x(x,y) = 2Dx + Ey$. Now the given homogeneous D.E. is exact if and only if $M_y(x,y) = N_x(x,y)$, i.e., if and only if $Bx + 2Cy = 2Dx + Ey$. But $Bx + 2Cy = 2Dx + Ey$ if and only if $B = 2D$ and $2C = E$. Therefore, we have it that the given homogeneous D.E. is exact if and only if $B = 2D$ and $E = 2C$.

23. (a) The given D.E. is of the form $(Ax^2 + Bxy + Cy^2)dx + (Dx^2 + Exy + Fy^2)dy = 0$, where $A = 1$, $B = 0$, $C = 2$, $D = 0$, $E = 4$, $F = -1$. From exercise 21(b), we know the D.E. is homogeneous. Also, since $B = 0 = 2D$ and $E = 4 = 2C$, we know from Exercise 21(b) that the given D.E. is also exact.

First, let us solve the given D.E. as a homogeneous D.E. We first write the D.E. in the form $\frac{dy}{dx} = \frac{x^2 + 2y^2}{y^2 - 4xy}$ or $\frac{dy}{dx} = \frac{1 + 2(y/x)^2}{(y/x)^2 - 4(y/x)}$. In this form we recognize that the D.E. is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = y/x$. We make these substitutions to obtain $v + x\frac{dv}{dx} = \frac{1 + 2v^2}{v^2 - 4v}$ or $x\frac{dv}{dx} = \frac{1 + 6v^2 - v^3}{v^2 - 4v}$. We now separate variables to obtain $\frac{(v^2 - 4v)\,dv}{1 + 6v^2 - v^3} = \frac{dx}{x}$. We integrate to obtain $-\frac{1}{3}\ln|1 + 6v^2 - v^3| = \ln|x| + \ln c_0$ or $-\frac{1}{3}\ln|1 + 6v^2 - v^3| = \ln c_0|x|$. We multiply by $-3$

and simplify to obtain $\ln|1 + 6v^2 - v^3| = -3 \ln c_0|x|$ or $\ln|1 + 6v^2 - v^3| = c_0^{-3}|x|^{-3}$ or $|1 + 6v^2 - v^3| = c|x|^{-3}$, where $c = c_0^{-3}$. We now resubstitute $v = y/x$ to obtain $|1 + 6(y/x)^2 - (y/x)^3| = c|x|^{-3}$. We multiply by $|x|^3$ to obtain $|x^3 + 6xy^2 - y^3| = c$. For suitable values of x and y, this may be expressed somewhat more simply as $x^3 + 6xy^2 - y^3 = c$, or $x^3 - y^3 + 6xy^2 = c$.

Now let us solve the given D.E. as an exact D.E. Here $M(x,y) = x^2 + 2y^2$, $N(x,y) = 4xy - y^2$. From these we find $M_y(x,y) = 4y = N_x(x,y)$, so the D.E. is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = x^2 + 2y^2$ and $F_y(x,y) = N(x,y) = 4xy - y^2$. From the first of these, $F(x,y) = \int M(x,y)\partial x + \phi(y) = \int (x^2 + y^2)\partial x + \phi(y) = \frac{x^3}{3} + 2xy^2 + \phi(y)$. From this, $F_y(x,y) = 4xy + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = 4xy - y^2$. Therefore $4xy + \phi'(y) = 4xy - y^2$ or $\phi'(y) = -y^2$. So $\phi(y) = -\frac{y^3}{3} + c_0$. Thus $F(x,y) = \frac{x^3}{3} + 2xy^2 - \frac{y^3}{3} + c_0$. The one-parameter family of solutions $F(x,y) = c_1$ is $\frac{x^3}{3} + 2xy^2 - \frac{y^3}{3} = c_2$, where $c_2 = c_1 - c_0$. We multiply by 3 to obtain $x^3 + 6xy^2 - y^3 = c$, where $c = 3c_2$. So the one-parameter family of solutions of the given D.E. is $x^3 - y^3 + 6xy^2 = c$.

25. Since the D.E. is homogeneous, it can be expressed in the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$. Let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{dy}{dx} = \frac{r\cos\theta\, d\theta + \sin\theta\, dr}{-r\sin\theta\, d\theta + \cos\theta\, dr} = \frac{r\cos\theta + \sin\theta\, dr/d\theta}{-r\sin\theta + \cos\theta\, dr/d\theta}$$

Substituting into $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$. The D.E. reduces successively to $\frac{r\cos\theta + \sin\theta\, dr/d\theta}{-r\sin\theta + \cos\theta\, dr/d\theta} = g(\tan\theta)$,

$r\cos\theta + \sin\theta\frac{dr}{d\theta} = g(\tan\theta)\left[-r\sin\theta + \cos\theta\frac{dr}{d\theta}\right]$,

$[\sin\theta - \cos\theta\, g(\tan\theta)]\frac{dr}{d\theta} = -[g(\tan\theta)\sin\theta + \cos\theta]r$,

$\frac{dr}{r} = \frac{\sin\theta\, g(\tan\theta) + \cos\theta}{\cos\theta\, g(\tan\theta) - \sin\theta}\, d\theta$, which is separable in r and $\theta$.

26. (a) The D.E. of Exercise 8 is $(x + y)\,dx - x\,dy = 0$. This can be written $\frac{dy}{dx} = 1 + y/x$, and so is homogeneous. Using the method of Exercise 25, we let $x = r\cos\theta$, $y = r\sin\theta$. Then $\frac{dy}{dx} = \frac{r\cos\theta\, d\theta + \sin\theta\, dr}{-r\sin\theta\, d\theta + \cos\theta\, dr}$. We substitute this into $\frac{dy}{dx} = 1 + y/x$. The D.E. reduces successively to

$$\frac{r\cos\theta\, d\theta + \sin\theta\, dr}{-r\sin\theta\, d\theta + \cos\theta\, dr} = 1 + \tan\theta,$$

$r\cos\theta\, d\theta + \sin\theta dr = (1 + \tan\theta)(-r\sin\theta\, d\theta + \cos\theta\, dr)$, $[\sin\theta - (1 + \tan\theta)\cos\theta]\, dr = -r[\cos\theta + (1 + \tan\theta)\sin\theta]\, d\theta$,

$$\frac{dr}{r} = \frac{\cos\theta + (1 + \tan\theta)\sin\theta}{(1 + \tan\theta)\cos\theta - \sin\theta}\,d\theta .$$

$$= \left[\frac{\cos\theta + \sin\theta + \tan\theta\sin\theta}{\cos\theta}\right] d\theta$$

$$= (1 + \tan\theta + \tan^2\theta)\,d\theta ,$$

$$= (\sec^2\theta + \tan\theta)\,d\theta ,$$

or finally

$$\frac{dr}{r} = (\sec^2\theta + \tan\theta)\,d\theta .$$

This is separable. Integrating, assuming $r > 0$, $\cos\theta > 0$, we obtain $\ln r = \tan\theta - \ln\cos\theta + \ln c$, where $c > 0$. Now resubstitute, according to $x = r\cos\theta$, $y = r\sin\theta$; that is, let $r = \sqrt{x^2 + y^2}$, $\tan\theta = y/x$. We obtain successively

$$\ln\sqrt{x^2 + y^2} = y/x - \ln(x/\sqrt{x^2 + y^2}) - \ln c ,$$

$$\ln\sqrt{x^2 + y^2} + \ln x/\sqrt{x^2 + y^2} + \ln c = y/x,$$

$\ln(cx) = y/x$, or finally $y = x\ln(cx)$.

**Section 2.3, Page 56**

The equations of Exercises 1 through 14 are linear. In solving, first express the equation in the standard form of equation (2.26) and then follow the procedure of Example 2.14.

1. The D.E. is already in the standard form (2.26), with $P(x) = 3/x$, $Q(x) = 6x^2$. An I.F. is $e^{\int P(x)dx} = e^{\int (3/x)dx} = e^{3\ln|x|} = e^{\ln|x|^3} = |x|^3 = \pm x^3$ (+ if $x \geq 0$, - if $x < 0$). We multiply the D.E. through by this I.F. to obtain $x^3\frac{dy}{dx} + 3x^2 = 6x^5$ or $\frac{d}{dx}(x^3y) = 6x^5$. Integrating, we obtain $x^3y = x^6 + c$ or $y = x^3 + cx^{-3}$.

3. The D.E. is already in the standard form (2.26), wih $P(x) = 3$, $Q(x) = 3x^2e^{-3x}$. An I.F. is $e^{\int P(x)\,dx} = e^{\int 3\,dx} = e^{3x}$. We multiply the D.E. through by the I.F. to obtain

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = 3x^2$$

or $\frac{d}{dx}\left(e^{3x}y\right) = 3x^2$. Integrating we obtain $e^{3x}y = x^3 + c$ or $y = \left(x^3 + c\right)e^{-3x}$.

6. This equation is linear in v. We divide through by $u^2 + 1$ to put it in the standard form $\frac{dv}{du} + \frac{4u}{u^2+1}v = \frac{3u}{u^2+1}$, with $P(u) = \frac{4u}{u^2+1}$ and $Q(u) = \frac{3u}{u^2+1}$. An I.F. is

$$e^{\int P(u)du} = \exp\int\frac{4u}{u^2+1}du = e^{2\ln(u^2+1)} = (u^2+1)^2.$$

We multiply the standard form equation through by this to obtain $(u^2+1)^2\frac{dv}{du} + 4u(u^2+1)v = 3u(u^2+1)$ or $\frac{d}{du}[(u^2+1)^2v] = 3u^3 + 3u$. Integrating, we obtain $(u^2+1)^2v = \frac{3u^4}{4} + \frac{3u^2}{2} + c$.

7. We first divide through by x to obtain $\frac{dy}{dx} + \frac{2x + 1}{x^2 + x} y = \frac{x - 1}{x}$, where $P(x) = \frac{2x + 1}{x^2 + x}$ and $Q(x) = \frac{x - 1}{x}$. An I.F. is $e^{\int P(x)dx} = \exp\left(\int \frac{2x + 1}{x^2 + x} dx\right) = e^{\ln|x^2+x|} = |x^2 + x| = \pm (x^2 + x)$ (+ if $x < -1$ or $x > 0$, - if $-1 < x < 0$). In any case, upon multiplying the standard form equation through by the I.F., we obtain $(x^2 + x) \frac{dy}{dx} + (2x + 1) y = (x + 1)(x - 1)$ or $(x^2 + x) \frac{dy}{dx} + (2x + 1) y = x^2 - 1$, that is, $\frac{d}{dx} [(x^2 + x)y] = x^2 - 1$. Integrating, we find $(x^2 + x)y = x^3/3 - x + c/3$ or $3(x^2 + x)y = x^3 - 3x + c$.

8. We first divide through by $x^2 + x - 2$ to put the equation in the standard form $\frac{dy}{dx} + \frac{3(x + 1)}{(x + 2)(x - 1)} y = \frac{1}{x + 2}$, where $P(x) = \frac{3(x + 1)}{(x + 2)(x - 1)}$ and $Q(x) = \frac{1}{x + 2}$. An I.F. is $e^{\int P(x)dx} = \exp \int \left[\frac{3(x + 1)}{(x + 2)(x - 1)}\right] dx = e^{\ln|x+2|+2\ln|x-1|} = |x + 2|(x - 1)^2 = \pm(x + 2)(x - 1)^2$, (+ if $x \geq -2$, - if $x \leq -2$), where partial fractions have been used to perform the integration. In either case ($x \geq -2$ or $x \leq -2$), upon multiplying the standard form equation through by the I.F., we obtain $(x + 2)(x - 1)^2 \frac{dy}{dx} + 3(x + 1)(x - 1)y = (x - 1)^2$, that is, $\frac{d}{dx} [(x + 2)(x -1)^2 y] = (x - 1)^2$. Integrating, we find $(x + 2)(x - 1)^2 y = \frac{(x - 1)^3}{3} + \frac{c}{3}$ or $3(x + 2)y = x - 1 + c(x - 1)^{-2}$.

9. We first put the equation in the form

$$\frac{dy}{dx} + \frac{xy + y - 1}{x} = 0,$$

and then in the standard form

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{1}{x},$$

where $P(x) = 1 + 1/x$ and $Q(x) = 1/x$. An I.F. is $e^{\int P(x)dx}$ $= e^{(1+1/x)\,dx} = e^{x+\ln|x|} = |x|e^x = \pm\, x e^x$, (+ if $x \geq 0$, - if $x < 0$). In either case, we multiply the standard form equation through by the I.F. and obtain

$$xe^x \frac{dy}{dy} + (x + 1)e^x y = e^x$$

or $\frac{d}{dx}[x\, e^x y] = e^x$. Integrating, we find $x\, e^x y = e^x + c$ or $y = x^{-1}(1 + c\, e^{-x})$.

10. This D.E. is linear in x (like Example 2.16). We divide through by y and dy to put it in the standard form

$$\frac{dx}{dy} + \frac{xy^2 + x - y}{y} = 0,$$

or $\frac{dx}{dy} + \left(y + \frac{1}{y}\right) x = 1$, where $P(y) = y + \frac{1}{y}$, $Q(y) = 1$. An I.F. is

$$e^{\int P(y)dy} = e^{\int (y+1/y)dy} = e^{y^2/2+\ln|y|} = |y|e^{y^2/2} =$$

$\pm\, y e^{y^2/2}$ (+ if $y \geq 0$; - if $y < 0$). Multiplying the standard form equation through by this, we obtain

$$ye^{y^2/2}\frac{dx}{dy} + (y^2 + 1)e^{y^2/2}x = ye^{y^2/2}$$

or $\frac{d}{dy}(ye^{y^2/2}x) = ye^{y^2/2}$. Integrating, we find $ye^{y^2/2}x = e^{y^2/2} + c$ or $xy = 1 + ce^{-y^2/2}$.

12. This D.E. is linear in r. We first divide through by $d\theta$ to obtain $\cos\theta\frac{dr}{d\theta} + r\sin\theta - \cos^4\theta = 0$ or $\cos\theta\frac{dr}{d\theta} + r\sin\theta = \cos^4\theta$. Next we divide through by $\cos\theta$ to put the equation in the standard form $\frac{dr}{d\theta} + (\tan\theta)r = \cos^3\theta$, where $P(\theta) = \tan\theta$ and $Q(\theta) = \cos^3\theta$. An I.F. is $e^{\int P(\theta)d\theta} = e^{\int\tan\theta\, d\theta} = e^{\ln|\sec\theta|} = |\sec\theta| = \pm\sec\theta$ (+ if $\frac{4k-1}{2}\pi < \theta < \frac{4k+1}{2}\pi$ for some integer k, and - if $\frac{4k+1}{2}\pi < \theta < \frac{4k+3}{2}\pi$ for some integer k). In any case, upon multiplying the standard form equation through by the I.F., we obtain $\sec\theta\frac{dr}{d\theta} + (\sec\theta\tan\theta)r = \cos^2\theta$, that is, $\frac{d}{d\theta}[r\sec\theta] = \cos^2\theta$. Upon integrating, we find $r\sec\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c_0$, where the right-side was integrated as follows: $\int\cos^2\theta\, d\theta = \int\frac{1+\cos 2\theta}{2}d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$. Substituting $2\sin\theta\cos\theta$ for $\sin 2\theta$, the solutions may be expressed as $r\sec\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + c_0$. Finally, we multiply through by $2\cos\theta$ to obtain

$$2r = (\theta + \sin\theta\cos\theta + c)\cos\theta, \text{ where } c = 2c_0.$$

13. We first put the D.E. in the form

$$(1 + \sin x)\frac{dy}{dx} + (\cos x)y = \cos^2 x,$$

and then in the standard form

$$\frac{dy}{dx} + \frac{\cos x}{1 + \sin x}\ y = \frac{\cos^2 x}{1 + \sin x},$$

where $P(x) = \frac{\cos x}{1 + \sin x}$, $Q(x) = \frac{\cos^2 x}{1 + \sin x}$. An I.F. is $e^{\int P(x)dx} = e^{\int \frac{\cos x\, dx}{1 + \sin x}} = e^{\ln|1+\sin x|} = |1 + \sin x| = 1 + \sin x$, since $1 + \sin x \geq 0$. Multiply the standard form equation through by this, to obtain

$$(1 + \sin x)\frac{dy}{dx} + (\cos x)y = \cos^2 x,$$

which is in fact the form preceding the standard form. We observe that this is $\frac{d}{dx}[(1 + \sin x)y] = \cos^2 x$. Integrating we obtain $(1 + \sin x)y = \frac{x}{2} + \frac{\sin 2x}{4} + c_1$ or $2(1 + \sin x)y = x + \sin x \cos x + c$, where $c = 2c_1$ and we have employed double-angle formulas.

14. This D.E. is linear in y. We first divide through by dy to obtain $y \sin 2x - \cos x + (1 + \sin^2 x)\frac{dy}{dx} = 0$ or $(1 + \sin^2 x)\frac{dy}{dx} + (\sin 2x)y = \cos x$. Next we divide through by $1 + \sin^2 x$ to put the equation in the standard

form $\frac{dy}{dx} + \frac{\sin 2x}{1 + \sin^2 x} y = \frac{\cos x}{1 + \sin^2 x}$, where $P(x) = \frac{\sin 2x}{1 + \sin^2 x}$ and $Q(x) = \frac{\cos x}{1 + \sin^2 x}$. An I.F. is

$$e^{\int P(x)dx} = \exp\left[\int \frac{\sin 2x}{1 + \sin^2 x} dx\right] = \exp\left[\int \frac{2 \sin x \cos x}{1 + \sin^2 x} dx\right]$$

$= e^{\ln(1+\sin^2 x)} = 1 + \sin^2 x$. Upon multiplying the standard form equation through by the I.F., we obtain $(1 + \sin^2 x)\frac{dy}{dx} + (\sin 2x)y = \cos x$, that is, $\frac{d}{dx}[(1 + \sin^2 x)y] = \cos x$. Integrating, we find $(1 + \sin^2 x)y = \sin x + c$.

15. This is a Bernoulli D.E., where $n = 2$. We multiply through by $y^{-2}$ to obtain $y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -\frac{1}{x}$. Let $v = y^{1-n} = y^{-1}$; then $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. The preceeding D.E. readily transforms into the linear equation $\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x}$. An I.F. is $e^{\int dx/x} = e^{\ln|x|} = |x| = \pm x$. Multiplying through by this, we find $x\frac{dv}{dx} + v = 1$ or $\frac{d}{dx}(xv) = 1$. Integrating, we find $xv = x + c$, from which $v = 1 + cx^{-1}$. But $v = 1/y$. Thus we obtain the solution in the form $1/y = 1 + cx^{-1}$.

18. This is a Bernoulli D.E. in the dependent variable x and independent variable t, where $n = -1$. We multiply through by x to obtain $x\frac{dx}{dt} + \frac{t + 1}{2t}x^2 = \frac{t + 1}{t}$. Let $v = x^{1-n} = x^2$; then $\frac{dv}{dt} = 2x\frac{dx}{dt}$. The preceeding D.E. readily

transforms into the linear equation $\frac{dv}{dt} + \frac{t+1}{t}v = 2\left(\frac{t+1}{t}\right)$. An I.F. is $e^{\int \frac{t+1}{1}dt} = e^{t+\ln|t|} = |t|e^t = \pm t e^t$. Multiplying through by this, we find $t e^t \frac{dv}{dt} + (t+1)e^t v = 2(t+1)e^t$ or $\frac{d}{dt}[t e^t v] = 2(t+1)e^t$. Integrating, we obtain $t e^t v = 2 t e^t + c$ or $v = 2 + c\, t^{-1} e^{-t}$. But $v = x^2$. Thus we obtain the solution in the form $x^2 = 2 + c\, t^{-1} e^{-t}$.

The equations of Exercises 19 through 24 are linear.

21. We divide thru by $(e^x + 1)$ and dx to put in the form $e^x[y(e^x + 1)^{-1} - 3(e^x + 1)] + \frac{dy}{dx} = 0$ and thence in the standard form

$$\frac{dy}{dx} + \frac{e^x}{e^x + 1}y = 3e^x(e^x + 1),$$

where $P(x) = \frac{e^x}{e^x + 1}$, $Q(x) = 3e^x(e^x + 1)$. An I.F. is $e^{\int P(x)dx} = \exp\int \frac{e^x}{e^x + 1}dx = e^{\ln(e^x+1)} = e^x + 1$. We multiply the standard form equation through by this, obtaining $(e^x + 1)\frac{dy}{dx} + e^x y = 3e^x(e^x + 1)^2$ or $\frac{d}{dx}[(e^x + 1)y] = 3e^x(e^x + 1)^2$. Integrating, we obtain $(e^x + 1)y = (e^x + 1)^3 + c$ or

$$y = (e^x + 1)^2 + c(e^x + 1)^{-1}. \qquad (*)$$

We apply the I.C. $y(0) = 4$: Let $x = 0, y = 4$ in (*) to obtain $4 = 4 + c/2$. Thus $c = 0$, and the particular solution of the stated I.V.P. is $y = (e^x + 1)^2$.

24. This equation is linear in the dependent variable x with t as the independent variable, and is already in the standard form, with $P(t) = -1$, $Q(t) = \sin 2t$. An I.F. is $e^{\int P(t)dt} = e^{\int(-1)dt} = e^{-t}$. We multiply the equation through by this to obtain $e^{-t}\frac{dx}{dt} - e^{-t}x = e^{-t}\sin 2t$ or $\frac{d}{dt}(e^{-t}x) = e^{-t}\sin 2t$. We next integrate, using integration by parts twice, or an integral table, on the right. We obtain

$$e^{-t}x = e^{-t}\,\frac{-\sin 2t - 2\cos 2t}{5} + c$$

or

$$x = \frac{-(\sin 2t + 2\cos 2t)}{5} + ce^{t}. \qquad (*)$$

We apply the I.C. $x(0) = 0$: Let $t = x = 0$ in (*). We obtain $0 = -1/5(0 + 2) + c$ from which $c = 2/5$. Thus the general solution of the stated I.V.P. is

$$x = \frac{2e^{t} - \sin 2t - 2\cos 2t}{5}.$$

25. This is a Bernoulli D.E. with $n = -3$. We first multiply through by $y^3$ to obtain $y^3\frac{dy}{dx} + \frac{y^4}{2x} = x$. Now let $v = y^{1-n} = y^4$. Then $\frac{dv}{dx} = 4y^3\frac{dy}{dx}$ and the preceeding D.E.

transforms into $\frac{1}{4}\frac{dv}{dx} + \frac{v}{2x} = x$, which is linear in v. In the standard form this linear equation is $\frac{dv}{dx} + \frac{2}{x}v = 4x$, with $P(x) = 2/x$, $Q(x) = 4x$. An I.F. is

$$e^{\int P(x)dx} = e^{\int (2/x)dx} = e^{2\ln|x|} = x^2.$$

We multiply the standard form through by this to obtain $x^2\frac{dv}{dx} + 2xv = 4x^3$ or $\frac{d}{dx}(x^2) = 4x^3$. We integrate to obtain $x^2v = x^4 + c$. But $v = y^4$. Hence we obtain the one-parameter family of solutions of the given Bernoulli Equation in the form

$$x^2y^4 = x^4 + c. \qquad (*)$$

We apply the I.C. $y(1) = 2$. Let $x = 1$, $y = 2$ in (*) to obtain $16 = 1 + c$, so $c = 15$. Thus the particular solution of the stated I.V.P. is $x^2y^4 = x^4 + 15$.

26. We first write the D.E. in the form $x\frac{dy}{dx} + y = x^{3/2}y^{3/2}$ and multiply through by $1/x$ to obtain $\frac{dy}{dx} + \frac{1}{x}y = x^{1/2}y^{3/2}$. We recognize this D.E. as a Bernoulli D.E. with $n = 3/2$. We now multiply through by $y^{-3/2}$ to obtain $y^{-3/2}\frac{dy}{dx} + \frac{1}{x}y^{-1/2} = x^{1/2}$. Now let $v = y^{1-n} = y^{-1/2}$. Then $\frac{dv}{dx} = -\frac{1}{2}y^{-3/2}\frac{dy}{dx}$ and the preceeding D.E. transforms into $-2\frac{dv}{dx} + \frac{1}{x}v = x^{1/2}$, which is linear in v. In the standard

form this linear D.E. is $\frac{dv}{dx} + \left(\frac{-1}{2x}\right)v = -\frac{x^{1/2}}{2}$, with $P(x) = -\frac{1}{2x}$, $Q(x) = -\frac{x^{1/2}}{2}$. An I.F. is $e^{\int P(x)dx} = e^{\int(-1/2x)dx} = e^{-1/2\ln|x|} = e^{\ln|x|^{-1/2}} = |x|^{-1/2} = x^{-1/2}$ (assuming $x > 0$). We multiply the standard form through by this I.F. to obtain $x^{-1/2}\frac{dv}{dx} - \frac{1}{2x^{3/2}}v = -\frac{1}{2}$ or $\frac{d}{dx}(x^{-1/2}v) = -\frac{1}{2}$. We integrate to obtain $x^{-1/2}v = -\frac{1}{2}x + c$. But $v = y^{-1/2}$. Hence we obtain the one-parameter family of solutions of the given Bernoulli equation in the form $x^{-1/2}y^{-1/2} = -\frac{1}{2}x + c$ or

$$\frac{1}{\sqrt{xy}} = -\frac{1}{2}x + c \qquad (*)$$

We apply the I.C. $y(1) = 4$. Let $x = 1$, $y = 4$ in (*) to obtain $\frac{1}{2} = -\frac{1}{2} + c$, so $c = 1$. Thus the particular solution of the stated I.V.P. is $\frac{1}{\sqrt{xy}} = -\frac{1}{2}x + 1$.

27. Here we actually have two I.V. problems:

$$\text{(I)}\begin{cases} \text{For } 0 \le x < 1, \\ \dfrac{dy}{dx} + y = 2, \\ y(0) = 0, \end{cases} \qquad \text{(II)}\begin{cases} \text{For } x \ge 1, \\ \dfrac{dy}{dx} + y = 0, \\ y(1) = a, \end{cases}$$

where a is the value of $\lim_{x \to 1-} \phi(x)$ and $\phi$ denotes the solution of (I). This is prescribed so that the solution of the entire problem will be continuous at $x = 1$.

We first solve (I). The D.E. of this problem is linear in standard form with $P(x) = 1$, $Q(x) = 2$. An I.F. is $e^{\int P(x)dx} = e^{\int (1)dx} = e^x$. We multiply the D.E. of (I) through by this to obtain $e^x \frac{dy}{dx} + e^x y = 2e^x$ or $\frac{d}{dx}[e^x y] = 2e^x$. We integrate to obtain $e^x y = 2e^x + c$ or $y = 2 + ce^{-x}$. We apply the I.C. of (I), $y(0) = 0$, to this, obtaining $0 = 2 + c$ or $c = -2$. Thus the particular solution of problem (I) is $y = 2 - 2e^{-x}$. This is valid for $0 \leq x < 1$.

Letting $\phi$ denote the solution just obtained, we note that $\lim_{x \to 1^-} \phi(x) = \lim_{x \to 1^-} (2 - 2e^{-x}) = 2 - 2e^{-1}$. This is the a of Problem (II); that is, the I.C. of (II) is $y(1) = 2 - 2e^{-1}$.

Now solve (II). The D.E. of this problem is linear is standard form with $P(x) = 1$, $Q(x) = 0$. An I.F., as in problem (I), is $e^x$. We multiply through by this to obtain $e^x \frac{dy}{dx} + e^x y = 0$ or $\frac{d}{dx}(e^x y) = 0$. We integrate to obtain $e^x y = c$ or $y = ce^{-x}$. Now apply the I.C. of (II), $y(1) = 2 - 2e^{-1}$, to this. Let $x = 1$, $y = 2 - 2e^{-1}$. We obtain $2 - 2e^{-1} = ce^{-1}$, from which $c = 2e - 2$. Thus the particular solution of problem (II) is $y = (2e - 2)e^{-x}$. This is valid for $x \geq 1$.

We write the solution of the entire problem, showing intervals where each part is valid, as

$$y = \begin{cases} 2(1 - e^{-x}), & 0 \leq x < 1, \\ 2(e - 1)e^{-x}, & x \geq 1 \end{cases}.$$

29. As in Exercise 27, we actually have two I.V. problems:

$$\text{(I)}\begin{cases}\text{For } 0 \le x < 2,\\ \dfrac{dy}{dx} + y = e^{-x},\\ y(0) = 1,\end{cases} \qquad \text{(II)}\begin{cases}\text{For } x \ge 2,\\ \dfrac{dy}{dx} + y = e^{-2},\\ y(2) = a,\end{cases}$$

where a is the value of $\lim_{x \to 2^-} \phi(x)$ and $\phi$ denotes the solution of (I). This is prescribed so that the solution of the entire problem will be continuous at $x = 2$.

We first solve (I). The D.E. of this problem is linear in standard form with $P(x) = 1$, $Q(x) = e^{-x}$. An I.F. is $e^{\int P(x)dx} = e^{\int (1)dx} = e^{x}$. We multiply the D.E. of (I) through by this to obtain $e^{x}\frac{dy}{dx} + e^{x}y = 1$ or $\frac{d}{dx}(e^{x}y) = 1$. Integrating we obtain $e^{x}y = x + c$ or $y = e^{-x}(x + c)$.

We apply the I.C. of (I), $y(0) = 1$ to this, obtaining $1 = 0 + c$, or $c = 1$. Thus the particular solution of problem (I) is $y = e^{-x}(x + 1)$. This is valid for $0 \le x < 2$.

Letting $\phi$ denote the solution just obtained, we find $\lim_{x \to 2^-} \phi(x) = \lim_{x \to 2^-} e^{-x}(x + 1) = 3e^{-2}$. This is the a of Problem (II); that is, the I.C. of (II) is $y(2) = 3e^{-2}$.

Now solve (II). The D.E. of this problem is linear in standard form with $P(x) = 1$, $Q(x) = e^{-2}$. An I.F., as in Problem (I), is $e^{x}$. We multiply through by this to obtain $e^{x}\frac{dy}{dx} + e^{x}y = e^{x-2}$ or $\frac{d}{dx}[e^{x}y] = e^{x-2}$. We integrate to

obtain $e^x y = e^{x-2} + c$ or $y = e^{-x}(e^{x-2} + c)$ or $y = ce^{-x} + e^{-2}$. Now apply the I.C. of (II), $y(2) = 3e^{-2}$, to this. Let $x = 2$, $y = 3e^{-2}$. We obtain $3e^{-2} = ce^{-2} + e^{-2}$, from which $c = 2$. Thus the particular solution of problem (II) is $y = 2e^{-x} + e^{-2}$. This is valid for $x \geq 2$.

We write the solution of the entire problem, showing intervals where each part is valid, as

$$y = \begin{cases} e^{-x}(x + 1), & 0 \leq x < 2, \\ 2e^{-x} + e^{-2}, & x \geq 2 \end{cases} .$$

30. As in Exercises 27 and 29, we actually have two I.V. problems:

$$\text{(I)} \begin{cases} \text{For } 0 \leq x < 3, \\ (x + 1)\dfrac{dy}{dx} + y = x, \\ y(0) = 1/2. \end{cases} \qquad \text{(II)} \begin{cases} \text{For } x \geq 3, \\ (x + 1)\dfrac{dy}{dx} + y = 3, \\ y(3) = a, \end{cases}$$

where a is the value of $\lim_{x \to 3^-} \phi(x)$ and $\phi$ denotes the solution of (I). This is prescribed so that the solution of the entire problem will be continuous at $x = 3$.

We first solve (I). We first express the D.E. in the standard form $\dfrac{dy}{dx} + \dfrac{1}{x + 1} y = \dfrac{x}{x + 1}$, where $P(x) = 1/(x + 1)$, $Q(x) = x/(x + 1)$. An I.F. is $e^{\int P(x)dx} = e^{\int dx/x+1} = e^{\ln|x+1|} = |x + 1|$. Since $x + 1 \geq 1 > 0$ in (I), we have $|x + 1| = x + 1$. We multiply the standard

form equation through by this to obtain $(x + 1)\frac{dy}{dx} + y = x$, which turns out to be exactly the equation that we started with. We note that this is $\frac{d}{dx}[(x + 1)y] = x$. Integrating we obtain $(x + 1)y = x^2/2 + c$. Applying the I.C. $y(0) = 1/2$, we obtain $c = 1/2$. Thus the particular solution of Problem (I) is $(x + 1)y = \frac{1}{2}(x^2 + 1)$ or $y = \frac{x^2 + 1}{2(x + 1)}$. This is valid for $0 \leq x < 3$.

Letting $\phi$ denote the solution just obtained, we find $\lim_{x \to 3^-} \phi(x) = \lim_{x \to 3^-} \left[\frac{x^2 + 1}{2(x + 1)}\right] = \frac{5}{4}$. This is the a of Problem (II); that is, the I.C. of (II) is $y(3) = 5/4$.

Now solve (II). We put the D.E. in standard form $\frac{dy}{dx} + \frac{1}{x + 1} y = \frac{3}{x + 1}$ and find, as in Problem (I), that an I.F. is $(x + 1)$. We multiply the standard form equation through by this to obtain $(x + 1)\frac{dy}{dx} + y = 3$, which is exactly what we started with here in Problem (II). We note that this is $\frac{d}{dx}[(x + 1)y] = 3$. Integrating we obtain $(x + 1)y = 3x + c$. Applying the I.C. $y(3) = 5/4$, we obtain $5 = 9 + c$ or $c = -4$. Thus the particular solution of Problem (II) is $(x + 1)y = 3x - 4$ or $y = \frac{3x - 4}{x + 1}$. This is valid for $x \geq 3$.

The solution of the entire problem is

$$y = \begin{cases} \dfrac{x^2 + 1}{2(x + 1)}, & 0 \le x < 3, \\ \dfrac{3x - 4}{x + 1}, & x \ge 3 \end{cases} .$$

31. (a) The D.E. is linear. In the standard form it is $\frac{dy}{dx} + \frac{b}{a}y = \frac{k}{a}e^{-\lambda x}$, with $P(x) = \frac{b}{a}$, $Q(x) = \frac{k}{a}e^{-\lambda x}$. An I.F. is $e^{\int P(x)dx} = e^{\int (b/a)dx} = e^{(b/a)x}$. Multiplying the standard form equation through by this, we obtain

$$e^{(b/a)x}\frac{dy}{dx} + \frac{b}{a}e^{(b/a)x}y = \frac{k}{a}e^{(b/a-\lambda)x}$$

$$\text{or } \frac{d}{dx}\left[e^{(b/a)x}y\right] = \frac{k}{a}e^{(b/a-\lambda)x} . \qquad (1)$$

We now consider two cases: (i) $\lambda \neq \frac{b}{a}$; and (ii) $\lambda = \frac{b}{a}$.

In case (i), where $\lambda \neq \frac{b}{a}$, integrating (1) we obtain $e^{(b/a)x}y = \frac{k}{a}\left(\frac{b}{a} - \lambda\right)^{-1} e^{(b/a-\lambda)x} + c$ or

$$y = \frac{ke^{-\lambda x}}{b - a\lambda} + ce^{-bx/a} . \qquad (2)$$

In case (ii), where $\lambda = \frac{b}{a}$, (1) becomes $\frac{d}{dx}\left[e^{(b/a)x}y\right] = \frac{k}{a}$. Integrating, we obtain $e^{(b/a)x}y = \frac{kx}{a} + c$ or

$$y = \frac{k \times e^{-bx/a}}{a} + ce^{-bx/a} . \qquad (3)$$

(b) Suppose $\lambda = 0$. Then since $\frac{b}{a} > 0$, we can have only case (i), $\lambda \neq \frac{b}{a}$, here. Then with $\lambda = 0$, (2) becomes $y = \frac{k}{b} + ce^{-bx/a}$. As $x \to \infty$, $e^{-bx/a} \to 0$ and $y \to \frac{k}{b}$.

Suppose $\lambda > 0$. Then either case (i) or case (ii) can occur. In case (i), y is given by (2); and since $e^{-\lambda x} \to 0$ and $e^{-bx/a} \to 0$ as $x \to \infty$, we have $y \to 0$ as $x \to \infty$. In case (ii), y is given by (3); and since $xe^{-bx/a} \to 0$ and $e^{-bx/a} \to 0$ as $x \to \infty$, we again have $y \to 0$ as $x \to \infty$.

33. (a) The function f such that $f(x) = 0$ for all $x \in I$ has derivative $f'(x) = 0$ for all $x \in I$. We substitute this $f(x)$ for you and $f'(x)$ for $\frac{dy}{dx}$ in the D.E. We obtain $f'(x) + P(x)f(x) = 0 + P(x)0 = 0$ for all $x \in I$, so $f(x)$ is a solution.

(b) By the Theorem 1.1, there is a unique solution f of the D.E. satisfying the I.C. $f(x_0) = 0$, where $x_0 \in I$. By (a), $\phi(x) = 0$ for all $x \in I$ is a solution of the D.E. and obviously it satisfies the I.C. Thus by the uniqueness of Theorem 1.1, we must have $f(x) = \phi(x)$, that is, $f(x) = 0$ for all $x \in I$.

(c) Since f and g are solutions of the D.E., so is their difference $h = f - g$ (by Exercise 32(a)). Also $h(x_0) = f(x_0) - g(x_0) = 0$. So h is a solution such that $h(x_0) = 0$ for some $x_0 \in I$. By part (b), $h(x) = 0$ for all $x \in I$. But this means $f(x) - g(x) = 0$ and hence $f(x) = g(x)$ for all $x \in I$.

34. (a) Since f is a solution of (A),

$$f'(x) + P(x)f(x) = Q(x). \qquad (1)$$

Since g is a solution of (A),

$$g'(x) + P(x)g(x) = Q(x). \qquad (2)$$

Then, subtracting (2) from (1), we have

$$f'(x) + P(x)f(x) - g'(x) - P(x)g(x) = Q(x) - Q(x)$$

or

$$f'(x) - g'(x) + P(x)[f(x) - g(x)] = 0$$

or, finally,

$$[f(x) - g(x)]' + P(x)[f(x) - g(x)] = 0.$$

This shows that f - g is a solution of the D.E.

$$\frac{dy}{dx} + P(x)y = 0.$$

(b) Since f is a solution of Equation (A), we have

$$f'(x) + P(x)f(x) = Q(x). \qquad (3)$$

By part (a), since f and g are solutions of (A), their difference f - g is a solution of dy/dx + P(x)y = 0. Thus we have

$$[f(x) - g(x)]' + P(x)[f(x) - g(x)] = 0.$$

Multiplying this by c, where c is an arbitrary constant, we have

$$c[f(x) - g(x)]' + cP(x)[f(x) - g(x)] = 0. \quad (4)$$

Adding (3) and (4), we have

$$\{f'(x) + P(x)f(x)\} + \{c[f(x) - g(x)]' + cP(x)[f(x) - g(x)]\} = Q(x).$$

Rearranging terms, this takes the form

$$\{c[f(x) - g(x)]' + f'(x)\} + P(x)\{c[f(x) - g(x)] + f(x)\} = Q(x).$$

or

$$\{c[f(x) - g(x)] + f(x)\}' + P(x)\{c[f(x) - g(x)] + f(x)\} = P(x).$$

We see from this that $c(f - g) + f$ satisfies equation (A). That is, since c is an arbitrary constant, $c(f - g) + f$ is a one-parameter family of solutions of (A).

37. (a) This is of the form (2.41) with $f(y) = \sin y$, $P(x) = \frac{1}{x}$, and $Q(x) = 1$. We let $v = f(y) = \sin y$, from which $\frac{dv}{dx} = (\cos y)\frac{dy}{dx}$. Substituting this into the stated D.E., it becomes $\frac{dv}{dx} + \frac{1}{x}v = 1$, which is linear in v. An I.F. is $e^{\int P(x)dx} = e^{\int (1/x)dx} = e^{\ln|x|} = |x|$.

Multiplying the linear equation through by this, we have $x\frac{dv}{dx} + v = x$ or $\frac{d}{dx}(xv) = x$. Integrating, we find $xv = \frac{x^2}{2} + c_0$, or $2xv = x^2 + c$, where $c = 2c_0$. Now replacing v by $\sin y$, we obtain the solution in the form $2x \sin y - x^2 = c$.

40. This is a Riccati Equation of the form (A) of Exercise 38, with $A(x) = -1$, $B(x) = x$, $C(x) = 1$. The solution $f(x) = x$ is given. By Exercise 38(b), we make the transformation $y = f(x) + \frac{1}{v} = x + \frac{1}{v}$, from which $\frac{dy}{dx} = 1 - \left(\frac{1}{v^2}\right)\frac{dv}{dx}$.

Substituting in the given D.E., it takes the form

$$1 - \frac{1}{v^2}\frac{dv}{dx} = -\left(x + \frac{1}{v}\right)^2 + x\left(x + \frac{1}{v}\right) + 1.$$

This reduces to $-\frac{1}{v^2}\frac{dv}{dx} = -\frac{1}{v^2} - \frac{x}{v}$ or $\frac{dv}{dx} - xv = 1$, which is linear in v, with $P(x) = -x$, $Q(x) = 1$. An I.F. is $e^{\int P(x)dx} = e^{\int(-x)dx} = e^{-x^2/2}$. Multiplying the linear D.E. through by this, we obtain

$$e^{-x^2/2}\frac{dv}{dx} - e^{-x^2/2}\,x\,v = e^{-x^2/2}$$

or

$$\frac{d}{dx}(e^{-x^2/2}v) = e^{-x^2/2}.$$

Integrating, we obtain $e^{-x^2/2}v = \int e^{-x^2/2}\,dx + c$. Since the integral on the right cannot be expressed as a finite sum of known elementary functions, we leave it as

indicated. But from the transformation $y = x + \frac{1}{v}$, we see that $v = \frac{1}{(y - x)}$. We replace v accordingly and thus obtain the solution,

$$\frac{e^{-x^2/2}}{y - x} = \int e^{-x^2/2}\, dx + c.$$

**Section 2.3, Miscellaneous Review Exercises, Page 59**

The solution of each of these review problems is given, but most of these solutions are presented in abbreviated form with many details omitted.

1. The D.E. is both separable and linear. Upon separating variables and integrating, we obtain $2 \ln (x^3 + 1) + \ln|c| = \ln|y|$, from which we find $y = c(x^3 + 1)^2$.

   The solution as a linear equation is almost as easy. The I.F. is $(x^3 + 1)^{-2}$.

2. The D.E. is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 2xy^3 - y$ and $F_y(x,y) = N(x,y) = 3x^2y^2 - x$. From the first of these, we find $F(x,y) = x^2y^3 - xy + \phi(y)$. From this $F_y(x,y) = 3x^2y^2 - x + \phi'(y)$. But since $F_y(x,y) = N(x,y) = 3x^2y^2 - x$, we find $\phi'(y) = 0$, $\phi(y) = c_0$. Thus $F(x,y) = x^2y^3 - xy + c_0$, and the family of solutions is $x^2y^3 - xy = c$.

3. The D.E. is both separable and linear. First consider it as a separable equation. Upon separating variables and using partial fractions (preparatory to integrating), we find $\left[\frac{1}{x} - \frac{1}{x+1}\right]dx + \frac{dy}{y-1} = 0$. Integration gives $\ln|x| - \ln|x+1| + \ln|y-1| = \ln c_1$. Upon simplifying and taking antilogs, we find $|x(y-1)| = |c_1(x+1)|$. Assuming $x > 0$, $y > 1$ and simplifying, we can write this as $xy = cx + (c-1)$, or $xy + 1 = c(x+1)$, where $c = 1 + c_1$.

   Alternately, consider the D.E. as a linear equation. The I.F. is $\exp\left[\frac{dx}{x(x+1)}\right] = \pm\frac{x}{x+1}$. Multiplying the "standard form" of the D.E. through by this we have $\left[\frac{xy}{x+1}\right]' = (x+1)^{-2}$. Integrating, we find $\frac{xy}{x+1} = -(x+1)^{-1} + c$. From this we find $xy + 1 = c(x+1)$, as before.

4. The D.E. is linear. The I.F. is $x$. Multiplying the "standard form" of the equation through by this, we find $x^2y' + 2xy = x^3$ and hence $(x^2y)' = x^3$. Integration and simplification at once give $y = \frac{x^2}{4} + \frac{c}{x^2}$.

5. Writing the D.E. in the form $\frac{dy}{dx} = \frac{5\left[\frac{y}{x}\right] - 3}{1 + \frac{y}{x}}$, we recognize that it is homogeneous. We let $y = vx$. Then the D.E. becomes $v + x\frac{dv}{dx} = \frac{5v-3}{1+v}$. Simplification yields

$\frac{(v + 1)dv}{v^2 - 4v + 3} = \frac{-dx}{x}$. Then upon applying partial fractions to the left member, we find

$$\left[-\frac{1}{v - 1} + \frac{2}{v - 3}\right]dv = -\frac{dx}{x}.$$

Integration gives $-\ln|v - 1| + 2\ln|v - 3| = -\ln|x| + \ln|c|$. Simplification and taking of antilogarthims results in $\frac{(v - 3)^2}{|v - 1|} = \left|\frac{c}{x}\right|$. But $v = \frac{y}{x}$. Upon resubstituting accordingly and simplifying, we find $(3x - y)^2 = |c(y - x)|$.

6. The D.E. is exact, separable, and linear. Considered as an exact equation, we seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = e^{2x}y^2$ and $F_y(x,y) = N(x,y) = e^{2x}y - 2y$. From the first of these, we find $F(x,y) = e^{2x}\frac{y^2}{2} + \phi(y)$. From this, $F_y(x,y) = e^{2x}y^2 + \phi'(y)$. But since $F_y(x,y) = N(x,y) = e^{2x}y - 2y$, we find $\phi'(y) = -2y$, $\phi(y) = -y^2 + c_0$. Thus $F(x,y) = \frac{e^{2x}y^2}{2} - y^2 + c_0$, and the family of solutions is $e^{2x}y^2 - 2y^2 = c$, [where we have set $F(x,y) = c_1$, multiplied through by 2, and let $c = 2(c_1 - c_0)$]. Assuming $e^{2x} - 2 \geq 0$, this can also be written as $y = c_2(e^{2x} - 2)^{-1/2}$ where $c_2 = \sqrt{c}$.

We also consider the D.E. as a linear equation. We rewrite it as $(e^{2x} - 2)y' + e^{2x}y = 0$ and then put it in the "standard form" $y' + e^{2x}y/(e^{2x} - 2) = 0$. From this we

find the I.F. is $(e^{2x} - 2)^{1/2}$, where we assume $e^{2x} - 2 \geq 0$. Multiplying the standard form through by the I.F., we find $(e^{2x} - 2)^{1/2}y' + e^{2x}y(e^{2x} - 2)^{-1/2} = 0$ or $[(e^{2x} - 2)^{1/2}y]' = 0$. Integration and simplification readily yield $y = c(e^{2x} - 2)^{-1/2}$.

7. The D.E. is both separable and linear. We first solve this as a separable equation. Separating variables, we have $\frac{4x^3dx}{x^4 + 1} + \frac{dy}{2y - 3} = 0$. Integration yields $\ln(x^4 + 1) + \frac{1}{2}\ln|2y - 3| = \ln|c_1|$. Simplify, assuming $2y - 3 \geq 0$, to obtain $\ln(x^4 + 1)(2y - 3)^{1/2} = \ln|c_1|$. From this we at once have $(x^4 + 1)^2(2y - 3) = c_1^2$. Then $y = 3/2 + c(x^4 + 1)^{-2}$, where $c = \frac{c_1^2}{2}$.

Consider the D.E. as a linear equation. In standard form the D.E. is $y' + \frac{8x^3y}{x^4 + 1} = \frac{12x^3}{x^4 + 1}$. The I.F. is $(x^4 + 1)^2$. Multiplying the standard form equation through by this we find $[(x^4 + 1)^2y]' = (x^4 + 1)(12x^3)$. Integration and division by $(x^4 + 1)^2$ then give $y = \frac{3}{2} + c(x^4 + 1)^{-2}$.

8. Writing the D.E. in the form $\frac{dy}{dx} = -\frac{\left[2 + \frac{y}{x} + \left[\frac{y}{x}\right]^2\right]}{2}$, we see that the D.E. is homogeneous. Let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$, and the D.E. becomes $v + x\frac{dv}{dx} = -1 - \frac{v}{2} - \frac{v^2}{2}$.

Simplification yields $x\frac{dv}{dx} = -\frac{(v^2 + 3v + 2)}{2}$, and hence $\frac{2\,dv}{v^2 + 3v + 2} = -\frac{dx}{x}$. Use of partial fractions gives $2\left[\frac{1}{v + 1} - \frac{1}{v + 2}\right]dv = -\frac{dx}{x}$. Integration and immediate simplification give $2 \ln\left|\frac{v + 1}{v + 2}\right| = \ln\left|\frac{c}{x}\right|$. Further simplification gives $\left|\frac{v + 1}{v + 2}\right|^2 = \left|\frac{c}{x}\right|$. But $v = \frac{y}{x}$. Resubstituting accordingly and again simplifying, we obtain $\left[\frac{y + x}{y + 2x}\right]^2 = \left|\frac{c}{x}\right|$.

9. Rewriting the D.E. in the form $(4x^3y^2 - 3x^2y)dx + (2x^4y - x^3)dy = 0$, we find that it is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 4x^3y^2 - 3x^2y$ and $F_y(x,y) = N(x,y) = 2x^4y - x^3$. From the first of these, $F(x,y) = x^4y^2 - x^3y + \phi(y)$. From this, $F_y(x,y) = 2x^4y - x^3 + \phi'(y)$. But since $F_y(x,y) = N(x,y) = 2x^4y - x^3$, we must have $\phi'(y) = 0$, $\phi(y) = c_0$. Thus $F(x,y) = x^4y^2 - x^3y + c_0$, and the solution is $x^4y^2 - x^3y = c$.

10. The D.E. is linear. The standard form is $y' + \frac{xy}{x + 1} = e^{-x}(x + 1)$, and the I.F. is $\pm e^x(x + 1)$. Multiplying the standard form through by this, we obtain the equivalent of $\left[\frac{e^xy}{x + 1}\right]' = (x + 1)^{-2}$. Integration and multiplication by $(x + 1)e^{-x}$ result in $y = e^{-x}[-1 + c(x + 1)]$.

11. Writing the D.E. in the form $\frac{dy}{dx} = \frac{2 - 7\left[\frac{y}{x}\right]}{3\left[\frac{y}{x}\right] - 8}$, we recognize that it is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The D.E. becomes $v + x\frac{dv}{dx} = \frac{2 - 7v}{3v - 8}$. Simplification gives $\frac{(3v - 8)dv}{3v^2 - v - 2} = -\frac{dx}{x}$. Partial fractions decomposition puts this in the form $\left[\frac{6}{3v + 2} - \frac{1}{v - 1}\right] dv = -\frac{dx}{x}$. Integration gives $2 \ln (3v + 2) - \ln|v - 1| = -\ln|x| + \ln|c|$. Simplification and taking of antilogarithms results in $\frac{(3v + 2)^2}{|v - 1|} = \left|\frac{c}{x}\right|$. But $v = \frac{y}{x}$. Resubstituting accordingly and simplifying gives $\frac{(3y + 2x)^2}{x^2} \cdot \frac{|x|}{|y - x|} = \left|\frac{c}{x}\right|$ and hence $(2x + 3y)^2 = |c(y - x)|$.

12. The D.E. is a Bernoulli Equation with $n = 3$. We multiply through by $y^{-3}$ to obtain $x^2y^{-3}\frac{dy}{dx} + xy^{-2} = x$. Let $v = y^{1-n} = y^{-2}$; then $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$. The preceeding D.E. readily transforms into the linear equation $\frac{dv}{dx} - \frac{2}{x}v = -\frac{2}{x}$. An I.F. is $e^{-\int 2/x\, dx} = \frac{1}{x^2}$. Multiplying through by this, we obtain $\frac{d}{dx}\left[\frac{v}{x^2}\right] = -\frac{2}{x^3}$. Integrating, we find $\frac{v}{x^2} = \frac{1}{x^2} + c$. Replacing v by $y^{-2}$ and simplifying, we find $y^2 = \frac{1}{1 + cx^2}$.

13. This is a linear equation. The standard form is $y' + \frac{6x^2y}{x^3 + 1} = \frac{6x^2}{x^3 + 1}$. An I.F. is $(x^3 + 1)^2$. Multiplying through by this we obtain $[(x^3 + 1)y]' = (x^3 + 1)(6x^2)$. Integration and division by $(x^3 + 1)^2$ then give the solutions $y = 1 + c(x^3 + 1)^{-2}$.

14. Writing the D.E. in the form $\frac{dy}{dx} = \frac{2 + \left[\frac{y}{x}\right]^2}{2\left[\frac{y}{x}\right] - 1}$, we recognize that it is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The D.E. becomes $v + x\frac{dv}{dx} = \frac{2 + v^2}{2v - 1}$. Simplification gives $\frac{(2v - 1)dv}{v^2 - v - 2} = -\frac{dx}{x}$. Integration gives $\ln|v^2 - v - 2| = \ln\left|\frac{c}{x}\right|$. Taking antilogarithms and simplifying slightly, we obtain $v^2 - v - 2 = c/x$. But $v = y/x$. Resubstituting accordingly and simplifying, we find $y^2 - xy - 2x^2 = cx$ or $(y - 2x)(y + x) = cx$.

15. The D.E. is both homogeneous and Bernoulli. We first consider it as a homogeneous equation, which form we recognize by writing it as $\frac{dy}{dx} = \frac{1 + \left[\frac{y}{x}\right]^2}{2\left[\frac{y}{x}\right]}$. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The D.E. becomes $v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$. Simplification quickly leads to $\frac{2vdv}{v^2 - 1} = -\frac{dx}{x}$, and then integration gives $\ln|v^2 - 1| = \ln\left|\frac{c}{x}\right|$. From this, we readily find $v^2 - 1 = \frac{c}{x}$. Resubstituting $v = \frac{y}{x}$ and simplifying, we find $y^2 - x^2 = cx$.

Now apply the initial condition $y(1) = 2$. Letting $x = 1$, $y = 2$, we get $3 = c$, and hence obtain the solution in the form $y^2 = x^2 + 3x$ or $y = \sqrt{x^2 + 3x}$.

Alternatively, we recognize that the given D.E. is a Bernoulli equation, with $n = -1$, by writing in the equivalent form $2x\frac{dy}{dx} - y = x^2y^{-1}$. We rewrite this as $2xy\frac{dy}{dx} - y^2 = x^2$; let $v = y^2$ and $\frac{dv}{dx} = 2y\frac{dy}{dx}$; and transform the D.E. into the linear equation $x\frac{dy}{dx} - v = x^2$. The standard form of this is $\frac{dv}{dx} - \frac{1}{x}v = x$, and an I.F. is $x^{-1}$. Multiplying the standard form equation through by this, we obtain $(x^{-1}v)' = 1$. Integration then gives $x^{-1}v = x + c$. Now resubstituting $v = y^2$ and simplifying slightly, we find $y^2 = x^2 + cx$. Application of the I.C. again gives the particular solution $y^2 = x^2 + 3x$ or $y = \sqrt{x^2 + 3x}$.

16. The D.E. is both a separate equation and a Bernoulli Equation. We first solve it as a separable equation. Separating variables, we get $\frac{2\,dx}{1 - x^2} + \frac{y\,dy}{y^2 + 4} = 0$. We can use partial fractions or an integral table to integrate the dx term. Integrating the equation and simplifying slightly, we find

$$\ln\left|\frac{1 + x}{1 - x}\right| + \ln(y^2 + 4)^{1/2} = \ln|c|.$$

Further simplification results in the one-parameter family of solutions

$$|(1 + x)(y^2 + 4)^{1/2}| = |c(1 - x)|.$$

Now apply the I.C. $y(3) = 0$. Letting $x = 3$, $y = 0$ in the preceeding gives $8 = 2|c|$, so $|c| = 4$. Substituting this value of $|c|$ back into the equation of the family of solutions, we get

$$|(1 + x)(y^2 + 4)^{1/2}| = 4|1 - x|.$$

Since the initial x value is $3 > 1$, we take $x > 1$ and $1 - x < 0$, so $|1 - x| = x - 1$. Thus we obtain the particular solution

$$(x + 1)(y^2 + 4)^{1/2} = 4(x - 1).$$

Alternately, we now solve the given D.E. as a Bernoulli Equation. We first rewrite it as $\frac{dy}{dx} + \frac{2}{1 - x^2} y = -\frac{8}{1 - x^2} y^{-1}$, in which we recognize that it is indeed a Bernoulli D.E. with $n = -1$. We thus let $v = y^{1-n} = y^2$, find $\frac{dv}{dx} = 2y\frac{dy}{dx}$, and transform the D.E. into

$$\frac{dv}{dx} + \frac{4}{1 - x^2} v = -\frac{16}{1 - x^2},$$

which is linear in v, with $P(x) = \frac{4}{1 - x^2}$, $Q(x) = \frac{-16}{1 - x^2}$. An I.F. is $e^{\int P(x)dx} = e^{\int 4dx/(1-x^2)} = e^{2 \ln |(1+x)/(1-x)|} = \left(\frac{1 + x}{1 - x}\right)^2$. Multiplying the linear D.E. through by this, we obtain

$$\frac{d}{dx}\left[\left(\frac{1+x}{1-x}\right)^2 v\right] = -\frac{16(1+x)}{(1-x)^3}.$$

To integrate the right member, we note that

$$-\frac{16(1+x)}{(1-x)^3} = -16\left[-\frac{1}{(1-x)^2} + \frac{2}{(1-x)^3}\right].$$

Integrating the D.E., we thus find

$$\left(\frac{1+x}{1-x}\right)^2 v = -16\left[-\frac{1}{1-x} + \frac{1}{(1-x)^2}\right] + c,$$

which readily simplifies into

$$\left(\frac{1+x}{1-x}\right)^2 v = \frac{-16x}{(1-x)^2} + c.$$

Application of the I.C. $x = 3$ at $y = 0$ gives $c = 12$. Substituting this back and multiplying by $(1-x)^2$, we obtain $(1+x)^2y^2 = -16x + 12(1-x)$. Adding $4(1+x)^2$ to both sides of this and simplifying, we find $(1+x)^2(y^2+4) = 16(x-1)^2$, from which the solution previously found is readily obtained.

17. This D.E., with $M(x,y) = e^{2x}y^2 - 2x$ and $N(x,y) = e^{2x}y$, is exact; for $M_y(x,y) = 2e^{2x}y = N_y(x,y)$. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = e^{2x}y^2 - 2x$ and $F_y(x,y) = N(x,y) = e^{2x}y$. From the first of these, $F(x,y) = e^{2x}\frac{y^2}{2} - x^2 + \phi(y)$. From this, $F_y(x,y) = e^{2x}y + \phi'(y)$. But this must

equal $N(x,y) = e^{2x}y$. Thus we must have $\phi'(y) = 0$ and hence $\phi(y) = c_0$, Thus $F(x,y) = e^{2x}\frac{y^2}{2} - x^2 + c_0$; and from this the solution $F(x,y) = c_1$ take the form $e^{2x}\frac{y^2}{2} - x^2 = c$, where $c = c_1 - c_0$. Now apply the given I.C. $y(0) = 2$ to this. Thus letting $x = 0$, $y = 2$, we find $c = 2$. Thus we obtain the solution $e^{2x}\frac{y^2}{2} - x^2 = 2$ or $y^2e^{2x} = 2x^2 + 4$ or, finally, $y = e^{-x}(2x^2 + 4)^{1/2}$.

18. This D.E., with $M(x,y) = 3x^2 + 2xy^2$ and $N(x,y) = 2x^2y + 6y^2$, is exact; for $M_y(x,y) = 4xy = N_x(x,y)$. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 3x^2 + 2xy^2$ and $F_y(x,y) = N(x,y) = 2x^2y + 6y^2$. From the first of these, $F(x,y) = x^3 + x^2y^2 + \phi(y)$. From this, $F_y(x,y) = 2x^2y + \phi(y)$. But this must equal $N(x,y) = 2x^2y + 6y^2$. Thus we must have $\phi'(y) = 6y^2$ and hence $\phi(y) = 2y^3 + c_0$. Thus $F(x,y) = x^3 + x^2y^2 + 2y^3 + c_0$; and from this the solutions $F(x,y) = c_1$ take the form $x^3 + x^2y^2 + 2y^3 = c$, where $c = c_1 - c_0$. Applying the I.C. $y(1) = 2$, we let $x = 1$, $y = 2$, to obtain $c = 21$. Thus we obtain the solution in the form $x^3 + x^2y^2 + 2y^3 = 21$.

19. This D.E. is both a separable equation and a Bernoulli Equation. We first solve it as a separable D.E., writing it as $\frac{4y\,dy}{y^2 + 1} = \frac{dx}{x}$. Integration then gives $2\ln(y^2 + 1) = \ln|x| + \ln|c|$. Simplifying we find $\ln(y^2 + 1)^2 = \ln|cx|$ and hence $(y^2 + 1)^2 = |cx|$. Applying the I.C. $y(2) = 1$,

we set $x = 2$, $y = 1$ in this to obtain $4 = 2|c|$ and hence $|c| = 2$. Thus we obtain the solution $(y^2 + 1)^2 = 2|x|$. Since the initial x value is $2 > 0$, we take $x > 0$ and write the solution as $(y^2 + 1)^2 = 2x$. Alternately, writing the D.E. as $4x\frac{dy}{dx} - y = y^{-1}$, we recognize that it is a Bernoulli Equation with $n = -1$. Thus we let $v = y^{1-n} = y^2$, $\frac{dv}{dx} = 2y\frac{dy}{dx}$, and substitute into the stated equation $4xy\frac{dy}{dx} = y^2 + 1$, obtaining $2x\frac{dv}{dx} = v + 1$ or $\frac{dv}{dx} - \frac{1}{2x}v = \frac{1}{2x}$. This is linear in v, with I.F. $e^{-\int(1/2x)dx} = x^{-1/2}$. Multiplying the standard form linear equation through by this, we find $\frac{d}{dx}(x^{-1/2}v) = \frac{x^{-3/2}}{2}$. Integrating, we obtain $x^{-1/2}v = -x^{-1/2} + c$ or $v = -1 + cx^{1/2}$. But $v = y^2$, so we have $y^2 = -1 + cx^{1/2}$. Applying the I.C. $y(2) = 1$ gives $c = \sqrt{2}$. Thus we obtain the solution $y^2 = -1 + \sqrt{2}x^{1/2}$, from which we readily obtain the solution $(y^2 + 1)^2 = 2x$ previously found.

20. Writing the D.E. in the form $\frac{dy}{dx} = \frac{2 + 7\left[\frac{y}{x}\right]}{2 - 2\left[\frac{y}{x}\right]}$, we see that it is homogeneous. Thus we let $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The D.E. successively reduces to $v + x\frac{dv}{dx} = \frac{2 + 7v}{2 - 2v}$, $x\frac{dv}{dx} = \frac{2v^2 + 5v + 2}{2 - 2v}$, and $\frac{(2v - 2)dv}{2v^2 + 5v + 2} = -\frac{dx}{x}$. This last equation is separable, with the variables separated. Using partial fractions, we find $\frac{2v - 2}{2v^2 + 5v + 2} = -\frac{2}{2v + 1} + \frac{2}{v + 2}$. We

thus have $\left[\frac{-2}{2v + 1} + \frac{2}{v + 2}\right]dv = -\frac{dx}{x}$. Integrating, we obtain $-\ln|2v + 1| + 2\ln|v + 2| = -\ln|x| + \ln|c|$, which reduces to

$$\ln\frac{(v + 2)^2}{|2v + 1|} = \ln\left|\frac{c}{x}\right|.$$

From this we find $\frac{(v + 2)^2}{|2v + 1|} = \left|\frac{c}{x}\right|$. But $v = \frac{y}{x}$; and resubstituting and simplifying, we find

$$\frac{(y + 2x)^2}{x^2} \cdot \frac{|x|}{|2y + x|} = \left|\frac{c}{x}\right|$$

or $(2x + y)^2 = |c(2y + x)|$. Applying the I.C. $y(1) = 2$, we let $x = 1$, $y = 2$, to obtain $16 = |c|5$, from which $|c| = \frac{16}{5}$. Replacing $|c|$ with this value, and taking $x + 2y \geq 0$, we express the solution as $5(2x + y)^2 = 16(x + 2y)$.

21. This equation is both separable and linear. Treating it as a separable equation, we have $\frac{dy}{y} = \frac{x\,dx}{x^2 + 1}$. Integrating then gives $2\ln|y| = \ln(x^2 + 1) + \ln|c|$, which quickly simplifies to $y^2 = c(x^2 + 1)$, with $c > 0$. Applying the I.C. $y(\sqrt{15}) = 2$, we let $x = \sqrt{15}$, $y = 2$, to obtain $16c = 4$, $c = \frac{1}{4}$. Thus we find the solution $y^2 = \frac{(x^2 + 1)}{4}$ or $y = \frac{(x^2 + 1)^{1/2}}{2}$. Now treating the equation as a linear equation, we have $\frac{dy}{dx} - \frac{x}{x^2 + 1}y = 0$. with $P(x) = \frac{-x}{x^2 + 1}$,

$Q(x) = 0$. An I.F. is $e^{\int P(x)dx} = e^{-\int [x/(x^2+1)]dx} = e^{-(1/2)\ln(x^2+1)} = (x^2+1)^{-1/2}$. Multiplying the D.E. through by this, we obtain $\frac{d}{dx}[(x^2+1)^{-1/2}y] = 0$. Integrating, we find $(x^2+1)^{-1/2}y = c$ or $y = c(x^2+1)^{1/2}$. Applying the I.C. $y(\sqrt{15}) = 2$ gives $c = \frac{1}{2}$. Thus we again obtain $y = \frac{(x^2+1)^{1/2}}{2}$.

22. Here we actually have two I.V. problems

$$\text{(I)}\begin{cases} \text{For } 0 \le x < 2, \\ \dfrac{dy}{dx} + y = 1, \\ y(0) = 0; \end{cases} \qquad \text{(II)}\begin{cases} \text{For } x \ge 2, \\ \dfrac{dy}{dx} + y = 0, \\ y(2) = a; \end{cases}$$

where a is the value of $\lim_{x \to 2^-} \phi(x)$ and $\phi$ denotes the solution of (I). This is prescribed so that the solution of the entire problem will be continuous at $x = 2$.

We first solve (I). The D.E. is linear in standard form, with $P(x) = 1$, $Q(x) = 1$. An I.F. is $e^{\int P(x)dx} = e^{\int dx} = e^x$. Multiplying the D.E. through by this we obtain $e^x\frac{dy}{dx} + e^x y = e^x$ or $\frac{d}{dx}(e^x y) = e^x$. Integrating, we find $e^x y = e^x + c$ or $y = 1 + ce^{-x}$. Applying the I.C. $y(0) = 0$ gives $0 = 1 + ce^0$, $c = -1$. Hence the solution of Problem (I) is $y = 1 - e^{-x}$, valid for $0 \le x < 2$.

Letting $\phi$ denote the solution just obtained, we note that $\lim_{x \to 2^-} \phi(x) = \lim_{x \to 2^-} (1 - e^{-x}) = 1 - e^{-2}$. This is the a of problem (II); that is, the I.C. of problem (II) is $y(2) = 1 - e^{-2}$.

We now solve problem (II). The D.E. is linear in standard form, with $P(x) = 1$, $Q(x) = 0$. An I.F., as in Problem (I), is $e^x$. Multiplying the D.E. through by this and integrating, we obtain $e^x \frac{dy}{dx} + e^x y = 0$ or $\frac{d}{dx}(e^x y) = 0$. Integrating, we find $e^x y = c$ or $y = ce^{-x}$. Now apply the I.C. of problem (II), namely, $y(2) = 1 - e^{-2}$. We have $1 - e^{-2} = ce^{-2}$, from which we find $c = e^2 - 1$. Thus the solution of problem (II) is $y = (e^2 - 1)e^{-x}$, valid for $x > 2$.

We write the solution of the entire problem, showing intervals where each part is valid, as

$$y = \begin{cases} 1 - e^{-x}, & 0 \leq x < 2, \\ (e^2 - 1)e^{-x}, & x > 2. \end{cases}$$

23. Here, as in problem 22, we actually have two I.V. problems:

$$\text{(I)} \begin{cases} \text{For } 0 \leq x < 2, \\ (x + 2)\frac{dy}{dx} + y = 2x, \\ y(0) = 4; \end{cases} \qquad \text{(II)} \begin{cases} \text{For } x > 2, \\ (x + 2)\frac{dy}{dx} + y = 4, \\ y(2) = a; \end{cases}$$

where a is the value of $\lim_{x \to 2^-} \phi(x)$ and $\phi$ denotes the solution of (I).

We first solve (I). The D.E. of this problem is linear. In standard form it is $\frac{dy}{dx} + \left(\frac{1}{x+2}\right)y = \frac{2x}{x+2}$ with $P(x) = \frac{1}{x+2}$, $Q(x) = \frac{2x}{x+2}$. An I.F. is $e^{\int P(x)dx} = e^{\int [1/(x+2)]dx} = e^{\ln|x+2|} = |x+2|$. Multiplying the standard form equation through by this, we obtain the originally stated D.E., $(x+2)\frac{dy}{dx} + y = 2x$, or $\frac{d}{dx}[(x+2)y] = 2x$. Integrating, we find $(x+2)y = x^2 + c$. Thus the particular solution of Problem (I) is $(x+2)y = x^2 + 8$, valid for $0 \leq x < 2$.

We can write this solution as $y = \frac{x^2+8}{x+2}$. Denoting this solution by $\phi$, we note that $\lim_{x \to 2^-} \phi(x) = \lim_{x \to 2^-} \left[\frac{x^2+8}{x+2}\right] = 3$. This is the a of problem (II); that is the I.C. of problem (II) is $y(2) = 3$.

We now solve (II). The D.E. is linear. In standard form it is $\frac{dy}{dx} + \left(\frac{1}{x+2}\right)y = 4$. Just as in (I), an I.F. is $|x+2|$. Multiplying the standard form equation through by this, we obtain the originally stated D.E., $(x+2)\frac{dy}{dx} + y = 4$ or $\frac{d}{dx}[(x+2)y] = 4$. Integrating, we find $(x+2)y = 4x + c$. Applying the I.C. of (II), $y(2) = 3$, we find $c = 4$. Thus the particular solution of problem (I) is $(x+2)y = 4x + 4$, valid for $x > 2$.

We write the solution of the entire problem as

$$\begin{cases} (x + 2)y = x^2 + 8, & 0 \leq x \leq 2, \\ (x + 2)y = 4x + 4, & x > 2. \end{cases}$$

24. This D.E. is both a Bernoulli D.E. and a homogeneous D.E. We first solve it as a Bernoulli Equation, with n = 3. We thus multiply through by $y^{-3}$, expressing the D.E. in the equivalent form $x^2y^{-3}\frac{dy}{dx} + xy^{-2} = x^{-1}$ or $-2y^{-3}\frac{dy}{dx} - \frac{2}{x}y^{-2} = -\frac{2}{x^3}$. We let $v = y^{1-n} = y^{-2}$, $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$. The D.E. then transforms into the linear D.E. in v, $\frac{dv}{dx} - \frac{2}{x}v = -\frac{2}{x^3}$. This linear equation is in standard form with $P(x) = -\frac{2}{x}$, $Q(x) = -\frac{2}{x^3}$. An I.F. is $e^{\int P(x)dx} = e^{-\int(2/x)dx} = e^{-2\ln|x|} = x^{-2}$. Multiplying the linear equation through by this we get $x^{-2}\frac{dv}{dx} - 2x^{-3}v = -2x^{-5}$ or $\frac{d}{dx}(x^{-2}v) = -2x^{-5}$. Integrating, we find $x^{-2}v = \frac{x^{-4}}{2} + c$ or $v = \frac{x^{-2}}{2} + cx^2$. But $v = y^{-2}$, thus we obtain $y^{-2} = \frac{x^{-2}}{2} + cx^2$. Applying the I.C. y(1) = 1 to this, we find c = 1/2. Thus we obtain the solution $y^{-2} = \frac{x^{-2}}{2} + \frac{x^2}{2}$, which can be written as $y^2 = \frac{2x^2}{x^4 + 1}$, or $y = \frac{\sqrt{2}\,x}{(x^4 + 1)^{1/2}}$.

Now we express the D.E. in the form $\frac{dy}{dx} = -\frac{y}{x} + \left(\frac{y}{x}\right)^3$ and solve it as a homogeneous equation. We let $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and obtain the transformed separable equation $v + x\frac{dv}{dx} = -v + v^3$, which reduces to $\frac{dv}{v(v^2 - 2)} = \frac{dx}{x}$. We rewrite the left member, using partial fractions, and thus have $\left(-\frac{1}{v} + \frac{v}{v^2 - 2}\right)dv = 2\frac{dx}{x}$.

Integration gives $-\ln|v| + \frac{\ln|v^2 - 2|}{2} = 2\ln|x| + \ln|c_1|$. This simplifies to

$$-\ln v^2 + \ln|v^2 - 2| = \ln x^4 + \ln c,$$

$\ln \frac{|v^2 - 2|}{v^2} = \ln cx^4$. From this, $|v^2 - 2| = cv^2x^4$. Resubstituting $v = \frac{y}{x}$, we find $|y^2 - 2x^2| = cy^2x^4$.

Applying the I.C. $y(1) = 1$, we find $c = 1$. Thus we find the solution $|y^2 - 2x^2| = y^2x^4$. Since $y^2 - 2x^2$ is initially $-1$, we take $|y^2 - 2x^2| = 2x^2 - y^2$, and write the solution as $2x^2 - y^2 = x^4y^2$ or $(x^4 + 1)y^2 = 2x^2$ or $y^2 = \frac{2x^2}{x^4 + 1}$, as we obtained before.

**Section 2.4, Page 67**

1. This equation is of the form (2.42) of Theorem 2.6 with $M(x,y) = 5xy + 4y^2 + 1$, $N(x,y) = x^2 + 2xy$. To apply that theorem, we first find

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{5x + 8y - 2x - 2y}{x^2 + 2xy} =$$

$$\frac{3(x + 2y)}{x(x + 2y)} = \frac{3}{x}.$$

This depends on x only, so

$$e^{\int (3/x)dx} = e^{3\ln|x|} = e^{\ln|x|^3} = |x|^3$$

is an I.F. of the given equation. Multiplying the given equation through by this and simplifying, we find $(5x^4y + 4x^3y^2 + x^3)dx + (x^5 + 2x^4y)dy = 0$. For this equation $M(x,y) = 5x^4y + 4x^3y^2 + x^3$, $N(x,y) = x^5 + 2x^4y$, and $M_y(x,y) = 5x^4 + 8x^3y = N_x(x,y)$. Thus this equation is exact. We seek F(x,y) such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$. From the first of these, F(x,y) $\int M(x,y)\partial x = \int (5x^4y + 4x^3y^2 + x^3)\partial x = x^5y + x^4y^2 + \frac{x^4}{4} + \phi(y)$. From this, $F_y(x,y) = x^5 + 2x^4y + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = x^5 + 2x^4y$. Thus $\phi'(y) = 0$, $\phi(y) = c_0$. Hence $F(x,y) = x^5y + x^4y^2 + \frac{x^4}{4} + c_0$. The family of solutions is then $F(x,y) = c_1$, and multiplying through by 4 we express this as $4x^5y + 4x^4y^2 + x^4 = c$, where $c = 4(c_1 - c_0)$.

2. This equation is of the form (2.42) of Theorem 2.6 with $M(x,y) = 2x + \tan y$, $N(x,y) = x - x^2 \tan y$. To apply that theorem, we use $1 + \tan^2 y = \sec^2 y$ and find

$$\frac{N_x(x,y) - M_y(x,y)}{M(x,y)} = \frac{1 - 2x\tan y - \sec^2 y}{2x + \tan y}$$

$$= \frac{-2x\tan y - \tan^2 y}{2x + \tan y} = -\tan y.$$

This depends on y only, so $e^{-\int \tan y\, dy} = e^{\ln|\cos y|} = |\cos y|$ is an I.F. of the given equation. We assume $\cos y \geq 0$, and take $|\cos y| = \cos y$. Multiplying the given equation through by this and simplifying, we find $(2x\cos y + \sin y)dx + (x\cos y - x^2 \sin y)dy = 0$. For this equation $M(x,y) = 2x\cos y + \sin y$, $N(x,y) = \cos y - x^2 \sin y$, and $M_y(x,y) = -2x\sin y + \cos y = N_x(x,y)$. Thus this equation is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$. From the first of these $F(x,y) = \int M(x,y)\,\partial x = \int (2x\cos y + \sin y)\,\partial x = x^2\cos y + x\sin y + \phi(y)$. From this, $F_y(x,y) = -x^2\sin y + x\cos y + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = x\cos y - x^2 \sin y$. Thus $\phi'(y) = 0$, $\phi(y) = c_0$. The family of solutions is then $F(x,y) = c_1$. We express this as $x^2\cos y + x\sin y = c$, when $c = c_1 - c_0$.

4. This equation is of the form (2.42) of Theorem 2.6 with $M(x,y) = 2xy^2 + y$, $N(x,y) = 2y^3 - x$. To apply that theorem, we first find $\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{2(2xy + 1)}{2y^3 - x}$; but

since this depends on y as well as x, we cannot proceed using this. We next find $\frac{N_x(x,y) - M_y(x,y)}{M(x,y)} = \frac{-2(1 + 2xy)}{y(1 + 2xy)} = -\frac{2}{y}$. This depends on y only, so

$$e^{-\int (2/y)dy} = e^{-2\ln|y|} = e^{\ln|y|^{-2}} = |y|^{-2} = \frac{1}{y^2}$$

is an I.F. of the given equation. Multiplying the given equation through by this and simplifying, we find

$$(2x + y^{-1})dx + (2y - xy^{-2})dy = 0.$$

For this equation $M(x,y) = 2x + y^{-1}$, $N(x,y) = 2y - xy^{-2}$, $M_y(x,y) = -y^{-2} = N_x(x,y)$. Thus this equation is exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$. From the first of these, $F(x,y) = \int M(x,y)\partial x = \int (2x + y^{-1})\partial x = x^2 + xy^{-1} + \phi(y)$. From this, $F_y(x,y) = -xy^{-2} + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = 2y - xy^{-2}$. Thus $\phi'(y) = 2y$, $\phi(y) = y^2 + c_0$. Hence $F(x,y) = x^2 + xy^{-1} + y^2 + c_0$. The family of solutions $F(x,y) = c_1$ is then expressed as $x^2 + xy^{-1} + y^2 = c$, where $c = c_1 - c_0$.

5. We first multiply the stated equation through by $x^p y^q$ to obtain $(4x^{p+1}y^{q+2} + 6x^p y^{q+1})dx + (5x^{p+2}y^{q+1} + 8x^{p+1}y^q)dy = 0$. For this equation, we have $M(x,y) = 4x^{p+1}y^{q+2} + 6x^p y^{q+1}$, $N(x,y) = 5x^{p+2}y^{q+1} + 8x^{p+1}y^q$, $M_y(x,y) = 4(q+2)x^{p+1}y^{q+1} + 6(q+1)x^p y^q$, and $N_x(x,y) = 5(p+2)x^{p+1}y^{q+1} + 8(p+1)x^p y^q$. In order for the equation to be exact, we must have $M_y(x,y) = N_x(x,y)$, and hence

$$\begin{cases} 4(q+2) = 5(p+2) \\ 6(q+1) = 8(p+1), \text{ that is,} \end{cases} \qquad \begin{cases} 5p - 4q = -2 \\ 8p - 6q = -2. \end{cases}$$

Solving these, we find $p = 2$, $q = 3$. Thus the desired I.F. of the form $x^p y^q$ is $x^2y^3$.

Now multiplying the original equation through by this I.F. $x^2y^3$, we obtain the equivalent exact equation $(4x^3y^5 + 6x^2y^4)dx + (5x^4y^4 + 8x^3y^3)dy = 0$. It is easy to check that this is indeed exact. We seek $F(x,y)$ such that $F_x(x,y) = M(x,y) = 4x^3y^5 + 6x^2y^4$ and $F_y(x,y) = N(x,y) = 5x^4y^4 + 8x^3y^3$. From the first of these,

$$F(x,y) = \int M(x,y)\,\partial x = \int (4x^3y^5 + 6x^2y^4)\,\partial x$$
$$= x^4y^5 + 2x^3y^4 + \phi(y).$$

From this, $F_y(x,y) = 5x^4y^4 + 8x^3y^3 + \phi'(y)$. But we must have $F_y(x,y) = N(x,y) = 5x^4y^4 + 8x^3y^3$. Thus $\phi'(y) = 0$, $\phi(y) = c_0$. Thus $F(x,y) = x^4y^5 + 2x^3y^4 + c_0$, and the

solution $F(x,y) = c_1$ is $x^4y^5 + 2x^3y^4 = c$ or $x^3y^4(xy + 2) = c$, where $c = c_1 - c_0$.

7. For this equation $a_1 = 5$, $b_1 = 2$, $a_2 = 2$, $b_2 = 1$, so

$$\frac{a_2}{a_1} = \frac{2}{5} \neq \frac{1}{2} = \frac{b_2}{b_1}.$$

Therefore this is Case 1 of Theorem 2.7. We make the transformation

$$\begin{cases} x = X + h \\ y = Y + k, \end{cases}$$

where $(h,k)$ is the solution of the system

$$\begin{cases} 5h + 2k + 1 = 0 \\ 2h + k + 1 = 0. \end{cases}$$

The solution of this system is $h = 1$, $k = -3$, and so the transformation is

$$\begin{cases} x = X + 1 \\ y = Y - 3. \end{cases}$$

This reduces the given equation to the homogeneous equation

$$(5X + 2Y)dX + (2X + Y)dY = 0.$$

We write this in the form

$$\frac{dY}{dX} = -\frac{5 + 2\left[\frac{Y}{X}\right]}{2 + \left[\frac{Y}{X}\right]}$$

and let $Y = vX$ to obtain

$$v + X\frac{dv}{dX} = -\frac{5 + 2v}{2 + v}.$$

This reduces to

$$X\frac{dv}{dX} = \frac{5 - v^2}{2 + v} \text{ or } \frac{(v + 2)dv}{v^2 + 4v + 5} = -\frac{dX}{X}.$$

Integrating we find

$$\ln|v^2 + 4v + 5| = -2\ln|X| + \ln|c_1|.$$

or

$$\ln|v^2 + 4v + 5| = \ln(|c_1|X^{-2}).$$

From this, $|v^2 + 4v + 5| = |c_1|X^{-2}$. Now replacing v by $\frac{Y}{X}$, and simplifying, we obtain

$$|5X^2 + 4XY + Y^2| = |c_1|.$$

Finally replacing X by $x - 1$ and Y by $y + 3$, we obtain the solutions of the original D.E. in the form

$$|5(x - 1)^2 + 4(x - 1)(y + 3) + (y + 3)^2| = |c_1|$$

or

$$|5x^2 + 4xy + y^2 + 2x + 2y + 14| = |c_1|.$$

Assuming $x > 0$, $y > 0$, and writing $c = c_1 - 14$, this takes the form

$$5x^2 + 4xy + y^2 + 2x + 2y = c.$$

8. Here $a_1 = 3$, $b_1 = -1$, $a_2 = -6$, $b_2 = 2$, and $\frac{a_2}{a_1} = -2 = \frac{b_2}{b_1}$. Therefore this is Case 2 of Theorem 2.7. We therefore let $z = 3x - y$. Then $dy = 3dx - dz$, and the given D.E. transforms into $(z + 1)dx - (2z - 3)(3dx - dz) = 0$ or $(-5z + 10)dx + (2z - 3)dz = 0$, which is separable. Separating variables, we obtain

$$dx + \frac{2z - 3}{-5z + 10}\,dz = 0$$

or

$$dx + \left(-\frac{2}{5} - \frac{1}{5(z - 2)}\right) dz = 0.$$

Integrating, we have

$$x - \frac{2}{5}z - \frac{1}{5}\ln|z - 2| = c_1.$$

or

$$5x - 2z - \ln|z - 2| = 5c_1.$$

Replacing z by 3x - y and simplifying, we obtain the solution of the given D.E. in the form $x - 2y + \ln|3x - y - 2| = c$, where $c = -5c_1$.

10. For this equation $a_1 = 10$, $b_1 = -4$, $a_2 = -1$, $b_2 = -5$, so $\frac{a_2}{a_1} = -\frac{1}{10} \neq \frac{5}{4} = \frac{b_2}{b_1}$. Therefore this is Case 1 of Theorem 2.7. We make the transformation

$$\begin{cases} x = X + h, \\ y = Y + k, \end{cases}$$

where (h,k) is the solution of the system

$$\begin{cases} 10h - 4k + 12 = 0, \\ h + 5k + 3 = 0. \end{cases}$$

The solution of this system is $h = -\frac{4}{3}$, $k = -\frac{1}{3}$, and so the transformation is

$$\begin{cases} x = X - \frac{4}{3}, \\ y = Y - \frac{1}{3}. \end{cases}$$

This reduces the given D.E. to the homogeneous equation

$$(10X - 4Y)dX - (X + 5Y)dY = 0.$$

We write this in the form

$$\frac{dY}{dX} = \frac{10 - 4(Y/X)}{1 + 5(Y/X)}$$

and let $Y = vX$ to obtain

$$v + X\frac{dv}{dX} = \frac{10 - 4v}{1 + 5v}.$$

This reduces to

$$X\frac{dv}{dX} = -\frac{5v^2 + 5v - 10}{5v + 1}$$

or

$$\frac{(5v + 1)dv}{v^2 + v - 2} = -5\frac{dX}{X}.$$

We write $v^2 + v - 2 = (v + 2)(v - 1)$ and use partial fractions to integrate the left member. We find

$$3 \ln|v + 2| + 2 \ln|v - 1| = -5 \ln|X| + \ln|c|$$

or

$$\ln|v + 2|^3(v - 1)^2 = \ln\frac{|c|}{|X|^5}$$

and hence

$$|v + 2|^3(v - 1)^2 = \frac{|c|}{|X|^5}.$$

Now replacing v by Y/X and simplifying, we find

$$|Y + 2X|^3(Y - X)^2 = |c|.$$

Finally, replacing X by $x + 4/3$ and Y by $y + 1/3$ and simplifying, we obtain

$$|2x + y + 3|^3(x - y + 1) = |c|.$$

We assume $2x + y + 3 \geq 0$, take $c \geq 0$, and express this in the form

$$(2x + y + 3)^3(x - y + 1)^2 = c.$$

12. The D.E. of this problem is of the form (2.52) of Theorem 2.7, within $a_1 = 3$, $b_1 = -1$, $a_2 = 1$, $b_2 = 1$, and hence $\frac{a_2}{a_1} = \frac{1}{3} \neq -1 = \frac{b_2}{b_1}$. Therefore this is Case 1 of the theorem.

We let $x = X + h$, $y = Y + k$, where $(h,k)$ is the solution of the system $\begin{cases} 3h - k - 6 = 0 \\ h + k + 2 = 0 \end{cases}$. The solution of this is $h = 1$, $k = -3$, and so the transformation is $\begin{cases} x = X + 1 \\ y = Y - 3 \end{cases}$.

This reduces the given equation to the homogeneous equation $(3X - Y)dX + (X + Y)dY = 0$. We write this in the form

$$\frac{dY}{dX} = \frac{(Y/X) - 3}{(Y/X) + 1}$$

and let $Y = vX$ to obtain

$$v + X\frac{dv}{dX} = \frac{v - 3}{v + 1}.$$

This reduces to

$$X\frac{dv}{dX} = -\frac{v^2 + 3}{v + 1} \text{ or } \frac{(v + 1)dv}{v^2 + 3} = -\frac{dX}{X}.$$

Integrating (tables may be needed!), we find

$$\frac{1}{2}\ln(v^2 + 3) + \frac{1}{\sqrt{3}}\arctan\frac{v}{\sqrt{3}} = -\ln|X| + c$$

or

$$\ln(v^2 + 3) + \frac{2}{\sqrt{3}}\arctan\frac{v}{\sqrt{3}} = -\ln X^2 + c_1.$$

Now replacing v by $\frac{Y}{X}$, we obtain

$$\ln[Y^2/X^2 + 3] + \frac{2}{\sqrt{3}}\arctan\frac{Y}{\sqrt{3X}} = -\ln X^2 + c_1.$$

Combining the ln terms, this takes the form

$$\ln(Y^2 + 3X^2) + \frac{2}{\sqrt{3}}\arctan\frac{Y}{\sqrt{3X}} = C_1.$$

Finally replacing X by x - 1 and Y by y + 3, we obtain the solutions of the original D.E. in the form

$$\ln[3(x - 1)^2 + (y + 3)^2] + \frac{2}{\sqrt{3}}\arctan\frac{y + 3}{\sqrt{3}(x - 1)} = c_1.$$

We now apply the I.C.y(2) = -2 to this, obtaining

$$\ln 4 + \frac{2}{\sqrt{3}}\arctan\frac{1}{\sqrt{3}} = c_1$$

or $\ln 4 + (2/\sqrt{3})(\pi/6) = c_1$. Thus we obtain the solution of the stated I.V. problem as

$$\ln[3(x - 1)^2 + (y + 3)^2] + \frac{2}{\sqrt{3}} \arctan \frac{y + 3}{\sqrt{3}(x - 1)} =$$

$$\ln 4 + \frac{\pi}{3\sqrt{3}}.$$

13. Here $a_1 = 2$, $b_1 = 3$, $a_2 = 4$, $b_2 = 6$, and $a_2/a_1 = 2 = b_2/b_1$. Thus this is Case 2 of Theorem 2.7. We let $z = 2x + 3y$. Then $dy = (dz - 2dx)/3$, and the given D.E. transforms into

$$(z + 1)dx + (2z + 1)\left(\frac{dz - 2\,dx}{3}\right) = 0$$

or

$$-(z - 1)dx + (2z + 1)dz = 0,$$

which is separable in x and z. Separating variables, we obtain

$$dx - \frac{2z + 1}{z - 1} dz = 0$$

or

$$dx - \left(2 + \frac{3}{z - 1}\right) dz = 0.$$

Integrating, we find

$$x - 2z + 3 \ln |z - 1| = c_1.$$

Replacing z by $2x + 3y$ and simplifying, we obtain

$$x + 2y - \ln |2x + 3y - 1| = c_2.$$

Applying the I.C. $y(-2) = 2$, we find $c_2 = 2$ and hence obtain

$$x + 2y - 2 - \ln|2x + 3y - 1| = 0.$$

21. (a) The D.E. is of the form (A) of Exercise 20, with $f(p) = p^2$.

Following the procedure outlined there, we differentiate $y = px + p^2$ with respect to x, to obtain $\frac{dy}{dx} = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$, since $p = \frac{dy}{dx}$, this is $[x + 2p]\frac{dp}{dx} = 0$, which is of form (B) of Exercise 20.

Assume $x + 2p \neq 0$, divide through by it, and we are left with just $\frac{dp}{dx} = 0$, from which $p = c$ (where c is an arbitrary constant). Then, returning to the given D.E., we find the one-parameter family of solutions $y = cx + c^2$.

(b) Now assume $x + 2p = 0$, and eliminate p between this and the given D.E. From this, $p = -x/2$. Substituting this into the given D.E. and simplifying, we obtain $y = -x^2/4$. This is the desired "extra" solution.

# Chapter 3

**Section 3.1, Page 78**

1. **Step 1.** We first find the D.E. of the given family $y = cx^3$. Differentiating, we obtain $\frac{dy}{dx} = 3cx^2$.

Eliminating the parameter c between the equation of the given family and its derived equation, we obtain the D.E. of the given family in the form $\frac{dy}{dx} = 3(y/x^3)x^2$ or $\frac{dy}{dx} = \frac{3y}{x}$.

**Step 2.** We now find the D.E. of the orthogonal trajectories by replacing $\frac{3y}{x}$ in the D.E. (of Step 1) by its negative reciprocal $-\frac{x}{3y}$, thus obtaining

$$\frac{dy}{dx} = -\frac{x}{3y}.$$

**Step 3.** We solve this last D.E. Separating variables, we have $3ydy = -x\,dx$. Integrating, we find $\frac{3y^2}{2} = -(x^2/2) + k_1$ or $x^2 + 3y^2 = k^2$, where $k^2 = -2k_1 \geq 0$. This is the family of orthogonal trajectories of the given family of cubics. It is a family of ellipses with centers at the origin and major axes along the x axis.

4. **Step 1.** We first find the D.E. of the given family $y = e^{cx}$. Differentiating, we obtain $\frac{dy}{dx} = ce^{cx}$. We eliminate the parameter c between the equation of the given family and its derived equation. From the given equation, we have $y > 0$ and $\ln y = cx$. Thus $c = \frac{\ln y}{x}$. Substituting in

the derived equation we find $\frac{dy}{dx} = \frac{y \ln y}{x}$. This is the D.E. of the given family.

**Step 2.** We now find the D.E. of the orthogonal trajectories by replacing $\frac{y \ln y}{x}$ by its negative reciprocal $-\frac{x}{y \ln y}$, thereby obtaining the D.E.

$$\frac{dy}{dx} = -\frac{x}{y \ln y}.$$

**Step 3.** We solve this last D.E. Separating variables, we have $y \ln y \, dy = -x \, dx$. Integrating (by parts), we find $\frac{y^2 \ln y}{2} - \frac{y^2}{4} = -\frac{x^2}{2} + k_1$ or $y^2(\ln y - 1/2) = -x^2 + k$.

This is the family of orthogonal trajectories of the given family of exponentials.

5. **Step 1.** We first find the D.E. of the given family $y = x - 1 + c e^{-x}$. Differentiating, we obtain $\frac{dy}{dx} = 1 - c e^{-x}$. Eliminating the parameter c between the equation of the given family and its derived equation, we obtain $\frac{dy}{dx} = 1 - (y - x + 1)$ or $\frac{dy}{dx} = x - y$.

**Step 2.** We now find the D.E. of the orthogonal trajectories by replacing $x - y$ by its negative reciprocal $\frac{1}{y - x}$, thereby obtaining the D.E. $\frac{dy}{dx} = \frac{1}{y - x}$.

**Step 3.** We solve this last D.E. We regard x as the dependent variable and write the equation in the form $\frac{dx}{dy} = y - x$ or $\frac{dx}{dy} + x = y$. This is linear in x, with I.F. $e^{\int 1 dy} = e^y$. Multiplying $\frac{dx}{dy} + x = y$ through by this, we have $e^y \frac{dx}{dy} + e^y x = y e^y$ or $\frac{d}{dy}[x e^y] = y e^y$. Integrating, we obtain $xe^y = e^y(y - 1) + k$ or $x = y - 1 + ke^{-y}$. This is the family of orthogonal trajectories of the given family.

6. **Step 1.** We first find the D.E. of the given family $y = cx^2/(x + 1)$. Differentiating we obtain, $\frac{dy}{dx} = c(x^2 + 2x)/(x + 1)^2$. From the given equation, $c = (x + 1)y/x^2$; and substituting this into the derived equation and simplifying, we obtain $\frac{dy}{dx} = \frac{(x + 2)y}{x(x + 1)}$. This is the D.E. of the given family.

**Step 2.** We now find the D.E. of the orthogonal trajectories by replacing $\frac{(x + 2)y}{x(x + 1)}$ in the D.E. of step 1 by its negative reciprocal $-\frac{x(x + 1)}{(x + 2)y}$, thereby obtaining the D.E. $\frac{dy}{dx} = -\frac{x(x + 1)}{(x + 2)y}$.

**Step 3.** We solve this last D.E. Separating variables, we have $y\,dy = -\frac{x(x + 1)}{x + 2}dx$, which can be rewritten as $y\,dy = \left(-x + 1 - \frac{2}{x + 2}\right)dx$. Integrating this, we have $\frac{y^2}{2} = -\frac{x^2}{2} + x - 2\ln|x + 2| + k_1$ or $x^2 + y^2 - 2x + \ln(x + 2)^4 = k$.

8. **Step 1.** We first find the D.E. of the given family $x^2 = 2y - 1 + ce^{-2y}$. Differentiating, we obtain $2x = 2y\frac{dy}{dx} - 2ce^{-2y}\frac{dy}{dx}$. We eliminate the parameter c between these two equations as follows. From the given equation, $c = (x^2 - 2y + 1)e^{2y}$. Substituting this into the second equation, we get

$$x = \frac{dy}{dx} - (x^2 - 2y + 1)\frac{dy}{dx} \text{ or } \frac{dy}{dx} = \frac{x}{2y - x^2}$$

This is the D.E. of the given family.

**Step 2.** We now find the D.E. of the orthogonal trajectories by replacing $\frac{x}{2y - x^2}$ by its negative reciprocal $\frac{x^2 - 2y}{x}$, thereby obtaining the D.E.

$$\frac{dy}{dx} = \frac{x^2 - 2y}{x}.$$

**Step 3.** We solve this last D.E. Writing it as $\frac{dy}{dx} + \frac{2}{x}y = x$, we recognize that it is a linear equation in standard form with $P(x) = \frac{2}{x}$ and $Q(x) = x$. An I.F. is $e^{\int P(x)dx} = e^{\int (2/x)dx} = e^{2\ln|x|} = x^2$. Multiplying through by this, we have $x^2\frac{dy}{dx} + 2xy = x^3$ or $\frac{d}{dx}[x^2y] = x^3$. Integrating, we find $x^2y = \frac{x^4}{4} + k$. This is the equation of the one-parameter family of orthogonal trajectories.

10. **Step 1.** We first find the D.E. of the given family $x^2 - y^2 = cx^3$. Differentiating, we obtain $2x - 2y\frac{dy}{dx} = 3cx^2$. From the given equation, $c = (x^2 - y^2)/x^3$. Substituting this in the derived equation and simplifying, we obtain successively $2x - 2y\frac{dy}{dx} = 3(x^2 - y^2)/x$ or $2y\frac{dy}{dx} = -x + 3\frac{y^2}{x}$ or $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$.

**Step 2.** We now find the D.E. of the orthogonal trajectories by replacing $\frac{3y^2 - x^2}{2xy}$ by its negative reciprocal $\frac{2xy}{x^2 - 3y^2}$, thereby obtaining the D.E.

$$\frac{dy}{dx} = \frac{2xy}{x^2 - 3y^2}$$

**Step 3.** We solve this last D.E. Writing it in the form $\frac{dy}{dx} = \frac{2(y/x)}{1 - 3(y/x)^2}$, we recognize that it is homogeneous. We let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$, and the D.E. becomes $v + x\frac{dv}{dx} = \frac{2v}{1 - 3v^2}$ or $x\frac{dv}{dx} = \frac{3v^3 + v}{1 - 3v^2}$. Separating variables, we have $\left[\frac{1 - 3v^2}{3v^3 + v}\right] dv = \frac{dx}{x}$. Using partial fractions, this becomes $\left[\frac{1}{v} - \frac{6v}{3v^2 + 1}\right] dv = \frac{dx}{x}$. Integrating, we find $\ln|v| - \ln(3v^2 + 1) = \ln|x| + \ln k$, where $k > 0$, or $\ln \frac{|v|}{3v^2 + 1} = \ln k|x|$. From this,

$\frac{|v|}{3v^2 + 1} = k|x|$. Now replacing v by y/x and taking $x > 0$ and $y > 0$, we have $\frac{y/x}{3y^2/x^2 + 1} = kx$, from which $y = k(x^2 + 3y^2)$.

11. **Step 1.** The given family of ellipse has the equation $\frac{x^2}{4c^2} + \frac{y^2}{3c^2} = 1$ or $\frac{x^2}{4} + \frac{y^2}{3} = c^2$. We first find the D.E. of this given family. Differentiating $\frac{x^2}{4} + \frac{y^2}{3} = c^2$, we find $\frac{x}{2} + \frac{2y}{3}\frac{dy}{dx} = 0$. From this, we have $\frac{dy}{dx} = -\frac{3x}{4y}$, which is the D.E. of the given family.

**Step 2.** The D.E. of the orthogonal trajectories is obtained from this by replacing $-3x/4y$ by its negative reciprocal $4y/3x$, and so it is $\frac{dy}{dx} = \frac{4y}{3x}$.

**Step 3.** We solve this lost D.E. Separating variables, we have $\frac{dy}{y} = \frac{4}{3}\frac{dx}{x}$. Integrating, we find $\ln|y| = \frac{4}{3}\ln|x| + \ln|k|$; and simplifying, we obtain $y = k\,x^{4/3}$.

13. We first find the D.E. of the given family of parabolas $y = c_1x^2 + K$. Differentiating, we obtain $\frac{dy}{dx} = 2c_1x$. Eliminating the parameter c between the given equation and the derived equation, we obtain the D.E. of the family of parabolas in the form $\frac{dy}{dx} = \frac{2y - 2K}{x}$. It will be convenient to rewrite this slightly, as follows: $\frac{dy}{dx} = \frac{4y - 4K}{2x}$. Then the D.E. of the family of orthogonal trajectories of the given family of parabolas is

$$\frac{dy}{dx} = \frac{2x}{4K - 4y}. \qquad (*)$$

We now find the D.E. of the given family of ellipses $x^2 + 2y^2 - y = c_2$. Differentiating, we obtain $2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$, from which we at once obtain the D.E. of the family of ellipses, that is,

$$\frac{dy}{dx} = \frac{2x}{1 - 4y}. \qquad (**)$$

Comparing (*) amd (**) we see that the two given families are orthogonal provided $K = \frac{1}{4}$.

14. **Step 1.** We first find the D.E. of the given family $y = \frac{x}{1 - c_2x}$. Differentiating, we obtain $\frac{dy}{dx} = 1/(1 - c_2x)^2$. From the given D.E., $1 - c_2x = \frac{x}{y}$. Substituting this into the derived equation, we have $\frac{dy}{dx} = \frac{y^2}{x^2}$. This is the D.E. of the given family.

**Step 2.** The D.E. of the orthogonal trajectories is obtained from this by replacing $y^2/x^2$ by its negative reciprocal $-x^2/y^2$, and so it is $\frac{dy}{dx} = -\frac{x^2}{y^2}$.

**Step 3.** We solve this last D.E. Separating variables, we have $y^2dy = -x^2dx$. Then integrating and simplifying, we find $x^3 + y^3 = c_1$. Thus the value of n is 3.

18. **Step 1.** We first find the D.E. of the given family $x + y = cx^2$. Differentiating, we obtain $1 + \frac{dy}{dx} = 2cx$. Eliminating the parameter c between the given equation and the derived equation, we obtain the D.E. of the given family in the form $\frac{dy}{dx} = \frac{x + 2y}{x}$.

**Step 2.** We replace $f(x,y) = \frac{x + 2y}{x}$ in this D.E. by

$$\frac{f(x,y) + \tan \alpha}{1 - f(x,y)\tan \alpha} = \frac{f(x,y) + 2}{1 - 2f(x,y)} = \frac{\frac{x + 2y}{x} + 2}{1 - 2\frac{(x + 2y)}{x}} = -\frac{3x + 2y}{x + 4y}.$$

We have the D.E. $\frac{dy}{dx} = -\frac{3x + 2y}{x + 4y}$ of the desired oblique trajectories.

**Step 3.** We now solve this D.E. Writing it in the form $\frac{dy}{dx} = -\frac{3 + 2(y/x)}{1 + 4(y/x)}$, we see that it is homogeneous. We let $y = vx$ to obtain $v + x\frac{dv}{dx} = -\frac{3 + 2v}{1 + 4v}$ or $\frac{(4v + 1)dv}{4v^2 + 3v + 3} = -\frac{dx}{x}$.

Integrating this (we recommend tables), we find $(1/2)\ln|4v^2 + 3v + 3| - (1/2)(2/\sqrt{39}) \arctan \frac{8v + 3}{\sqrt{39}} = -\ln x + (1/2)c$, where the final 1/2 is for convenience. We multiply through by 2 and replace v by y/x to obtain

$$\ln\left|\frac{4y^2 + 3xy + 3x^2}{x^2}\right| - \left(\frac{2}{\sqrt{39}}\right) \arctan\left[\frac{8y + 3x}{\sqrt{39}\,x}\right] = -\ln x^2 + c.$$

Thus we find the desired family of oblique trajectories in the form $\ln|3x^2 + 3xy + 4y^2| = -\left(\frac{2}{\sqrt{39}}\right) \arctan\left[\frac{3x + 8y}{\sqrt{39}\,x}\right] = c.$

**Section 3.2, Page 88**

1. We choose the positive x axis vertically downward along the path of the stone and the origin at the point from which the body fell. The forces acting on the stone are: (1) $F_1$, its weight, 4 lbs., which acts downward and so is positive, and (2) $F_2$, the air resistance, numerically equally to v/2, which acts upward and so is the negative quantity -v/2. Newton's Second Law gives $m\frac{dv}{dt} = F_1 + F_2$. Using $g = 32$ and $m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$, this becomes $\frac{1}{8}\frac{dv}{dt} = 4 - \frac{v}{2}$.

   The initial condition is $v(0) = 0$.

   The D.E. is separable. We write it as

$$\frac{dv}{4 - v/2} = 8\,dt\,.$$

   Integrating we find $-2\ln|4 - v/2| = 8t + c_0$ which reduces to $|4 - v/2| = c_1e^{-4t}$. Applying the I.C. $v(0) = 0$ to this, we find $c_1 = 4$. Thus the velocity at time t is given by $v = 8 - 8e^{-4t}$. Writing this as $\frac{dx}{dt} = 8 - 8e^{-4t}$ and integrating, we find $x = 8t + 2e^{-4t} + c_2$. Applying the I.C. $x(0) = 0$, we find $c_2 = -2$. Thus the distance fallen is $x = 8t + 2e^{-4t} - 2$. Thus the answers to part (a) are: $v = 8(1 - e^{-4t})$ and $x = 2(4t + e^{-4t} - 1)$.

To answer part (b), we simply let $t = 5$ in these expressions for v and x. Thus $v(5) = 8(1 - e^{-20}) \approx 8$ ft/sec and $x(5) = 2(19 + e^{-20}) \approx 38$ feet.

3. We choose the positive x axis vertically upward along the path of the ball and the origin at the ground level. The forces acting on the ball are: (1) $F_1$, its weight, (3/4) lb, which acts downward and so is negative and (2) $F_2$, the air resistance, numerically equal to v/64, which also acts downward and so is the negative quantity $-v/64$. Newton's Second Law gives $m\frac{dv}{dt} = F_1 + F_2$. Using $g = 32$, and $m = w/g = \frac{3/4}{32} = \frac{3}{128}$, this becomes $\frac{3}{128}\frac{dv}{dt} = -\frac{3}{4} - \frac{v}{64}$. The initial condition is $v(0) = 20$.

The D.E. is separable. We write it as

$$\frac{3dv}{2v + 96} = -dt.$$

Integrating we find $(3/2)\ln|2v + 96| = -t + c_0$ which reduces to $|v + 48| = c_1 e^{-2t/3}$. Applying the I.C. $v(0) = 20$ to this, we find $c_1 = 68$. Thus the velocity at time t is given by $v = 68e^{-2t/3} - 48$.

From this we obtain $x = -102e^{-2t/3} - 48t + c_2$. Applying the I.C. $x(0) = 6$, we obtain $c_2 = 108$. Thus we have the distance $x = -102e^{-2t/3} - 48t + 108$.

The question asks how high the ball will rise. It will stop rising and start falling when the velocity $v = 0$. Thus we set $68e^{-2t/3} - 48 = 0$, obtaining $e^{-2t/3} = \frac{12}{17}$ and $t \approx 0.5225$ (seconds). For this value of t, we find $x \approx 10.92$, which is the height above the ground that the ball will rise.

4. The forces acting on the ship are: (1) $F_1$, the constant propeller thrust of 100,000 lb., which moves the ship forward and so is positive, and (2) $F_2$, the resistence, numerically equal to 8000v, which acts against the forward motion of the ship and so is the negative quantity -8000v. Newton's Second Law gives $m\frac{dv}{dt} = F_1 + F_2$. Using $g = 32$ and $m = w/g = (32000 \text{ tons}) \frac{(2000 \text{ lbs/ton})}{32} = 2{,}000{,}000$, this becomes

$$2{,}000{,}000\frac{dv}{dt} = 100{,}000 - 8000v.$$

The initial condition is $v(0) = 0$.

The D.E. is separable. We write it as

$$\frac{dv}{dt} = \frac{1}{20} - \frac{v}{250} \text{ or } \frac{dv}{25 - 2v} = \frac{dt}{500}.$$

Integrating we find $\ln|25 - 2v| = -\frac{t}{250} + c_0$ which reduces to $|25 - 2v| = c_1 e^{-t/250}$. Applying the I.C. $v(0) = 0$ to this, we find $c_1 = 25$. Thus the velocity at time t is given by

$$v = \frac{25}{2}(1 - e^{-t/250}) \qquad (*)$$

This is the answer to part (a). The answer to part (b) is then $\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{25}{2}(1 - e^{-t/250}) = 12.5$(ft/sec). To answer part (c), let $v = (0.80)(12.5)$ in (*) and solve for t. We obtain $0.80 = 1 - e^{-t/250}$ and hence $e^{-t/250} = 1/5$. Thus $-\frac{t}{250} = \ln(1/5)$, from which $t = 402$ (sec.).

6. We break the problem into two parts: (1) while the body is rising, and (2) while the body is falling back toward earth.

We consider part (1). We choose the positive x-axis upward along the path of the object. The forces acting on the object are: (1)$F_1$, its weight, - 98000 dynes; and (2) $F_2$, air resistance, - 200 v dynes. Each is negative, since each acts in the downward direction. Newton's Second Law gives $m\frac{dv}{dt} = F_1 + F_2$ or $100\frac{dv}{dt} = -98000 - 200v$. The I.C. is $v(0) = 150$.

We simplify the D.E. to read $\frac{dv}{dt} = -(2v + 980)$. It is separable; and separating variables, we have $\frac{dv}{2v + 980} = -dt$. Integrating, we find $\ln|2v + 980| = -2t + c_0$. Since $v \geq 0$ in part (1), this reduces to $2v + 980 = c_1 e^{-2t}$ or $v = -490 + c\,e^{-2t}$. Applying the I.C., $v(0) = 150$ to this, we find $c = 640$. Hence

$$v = -490 + 640\,e^{-2t}. \qquad (*)$$

The object stops rising and starts falling when $v = 0$. Thus this happens at $t$ such that $e^{-2t} = \frac{49}{64}$. We find $t \approx 0.1336$ (sec.).

In (a) we seek the velocity 0.1 second after the object is thrown. Since $0.1 < 0.1336$, we let $t = 0.1$ in (*), and find $v = -490 + 640\, e^{-1.2} \approx 33.99$ cm/sec.

Now consider part (2). We choose the positive x-axis vertically downward from the highest point reached. Now the weight is + 98000, since it acts in this downward direction. The D.E. is now $100\frac{dv}{dt} = 98000 - 200v$, and the I.C. is $v(0) = 0$. Simplifying the D.E. and separating variables, we have $\frac{dv}{980 - 2v} = dt$. Integrating, we find $\ln|980 - 2v| = -2t + c_0$. Since $980 - 2v \geq 0$, upon simplifying we have $v = 490 - c\, e^{-2t}$. The I.C. gives $c = 490$, and so

$$v = 490(1 - e^{-2t}). \qquad (**)$$

In (b) we seek the velocity 0.1 second after the object stops rising and starts falling. So we let $t = 0.1$ in (**) and obtain $v = 490(1 - e^{-0.2}) \approx 88.82$ cm./sec.

8. We choose the positive x-axis horizontally along the given direction of motion. The forces acting on the boat and rider are: (1) $F_1$, the constant force of 12 lb., which acts in the given direction and so is positive, and (2) $F_2$, the resistance force, numerically equal to 2v, which

acts opposite to the given direction and so is the negative quantity $-2v$.

Newton's Second Law gives $m\frac{dv}{dt} = F_1 + F_2$. Using $g = 32$ and $m = \frac{w}{g} = \frac{150 + 170}{32} = 10$, this becomes

$$10\frac{dv}{dt} = 12 - 2v.$$

The initial velocity is 20 m.p.h. $= \frac{88}{3}$ ft/sec. Thus the I.C. is $v(0) = \frac{88}{3}$.

The D.E. is separable. We write it as

$$\frac{5\ dv}{6 - v} = dt.$$

Integrating we find $5\ \ln|v - 6| = -t + c_0$ which reduces to $|v - 6| = c\,e^{-t/5}$. Applying the I.C. $v(0) = \frac{88}{3}$ to this, we find $c = \frac{70}{3}$. Thus the velocity at time t is given by

$$v = 6 + \left(\frac{70}{3}\right)e^{-t/5}. \qquad (*)$$

To answer part (a), we let $t = 15$ in (*) to obtain $v(15) = 6 + \left(\frac{70}{3}\right)e^{-3} \approx 7.16$ ft/sec. To answer part (b), let $v = \left(\frac{1}{2}\right)\left(\frac{88}{3}\right) = \frac{44}{3}$ in (*) and solve for t. We have $6 + \left(\frac{70}{3}\right)e^{-t/5} = (44)(3)$, from which $e^{-t/5} = \frac{13}{35}$. From this, $-\frac{t}{5} \approx -0.99$ and $t \approx 4.95$ (seconds).

10. We choose the positive x-axis vertically upward along the path of the shell with the origin at the earth's surface. The forces acting on the shell are: (1) $F_1$, its weight, 1 lb., which acts downward and so is the negative quantity $-1$; and (2) $F_2$, the air resistance numerically equal to $10^{-4}v^2$, which also acts downward (against the rising shell) and so is the negative quantity $-10^{-4}v^2$. Newton's Second Law gives $m\frac{dv}{dt} = F_1 + F_2$. Using $g = 32$ and $m = \frac{w}{g} = \frac{1}{32}$, this becomes

$$\frac{1}{32}\frac{dv}{dt} = -1 - 10^{-4}v^2.$$

The initial condition is $v(0) = 1000$.

The D.E. is separable. We write it as

$$\frac{1}{32}\frac{dv}{dt} = -\frac{10^4 + v^2}{10^4} \text{ or } \frac{dv}{v^2 + 10^4} = -\frac{32}{10^4}dt .$$

Integrating we find $\left(\frac{1}{100}\right) \arctan\left(\frac{v}{100}\right) = -\frac{32t}{10^4} + c$ or $\arctan\left(\frac{v}{100}\right) = -\frac{32t}{100} + c$. Application of the initial condition at once gives $c = \arctan 10$. Thus we obtain the solution in the form

$$\arctan\left(\frac{v}{100}\right) = \arctan 10 - \left(\frac{32t}{100}\right).$$

Taking the tan of each side and multiplying by 100 gives

$$v = 100 \tan(\arctan 10 - 0.32t).$$

This is the answer to part (a). To answer part (b), note that the shell will stop rising when $v = 0$. Setting $v = 0$, we at once have $\arctan 10 - 0.32t = 0$, and thus $t = \arctan 10/(0.32) \approx 4.60$ (sec).

13. We choose the positive x-axis horizontally along the given direction of motion and the origin at the point at which the man stops pushing. The forces acting on the loaded sled as it continues are: (1) $F_1$, the air resistance, numerically equal to $\frac{3v}{4}$, which acts opposite to the direction of motion and so is given by $-\frac{3v}{4}$, and (2) $F_2$, the frictional force, having numerical value $\mu N = (0.04)(80)$, which also acts opposite to the direction of motion and so is given by $-(0.04)(80) = -\frac{16}{5}$. Newton's Second Law gives $m\frac{dv}{dt} = F_1 + F_2$. Using $g = 32$ and $m = \frac{w}{g} = \frac{80}{32} = \frac{5}{2}$, this becomes

$$\frac{5}{2}\frac{dv}{dt} = -\frac{3v}{4} - \frac{16}{5}.$$

The initial velocity is 10 ft/sec., so the I.C. is $v(0) = 10$. The D.E. is separable. We write it as

$$\frac{dv}{15v + 64} = -\frac{dt}{50}.$$

Integrating, we find $\ln|15v + 64| = -\frac{3t}{10} + c_0$ which reduces to $|15v + 64| = c\,e^{-3t/10}$. Applying the I.C. $v(0) = 10$ to this, we find $c = 214$. Thus the velocity is given by

$$v = \left(\frac{214}{15}\right)e^{-3t/10} - \frac{64}{15}.$$

From this, integration and simplification give $x = -\left(\frac{428}{9}\right)e^{-3t/10} - \frac{64t}{15} + c_1$. Application of the condition $x(0) = 0$ then gives $c_1 = \frac{428}{9}$. Hence the distance x is given by

$$x = \left(\frac{428}{9}\right)(1 - e^{-3t/10}) - \frac{64t}{15}.$$

To answer the stated question, note that the sled will continue until the velocity $v = 0$. Thus we set $v = 0$ and solve for t, finding $e^{-3t/10} = \frac{32}{107}$ from which $t \approx 4.02$. We now evaluate x at $t \approx 4.02$ to determine the distance which the sled will continue. We find

$$x(4.02) \approx \left(\frac{428}{9}\right)\left(1 - \frac{32}{107}\right) - \frac{64(4.02)}{15} \approx 16.18 \text{ feet}.$$

15. We choose the positive x direction down the slide with the origin at the top. The forces acting on the case are: (1) $F_1$, the component of its weight parallel to the slide, (2) $F_2$, the frictional force; and (3) $F_3$, the air resistance. The case weighs 24 lbs., and the component parallel to the slide has numerical value $24 \sin 45^0 = \frac{24}{\sqrt{2}}$. Since this acts in the positive (downward) direction along the slide, $F_1 = \frac{24}{\sqrt{2}}$. The frictional force $F_2$, has numerical value $\mu N$, where $\mu$ is the coefficient of the friction and N is the normal force. Here $\mu = 0.4$ and the

magnitude of N is $24\cos 45^0 = \frac{24}{\sqrt{2}}$. Since the force $F_2$ acts in the negative (upward) direction along the slide, we have $F_2 = -(0.4)\left(\frac{24}{\sqrt{2}}\right)$. Finally, the air resistance $F_3$ has numerical value $\frac{v}{3}$, Since this also acts in the negative direction, we thus have $F_3 = -\frac{v}{3}$. Newton's Second Law now gives $m\frac{dv}{dt} = F_1 + F_2 + F_3$. With $g = 32$, $m = \frac{w}{g} = \frac{24}{32} = \frac{3}{4}$ and the above forces, this becomes

$$\left(\frac{3}{4}\right)\left(\frac{dv}{dt}\right) = \left(\frac{24}{\sqrt{2}}\right) - (0.4)\left(\frac{24}{\sqrt{2}}\right) - \frac{v}{3}$$

The initial condition is $v(0) = 0$.

The D.E. is separable. We write it as $\left(\frac{3}{4}\right)\left(\frac{dv}{dt}\right) = \frac{(-5v + 108\sqrt{2})}{15}$ or $\frac{dv}{5v - 108\sqrt{2}} = -\frac{4}{45}dt$. Integrating we find $\ln|5v - 108\sqrt{2}| = -\frac{4t}{9} + c_0$ or $|5v - 108\sqrt{2}| = c\,e^{-4t/9}$. Applying the initial condition, we at once find $c = 108\sqrt{2}$. Thus the velocity is given by

$$v = \frac{108\sqrt{2}}{5}\left(1 - e^{-4t/9}\right). \qquad (*)$$

To answer (a), simply let $t = 1$ in (*). We obtain $v(1) = \left(\frac{108\sqrt{2}}{5}\right)\left(1 - \frac{e^{-4}}{9}\right) \approx 10.96$(ft/sec). To answer (b), more work is required. We first integrate (*) to find the distance x from the top of the slide. We have

$$x = \left(\frac{108\sqrt{2}}{5}\right)t + \left(\frac{243\sqrt{2}}{5}\right)e^{-4t/9} + c.$$

Since $x = 0$ at $t = 0$, we find $c = -\frac{243\sqrt{2}}{5}$, and hence

$$x = \left(\frac{108\sqrt{2}}{5}\right)t + \left(\frac{243\sqrt{2}}{5}\right)e^{-4t/9} - \frac{243\sqrt{2}}{5}.$$

The slide being 30 ft. long, we let $x = 30$ in this to determine the time t at which the case reaches the bottom. That is, we must find t such that

$$30 = \left(\frac{108\sqrt{2}}{5}\right)t + \left(\frac{243\sqrt{2}}{5}\right)e^{-4t/9} - \frac{243\sqrt{2}}{5}.$$

We simplify this so that it takes the form

$$68.72\, e^{-4t/9} = -30.54t + 98.72.$$

A little trial-and-error calculation with a hand calculator shows that the two sides of this are approximately equal for $t = 2.49$. This is the time at which the case reaches the bottom. Letting $t = 2.49$ in (*), we find the velocity at that time to be approximately 20.46 (ft/sec).

16. We choose the positive x direction down the hill with the origin at the starting point. The forces acting on the boy and sled are: (1) $F_1$, the component of their weight parallel to the hill; (2) $F_2$, the frictional force; and $F_3$, the air resistance. The boy and sled weigh 72 lbs.,

and the component parallel to the hill has numerical value $72 \sin 30^0 = 36$. Since this acts in the positive (downward) direction on the hill, $F_1 = 36$. The frictional force $F_2$ has numerical value $\mu N$, where $\mu > 0$ is the coefficient of friction and N is the normal force. The magnitude of N is $72 \cos 30^0 = 36\sqrt{3}$; the $\mu$ is an unknown which will be determined in due course from the given data of the problem. Since the force $F_2$ acts in the negative (upward) direction on the hill, we have $F_2 = -\mu(36\sqrt{3})$. Finally, the air resistance $F_3$ has numerical value 2v. Since this also acts in the negative direction, we thus have $F_3 = -2v$. Newton's Second Law now gives $m\frac{dv}{dt} = F_1 + F_2 + F_3$. With $g = 32$, $m = \frac{w}{g} = \frac{72}{32} = \frac{9}{4}$, and the above forces, this becomes

$$\left(\frac{9}{4}\right)\left(\frac{dv}{dt}\right) = 36 - \mu 36\sqrt{3} - 2v.$$

The initial condition is $v(0) = 0$. Another condition is also given, that is $v(5) = 10$; and this extra condition will be sufficient for us to eventually determine $\mu$.

The D.E. is separable. We write it as

$$\frac{dv}{dt} = \left(\frac{8}{9}\right)(18 - \mu 18\sqrt{3} - v)$$

or

$$\frac{dv}{v + 18(\mu\sqrt{3} - 1)} = -\frac{8}{9}dt .$$

Integrating we find $\ln|v + 18(\mu\sqrt{3} - 1)| = -\frac{8t}{9} + c_0$ or $|v + 18(\mu\sqrt{3} - 1)| = c\,e^{-8t/9}$. Applying the initial condition, we at once find that $c = |18(\mu\sqrt{3} - 1)|$. Thus we obtain the solution in the form $v + 18(\mu\sqrt{3} - 1) = 18(\mu\sqrt{3} - 1)e^{-8t/9}$ or $v = 18(1 - \mu\sqrt{3})(1 - e^{-8t/9})$. Now we apply the extra condition $v(5) = 10$ to this to determine $\mu$. We have $10 = 18(1 - \mu\sqrt{3})(1 - e^{-40/9})$ and hence $1 - \mu\sqrt{3} = \left(\frac{5}{9}\right)(0.988)^{-1}$, from which we obtain $\mu = 0.25$.

17. This problem has two parts: (A), *before* the object reaches the surface of the lake; and (B), *after* it passes beneath the surface. We consider (A) first. We take the positive x-axis vertically downward with the origin at the point of release of the object. The forces acting on the body are: (1) $F_1$, its weight, 32 lbs., which acts downward and so is positive; and (2) $F_2$, the air resistance, numerically equal to 2v, which acts upward and so is the negative quantity -2v. Applying Newton's Second Law, with $m\,\frac{w}{g} = \frac{32}{32} = 1$, we at once obtain the D.E.

$$\frac{dv}{dt} = 32 - 2v.$$

The initial condition is $v(0) = 0$. The D.E. is separable. We write it as $\frac{dv}{32 - 2v} = dt$. Integrating we find $\ln|32 - 2v| = -2t + c_0$ or $|32 - 2v| = c_1 e^{-2t}$. Applying the initial condition we find $c = 32$. With this, we at once have

$$v = 16(1 - e^{-2t}) \qquad (*)$$

This gives the velocity at each time t before the object reaches the surface of the lake. To solve problem (B), we will need to know the velocity at the instant when the object reaches the surface. To find out when this is, we need to know distance fallen as a function of time. This is found by integrating (*). We at once obtain

$$x = 16t + 8e^{-2t} + k.$$

Since $x(0) = 0$, we have $k = -8$, and hence the distance fallen (*before* striking the water) is given by

$$x = 16t + 8e^{-2t} - 8.$$

Since the point of release was 50 feet above the water, if we let $x = 50$ in this, it will determine the time at which the object hits the surface. Thus we must solve

$$50 = 16t + 8e^{-2t} - 8$$

for t. We write this in the form $58 - 16t = 8e^{-2t}$. A little trial-and-error calculation with a hand calculator leads to the approximate solution $t = 3.62$. The velocity at this instant is then found by letting $t = 3.62$ in (*). The result is approximately 15.99 (ft/sec).

We now turn to problem (B). We again take the positive x-axis vertically downward, but now we take the origin at the point where the object hits the surface of the lake. The forces now acting on the body are given by

$F_1 = 32$, $F_2 = -6v$, and $F_3 = -8$. The last two (the water resistance and the buoyancy) have negative signs since they act upward. Newton's Second Law leads to the D.E.

$$\frac{dv}{dt} = 32 - 6v - 8.$$

By part (A), the velocity of the object at the surface of the lake is 15.99 (ft/sec). We take this as the I.C. here: $v(0) = 15.99$.

Separating variables, the D.E. becomes $\frac{dv}{24 - 6v} = dt$. Integrating, we find $\ln|24 - 6v| = -6t + c_1$ or $|24 - 6v| = ce^{-6t}$. Applying the I.C. gives $c = 71.94$. Thus we obtain $-(24 - 6v) = 71.94\, e^{-6t}$ and hence $v = 4 + 11.99\, e^{-6t}$. This is the velocity after the object passes beneath the surface. We want to know what this is 2 sec. after. Hence we let $t = 2$ in this to obtain $v \approx 4.00$ (ft/sec).

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1. Let x be the amount of radioactive nuclei present after t years. Then x satisfies the D.E. $\frac{dx}{dt} = -kx$, where $k > 0$. Letting $x_0$ denote the amount initially present, we have the I.C. $x(0) = x_0$. Also, since 10% of the original number have undergone disintegration in 100 years, 90% remain, and so we have the additional condition $x(100) = \frac{9x_0}{10}$. The D.E. is separable; and separating variables and integrating, we at once obtain $x = c\, e^{-kt}$. Application of the I.C. immediately gives $x_0 = c$. Hence we have

$x = x_0 e^{-kt}$. Now apply the additional condition to determine k. We have $\frac{9x_0}{10} = x_0 e^{-100k}$, which reduces to $e^{-k} = \left(\frac{9}{10}\right)^{1/100}$. Thus the solution takes the form

$$x = x_0 \left(\frac{9}{10}\right)^{t/100} \qquad (*)$$

To answer question (a), we let t = 1000 in (*). We find $x(1000) = x_0 \left(\frac{9}{10}\right)^{10} \approx 0.3487\, x_0$. Thus the answer to 34.87%. To answer question (b), we let $x = \frac{x_0}{4}$ in (*). We have $\frac{x_0}{4} = x_0 \left(\frac{9}{10}\right)^{t/100}$. From this, $\frac{1}{100} \ln\left(\frac{9}{10}\right) = \ln\left(\frac{1}{4}\right)$, from which $t = 100 \left[\frac{\ln \frac{1}{4}}{\ln \frac{9}{10}}\right] \approx 1315.28$. Thus $t \approx 1315$ years.

4. Let x = the amount of the first chemical present. Then x satisfies the D.E. $\frac{dx}{dt} = -kx$, where $k > 0$. Two conditions are given: $x(1) = \frac{2}{3}$ and $x(4) = \frac{1}{3}$. The D.E. is separable; and separating variables and integrating, we at once obtain $x = ce^{-kt}$. Applying the two conditions, we obtain respectively $ce^{-k} = \frac{2}{3}$ and $ce^{-4k} = \frac{1}{3}$. These two equations will determine c and k. Dividing the first by the second gives $e^{3k} = 2$, from which $e^{k} = 2^{1/3}$. Then from the second equation for c and k, $c = \left(\frac{1}{3}\right)e^{4k} = \frac{2^{4/3}}{3}$. Thus the solution of the D.E. which satisfies the two given

conditions is $x = \left(\frac{2^{4/3}}{3}\right)(e^k)^{-t} = \left(\frac{2^{4/3}}{3}\right)(2^{1/3})^{-t}$ or $x = \frac{2^{(4-t)/3}}{3}$. (*)

To answer question (a), let $t = 7$ in (*). This gives $x(7) = \frac{2^{-1}}{3} = \frac{1}{6}$ (kg). Now compare this with the orignial amount, which is $x(0) = \frac{2^{4/3}}{3}$ (kg). We have $\frac{x(7)}{x(0)} = \frac{\frac{1}{6}}{\frac{2^{4/3}}{3}} = \frac{1}{4\sqrt[3]{2}} \approx 0.1984$. Thus 19.8% of the first chemical remains at the end of seven hours.

To answer question (b), we first note that one tenth of the first chemical is $\frac{x(0)}{10} = \frac{2^{4/3}}{30}$ (kg). We thus let $x = \frac{2^{4/3}}{30}$ in (*) and solve for t. We have $\frac{2^{(4-t)/3}}{3} = \frac{2^{4/3}}{30}$. From this $\left[\frac{4 - t}{3}\right][\ln 2] = \ln \frac{2^{4/3}}{10}$; and solving for t, we find $t \approx 9.97$. Thus the answer is 9 hours, 58 minutes.

7. Let x = the temperature of the body at time t. Assuming Newton's Law of Cooling, we have the D.E. $\frac{dx}{dt} = k(x - 40)$. We also have the I.C. $x(0) = 100$ and the additional condition $x(10) = 90$. The D.E. is separable. We write it in the form $\frac{dx}{x - 40} = k\,dt$. Integrating, we obtain $\ln|x - 40| = kt + c_0$ and hence $|x - 40| = c\,e^{kt}$. Since $x \geq 40$, $|x - 40| = x - 40$, so we have $x = 40 + c\,e^{kt}$. Application of I.C. $x(0) = 100$ gives $100 = 40 + c$ or $c = 60$. Thus $x = 40 + 60\,e^{kt}$. Then application of the

additional condition $x(10) = 90$ gives $90 = 40 + 60\,e^{10k}$ from which $e^{10k} = \frac{5}{6}$ or $e^{k} = \left(\frac{5}{6}\right)^{1/10}$. Thus $e^{kt} = \left(\frac{5}{6}\right)^{t/10}$, and we have the solution

$$x = 40 + 60\left(\frac{5}{6}\right)^{t/10}. \qquad (*)$$

(a) Let $t = 30$ in (*). We find $x(30) = 40 + 60\left(\frac{5}{6}\right)^{3} \approx 74.72^{0}$ F.

(b) Let $x = 50$ in (*). We have $50 = 40 + 60\left(\frac{5}{6}\right)^{t/10}$, from which $\left(\frac{5}{6}\right)^{t/10} = \frac{1}{6}$. Then $t = \frac{10\ \ln(1/6)}{\ln\ (5/6)} \approx 98.29$. Hence $t \approx 98$ minutes, 17 seconds.

9. Let $x$ = the temperature of the pie at time $t$. Assuming Newton's Law of Cooling, we have the D.E. $\frac{dx}{dt} = k(x - 80)$. We also have the I.C. $x(0) = 350$ and the additional condition $x(5) = 300$. Separating variables in the D.E., we have $\frac{dx}{x - 80} = k\,dt$; and integrating, we find $\ln|x - 80| = kt + c_0$, and hence $|x - 80| = c\,e^{kt}$. Since $x \geq 80$, this simplifies to $x = 80 + c\,e^{kt}$. Applying the I.C. $x(0) = 350$, we have $350 = 80 + c$, so $c = 270$. Thus $x = 80 + 270\,e^{kt}$. Application of the additional condition $x(5) = 300$ gives $300 = 80 + 270\,e^{5k}$, from which $e^{5k} = \frac{220}{270}$ or $e^{k} = \left(\frac{22}{27}\right)^{1/5}$. Thus we obtain the solution

$$x = 80 + 270\left(\frac{22}{27}\right)^{t/5}. \qquad (*)$$

(a) Let $t = 10$ in (*). We find $x(10) = 80 + 270\left(\frac{22}{27}\right)^2 \approx 259.26^0$.

(b) Let $x = 100$ in (*). We have $100 = 80 + 270\left(\frac{22}{27}\right)^{t/5}$, from which $\left(\frac{22}{27}\right)^{t/5} = \frac{2}{27}$. Then $t = 5\,\frac{\ln(2/27)}{\ln(22/27)} \approx 63.54$. Hence $t \approx 63$ minutes, 32 seconds.

10. Let $x$ = the temperature of the coffee at time $t$. Assuming Newton's Law of Cooling, we have the D.E. $\frac{dx}{dt} = k(x - 70)$. We also have the I.C. $x(0) = 180$ and the additional condition $x(10) = 160$. Separating variables in the D.E., we have $\frac{dx}{x - 70} = k\,dt$; and integrating, we find $\ln|x - 70| = kt + c_0$, and hence $|x - 70| = c\,e^{kt}$. Since $x \geq 70$, this simplifies to $x = 70 + c\,e^{kt}$. Applying the I.C. $x(0) = 180$, we have $180 = 70 + c$, so $c = 110$. Thus $x = 70 + 110\,e^{kt}$. Application of the additional condition $x(10) = 160$ gives $160 = 70 + 110\,e^{10k}$, from which $e^{10k} = \frac{90}{110}$ or $e^k = \left(\frac{9}{11}\right)^{1/10}$. Thus we obtain the solution

$$x = 70 + 110\left(\frac{9}{11}\right)^{t/10} \qquad (*).$$

(a) Let $t = 15$ in (*). We have $x(15) = 70 + 110\left(\frac{9}{11}\right)^{3/2} \approx 151.41^0$.

(b) Let $x = 140$ in (*). We have $140 = 70 + 110\left(\frac{9}{11}\right)^{t/10}$, from which $\left(\frac{9}{11}\right)^{t/10} = \frac{7}{11}$. Then $t = \frac{10 \ln(7/11)}{\ln(9/11)} \approx 22.52$. So $x = 140$ at $t \approx 22$ minutes, 31 seconds. Now let $x = 130$ in (*). We have $130 = 70 + 110\left(\frac{9}{11}\right)^{t/10}$, from which $\left(\frac{9}{11}\right)^{t/10} = \frac{6}{11}$. Then $t = \frac{10 \ln(6/11)}{\ln(9/11)} \approx 30.20$. So $x = 130$ at $t \approx 30$ minutes, 12 seconds. Thus the woman should have drunk the coffee between 22 minutes, 31 seconds past 10 and 30 minutes, 12 seconds past 10.

11. Let $x$ = the population at time $t$. Then we at once have the D.E. $\frac{dx}{dt} = kx$. Letting $x_0$ denote the population at the start of the given 40 year period, we have the I.C. $x(0) = x_0$. Since the population doubles in 40 years, we have the additional condition $x(40) = 2x_0$. The solution of the D.E. is $x = c\,e^{kt}$. Applying the I.C. to this we at once have $c = x_0$, and hence $x = x_0 e^{kt}$. Now applying the additional condition to this, we have $2x_0 = x_0 e^{40k}$, from which $e^{40k} = 2$ and $k = \ln \frac{2}{40} \approx 0.0173$. Thus the solution of the D.E. which satisfies the two given conditions is $x = x_0 e^{0.0173t}$. To answer the stated question, we let $x = 3x_0$ in this, from which $e^{0.0173t} = 3$ and $0.0173t = \ln 3$. From this we find $t \approx 63.5$. Thus the population triples in approximately 63.5 years.

12. Let $x$ = the population at time $t$. We at once have the D.E. $\frac{dx}{dt} = kx$. We take 1970 as the zeroth year of the problem and have the I.C. $x(0) = 30{,}000$. Then 1980 is the tenth year, and we have the additional condition $x(10) = 35{,}000$. The one-parameter family of solutions of the D.E. is $x = c\,e^{kt}$. Applying the I.C. to this, we have $c = 30{,}000$. Thus $x = 30{,}000\,e^{kt}$. We apply the additional condition to this, obtaining $35{,}000 = 30{,}000\,e^{10k}$. From this, $e^{k} = \left(\frac{7}{6}\right)^{1/10}$. Thus we obtain the solution $x = 30{,}000\left(\frac{7}{6}\right)^{t/10}$. We seek the population in 1990, which is the twentieth year on or scale. Letting $t = 20$ in the solution, we find $x = 30{,}000\left(\frac{7}{6}\right)^{2} \approx 40{,}833$.

15. The population $x$ satisfies the D.E.

$$\frac{dx}{dt} = \frac{3x}{100} - \frac{3x^2}{10^8}.$$

The I.C. is $x(1980) = 200{,}000$. The D.E. is separable. We write it in the form

$$\frac{10^6 dx}{x(x - 10^6)} = -\frac{3\,dt}{100}.$$

To integrate the left member, we use partial fractions. Thus we obtain

$$\left[\frac{1}{x - 10^6} - \frac{1}{x}\right]dx = -\frac{3}{100}\,dt.$$

Integrating we find $\ln|x - 10^6| - \ln|x| = -\frac{3t}{100} + c_0$ or $\ln\left|\frac{x - 10^6}{x}\right| = -\frac{3t}{100} + c_0$. From this we obtain $\frac{x - 10^6}{x} = ce^{-3t/100}$. We apply the I.C. $x(1980) = 200{,}000 = (2)(10^5)$ to this. We have $\frac{[(2)(10^5) - 10^6]}{2(10)^5} = ce^{-3(198)/10}$, or $c = -4e^{59.4}$. Thus we obtain the solution in the form $\frac{x - 10^6}{x} = -4e^{59.4-3t/100}$. We solve this for x, obtaining $x + 4x\,e^{59.4-3t/100} = 10^6$ and hence

$$x = \frac{10^6}{1 + 4e^{59.4-3t/100}} \qquad (*)$$

This is the answer to part (a). To answer part (b), we let $t = 2000$ in (*). We find $x(2000) = \frac{10^6}{[1 + 4e^{0.6}]} \approx 312{,}966$. To answer (c), simply find $\lim_{t \to \infty} x$, where x is given by (*). Since $\lim_{t \to \infty} e^{59.4-3t/100} = 0$, we have

$$\lim_{t \to \infty} x = \frac{10^6}{1} = 1{,}000{,}000.$$

16. (a) We have the initial-value problem $\frac{dx}{dt} = kx - \lambda x^2$, $x(t_0) = x_0$, where we assume $kx - \lambda x^2 > 0$. The D.E. is separable. We separate variables and use partial fractions to obtain $\frac{1}{k}\left[\frac{1}{x} + \frac{\lambda}{k - \lambda x}\right]dx = dt$. Then integration gives $\frac{1}{k}[\ln x - \ln(k - \lambda x)] = t + \ln c_0$. Simplifying, we obtain $\frac{x}{k - \lambda x} = c\,e^{kt}$. Applying the

I.C. $x(t_0) = x_0$ to this, we find $c = \dfrac{x_0 e^{-kt_0}}{k - \lambda x_0}$. Before using this, we solve the one-parameter family of solutions for x. We have $x = k\,c\,e^{kt} - \lambda\,c\,x\,e^{kt}$, from which $(1 + \lambda\,c\,e^{kt})x = k\,c\,e^{kt}$ and hence $x = \dfrac{k\,c\,e^{kt}}{1 + \lambda\,c\,e^{kt}}$ or finally $x = \dfrac{kc}{\lambda c + e^{-kt}}$. We now substitute into this the value of c already determined from the initial condition. We find

$$x = \frac{k\left[\dfrac{x_0 e^{-k\,t_0}}{k - \lambda\,x_0}\right]}{\lambda\left[\dfrac{x_0 e^{-k\,t_0}}{k - \lambda\,x_0}\right] + e^{-kt}}.$$

We simplify this by multiplying numerator and denominator through by $(k - \lambda\,x_0)e^{k\,t_0}$. We obtain

$$x = \frac{k\,x_0}{\lambda\,x_0 + (k - \lambda\,x_0)e^{-k(t - t_0)}}.$$

17. Let x = the population at time t. Since 100 people leave the island every year, the D.E. (3.58) is correctly modified by subtracting 100 from its right member. Thus we have the D.E.

$$\frac{dx}{dt} = \left(\frac{1}{400}\right)x - 10^{-8}x^2 - 100.$$

We also have the I.C. x(1980) = 20,000. The D.E. is separable. We write it in the successive forms

$$\frac{dx}{dt} = \frac{(10)^6 x - 4x^2 - 4(10)^{10}}{4(10)^8},$$

$$\frac{dx}{[x^2 - 25(10)^4 x + (10)^{10}]} = -\frac{dt}{(10)^8},$$

$$\frac{dx}{[x - 5(10)^4][x - 20(10)^4]} = -\frac{dt}{(10)^8}.$$

We now apply partial fractions to the left member and multiply through by $(15)(10)^4$ to obtain

$$\left[\frac{1}{x - 20(10)^4} - \frac{1}{x - 5(10)^4}\right] dx = -\frac{15\ dt}{(10)^4}.$$

Integrating and simplification give

$$\ln\left[\frac{x - 20(10)^4}{x - 5(10)^4}\right] = -\frac{15t}{(10)^4} + c_0$$

or

$$\frac{x - 20(10)^4}{x - 5(10)^4} = c\, e^{-15t/(10)^4}.$$

We apply the I.C. $x(1980) = 20{,}000 = (20)(10)^3$ to this. With $x = (20)(10)^3$, the left member reduces thus:

$$\frac{(20)(10)^3(1 - 10)}{(5)(10)^3(4 - 10)} = 6.$$

Thus we find $6 = c\, e^{-(15)(1980)/(10)^4}$, and so $c = 6\, e^{(15)(1980)/(10)^4}$. Thus the solution takes the form

$$\frac{x - (20)(10)^4}{x - 5(10)^4} = 6\, e^{15(1980-t)/(10)^4}.$$

We must solve this for x. After some algebraic manipulations, we obtain the desired result

$$x = \frac{(10)^5[3\, e^{15(1980-t)/(10)^4} - 2]}{6\, e^{15(1980-t)/(10)^4} - 1}.$$

18. This falls into two problems: (A) Before the cats arrived, and (B) after the cats arrived. We let x = the number of mice on the island at time t and proceed to solve problem (A). During this time, in the absence of cats killing mice, the D.E. is simply $\frac{dx}{dt} = kx$, with solution $x = c\, e^{kt}$. Choosing January 1, 1970 as time t = 0, the I.C. is x(0) = 50,000. Then, measuring t in years, Jan. 1, 1980 is time t = 10; and we have the additional condition x(10) = 100,000. Applying the I.C. x(0) = 50,000, we at once find c = 50,000 and hence $x = 50{,}000\, e^{kt}$. The additional condition now gives $e^{10k} = 2$, from which $k = \ln \frac{2}{10}$. With this, the solution takes the form $x = 50{,}000\, e^{t \ln(2/10)}$. But this solution only holds until 1980 and so will not help us to answer the stated question. It is the number k, which represents the natural rate of population increase, which we need here to solve problem (B) and answer the stated question.

Turning to problem (B), since the cats kill 1000 mice/month = 12,000 mice/year, the D.E. of problem (A) must be modified by subtracting 12,000 from its right member.

Thus we have $\frac{dx}{dt} = kx - 12{,}000$, where $k = \ln \frac{2}{10}$ was found in problem (A). The D.E. is separable, and we write it in the form $\frac{dx}{kx - 12{,}000} = dt$. Integrating we obtain $kx = 12{,}000 + c\,e^{kt}$. Now we have an I.C. here, for on Jan. 1, 1980, there were 100,000 mice. Taking this date as time $t = 0$ for problem (B), we thus have the I.C. $x(0) = 100{,}000$. Application of this gives $c = 100{,}000\,k - 12{,}000$. With this value of c and $k = \ln \frac{2}{10}$, we obtain the solution of problem (B):

$$x = \left(\frac{10}{\ln 2}\right)\left[12{,}000 + (10{,}000 \ln 2 - 12{,}000)e^{t \ln 2/10}\right]$$

The answer to the stated question is now found by letting $t = 1$ in this expression. We have

$$\begin{aligned} x(1) &= (14.427)[12{,}000 + (6931 - 12{,}000)(1.0718)] \\ &= 94{,}742. \end{aligned}$$

19. Let x = the amount of money at time t. We have the D.E. $\frac{dx}{dt} = kx$. Since the annual rate is 6%, we let $k = .06$ and so have the D.E. $\frac{dx}{dt} = .06x$. We also have the I.C. $x(0) = 1000$. Solving the D.E., we find $x = c\,e^{.06t}$. The I.C. gives $c = 1000$, so we have the solution $x = 1000\,e^{.06t}$.

(a) We let $t = 10$ in the solution. We find $x = 1000\,e^{0.6} \approx 1822.12$. Thus there will be \$1822.12.

(b) We let $x = 2000$ in the solution. We obtain $2000 = 1000\,e^{.06t}$ or $e^{.06t} = 2$. From this $.06t = \ln 2$ and $t \approx 11.55$. Thus the amount will double in approximately 11.55 years.

20. From exercise 19, we have the D.E. $\frac{dx}{dt} = kx$, where k gives the annual rate. We have the I.C. $x(0) = x_0$, where $x_0$ denotes the original amount. Solving the D.E., we find $x = c\,e^{kt}$. Applying the I.C. to this, we find $c = x_0$ and so have the solution $x = x_0 e^{kt}$.

(a) If the original amount doubles in two years, then $x = 2x_0$ when $t = 2$. Applying this to the solution, we obtain $2x_0 = x_0 e^{2k}$. From this $k = \frac{\ln 2}{2} \approx 0.3466$. Thus the annual rate is a remarkable 34.66%.

(b) Here we have the additional condition $x(1/2) = 3x_0/2$. We apply this to the solution. We have $3x_0/2 = x_0 e^{k/2}$, from which $e^k = 9/4$. Then the solution becomes $x = x_0\left(\frac{9}{4}\right)^t$. We ask how long it will take the original amount to double, so we let $x = 2x_0$ in this solution and solve for t. We find $\left(\frac{9}{4}\right)^t = 2$, from which $t = \frac{\ln 2}{\ln(9/4)} \approx 0.8547$. So the amount doubles in approximately 0.85 years.

21. Let x = the amount of salt at time t. We use the basic equation (3.63), $\frac{dx}{dt}$ = IN - OUT. We find IN = (3 lb./gal) (4 gal./min.) = 12(lb./min.); and OUT = (C lb./gal.) (4 gal./min.) where C lb./gal. is the concentration. Since the rates of inflow and outflow are the same, there is always 100 gal. in the tank and so this is simply $\frac{x}{100}$. Thus we have OUT = $\frac{4x}{100}$ (lb./min.) Hence the D.E. of the problem is $\frac{dx}{dt} = 12 - \frac{4x}{100}$. The I.C. is x(0) = 20.

The D.E. is separable. We write it as $\frac{dx}{dt} = \frac{300 - x}{25}$ or $\frac{dx}{300 - x} = \frac{dt}{25}$. Integrating, we find $\ln|300 - x| = -\frac{t}{25} + c_0$ or $|300 - x| = c\,e^{-t/25}$. Applying the I.C., we find 280 = c; and since 300 - x > 0, we take $|300 - x| = 300 - x$. Thus we obtain the solution in the form $300 - x = 280\,e^{-t/25}$ or

$$x = 300 - 280\,e^{-t/25} \qquad (*)$$

To answer question (a), let t = 10 in (*). We find x ≈ 112.31 (lb.). To answer question (b), let x = 160 in (*). We have $280\,e^{-t/25} = 140$ or $e^{-t/25} = 0.5$. Then $-\frac{t}{25} = \ln 0.5$, from which we find t ≈ 17.33 (min.).

22. Let x = the amount of salt at time t. We use the basic equation (3.63), $\frac{dx}{dt}$ = IN - OUT. We have IN = 0, since pure water flows into the tank; and we find OUT = (C lb./gal.)(2gal/min), where c lb./gal. is the concentration. Since water flows in at 5 gal/min. and the mixture flows

out at 2 gal/min., there is a gain of 3 gal/min of brine in the tank. Since there was initially 100 gal. there, the amount of brine at time t is 100 + 3t. Thus the concentration is $\frac{x}{100 + 3t}$. Thus OUT = $\frac{2x}{100 + 3t}$ (lb./min.). Hence the D.E. of the problem is $\frac{dx}{dt} = -\frac{2x}{100 + 3t}$. The I.C. is $x(0) = 10$.

The D.E. is separable. We write it as $\frac{dx}{x} = -\frac{2\,dt}{3t + 100}$. Integrating, we find $\ln x = -\frac{2}{3}\ln(3t + 100) + \ln c_0$, from which $x = \frac{c}{(3t + 100)^{2/3}}$. We now apply the I.C., obtaining $10 = \frac{c}{(100)^{2/3}}$ or $c = 10(100)^{2/3}$. Thus we have the solution $x = \frac{10(100)^{2/3}}{(3t + 100)^{2/3}}$.

(a) We let $t = 15$ in the solution. We find $x = \frac{10(100)^{2/3}}{(145)^{2/3}} \approx 7.81$. So there is approximately 7.81 lb. The concentration is $\frac{7.81}{145} \approx 0.0539$ lb./gal.

(b) Since there was initially 100 gallons of brine in the tank and the amount is increasing at the rate of 3 gal./min., the number of minutes t needed to obtain a full tank of 250 gallons is given by $100 + 3t = 250$; so $t = 50$. We let $t = 50$ in the solution and find $x = \frac{10(100)^{2/3}}{(250)^{2/3}} \approx 5.4279$. The concentration is then $\frac{5.4279}{250} \approx 0.0217$ lb./gal.

25. Let x = the amount of salt at time t. We use the basic equation $\frac{dx}{dt}$ = IN - OUT. We find IN = (30 gm/liter)(4 liters/min) = 120 (gm/min); and OUT = (C gm/liter) $\left(\frac{5}{2} \text{ liters/min}\right)$, where C gm/liter is the concentration. The rate of inflow is 4 liters/min and that of outflow is $\frac{5}{2}$ liters/min, so there is a net gain of $\frac{3}{2}$ liters/min of fluid in the tank. Hence at the end of t minutes the amount of fluid in the tank is $300 + \frac{3t}{2}$ liters. Thus the concentration at time t is $\frac{x}{300 + \frac{3t}{2}}$ gm/liter, and so OUT = $\frac{5x}{2\left[300 + \frac{3t}{2}\right]} = \frac{5x}{3t + 600}$ (gm/min). Hence the D.E. of the problem is $\frac{dx}{dt} = 120 - \frac{5x}{3t + 600}$. The I.C. is x(0) = 50.

The D.E. is linear. We write it in the standard form

$$\frac{dx}{dt} + \frac{5}{3t + 600} x = 120.$$

An I.F. is $e^{\int [(5/(3t+600)]dt} = e^{(5/3)\ln(3t+600)} = (3t + 600)^{5/3}$. Multiplying through by this we have

$$(3t + 600)^{5/3} \frac{dx}{dt} + 5(3t + 600)^{2/3} x = 120(3t + 600)^{5/3}$$

or

$$\frac{d}{dt} [(3t + 600)^{5/3} x] = 120(3t + 600)^{5/3}.$$

Integration gives $(3t + 600)^{5/3} x = 15(3t + 600)^{8/3} + c$ or

$$x = 15(3t + 600) + c(3t + 600)^{-5/3}.$$

Applying the I.C. $x(0) = 50$ gives $50 = (15)(600) + c(600)^{-5/3}$, from which $c = -8950(600)^{5/3}$. Thus the solution is given by

$$x = 15(3t + 600) - (8950)(600)^{5/3}(3t + 600)^{-5/3}. \quad (*)$$

The stated question asks for x at the instant when the tank overflows. Since the amount of fluid increases at the rate of $\frac{3}{2}$ liter/min and the 500 liter tank originally had 300 liters in it, this time t is given by $\frac{3t}{2} = 200$ or $t = \frac{400}{3}$. We thus let $t = \frac{400}{3}$ in (*) and obtain $x\left(\frac{400}{3}\right) = 1500(1000) - (8950)(600)^{5/3}(1000)^{5/3} = 15000 - 8950\left(\frac{3}{5}\right)^{5/3} \approx 11{,}179.96$ (grams).

26. Let x = the amount of salt at time t. We use the basic equation $\frac{dx}{dt}$ = IN - OUT. We find IN = (50 gm/liter)(5 liters/min) = $\frac{250}{\text{gm/min}}$; and OUT = (C gm/liter)(7 liter/min), where C gm/liter is the concentration. The rate of inflow is 5 liters/min. and that of outflow is 7 liters/min., so there is a net loss of 2 liters/min of fluid in the tank. Hence at the end of t minutes the amount of fluid in the tank is 200 - 2t liters. Thus the concentration at time t is $\frac{x}{200 - 2t}$ gm/liter, and so OUT = $\frac{7x}{200 - 2t}$ (gm/min). Hence the D.E. is $\frac{dx}{dt} = 250 - \frac{7x}{200 - 2t}$. The I.C. is $x(0) = 40$.

The D.E. is linear. We write it in the standard form

$$\frac{dx}{dt} + \frac{7}{200 - 2t}x = 250.$$

An I.F. is $e^{\int [7/(200-2t)]dt} = e^{-(7/2)\ln(200-2t)} = (200 - 2t)^{-7/2}$. Multiplying through by this, we have $(200 - 2t)^{-7/2}\frac{dx}{dt} + 7(200 - 2t)^{-9/2}x = 250(200 - 2t)^{-7/2}$ or

$$\frac{d}{dt}\left[(200 - 2t)^{-7/2}x\right] = 250(200 - 2t)^{-7/2}.$$

Integration gives $(200 - 2t)^{-7/2}x = 50(200 - 2t)^{-5/2} + c$ or

$$x = 50(200 - 2t) + c(200 - 2t)^{7/2}.$$

Applying the I.C. $x(0) = 40$, we find $40 = 10{,}000 + c(200)^{7/2}$, from which $c = -9960(200)^{-7/2}$. Thus the solution is given by

$$x = 50(200 - 2t) - 9960\left(1 - \frac{t}{100}\right)^{7/2}. \qquad (*)$$

The tank will be half full when it contains 100 liters. Since it initially contained 200 liters and it loses 2 liters/min., it will contain 100 liters after 50 min. Thus we let $t = 50$ in (*). We find

$$x(50) = 5000 - 9960\left(\frac{1}{2}\right)^{7/2} \approx 4119.65 \text{ (gm)}.$$

30. Let $x$ = the number of people who have the disease after $t$ days. Then $10{,}000 - x$ people do not have it, and we have the D.E. $\frac{dx}{dt} = kx(10{,}000 - x)$. We also have the I.C. $x(0) = 1$. The D.E. is separable; and separating variables, we have $\frac{dx}{x(10{,}000 - x)} = k\,dt$. Using partial fractions and integrating, we find $\frac{1}{10000}\,[\ln|x| - \ln|10{,}000 - x|] = kt + c_0$. Noting that $x \geq 0$ and $10{,}000 - x \geq 0$, we simplify and obtain $\ln \frac{x}{10{,}000 - x} = 10{,}000\,kt + c_1$. Then we have $\frac{x}{10{,}000 - x} = c\,e^{10{,}000\,kt}$. We apply the I.C. to obtain $c = 1/9999$. Thus we obtain the solution $\frac{x}{10{,}000 - x} = \frac{e^{10{,}000\,kt}}{9999}$.

We solve this for $x$ in terms of $t$. We find $\left(1 + \frac{e^{10{,}000\,kt}}{9999}\right)x = \frac{10{,}000}{9999}\,e^{10{,}000\,kt}$ and hence

$$x = \frac{10{,}000}{1 + 9999\,e^{-10{,}000\,kt}}.$$

Now besides the I.C., we also have the additional condition $x(5) = 50$. We apply this to the preceeding solution. We have

$$50 = \frac{10{,}000}{1 + 9999\,e^{-50{,}000\,k}}.$$

Then $(50)(9999)e^{-50{,}000\,k} = 9950$, $e^{-50{,}000\,k} = \frac{199}{9999}$, and so $e^{-10{,}000\,kt} = \left(\frac{199}{9999}\right)^{t/5}$. Thus the solution takes the form

$$x = \frac{10{,}000}{1 + 9999\left[\frac{199}{9999}\right]^{t/5}}.$$

We are asked how many people have the disease after 10 days. We let $t = 10$ in this to obtain

$$x(10) = \frac{10{,}000}{1 + 9999\left[\frac{199}{9999}\right]^{2}} \approx 478.88.$$

Thus about 479 people have the disease at that time.

31. Let $x$ = the number of lbs. of $c_3$ formed in time $t$. We have the D.E.

$$\frac{dx}{dt} = k\left(10 - \frac{x}{4}\right)\left(15 - \frac{3x}{4}\right),$$

the I.C. $x(0) = 0$, and the additional condition $x(15) = 5$. Separating variables and simplifying, the D.E. takes the form $\frac{16\,dx}{3(x - 40)(x - 20)} = k\,dt$.

Applying partial fractions, we have $\frac{4}{15}\left[\frac{1}{x - 40} - \frac{1}{x - 20}\right]dx = k\,dt$. Integrating, we obtain $\frac{4}{15}\,[\ln(x - 40) - \ln(x - 20)] = kt + c_0$ or $\ln\frac{x - 40}{x - 20} = \frac{15}{4}\,kt + c_1$. From this,

$$\frac{x - 40}{x - 20} = c\,e^{\frac{15}{4}kt}. \qquad (*)$$

We apply the I.C. $x(0) = 0$ to this, and at once obtain $c = 2$. We set $c = 2$ in (*), and apply the additional condition $x(15) = 5$.

We have $\frac{35}{15} = 2\left(e^{\frac{15}{4}k}\right)^{15}$ and hence $e^{\frac{15}{4}k} = \left(\frac{7}{6}\right)^{1/15}$.

Thus we obtain the solution

$$\frac{x - 40}{x - 20} = 2\left(\frac{7}{6}\right)^{t/15}.$$

We solve this for x in terms of t, obtaining

$$x = \frac{40\left[\left(\frac{7}{6}\right)^{t/15} - 1\right]}{2\left(\frac{7}{6}\right)^{t/15} - 1}.$$

This is the answer to part (a). To answer (b), let $t = 60$ in this. We have

$$x(60) = \frac{40\left[\left(\frac{7}{6}\right)^{4} - 1\right]}{2\left(\frac{7}{6}\right)^{4} - 1} \approx \frac{40(0.8526)}{2.7052} \approx 12.6068$$

Thus we find $x \approx 12.61$ (1b).

# Chapter 4

**Section 4.1.B, Page 122**

6. (a) Suppose $c_1e^x + c_2e^{3x} = 0$ for all x on $a \le x \le b$. Differentiating, we obtain $c_1e^x + 3c_2e^{3x} = 0$ for all x on $a \le x \le b$. Subtracting the first of these from the second, we have $2c_2e^{3x} = 0$ for all x on $a \le x \le b$. Since $2e^{3x} \neq 0$ for all such x, we must have $c_2 = 0$. Then $c_1e^x + c_2e^{3x} = 0$ reduces to $c_1e^x = 0$ for all x on $a \le x \le b$. Since $e^x \neq 0$ for all such x, we must have $c_1 = 0$. Thus $c_1e^x + c_2e^{3x} = 0$ for all x on $a \le x \le b$ implies that $c_1 = c_2 = 0$, so $e^x$ and $e^{3x}$ are linearly independent on $a \le x \le b$.

(b) $W(e^x,e^{3x}) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 2e^{4x} \neq 0$ on $a \le x \le b$, so $e^x$ and $e^{3x}$ are linearly independent on this interval.

7. (a) One readily verifies by direct substitution into the D.E. that each of these functions is indeed a solution. To show that they are linearly independent, we apply Theorem 4.4. We have

$$W(e^{2x},e^{3x}) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0$$

for all x, $-\infty < x < \infty$. Thus the two solutions are linearly independent on $-\infty < x < \infty$.

(b) $y = c_1e^{2x} + c_2e^{3x}$.

(c) To satisfy the condition $y(0) = 2$, we let $x = 0$, $y = 2$ in the general solution of part (b). We have

$$c_1 + c_2 = 2. \qquad (*)$$

Now we differentiate the general solution, obtaining $y' = 2c_1e^{2x} + 3c_2e^{3x}$. To satisfy the condition $y'(0) = 3$, we let $x = 0$, $y' = 3$ in this derived equation. We have

$$2c_1 + 3c_2 = 3. \qquad (**)$$

Solving the two equations (*) and (**) for $c_1$ and $c_2$, - we find $c_1 = 3$, $c_2 = -1$. Substituting these values of $c_1$ and $c_2$ into the general solution of part (b), we have the desired particular solution $y = 3e^{2x} - e^{3x}$. This is unique by Theorem 4.1; and it is defined on $-\infty < x < \infty$.

8. (a) One readily verifies by direct substitution into the D.E. that each of these functions is indeed a solution. To show that they are linearly independent, we apply Theorem 4.4. We have

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x} \neq 0$$

for all $x$, $-\infty < x < \infty$. Thus the two solutions are linearly independent on the interval $-\infty < x < \infty$.

(b) $y = c_1 e^x + c_2 x e^x$.

(c) To satisfy the condition $y(0) = 1$, we let $x = 0$, $y = 1$ in the general solution of part (b). We have $c_1 = 1$. Now we differentiate the general solution, obtaining $y' = c_1 e^x + c_2(x+1)e^x$. To satisfy the condition $y'(0) = 4$, we let $x = 0$, $y' = 4$ in the derived equation. We have $c_1 + c_2 = 4$. Since $c_1 = 1$, this yields $c_2 = 3$. Substituting these values of $c_1$ and $c_2$ into the general solution, we have the desired particular solution $y = e^x + 3xe^x$. This is unique by Theorem 4.1; and it is defined on $-\infty < x < \infty$.

10. (a) One readily verifies by direct substitution into the D.E. that each of these functions is indeed a solution. To show that they are linearly independent, we apply Theorem 4.4. We have

$$W\left(x^2, \frac{1}{x^2}\right) = \begin{vmatrix} x^2 & \frac{1}{x^2} \\ 2x & -\frac{2}{x^3} \end{vmatrix} = -\frac{2}{x} - \frac{2}{x} = -\frac{4}{x} \neq 0$$

for all $x$, $0 < x < \infty$. Thus the two solutions are linearly independent on $0 < x < \infty$.

(b) $y = c_1x^2 + \dfrac{c_2}{x^2}$.

(c) To satisfy the condition $y(2) = 3$, we let $x = 2$, $y = 3$ in the general solution of part (b). We have

$$4c_1 + \frac{c_2}{4} = 3. \qquad (*)$$

Now we differentiate the general solution, obtaining $y' = c_1e^x + c_2(x + 1)e^x$. To satisfy the condition $y'(0) = 4$, we let $x = 0$, $y' = 4$ in the derived equation. We have $c_1 + c_2 = 4$. Since $c_1 = 1$, this yields $c_2 = 3$. Substituting these values of $c_1$ and $c_2$ into the general solution, we have the desired particular solution $y = e^x + 3xe^x$. This is unique by Theorem 4.1; and it is defined on $-\infty < x < \infty$.

10. (a) One readily verifies by direct substitution into the D.E. that each of these functions is indeed a solution. To show that they are linearly independent, we apply Theorem 4.4. We have

$$W\left(x^2, \frac{1}{x^2}\right) = \begin{vmatrix} x^2 & \dfrac{1}{x^2} \\ 2x & -\dfrac{2}{x^3} \end{vmatrix} = -\frac{2}{x} - \frac{2}{x} = -\frac{4}{x} \neq 0$$

for all $x$, $0 < x < \infty$. Thus the two solutions are linearly independent on $0 < x < \infty$.

(b) $y = c_1x^2 + \dfrac{c_2}{x^2}$.

(c) To satisfy the condition $y(2) = 3$, we let $x = 2$, $y = 3$ in the general solution of part (b). We have

$$4c_1 + \frac{c_2}{4} = 3 \qquad (*).$$

Now we differentiate the general solution, obtaining $y' = 2c_1x - 2c_2x^{-3}$. To satisfy the condition $y'(2) = -1$, we let $x = 2$, $y' = -1$ in this derived equation. We have

$$4c_1 - \frac{c_2}{4} = -1 \qquad (**).$$

Solving the two equations (*) and (**) for $c_1$ and $c_2$, we find $c_1 = \frac{1}{4}$, $c_2 = 8$. Substituting these values of $c_1$ and $c_2$ into the general solution of part (b) we have the desired particular solution $y = \left(\frac{1}{4}\right) x^2 + \dfrac{8}{x^2}$. This is unique by Theorem 4.1; and it is defined on $0 < x < \infty$.

12. We use Theorem 4.4. We have

$$W(e^{-x},e^{3x},e^{4x}) = \begin{vmatrix} e^{-x} & e^{3x} & e^{4x} \\ -e^{-x} & 3e^{3x} & 4e^{4x} \\ e^{-x} & 9e^{3x} & 16e^{4x} \end{vmatrix}$$

$$= e^{-x}e^{3x}e^{4x} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & 4 \\ 1 & 9 & 16 \end{vmatrix}$$

$$= e^{6x}\left\{\begin{vmatrix} 3 & 4 \\ 9 & 16 \end{vmatrix} - \begin{vmatrix} -1 & 4 \\ 1 & 16 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 1 & 9 \end{vmatrix}\right\}$$

$$= e^{6x}[12 + 20 - 12] = 20e^{6x} \neq 0$$

for all $x$, $-\infty < x < \infty$. Thus the three solutions are linearly independent on $-\infty < x < \infty$. The general solution is $y = c_1e^{-x} + c_2e^{3x} + c_3e^{4x}$.

13. We have

$$W(x,x^2,x^4) = \begin{vmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix}$$

$$= x\begin{vmatrix} 2x & 4x^3 \\ 2 & 12x^2 \end{vmatrix} - \begin{vmatrix} x^2 & x^4 \\ 2 & 12x^2 \end{vmatrix}$$

$$= 16x^4 - 10x^4 = 6x^4 \neq 0$$

on $0 < x < \infty$. Thus by Theorem 4.4, the three given solutions are linearly independent on $0 < x < \infty$. The general solution is $y = c_1x + c_2x^2 + c_3x^4$, where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**Section 4.1.D, Page 132**

1. Let $y = vx$. Then $y' = xv' + v$ and $y'' = xv'' + 2v'$. Substituting these into the given D.E., we obtain $x^2(xv'' + 2v') - 4x(xv' + v) + 4xv = 0$ or $x^3v'' - 2x^2v' = 0$ or $xv'' - 2v' = 0$. Letting $w = v'$, we obtain $x\frac{dw}{dx} - 2w = 0$ or $\frac{dw}{w} = 2\frac{dx}{x}$. Integrating, we find $\ln|w| = 2\ln|x| + \ln|c|$ or $w = cx^2$. We choose $c = 1$, recall $v' = w$, and integrate to obtain $v = \frac{x^3}{3}$. Now forming $y = xv$, we obtain $y = \frac{x^4}{3}$. This or any nonzero constant multiple thereof serves as the desired linearly independent solution. Choosing multiple 3, we have $y = x^4$. The general solution is then $y = c_1x + c_2x^4$.

4. Let $y = xv$. Then $y' = xv' + v$ and $y'' = xv'' + 2v'$. Substituting these into the given D.E., we obtain $(x^2 - x + 1)(xv'' + 2v') - (x^2 + x)(xv' + v) + (x + 1)xv = 0$ or $x(x^2 - x + 1)v'' + (-x^3 + x^2 - 2x + 2)v' = 0$ or $x(x^2 - x + 1)v'' + (x^2 + 2)(1 - x)v' = 0$. Letting $w = v'$, we obtain $x(x^2 - x + 1)\frac{dw}{dx} + (x^2 + 2)(1 - x)w = 0$, or $\frac{dw}{w} = \frac{(x - 1)(x^2 + 2)}{x(x^2 - x + 1)}dx$. Using long division and partial

fractions, we express this in the form $\frac{dw}{w} = \left[1 - \frac{2}{x} + \frac{2x - 1}{x^2 - x + 1}\right]dx$. Integration now yields

$$\ln|w| = x - 2\ln|x| + \ln|x^2 - x + 1| + \ln|c|$$

and so

$$\ln|w| = x + \ln\frac{|c(x^2 - x + 1)|}{x^2}.$$

We choose $c = 1$, recall $w = v'$, and obtain

$$v' = \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)e^x.$$

Thus $v = e^x - \int\frac{e^x}{x}\,dx + \int\frac{1}{x^2}\,e^x\,dx$. Integrating $\int\frac{1}{x^2}\,e^x\,dx$ by parts, we obtain $-\frac{e^x}{x} + \int\frac{e^x}{x}\,dx$. Thus $v = e^x - \frac{e^x}{x}$ or $y = e^x\left(1 - \frac{1}{x}\right)$. Now forming $y = xv$, we find the desired linearly independent solution $y = (x - 1)e^x$. Thus the general solution is $y = c_1x + c_2(x - 1)e^x$.

5. Let $y = e^{2x}v$. Then $y' = e^{2x}v' + 2e^{2x}v$ and $y'' = e^{2x}v'' + 4e^{2x}v' + 4e^{2x}v$. Substituting this into the given D.E., we obtain $(2x + 1)(e^{2x}v'' + 4e^{2x}v' + 4e^{2x}v)$

$$-4(x + 1)(e^{2x}v' + 2e^{2x}v) + 4e^{2x}v = 0$$

or

$$(2x + 1)e^{2x}v'' + 4xe^{2x}v' = 0$$

or

$$(2x + 1)v'' + 4xv' = 0.$$

Letting $w = v'$, we obtain $(2x + 1)\frac{dw}{dx} + 4xw = 0$ or $\frac{dw}{w} = -4x\frac{dx}{2x + 1}$. Using long division on the right member, we rewrite this as $\frac{dw}{w} = \left(-2 + \frac{2}{2x + 1}\right)dx$. Integrating, we have $\ln|w| = -2x + \ln|2x + 1| + \ln|c|$ or $w = c(2x + 1)e^{-2x}$. We choose $c = 1$ and recall $w = v'$ to obtain $\frac{dv}{dx} = (2x + 1)e^{-2x}$. Integrating this, we obtain $v = -(x + 1)e^{-2x}$. Now forming $y = e^{2x}v$, we obtain $y = -(x + 1)$. This or any nonzero constant multiple thereof serves as the desired linearly independent solution. Choosing multiple $-1$, we have $y = x + 1$. The general solution is then $y = c_1e^{2x} + c_2(x + 1)$.

8. Let $y = e^xv$. Then $y' = e^xv' + e^xv$ and $y'' = e^xv'' + 2e^xv' + e^xv$. Substituting into the given D.E., we obtain $(x^2 + x)(e^xv'' + 2e^xv' + e^xv) - (x^2 - 2)(e^xv' + e^xv) - (x + 2)(e^xv) = 0$ or $(x^2 + x)v'' + (x^2 + 2x + 2)v' = 0$. Letting $w = v'$, we obtain $(x^2 + x) + (x^2 + 2x + 2)w = 0$ or $\frac{dw}{w} = -\frac{x^2 + 2x + 2}{x^2 + x}dx$. Using long division and partial fractions, we express this in the form $\frac{dw}{w} = -\left[1 + \frac{2}{x} - \frac{1}{x + 1}\right]dx$. Integration now yields

$\ln|w| = -x - 2\ln|x| + \ln|x + 1| + \ln|c|$ and so

$$\ln|w| = -x + \ln\frac{|c(x + 1)|}{x^2}.$$

We choose $c = 1$, recall $w = v'$, and obtain

$$v' = \left(\frac{1}{x} + \frac{1}{x^2}\right)e^{-x}.$$

Thus

$$v = \int\frac{e^{-x}}{x}\,dx + \int\frac{e^{-x}}{x^2}\,dx.$$

Integrating $\int\frac{1}{x}e^{-x}dx$ by parts, we obtain $-\frac{e^{-x}}{x} - \int\frac{e^{-x}}{x^2}\,dx$. Thus we have $v = -\frac{e^{-x}}{x}$. Now forming $y = e^x v$, we find the desired linearly independent solution $y = -1/x$. Thus the general solution is $y = c_1e^x + c_2/x$.

11. We first apply Theorem 4.10 to obtain the solution of the D.E.

$$y'' - 5y' + 6y = 2 - 12x. \qquad (*)$$

From the first two given D.E.'s, $F_1(x) = 1$, $F_2(x) = x$, with respective given particular integrals $f_1(x) = 1/6$, $f_2(x) = x/6 + 5/36$. The right member of (*) is $2F_1(x) - 12F_2(x)$. Then by the theorem, a particular integral of (*) is $y = 2f_1(x) - 12f_2(x) = 1/3 - 2x - 5/3 = -2x - 4/3$.

We now apply the theorem over again, this time to the third given D.E. and the D.E.(*), with $F_1(x) = 2 - 12x$, $F_2(x) = e^x$, $f_1(x) = -2x - 4/3$, $f_2(x) = e^x/2$, $k_1 = 1$, and $k_2 = 6$. Doing so, we find the particular integral $y = -2x - 4/3 + 3e^x$ of the fourth given D.E.

**Section 4.2, Page 143**

1. The auxiliary equation is $m^2 - 5m + 6 = 0$. Its roots are 2 and 3 (real and distinct). The G.S. is $y = c_1e^{2x} + c_2e^{3x}$.

4. The auxiliary equation is $3m^2 - 14m - 5 = 0$. Its roots are 5 and $-\frac{1}{3}$ (real and distinct). The G.S is $y = c_1e^{5x} + c_2e^{-x/3}$.

6. The auxiliary equation is $2m^2 + 3m - 2 = 0$. Its roots are $\frac{1}{2}$ and -2 (real and distinct). The G.S. is $y = c_1e^{x/2} + c_2e^{-2x}$.

7. The auxiliary equation is $4m^2 - 4m + 1 = 0$. Its roots are $\frac{1}{2}$ and $\frac{1}{2}$ (a double real root). The G.S. is $y = (c_1 + c_2x)e^{x/2}$.

10. The auxiliary equation is $16m^2 + 32m + 25 = 0$. We solve,using the quadratic formula, and obtain

$$m = \frac{-32 \pm \sqrt{(32)^2 - (4)(16)(25)}}{2(32)} = \frac{-32 \pm \sqrt{64(-9)}}{32}$$

$$= \frac{-32 \pm 24i}{32} = -1 \pm \frac{3}{4}\, i.$$

Thus these conjugate complex members are the roots of the auxiliary equation. The G.S. is
$y = e^{-x}(c_1 \sin 3x/4 + c_2 \cos 3x/4)$.

11. The auxiliary equation is $m^3 - 3m^2 - m + 3 = 0$, which can be written $m^2(m - 3) - (m - 3) = 0$ or $(m + 1)(m - 1)(m - 3) = 0$. Hence its roots are 1, -1, and 3 (real and distinct). The G.S. is $y = c_1 e^x + c_2 e^{-x} + c_3 e^{3x}$.

14. The auxiliary equation is $4m^3 + 4m^2 - 7m + 2 = 0$. Observe by inspection that -2 is a root. Then by synthetic division,

$$\begin{array}{r|cccc} -2 & 4 & 4 & -7 & 2 \\ & & -8 & 8 & -2 \\ \hline & 4 & -4 & 1 & \boxed{0} \end{array},$$

Find the factorization $(m + 2)(4m^2 - 4m + 1) = 0$ and hence $(m + 2)(2m - 1)^2 = 0$. Thus the roots are -2 (real simple root) $\frac{1}{2}$, $\frac{1}{2}$ (real double root). The G.S. is

$$y = c_1e^{-2x} + (c_2 + c_3x)\, e^{(1/2)x}$$

or

$$y = c_1e^{-2x} + c_2e^{(1/2)x} + c_3xe^{(1/2)x}.$$

15. The auxiliary equation is $m^3 - m^2 + m - 1 = 0$, which can be written $m^2(m - 1) + (m - 1) = 0$ or $(m^2 + 1)(m - 1) = 0$. Thus the roots are the real simple root 1 and the conjugate complex pair $\pm i$. The G.S. is $y = c_1e^x + c_2\sin x + c_3\cos x$.

16. The auxiliary equation is $m^3 + 4m^2 + 5m + 6 = 0$. Observe by inspection that $-3$ is a root. Then by synthetic division,

| | | | | |
|---|---|---|---|---|
| $-3$ | 1 | 4 | 5 | 6 |
| | | $-3$ | $-3$ | $-6$ |
| | 1 | 1 | 2 | 0 |

find the factorization $(m + 3)(m^2 + m + 2) = 0$. From $m^2 + m + 2 = 0$, obtain

$$m = \frac{-1 \pm \sqrt{1 - 8}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}\, i.$$

Thus the roots are the real simple root $-3$ and the conjugate complex roots $-\frac{1}{2} \pm \left(\frac{\sqrt{7}}{2}\right)i$. The G.S. is

$$y = c_1e^{-3x} + e^{-x/2}\left[c_2\sin\left(\frac{\sqrt{7}}{2}\right)x + c_3\cos\left(\frac{\sqrt{7}}{2}\right)x\right].$$

17. The auxiliary equation is $m^2 - 8m + 16 = 0$. Its roots are 4, 4 (real, double root). The G.S is

$$y = (c_1 + c_2x)e^{4x}$$

or

$$y = c_1e^{4x} + c_2xe^{4x}.$$

19. The auxiliary equation is $m^2 - 4m + 13 = 0$. Solving it, obtain

$$m = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i.$$

So the roots are the conjugate complex numbers $2 \pm 3i$. The G.S. is $y = e^{2x}(c_1 \sin 3x + c_2 \cos 3x)$.

22. The auxiliary equation is $4m^2 + 1 = 0$. Its roots are $m = \pm \left(\frac{1}{2}\right)i$. The G.S. is $y = c_1 \sin\left(\frac{1}{2}\right)x + c_2 \cos\left(\frac{1}{2}\right)x$.

24. The auxiliary equation is $8m^3 + 12m^2 + 6m + 1 = 0$. By inspection, we find that $-1/2$ is a root. Then by synthetic division,

$$\begin{array}{r|cccc} -1/2 & 8 & 12 & 6 & 1 \\ & & -4 & -4 & -1 \\ \hline & 8 & 8 & 2 & 0 \end{array},$$

We find the factorization $(m + 1/2)^2(8m^2 + 8m + 2) = 0$ or $2(m + 1/2)^3 = 0$. Thus $-1/2$ is a triple real root. The G.S. is $y = (c_1 + c_2x + c_3x^2)e^{-x/2}$.

27. The auxiliary equation is $m^4 + 8m^2 + 16 = 0$ or $(m^2 + 4)^2 = 0$. The roots are $\pm 2i$, $\pm 2i$. That is, each of the conjugate complex pair $\pm 2i$ is a double root. The G.S. is $y = (c_1 + c_2x)\sin 2x + (c_3 + c_4x)\cos 2x$.

28. The auxiliary equation is $m^4 - m^3 - 3m^2 + m + 2 = 0$. Observe by inspection that $-1$ is a root. Then by synthetic division

$$\begin{array}{c|ccccc} -1 & 1 & -1 & -3 & 1 & 2 \\ & & -1 & 2 & 1 & -2 \\ \hline & 1 & -2 & -1 & 2 & 0 \end{array}$$

we find the factorization $(m + 1)(m^3 - 2m^2 - m + 2) = 0$ which can be written $(m + 1)[m^2(m - 2) - (m - 2)] = 0$ or $(m + 1)(m^2 - 1)(m - 2) = 0$ or $(m + 1)^2(m - 1)(m - 2) = 0$. Thus the roots are 1, 2 (real and distinct) and $-1$, $-1$ (real double root). The G.S. is

$$y = c_1e^x + c_2e^{2x} + (c_3 + c_4x)e^{-x}$$

or

$$y = c_1e^x + c_2e^{2x} + c_3e^{-x} + c_4xe^{-x}.$$

30. The auxiliary equation is $m^4 + 6m^3 + 15m^2 + 20m + 12 = 0$. Observe by inspection that $-2$ is a root. Then by synthetic division,

$$\begin{array}{c|ccccc} -2 & 1 & 6 & 15 & 20 & 12 \\ & & -2 & -8 & -14 & -12 \\ \hline & 1 & 4 & 7 & 6 & 0 \end{array}$$

find the factorization $(m + 2)(m^3 + 4m^2 + 7m + 6) = 0$. Now observe by inspection that $-2$ is also a root of the reduced cubic $m^3 + 4m^2 + 7m + 6 = 0$. Then by synthetic division,

$$\begin{array}{r|cccc} -2 & 1 & 4 & 7 & 6 \\ & & -2 & -4 & -6 \\ \hline & 1 & 2 & 3 & 0 \end{array}$$

find the factorization $(m + 2)(m^2 + 2m + 3) = 0$ of the reduced cubic. Hence the auxiliary equation has the factored form $(m + 2)^2(m^2 + 2m + 3) = 0$. The factor $(m + 2)^2$ yields the real double root $-2$, $-2$. The factor $m^2 + 2m + 3 = 0$ gives the conjugate complex roots

$$m = \frac{-2 \pm \sqrt{4 - 12}}{2} = \frac{-2 \pm 2\sqrt{2}i}{2} = -1 \pm \sqrt{2}i.$$

Thus the roots are $-2$, $-2$, $-1 \pm \sqrt{2}i$. The G.S. is

$$y = (c_1 + c_2x)e^{-2x} + e^{-x}(c_3\sin\sqrt{2}x + c_4\cos\sqrt{2}x)$$

or

$$y = c_1e^{-2x} + c_2xe^{-2x} + e^{-x}(c_3\sin\sqrt{2}x + c_4\cos\sqrt{2}x).$$

31. The auxiliary equation is $m^5 - 2m^4 + m^3 = 0$. Factoring gives $m^3(m^2 - 2m + 1) = 0$ or $m^3(m - 1)^2 = 0$. Thus the roots are 0, 0, 0 [triple root, from factor $m^3$], 1, 1, [double root, from factor $(m - 1)^2$]. The G.S. is

$$y = (c_1 + c_2x + c_3x^2)e^{0x} + (c_4 + c_5x)e^x,$$

that is,

$$y = c_1 + c_2x + c_3x^2 + (c_4 + c_5x)e^x$$

or

$$y = c_1 + c_2x + c_3x^2 + c_4e^x + c_5xe^x.$$

32. The auxiliary equation is $m^5 + 5m^4 + 10m^3 + 10m^2 + 5m + 1 = 0$. We recognize this as a binomial expansion, and write it as $(m + 1)^5 = 0$. Thus $-1$ is a five-fold root of the auxiliary equation. The general solution of the D.E. is $y = (c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4)e^{-x}$.

34. The auxiliary equation is $m^6 - 2m^3 + 1 = 0$ or $(m^3 - 1)^2 = 0$ or $(m - 1)^2(m^2 + m + 1)^2 = 0$. From $(m - 1)^2 = 0$, we find the double real root 1. Solving the quadratic $m^2 + m + 1 = 0$, we obtain

$$m = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i,$$

which is a conjugate complex pair. Thus the roots of the auxiliary equation are $1, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i, -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i, -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i, -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$. Note that each root is double. The general solution of the D.E. is $y = (c_1 + c_2x)e^x + e^{-x/2}\left[(c_3 + c_4x)\sin\frac{\sqrt{3}}{2}x + (c_5 + c_6x)\cos\frac{\sqrt{3}}{2}x\right]$.

35. The auxiliary equation is $m^4 + 1 = 0$. Hence we seek m such that $m^4 = -1$; that is, we seek the four fourth roots of $z = -1$. We express this in the so-called polar form $z = 1(\cos \pi + i \sin \pi)$. From complex number theory, we know that the n nth roots of $z = r(\cos \theta + i \sin \theta)$ are given by the formula

$$z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$

$$k = 0,1,2,\cdots,n - 1. \qquad (*)$$

Since $z = 1(\cos \pi + i \sin \pi)$, we have $r = 1$, $\theta = \pi$ here; and since we want the four fourth roots, we take $n = 4$. Thus the formula (*) becomes

$$z^{1/4} = \sqrt[4]{1}\left[\cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right)\right], \quad k = 0,1,2,3.$$

Letting k = 0, 1, 2, and 3 successively in this, we find

$$\begin{cases} \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}, \\ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}, \\ \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}, \\ \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}, \end{cases}$$

respectively. These are the four fourth roots of -1. Thus they are the four roots of the auxiliary equation

$m^4 + 1 = 0$. Observe that they are in fact the two pairs of conjugate complex numbers $\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$.

Alternatively, the auxiliary equation is $m^4 + 1 = 0$, which can be written as $(m^4 + 2m^2 + 1) - 2m^2 = 0$ or $(m^2 + 1)^2 - (\sqrt{2}m)^2 = 0$. Factoring this as a difference of squares gives $(m^2 + 1 - \sqrt{2}m)(m^2 + 1 + \sqrt{2}m) = 0$ or $(m^2 - \sqrt{2}m + 1)(m^2 + \sqrt{2}m + 1) = 0$. The factor $m^2 - \sqrt{2}m + 1$ gives the conjugate complex roots

$$m = \frac{\sqrt{2} \pm \sqrt{2 - 4}}{2} = \frac{\sqrt{2} \pm \sqrt{2}i}{2} = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i,$$

while the factor $m^2 + \sqrt{2}m + 1$ gives the conjugate complex roots

$$m = \frac{-\sqrt{2} \pm \sqrt{2 - 4}}{2} = \frac{-\sqrt{2} \pm \sqrt{2}i}{2} = -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i.$$

Thus the roots are $\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$.

Thus the G.S. of the D.E. is

$$y = e^{\frac{\sqrt{2}}{2}x}\left(c_1 \sin \frac{\sqrt{2}}{2}x + c_2 \cos \frac{\sqrt{2}}{2}x\right) + e^{-\frac{\sqrt{2}}{2}x}\left(c_3 \sin \frac{\sqrt{2}}{2}x + c_4 \cos \frac{\sqrt{2}}{2}x\right).$$

36. The auxiliary equation is $m^6 + 64 = 0$. So we seek m such that $m^6 = -64$; that is, we seek the six sixth roots of $z = -64$. In the so-called polar form, this is $z = 64(\cos \pi + i \sin \pi)$. We use the formula (*) given in the solution of Exercise 35, with $r = 64$, $\theta = \pi$, and $n = 6$. The formula becomes

$$z^{1/6} = \sqrt[6]{64}\left[\cos\left(\frac{\pi + 2k\pi}{6}\right) + i \sin\left(\frac{\pi + 2k\pi}{6}\right)\right], \qquad k = 0,1,2,\cdots,5.$$

Letting k = 0, 1, 2, 3, 4, and 5 successively in this, we find

$$2\left[\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right] = 2\left[\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right] = \sqrt{3} + i,$$

$$2\left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right] = 2[0 + i(1)] = 2i,$$

$$2\left[\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right] = 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right] = -\sqrt{3} + i,$$

$$2\left[\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right] = 2\left[-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right] = -\sqrt{3} - i,$$

$$2\left[\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right] = 2[0 + i(-1)] = -2i,$$

$$2\left[\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right] = 2\left[\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right] = \sqrt{3} - i,$$

respectively. These are the six sixth roots of -64. Thus they are the six roots of the auxiliary equation $m^6 + 64 = 0$. Note that they are in fact the three pairs of

conjugate complex numbers $\pm 2i$, $\sqrt{3} \pm i$, and $-\sqrt{3} \pm i$. Thus the G.S. of the D.E. is

$$y = c_1 \sin 2x + c_2 \cos 2x + e^{\sqrt{3}x}(c_3 \sin x + c_4 \cos x) + e^{-\sqrt{3}x}(c_5 \sin x + c_6 \cos x).$$

37. The auxiliary equation is $m^2 - m - 12 = 0$. Its roots are 4, -3 (real and distinct). The G.S. is

$$y = c_1 e^{4x} + c_2 e^{-3x}. \qquad (A)$$

From this,

$$y' = 4c_1 e^{4x} - 3c_2 e^{-3x}. \qquad (B)$$

Apply condition $y(0) = 3$ to (A) to obtain $c_1 + c_2 = 3$.
Apply condition $y'(0) = 5$ to (B) to obtain $4c_1 - 3c_2 = 5$.
From these two equations in $c_1$ and $c_2$, find $c_1 = 2$, $c_2 = 1$. Thus the solution of the stated I.V.P. is $y = 2e^{4x} + e^{-3x}$.

40. The auxiliary equation is $3m^2 + 4m - 4 = 0$, which can be written $(3m - 2)(m + 2) = 0$. Its roots are 2/3, -2 (real and distinct). The G.S. is

$$y = c_1 e^{\frac{2}{3}x} + c_2 e^{-2x}. \qquad (A)$$

From this,

$$y' = \frac{2}{3} c_1 e^{\frac{2}{3}x} - 2c_2 e^{-2x}. \qquad (B)$$

Apply condition $y(0) = 2$ to (A) to obtain $c_1 + c_2 = 2$. Apply condition $y'(0) = -4$ to (B) to obtain $\frac{2}{3} c_1 - 2c_2 = -4$. From these two equations in $c_1$ and $c_2$, find $c_1 = 0$, $c_2 = 2$. Thus the solution of the stated I.V.P. is $y = 2e^{-2x}$.

41. The auxiliary equation is $m^2 + 6m + 9 = 0$. Its roots are $-3, -3$ (real double root). The G.S. is

$$y = (c_1 + c_2 x)e^{-3x}. \qquad (A)$$

From this,

$$y' = (-3c_1 + c_2 - 3c_2 x)e^{-3x}. \qquad (B)$$

Apply condition $y(0) = 2$ to (A) to obtain $c_1 = 2$. Apply condition $y'(0) = -3$ to (B) to obtain $-3c_1 + c_2 = -3$. From this, find $c_2 = 3$. Thus the solution of the stated I.V.P. is $y = (2 + 3x)e^{-3x}$.

44. The auxiliary equation is $9m^2 - 6m + 1 = 0$, which can be written $(3m - 1)^2 = 0$. Its roots are $1/3, 1/3$ (real double root). The G.S. is

$$y = (c_1 + c_2x)e^{\frac{1}{3}x} \qquad (A)$$

From this,

$$y' = \left(\frac{1}{3}\,c_1 + c_2 + \frac{1}{3}\,c_2x\right)e^{\frac{1}{3}x} \qquad (B)$$

Apply condition $y(0) = 3$ to (A) to obtain $c_1 = 3$. Apply condition $y'(0) = -1$ to (B) to obtain $\frac{1}{3}\,c_1 + c_2 = -1$. From this, find $c_2 = -2$. Thus the solution of the stated I.V.P is $y = (3 - 2x)e^{\frac{1}{3}x}$.

45. The auxiliary equation is $m^2 - 4m + 29 = 0$. Solving, obtain the conjugate complex roots

$$m = \frac{4 \pm \sqrt{16 - 116}}{2} = \frac{4 \pm 10i}{2} = 2 \pm 5i.$$

The G.S. is

$$y = e^{2x}(c_1 \sin 5x + c_2 \cos 5x) \qquad (A)$$

From this,

$$y' = e^{2x}[(2c_1 - 5c_2)\sin 5x + (5c_1 + 2c_2)\cos 5x]. \qquad (B)$$

Apply condition $y(0) = 0$ to (A) to obtain

$$e^0(c_1 \sin 0 + c_2 \cos 0) = 0 \quad \text{or} \quad c_2 = 0.$$

Apply condition $y'(0) = 5$ to (B) to obtain

$$e^0[(2c_1 - 5c_2)\sin 0 + (5c_1 + 2c_2)\cos 0] = 5$$

or

$$5c_1 + 2c_2 = 5.$$

From this, $c_1 = 1$. Thus the solution of the stated I.V.P. is $y = e^{2x}\sin 5x$.

49. The auxiliary equation is $9m^2 + 6m + 5 = 0$. Solving, obtain the conjugate complex roots

$$m = \frac{-6 \pm \sqrt{36 - (36)(5)}}{18} = \frac{-6 \pm 12i}{18} = -\frac{1}{3} \pm \frac{2}{3}\,i.$$

The G.S. is

$$y = e^{-\frac{1}{3}x}\left(c_1\sin\frac{2}{3}x + c_2\cos\frac{2}{3}x\right). \qquad (A)$$

From this,

$$y' = e^{-\frac{1}{3}x}\left[\left(-\frac{1}{3}c_1 - \frac{2}{3}c_2\right)\sin\frac{2}{3}x + \left(\frac{2}{3}c_1 - \frac{1}{3}c_2\right)\cos\frac{2}{3}x\right]. \qquad (B)$$

Apply condition $y(0) = 6$ to (A) to obtain $c_2 = 6$. Apply condition $y'(0) = 0$ to (B) to obtain $\frac{2}{3}c_1 - \frac{1}{3}c_2 = 0$.

From this, $c_1 = 3$. Thus the solution of the stated I.V.P. is

$$y = e^{-\frac{1}{3}x}\left(3\sin\frac{2}{3}x + 6\cos\frac{2}{3}x\right)$$

or

$$y = 3e^{-\frac{1}{3}x}\left(\sin\frac{2}{3}x + 2\cos\frac{2}{3}x\right).$$

50. The auxiliary equation is $4m^2 + 4m + 37 = 0$. Solving, obtain the conjugate complex roots

$$m = \frac{-4 \pm \sqrt{16 - 16(37)}}{8} = \frac{-4 \pm 24i}{8} = -\frac{1}{2} \pm 3i.$$

The G.S. is $y = e^{-x/2}(c_1\sin 3x + c_2\cos 3x)$. (A). From this, $y' = e^{-x/2}\left[\left(-\frac{1}{2}c_1 - 3c_2\right)\sin 3x + \left(3c_1 - \frac{c_2}{2}\right)\cos 3x\right]$. (B). Apply condition $y(0) = 2$ to (A) to obtain $c_2 = 2$. Apply condition $y'(0) = -4$ to (B) to obtain $3c_1 - \frac{c_2}{2} = -4$. From this, $c_1 = -1$. Thus the solution of the stated I.V.P. is $y = e^{-x/2}(-\sin 3x + 2\cos 3x)$.

51. The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$. Observe by inspection that 1 is a root. Then by synthetic division,

| | | | | |
|---|---|---|---|---|
| 1 | 1 | -6 | 11 | -6 |
| | | 1 | -5 | 6 |
| | 1 | -5 | 6 | 0 |

find the factorization $(m - 1)(m^2 - 5m + 6) = 0$, which can be written $(m - 1)(m - 2)(m - 3) = 0$. Thus the roots are 1, 2, 3 (real and distinct). The G.S. is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} \qquad (A)$$

From this,

$$y' = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x} \qquad (B)$$

$$y'' = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x} \qquad (C)$$

Apply condition $y(0) = 0$ to (A) to obtain $c_1 + c_2 + c_3 = 0$. Apply condition $y'(0) = 0$ to (B) to obtain $c_1 + 2c_2 + 3c_3 = 0$. Apply condition $y''(0) = 2$ to (C) to obtain $c_1 + 4c_2 + 9c_3 = 2$. The solution of these three equations in $c_1$, $c_2$, $c_3$ is $c_1 = 1$, $c_2 = -2$, $c_3 = 1$. Thus the solution of the stated I.V.P. is $y = e^x - 2e^{2x} + e^{3x}$.

53. The auxiliary equation is $m^3 - 3m^2 + 4 = 0$. Observe by inspection that $-1$ is a root. Then by synthetic division,

| | | | | |
|---|---|---|---|---|
| $-1$ | 1 | $-3$ | 0 | 4 |
| | | $-1$ | 4 | $-4$ |
| | 1 | $-4$ | 4 | 0 |

find the factorization $(m + 1)(m^2 - 4m + 4) = 0$, which can be written $(m + 1)(m - 2)^2 = 0$. Thus the roots are -1 (real simple root) and 2, 2 (real double root). The G.S. is

$$y = c_1e^{-x} + (c_2 + c_3x)e^{2x} \qquad (A)$$

From this,

$$y' = -c_1e^{-x} + (2c_2 + c_3 + 2c_3x)e^{2x} \qquad (B)$$

$$y'' = c_1e^{-x} + (4c_2 + 4c_3 + 4c_3x)e^{2x} \qquad (C)$$

Apply condition $y(0) = 1$ to (A) to obtain $c_1 + c_2 = 1$. Apply condition $y'(0) = -8$ to (B) to obtain $-c_1 + 2c_2 + c_3 = -8$. Apply condition $y''(0) = -4$ to (C) to obtain $c_1 + 4c_2 + 4c_3 = -4$. The solution of these three equations in $c_1$, $c_2$, $c_3$ is $c_1 = 32/9$, $c_2 = -23/9$, $c_3 = 2/3$. Thus the solution of the stated I.V.P. is

$$y = \frac{32}{9}e^{-x} + \left(-\frac{23}{9} + \frac{2}{3}x\right)e^{2x}$$

or

$$y = \frac{32}{9}e^{-x} - \frac{23}{9}e^{2x} + \frac{2}{3}xe^{2x}.$$

54. The auxiliary equation is $m^3 - 5m^2 + 9m - 5 = 0$. Observe by inspection that 1 is a root. Then by synthetic division,

| | | | | |
|---|---|---|---|---|
| 1 | 1 | -5 | 9 | -5 |
| | | 1 | -4 | 5 |
| | 1 | -4 | 5 | 0 |

Find the factorization $(m - 1)(m^2 - 4m + 5) = 0$. From $m^2 - 4m + 5 = 0$, obtain

$$m = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i.$$

Thus the roots are the real simple root 1 and the conjugate complex roots $2 \pm i$. The G.S. is

$$y = c_1 e^x + e^{2x}(c_2 \sin x + c_3 \cos x). \tag{A}$$

From this,

$$y' = c_1 e^x + e^{2x}[(2c_2 - c_3)\sin x + (c_2 + 2c_3)\cos x], \tag{B}$$

$$y'' = c_1 e^x + e^{2x}[(3c_2 - 4c_3)\sin x + (4c_2 + 3c_3)\cos x] \tag{C}$$

Apply condition $y(0) = 0$ to (A) to obtain $c_1 + c_3 = 0$. Apply condition $y'(0) = 1$ to (B) to obtain $c_1 + c_2 + 2c_3 = 1$. Apply condition $y''(0) = 6$ to (C) to obtain $c_1 + 4c_2 + 3c_3 = 6$. The solution of these three equations in $c_1$, $c_2$, $c_3$ is $c_1 = 1$, $c_2 = 2$, $c_3 = -1$. Thus the solution of the stated I.V.P. is $y = e^x + e^{2x}(2 \sin x - \cos x)$.

55. The auxiliary equation is $m^4 - 3m^3 + 2m^2 = 0$ or $m^2(m - 1)(m - 2)$, and so its roots are $m = 0, 0, 1, 2$. Thus the general solution of the D.E. is

$$y = c_1 + c_2x + c_3e^x + c_4e^{2x}. \tag{1}$$

Its first three derivatives are

$$y' = c_2 + c_3e^x + 2c_4e^{2x}, \tag{2}$$

$$y'' = c_3e^x + 4c_4e^{2x}, \tag{3}$$

and

$$y''' = c_3e^x + 8c_4e^{2x}, \tag{4}$$

respectively. Apply I.C. $y(0) = 2$ to (1), I.C. $y'(0) = 0$ to (2), I.C. $y''(0) = 2$ to (3), and I.C. $y'''(0) = 2$ to (4). We obtain respectively

$$c_1 + c_3 + c_4 = 2, \qquad c_2 + c_3 + 2c_4 = 0,$$
$$c_3 + 4c_4 = 2, \qquad \text{and} \qquad c_3 + 8c_4 = 2.$$

From the last two of these four, $c_3 = 2$, $c_4 = 0$. Then the second of the four gives $c_2 = -2 + 0 = -2$. Finally, the first equation gives $c_1 = 2 - 2 + 0 = 0$. Substituting these values into (1), we have the solution $y = -2x + 2e^x$.

59. Since 4 is the 4-fold real root of the auxiliary equation, the part of the general solution corresponding to it is

$$(c_1 + c_2x + c_3x^2 + c_4x^3)e^{4x}.$$

Since the conjugate complex numbers $2 + 3i$ and $2 - 3i$ are each 3-fold roots of the auxiliary equation, the corresponding part of the general solution is

$$e^{2x}[(c_5 + c_6x + c_7x^2)\sin 3x + (c_8 + c_9x + c_{10}x^2)\cos 3x].$$

Thus the general solution is

$$\begin{aligned} y = {} & (c_1 + c_2x + c_3x^2 + c_4x^3)e^{4x} \\ & + e^{2x}[(c_5 + c_6x + c_7x^2)\sin 3x \\ & + (c_8 + c_9x + c_{10}x^2)\cos 3x]. \end{aligned}$$

61. Since $\sin x$ is a solution of the D.E., $m = \pm i$ must be roots of the auxiliary equation $m^4 + 2m^3 + 6m^2 + 2m + 5 = 0$, and hence $(m - i)(m + i) = m^2 + 1$ must be a factor of $m^4 + 2m^3 + 6m^2 + 2m + 5$. By long division, we find the other factor is $m^2 + 2m + 5$. Hence in factored form the auxiliary equation is

$$(m^2 + 1)(m^2 + 2m + 5) = 0.$$

The factor $m^2 + 2m + 5 = 0$ gives

$$m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

Thus the roots of the auxiliary equation are the two pairs of conjugate complex numbers $\pm i$ and $-1 \pm 2i$. The G.S. of the D.E. is

$$y = c_1\sin x + c_2\cos x + e^{-x}(c_3\sin 2x + c_4\cos 2x).$$

62. Since $e^x \sin 2x$ is a solution of the D.E., $m = 1 \pm 2i$ must be roots of the auxiliary equation $m^4 + 3m^3 + m^2 + 13m + 30 = 0$, and hence $[m - (1 + 2i)][m - (1 - 2i)] = m^2 - 2m + 5$ must be a factor of $m^4 + 3m^3 + m^2 + 13m + 30$. By long division, we find the other factor is $m^2 + 5m + 6$. Since $m^2 + 5m + 6 = (m + 2)(m + 3)$, the auxiliary equation in factored form is $(m + 2)(m + 3)(m^2 - 2m + 5) = 0$; and its roots are $m = -2, -3, 1 \pm 2i$. Thus the general solution is $y = c_1e^{-2x} + c_2e^{-3x} + e^x(c_3 \sin 2x + c_4 \cos 2x)$.

**Section 4.3, Page 159**

Exercises 1 through 24 follow the pattern of Examples 4.36, 4.37 and 4.38 on Pages 154-159 of the text. The five steps of the method outlined on pg. 152 are indicated in each solution.

1. The corresponding homogeneous D.E. is $y'' - 3y' + 2y = 0$. The auxiliary equation is $m^2 - 3m + 2 = 0$, with roots 1, 2. The complementary function is $y_c = c_1e^x + c_2e^{2x}$. The NH term is a constant multiple of the UC function given by $x^2$.

**Step 1**: Form the UC set of $x^2$. It is $S_1 = \{x^2, x, 1\}$.

**Step 2**: This step does not apply, since there is only one UC set present.

**Step 3**: An examination of the complementary function shows that none of the functions in $S_1$ is a solution of

the corresponding homogeneous D.E. Hence $S_1$ does not need revision.

**Step 4**: Thus the original set $S_1$ remains. Form a linear combination of its three members.

**Step 5**: Thus we take

$$y_p = Ax^2 + Bx + C$$

as a particular solution. Then

$$y_p' = 2Ax + B, \quad y_p'' = 2A.$$

We substitute in the D.E., obtaining

$$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2 \quad \text{or}$$

$$2Ax^2 + (-6A + 2B)x + (2A - 3B + 2C) = 4x^2.$$

We equate coefficients of like terms on both sides of this to obtain

$$2A = 4, \ -6A + 2B = 0, \ 2A - 3B + 2C = 0.$$

From these, we find $A = 2$, $B = 6$, $C = 7$. Thus we obtain the particular integral

$$y_p = 2x^2 + 6x + 7.$$

The G.S. of the D.E.is

$$y = c_1e^x + c_2e^{2x} + 2x^2 + 6x + 7.$$

4. The corresponding homogeneous D.E. is $y'' + 2y' + 2y = 0$. The auxiliary equation is $m^2 + 2m + 2 = 0$, with roots $-1 \pm i$. The complementary function is $y_c = e^{-x}(c_1 \sin x + c_2 \cos x)$. The NH term is a constant multiple of the UC function given by $\sin 4x$.

**Step 1:** Form the UC set of $\sin 4x$. It is $S_1 = \{\sin 4x, \cos 4x\}$.

**Step 2:** This step does not apply.

**Step 3:** An examination of the complementary function shows that none of the functions in $S_1$ is a solution of the corresponding homogeneous D.E. Hence $S_1$ does not need revision.

**Step 4:** Thus the original set $S_1$ remains. Form a linear combination of its two members.

**Step 5:** Thus we take $y_p = A \sin 4x + B \cos 4x$ as a particular solution. Then

$$y_p' = 4A \cos 4x - 4B \sin 4x.$$
$$y_p'' = -16A \sin 4x - 16B \cos 4x.$$

We substitute in the D.E., obtaining

$-16A \sin 4x - 16B \cos 4x + 8A \cos 4x - 8B \sin 4x$

$+ 2A \sin 4x + 2B \cos 4x = 10 \sin 4x$ or

$(-14A - 8B)\sin 4x + (8A - 14B)\cos 4x = 10 \sin 4x.$

We equate coefficients of like terms on both sides of this to obtain

$$-14A - 8B = 10, \quad 8A - 14B = 0.$$

From these, we find $A = -\frac{7}{13}$, $B = -\frac{4}{13}$. Thus we obtain the particular integral

$$y_p = -\left(\frac{7}{13}\right) \sin 4x - \left(\frac{4}{13}\right) \cos 4x.$$

The G.S. of the D.E.is

$$y = e^{-x}(c_1 \sin x + c_2 \cos x) - \left(\frac{7}{13}\right) \sin 4x - \left(\frac{4}{13}\right) \cos 4x.$$

8. The corresponding homogeneous D.E. is $y'' + 2y' + 10y = 0$. The auxiliary equation is $m^2 + 2m + 10 = 0$, with roots $-1 \pm 3i$. The complementary function is $y_c = e^{-x}(c_1 \sin 3x + c_2 \cos 3x)$. The NH term is a constant multiple of the UC function given by $xe^{-2x}$

**Step 1**: Form the UC set of $xe^{-2x}$. It is

$S_1 = \{xe^{-2x}, e^{-2x}\}$.

**Step 2**: This step does not apply.

**Step 3**: An examination of the complementary function shows that none of the functions in $S_1$ is a solution of the corresponding homogeneous D.E. Hence $S_1$ does not need revision.

**Step 4**: Thus the original set $S_1$ remains. Form a linear combination of its two members.

**Step 5**: Thus we take $y_p = Axe^{-2x} + Be^{-2x}$ as a particular solution. Then

$$y_p' = -2Axe^{-2x} + (A - 2B)e^{-2x},$$

$$y_p'' = 4Axe^{-2x} + (-4A + 4B)e^{-2x}.$$

We substitute into the D.E., obtaining

$$4Axe^{-2x} + (-4A + 4B)e^{-2x} - 4Axe^{-2x} + (2A - 4B)e^{-2x} + 10Axe^{-2x} + 10Be^{-2x} = 5xe^{-2x}$$

or

$$10Axe^{-2x} + (-2A + 10B)e^{-2x} = 5xe^{-2x}.$$

We equate coefficients of like terms on both sides of this to obtain $10A = 5$, $-2A + 10B = 0$. From these, we find $A = \frac{1}{2}$, $B = \frac{1}{10}$. Thus we obtain the particular integral

$$y_p = \frac{xe^{-2x}}{2} + \frac{e^{-2x}}{10}.$$

The G.S. of the D.E.is

$$y = e^{-x}(c_1 \sin 3x + c_2 \cos 3x) + \frac{xe^{-2x}}{2} + \frac{e^{-2x}}{10}.$$

11. The corresponding homogeneous D.E. is $y'' + 4y = 0$. The auxiliary equation is $m^2 + 4 = 0$, with conjugate complex roots $m = \pm 2i$. The complementary function is $y_c = c_1 \sin 2x + c_2 \cos 2x$. The NH member is a linear combination of the UC functions $\sin 2x$ and $\cos 2x$.

**Step 1:** Form the UC set of each of these two UC functions:
$S_1 = \{\sin 2x, \cos 2x\}$, $S_2 = \{\cos 2x, \sin 2x\}$.

**Step 2:** Each set is identical with the other, so we only keep one of them, say $S_1$.

**Step 3:** Observe that each member of $S_1$ is included in the complementary function and so is a solution of the corresponding homogeneous equation. Thus we multiply each member of $S_1$ by $x$ to obtain the revised set $S_1' = \{x \sin 2x, x \cos 2x\}$, whose members are not solutions of the homogeneous D.E.

**Step 4:** We now have only the revised set $S_1' = \{x \sin 2x, x \cos 2x\}$. We form a linear combination $Ax \sin 2x + Bx \cos 2x$ of its two elements.

**Step 5:** Thus we take $y_p = Ax \sin 2x + Bx \cos 2x$ as a particular integral. Then

$$y_p' = 2Ax \cos 2x + A \sin 2x - 2Bx \sin 2x + B \cos 2x,$$
$$y_p'' = -4Ax \sin 2x + 4A \cos 2x - 4Bx \cos 2x - 4B \sin 2x.$$

We substitute into the D.E., obtaining

$$[-4Ax \sin 2x + 4A \cos 2x - 4Bx \cos 2x - 4B \sin 2x] + 4[Ax \sin 2x + Bx \cos 2x] = 4 \sin 2x + 8 \cos 2x$$

or $$4A \cos 2x - 4B \sin 2x = 4 \sin 2x + 8 \cos 2x.$$

We equate coefficients of like terms on both sides of this to obtain $4A = 8$, $-4B = 4$. From this, $A = 2$, $B = -1$. Thus we obtain the particular integral $y_p = 2x \sin 2x - x \cos 2x$. The G.S. is

$$y = c_1 \sin 2x + c_2 \cos 2x + 2x \sin 2x - x \cos 2x.$$

12. The corresponding homogeneous D.E. is $y'' - 4y = 0$. The auxiliary equation is $m^2 - 4 = 0$ or $(m + 2)(m - 2) = 0$, with roots $m = 2, -2$. The complementary function is $y_c = c_1 e^{2x} + c_2 e^{-2x}$. The NH member is a constant multiple of the UC functions $xe^{2x}$.

**Step 1**: Form the UC set of $xe^{2x}$: $S_1 = \{xe^{2x}, e^{2x}\}$.

**Step 2**: This step does not apply.

**Step 3**: Observe that the member of $e^{2x}$ of $S_1$ is included in the complementary function and so is a solution of the corresponding homogeneous equation. Thus we multiply each member of $S_1$ by $x$ to obtain the revised set $S_1' = \{x^2 e^{2x}, xe^{2x}\}$, whose members are not solutions of the homogeneous D.E.

**Step 4:** We now have only the revised set $S_1' = \{x^2e^{2x}, xe^{2x}\}$. We form a linear combination $Ax^2e^{2x} + Bxe^{2x}$ of its two elements.

**Step 5:** Thus we take $y_p = Ax^2e^{2x} + Bxe^{2x}$ as a particular integral. Then

$$y_p' = 2Ax^2e^{2x} + (2A + 2B)xe^{2x} + Be^{2x},$$
$$y_p'' = 4Ax^2e^{2x} + (8A + 4B)xe^{2x} + (2A + 4B)e^{2x}.$$

We substitute into the D.E., obtaining

$$[4Ax^2e^{2x} + (8A + 4B)xe^{2x} + (2A + 4B)e^{2x}] - 4[Ax^2e^{2x} + Bxe^{2x}] = 16xe^{2x} \quad \text{or}$$
$$8Axe^{2x} + (2A + 4B)e^{2x} = 16xe^{2x}.$$

We equate coefficients of like terms on both sides of this to obtain $8A = 16$, $2A + 4B = 0$. From this, we find $A = 2$, $B = -1$. Thus we obtain the particular integral $y_p = 2x^2e^{2x} - xe^{2x}$. The G.S. is $y = c_1e^{2x} + c_2e^{-2x} + 2x^2e^{2x} - xe^{2x}$.

15. The corresponding homogeneous D.E. is $y''' + 4y'' + y' - 6y = 0$. The auxiliary equation is $m^3 + 4m^2 + m - 6 = 0$. By inspection note that $m = 1$ is a root. From this (by synthetic division or otherwise) we obtain the factored form $(m - 1)(m + 2)(m + 3)$. Thus the roots of the auxiliary equation are $m = 1, -2, -3$. The complementary function is $y_c = c_1e^x + c_2e^{-2x} + c_3e^{-3x}$. The NH term is a linear combination of the UC functions $x^2$ and 1.

**Step 1:** Form the UC set of each of these two UC functions: $S_1 = \{x^2, x, 1\}$, $S_2 = \{1\}$.

**Step 2:** Set $S_2$ is completely included in $S_1$, so $S_2$ is omitted, leaving just $S_1$.

**Step 3:** An examination of the complementary function shows that none of the functions in $S_1$ is a solution of the corresponding homogeneous D.E. Hence $S_1$ does not need revision.

**Step 4:** Thus the original set $S_1$ remains. We form a linear combination of its three members.

**Step 5:** Thus we take $y_p = Ax^2 + Bx + C$. Then $y_p' = 2Ax + B$, $y_p'' = 2A$, $y_p''' = 0$. We substitute into the D.E., obtaining

$$0 + 4(2A) + (2Ax + B) - 6(Ax^2 + Bx + C) = -18x^2 + 1$$

or

$$-6Ax^2 + (2A - 6B)x + (8A + B - 6C) = -18x^2 + 1.$$

We equate coefficients of like terms on both sides of this to obtain $-6A = -18$, $2A - 6B = 0$, $8A + B - 6C = 1$. From these, we find $A = 3$, $B = 1$, $C = 4$. Thus we obtain the particular solution

$$y_p = 3x^2 + x + 4.$$

The G.S. of the D.E. is

$$y = c_1e^x + c_2e^{-2x} + c_3e^{-3x} + 3x^2 + x + 4.$$

17. The corresponding homogeneous D.E. is $y''' + y'' + 3y' - 5y = 0$. The auxiliary equation is $m^3 + m^2 + 3m - 5 = 0$. By inspection note that $m = 1$ is a root. From this (by synthetic division or otherwise) we obtain the factored form $(m - 1)(m^2 + 2m + 5) = 0$. From this, the roots of the auxiliary equation are 1 and $-1 \pm 2i$. The complementary function is $y_c = c_1e^x + e^{-x}(c_2\sin 2x + c_3\cos 2x)$. The NH term is a linear combination of the UC functions given by $\sin 2x$, $x^2$, $x$, and 1.

**Step 1:** Form the UC set of each of these four UC functions: $S_1 = \{\sin 2x, \cos 2x\}$, $S_2 = \{x^2, x, 1\}$, $S_3 = \{x, 1\}$, $S_4 = \{1\}$.

**Step 2:** Sets $S_3$ and $S_4$ are completely included in $S_2$; so $S_3$ and $S_4$ are omitted, leaving the two sets $S_1$ and $S_2$.

**Step 3:** An examination of the complementary function shows that none of the functions in $S_1$ or $S_2$ is a solution of the corresponding homogeneous D.E. Hence neither $S_1$ nor $S_2$ needs revision.

**Step 4:** Thus the original sets $S_1$ and $S_2$ remain. We form a linear combination of their five members.

**Step 5:** Thus we take

$$y_p = A \sin 2x + B \cos 2x + Cx^2 + Dx + E$$

as a particular solution. Then

$$y_p' = 2A \cos 2x - 2B \sin 2x + 2Cx + D,$$

$$y_p'' = -4A \sin 2x - 4B \cos 2x + 2C,$$

$$y_p''' = -8A \cos 2x + 8B \sin 2x.$$

We substitute in the D.E., obtaining

$$\begin{aligned}
&(-8A \cos 2x + 8B \sin 2x) + (-4A \sin 2x - 4B \cos 2x + C) \\
&\quad + 3(2A \cos 2x - 2B \sin 2x + 2Cx + D) \\
&\quad - 5(A \sin 2x + B \cos 2x + Cx^2 + Dx + E) \\
&= 5 \sin 2x + 10x^2 + 3x + 7 \qquad \text{or} \\
&(-9A + 2B)\sin 2x + (-2A - 9B)\cos 2x \\
&\quad - 5Cx^2 + (6C - 5D)x + (2C + 3D - 5E) \\
&= 5 \sin 2x + 10x^2 + 3x + 7.
\end{aligned}$$

We equate coefficients of like terms on both sides of this to obtain $-9A + 2B = 5$, $-2A - 9B = 0$, $-5C = 10$, $6C - 5D = 3$, $2C + 3D - 5E = 7$. From these, we find $A = -\frac{9}{17}$, $B = \frac{2}{17}$, $C = -2$, $D = -3$, $E = -4$. Thus we obtain the particular integral

$$y_p = -\frac{9 \sin 2x}{17} + \frac{2 \cos 2x}{17} - 2x^2 - \frac{9x}{5} - \frac{82}{25}.$$

The G.S. of the D.E. is

$$y = c_1e^x + e^{-x}(c_2\sin 2x + c_3\cos 2x) - \frac{9\sin 2x}{17} + \frac{2\cos 2x}{17} - 2x^2 - 3x - 4.$$

19. The corresponding homogeneous D.E. is $y'' + y' - 6y = 0$. The auxiliary equation is $m^2 + m - 6 = 0$, with roots 2, -3. The complementary function is $y_c = c_1e^{2x} + c_2e^{-3x}$. The NH term is a linear combination of the UC functions $e^{2x}$, $e^{3x}$, x and 1.

**Step 1**: Form the UC set of each of these four functions: $S_1 = \{e^{2x}\}$, $S_2 = \{e^{3x}\}$, $S_3 = \{x,1\}$, $S_4 = \{1\}$.

**Step 2**: Set $S_4$ is completely included in $S_3$; so $S_4$ is omitted, leaving the three sets $S_1$, $S_2$, and $S_3$.

**Step 3**: Observe that the only member $e^{2x}$ of $S_1 = \{e^{2x}\}$ is included in the complementary function and so is a solution of the corresponding homogeneous D.E. Thus we multiply the member $e^{2x}$ of $S_1$ by x to obtain the revised set

$$S_1' = \{xe^{2x}\},$$

whose only member is not a solution of the homogeneous D.E.

**Step 4**: We now have the three sets $S_1' = \{xe^{2x}\}$, $S_2 = \{e^{3x}\}$, $S_3 = \{x,1\}$. We form a linear combination of their four elements.

**Step 5**: Thus we take

$$y_p = Axe^{2x} + Be^{3x} + Cx + D$$

as a particular solution. Then

$$y_p' = 2Axe^{2x} + Ae^{2x} + 3Be^{3x} + C,$$
$$y_p'' = 4Axe^{2x} + 4Ae^{2x} + 9Be^{3x},$$

We substitute in the D.E., obtaining

$$\begin{aligned}
&(4Axe^{2x} + 4Ae^{2x} + 9Be^{3x}) \\
&\quad + (2Axe^{2x} + Ae^{2x} + 3Be^{3x} + C) \\
&\quad - 6(Axe^{2x} + Be^{3x} + Cx + D) \\
&= 10e^{2x} - 18e^{3x} - 6x - 11 \qquad \text{or} \\
&5Ae^{2x} + 6Be^{3x} - 6Cx + (C - 6D) \\
&= 10e^{2x} - 18e^{3x} - 6x - 11.
\end{aligned}$$

We equate coefficients of like terms on both sides of this to obtain $5A = 10$, $6B = -18$, $-6C = -6$, $C - 6D = -11$. From this, we find $A = 2$, $B = -3$, $C = 1$, $D = 2$. We thus obtain the particular integral

$$y_p = 2xe^{2x} - 3e^{3x} + x + 2.$$

The G.S. of the D.E. is

$$y = c_1e^{2x} + c_2e^{-3x} + 2xe^{2x} - 3e^{3x} + x + 2.$$

22. The corresponding homogeneous D.E. is $y'' - 4y' + 5y = 0$. The auxiliary equation is $m^2 - 4m + 5 = 0$, with roots $2 \pm i$. The complementary function is $y_c = e^{2x}(c_1 \sin x + c_2 \cos x)$. The NH term is a constant multiple of the UC function given by $e^{2x}\cos x$.

**Step 1:** Form the UC set of $e^{2x}\cos x$. It is
$S_1 = \{e^{2x}\cos x,\ e^{2x}\sin x\}$.

**Step 2:** This step does not apply.

**Step 3:** Both members $e^{2x}\cos x$ and $e^{2x}\sin x$ of $S_1$ are included in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of set $S_1$ by x to obtain the revised set $S_1' = \{xe^{2x}\cos x,\ xe^{2x}\sin x\}$, which has no members which are solutions of the homogeneous D.E.

**Step 4:** We now have the set $S_1' = \{xe^{2x}\cos x,\ xe^{2x}\sin x\}$, and we form a linear combination of its two members.

**Step 5** Thus we take

$$y_p = Axe^{2x}\sin x + Bxe^{2x}\cos x$$

as a particular solution. Then

$$\begin{aligned} y_p' &= (2A - B)xe^{2x}\sin x + (A + 2B)xe^{2x}\cos x \\ &\quad + Ae^{2x}\sin x + Be^{2x}\cos x, \\ y_p'' &= (3A - 4B)xe^{2x}\sin x + (4A + 3B)xe^{2x}\cos x \\ &\quad + (4A - 2B)e^{2x}\sin x + (2A + 4B)e^{2x}\cos x. \end{aligned}$$

We substitute in the D.E., obtaining

$$\begin{aligned}
&[(3A - 4B)xe^{2x}\sin x + (4A + 3B)xe^{2x}\cos x \\
&\quad + (4A - 2B)e^{2x}\sin x + (2A + 4B)e^{2x}\cos x] \\
&\quad - 4[(2A - B)xe^{2x}\sin x + (A + 2B)xe^{2x}\cos x \\
&\quad + Ae^{2x}\sin x + Be^{2x}\cos x] \\
&\quad + 5[Axe^{2x}\sin x + Bxe^{2x}\cos x] = 6e^{2x}\cos x \quad \text{or} \\
&(3A - 4B - 8A + 4B + 5A)xe^{2x}\sin x \\
&\quad + (4A + 3B - 4A - 8B + 5B)xe^{2x}\cos x \\
&\quad + (4A - 2B - 4A)e^{2x}\sin x + (2A + 4B - 4B)e^{2x}\cos x \\
&= 6e^{2x}\cos x \qquad \text{or simply}
\end{aligned}$$

$$-2Be^{2x}\sin x + 2Ae^{2x}\cos x = 6e^{2x}\cos x.$$

We equate coefficients of like terms on both sides of this to obtain

$$-2B = 0, \quad 2A = 6.$$

From this, we find $A = 3$, $B = 0$. We thus obtain the particular solution

$$y_p = 3xe^{2x}\sin x.$$

The G.S. of the D.E. is

$$y = e^{2x}(c_1\sin x + c_2\cos x) + 3xe^{2x}\sin x.$$

24. The corresponding homogeneous D.E. is $y''' - 2y'' - y' + 2y = 0$. The auxiliary equation is $m^3 - 2m^2 - m + 2 = 0$, that is $(m^2 - 1)(m - 2) = 0$. From this, the roots are 1, -1, 2. The complementary function is $y_c = c_1e^x + c_2e^{-x} + c_3e^{2x}$. The NH term is a linear combination of the UC functions given by $e^{2x}$ and $e^{3x}$.

**Step 1:** Form the UC set of each of these two functions: $S_1 = \{e^{2x}\}$, $S_2 = \{e^{3x}\}$.

**Step 2:** Neither set is identical with nor included in the other, so both are retained.

**Step 3:** Observe that the only member $e^{2x}$ of $S_1 = \{e^{2x}\}$ is included in the complementary function, and so is a solution of the corresponding homogeneous D.E. Thus we multiply the member $e^{2x}$ of $S_1$ by x to obtain the revised set $S_1' = \{xe^{2x}\}$ whose only member is not a solution of the homogeneous D.E.

**Step 4:** We now have the two sets $S_1' = \{xe^{2x}\}$, $S_2 = \{e^{3x}\}$. We form a linear combination of their two elements.

**Step 5:** Thus we take $y_p = Axe^{2x} + Be^{3x}$ as a particular solution. Then $y_p' = 2Axe^{2x} + Ae^{2x} + 3Be^{3x}$.

$$y_p'' = 4Axe^{2x} + 4Ae^{2x} + 9Be^{3x},$$

$$y_p''' = 8Axe^{2x} + 12Ae^{2x} + 27Be^{3x}.$$

We substitute in the D.E., obtaining

$$(8Axe^{2x} + 12Ae^{2x} + 27e^{3x}) - 2(4Axe^{2x} + 4Ae^{2x} + 9Be^{3x})$$
$$- (2Axe^{2x} + Ae^{2x} + 3Be^{3x}) + 2(Axe^{2x} + Be^{3x})$$
$$= 9e^{2x} - 8e^{3x} \qquad \text{or}$$
$$3Ae^{2x} + 8Be^{3x} = 9e^{2x} - 8e^{3x}.$$

We equate coefficients of like terms on both sides of this to obtain $3A = 9$, $8B = -8$, and hence $A = 3$, $B = -1$. Thus we obtain the particular integral $y_p = 3xe^{2x} - e^{3x}$. The G.S. of the D.E. is

$$y = c_1e^x + c_2e^{-x} + c_3e^{2x} + 3xe^{2x} - e^{3x}.$$

25. The corresponding homogeneous D.E. is $y''' + y' = 0$. The auxiliary equation is $m^3 + m = 0$, with roots $m = 0, \pm i$. The complementary function is $y_c = c_1 + c_2\sin x + c_3\cos x$. The NH term is a linear combination of the UC functions given by $x^2$ and $\sin x$.

**Step 1**: Form the UC set of each of these two UC functions: $S_1 = \{x^2, x, 1\}$, $S_2 = \{\sin x, \cos x\}$.

**Step 2**: Neither set is identical with nor included in the other, so both are retained.

**Step 3**: Observe that the member 1 of $S_1$ is included in the complementary function (in the $c_1 = c_1 \cdot 1$ term) and so is a solution of the corresponding homogeneous D.E. Thus we

multiply each member of set $S_1$ by x to obtain the revised set $S_1' = \{x^3, x^2, x\}$, which has no members which are members of $S_2$ are included in the complementary function. Thus we multiply each member of $S_2$ by x to obtain the revised set $S_2' = \{x \sin x, x \cos x\}$, which has no members which are solutions of the homogeneous D.E.

**Step 4:** We now have the two sets $S_1' = \{x^3, x^2, x\}$, $S_2' = \{x \sin x, x \cos x\}$. We form a linear combination of their five elements.

**Step 5:** Thus we take $y_p = Ax^3 + Bx^2 + Cx + Dx \sin x + Ex \cos x$. Then

$$y_p' = 3Ax^2 + 2Bx + C + Dx \cos x - Ex \sin x + D \sin x + E \cos x,$$

$$y_p'' = 6Ax + 2B - Dx \sin x - Ex \cos x + 2D \cos x - 2E \sin x,$$

$$y_p''' = 6A - Dx \cos x + Ex \sin x - 3D \sin x - 3E \cos x.$$

We substitute in the D.E., obtaining

$$(6A - Dx \cos x + Ex \sin x - 3D \sin x - 3E \cos x) + (3Ax^2 + 2Bx + C + Dx \cos x - Ex \sin x + D \sin x + E \cos x) = 2x^2 + 4 \sin x \quad \text{or}$$

$$3Ax^2 + 2Bx + (6A + C) - 2D \sin x - 2E \cos x = 2x^2 + 4 \sin x.$$

We equate coefficients of like terms on both sides of this to obtain $3A = 2$, $2B = 0$, $6A + C = 0$, $-2D = 4$, $-2E = 0$, and hence $A = \frac{2}{3}$, $B = 0$, $C = -4$, $D = -2$, $E = 0$. Thus we obtain the particular integral $y_p = \left(\frac{2}{3}\right)x^3 - 4x - 2x\sin x$. The G.S. of the D.E. is

$$y = c_1 + c_2\sin x + c_3\cos x + \left(\frac{2}{3}\right)x^3 - 4x - 2x\sin x.$$

26. The corresponding homogeneous D.E. is $y^{iv} - y''' + 2y'' = 0$. The auxiliary equation is $m^4 - 3m^3 + 2m^2 = 0$, or $m^2(m - 1)(m - 2) = 0$, with roots 0, 0 (real double root), 1, 2. The complementary function is $y_c = c_1 + c_2x + c_3e^x + c_4e^{2x}$. The NH term is a linear combination of the UC functions given by $e^{-x}$, $e^{2x}$, and $x$.

**Step 1**: Form the UC set of each of these three UC functions: $S_1 = \{e^{-x}\}$, $S_2 = \{e^{2x}\}$, $S_3 = \{x,1\}$.

**Step 2**: No set is identical with nor included in any other, so each is retained.

**Step 3**: Observe that the member $e^{2x}$ of $S_2$ is included in the complementary function and so is a solution of the corresponding homogeneous D.E. Thus we multiply each member of $S_2$ by $x$ to obtain the revised set $S_2' = \{xe^{2x}\}$, whose member is not a solution of the homogeneous D.E. Next observe that both members $x$ and 1 of $S_3$ are included in the complementary function. Thus we multiply each member of $S_3$ by $x^2$ to obtain the revised set

$S_3' = \{x^3, x^2\}$, whose members are not solutions of the homogeneous D.E. (note that multiplication by x, instead of $x^2$, is not sufficient here).

**Step 4**: We now have the three sets $S_1' = \{e^{-x}\}$, $S_2' = \{xe^{2x}\}$, $S_3' = \{x^3, x^2\}$. We form a linear combination of their four elements.

**Step 5**: Thus we take $y_p = Ae^{-x} + Bxe^{2x} + Cx^3 + Dx^2$ as a particular solution. Then

$$y_p' = -Ae^{-x} + 2Bxe^{2x} + Be^{2x} + 3Cx^2 + 2Dx,$$

$$y_p'' = Ae^{-x} + 4Bxe^{2x} + 4Be^{2x} + 6Cx + 2D,$$

$$y_p''' = -Ae^{-x} + 8Bxe^{2x} + 12Be^{2x} + 6C,$$

$$y_p^{iv} = Ae^{-x} + 16Bxe^{2x} + 32Be^{2x}.$$

We substitute into the D.E., obtaining

$$(Ae^{-x} + 16Bxe^{2x} + 32Be^{2x}) - 3(-Ae^{-x} + 8Bxe^{2x} + 12Be^{2x} + 6C) + 2(Ae^{-x} + 4Bxe^{2x} + 4Be^{2x} + 6Cx + 2D) = 3e^{-x} + 6e^{2x} - 6x \text{ or}$$

$$6Ae^{-x} + 4Be^{2x} + 12Cx + (-18C + 4D) = 3e^{-x} + 6e^{2x} - 6x.$$

We equate coefficients of like terms on both sides of this to obtain $6A = 3$, $4B = 6$, $12C = -6$, $-18C + 4D = 0$, and

hence $A = \frac{1}{2}$, $B = \frac{3}{2}$, $C = -\frac{1}{2}$, $D = -\frac{9}{4}$. Thus we obtain the particular integral $y_p = \frac{1}{2}e^{-x} + \frac{3}{2}xe^{2x} - \frac{1}{2}x^3 - \frac{9}{4}x^2$.

The G.S. of the D.E. is $y = c_1 + c_2x + c_3e^x + c_4e^{2x} + \frac{1}{2}e^{-x} + \frac{3}{2}xe^{2x} - \frac{1}{2}x^3 - \frac{9}{4}x^2$.

27. The corresponding homogeneous D.E. is $y''' - 6y'' + 11y' - 6y = 0$. The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$. By inspection we find that $m = 1$ is a root. Then by synthetic division, we have $(m - 1)(m^2 - 5m + 6) = 0$ or $(m - 1)(m - 2)(m - 3) = 0$. Thus the roots are $m = 1, 2, 3$. The complementary function is $y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$. The NH member is a linear combination of the UC functions $xe^x$, $e^{2x}$, and $e^{4x}$.

**Step 1**: Form the UC set of each of these three UC functions: $S_1 = \{xe^x, e^x\}$, $S_2 = \{e^{2x}\}$, $S_3 = \{e^{4x}\}$.

**Step 2**: No set is identical with nor included in another, so each is retained.

**Step 3**: Observe that member $e^x$ of $S_1$ is included in the complementary function and so is a solution of the corresponding homogeneous equation. Thus we multiply each member of $S_1$ by x to obtain the revised set $S_1' = \{x^2e^x, xe^x\}$, whose members are not solutions of the homogeneous D.E. Similarly, member $e^{2x}$ of $S_2$ is included in $y_c$, so we multiply each member of $S_2$ by x to obtain the revised set $S_2' = \{xe^{2x}\}$.

**Step 4:** We now have the two revised sets $S_1' = \{x^2e^x, xe^x\}$ and $S_2' = \{xe^{2x}\}$ and also the original set $S_3 = \{e^{4x}\}$. We form a linear combination $Ax^2e^x + Bxe^x + Cxe^{2x} + De^{4x}$ of their four elements.

**Step 5:** Thus we take $y_p = Ax^2e^x + Bxe^x + Cxe^{2x} + De^{4x}$ as a particular solution. Then

$$\begin{aligned} y_p' &= Ax^2e^x + (2A + B)xe^x + Be^x + 2Cxe^{2x} \\ &\quad + Ce^{2x} + 4De^{4x}, \\ y_p'' &= Ax^2e^x + (4A + B)xe^x + (2A + 2B)e^x \\ &\quad + 4Cxe^{2x} + 4Ce^{2x} + 16De^{4x}, \\ y_p''' &= Ax^2e^x + (6A + B)xe^x + (6A + 3B)e^x \\ &\quad + 8Cxe^{2x} + 12Ce^{2x} + 64De^{4x}. \end{aligned}$$

We substitute in the D.E., obtaining

$$\begin{aligned} &[Ax^2e^x + (6A + B)xe^x + (6A + 3B)e^x + 8Cxe^{2x} \\ &\quad + 12Ce^{2x} + 64De^{4x}] - 6[Ax^2e^x + (4A + B)xe^x \\ &\quad + (2A + 2B)e^x + 4Cxe^{2x} + 4Ce^{2x} + 16De^{4x}] \\ &\quad + 11[Ax^2e^x + (2A + B)xe^x + Be^x + 2Cxe^{2x} \\ &\quad + Ce^{2x} + 4De^{4x}] - 6[Ax^2e^x + Bxe^x \\ &\quad + Cxe^{2x} + De^{4x}] = xe^x - 4e^{2x} + 6e^{4x} \qquad \text{or} \\ &4Axe^x + (-6A + 2B)e^x - Ce^{2x} + 6De^{4x} \\ &\quad = xe^x - 4e^{2x} + 6e^{4x}. \end{aligned}$$

We equate coefficients of like terms on both sides of this to obtain $4A = 1$, $-6A + 2B = 0$, $-C = -4$, $6D = 6$. From these equations, we find $A = \frac{1}{4}$, $B = \frac{3}{4}$, $C = 4$, $D = 1$. Thus we obtain the particular integral

$$y_p = \frac{1}{4}x^2e^x + \frac{3}{4}xe^x + 4xe^{2x} + e^{4x}.$$

The G.S. of the D.E. is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{1}{4}x^2e^x + \frac{3}{4}xe^x + 4xe^{2x} + e^{4x}.$$

28. The corresponding homogeneous D.E. is $y''' - 4y'' + 5y' - 2y = 0$. The auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$. By inspection we find that $m = 1$ is a root. Then by synthetic division, we have $(m - 1)(m^2 - 3m + 2) = 0$ or $(m - 1)^2(m - 2) = 0$. Thus the roots are $m = 1, 1$ (real double root), and 2. The complementary function is $y_c = (c_1 + c_2x)e^x + c_3e^{2x}$. The NH member is a linear combination of the UC functions $x^2e^x$ and $e^x$.

**Step 1:** Form the UC set of each of these two UC functions: $S_1 = \{x^2e^x, xe^x, e^x\}$, $S_2 = \{e^x\}$.

**Step 2:** We have $S_2 \subset S_1$, so we discard $S_2$, leaving only $S_1$.

**Step 3:** Observe that the members $e^x$ and $xe^x$ of $S_1$ are both included in the complementary function and so are solutions of the corresponding homogeneous equation. Thus we multiply each member of $S_1$ by $x^2$ to obtain the revised set $S_1' = \{x^4e^x, x^3e^x, x^2e^x\}$, whose members are not solutions of the homogeneous D.E.

**Step 4:** We now have the set $S_1' = \{x^4e^x, x^3e^x, x^2e^x\}$, which was revised in Step 3. We form the linear combination $Ax^4e^x + Bx^3e^x + Cx^2e^x$ of the three elements of $S_1'$.

**Step 5:** Thus we take $y_p = Ax^4e^x + Bx^3e^x + Cx^2e^x$ as a particular integral. Then

$$y_p' = Ax^4e^x + (4A + B)x^3e^x + (3B + C)x^2e^x + 2Cxe^x,$$

$$y_p'' = Ax^4e^x + (8A + B)x^3e^x + (12A + 6B + C)x^2e^x + (6B + 4C)xe^x + 2Ce^x,$$

$$y_p''' = Ax^4e^x + (12A + B)x^3e^x + (36A + 9B + C)x^2e^x + (24A + 18B + 6C)xe^x + (6B + 6C)e^x.$$

We substitute into the D.E., obtaining

$$\begin{aligned}&[Ax^4e^x + (12A + B)x^3e^x + (36A + 9B + C)x^2e^x \\ &+ (24A + 18B + 6C)xe^x + (6B + 6C)e^x] - 4[Ax^4e^x \\ &+ (8A + B)x^3e^x + (12A + 6B + C)x^2e^x + (6B + 4C)xe^x \\ &+ 2Ce^x] + 5[Ax^4e^x + (4A + B)x^3e^x + (3B + C)x^2e^x \\ &+ 2Cxe^x] - 2[Ax^4e^x + Bx^3e^x + Cx^2e^x] \\ &= 3x^2e^x - 7e^x\end{aligned}$$

or

$$-12Ax^2e^x + (24A - 6B)xe^x + (6B - 2C)e^x = 3x^2e^x - 7e^x.$$

We equate coefficients of like terms on both sides of this to obtain $-12A = 3$, $24A - 6B = 0$, $6B - 2C = -7$. From these equations, we find $A = -1/4$, $B = -1$, $C = 1/2$. Thus we obtain the particular integral

$$y_p = -\frac{1}{4}x^4e^x - x^3e^x + \frac{1}{2}x^2e^x.$$

The G.S. of the D.E. is

$$y = (c_1 + c_2x)\,e^x + c_3e^{2x} - \frac{1}{4}x^4e^x - x^3e^x + \frac{1}{2}x^2e^x.$$

31. The corresponding homogeneous D.E. is $y'' + y = 0$. The auxiliary equation is $m^2 + 1 = 0$, with roots $\pm i$. The complementary function is $y_c = c_1\sin x + c_2\cos x$. The NH term is the UC function $x\sin x$.

**Step 1**: Form the UC set of this UC function. It is $S_1 = \{x\sin x,\ x\cos x,\ \sin x,\ \cos x\}$.

**Step 2**: This step does not apply.

**Step 3**: Observe that the members $\sin x$ and $\cos x$ of $S_1$ are included in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by $x$ to obtain the revised set $S_1' = \{x^2\sin x,\ x^2\cos x,\ x\sin x,\ x\cos x\}$ whose members are not solutions of the homogeneous D.E.

**Step 4:** We now have the set $S_1'$. We form a linear combination of its four elements $x^2\sin x$, $x^2\cos x$, $x\sin x$, and $x\cos x$.

**Step 5:** Thus we take

$$y_p = Ax^2\sin x + Bx^2\cos x + Cx\sin x + Dx\cos x$$

as a particular solution. Then

$$y_p' = Ax^2\cos x - Bx^2\sin x + (2A - D)x\sin x + (2B + C)x\cos x + C\sin x + D\cos x \quad \text{and}$$

$$y_p'' = -Ax^2\sin x - Bx^2\cos x + (-4B - C)x\sin x + (4A - D)x\cos x + (2A - 2D)\sin x + (2B + 2C)\cos x.$$

We substitute into the D.E., obtaining

$$-4Bx\sin x + 4Ax\cos x + (2A - 2D)\sin x + (2B + 2C)\cos x = x\sin x.$$

We equate coefficients of like terms on both sides of this to obtain $-4B = 1$, $4A = 0$, $2A - 2D = 0$, $2B + C = 0$. From this, $A = 0$, $B = -\frac{1}{4}$, $C = \frac{1}{4}$, $D = 0$. Thus we obtain the particular integral $y_p = -\frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x$. The G.S. of the D.E. is

$$y = c_1\sin x + c_2\cos x - \frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x.$$

33. The corresponding homogeneous D.E. is $y^{iv} + 2y''' - 3y'' = .$ The auxiliary equation is $m^4 + 2m^3 - 3m^2 = 0$ or $m^2(m - 1)(m + 3) = 0$, with roots 0, 0 (real double root), 1, -3. The complementary function is $y_c = c_1 + c_2x + c_3e^x + c_4e^{-3x}$. The NH term is a linear combination of the UC functions given by $x^2$, $xe^x$, $e^{3x}$, and 1.

**Step 1:** Form the UC set of each of these four UC functions: $S_1 = \{x^2,x,1\}$, $S_2 = \{xe^x,e^x\}$, $S_3 = \{e^{3x}\}$, $S_4 = \{1\}$.

**Step 2:** Set $S_4$ is completely included in $S_1$, so $S_4$ is omitted, leaving the three sets $S_1$, $S_2$, $S_3$.

**Step 3:** Observe that members x and 1 of $S_1$ are included in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by $x^2$ to obtain the revised set $S_1' = \{x^4,x^3,x^2\}$, whose members are not solutions of the homogeneous D.E. (Note that multiplication by x, instead of $x^2$, is not sufficient here.) Next observe that the member $e^x$ of $S_2$ is included in the complementary function. Thus we multiply each member of $S_2$ by x to obtain the revised set $S_2' = \{x^2e^x,xe^x\}$, whose members are not solutions of the homogeneous D.E.

**Step 4:** We now have the three sets $S_1' = \{x^4,x^3,x^2\}$, $S_2' = \{x^2e^x,xe^x\}$, $S_3 = \{e^{3x}\}$. We form the linear

combination $x^4 + Bx^3 + Cx^2 + Dx^2e^x + Exe^x + Fe^{3x}$ of their six elements.

**Step 5:** Thus we take

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx^2e^x + Exe^x + Fe^{3x}.$$

Then

$$\begin{aligned}
y_p' &= 4Ax^3 + 3Bx^2 + 2Cx + Dx^2e^x \\
&\quad + (2D + E)xe^x + Ee^x + 3Fe^{3x}, \\
y_p'' &= 12Ax^2 + 6Bx + 2C + Dx^2e^x \\
&\quad + (4D + E)xe^x + (2D + 2E)e^x + 9Fe^{3x}, \\
y_p''' &= 24Ax + 6B + Dx^2e^x + (6D + E)xe^x \\
&\quad + (6D + 3E)e^x + 27Fe^{3x}, \\
y_p^{iv} &= 24A + Dx^2e^x + (8D + E)xe^x \\
&\quad + (12D + 4E)e^x + 81Fe^{3x}.
\end{aligned}$$

We substitute into the D.E., obtaining

$$\begin{aligned}
&24A + Dx^2e^x + (8D + E)xe^x + (12D + 4E)e^x \\
&\quad + 81Fe^{3x} + 2[24Ax + 6B + Dx^2e^x + (6D + E)xe^x \\
&\quad + (6D + 3E)e^x + 27Fe^{3x}] - 3[12Ax^2 + 6Bx + 2C \\
&\quad + Dx^2e^x + (4D + E)xe^x + (2D + 2E)e^x + 9Fe^{3x}] \\
&\quad = 18x^2 + 16xe^x + 4e^{3x} - 9
\end{aligned}$$

or

$$\begin{aligned}
&-36Ax^2 + (48A - 18B)x + (24A + 12B - 6C) \\
&\quad + 8Dxe^x + (18D + 4E)e^x + 108Fe^{3x} \\
&\quad = 18x^2 + 16xe^x + 4e^{3x} - 9.
\end{aligned}$$

We equate coefficients of like terms on both sides of this to obtain $-36A = 18$, $48A - 18B = 0$, $24A + 12B - 6C = -9$, $8D = 16$, $18D + 4E = 0$, $108F = 4$. From this, we find $A = -1/2$, $B = -4/3$, $C = -19/6$, $D = 2$, $E = -9$, $F = 1/27$. Thus we obtain the particular integral

$$y_p = -\frac{1}{2}x^4 - \frac{4}{3}x^3 - \frac{19}{6}x^2 + 2x^2e^x - 9xe^x + \frac{1}{27}e^{3x}.$$

The G.S. of the D.E. is

$$y = c_1 + c_2x + c_3e^x + c_4e^{-3x} - \frac{1}{2}x^4 - \frac{4}{3}x^3 - \frac{19}{6}x^2 + 2x^2e^x - 9xe^x + \frac{1}{27}e^{3x}.$$

34. The corresponding homogeneous D.E. is $y^{iv} - 5y''' + 7y'' - 5y' + 6y = 0$. The auxiliary equation is $m^4 - 5m^3 + 7m^2 - 5m + 6 = 0$. By inspection we find that $m = 2$ is a root. Then by synthetic division, we have $(m - 2)(m^3 - 3m^2 + m - 3) = 0$ or $(m - 2)(m - 3)(m^2 + 1) = 0$. Thus the roots are $m = 2, 3, \pm i$. The complementary function is $y_c = c_1e^{2x} + c_2e^{3x} + c_3\sin x + c_4\cos x$. The NH term is a linear combination of the UC functions given by $\sin x$ and $\sin 2x$.

**Step 1:** Form the UC set of each of these two UC functions: $S_1 = \{\sin x, \cos x\}$, $S_2 = \{\sin 2x, \cos 2x\}$.

**Step 2:** Neither set is identical with nor included in the other, so both are retained.

**Step 3:** Observe that both members of $S_1$ are included in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by x to obtain the revised set $S_1' = \{x \sin x, x \cos x\}$, whose members are not solutions of the homogeneous D.E.

**Step 4:** We now have the two sets $S_1' = \{x \sin x, x \cos x\}$, $S_2 = \{\sin 2x, \cos 2x\}$. We form a linear combination of their four elements.

**Step 5:** Thus we take $y_p = Ax \sin x + Bx \cos x + C \sin 2x + D \cos 2x$ as a particular solution. Then $y_p' = Ax \cos x - Bx \sin x + A \sin x + B \cos x + 2C \cos 2x - 2D \sin 2x$, $y_p'' = -Ax \sin x - Bx \cos x + 2A \cos x - 2B \sin x - 4C \sin 2x - 4D \cos 2x$, $y_p''' = -Ax \cos x + Bx \sin x - 3A \sin x - 3B \cos x - 8C \cos 2x + 8D \sin 2x$, $y_p^{iv} = Ax \sin x + Bx \cos x - 4A \cos x + 4B \sin x + 16C \sin 2x + 16D \cos 2x$.

We 9ubstitute into the D.E., obtaining $(Ax \sin x + Bx \cos x - 4A \cos x + 4B \sin x + 16C \sin 2x + 16D \cos 2x) - 5(-Ax \cos x + Bx \sin x - 3A \sin x - 3B \cos x - 8C \cos 2x + 8D \sin 2x) + 7(-Ax \sin x - Bx \cos x + 2A \cos x - 2B \sin x - 4C \sin 2x - 4D \cos 2x) - 5(Ax \cos x - Bx \sin x + A \sin x + B \cos x + 2C \cos 2x - 2D \sin 2x) + 6(Ax \sin x + Bx \cos x + C \sin 2x + D \cos 2x) = 5 \sin x - 12 \sin 2x$

or $(10A - 10B)\sin x + (10A + 10B)\cos x + (-6C - 30D)\sin 2x + (30C - 6D)\cos 2x = 5 \sin x - 12 \sin 2x$. We equate coefficients of like terms

on both sides of this to obtain
$10A - 10B = 5$, $10A + 10B = 0$, $-6C - 30D = -12$, $30C - 6D = 0$. The equations in A and B are equivalent to $2A - 2B = 1$, $A + B = 0$, from which $A = \frac{1}{4}$, $B = -\frac{1}{4}$. The equations in C and D are equivalent to $C + 5D = 2$, $5C - D = 0$, from which $C = \frac{1}{13}$, $D = \frac{5}{13}$. Thus we obtain the particular integral $y_p = \frac{1}{4}x\sin x - \frac{1}{4}x\cos x + \frac{1}{13}\sin 2x + \frac{5}{13}\cos 2x$. The G.S. of the D.E. is

$$y = c_1e^{2x} + c_2e^{3x} + c_3\sin x + c_4\cos x + \frac{1}{4}x\sin x - \frac{1}{4}x\cos x + \frac{1}{13}\sin 2x. + \frac{5}{13}\cos 2x.$$

37. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - 8m + 15 = 0$, with roots 3, 5. The complementary function is $y_c = c_1e^{3x} + c_2e^{5x}$. The UC set of UC function $xe^{2x}$ is $S_1 = \{xe^{2x}, e^{2x}\}$. This does not need revision, so we take $y_p = Axe^{2x} + Be^{2x}$ as a particular integral. Then $y_p' = 2Axe^{2x} + (A + 2B)e^{2x}$ and $y_p'' = 4Axe^{2x} + (4A + 4B)e^{2x}$. We substitute in the D.E. obtaining

$$4Axe^{2x} + (4A + 4B)e^{2x} - 8[2Axe^{2x} + (A + 2B)e^{2x}] + 15(Axe^{2x} + Be^{2x} = 9xe^{2x}$$

or

$$3Axe^{2x} + (-4A + 3B)e^{2x} = 9xe^{2x}.$$

We equate coefficients of like terms on both sides of this to obtain $3A = 9$, $-4A + 3B = 0$. From these, we find $A = 3$, $B = 4$. Thus we obtain the particular integral $y_p = 3xe^{2x} + 4e^{2x}$. The G.S. of the D.E. is

$$y = c_1e^{2x} + c_2e^{5x} + 3xe^{2x} + 4e^{2x}.$$

We apply the I.C. $y(0) = 5$ to this. We find $c_1 + c_2 + 4 = 5$ or $c_1 + c_2 = 1$. (*) We next differentiate the G.S. to obtain $y' = 3c_1e^{3x} + 5c_2e^{5x} + 6xe^{2x} + 11e^{2x}$. We apply the I.C. $y'(0) = 10$ to this. We find $3c_1 + 5c_2 + 11 = 10$ or $3c_1 + 5c_2 = -1$. (**) From (*) and (**) we find $c_1 = 3$, $c_2 = -2$. Thus we obtain the particular solution

$$y = 3e^{3x} - 2e^{5x} + 3xe^{2x} + 4e^{2x}.$$

38. The corresponding homogeneous equation is $y'' + 7y' + 10y = 0$; and the auxiliary equation is $m^2 + 7m + 10 = 0$, with roots $m = -2, -5$. Thus the complementary function is $y_c = c_1e^{-2x} + c_2e^{-5x}$. The UC set of UC function $4xe^{-3x}$ is $\{xe^{-3x}, e^{-3x}\}$. We assume a particular integral $y_p = Axe^{-3x} + Be^{-3x}$. Then $y_p' = -3Axe^{-3x} + (A - 3B)e^{-3x}$, $y_p'' = 9Axe^{-3x} + (-6A + 9B)e^{-3x}$. Substituting in the given D.E., we obtain

$$[9Axe^{-3x} + (-6A + 9B)e^{-3x}] + 7[-3Axe^{-3x} + (A - 3B)e^{-3x}] + 10[Axe^{-3x} + Be^{-3x}] = 4xe^{-3x}.$$

Simplifying, we find

$$-2Axe^{-3x} - (A + 2B)e^{-3x} = 4xe^{-3x}.$$

From this, we have the equations $-2A = 4$, $A - 2B = 0$, with the solution $A = -2$, $B = -1$. Thus we find the particular integral $y_p = -2xe^{-3x} - e^{-3x}$ and the general solution

$$y = c_1e^{-2x} + c_2e^{-5x} - 2xe^{-3x} - e^{-3x}.$$

We now apply the initial condition $y(0) = 0$ to this. We have

$$c_1 + c_2 = 1.$$

Differentiating the general solution, we find

$$y' = -2c_1e^{-2x} - 5c_2e^{-5x} + 6xe^{-3x} + e^{-3x}.$$

Applying the initial condition $y'(0) = -1$ to this, we find

$$2c_1 + 5c_2 = 2.$$

The solution of the two equations in $c_1$ and $c_2$ is $c_1 = 1$, $c_2 = 0$. Thus the solution is

$$y = e^{-2x} - 2xe^{-3x} - e^{-3x}.$$

39. The auxiliary equation of the corresponding homogeneous equation is $m^2 + 8m + 16 = 0$ or $(m + 4)^2 = 0$, with double real root $-4$. Thus the complementary function is $y_c = (c_1 + c_2x)e^{-4x}$. The UC set of UC function $8e^{-2x}$ is $\{e^{-2x}\}$. We assume the particular integral $y_p = Ae^{-2x}$. Then $y_p' = -2Ae^{-2x}$, $y_p'' = 4Ae^{-2x}$. Substituting into the given D.E., we find

$$4Ae^{-2x} - 16Ae^{-2x} + 16Ae^{-2x} = 8e^{-2x}$$

or $4Ae^{-2x} = 8e^{-2x}$. From this $4A = 8$, and so $A = 2$. Thus we obtain the particular integral $y_p = 2e^{-2x}$ and the general solution $y = (c_1 + c_2x)e^{-4x} + 2e^{-2x}$.

We apply the initial condition $y(0) = 2$ to this, obtaining $c_1 = 0$. We differentiate the general solution to obtain $y' = (-4c_1 + c_2 - 4c_2xe^{-2x})\ e^{-4x} - 4e^{-2x}$. From this, $(-4c_1 + c_2) - 4 = 0$. Since $c_1 = 0$, this gives $c_2 = 4$. Thus we obtain

$$y = 4xe^{-4x} + 2e^{-2x}.$$

40. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 + 6m + 9 = 0$, with double real root $-3$. The complementary function is $y = (c_1 + c_2x)e^{-3x}$. The UC set of the UC function $27e^{-6x}$ is $\{e^{-6x}\}$. We assume the

particular integral $y_p = Ae^{-6x}$. Then $y_p' = -6Ae^{-6x}$, $y_p'' = 36Ae^{-6x}$. Substituting in the given D.E., we obtain

$$36Ae^{-6x} - 36Ae^{-6x} + 9Ae^{-6x} = 27e^{-6x}$$

or $9A = 27e^{-6x}$. From this, $9A = 27$, and so $A = 3$. Thus we obtain the particular integral $y_p = 3e^{-6x}$ and the general solution $y = (c_1 + c_2x)e^{-3x} + 3e^{-6x}$. Now apply the I.C. $y(0) = -2$ to the general solution. We obtain $c_1 + 3 = -2$, so $c_1 = -5$. Differentiate the general solution to obtain $y' = (-3c_1 + c_2 - 3c_2x)e^{-3x} - 18e^{-6x}$. Apply the I.C. $y'(0) = 0$ to this. We obtain $-3c_1 + c_2 - 18 = 0$. Since $c_1 = -5$, we find $c_2 = 3$. Thus we obtain the solution $y = (3x - 5)e^{-3x} + 3e^{-6x}$.

42. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - 10m + 29 = 0$, with roots $5 \pm 2i$. The complementary function is $y_c = e^{5x}(c_1 \sin 2x + c_2 \cos 2x)$. The UC set of the UC function $e^{5x}$ is $S_1 = \{e^{5x}\}$. This does not need revision, so we take $y_p = Ae^{5x}$ as a particular integral. Then $y_p' = 5Ae^{5x}$ and $y_p'' = 25Ae^{5x}$. We substitute in the D.E., obtaining

$$25Ae^{5x} - 10(5Ae^{5x}) + 29Ae^{5x} = 8e^{5x}$$

or $4Ae^{5x} = 8e^{5x}$. We at once see that $A = 2$ and hence

obtain the particular integral $y_p = 2e^{5x}$. The G.S. of the D.E. is

$$y = e^{5x}(c_1 \sin 2x + c_2 \cos 2x) + 2e^{5x}.$$

We apply the I.C. $y(0) = 0$ to this. We find $c_2 + 2 = 0$, so $c_2 = -2$. We next differentiate the G.S. to obtain

$$y' = e^{5x}[(5c_1 - 2c_2)\sin 2x + (2c_1 + 5c_2)\cos 2x] + 10e^{5x}.$$

We apply the I.C. $y'(0) = 8$ to this. We find $2c_1 + 5c_2 + 10 = 8$ or $2c_1 + 5c_2 = -2$. Since $c_2 = -2$, this gives $c_1 = 4$. Thus we obtain the particular solution $y = 2e^{5x}(2 \sin 2x - \cos 2x + 1)$.

44. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - m - 6 = 0$ or $(m - 3)(m + 2) = 0$, with roots 3, -2. The complementary function is $y_c = c_1 e^{3x} + c_2 e^{-2x}$. The UC sets of the UC functions in the right member of the D.E. are $S_1 = \{e^{2x}\}$, $S_2 = \{e^{3x}\}$. Neither is completely contained in the other so both are retained. The member $e^{3x}$ of $S_2$ is contained in the complementary function and so is a solution of the corresponding homogeneous D.E. Thus we multiply the member of $S_2$ by x to obtain the revised set $S_2' = \{xe^{3x}\}$, whose member is not a solution of the homogeneous D.E.

Now we have the two sets $S_1 = \{e^{2x}\}$, $S_2' = \{xe^{3x}\}$. We take a linear combination of their two members as a particular integral. That is, we take $y_p = Ae^{2x} + Bxe^{3x}$. Then $y_p' = 2Ae^{2x} + Be^{3x} + 3Bxe^{3x}$ and $y_p'' = 4Ae^{2x} + 6Be^{3x} + 9Bxe^{3x}$. We substitute into the D.E., obtaining

$$4Ae^{2x} + 6Be^{3x} + 9Bxe^{3x} - (2Ae^{2x} + Be^{3x} + 3Bxe^{3x}) - 6(Ae^{2x} + Bxe^{3x}) = 8e^{2x} - 5e^{3x} \quad \text{or}$$
$$-4Ae^{2x} + 5Be^{3x} = 8e^{2x} - 5e^{3x}.$$

We equate coefficients of like terms on both sides of this to obtain $-4A = 8$, $5B = -5$. From these, we find $A = -2$, $B = -1$. Thus we obtain the particular integral $y_p = -2e^{2x} - xe^{3x}$. The G.S. of the D.E. is

$$y = c_1e^{3x} + c_2e^{-2x} - 2e^{2x} - xe^{3x}.$$

We apply the I.C. $y(0) = 3$ to this. We find $c_1 + c_2 - 2 = 3$ or $c_1 + c_2 = 5$ (*). We next differentiate the G.S. to obtain $y' = 3c_1e^{3x} - 2c_2e^{-2x} - 4e^{2x} - 3xe^{3x} - e^{3x}$. We apply the I.C. $y'(0) = 5$ to this. We find

$$3c_1 - 2c_2 - 4 - 1 = 5$$

or

$$3c_1 - 2c_2 = 10 \qquad (**).$$

From the two equations (*) and (**) in the unknowns $c_1$ and $c_2$, we find that $c_1 = 4$, $c_2 = 1$. Thus we obtain the particular solution $y = 4e^{3x} + e^{-2x} - 2e^{2x} - xe^{3x}$.

45. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - 2m + 1 = 0$ or $(m - 1)^2 = 0$, with roots 1, 1 (double root). The complementary function is $y_c = c_1e^x + c_2xe^x$. The UC sets of the UC function in the right member of the D.E. are $S_1 = \{xe^{2x}, e^{2x}\}$, $S_2 = \{e^x\}$. Neither is completely contained in the other, so both are retained. The member $e^x$ of $S_2$ is included in the complementary function and so is a solution of the corresponding homogeneous D.E. Thus we multiply the member of $S_2$ by $x^2$ to obtain the revised set $S_2' = \{x^2e^x\}$, whose member is not a solution of the homogeneous D.E. (note that multiplication by x, instead of $x^2$, is not sufficient here).

Now we have the two sets $S_1 = \{xe^{2x}, e^{2x}\}$, $S_2' = \{x^2e^x\}$. We take a linear combination of their three members as a particular integral. That is, we take $y_p = Axe^{2x} + Be^{2x} + Cx^2e^x$. Then $y_p' = 2Axe^{2x} + (A + 2B)e^{2x} + Cx^2e^x + 2Cxe^x$ and $y_p'' = 4Axe^{2x} + (4A + 4B)e^{2x} + Cx^2e^x + 4Cxe^x + 2Ce^x$. We substitute into the D.E., obtaining

$$4Axe^{2x} + (4A + 4B)e^{2x} + Cx^2e^x + 4Cxe^x + 2Ce^x$$
$$- 2[2Axe^{2x} + (A + 2B)e^{2x} + Cx^2e^x + 2Cxe^x]$$
$$+ Axe^{2x} + Be^{2x} + Cx^2e^x = 2xe^{2x} + 6e^x \quad \text{or}$$
$$Axe^{2x} + (2A + B)e^{2x} + 2Ce^x = 2xe^{2x} + 6e^x.$$

We equate coefficients of like terms on both sides of this to obtain $A = 2$, $2A + B = 0$, $2C = 6$. From these, we find $A = 2$, $B = -4$, $C = 3$. Thus we obtain the particular integral $y_p = 2xe^{2x} - 4e^{2x} + 3x^2e^x$. The G.S. of the D.E. is $y = c_1e^x + c_2xe^x + 2xe^{2x} - 4e^{2x} + 3x^2e^x$.

We apply the I.C. $y(0) = 1$ to this. We find $c_1 - 4 = 1$ or $c_1 = 5$. We next differentiate the G.S. to obtain $y' = c_1e^x + c_2xe^x + c_2e^x + 4xe^{2x} + 2e^{2x} - 8e^{2x} + 3x^2e^x + 6xe^x$. We apply the I.C. $y'(0) = 0$ to this. We find $c_1 + c_2 + 2 - 8 = 0$ or $c_1 + c_2 = 6$. Since $c_1 = 5$, this gives $c_2 = 1$. Thus we obtain the particular solution

$$y = 5e^x + xe^x + 2xe^{2x} - 4e^{2x} + 3x^2e^x$$
$$= (x + 5)e^x + 3x^2e^x + 2xe^{2x} - 4e^{2x}.$$

47. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 + 1 = 0$, with roots $\pm i$. The complementary function is $y_c = c_1\sin x + c_2\cos x$. The UC sets of the UC functions in the right member of the D.E. are $S_1 = \{x^2, x, 1\}$, $S_2 = \{\sin x, \cos x\}$. Neither is completely contained in the other, so both are retained. The members $\sin x$ and $\cos x$ of $S_2$ are included in the complementary

function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_2$ by $x$ to obtain the revised set $S_2' = \{x \sin x, x \cos x\}$, whose members are not solutions of the homogeneous D.E.

Now we have the two sets $S_1 = \{x^2, x, 1\}$, $S_2' = \{x \sin x, x \cos x\}$. We take a linear combination of their five members as a particular integral. That is, we take $y_p = Ax^2 + Bx + C + Dx \sin x + Ex \cos x$. then $y_p' = 2Ax + B + Dx \cos x + D \sin x - Ex \sin x + E \cos x$ and $y_p'' = 2A - Dx \sin x + 2D \cos x - Ex \cos x - 2E \sin x$. We substitute into the D.E., obtaining

$$2A - Dx \sin x + 2D \cos x - Ex \cos x - 2E \sin x + Ax^2 + Bx + C + Dx \sin x + Ex \cos x = 3x^2 - 4 \sin x \quad \text{or}$$

$$Ax^2 + Bx + (2A + C) + 2D \cos x - 2E \sin x = 3x^2 - 4 \sin x.$$

We equate coefficients of like terms on both sides of this to obtain $A = 3$, $B = 0$, $2A + C = 0$, $2D = 0$, $-2E = -4$. From these we find $A = 3$, $B = 0$, $C = -6$, $D = 0$, $E = 2$. Thus we obtain the particular integral $y_p = 3x^2 - 6 + 2x \cos x$. The G.S. of the D.E. is $y = c_1 \sin x + c_2 \cos x + 3x^2 - 6 + 2x \cos x$.

We apply the I.C. $y(0) = 0$ to this. We find $c_2 - 6 = 0$ or $c_2 = 6$. We next differentiate the G.S. to obtain $y' = c_1 \cos x - c_2 \sin x + 6x - 2x \sin x + 2 \cos x$. We apply the I.C. $y'(0) = 1$ to this. We find $c_1 + 2 = 1$ or $c_1 = -1$. Thus we obtain the particular solution $y = 6 \cos x - \sin x + 3x^2 - 6 + 2x \cos x$.

48. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 + 4 = 0$, with pure imaginary conjugate complex roots $\pm 2i$. Thus the complementary function is $y_c = c_1 \sin 2x + c_2 \cos 2x$. The UC set of the UC function $8 \sin 2x$ is $S = \{\sin 2x, \cos 2x\}$. Since each member of this set is a solution of the corresponding homogeneous equation, we multiply each by x and replace S by $S' = \{x \sin 2x, x \cos 2x\}$. We then form the particular integral $y_p = Ax \sin 2x + Bx \cos 2x$. Then

$y_p' = -2Bx \sin 2x + 2Ax \cos 2x + A \sin 2x + B \cos 2x$ and

$y_p'' = -4Ax \sin 2x - 4Bx \cos 2x - 4B \sin 2x + 4A \cos 2x$.

Substituting in the given D.E., we obtain

$$\begin{aligned}(-4Ax \sin 2x &- 4Bx \cos 2x - 4B \sin 2x \\ &+ 4A \cos 2x) + (4Ax \sin 2x + 4Bx \cos 2x) \\ &= 8 \sin 2x \qquad \text{or} \\ -4B \sin 2x &+ 4A \cos 2x = 8 \sin 2x.\end{aligned}$$

From this, $4A = 0$, $-4B = 8$, and hence $A = 0$, $B = -2$. Thus we obtain the particular integral $y_p = -2x \cos 2x$ and the general solution $y = c_1 \sin 2x + c_2 \cos 2x - 2x \cos 2x$.

We apply the I.C. $y(0) = 6$ to this, obtaining $c_2 = 6$. Differentiating the general solution, we obtain $y' = 2c_1\cos 2x - 2c_2\sin 2x + 4x\sin 2x - 2\cos 2x$. Applying the I.C. $y'(0) = 8$ to this, we find $2c_1 - 2 = 8$, from which $c_1 = 5$. Thus we obtain the solution $y = 5\sin 2x + 6\cos 2x - 2x\cos 2x$.

49. The auxiliary equation of the corresponding homogeneous D.E. is $m^3 - 4m^2 + m + 6 = 0$ or $(m + 1)(m - 3)(m - 2) = 0$, with roots $-1, 2, 3$. The complementary function is $y_c = c_1e^{-x} + c_2e^{2x} + c_3e^{3x}$. The UC sets of the UC functions in the right member of the D.E. are $S_1 = \{xe^x, e^x\}$, $S_2 = \{e^x\}$, and $S_3 = \{\sin x, \cos x\}$. Since $S_1 \supset S_2$, we omit $S_2$, retaining $S_1$ and $S_3$. Neither of these need revision, so we take a linear combination of their four members as a particular integral. That is, we take $y_p = Axe^x + Be^x + C\sin x + D\cos x$. Then $y_p' = Axe^x + (A + B)e^x + C\cos x - D\sin x$, $y_p'' = Axe^x + (2A + B)e^x - C\sin x - D\cos x$, $y_p''' = Axe^x + (3A + B)e^x - C\cos x + D\sin x$. We substitute into the D.E., obtaining $Axe^x + (3A + B)e^x - C\cos x + D\sin x - 4[Axe^x + (2A + B)e^x - C\sin x - D\cos x] + [Axe^x + (A + B)e^x + C\cos x - D\sin x] + 6[Axe^x + Be^x + C\sin x + D\cos x] = 3xe^x + 2e^x - \sin x$ or $4Axe^x + (-4A + 4B)e^x + 10C\sin x + 10D\cos x = 3xe^x + 2e^x - \sin x$. We equate coefficients of like terms on both sides of this to obtain $4A = 3$, $-4A + 4B = 2$, $10C = -1$, $10D = 0$. From

these, we find A = 3/4, B = 5/4, C = -1/10, D = 0. Thus we obtain the particular integral,

$$y_p = \frac{3}{4}xe^x + \frac{5}{4}e^x - \frac{1}{10}\sin x.$$

The G.S. of the D.E. is

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{3x} + \frac{3}{4}xe^x + \frac{5}{4}e^x - \frac{1}{10}\sin x.$$

We apply the I.C. y(0) = 33/40 to this. We find $c_1 + c_2 + c_3 + 5/4 = 33/40$ or

$$c_1 + c_2 + c_3 = -17/40. \qquad (*)$$

We next differentiate the G.S. to obtain

$$y' = -c_1e^{-x} + 2c_2e^{2x} + 3c_3e^{3x} + \frac{3}{4}xe^x + 2e^x - \frac{1}{10}\cos x.$$

We apply the I.C. y'(0) = 0 to this. We find

$$-c_1 + 2c_2 + 3c_3 + 2 - \frac{1}{10} = 0$$

or

$$-c_1 + 2c_2 + 3c_3 = -\frac{19}{10}. \qquad (**)$$

We differentiate once more, obtaining

$$y'' = c_1 e^{-x} + 4c_2 e^{2x} + 9c_3 e^{3x} + \frac{3}{4} x e^{x} + \frac{11}{4} e^{x} + \frac{1}{10} \sin x.$$

We apply the I.C. $y''(0) = 0$ to this. We find

$$c_1 + 4c_2 + 9c_3 + \frac{11}{4} = 0$$

or

$$c_1 + 4c_2 + 9c_3 = -\frac{11}{4}. \qquad (***)$$

The three equations (*), (**), (***) determine $c_1$, $c_2$, $c_3$. Adding (*) and (**), we have $3c_2 + 4c_3 = -93/40$. Adding (**) and (***), we have $6c_2 + 12c_3 = -93/20$. Solving these two resulting equations in $c_2$ and $c_3$, we find $c_2 = -31/40$, $c_3 = 0$. Then (*) gives $c_1 = -c_2 - c_3 - 17/40 = 7/20$. Thus we obtain the desired particular solution

$$y = \frac{7}{20} e^{-x} - \frac{31}{40} e^{2x} + \frac{3}{4} x e^{x} + \frac{5}{4} e^{x} - \frac{1}{10} \sin x.$$

50. The auxiliary equation of the corresponding homogeneous D.E. is $m^3 - 6m^2 + 9m - 4 = 0$. Observe by inspection that 1 is a root. Then, by synthetic division, we obtain

| | | | | |
|---|---|---|---|---|
| 1 | 1 | -6 | 9 | -4 |
| | | 1 | -5 | 4 |
| | 1 | -5 | 4 | 0 |

.

Thus we find the factorization $(m - 1)(m^2 - 5m + 4) = 0$ or $(m - 1)^2(m - 4) = 0$. Thus the roots are 1, 1 (double real root) and the simple real root 4. The complementary function is $y_c = (c_1 + c_2x)e^x + c_3e^{4x}$. The UC sets of the UC functions in the right member of the D.E. are $S_1 = \{x^2, x, 1\}$, $S_2 = \{1\}$, $S_3 = \{e^{2x}\}$. Since $S_1 \supset S_2$, we omit $S_2$, retaining $S_1$ and $S_3$. Neither of these need revision, so we take a linear combination of their four members as a particular integral. That is, we take $y_p = Ax^2 + Bx + C + De^{2x}$. Then $y_p' = 2Ax + B + 2De^{2x}$, $y_p'' = 2A + 4De^{2x}$, $y_p''' = 8De^{2x}$. We substitute into the D.E, obtaining

$$8De^{2x} - 6(2A + 4De^{2x}) + 9(2Ax + B + 2De^{2x}) - 4(Ax^2 + Bx + C + De^{2x}) = 8x^2 + 3 - 6e^{2x}$$

or

$$-4Ax^2 + (18A - 4B)x + (-12A + 9B - 4C) - 2De^{2x} = 8x^2 + 3 - 6e^{2x}.$$

We equate coefficients of like terms on both sides of this to obtain $-4A = 8$, $18A - 4B = 0$, $-12A + 9B - 4C = 3$, $-2D = -6$. From these, we find $A = -2$, $B = -9$, $C = -15$, $D = 3$. Thus we obtain the particular integral $y_p = -2x^2 - 9x - 15 + 3e^{2x}$. The G.S. of the D.E. is

$$y = (c_1 + c_2x)e^x + c_3e^{4x} - 2x^2 - 9x - 15 + 3e^{2x}.$$

We apply the I.C. $y(0) = 1$ to this. We find

$$c_1 + c_3 - 15 + 3 = 1$$

or

$$c_1 + c_3 = 13. \qquad (*)$$

We next differentiate the G.S. to obtain

$$y' = (c_1 + c_2 + c_2x)e^x + 4c_3e^{4x} - 4x - 9 + 6e^{2x}.$$

We apply the I.C. $y'(0) = 7$ to this. We find

$$c_1 + c_2 + 4c_3 - 9 + 6 = 7$$

or

$$c_1 + c_2 + 4c_3 = 10. \qquad (**)$$

We differentiate once more, obtaining

$$y'' = (c_1 + 2c_2 + c_2x)e^x + 16c_3e^{4x} - 4 + 12e^{2x}.$$

We apply the I.C. $y''(0) = 10$ to this. We find

$$c_1 + 2c_2 + 16c_3 - 4 + 12 = 10$$

or

$$c_1 + 2c_2 + 16c_3 = 2. \qquad (***)$$

Solving the three equations (*), (**), (***) for $c_1$, $c_2$, $c_3$, we find $c_1 = 122/9$, $c_2 = -4/3$, $c_3 = -5/9$.

Thus we obtain the desired particular solution

$$y = \left(\frac{122}{9} - \frac{4}{3}x\right)e^{x} - \frac{5}{9}e^{4x} - 2x^{2} - 9x - 15 + 3e^{2x}.$$

53. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 + 4m + 5 = 0$ with conjugate complex roots $-2 \pm i$. The complementary function is $y_c = e^{-2x}(c_1 \sin x + c_2 \cos x)$. The NH member can be written as the linear combination $e^{-2x} + e^{-2x}\cos x$ of the two UC functions $e^{-2x}$ and $e^{-2x}\cos x$. The UC sets of these functions are, respectively, $S_1 = \{e^{-2x}\}$ and $S_2 = \{e^{-2x}\sin x, e^{-2x}\cos x\}$. Neither is contained in the other, so each is retained.

Both members of $S_2$ are contained in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_2$ by the lowest integral power of x so that the resulting revised set will contain no members that are solutions of the corresponding homogeneous D.E. It turns out that the first power of x, x itself, will accomplish this. So we multiply each member of $S_2$ by x to obtain the revised set $S_2' = \{xe^{-2x}\sin x, xe^{-2x}\cos x\}$, whose members are not solutions of the homogeneous D.E.

We now have sets $S_1 = \{e^{-2x}\}$ and $S_2' = \{xe^{-2x}\sin x, xe^{-2x}\cos x\}$. We form a linear combination of their members

$$y_p = Ae^{-2x} + Bxe^{-2x}\sin x + Cxe^{-2x}\cos x.$$

54. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - 6m + 9 = 0$, with roots 3, 3 (double root). The complementary function is $y_c = (c_1 + c_2x)e^{3x}$. The NH member is a linear combination of the three UC functions $x^4e^x$, $x^3e^{2x}$, $x^2e^{3x}$. The UC sets of these functions are

$$S_1 = \{x^4e^x, x^3e^x, x^2e^x, xe^x, e^x\},$$

$$S_2 = \{x^3e^{2x}, x^2e^{2x}, xe^{2x}, e^{2x}\}, \quad \text{and}$$

$$S_3 = \{x^2e^{3x}, xe^{3x}, e^{3x}\}, \quad \text{respectively.}$$

None is completely contained in any other, so each is retained. The members $xe^{3x}$ and $e^{3x}$ of $S_3$ are included in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_3$ by $x^2$ to obtain the revised set $S_3' = \{x^4e^{3x}, x^3e^{3x}, x^2e^{3x}\}$, whose members are not solutions of the homogeneous D.E. (Note that multiplication by x, instead of $x^2$, is not sufficient here.) We now have the three sets $S_1$, $S_2$, and $S_3'$. We form a linear combination of their twelve members. Thus:

$$\begin{aligned} y_p = {} & Ax^4e^x + Bx^3e^x + Cx^2e^x + Dxe^x + Ee^x \\ & + Fx^3e^{2x} + Gx^2e^{2x} + Hxe^{2x} + Ie^{2x} \\ & + Jx^4e^{3x} + Kx^3e^{3x} + Lx^2e^{3x}. \end{aligned}$$

55. The auxiliary equation of the corresponding homogeneous D.E. is $m^2 + 6m + 13 = 0$, with roots $-3 \pm 2i$. The complementary function is $y_c = e^{-3x}(c_1 \sin 2x + c_2 \cos 2x)$. The NH member is a linear combination of the two UC functions $xe^{-3x}\sin 2x$ and $x^2e^{-2x}\sin 3x$. The UC sets of these functions are

$$S_1 = \{xe^{-3x}\sin 2x,\ xe^{-3x}\cos 2x,\ e^{-3x}\sin 2x,\ e^{-3x}\cos 2x\}$$

and

$$S_2 = \{x^2e^{-2x}\sin 3x,\ x^2e^{-2x}\cos 3x,\ xe^{-2x}\sin 3x,\ xe^{-2x}\cos 3x,\ e^{-2x}\sin 3x,\ e^{-2x}\cos 3x\},$$

respectively. Neither is completely contained in the other, so each is retained.

The members $e^{-3x}\sin 2x$ and $e^{-3x}\cos 2x$ of $S_1$ are included in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by x to obtain the revised set.

$$S_1' = \{x^2e^{-3x}\sin 2x,\ x^2e^{-3x}\cos 2x,\ xe^{-3x}\sin 2x,\ xe^{-3x}\cos 2x\},$$

whose members are not solutions of the homogeneous D.E.

We now have the two sets $S_1'$ and $S_2$. We form a linear combination of their ten members. Thus:

$$y_p = Ax^2e^{-3x}\sin 2x + Bx^2e^{-3x}\cos 2x + Cxe^{-3x}\sin 2x + Dxe^{-3x}\cos 2x + Ex^2e^{-2x}\sin 3x + Fx^2e^{-2x}\cos 3x + Gxe^{-2x}\sin 3x + Hxe^{-2x}\cos 3x + Ie^{-2x}\sin 3x + Je^{-2x}\cos 3x.$$

57. The auxiliary equation of the corresponding homogeneous D.E. is $m^3 - 6m^2 + 12m - 8 = 0$ or $(m - 2)^3 = 0$, with roots 2, 2, 2 (three-fold root). The complementary function is $y_c = (c_1 + c_2x + c_3x^2)e^{2x}$. The NH member is a linear combination of the two UC functions $xe^{2x}$ and $x^2e^{3x}$. The UC sets of these functions are, respectively, $S_1 = \{xe^{2x}, e^{2x}\}$ and $S_2 = \{x^2e^{3x}, xe^{3x}, e^{3x}\}$. Neither is contained in the other, so each is retained.

Both members of $S_1$ are contained in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by the lowest integral power of x so that the resulting revised set will contain no members that are solutions of the corresponding homogeneous D.E. By trial, we see that multiplying by x or $x^2$ will *not* accomplish this. Rather, we must multiply each member of $S_1$ by $x^3$, obtaining $S_1' = \{x^4e^{2x}, x^3e^{2x}\}$, whose members are not solutions of the homogeneous D.E.

We now have sets $S_1' = \{x^4e^{2x}, x^3e^{2x}\}$ and $S_2 = \{x^2e^{3x}, xe^{3x}, e^{3x}\}$. We form a linear combination of

their five members $y_p = Ax^4e^{2x} + Bx^3e^{2x} + Cx^2e^{3x} + Dxe^{3x} + Ee^{3x}$.

60. The auxiliary equation of the corresponding homogeneous D.E. is $m^6 + 2m^5 + 5m^4 = 0$, with roots 0, 0, 0, 0 (four-fold root) and $-1 \pm 2i$. The complementary function is $y_c = c_1 + c_2x + c_3x^2 + c_4x^3 + e^{-x}(c_5 \sin 2x + c_6 \cos 2x)$. The NH member is a linear combination of the three UC functions $x^3$, $x^2e^{-x}$, and $e^{-x}\sin 2x$. The UC sets of these functions are $S_1 = \{x^3, x^2, x, 1\}$, $S_2 = \{x^2e^{-x}, xe^{-x}, e^{-x}\}$, and $S_3 = \{e^{-x}\sin 2x,\ e^{-x}\cos 2x\}$. None is completely contained in any other, so all three are retained.

All four members of $S_1$ are contained in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by the lowest positive integral power of x so that the resulting revised set will contain no members that are solutions of the corresponding homogeneous D.E. By actual trial, we see that multiplying by x, $x^2$, or $x^3$ will *not* accomplish this. Rather, we must multiply each member of $S_1$ by $x^4$, obtaining

$$S_1' = \{x^7, x^6, x^5, x^4\}$$

whose members are not solutions of the homogeneous D.E.

Also, both members of $S_3$ are included in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_3$ by x to obtain the revised set

$$S_3' = \{xe^{-x}\sin 2x,\ xe^{-x}\cos 2x\}.$$

whose members are not solutions of the homogeneous D.E. We now have the three sets $S_1'$, $S_2$, and $S_3'$. We form a linear combination of their nine members. Thus:

$$y_p = Ax^7 + Bx^6 + Cx^5 + Dx^4 + Ex^2e^{-x} + Fxe^{-x} + Ge^{-x} + Hxe^{-x}\sin 2x + Ixe^{-x}\cos 2x.$$

63. The auxiliary equation of the corresponding homogeneous D.E. is $m^4 + 3m^2 - 4 = 0$ or $(m^2 + 4)(m^2 - 1) = 0$, with roots 1, -1, $\pm 2i$. The complementary function is $y_c = c_1e^x + c_2e^{-x} + c_3\sin 2x + c_4\cos 2x$. The NH member is a linear combination of $\cos^2 x$ and $\cosh x$. Since $\cos^2 x = \frac{1 + \cos 2x}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$, we see that the NH member is in fact the linear combination

$$\frac{1}{2} + \frac{1}{2}\cos 2x - \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

of the four UC functions 1, $\cos 2x$, $e^x$, and $e^{-x}$. The UC sets of these are $S_1 = \{1\}$, $S_2 = \{\cos 2x, \sin 2x\}$, $S_3 = \{e^x\}$, and $S_4 = \{e^{-x}\}$, respectively.

Both members $\cos 2x$ and $\sin 2x$ of $S_2$ are contained in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_2$ by $x$ to obtain the revised set $S_2' = \{x \cos 2x, x \sin 2x\}$, whose members are not solutions of the homogeneous D.E. Thus we replace $S_2$ by the revised set $S_2'$. The situation is exactly the same for each of $S_3$ and $S_4$, so we replace $S_3$ by the revised set $S_3' = \{xe^x\}$ and $S_4$ by the revised set $S_4' = \{xe^{-x}\}$. Thus we have the sets $S_1 = \{1\}$, $S_2' = \{x \sin 2x, x \cos 2x\}$, $S_3' = \{xe^x\}$, and $S_4' = \{xe^{-x}\}$. We form a linear combination of their five members. Thus

$$y_p = A + Bx \sin 2x + Cx \cos 2x + Dxe^x + Exe^{-x}.$$

Alternatively, we regard the NH member of the D.E. as a linear combination of the UC function $\cos^2 x = (\cos x)(\cos x)$ and the UC combination $\cos h\, x = \frac{e^x + e^{-x}}{2}$. The UC sets of these are respectively $S_1 = \{\sin^2 x, \cos^2 x, \sin x \cos x\}$ and $S_2 = \{\sinh x, \cosh x\}$. One finds by direct substitution that the member $\sin x \cos x$ of $S_1$ is a solution of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by $x$ to obtain the revised set $S_1' = \{x \sin^2 x, x \cos^2 x, x \sin x \cos x\}$, whose members are not solutions of the homogeneous D.E. So we replace $S_1$ by the revised set $S_1'$. In like manner, one finds $\cosh x$ is

a solution of the homogeneous D.E.; and so we replace $S_2$ by the revised set $S_2' = \{x \sinh x,\ x \cosh x\}$. So we now have the two revised sets $S_1'$ and $S_2'$. We form a linear combination of their five members. Thus:

$$y_p = Ax \sin^2 x + Bx \cos^2 x + Cx \sin x \cos x + Dx \sinh x + Ex \cosh x.$$

64. The auxiliary equation of the corresponding homogeneous D.E. is $m^4 + 10m^2 + 9 = 0$ or $(m^2 + 9)(m^2 + 1) = 0$ with roots $\pm i, \pm 3i$. The complementary function is $y_c = c_1 \sin x + c_2 \cos x + c_3 \sin 3x + c_4 \cos 3x$. The NH member is the UC function $\sin x \sin 2x$. The UC set of this function is $S_1 = \{\sin x \sin 2x,\ \sin x \cos 2x,\ \cos x \sin 2x,\ \cos x \cos 2x\}$. One finds by direct substitution that the member $\sin x \sin 2x$ of $S_1$ is a solution of the corresponding homogeneous D.E. Thus we multiply each member of $S_1$ by x to obtain the revised set $S_1' = \{x \sin x \sin 2x,\ x \sin x \cos 2x,\ x \cos x \sin 2x,\ x \cos x \cos 2x\}$, whose members are not solutions of the homogeneous D.E. So we replace $S_1$ by the revised set $S_1'$. We form a linear combination of its four members. Thus:

$$y_p = Ax \sin x \sin 2x + Bx \sin x \cos 2x + Cx \cos x \sin 2x + Dx \cos x \cos 2x. \quad (*)$$

Alternatively, using the trigonometric identify $\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$ with $u = 2x$,

$v = x$, we see that the NH member is equal to $\frac{1}{2}[\cos x - \cos 3x]$ and thus is a linear combination of the simple UC functions $\cos x$ and $\cos 3x$. The UC sets of these two functions are $S_2 = \{\cos x, \sin x\}$ and $S_3 = \{\cos 3x, \sin 3x\}$, respectively.

Both members of $S_2$ are contained in the complementary function and so are solutions of the corresponding homogeneous D.E. Thus we multiply each member of $S_2$ by $x$ to obtain the revised set $S_2' = \{x \cos x, x \sin x\}$ whose members are not solutions of the homogeneous D.E. Thus we replace $S_2$ by the revised set $S_2'$. The situation is exactly the same for the set $S_3$, so we replace $S_3$ by the revised set $S_3' = \{x \cos 3x, x \sin 3x\}$. We now have the two revised sets $S_2'$ and $S_3'$. We form a linear combination of their four members. Thus:

$$y_p = Ex \cos x + Fx \sin x + Gx \cos 3x + Hx \sin 3x \qquad (**)$$

The student should convince himself that the $y_p$ given by (*) can be expressed in the form given by (**), and vice versa.

**Section 4.4, Page 169.**

3. The complementary function is defined by $y_c(x) = c_1 \sin x + c_2 \cos x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)\sin x + v_2(x)\cos x. \qquad (1)$$

Then $y_p'(x) = v_1(x)\cos x - v_2(x)\sin x + v_1'(x)\sin x + v_2'(x)\cos x$. We impose the condition

$$v_1'(x)\sin x + v_2'(x)\cos x = 0, \qquad (2)$$

leaving $y_p' = v_1(x)\cos x - v_2(x)\sin x$. Then from this

$$y_p''(x) = -v_1(x)\sin x - v_2(x)\cos x + v_1'(x)\cos x - v_2'(x)\sin x. \qquad (3)$$

Substituting (1) and (3) into the given D.E., we obtain

$$v_1'(x)\cos x - v_2'(x)\sin x = \sec x. \qquad (4)$$

We now have conditions (2) and (4) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} \sin x\ v_1'(x) + \cos x\ v_2'(x) = 0, \\ \cos x\ v_1'(x) - \sin x\ v_2'(x) = \sec x. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\sec x \cos x}{-1} = 1.$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sec x \end{vmatrix}} = \frac{\sin x \sec x}{-1} = -\tan x.$$

Integrating, we find $v_1(x) = x$, $v_2(x) = \ln|\cos x|$. Substituting into $y_p(x) = v_1(x)\sin x + v_2(x)\cos x$, we find $y_p(x) = x \sin x + \cos x\,[\ln|\cos x|]$. The G.S. of the D.E. is

$$y = c_1 \sin x + c_2 \cos x + x \sin x + \cos x\,[\ln|\cos x|].$$

6. The complementary function is defined by $y_c(x) = c_1 \sin x + c_2 \cos x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)\sin x + v_2(x)\cos x. \qquad (1)$$

Then $y_p'(x) = v_1(x)\cos x - v_2(x)\sin x + v_1'(x)\sin x + v_2'(x)\cos x$. We impose the condition

$$v_1'(x)\sin x + v_2'(x)\cos x = 0. \qquad (2)$$

leaving $y_p'(x) = v_1(x)\cos x - v_2(x)\sin x$. Then from this

$$y_p''(x) = -v_1(x)\sin x - v_2(x)\cos x + v_1'(x)\cos x - v_2'(x)\sin x. \tag{3}$$

Substituting (1) and (3) into the given D.E., we obtain

$$v_1'(x)\cos x - v_2'(x)\sin x = \tan x \sec x. \tag{4}$$

We now have conditions (2) and (4) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} \sin x\ v_1'(x) + \cos x\ v_2'(x) = 0 \\ \cos x\ v_1'(x) - \sin x\ v_2'(x) = \tan x \sec x. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \tan x \sec x}{-1}$$

$$= \tan x.$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \sec x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \tan x \sec x}{-1}$$

$$= -\tan^2 x = 1 - \sec^2 x.$$

Integrating, we find $v_1(x) = -\ln|\cos x|$, $v_2(x) = x - \tan x$. Substituting into $y_p(x) = v_1(x)\sin x + v_2(x)\cos x$, we find $y_p(x) = -\sin x \ln|\cos x| + (x - \tan x)\cos x = -\sin x \ln|\cos x| + x\cos x - \sin x$. The G.S. of the D.E. is

$$y = c_1 \sin x + c_2 \cos x - \sin x \ln|\cos x| + x\cos x - \sin x,$$

which may be more simply written as

$$y = c_0 \sin x + c_2 \cos x - \sin x \ln|\cos x| + x\cos x,$$

where $c_0 = c_1 - 1$.

7. The auxiliary equation $m^2 + 4m + 5 = 0$ has the conjugate complex roots $m = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$, so the complementary function is $y_c(x) = c_1 e^{-2x}\sin x + c_2 e^{-2x}\cos x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)e^{-2x}\sin x + v_2(x)e^{-2x}\cos x. \qquad (1)$$

Then $y_p'(x) = v_1(x)e^{-2x}\cos x - 2v_1(x)e^{-2x}\sin x + v_1'(x)e^{-2x}\sin x - v_2(x)e^{-2x}\sin x - 2v_2(x)e^{-2x}\cos x + v_2'(x)e^{-2x}\cos x$. We impose the condition

$$v_1'(x)e^{-2x}\sin x + v_2'(x)e^{-2x}\cos x = 0, \qquad (2)$$

leaving

$$y_p'(x) = v_1(x)e^{-2x}\cos x - 2v_1(x)e^{-2x}\sin x - v_2(x)e^{-2x}\sin x - 2v_2(x)e^{-2x}\cos x. \quad (3)$$

Then, differentiating this and collecting like terms, we find

$$\begin{aligned} y_p''(x) = {} & 3v_1(x)e^{-2x}\sin x - 4v_1(x)e^{-2x}\cos x \\ & + v_1'(x)[e^{-2x}\cos x - 2e^{-2x}\sin x] \\ & + 3v_2(x)e^{-2x}\cos x + 4v_2(x)e^{-2x}\sin x \\ & - v_2'(x)[e^{-2x}\sin x + 2e^{-2x}\cos x]. \end{aligned} \quad (4)$$

Then substituting (1), (3), and (4) into the given D.E. and collecting like terms, we obtain

$$v_1'(x)[e^{-2x}\cos x - 2e^{-2x}\sin x] - v_2'(x)[e^{-2x}\sin x + 2e^{-2x}\cos x] = e^{-2x}\sec x. \quad (5)$$

We now have conditions (2) and (5) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{aligned} & e^{-2x}\sin x\ v_1'(x) + e^{-2x}\cos x\ v_2'(x) = 0, \\ & e^{-2x}(\cos x - 2\sin x)\ v_1'(x) + e^{-2x}(-\sin x - 2\cos x)\ v_2'(x) = e^{-2x}\sec x. \end{aligned}$$

Solving, we find:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{-2x}\cos x \\ e^{-2x}\sec x & e^{-2x}(-\sin x - 2\cos x) \end{vmatrix}}{\begin{vmatrix} e^{-2x}\sin x & e^{-2x}\cos x \\ e^{-2x}(\cos x - 2\sin x) & e^{-2x}(-\sin x - 2\cos x) \end{vmatrix}}$$

$$= \frac{-e^{-4x}\cos x \sec x}{e^{-4x}(-1)} = 1.$$

$$v_2'(x) = \frac{\begin{vmatrix} e^{-2x}\sin x & 0 \\ e^{-2x}(\cos x - 2\sin x) & e^{-2x}\sec x \end{vmatrix}}{\begin{vmatrix} e^{-2x}\sin x & e^{-2x}\cos x \\ e^{-2x}(\cos x - 2\sin x) & e^{-2x}(-\sin x - 2\cos x) \end{vmatrix}}$$

$$= \frac{e^{-4x}\sec x \sin x}{e^{-4x}(-1)} = -\tan x.$$

Integrating, we find $v_1(x) = x$, $v_2(x) = \ln|\cos x|$. Substituting into $y_p(x) = v_1(x)e^{-2x}\sin x + v_2(x)e^{-2x}\cos x$, we find $y_p(x) = xe^{-2x}\sin x + [\ln|\cos x|]\, e^{-2x}\cos x$. The G.S. of the D.E. is

$$y = e^{-2x}(c_1\sin x + c_2\cos x) + xe^{-2x}\sin x + [\ln|\cos x|]\, e^{-2x}\cos x.$$

10. The auxiliary equation $m^2 - 2m + 1 = 0$ has roots $-1, -1$, so the complementary function is $y_c(x) = (c_1 + c_2x)e^x$,

which we rewrite slightly as $y_c(x) = c_1e^x + c_2xe^x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)e^x + v_2(x)xe^x. \tag{1}$$

Then $y'_p(x) = v_1(x)e^x + v_2(x)[(x + 1)e^x] + v_1'(x)e^x + v_2'(x)xe^x$. We impose the condition

$$v_1'(x)e^x + v_2'(x)xe^x = 0 \tag{2}$$

leaving

$$y_p'(x) = v_1(x)e^x + v_2(x)[(x + 1)e^x]. \tag{3}$$

Then

$$y_p''(x) = v_1(x)e^x + v_2(x)[(x + 2)e^x] + v_1'(x)e^x + v_2'(x)[(x + 1)e^x]. \tag{4}$$

Then substituting (1), (3), and (4) into the given D.E., we obtain

$$\begin{aligned} &v_1(x)e^x + v_2(x)[(x + 2)e^x] + v_1'(x)e^x \\ &\quad + v_2'(x)[(x + 1)e^x] - 2\{v_1(x)e^x \\ &\quad + v_2(x)[(x + 1)e^x]\} + v_1(x)e^x + v_2(x)xe^x \\ &\quad = xe^x \ln x \end{aligned}$$

or

$$v_1'(x)e^x + v_2'(x)[(x + 1)e^x] = xe^x \ln x. \qquad (5)$$

We now have conditions(2) and (5) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} v_1'(x)e^x + v_2'(x)xe^x = 0, \\ v_1'(x)e^x + v_2'(x)[(x + 1)e^x] = xe^x \ln x. \end{cases}$$

Dividing out $e^x \neq 0$, these simplify to

$$\begin{cases} v_1'(x) + xv_2'(x) = 0, \\ v_1'(x) + (x + 1)v_2'(x) = x \ln x. \end{cases}$$

Subtracting the first from the second, we find

$$v_2'(x) = x \ln x;$$

and then the first equation gives

$$v_1'(x) = -x^2 \ln x.$$

Integrating (using either tables or integration by parts), we find

$$v_1(x) = \frac{1}{9} x^3 - \frac{1}{3} x^3 \ln x$$

$$v_2(x) = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$$

Substituting into $y_p(x) = v_1(x)e^x + v_2(x)xe^x$, we find $y_p = \left(\frac{1}{9}x^3 - \frac{1}{3}x^3\ln x\right)e^x + \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)xe^x$ or $y_p = -\frac{5}{36}x^3e^x + \frac{1}{6}x^3e^x\ln x$. The G.S. of the D.E. is $y = c_1e^x + c_2xe^x - \frac{5}{36}x^3e^x + \frac{1}{6}x^3e^x\ln x$.

12. The complementary function is defined by $y_c(x) = c_1\sin x + c_2\cos x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)\sin x + v_2(x)\cos x. \qquad (1)$$

Then $y_p'(x) = v_1(x)\cos x - v_2(x)\sin x + v_1'(x)\sin x + v_2'(x)\cos x$. We impose the condition

$$v_1'(x)\sin x + v_2'(x)\cos x = 0, \qquad (2)$$

leaving $y_p'(x) = v_1(x)\cos x - v_2(x)\sin x$. Then from this,

$$y_p''(x) = -v_1(x)\sin x - v_2(x)\cos x + v_1'(x)\cos x - v_2'(x)\sin x. \qquad (3)$$

Substituting (1) and (3) into the given D.E., we obtain

$$v_1'(x)\cos x - v_2'(x)\sin x = \tan^3 x. \qquad (4)$$

We now have conditions (2) and (4) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} \sin x\ v_1'(x) + \cos x\ v_2'(x) = 0, \\ \cos x\ v_1'(x) - \sin x\ v_2'(x) = \tan^3 x. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \tan^3 x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\tan^3 x \cos x}{-1}$$

$$= \frac{(1 - \cos^2 x)\sin x}{\cos^2 x} = [(\cos x)^{-2} - 1] \sin x,$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan^3 x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\tan^3 x \sin x}{-1} = -\frac{\sin^4 x}{\cos^3 x}$$

$$= -\frac{(1 - \cos^2 x)^2}{\cos^3 x} = -\frac{1}{\cos^3 x} + \frac{2}{\cos x} - \cos x$$

$$= -\sec^3 x + 2 \sec x - \cos x.$$

Integrating (we recommend using tables), we find

$$v_1(x) = (\cos x)^{-1} + \cos x,$$

$$v_2(x) = -\frac{1}{2} \tan x \sec x + \frac{3}{2} \ln|\sec x + \tan x| - \sin x.$$

Substituting into $y_p(x) = v_1(x)\sin x + v_2(x)\cos x$, we find

$$y_p(x) = [(\cos x)^{-1} + \cos x]\sin x + \left[\frac{1}{2}\tan x \sec x + \frac{3}{2}\ln|\sec x + \tan x| - \sin x\right]\cos x = \frac{\tan x}{2} + \frac{3}{2}\cos x \ln|\sec x + \tan x|.$$

The G.S. of the D.E. is

$$y = c_1 \sin x + c_2 \cos x + \frac{\tan x}{2} + \frac{3}{2}\cos x \ln|\sec x + \tan x|.$$

13. The auxiliary equation $m^2 + 3m + 2 = 0$ has roots $-1$, $-2$, so the complementary function is $y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$. We assume a particular integral of the form

$$y_p(x) = v_1(x)e^{-x} + v_2(x)e^{-2x} \qquad (1)$$

Then $y_p'(x) = -v_1(x)e^{-x} - 2v_2(x)e^{-2x} + v_1'(x)e^{-x} + v_2'(x)e^{-2x}$. We impose the condition

$$v_1'(x)e^{-x} + v_2'(x)e^{-2x} = 0 \qquad (2)$$

leaving

$$y_p'(x) = -v_1(x)e^{-x} - 2v_2(x)e^{-2x}. \qquad (3)$$

From this,

$$y_p''(x) = v_1(x)e^{-x} + 4v_2(x)e^{-2x} - v_1'(x)e^{-x} - 2v_2'(x)e^{-2x} \tag{4}$$

Substituting (1), (3), and (4) into the given D.E., we obtain

$$v_1(x)e^{-x} + 4v_2(x)e^{-2x} - v_1'(x)e^{-x} - 2v_2'(x)e^{-2x} + 3[-v_1(x)e^{-x} - 2v_2(x)e^{-2x}] + 2[v_1(x)e^{-x} + v_2(x)e^{-2x}] = \frac{1}{1 + e^x}$$

or

$$-v_1'(x)e^{-x} - 2v_2'(x)e^{-2x} = \frac{1}{1 + e^x} \tag{5}$$

We now have conditions (2) and (5) from which to obtain $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} e^{-x}v_1'(x) + e^{-2x}v_2'(x) = 0, \\ -e^{-x}v_1'(x) - 2e^{-2x}v_2'(x) = \dfrac{1}{1 + e^x}. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{-2x} \\ \dfrac{1}{1+e^x} & -2e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}} = \frac{e^x}{1+e^x},$$

$$v_2'(x) = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \dfrac{1}{1+e^x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}} = -\frac{e^{2x}}{1+e^x}$$

Integrating, we find

$$v_1(x) = \ln(1+e^x)$$

$$v_2(x) = \int \frac{-e^{2x}}{1+e^x}\,dx = \int \left(\frac{e^x}{1+e^x} - e^x\right)dx$$

$$= \ln(1+e^x) - e^x.$$

Substituting into $y_p(x) = v_1(x)e^{-x} + v_2(x)e^{-2x}$, we find $y_p(x) = [\ln(1+e^x)]e^{-x} + [\ln(1+e^x) - e^x]e^{-2x}$. The G.S. of the D.E. is

$$y = c_1e^{-x} + c_2e^{-2x} + [\ln(1+e^x)]e^{-x} + [\ln(1+e^x) - e^x]e^{-2x}$$

or

$$y = c_3e^{-x} + c_2e^{-2x} + (e^{-x} + e^{-2x})[\ln(1 + e^x)],$$

where $c_3 = c_1 - 1$.

15. The complementary function is defined by $y_c(x) = c_1 \sin x + c_2 \cos x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)\sin x + v_2(x)\cos x. \qquad (1)$$

Then $y_p'(x) = v_1(x)\cos x - v_2(x)\sin x + v_1'(x)\sin x + v_2'(x)\cos x$. We impose the condition

$$v_1'(x)\sin x + v_2'(x)\cos x = 0, \qquad (2)$$

leaving $y_p'(x) = v_1(x)\cos x - v_2(x)\sin x$. Then from this

$$y_p''(x) = -v_1(x)\sin x - v_2(x)\cos x + v_1'(x)\cos x - v_2'(x)\sin x. \qquad (3)$$

Substituting (1) and (3) into the given D.E., we obtain

$$v_1'(x)\cos x - v_2'(x)\sin x = \frac{1}{1 + \sin x}. \qquad (4)$$

We now have conditions (2) and (4) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} \sin x \; v_1'(x) + \cos x \; v_2'(x) = 0, \\ \cos x \; v_1'(x) - \sin x \; v_2'(x) = \dfrac{1}{1 + \sin x}. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \dfrac{1}{1 + \sin x} & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\cos x}{1 + \sin x},$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \dfrac{1}{1 + \sin x} \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = -\frac{\sin x}{1 + \sin x}.$$

Integrating, we find

$$v_1(x) = \ln(1 + \sin x),$$

$$v_2(x) = -\int \frac{\sin x \, dx}{1 + \sin x} = \int \left(-1 + \frac{1}{1 + \sin x}\right) dx$$

$$= -x - \frac{\cos x}{1 + \sin x}.$$

Substituting into $y_p(x) = v_1(x)\sin x + v_2(x)\cos x$, we find

$$y_p(x) = \sin x \left[\ln(1 + \sin x)\right] - x \cos x - \frac{\cos^2 x}{1 + \sin x}.$$

The G.S. of the D.E. is

$$y = c_1 \sin x + c_2 \cos x + \sin x \left[\ln(1 + \sin x)\right] - x \cos x - \frac{\cos^2 x}{1 + \sin x}.$$

16. The auxiliary equation $m^2 - 2m + 1 = 0$ has roots 1, 1 (double root), so the complementary function is $y_c(x) = (c_1 + c_2 x)e^x$, which we rewrite slightly as $y_c(x) = c_1 e^x + c_2 x e^x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)e^x + v_2(x)xe^x. \tag{1}$$

Then

$$y_p'(x) = v_1(x)e^x + v_2(x)[(x + 1)e^x] + v_1'(x)e^x + v_2'(x)xe^x.$$

We impose the condition

$$v_1'(x)e^x + v_2'(x)xe^x = 0 \tag{2}$$

leaving

$$y_p'(x) = v_1(x)e^x + v_2(x)[(x + 1)e^x]. \tag{3}$$

Then

$$y_p''(x) = v_1(x)e^x + v_2(x)[(x + 2)e^x] + v_1'(x)e^x + v_2'(x)[(x + 1)e^x]. \tag{4}$$

Then substituting (1), (3), and (4) into the given D.E., we obtain

$$\begin{aligned} & v_1(x)e^x + v_2(x)[(x+2)e^x] + v_1'(x)e^x \\ & \quad + v_2'(x)[(x+1)e^x] - 2\{v_1(x)e^x \\ & \quad + v_2(x)[(x+1)e^x]\} + v_1(x)e^x + v_2(x)xe^x \\ & \quad = e^x \sin^{-1}x \end{aligned}$$

or

$$v_1'(x)e^x + v_2'(x)[(x+1)e^x] = e^x \sin^{-1}x. \qquad (5)$$

We now have conditions (2) and (5) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} e^x v_1'(x) + xe^x v_2'(x) = 0, \\ e^x v_1'(x) + (x+1)e^x v_2'(x) = e \sin^{-1}x. \end{cases}$$

Dividing out $e^x \neq 0$, these simplify to

$$\begin{cases} v_1'(x) + x v_2'(x) = 0, \\ v_1'(x) + (x+1)v_2'(x) = \sin^{-1}x. \end{cases}$$

Subtracting the first from the second, we find

$$v_2'(x) = \sin^{-1}x;$$

and then the first equation gives

$$v_1'(x) = -xv_2'(x) = -x \sin^{-1}x.$$

Integrating (using tables), we find

$$v_1(x) = -\left(\frac{x^2}{2}\right)\sin^{-1}x + \left(\frac{1}{4}\right)\sin^{-1}x - \left(\frac{x}{4}\right)\sqrt{1 - x^2},$$

$$v_2(x) = x \sin^{-1}x + \sqrt{1 - x^2}.$$

Substituting into $y_p(x) = v_1(x)e^x + v_2(x)xe^x$, we find

$$\begin{aligned} y_p(x) &= -\left(\frac{x^2e^x}{2}\right)\sin^{-1}x + \left(\frac{1}{4}\right)xe^x\sin^{-1}x \\ &\quad - \left(\frac{1}{4}\right)xe^x\sqrt{1 - x^2} + x^2e^x\sin^{-1}x \\ &\quad + xe^x\sqrt{1 - x^2} \\ &= \frac{e^x\sin^{-1}x}{4} + \frac{x^2e^x\sin^{-1}x}{2} + \frac{3xe^x\sqrt{1 - x^2}}{4}. \end{aligned}$$

The G.S. of the D.E. is

$$\begin{aligned} y = (c_1 + c_2x)e^x &+ \frac{e^x\sin^{-1}x}{4} + \frac{x^2e^x\sin^{-1}x}{2} \\ &+ \frac{3xe^x\sqrt{1 - x^2}}{4}. \end{aligned}$$

17. The auxiliary equation $m^2 + 3m + 2 = 0$ has the roots $-1$, $-2$, so the complementary function is $y_c(x) = c_1e^{-x} + c_2e^{-2x}$. We assume a particular integral of the form

$$y_p(x) = v_1(x)e^{-x} + v_2(x)e^{-2x}. \qquad (1)$$

Then

$$y_p'(x) = -v_1(x)e^{-x} - 2v_2(x)e^{-2x} + v_1'(x)e^{-x} + v_2'(x)e^{-2x}.$$

We impose the condition

$$v_1'(x)e^{-x} + v_2'(x)e^{-2x} = 0 \qquad (2)$$

leaving

$$y_p'(x) = -v_1(x)e^{-x} - 2v_2(x)e^{-2x}. \qquad (3)$$

Then from this,

$$y_p''(x) = v_1(x)e^{-x} + 4v_2(x)e^{-2x} - v_1'(x)e^{-x} - 2v_2'(x)e^{-2x}. \qquad (4)$$

Substituting (1), (3), and (4) into the given D.E., we obtain

$$v_1(x)e^{-x} + 4v_2(x)e^{-2x} - v_1'(x)e^{-x} - 2v_2'(x)e^{-2x} + 3[-v_1(x)e^{-x} - 2v_2(x)e^{-2x}] + 2[v_1(x)e^{-x} + v_2(x)e^{-2x}] = \frac{e^{-x}}{x}$$

or

$$-v_1'(x)e^{-x} - 2v_2'(x)e^{-2x} = \frac{e^{-x}}{x}. \qquad (5)$$

We now have conditions (2) and (5) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} e^{-x}v_1'(x) + e^{-2x}v_2'(x) = 0 \\ -e^{-x}v_1'(x) - 2e^{-2x}v_2'(x) = \dfrac{e^{-x}}{x}. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{-x}}{x} & -2e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}} = \frac{1}{x},$$

$$v_2'(x) = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{e^{-x}}{x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}} = -\frac{e^{x}}{x}.$$

Integrating, we find $v_1(x) = \ln|x|$. We also have $v_2(x) = -\int\left(\frac{e^x}{x}\right)dx$. Since $\int\left(\frac{e^x}{x}\right)dx$ cannot be expressed in closed form in terms of a finite number of elementary functions, we simply leave it as indicated. Substituting into $y_p(x) = v_1(x)e^{-x} + v_2(x)e^{-2x}$, we find $y_p(x) = e^{-x}\ln|x| - e^{-2x}\int\left(\frac{e^x}{x}\right)dx$. The G.S. of the D.E. is

$$y = c_1e^{-x} + c_2e^{-2x} + e^{-x}\ln|x| - e^{-2x}\int\left(\frac{e^x}{x}\right)dx.$$

18. The auxiliary equation $m^2 - 2m + 1 = 0$ has the roots 1, 1 (double root), so the complementary function is $y_c(x) = (c_1 + c_2x)e^x$, which we rewrite slightly as $y_c(x) = c_1e^x + c_2xe^x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)e^x + v_2(x)xe^x. \tag{1}$$

Then

$$y_p'(x) = v_1(x)e^x + v_2(x)[(x + 1)e^x] + v_1'(x)e^x + v_2'(x)xe^x.$$

We impose the condition

$$v_1'(x)e^x + v_2'(x)xe^x = 0, \tag{2}$$

leaving

$$y_p'(x) = v_1(x)e^x + v_2(x)[(x + 1)e^x]. \tag{3}$$

Then

$$y_p''(x) = v_1(x)e^x + v_2(x)[(x + 2)e^x] + v_1'(x)e^x + v_2'(x)[(x + 1)e^x]. \tag{4}$$

Then substituting (1), (3), and (4) into the given D.E., we obtain

$$\begin{aligned} &v_1(x)e^x + v_2(x)[(x+2)e^x] + v_1'(x)e^x \\ &\quad + v_2'(x)[(x+1)e^x] - 2\{v_1(x)e^x \\ &\quad + v_2(x)[(x+1)e^x]\} \\ &\quad + v_1(x)e^x + v_2(x)xe^x = x \ln x \end{aligned}$$

or

$$v_1'(x)e^x + v_2'(x)[(x+1)e^x] = x \ln x. \qquad (5)$$

We now have conditions (2) and (5) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} e^x v_1'(x) + xe^x v_2'(x) = 0, \\ e^x v_1'(x) + (x+1)e^x v_2'(x) = x \ln x. \end{cases}$$

Dividing out $e^x \neq 0$, these simplify to

$$\begin{aligned} v_1'(x) + xv_2'(x) &= 0, \\ v_1'(x) + (x+1)v_2'(x) &= e^{-x} x \ln x. \end{aligned}$$

Subtracting the first from the second, we find

$$v_2'(x) = e^{-x} x \ln x;$$

and then the first equation gives

$$v_1'(x) = -e^{-x} x^2 \ln x.$$

Thus we have $v_1(x) = -\int \frac{x^2 \ln x}{e^x}\,dx$ and $v_2(x) = \int \frac{x \ln x}{e^x}\,dx$; and since neither of these integrals can be expressed in closed form in terms of a finite number of elementary functions, we simply leave them as indicated. Substituting into

$$y_p(x) = v_1(x)e^x + v_2(x)xe^x,$$

we find

$$y_p(x) = -e^x \int \frac{x^2 \ln x}{e^x}\,dx + xe^x \int \frac{x \ln x}{e^x}\,dx.$$

The G.S. of the D.E. is

$$y = (c_1 + c_2 x)e^x - e^x \int \frac{x^2 \ln x}{e^x}\,dx + xe^x \int \frac{x \ln x}{e^x}\,dx.$$

21. Since $x + 1$ and $x^2$ are linearly independent solutions of the corresponding homogeneous equation, the complementary function is defined by $y_c(x) = c_1(x + 1) + c_2 x^2$. We assume a particular integral of the form

$$y_p(x) = v_1(x)(x + 1) + v_2(x)x^2. \qquad (1)$$

Then

$$y_p'(x) = v_1(x) + 2v_2(x)x + v_1'(x)(x + 1) + v_2'(x)x^2.$$

We impose the condition

$$v_1'(x)(x + 1) + v_2'(x)x^2 = 0, \tag{2}$$

leaving

$$y_p'(x) = v_1(x) + 2v_2(x)x. \tag{3}$$

Then from this,

$$y_p''(x) = v_1'(x) + 2v_2(x) + 2v_2'(x)x. \tag{4}$$

Substituting (1), (3), and (4) into the given D.E., we obtain

$$\begin{aligned}(x^2 + 2x)[v_1'(x) + 2v_2(x) + 2v_2'(x)x] \\ - 2(x + 1)[v_1(x) + 2v_2(x)x] + 2[v_1(x)(x + 1) \\ + v_2(x)x^2] = (x + 2)^2,\end{aligned}$$

or

$$(x^2 + 2x)[v_1'(x) + 2v_2'(x)x] = (x + 2)^2,$$

or finally,

$$v_1'(x) + 2v_2'(x)x = \frac{x + 2}{x}. \tag{5}$$

We now have conditions (2) and (5) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} (x+1)v_1'(x) + x^2 v_2'(x) = 0, \\ v_1'(x) + 2x\, v_2'(x) = \dfrac{x+2}{x}. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & x^2 \\ \frac{x+2}{x} & 2x \end{vmatrix}}{\begin{vmatrix} x+1 & x^2 \\ 1 & 2x \end{vmatrix}} = \frac{-x(x+2)}{x(x+2)} = -1,$$

$$v_2'(x) = \frac{\begin{vmatrix} x+1 & 0 \\ 1 & \frac{x+2}{x} \end{vmatrix}}{\begin{vmatrix} x+1 & x^2 \\ 1 & 2x \end{vmatrix}} = \frac{\left[\frac{(x+1)(x+2)}{x}\right]}{x(x+2)} = \frac{1}{x} + \frac{1}{x^2}.$$

Integrating we find $v_1(x) = -x$, $v_2(x) = \ln|x| - x^{-1}$.

Substituting into $y_p(x) = v_1(x)(x+1) + v_2(x)x^2$, we have $y_p(x) = -x(x+1) + (\ln|x| - x^{-1})x^2 = -x^2 - 2x + x^2\ln|x|$.

The G.S. of the D.E. is

$$y = c_1(x+1) + c_2x^2 - x^2 - 2x + x^2\ln|x|.$$

22. Since $x$ and $xe^x$ are linearly independent solutions of the corresponding homogeneous equation, the complementary function is defined by $y_c(x) = c_1x + c_2xe^x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)x + v_2(x)xe^x. \tag{1}$$

Then

$$y_p'(x) = v_1(x) + v_2(x)[(x + 1)e^x] + v_1'(x)x + v_2'(x)xe^x.$$

We impose the condition

$$v_1'(x)x + v_2'(x)xe^x = 0, \tag{2}$$

leaving

$$y_p'(x) = v_1(x) + v_2(x)[(x + 1)e^x]. \tag{3}$$

Then from this,

$$\begin{aligned} y_p''(x) &= v_1'(x) + v_2(x)[(x + 2)e^x] \\ &\quad + v_2'(x)[(x + 1)e^x]. \end{aligned} \tag{4}$$

Substituting (1), (3), and (4) into the given D.E., we obtain

$$\begin{aligned} &x^2\{v_1'(x) + v_2(x)[(x + 2)e^x] + v_2'(x)[(x + 1)e^x]\} \\ &\quad - x(x + 2)\{v_1(x) + v_2(x)[(x + 1)e^x]\} \\ &\quad + (x + 2)[v_1(x)x + v_2(x)xe^x] = x^3 \end{aligned}$$

or

$$x^2\{v_1'(x) + v_2'(x)[(x + 1)e^x]\} = x^3$$

or finally,

$$v_1'(x) + v_2'(x)[(x + 1)e^x] = x. \qquad (5)$$

We now have conditions (2) and (5) from which to obtain $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} v_1'(x)x + v_2'(x)xe^x = 0, \\ v_1'(x) + v_2'(x)[(x + 1)e^x] = x. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & xe^x \\ x & (x + 1)e^x \end{vmatrix}}{\begin{vmatrix} x & xe^x \\ 1 & (x + 1)e^x \end{vmatrix}} = -1,$$

$$v_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & x \end{vmatrix}}{\begin{vmatrix} x & xe^x \\ 1 & (x + 1)e^x \end{vmatrix}} = e^{-x}.$$

Integrating, we find $v_1(x) = -x$, $v_2(x) = -e^{-x}$. Substituting into $y_p(x) = v_1(x)x + v_2(x)xe^x$, we have $y_p(x) = -x^2 - x$. The G.S. of the D.E. is

$$y = c_1x + c_2xe^x - x^2 - x.$$

24. Since x and $(x + 1)^{-1}$ are linearly independent solutions of the corresponding homogeneous equation, the complementary function is defined by $y_c(x) = c_1x + c_2(x + 1)^{-1}$. We assume a particular integral of the form

$$y_p(x) = v_1(x)x + v_2(x)(x + 1)^{-1}. \qquad (1)$$

Then

$$y_p'(x) = v_1(x) - v_2(x)(x + 1)^{-2} + v_1'(x)x + v_2'(x)(x + 1)^{-1}.$$

We impose the condition

$$v_1'(x)x + v_2'(x)(x + 1)^{-1} = 0, \qquad (2)$$

leaving

$$y_p'(x) = v_1(x) - v_2(x)(x + 1)^{-2}. \qquad (3)$$

Then from this,

$$y_p''(x) = v_1'(x) + 2v_2(x)(x + 1)^{-3} - v_2'(x)(x + 1)^{-2}. \qquad (4)$$

Substituting (1), (3), and (4) into the given D.E., we obtain

$$(2x + 1)(x + 1)[v_1'(x) + 2v_2(x)(x + 1)^{-3}$$
$$- v_2'(x)(x + 1)^{-2}] + 2x[v_1(x) - v_2(x)(x + 1)^{-2}]$$
$$- 2[v_1(x)x + v_2(x)(x + 1)^{-1}] = (2x + 1)^2,$$

or

$$(2x + 1)(x + 1)[v_1'(x) - v_2'(x)(x + 1)^{-2}] = (2x + 1)^2,$$

or finally,

$$v_1'(x) - v_2'(x)(x + 1)^{-2} = (2x + 1)(x + 1)^{-1}. \quad (5)$$

We now have conditions (2) and (5) from which to determine $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} x\,v_1'(x) + (x + 1)^{-1} v_2'(x) = 0, \\ v_1'(x) - (x + 1)^{-2} v_2'(x) = (2x + 1)(x + 1)^{-1}. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & (x + 1)^{-1} \\ (2x + 1)(x + 1)^{-1} & -(x + 1)^{-2} \end{vmatrix}}{\begin{vmatrix} x & (x + 1)^{-1} \\ 1 & -(x + 1)^{-2} \end{vmatrix}}$$

$$= \frac{-(2x + 1)(x + 1)^{-2}}{-x(x + 1)^{-2} - (x + 1)^{-1}} = 1,$$

$$v_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & (2x+1)(x+1)^{-1} \end{vmatrix}}{\begin{vmatrix} x & (x+1)^{-1} \\ 1 & -(x+1)^{-2} \end{vmatrix}}$$

$$= \frac{x(2x+1)(x+1)^{-1}}{-x(x+1)^{-2} - (x+1)^{-1}} = -x(x+1).$$

Integrating, we find $v_1(x) = x$, $v_2(x) = -\frac{x^3}{3} - \frac{x^2}{2}$. Substituting into

$$y_p(x) = v_1(x)x + v_2(x)(x+1)^{-1},$$

we have

$$y_p(x) = x^2 - \frac{(2x^3 + 3x^2)(x+1)^{-1}}{6}.$$

The G.S. of the D.E. is

$$y = c_1x + c_2(x+1)^{-1} + x^2 - \frac{(2x^3 + 3x^2)(x+1)^{-1}}{6}.$$

25. Since $\sin x$ and $x \sin x$ are linearly independent solutions of the corresponding homogeneous equation, the complementary function is defined by $y_c(x) = c_1 \sin x + c_2 x \sin x$. We assume a particular integral of the form

$$y_p(x) = v_1(x)\sin x + v_2(x)x \sin x. \qquad (1)$$

Then

$$y_p'(x) = v_1(x)\cos x + v_2(x)[x\cos x + \sin x] + v_1'(x)\sin x + v_2'(x)x\sin x.$$

We impose the condition

$$v_1'(x)\sin x + v_2'(x)x\sin x = 0, \tag{2}$$

leaving

$$y_p'(x) = v_1(x)\cos x + v_2(x)[x\cos x + \sin x]. \tag{3}$$

Then from this,

$$\begin{aligned} y_p''(x) = {} & -v_1(x)\sin x + v_1'(x)\cos x \\ & + v_2(x)[-x\sin x + 2\cos x] \\ & + v_2'(x)[x\cos x + \sin x]. \end{aligned} \tag{4}$$

Substituting (1), (3), and (4) into the given D.E., we obtain

$$\begin{aligned} & \sin^2 x\{-v_1(x)\sin x + v_1'(x)\cos x \\ & \quad + v_2(x)[-x\sin x + 2\cos x] \\ & \quad + v_2'(x)[x\cos x + \sin x]\} \\ & \quad - 2\sin x\cos x\{v_1(x)\cos x \end{aligned}$$

$$+ v_2(x)[x\cos x + \sin x]\}$$
$$+ (\cos^2 x + 1)[v_1(x)\sin x$$
$$+ v_2(x)x\sin x] = \sin^3 x,$$

or

$$\sin^2 x\{v_1'(x)\cos x + v_2'(x)[x\cos x + \sin x]\} = \sin^3 x,$$

or finally

$$v_1'(x)\cos x + v_2'(x)[x\cos x + \sin x] = \sin x. \quad (5)$$

We now have conditions (2) and (5) from which to obtain $v_1'(x)$ and $v_2'(x)$:

$$\begin{cases} v_1'(x)\sin x + v_2'(x)x\sin x = 0, \\ v_1'(x)\cos x + v_2'(x)[x\cos x + \sin x] = \sin x. \end{cases}$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & x\sin x \\ \sin x & x\cos x + \sin x \end{vmatrix}}{\begin{vmatrix} \sin x & x\sin x \\ \cos x & x\cos x + \sin x \end{vmatrix}} = -x,$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sin x \end{vmatrix}}{\begin{vmatrix} \sin x & x\sin x \\ \cos x & x\cos x + \sin x \end{vmatrix}} = 1.$$

Integrating, we find $v_1(x) = -\frac{x^2}{2}$, $v_2(x) = x$.

Substituting into $y_p(x) = v_1(x)\sin x + v_2(x)x\sin x$, we have

$$y_p(x) = -\frac{x^2\sin x}{2} + x^2\sin x$$

or

$$y_p(x) = \frac{x^2\sin x}{2}.$$

The G.S. of the D.E. is

$$y = c_1\sin x + c_2 x\sin x + \frac{x^2\sin x}{2}.$$

**Section 4.5, Page 176**

1. Let $x = e^t$; then $t = \ln x$; and (as on page 172 of the text)

$$x\frac{dy}{dx} = \frac{dy}{dt}, \qquad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

The D.E. transforms into

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 3y = 0$$

or $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0$. The auxiliary equation of this is $m^2 - 4m + 3 = 0$, and it has the roots $m = 1, 3$. Thus the general solution of the D.E. in y and t is $y = c_1e^t + c_2e^{3t}$. We return to the original independent

variable x and replace $e^t$ by x and $e^{3t}$ by $x^3$. Doing this, we find that the general solution of the given D.E. is $y = c_1x + c_2x^3$.

3. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into $4\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 4\frac{dy}{dt} + 3y = 0$ or $4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 3y = 0$. The auxiliary equation of this is $4m^2 - 8m + 3 = 0$, and it has the roots $m = \frac{1}{2}, \frac{3}{2}$. Thus the general solution of the D.E. in y and t is $y = c_1e^{t/2} + c_2e^{3t/2}$. We return to the original independent variable x and replace $e^{t/2}$ by $x^{1/2}$ and $e^{3t/2}$ by $x^{3/2}$. Doing this, we find that the general solution of the given D.E. is $y = c_1x^{1/2} + c_2x^{3/2}$.

6. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 13y = 0$ or $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0$. The auxiliary equation of this is $m^2 - 4m + 13 = 0$, and it has the conjugate complex roots $2 \pm 3i$. The general solution of the D.E.in y and t is $y = e^{2t}(c_1\sin 3t + c_2\cos 3t)$. We return to the original independent variable x and replace $e^{2t}$ by $x^2$ and t by $\ln x$. Doing this, we find that the general solution of the given D.E. is $y = x^2[c_1\sin(3\ln x) + c_2\cos(3\ln x)]$, or $y = x^2[c_1\sin(\ln x^3) + c_2\cos(\ln x^3)]$.

8. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

The D.E. transforms into $\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + 9y = 0$ or $\frac{d^2y}{dt^2} + 9y = 0$. The auxiliary equation of this is $m^2 + 9 = 0$, and it has roots $\pm 3i$. Thus the general solution of the D.E. in y and t is $y = c_1 \sin 3t + c_2 \cos 3t$. We return to the original independent variable x and replace t by $\ln x$. Doing this, we find that the general solution of the given D.E. is $y = c_1 \sin(3 \ln x) + c_2 \cos(3 \ln x)$ or $y = c_1 \sin(\ln x^3) + c_2 \cos(\ln x^3)$.

9. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

The D.E. transforms into $9\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 3\frac{dy}{dt} + y = 0$ or $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 0$. The auxiliary equation of this is $9m^2 - 6m + 1 = 0$, and it has the real double root $\frac{1}{3}$.

The general solution of the D.E. in y and t is $y = (c_1 + c_2 t)e^{t/3}$. We return to the original independent variable x and replace $e^{t/3}$ by $x^{1/3}$ and t by $\ln x$. Doing this, we find that the general solution of the given D.E. is $y = (c_1 + c_2 \ln x)\, x^{1/3}$.

11. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$, and (as on page 174 of the text) $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$. The D.E. transforms into $\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 6\frac{dy}{dt} - 6y = 0$ or $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 0$. The auxiliary equation of this is $m^3 - 6m^2 + 11m - 6 = 0$, and its roots are 1, 2, 3. The general solution of the D.E. in y and t is $y = c_1e^t + c_2e^{2t} + c_3e^{3t}$. We return to the original independent variable x and replace $e^t$ by x, $e^{2t}$ by $x^2$, and $e^{3t}$ by $x^3$. Doing this, we find that the general solution of the given D.E. is $y = c_1x + c_2x^2 + c_3x^3$.

14. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. To transform $x^4\frac{d^4y}{dx^4}$, we apply the four-step procedure outlined in the Remarks on page 175 of the text. With n = 4, we determine $r(r - 1)(r - 2)(r - 3)$. We expand this to obtain $r^4 - 6r^3 + 11r^2 - 6r$. Replacing $r^k$ by $\frac{d^ky}{dt^k}$ for k = 1, 2, 3, 4, we have $\frac{d^4y}{dt^4} - 6\frac{d^3y}{dt^3} + 11\frac{d^2y}{dt^2} - 6\frac{dy}{dt}$. This is $x^4\frac{d^4y}{dx^4}$. That is, $x^4\frac{d^4y}{dx^4} = \frac{d^4y}{dt^4} - 6\frac{d^3y}{dt^3} + 11\frac{d^2y}{dt^2} - 6\frac{dy}{dt}$.

The D.E. transforms into $\frac{d^4y}{dt^4} - 6\frac{d^3y}{dt^3} + 11\frac{d^2y}{dt^2} - 6\frac{dy}{dt} - 4\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right] + 8\frac{dy}{dt} - 8y = 0$ or $\frac{d^4y}{dt^4} - 6\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 6\frac{dy}{dt} - 8y = 0$. The auxiliary equation of this is $m^4 - 6m^3 + 7m^2 + 6m - 8 = 0$, and its roots are 1, 2, 4, and -1. The G.S. of the D.E. in y and t is $y = c_1e^t + c_2e^{2t} + c_3e^{4t} + c_4e^{-t}$. We return to the original variable x and replace $e^t$ by x, $e^{2t}$ by $x^2$, $e^{4t}$ by $x^4$, and $e^{-t}$ by $x^{-1}$. Doing this, we find that the G.S. of the given D.E. is $y = c_1x + c_2x^2 + c_3x^4 + c_4x^{-1}$.

15. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

The D.E. transforms into

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 4e^t - 6 \qquad (1)$$

(note that x in the right member has transformed into $e^t$; see Remark 1, page 174 of text). The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - 5m + 6 = 0$, with roots m = 2, 3. Thus the complementary function of (1) is $y_c = c_1e^{2t} + c_2e^{3t}$. We find a particular integral by the method of undetermined coefficients. We assume $y_p = Ae^t + B$. Then $y_p' = Ae^t$, $y_p'' = Ae^t$, and substituting into D.E. (1), we quickly have $2Ae^t + 6B = 4e^t - 6$. From this, 2A = 4, 6B = -6, and hence A = 2, B = -1. Thus the particular integral of (1) is $y_p = 2e^t - 1$, and its

general solution is $y = c_1e^{2t} + c_2e^{3t} + 2e^t - 1$.

Returning to the original independent variable x by replacing $e^t$ by x, etc., we find the general solution of the given D.E.: $y = c_1x^2 + c_2x^3 + 2x - 1$.

16. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

The D.E. transforms into

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = 2e^{3t}. \qquad (1)$$

The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - 6m + 8 = 0$, with roots $m = 2, 4$. Thus the complementary function of (1) is $y_c = c_1e^{2t} + c_2e^{4t}$. We find a particular integral by the method of undetermined coefficients. We assume $y_p = Ae^{3t}$. Then $y_p' = 3Ae^{3t}$, $y_p'' = 9Ae^{3t}$, and substituting into D.E. (1), we quickly have $-Ae^{3t} = 2e^{3t}$. From this, $-A = 2$ and so $A = -2$. Thus the particular integral of (1) is $y_p = -2e^{3t}$, and its general solution is $y = c_1e^{2t} + c_2e^{4t} - 2e^{3t}$. Returning to the original independent variable x by replacing $e^t$ by x, etc., we find the general solution of the given D.E.: $y = c_1x^2 + c_2x^4 - 2x^3$.

18. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into

$$\frac{d^2y}{dt^2} + 4y = 2te^t. \qquad (1)$$

The auxiliary equation of the corresponding homogeneous D.E. is $m^2 + 4 = 0$ with roots $m = \pm 2i$. Thus the complementary function of (1) is $y_c = c_1 \sin 2t + c_2 \cos 2t$. We find a particular integral by the method of undetermined coefficients. We assume $y_p = Ate^t + Be^t$. Then $y_p' = Ate^t + (A + B)e^t$, $y_p'' = Ate^t + (2A + B)e^t$, and substituting into the D.E. (1), we have $5Ate^t + (2A + 5B)e^t = 2te^t$. From this $5A = 2$, $2A + 5B = 0$, hence $A = \frac{2}{5}$, $B = \frac{-4}{25}$. Thus the particular integral of (1) is $y_p = \frac{2}{5}te^t - \frac{4}{25}e^t$, and its general solution is $y = c_1 \sin 2t + c_2 \cos 2t + \frac{2}{5}te^t - \frac{4}{25}e^t$.

Returning to the original independent variable by replacing $e^t$ by $x$, $t$ by $\ln x$, we find the general solution of the given D.E.:

$$y = c_1 \sin(2 \ln x) + c_2 \cos(2 \ln x) + \frac{2}{5} x \ln x - \frac{4}{25} x$$

or

$$y = c_1 \sin(\ln x^2) + c_2 \cos(\ln x^2) + \frac{x \ln x^2}{5} - \frac{4x}{25}.$$

20. Let $x = e^t$; then $t = \ln x$, $x \frac{dy}{dx} = \frac{dy}{dt}$, and, $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 5y = 5e^{2t} \qquad \text{or}$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 5e^{2t}. \qquad (1)$$

The homogeneous equation corresponding to this has auxiliary equation $m^2 - 4m + 5 = 0$, with roots $2 \pm i$. The complementary function of (1) is $y_c = e^{2t}(c_1 \sin t + c_2 \cos t)$. We find a particular integral of (1) by the method of undetermined coefficients. We assume $y_p = Ae^{2t}$ and readily find $A = 5$. Thus the G.S. of (1) is $y = e^{2t}(c_1 \sin t + c_2 \cos t) + 5e^{2t}$. We return to the original independent variable x by replacing $e^{2t}$ by $x^2$ and t by $\ln x$. Thus the G.S. of the given D.E. is $y = x^2[c_1 \sin(\ln x) + c_2 \cos(\ln x)] + 5x^2$.

21. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$, and (as on page 174 of the text) $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$. The D.E. transforms into $\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 8\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 28\frac{dy}{dt} - 40y = -9e^{-t}$ or

$$\frac{d^3y}{dt^3} - 11\frac{d^2y}{dt^2} + 38\frac{dy}{dt} - 40y = -9e^{-t}. \qquad (1)$$

The auxiliary equation of the homogeneous D.E. corresponding to this is $m^3 - 11m^2 + 38m - 40 = 0$, with roots 2, 4, 5. The complementary function of (1) is $y_c = c_1e^{2t} + c_2e^{4t} + c_3e^{5t}$. We find a particular integral of (1) by the method of undetermined coefficients. We assume $y_p = Ae^{-t}$. Differentiating and substituting into (1), we find $-A - 11A - 38A - 40A = -9$ or $-90A = -9$; and so $A = \frac{1}{10}$. Thus the G.S. of (1) is $y = c_1e^{2t} + c_2e^{4t} + c_3e^{5t} + \frac{1}{10}e^{-t}$. Returning to the original independent variable x by replacing $e^{2t}$ by $x^2$, etc., we have the G.S. of the given D.E., that is,

$$y = c_1x^2 + c_2x^4 + c_3x^5 + \frac{1}{10}x^{-1}.$$

23. Let $x = e^t$; then $t = \ln x$, $x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 10y = 0$. The auxiliary equation of this is $m^2 - 3m - 10 = 0$, with roots $m = 5, -2$. The general solution of the D.E. in y and t is $y = c_1e^{5t} + c_2e^{-2t}$. Returning to the original independent variable x, we replace $e^t$ by x, etc., and obtain the general solution of the given D.E. in the form

$$y = c_1x^5 + c_2x^{-2}. \qquad (1)$$

Differentiating (1), we find

$$y' = 5c_1x^4 - 2c_2x^{-3}. \qquad (2)$$

Now apply the I.C.'s. Applying $y(1) = 5$ to (1), we have $c_1 + c_2 = 5$; and applying $y'(1) = 4$ to (2), we have $5c_1 - 2c_2 = 4$. Solving these two equations in $c_1$ and $c_2$, we find $c_1 = 2$, $c_2 = 3$. Substituting these values back into (1), we find the particular solution of the stated I.V.P.: $y = 2x^5 + 3x^{-2}$.

26. Let $x = e^t$; then $t = \ln x$, $x \frac{dy}{dx} = \frac{dy}{dt}$, and $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 4e^t - 8. \qquad (1)$$

The auxiliary equation of the corresponding homogeneous D.E. is $m^2 - m - 2 = 0$ with roots $m = -1, 2$. Thus the complementary function of (1) is $y_c = c_1e^{-t} + c_2e^{2t}$. We find a particular integral by the method of undetermined coefficients. We assume $y_p = Ae^t + B$. Then $y_p' = Ae^t$, $y_p'' = Ae^t$, and substituting into the D.E. (1), we have $-2Ae^t - 2B = 4e^t - 8$. From this, $A = -2$, $B = 4$. Thus the particular integral of (1) is $y_p = -2e^t + 4$, and its general solution is $y = c_1e^{-t} + c_2e^{2t} - 2e^t + 4$. Returning to the original variable by replacing $e^t$ by $x$, etc., we find the general solution of the given D.E.

$$y = c_1x^{-1} + c_2x^2 - 2x + 4. \qquad (2)$$

Differentiating (2), we find:

$$y' = -c_1 x^{-2} + 2c_2 x - 2. \qquad (3)$$

Now apply the I.C.'s. Applying $y(1) = 4$ to (2), we have $c_1 + c_2 = 2$; and applying $y'(1) = -1$ to (3), we have $-c_1 + 2c_2 = 1$. Solving these two equations in $c_1$ and $c_2$, we find $c_1 = 1$, $c_2 = 1$. Substituting these values back into (2), we find the particular solution of the stated I.V.P.: $y = x^{-1} + x^2 - 2x + 4$.

27. Let $x = e^t$; then $t = \ln x$, $x \frac{dy}{dx} = \frac{dy}{dt}$, and $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 4\frac{dy}{dt} + 4y = 4e^{2t} - 6e^{3t} \qquad \text{or}$$

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = 4e^{2t} - 6e^{3t}. \qquad (1)$$

The auxiliary equation of the homogeneous D.E. corresponding to (1) is $m^2 - 5m + 4 = 0$ with roots 1, 4. The complementary function of (1) is $y_c = c_1 e^t + c_2 e^{4t}$.

We find a particular integral of (1) by the method of undetermined coefficients. We assume $y_p = Ae^{2t} + Be^{3t}$.

Differentiating, substituting into (1), and simplifying, we obtain $-2Ae^{2t} - 2Be^{3t} = 4e^{2t} - 6e^{3t}$. Thus $A = -2$, $B = 3$. The G.S. of (1) is $y = c_1 e^t + c_2 e^{4t} - 2e^{2t} + 3e^{3t}$.

Returning to the original independent variable x, we

replace $e^t$ by x, etc., and obtain the G.S. of the given D.E.,

$$y = c_1x + c_2x^4 - 2x^2 + 3x^3. \qquad (2)$$

Applying the I.C. $y(2) = 4$ to this, we obtain $2c_1 + 16c_2 = -12$. Differentiating (2), we have $y' = c_1 + 4c_2x^3 - 4x + 9x^2$. Applying the I.C. $y'(2) = -1$ to this, we obtain $c_1 + 32c_2 = -29$. The system of equations $\begin{cases} c_1 + 8c_2 = -6 \\ c_1 + 32c_2 = -29 \end{cases}$ has the solution $c_1 = 5/3$, $c_2 = -23/24$. Thus we obtain the solution

$$y = \frac{5}{3}x - 2x^2 + 3x^3 - \frac{23}{24}x^4.$$

30. Let $x = e^t$; then $t = \ln x$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = t. \qquad (1)$$

The auxiliary equation of the corresponding homogeneous equation is $m^2 - m - 6 = 0$, with roots 3, -2. The complementary function of (1) is $y_c = c_1e^{3t} + c_2e^{-2t}$. We find a particular integral of (1) by the method of undetermined coefficients. We assume $y_p = At + B$. Differentiating and substituting into (1), we find $-A - 6At - 6B = t$. From this, $-6A = 1$, $-A - 6B = 0$.

Thus $A = -\frac{1}{6}$, $B = \frac{1}{36}$. The G.S. of (1) is $y = c_1e^{3t} + c_2e^{-2t} - \frac{1}{6}t + \frac{1}{36}$. Returning to the original independent variable x, we replace $e^t$ by x, etc., and obtain the G.S. of the given D.E.

$$y = c_1x^3 + c_2x^{-2} - \frac{1}{6}\ln x + \frac{1}{36}. \qquad (2)$$

We apply the I.C. $y(1) = 1/6$ to (2), obtaining $c_1 + c_2 = \frac{5}{36}$. Differentiating (2), we have $y' = 3c_1x^2 - 2c_2x^{-3} - \frac{1}{6x}$. Applying the I.C. $y'(1) = -1/6$ to this, we obtain $3c_1 - 2c_2 = 0$. From the pair $c_1 + c_2 = 5/36$, $3c_1 - 2c_2 = 0$, we find $c_1 = 1/18$, $c_2 = 1/12$. Thus we obtain the solution

$$y = \frac{1}{18}x^3 + \frac{1}{12}x^{-2} - \frac{1}{6}\ln x + \frac{1}{36} \qquad \text{or}$$

$$y = \frac{1}{6}\left(\frac{x^3}{3} + \frac{x^{-2}}{2} - \ln x + \frac{1}{6}\right).$$

31. Although this is not a Cauchy-Euler equation, it is similar to one. Let $x + 2 = e^t$. Then $t = \ln(x + 2)$, $(x + 2)\frac{dy}{dx} = \frac{dy}{dt}$, and $(x + 2)^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. The D.E. transforms into $\frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} - 3y = 0$ or $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. The corresponding auxiliary equation is $m^2 - 2m - 3 = 0$, with roots 3, -1. The G.S. of the D.E. in y and t is $y = c_1e^{3t} + c_2e^{-t}$. Returning to the

original independent variable x, we replace $e^{3t}$ by $(x + 2)^3$ and $e^{-t}$ by $(x + 2)^{-1}$. Thus the G.S. of the given D.E. is $y = c_1(x + 2)^3 + c_2(x + 2)^{-1}$.

# Chapter 5

**Section 5.2, Page 197**

1. This is an example of free, undamped motion; and equation (5.8) applies. Since the 12 lb. weight stretches the spring 1.5 in. = $\frac{1}{8}$ ft., Hooke's Law F = ks gives 12 = $\left(\frac{1}{8}\right)$, so k = 96 lb./ft. Also, m = $\frac{w}{g} = \frac{12}{32} = \frac{3}{8}$ (slug). Thus by (5.8) we have the D.E.

$$\frac{3}{8}x'' + 96x = 0 \quad \text{or} \quad x'' + 256x = 0. \qquad (1)$$

Since the weight was released from rest from a position 2 in. = $\frac{1}{6}$ ft. below its equilibrium position, we also have the I.C. $x(0) = \frac{1}{6}$, $x'(0) = 0$. The auxiliary equation of the D.E. (1) is $r^2 + 256 = 0$ with roots $r = \pm 16i$. The G.S. of the D.E. is

$$x = c_1 \sin 16t + c_2 \cos 16t. \qquad (2)$$

Differentiating this, we obtain

$$x' = 16c_1 \cos 16t - 16c_2 \sin 16t. \qquad (3)$$

Applying the first I.C. to (2), we find $c_2 = \frac{1}{6}$; and applying the second to (3) gives $c_1 = 0$. Thus the solution of the D.E. satisfying the stated I.C.'s is

$x = \left(\frac{1}{6}\right) \cos 16t$. The amplitude is $\frac{1}{6}$ (ft.); the period $\frac{2\pi}{16} = \frac{\pi}{8}$ (sec); and the frequency is $\frac{1}{(\pi/8)} = \frac{8}{\pi}$ oscillations/sec.

2. This is an example of free, undamped motion; and equation (5.8) applies. Since the 16 lb. weight stretches the spring 6 in. $= \frac{1}{2}$ foot, Hooke's Law $F = ks$ gives $16 = k\left(\frac{1}{2}\right)$, so $k = 32$ lb./ft. Also, $m = \frac{w}{g} = \frac{16}{32} = \frac{1}{2}$ (slug). Thus by (5.8), we have the D.E.

$$\frac{1}{2} x'' + 32x = 0 \quad \text{or} \quad x'' + 64x = 0. \qquad (1)$$

The auxiliary equation corresponding to (1) is $r^2 + 64 = 0$ with roots $r = \pm 8i$. The G.S. of the D.E. (1) is

$$x = c_1 \sin 8t + c_2 \cos 8t. \qquad (2)$$

Differentiating this, we obtain

$$x' = 8c_1 \cos 8t - 8c_2 \sin 8t. \qquad (3)$$

The three cases (a), (b), (c) lead to different I.C.'s.

In (a), the weight is released from a position 4 in. $= \frac{1}{3}$ ft. *below* its equilibrium position, so we have the first I.C. $x(0) = \frac{1}{3}$. Since it is released with initial velocity of 2 ft./sec., directed *downward*, we have the second I.C. $x'(0) = 2$. Applying the first I.C. to (2), we find $c_2 = \frac{1}{3}$; and applying the second to (3), we have $8c_1 = 2$, so

$c_1 = \frac{1}{4}$. Thus we obtain the particular solution

$$x = \left(\frac{1}{4}\right) \sin 8t + \left(\frac{1}{3}\right) \cos 8t.$$

In (b), we again have the first I.C. $x(0) = \frac{1}{3}$. But here the weight is released with initial velocity of 2 ft./sec., directed *upward*, so the second I.C. is $x'(0) = -2$. We again find $c_2 = \frac{1}{3}$; but applying the second I.C. to (3) gives $c_1 = -\frac{1}{4}$. Thus we obtain the particular solution $x = -\left(\frac{1}{4}\right) \sin 8t + \left(\frac{1}{3}\right) \cos 8t$.

In (c), the weight is released from a position 4 in. = $\frac{1}{3}$ ft. *above* its equilibrium position, so we have the first I.C. $x(0) = -\frac{1}{3}$. Since it is released with initial velocity of 2 ft./sec., directed *downward*, we have the second I.C. $x'(0) = 2$, just as in part (a). Applying the first I.C. to (2), we find $c_2 = -\frac{1}{3}$, and just as in part (a), the second I.C. gives $c_1 = \frac{1}{4}$. Thus we obtain the particular solution

$$x = \left(\frac{1}{4}\right) \sin 8t - \left(\frac{1}{3}\right) \cos 8t.$$

5. Equation (5.8) applies. Since the 4 lb. weight stretches the spring 6 in. = $\frac{1}{2}$ ft., Hooke's law $F = ks$ gives $4 = k\left(\frac{1}{2}\right)$, so $k = 8$ lb./ft. Also, $m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$ (slug). By (5.8) we have the D.E.

$$\frac{1}{8}\,x'' + 8x = 0 \quad \text{or} \quad x'' + 64x = 0. \tag{1}$$

Since the weight was released from its equilibrium position with an initial velocity of 2 ft./sec., directed downward, we have the I.C.'s $x(0) = 0$, $x'(0) = 2$. The auxiliary equation of the D.E. (1) is $r^2 + 64 = 0$ with roots $r = \pm 8\,i$. The G.S. of the D.E. is

$$x = c_1 \sin 8t + c_2 \cos 8t. \tag{2}$$

Differentiating this, we obtain

$$x' = 8c_1 \cos 8t - 8c_2 \sin 8t. \tag{3}$$

Applying the first I.C. $x(0) = 0$ to (2), we find $c_2 = 0$; and applying the second $x'(0) = 2$ to (3), we get $8c_1 = 2$, so $c_1 = \frac{1}{4}$.

Thus the solution of the D.E. satisfying the given I.C.'s is

$$x = \left(\frac{1}{4}\right) \sin 8t, \tag{4}$$

and its derivative is

$$x' = 2 \cos 8t. \tag{5}$$

These are the displacement and velocity, respectively, and hence provide the answer to part (a). From the solution

(4), we see that the amplitude is $\frac{1}{4}$ (ft.), the period is $\frac{2\pi}{8} = \frac{\pi}{4}$ (sec.), and the frequency is $\frac{1}{(\pi/4)} = \frac{4}{\pi}$ oscillations/sec. These are the answers to (b).

To answer (c), we seek times t at which x = 1.5 in. = $\frac{1}{8}$ ft. and $x' > 0$. Thus we first let $x = \frac{1}{8}$ in solution (4) and find $\sin 8t = \frac{1}{2}$. From this, $8t = \frac{\pi}{6} + 2n\pi$ or $8t = \frac{5\pi}{6} + 2n\pi$, and hence $t = \frac{\pi}{48} + \frac{n\pi}{4}$ or $t = \frac{5\pi}{48} + \frac{n\pi}{4}$, where n = 0, 1, 2,.... We must choose these t for which $x' > 0$. From (5), we see that $x'\left(\frac{\pi}{48} + \frac{n\pi}{4}\right) = 2\cos\left(\frac{\pi}{6} + 2n\pi\right) > 0$, but $x'\left(\frac{5\pi}{48} + \frac{n\pi}{4}\right) = 2\cos\left(\frac{5\pi}{6} + 2n\pi\right) < 0$. Thus the answer to part (c) is $t = \frac{\pi}{48} + \frac{n\pi}{4}$. (n = 0, 1, 2, ...). To answer (d), we seek times t at which x = 1.5 in. = $\frac{1}{8}$ ft. and $x' < 0$. From our work in (c), we see these are given by $t = \frac{5\pi}{48} + \frac{n\pi}{4}$, (n = 0, 1, 2, ...).

8. Equation (5.8) applies, with $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$ (slug) and the spring constant $k > 0$ to be determined. Thus (5.8) becomes

$$\frac{1}{4}x'' + kx = 0 \quad \text{or} \quad x'' + 4kx = 0. \qquad (1)$$

Since the weight is released from a position A ft. *below* its equilibrium position with an initial velocity of 3

ft./sec., directed *downward*, we have the I.C.'s $x(0) = A > 0$, $x'(0) = 3$. The auxiliary equation of D.E. (1) is $r^2 + 4k = 0$ with roots $r = \pm 2\sqrt{k}i$. The G.S. of the D.E. is

$$x = c_1 \sin 2\sqrt{k}t + c_2 \cos 2\sqrt{k}t. \qquad (2)$$

Differentiating we obtain

$$x' = 2\sqrt{k}c_1 \cos 2\sqrt{k}t - 2\sqrt{k}c_2 \sin 2\sqrt{k}t. \qquad (3)$$

Applying the first I.C. $x(0) = A$ to (2), we find $c_2 = A$; and applying the second $x'(0) = 3$ to (3), we have $2\sqrt{k}\, c_1 = 3$, so $c_1 = \dfrac{3}{2\sqrt{k}}$. Thus the solution of the D.E. satisfying the given I.C.'s is $x = \left(\dfrac{3}{2\sqrt{k}}\right) \sin 2\sqrt{k}\, t + A \cos 2\sqrt{k}\, t$. We express this in the form (5.18) of the text. Multiplying and dividing by

$$c = \sqrt{\left(\frac{3}{2\sqrt{k}}\right)^2 + A^2} = \frac{\sqrt{9 + 4kA^2}}{2\sqrt{k}},$$

we have $x = c\left[\left(\dfrac{3}{2\sqrt{k}\, c}\right) \sin 2\sqrt{k}\, t + \left(\dfrac{A}{C}\right) \cos 2\sqrt{k}\, t\right]$. Then letting $\dfrac{A}{C} = \cos\phi$, $\dfrac{3}{2\sqrt{k}\, c} = -\sin\phi$, we have $x = c\cos(2\sqrt{k}\, t + \phi)$. From this, the period is $\dfrac{2\pi}{2\sqrt{k}} = \dfrac{\pi}{\sqrt{k}}$; but the period is *given* to be $\dfrac{\pi}{2}$. Thus $\sqrt{k} = 2$, and hence $k = 4$. The amplitude is $c = \sqrt{\left(\dfrac{3}{2\sqrt{k}}\right)^2 + A^2} = \sqrt{\left(\dfrac{3}{4}\right)^2 + A^2}$; but

the amplitude is *given* to be $\sqrt{\frac{10}{2}} = \sqrt{5}$. Hence $\left(\frac{3}{4}\right)^2 + A^2 = 5$, from which $A = \frac{\sqrt{71}}{4}$.

9. There are two different D.E.'s of form (5.8) here, one involving the 8 lb. weight and the other involving the other weight. Concerning the 8 lb. weight, $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$ (slug), and the corresponding D.E. of form (5.8) is

$$\frac{1}{4}x'' + kx = 0 \quad \text{or} \quad x'' + 4kx = 0,$$

where $k > 0$ is the spring constant. The auxiliary equation is $r^2 + 4k = 0$ with roots $r = \pm 2\sqrt{k}\, i$. The G.S. of the D.E. is $x = c_1 \sin 2\sqrt{k}\, t + c_2 \cos 2\sqrt{k}\, t$. From this, the period of the motion is $\frac{2\pi}{2\sqrt{k}}$. But the period is given as 4. Thus $\frac{2\pi}{2\sqrt{k}} = 4$, from which $k = \frac{\pi^2}{16}$.

Now let w be the other weight. For this, $m = \frac{w}{g} = \frac{w}{32}$ (slugs), and the corresponding D.E. of form (5.8) is

$$\frac{w}{32}x'' + \frac{\pi^2}{16}x = 0 \quad \text{or} \quad x'' + \frac{2\pi^2}{w}x = 0.$$

The auxiliary equation is $r^2 + \frac{2\pi^2}{w} = 0$ with roots $r = \pm\sqrt{\frac{2}{w}}\,\pi\, i$. The G.S. of the D.E. is

$$x = c_1 \sin \sqrt{\frac{2}{w}}\,\pi\, t + c_2 \cos \sqrt{\frac{2}{w}}\,\pi\, t.$$

From this the period of the motion is $\frac{2\pi}{\sqrt{\frac{2}{w}}} \pi = \sqrt{2w}$. But the period of this motion is given to be 6. Thus $\sqrt{2w} = 6$, from which we find $w = 18$ (lb.).

**Section 5.3, Page 208**

1. (a) This is a free damped motion, and Equation (5.27) applies. Since the 8 lb. weight stretches the spring 0.4 ft., Hooke's Law $F = ks$ gives $8 = k(0.4)$, so $k = 20$ lb./ft. Also, $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$ (slug), and $a = 2$. Thus equation (5.27) becomes

$$\frac{1}{4} x'' + 2 x' + 20x = 0. \qquad (1)$$

Since the weight is then pulled down 6 in. $= \frac{1}{2}$ ft. below its equilibrium position and released from rest at $t = 0$, we have the I.C.'s

$$x(0) = \frac{1}{2}, \; x'(0) = 0. \qquad (2)$$

(b) The D.E. (1) may be written as $x'' + 8 x' + 80x = 0$. The auxiliary equation is $r^2 + 8r + 80 = 0$, with roots $r = -4 \pm 8i$. The G.S. of the D.E. is

$$x = e^{-4t}(c_1 \sin 8t + c_2 \cos 8t). \qquad (3)$$

Differentiating this, we find

$$x' = e^{-4t}[(-4c_1 - 8c_2) \sin 8t + (8c_1 - 4c_2) \cos 8t]. \qquad (4)$$

Applying the first I.C. (2) to (3), we find $c_2 = \frac{1}{2}$, and applying the second to (4), we have $8c_1 - 4c_2 = 0$, from which $c_1 = \frac{c_2}{2} = \frac{1}{4}$. Thus the solution is

$$x = e^{-4t}\left[\left(\frac{1}{4}\right)\sin 8t + \left(\frac{1}{2}\right)\cos 8t\right] \qquad (5)$$

(c) We first multiply and divide (5) by $c = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{4}$, obtaining $x = \left(\frac{\sqrt{5}}{4}\right) e^{-4t}\left[\left(\frac{1}{\sqrt{5}}\right)\sin 8t + \left(\frac{2}{\sqrt{5}}\right)\cos 8t\right]$. We can now write this as $x = \left(\frac{\sqrt{5}}{4}\right) e^{-4t}\cos(8t - \phi)$, where $\phi$ is such that $\cos\phi = \frac{2}{\sqrt{5}}$, $\sin\phi = \frac{1}{\sqrt{5}}$, and hence $\phi \approx 0.46$ (rad.).

(d) The period is $\frac{2\pi}{8} = \frac{\pi}{4}$ (sec.).

3. This is a free damped motion, and Equation (5.27) applies. Since the 8 lb. weight stretches the spring 6 in. $= \frac{1}{2}$ ft., Hooke's Law $F = ks$ gives $8 = k\left(\frac{1}{2}\right)$, so $k = 16$ lb./ft. Also, $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$ (slug), and $a = 4$. Thus equation (5.27) becomes

$$\frac{1}{4}x'' + 4x' + 16x = 0.$$

The I.C.'s are $x(0) = \frac{3}{4}$, $x'(0) = 0$. The D.E. may be written in the form

$$x'' + 16x' + 64x = 0.$$

The auxiliary equation is $r^2 + 16r + 64 = 0$ with roots $-8$, $-8$ (double root). The G.S. of the D.E. is $x = (c_1 + c_2 t)e^{-8t}$. Differentiating this, we find $\frac{dx}{dt} = (-8c_1 - 8c_2 t + c_2)e^{-8t}$. Applying the first I.C. to the G.S., we find $c_1 = \frac{3}{4}$; and applying the second to its derivative, we have $-8c_1 + c_2 = 0$, from which $c_2 = 6$. Thus we find the solution $x = \left(\frac{3}{4} + 6t\right) e^{-8t}$.

6. This is a free damped motion, and Equation (5.27) applies. The mks system of units is used. Since a force of 4 newtons stretches the spring 5 cm = 0.05 meters, Hooke's Law $F = ks$ gives $4 = k(0.05)$, so $k = 80$ newtons/meter. Also, we are given that $m = 2$ and $a = 16$. Thus equation (5.27) becomes

$$2x'' + 16x' + 80x = 0$$

The I.C.'s are $x(0) = 0.02$ (in meters), $x'(0) = 0.04$ (in meters). Dividing the D.E. by 2, we have the auxiliary equation $r^2 + 8r + 40 = 0$, with roots $r = -4 \pm 2\sqrt{6}\, i$. The G.S. of the D.E. is

$$x = e^{-4t}(c_1 \sin 2\sqrt{6}\, t + c_2 \cos 2\sqrt{6}\, t). \qquad (1)$$

Differentiating this, we find

$$x' = e^{-4t}[(-4c_1 - 2\sqrt{6}\ c_2)\sin 2\sqrt{6}\ t + (2\sqrt{6}\ c_1 - 4c_2)\cos 2\sqrt{6}\ t]. \qquad (2)$$

Applying the first I.C. $x(0) = 0.02$ to (1), we find $c_2 = 2/100$; and applying the second, $x'(0) = 0.04$, to (2), we find $2\sqrt{6}\ c_1 - 4\ c_2 = 4/100$, from which $c_1 = \sqrt{6}/100$. Thus the solution is

$$x = \frac{e^{-4t}}{100}\ [\sqrt{6}\ \sin 2\sqrt{6}\ t + 2\ \cos 2\sqrt{6}\ t].$$

7. This is a free damped motion, and Equation (5.27) applies. Since a force of 20 lb. would stretch the spring 6 in. $= \frac{1}{2}$ ft., Hooke's Law $F = ks$ gives $20 = k\left(\frac{1}{2}\right)$, so $k = 40$ lb./ft. The weight is 4 lb., so $m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$ (slug); and $a = 2$. Thus equation (5.27) becomes

$$\frac{1}{8}\ x'' + 2\ x' + 40x = 0.$$

Since the weight is released from rest from a position 8 inches $= \frac{2}{3}$ ft. below its equilibrium position, we have the I.C.'s $x(0) = \frac{2}{3}$, $x'(0) = 0$. The D.E. may be written $x'' + 16\ x' + 320x = 0$. The auxiliary equation is $r^2 + 16r + 320 = 0$, with roots $-8 \pm 16i$. The G.S. of the D.E. is

$$x = e^{-8t}(c_1\ \sin 16t + c_2\ \cos 16t).$$

Differentiating we obtain

$$x' = e^{-8t}[(-8c_1 - 16c_2)\sin 16t + (16c_1 - 8c_2)\cos 16t].$$

Applying the first I.C. to the G.S., we find $c_2 = \frac{2}{3}$; and applying the second to the derived equation, we have $16c_1 - 8c_2 = 0$, from which $c_1 = \frac{c_2}{2} = \frac{1}{3}$. Thus we have the displacement

$$x = \left(\frac{e^{-8t}}{3}\right)(\sin 16t + 2\cos 16t).$$

To put this in the form (5.32), we multiply and divide by $c = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$, obtaining $x = \left(\frac{\sqrt{5}e^{-8t}}{3}\right)\left[\left(\frac{1}{\sqrt{5}}\right)\sin 16t + \left(\frac{2}{\sqrt{5}}\right)\cos 16t\right]$. We can now write this as

$$x = \left(\frac{\sqrt{5}e^{-8t}}{3}\right)\cos(16t - \phi). \qquad (1)$$

where $\cos\phi = \frac{2}{\sqrt{5}}$, $\sin\phi = \frac{1}{\sqrt{5}}$, and hence $\phi \approx 0.46$ (rad.). We have thus answered part (a).

To answer part (b), we see from (1) that the period is $\frac{2\pi}{16} = \frac{\pi}{8}$ (sec.). In the notation of page 202 of the text,

the logarithmic decrement is $\frac{2\pi b}{\sqrt{\lambda^2 - b^2}}$. Here $b = \frac{a}{2m} = \frac{2}{(2)\left(\frac{1}{8}\right)} = 8$ and $\lambda^2 = \frac{k}{m} = \frac{40}{\left(\frac{1}{8}\right)} = 320$. Thus we find the logarithmic decrement is $\frac{(2\pi)8}{\sqrt{256}} = \pi$.

To answer part (c), we let $x = 0$ in (1) and solve for t. We have $\cos(16t - \phi) = 0$, so $16t - \phi = \frac{\pi}{2}$, from which $t = \frac{\phi + \frac{\pi}{2}}{16}$. With $\phi = 0.46$, this gives $t = 0.127$ (sec.).

8. This is a free damped motion, and Equation (5.27) applies. Since the 24 lb. weight stretches the spring 1 ft., Hooke's Law $F = ks$ gives $24 = k(1)$, so $k = 24$ lb./ft. Also, $m = \frac{w}{g} = \frac{24}{32} = \frac{3}{4}$ (slug), and $a = 6$. Thus equation (5.27) becomes

$$\frac{3}{4}x'' + 6x' + 24x = 0.$$

or

$$x'' + 8x' + 32x = 0.$$

The I.C.'s are $x(0) = 1$, $x'(0) = 0$. The auxiliary equation is $m^2 + 8m + 32 = 0$, with roots $m = -4 \pm 4i$. The G.S. of the D.E. is

$$x = e^{-4t}(c_1 \sin 4t + c_2 \cos 4t). \qquad (1)$$

Differentiating this, we find

$$x' = e^{-4t}[(-4c_1 - 4c_2)\sin 4t + (4c_1 - 4c_2)\cos 4t]. \qquad (2)$$

Applying the first I.C. $x(0) = 1$, to (1), we find $c_2 = 1$; and applying the second, $x'(0) = 0$, to (2), we have $4c_1 - 4c_2 = 0$, from which $c_1 = 1$. Thus the solution is

$$x = e^{-4t}(\sin 4t + \cos 4t). \qquad (3)$$

This is the first answer to part (a). Top determine the alternate form (5.32), we multiply and divide (3) by $c = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$, obtaining $x = \sqrt{2}\, e^{-4t}\left(\frac{1}{\sqrt{2}}\sin 4t + \frac{1}{\sqrt{2}}\cos 4t\right)$. We can express this as

$$x = \sqrt{2}\, e^{-4t}\cos(4t - \pi/4). \qquad (4)$$

This is the displacement in the desired alternate form.

(b) The quasi period is $2\pi/4 = \pi/2$, and the time-varying amplitude is $\sqrt{2}\, e^{-4t}$

(c) The weight first attains a relative maximum displacement above its equilibrium position at the first time $t > 0$ at which the derivative $x'$ of (4) equals zero. Differentiating (4) and equating it to zero, we obtain

$$-4\sqrt{2}\, e^{-4t}\,[\sin(4t - \pi/4) + \cos(4t - \pi/4)] = 0. \qquad (5)$$

This holds when $\sin(4t - \pi/4) + \cos(4t - \pi/4) = 0$ and hence when $\tan(4t - \pi/4) = -1$. Thus (5) is satisfied at $4t - \pi/4 = -\pi/4 + k\pi$, that is, $t = k\pi/4$, where k is an integer. The first $t > 0$ for which this is valid is for $k = 1$, and this gives $t = \pi/4$. This is the time desired. The displacement at this time is found by letting $t = \pi/4$ in (4). Doing so, we find $x = \sqrt{2}\, e^{-\pi} \cos 3\pi/4 \approx -0.0432$. The minus sign simply indicates that the mass is above the equilibrium position. Thus the first maximum displacement above it is approximately 0.04 feet.

11. Equation (5.27) applies, with $m = \frac{w}{g} = \frac{10}{32} = \frac{5}{16}$ (slug), $a > 0$, and $k = 20$ lb./ft. Thus we have the D.E.

$$\frac{5}{16} x'' + a x' + 20x = 0.$$

Writing this as $5 x'' + 16ax' + 320x = 0$, the auxiliary equation is $5r^2 + 16ar + 320 = 0$. The roots are given by

$$r = \frac{-16a \pm \sqrt{256a^2 - 6400}}{10} \qquad (1)$$

In part (a), we seek the smallest value of a for which damping is nonoscillatory. This is the value of a for which damping is critical. It occurs when the two roots given by (1) are real and equal. Thus we set $256a^2 - 6400 = 0$ and find $a^2 = \frac{6400}{256}$ and hence $a = \frac{80}{16} = 5$.

In part (b), we let $a = 5$ in (1) and obtain the roots $r = -8, -8$ (double root). Then the displacement is given

by $x = (c_1 + c_2 t)e^{-8t}$. Differentiating this, we have $x' = (-8c_1 - 8c_2 t + c_2)e^{-8t}$. The I.C.'s are $x(0) = \frac{1}{2}$, $x'(0) = 1$. Applying them to the preceding expressions for x and x′, we find $c_1 = \frac{1}{2}$ and $-8c_1 + c_2 = 1$, from which $c_2 = 5$. Thus we obtain the displacement $x = \left(\frac{1}{2} + 5t\right)e^{-8t}$.

For part (c), we note that the extrema of the displacement are found by setting $x' = (1 - 40t)e^{-8t}$ equal to zero and solving for t. We at once have $t = \frac{1}{40}$. Then $x\left(\frac{1}{40}\right) = \left(\frac{5}{8}\right)e^{-1/5} \approx 0.51$(sec.). For $t > \frac{1}{40}$, $1 - 40t < 0$ and $x' < 0$. Also using L'Hospital's Rule, $\lim_{t \to \infty} x' = 0$. Thus the weight approaches its position monotonically.

12. This is a free damped motion, and Equation (5.27) applies. Since the 64 lb. weight stretches the spring 4/3 foot, Hooke's Law F = ks gives $64 = k(4/3)$, so $k = 48$ lb./ft. Also, $m = \frac{w}{g} = \frac{64}{32} = 2$ (slugs). The damping constant is $a > 0$. Thus equation (5.27) becomes

$$2x'' + ax' + 48x = 0. \qquad (1)$$

The I.C.'s are $x(0) = 2$, $x'(0) = 0$. The auxiliary equation is $2m^2 + am + 48 = 0$, with roots

$$m = \frac{-a \pm \sqrt{a^2 - 384}}{4}. \qquad (2)$$

(a) We must determine a so that the resulting motion is critically damped. This occurs when the auxiliary

equation has a double root. This occurs if and only if $a^2 - 384 = 0$ and hence if $a = \sqrt{384} = 8\sqrt{6}$ (recall $a > 0$). In this case the auxiliary equation has the double root $m = -2\sqrt{6}$, and the G.S. of the D.E. (1) is

$$x = (c_1 + c_2 t)e^{-2\sqrt{6}\,t}. \tag{3}$$

Differentiating this, we obtain

$$x' = [(c_2 - 2\sqrt{6}\,c_1) - 2\sqrt{6}\,c_2\,t]e^{-2\sqrt{6}\,t}. \tag{4}$$

Applying the I.C. $x(0) = 2$ to (3), we find $c_1 = 2$; and applying the I.C. $x'(0) = 0$ to (4), we have $c_2 - 2\sqrt{6}\,c_1 = 0$, from which $c_2 = 4\sqrt{6}$. Thus the displacement is given by

$$x = (2 + 4\sqrt{6}\,t)e^{-2\sqrt{6}\,t},$$

which is positive for all $t > 0$. The derivative of this is

$$x' = -48\,t\,e^{-2\sqrt{6}\,t}.$$

Since $x' < 0$ for all $t > 0$, x decreases monotonically for all $t > 0$.

(b) The motion is underdamped if $0 < a < 8\sqrt{6}$. Then $a^2 - 384 < 0$, the roots (2) are the conjugate complex

numbers $\left(-a \pm \sqrt{384 - a^2}\, i\right)/4$, and the G.S. of D.E. (1) is of the form

$$x = e^{-at/4}\left[c_1 \sin\frac{\sqrt{384 - a^2}}{4}\, t + c_2 \cos\frac{\sqrt{384 - a^2}}{4}\, t\right]. \qquad (5)$$

From the discussion on pages 200-201 of the text, we see that the quasi period is given by $\dfrac{2\pi}{\left(\dfrac{\sqrt{384 - a^2}}{4}\right)}$.

Equating this to the given value $\pi/2$ in the present case, we obtain $\sqrt{384 - a^2} = 16$, from which $a = 8\sqrt{2}$. This is the desired value of a in part (b).

With this value of a, the solution (5) becomes

$$x = e^{-2\sqrt{2}t}(c_1 \sin 4t + c_2 \cos 4t), \qquad (6)$$

with derivative

$$x' = e^{-2\sqrt{2}t}[(-2\sqrt{2}\, c_1 - 4\, c_2)\sin 4t + (4\, c_1 - 2\sqrt{2}\, c_2)\cos 4t].$$

Applying the I.C.'s $x(0) = 2$, $x'(0) = 0$ to these gives $c_2 = 2$, $4\, c_1 - 2\sqrt{2}\, c_2 = 0$, from which $c_1 = \sqrt{2}$. Thus the solution (6) becomes

$$x = e^{-2\sqrt{2}t}(\sqrt{2}\sin 4t + 2\cos 4t)$$

$$= \sqrt{6}\, e^{-2\sqrt{2}t}\left(\frac{\sqrt{2}}{\sqrt{6}}\sin 4t + \frac{2}{\sqrt{6}}\cos 4t\right)$$

$$= \sqrt{6}\, e^{-2\sqrt{2}t}\cos(4t - \phi),$$

where $\cos\phi = 2/\sqrt{6}$, $\sin\phi = \sqrt{2}/\sqrt{6}$. From this we see that the time-varying amplitude is $\sqrt{6}e^{-2\sqrt{2}t}$.

13. Here $m = \frac{w}{g} = \frac{32}{32} = 1$ (slug). If there were no resistance, the D.E. would be $m\,x'' + kx = 0$, that is, $x'' + kx = 0$, where $k > 0$. The auxiliary equation of this is $r^2 + k = 0$, with roots $\pm\sqrt{k}i$. The G.S. of this D.E. is then $x = c_1 \sin\sqrt{k}t + c_2 \cos\sqrt{k}t$. The period of this undamped motion is $\frac{2\pi}{\sqrt{k}}$, and hence the natural frequency is $\frac{\sqrt{k}}{2\pi}$. But this is given to be $\frac{4}{\pi}$. Hence we have $\frac{\sqrt{k}}{2\pi} = \frac{4}{\pi}$, from which $\sqrt{k} = 8$ and $k = 64$. This answers (a).

To answer (b), we take the resistance into account and have the D.E. $mx'' + ax' + kx = 0$, that is, $x'' + ax' + 64x = 0$. The auxiliary equation of this is $r^2 + ar + 64 = 0$ with roots $r = \frac{-a \pm \sqrt{a^2 - 256}}{2}$, where $a^2 < 256$. The G.S. of this D.E. is

$$x = e^{-at/2}\left(c_1 \sin\sqrt{256 - a^2}\,\frac{t}{2} + c_2 \cos\sqrt{256 - a^2}\,\frac{t}{2}\right).$$

The period of the trigonometric factor of this is $\frac{2\pi}{\left[\frac{\sqrt{256 - a^2}}{2}\right]} = \frac{4\pi}{\sqrt{256 - a^2}}$. Hence the frequency of this motion is $\frac{\sqrt{256 - a^2}}{4\pi}$. But the frequency of this damped motion is given as half the natural frequency $\frac{4}{\pi}$ and so is $\frac{2}{\pi}$. Thus $\frac{\sqrt{256 - a^2}}{4\pi} = \frac{2}{\pi}$. Thus $256 - a^2 = 64$, from which $a = 8\sqrt{3}$.

**Section 5.4, Page 217**

1. The D.E. is of the form (5.58) of the text, with $m = \frac{w}{g} = \frac{6}{32} = \frac{3}{16}$ (slugs), $a = 0$ (since damping is negligible), $k = 27$ lb./ft., and $F(t) = 12 \cos 20t$. Thus we have

$$\frac{3}{16} x'' + 27 x = 12 \cos 20t$$

or

$$x'' + 144 x = 64 \cos 20t. \qquad (1)$$

The I.C.'s are $x(0) = 0$, $x'(0) = 0$. The auxiliary equation of the homogeneous D.E. corresponding to (1) is $r^2 + 144 = 0$, with pure imaginary conjugate complex roots $r = \pm 12i$. Thus the complementary function of (1) is $x = c_1 \sin 12t + c_2 \cos 12t$. We use undetermined coefficients to find a particular integral of (1). We let $x_p = A \sin 20t + B \cos 20t$. Then $x_p' = 20A \cos 20t - 20B \sin 20t$, $x_p'' = -400A \sin 20t - 400B \cos 20t$. Substituting into (1), we obtain $-256A \sin 20t - 256B \cos 20t = 64 \cos 20t$.

From this, we find A = 0, B = -1/4; and hence we have the particular integral $x_p = -\frac{1}{4}\cos 20t$ of (1). Thus the G.S. of (1) is

$$x = c_1 \sin 12t + c_2 \cos 12t - \frac{1}{4}\cos 20t. \qquad (2)$$

Differentiating, we find

$$x' = 12\, c_1 \cos 12t - 12\, c_2 \sin 12t + 5 \sin 20t. \qquad (3)$$

Applying the I.C. x(0) = 0 to (2), we have $c_2 - (1/4) = 0$, so $c_2 = 1/4$. Applying the I.C. x'(0) = 0 to (3), $12\, c_1 = 0$, so $c_1 = 0$. Thus we have

$$x = \frac{\cos 12t - \cos 20t}{4}$$

3. The D.E. is of the form (5.58) of the text, with $m = \frac{w}{g} = \frac{10}{32} = \frac{5}{16}$ (slug), a = 5, k = 20, and $F(t) = 10 \cos 8t$. Thus we have

$$\frac{5}{16} x'' + 5 x' + 20x = 10 \cos 8t$$

or

$$x'' + 16 x' + 64x = 32 \cos 8t. \qquad (1)$$

The I.C.'s are x(0) = 0, x'(0) = 0. The auxiliary equation of the homogeneous D.E. corresponding to (1) is $r^2 + 16r + 64 = 0$ or $(r + 8)^2 = 0$, with roots -8, -8 (double root). Thus the complementary function of (1) is

$x_c = (c_1 + c_2 t)e^{-8t}$. We use undetermined coefficients to find a particular integral. We let $x_p = A \cos 8t + B \sin 8t$. Then $x_p' = -8A \sin 8t + 8B \cos 8t$ and $x_p'' = -64A \cos 8t - 64B \sin 8t$. Substituting into (1) and simplifying, we find $128B \cos 8t - 128A \sin 8t = 32 \cos 8t$. Hence $-128A = 0$, $128B = 32$, so $A = 0$, $B = \frac{1}{4}$. Thus we have the particular integral $x_p = \left(\frac{1}{4}\right) \sin 8t$, and the G.S. of D.E. (1) is $x = (c_1 + c_2 t)e^{-8t} + \left(\frac{1}{4}\right) \sin 8t$. Differentiating this, we obtain $x' = (-8c_1 - 8c_2 t + c_2)e^{-8t} + 2 \cos 8t$. Applying the stated I.C. to these, we have $c_1 = 0$, $-8c_1 + c_2 + 2 = 0$, and hence $c_2 = -2$. Thus we obtain the solution $x = -2te^{-8t} + \left(\frac{1}{4}\right) \sin 8t$.

4. The D.E. is of the form (5.58) of the text, with $m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$ (slug), $a = 2$, and $F(t) = 13 \sin 4t$. Also, by Hooke's Law $F = ks$, we have $4 = k\left(\frac{1}{4}\right)$, so $k = 16$ lb./ft. Thus we have

$$\frac{1}{8} x'' + 2x' + 16x = 13 \sin 4t,$$

or

$$x'' + 16x' + 128x = 104 \sin 4t. \qquad (1)$$

The I.C.'s are $x(0) = \frac{1}{2}$ and $x'(0) = 0$. The auxiliary equation of the homogeneous D.E. corresponding to (1) is $r^2 + 16r + 128 = 0$ with roots $r = -8 \pm 8i$. Thus the complementary function of (1) is

$$x = e^{-8t}(c_1 \sin 8t + c_2 \cos 8t).$$

We use undetermined coefficients to find a particular integral. We let $x_p = A \sin 4t + B \cos 4t$. Then $x_p' = 4A \cos 4t - 4B \sin 4t$, $x_p'' = -16A \sin 4t - 16B \cos 4t$. Substituting into (1) and simplifying, we find $(112A - 64B) \sin 4t + (64A + 112B) \cos 4t = 104 \sin 4t$. Thus we have the equations $112A - 64B = 104$ and $64A + 112B = 0$. These reduce to $14A - 8B = 13$ and $8A + 14B = 0$, from which we find $A = \frac{7}{10}$, $B = -\frac{2}{5}$. Thus we have the particular integral $x_p = \left(\frac{7}{10}\right) \sin 4t - \left(\frac{2}{5}\right) \cos 4t$ and the G.S. of the D.E. (1) is

$$x = e^{-8t}(c_1 \sin 8t + c_2 \cos 8t) + \left(\frac{7}{10}\right) \sin 4t - \left(\frac{2}{5}\right) \cos 4t.$$

Differentiating this, we obtain

$$x' = e^{-8t}[(-8c_1 - 8c_2) \sin 8t + (8c_1 - 8c_2) \cos 8t]$$

$$+ \left(\frac{14}{5}\right) \cos 4t + \left(\frac{8}{5}\right) \sin 4t.$$

Applying the stated I.C. to these, we have $c_2 - \frac{2}{5} = \frac{1}{2}$ and $8c_1 - 8c_2 + \frac{14}{5} = 0$. Hence $c_2 = \frac{9}{10}$ and $c_1 = \frac{11}{20}$. Thus we obtain the solution

$$x = e^{-8t}\left[\left(\frac{11}{20}\right) \sin 8t + \left(\frac{9}{10}\right) \cos 8t\right]$$

$$+ \left(\frac{7}{10}\right) \sin 4t - \left(\frac{2}{5}\right) \cos 4t.$$

This is the answer to part (a). Concerning part (b), we note that the steady-state term is $\left(\frac{7}{10}\right)\sin 4t - \left(\frac{2}{5}\right)\cos 4t$. The amplitude of this is given by $c = \sqrt{\left(\frac{7}{10}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{\sqrt{65}}{10}$.

6. The D.E. is of the form (5.58) of the text, with $m = \frac{w}{g} = \frac{32}{32} = 1$ (slug), $a = 2$, $k = 10$ lb./ft., and $F(t) = \sin t + \frac{1}{4}\sin 2t + \frac{1}{9}\sin 3t$. Thus we have

$$x'' + 2x' + 10x = \sin t + \frac{1}{4}\sin 2t + \frac{1}{9}\sin 3t . \qquad (1)$$

The I.C.'s are $x(0) = 0$, $x'(0) = 0$. The auxiliary equation of the homogeneous D.E. corresponding to (1) is $m^2 + 2m + 10 = 0$, with conjugate complex roots $-1 \pm 3i$. Thus the complementary function of (1) is

$$x_c = e^{-t}(c_1 \sin 3t + c_2 \cos 3t). \qquad (2)$$

The problem tells us to find $x_p$ using Theorem 4.10 of Chapter 4. We first find the particular integral $x_{p,k}$ of

$$x'' + 2x' + 10x = \sin kt, \quad (k \text{ constant}), \qquad (3)$$

by the method of undetermined coefficients. We let $x_{p,k} = A\sin kt + B\cos kt$. Then $x'_{p,k} = kA\cos kt - kB\sin kt$, $x''_{p,k} = -k^2A\sin kt - k^2B\cos kt$. Substituting into (3) and simplifying, we obtain $[(10 - k^2)A - 2kB]\sin kt + [2kA + (10 - k^2)B]\cos kt = \sin kt$. From this,

$$A = \frac{10 - k^2}{(10 - k^2)^2 + 4k^2}, \qquad B = \frac{-2k}{(10 - k^2)^2 + 4k^2}.$$

Thus we find

$$x_{p,k} = \frac{10 - k^2}{(10 - k^2)^2 + 4k^2} \sin kt - \frac{2k}{(10 - k^2)^2 + 4k^2} \cos kt \qquad (4)$$

as the particular integral of (3). Letting $k = 1$ in (4), we have the particular integral

$$x_{p,1} = \frac{9}{85} \sin t - \frac{2}{85} \cos t \qquad (5)$$

of the D.E. $x'' + 2x' + 10x = \sin t$. Letting $k = 2$ in (4), we have the particular integral

$$x_{p,2} = \frac{3}{26} \sin 2t - \frac{1}{13} \cos 2t \qquad (6)$$

of the D.E. $x'' + 2x' + 10x = \sin 2t$. Letting $k = 3$ in (4), we have the particular integral

$$x_{p,3} = \frac{1}{37} \sin 3t - \frac{6}{37} \cos 3t \qquad (7)$$

of the D.E. $x'' + 2x' + 10x = \sin 3t$.

We are now ready to apply Theorem 4.10 of Chapter 4, and we essentially do so twice. We first use the theorem with (5) and (6) to obtain the particular integral

$$x_p = \frac{9}{85}\sin t - \frac{2}{85}\cos t + \frac{1}{4}\left[\frac{3}{26}\sin 2t - \frac{1}{13}\cos 2t\right] \qquad (8)$$

of the D.E. $x'' + 2x' + 10x = \sin t + \frac{1}{4}\sin 2t$. We then use the theorem with (7) and (8) to obtain the particular integral

$$x_p = \left[\frac{9}{85}\sin t - \frac{2}{85}\cos t + \frac{1}{4}\left(\frac{3}{26}\sin 2t - \frac{1}{13}\cos 2t\right)\right] + \frac{1}{9}\left[\frac{1}{37}\sin 3t - \frac{6}{37}\cos 3t\right]$$

of the D.E.

$$x'' + 2x' + 10x = \sin t + \frac{1}{4}\sin 2t + \frac{1}{9}\sin 3t. \qquad (1)$$

Thus the G.S. of (1) is

$$x = e^{-t}(c_1 \sin 3t + c_2 \cos 3t) + \frac{9}{85}\sin t - \frac{2}{85}\cos t + \frac{3}{104}\sin 2t - \frac{1}{52}\cos 2t + \frac{1}{333}\sin 3t - \frac{6}{333}\cos 3t. \qquad (9)$$

Differentiating this, we obtain

$$x' = e^{-t}[(-c_1 - 3c_2)\sin 3t + (3c_1 - c_2)\cos 3t] + \frac{9}{85}\cos t + \frac{2}{85}\sin t + \frac{3}{52}\cos 2t + \frac{1}{26}\sin 2t + \frac{1}{111}\cos 3t + \frac{6}{111}\sin 3t. \qquad (10)$$

Applying the I.C. $x(0) = 0$ to (9), we obtain

$$c_2 - \frac{2}{85} - \frac{1}{52} - \frac{6}{333} = 0, \qquad \text{so} \qquad c_2 = \frac{29,819}{490,620}.$$

Applying the I.C. $x'(0) = 0$ to (10), we obtain $3c_1 - c_2 + \frac{9}{85} + \frac{3}{52} + \frac{1}{111} = 0$, from which $c_1 = -\frac{54,854}{1,471,860}$. Substituting these values of $c_1$ and $c_2$ into (9) gives the desired solution.

7. There are two problems here, one for $0 \leq t \leq \pi$, and the other for $t > \pi$. We first consider that for which $0 \leq t \leq \pi$. Here the D.E. is of the form (5.58), with $m = \frac{w}{g} = \frac{32}{32} = 1$ (slug), $a = 4$, $k = 20$, and $F(t) = 40 \cos 2t$. Thus we have

$$x'' + 4x' + 20x = 40 \cos 2t. \qquad (1)$$

The I.C.'s are $x(0) = 0$, $x'(0) = 0$. The auxiliary equation of the homogeneous D.E. corresponding to (1) is $r^2 + 4r + 20 = 0$, with roots $-2 \pm 4i$. Thus the complementary function of (1) is $x = e^{-2t}(c_1 \sin 4t + c_2 \cos 4t)$. We use undetermined coefficients to find a particular integral. We let $x_p = A \sin 2t + B \cos 2t$. Then $x_p' = 2A \cos 2t - 2B \sin 2t$, $x_p'' = -4A \sin 2t - 4B \cos 2t$. Substituting into (1) and simplifying, we find $(16A - 8B)\sin 2t + (8A + 16B) \cos 2t = 40 \cos 2t$. Thus we have $16A - 8B = 0$ and $8A + 16B = 40$, from which we find $A = 1$, $B = 2$. Thus we have the particular integral of (1), $x_p = \sin 2t + 2 \cos 2t$; and the G.S. of D.E. (1) is

$$x = e^{-2t}(c_1 \sin 4t + c_2 \cos 4t) + \sin 2t + 2 \cos 2t.$$

Differentiating this, we obtain

$$x' = e^{-2t}[(-2c_1 - 4c_2) \sin 4t + (4c_1 - 2c_2) \cos 4t] + 2 \cos 2t - 4 \sin 2t.$$

Applying the stated I.C. to these, we have $c_2 + 2 = 0$ and $4c_1 - 2c_2 + 2 = 0$, from which we find $c_1 = -\frac{3}{2}$, $c_2 = -2$. Thus we obtain the solution

$$x = e^{-2t}\left[\left(-\frac{3}{2}\right) \sin 4t - 2 \cos 4t\right] + \sin 2t + 2 \cos 2t \qquad (2)$$

valid for $0 \leq t \leq \pi$.

Now we consider the problem for which $t > \pi$. The D.E. is again of the form (5.58), where $m = 1$, $a = 4$, $k = 20$, but here $F(t) = 0$. Thus we have

$$x'' + 4x' + 20x = 0. \qquad (3)$$

Assuming the displacement is continuous at $t = \pi$, the solution of (3) must take the value given by (2) at $t = \pi$. That is, we must impose the I.C.

$$x(\pi) = -2e^{-2\pi} + 2 \qquad (4)$$

on the solution of (3). Similarly, assuming the velocity is continuous at $t = \pi$, the derivative of the solution of

(3) must take the value given by the derivative of (2) at $t = \pi$. The derivative of (2) is $x' = e^{-2t}[11 \sin 4t - 2 \cos 4t] + 2 \cos 2t - 4 \sin 2t$. From this, we see that we must therefore impose the I.C.

$$x'(\pi) = -2e^{-2\pi} + 2 \qquad (5)$$

on the solution of (3). The auxiliary equation of (3) is $r^2 + 4r + 20 = 0$ with roots $-2 \pm 4i$. Thus the G.S. of (3) is $x = e^{-2t}(k_1 \sin 4t + k_2 \cos 4t)$, and its derivative is

$$x' = e^{-2t}[(-2k_1 - 4k_2) \sin 4t + (4k_1 - 2k_2) \cos 4t].$$

Applying the I.C. (4) to this G.S., we have $k_2 e^{-2\pi} = -2e^{-2\pi} + 2$, from which $k_2 = 2(e^{2\pi} - 1)$. Applying the I.C. (5) to the derivative of this G.S., we have $(4k_1 - 2k_2)e^{-2\pi} = -2e^{-2\pi} + 2$, from which $k_1 = \left(\frac{3}{2}\right)(e^{2\pi} - 1)$. Thus we obtain the solution

$$x = (e^{2\pi} - 1)e^{-2t}\left[\left(\frac{3}{2}\right) \sin 4t + 2 \cos 4t\right],$$

valid for $t \geq \pi$.

**Section 5.5, Page 224**

1. (a) Since the 12 lb. weight stretches the spring 6 in. $= \frac{1}{2}$ ft., Hooke's Law, $F = ks$, gives $12 = k\left(\frac{1}{2}\right)$, so $k = 24$ lb./ft. Then since $m = \frac{w}{g} = \frac{12}{32} = \frac{3}{8}$ (slugs), $a = 3$, and $F(t) = 2 \cos \omega t$, The D.E. is

$$\frac{3}{8}x'' + 3x' + 24x = 2\cos \omega t. \qquad (1)$$

The I.C.'s are $x(0) = 0$, $x'(0) = 0$. The resonance frequency is given by formula (5.69) of the text. It is

$$\frac{1}{2\pi}\sqrt{\frac{24}{\frac{3}{8}} - \frac{9}{2\left(\frac{3}{8}\right)^2}} = \frac{2\sqrt{2}}{\pi}.$$

This is $\frac{\omega_1}{2\pi}$, where $\omega_1$ is the value of $\omega$ for which the forcing function is in resonance with the system. Thus $\frac{\omega_1}{2\pi} = \frac{2\sqrt{2}}{\pi}$, and so $\omega_1 = 4\sqrt{2}$.

We now let $\omega = \omega_1 = 4\sqrt{2}$ in (1), obtaining

$$\frac{3}{8}x'' + 3x' + 24x = 2\cos 4\sqrt{2}t$$

or

$$x'' + 8x' + 64x = \frac{16}{3}\cos 4\sqrt{2}t. \qquad (2)$$

The auxiliary equation of the corresponding homogeneous D.E. is $r^2 + 8r + 64 = 0$, with roots $r = -4 \pm 4\sqrt{3}i$. Thus the complementary function of (2) is

$$x_c = e^{-4t}(c_1 \sin 4\sqrt{3}t + c_2 \cos 4\sqrt{3}t).$$

We use undetermined coefficients to find a particular integral. We let

$$x_p = A \sin 4\sqrt{2}t + B \cos 4\sqrt{2}t.$$

Differentiating twice and substituting into (1), we find

$$\begin{cases} 32A - 32\sqrt{2}B = 0, \\ 32\sqrt{2}A + 32B = \dfrac{16}{3}; \end{cases}$$

and from these $A = \dfrac{\sqrt{2}}{18}$, $B = \dfrac{1}{18}$. Thus we obtain the general solution of (2) in the form

$$x = e^{-4t}(c_1 \sin 4\sqrt{3}t + c_2 \cos 4\sqrt{3}t) + \frac{\sqrt{2}}{18} \sin 4\sqrt{2}t + \frac{1}{18} \cos 4\sqrt{2}t.$$

Differentiating, we find

$$x' = e^{-4t}[(-4c_1 - 4\sqrt{3}c_2) \sin 4\sqrt{3}t + (4\sqrt{3}c_1 - 4c_2) \cos 4\sqrt{3}t] + \frac{4}{9} \cos 4\sqrt{2}t - \frac{2\sqrt{2}}{9} \sin 4\sqrt{2}t.$$

Applying the I.C.'s to these expressions for x and x', we obtain $c_2 + \dfrac{1}{18} = 0$, $4\sqrt{3}c_1 - 4c_2 + \dfrac{4}{9} = 0$. From these, we find $c_1 = \dfrac{-\sqrt{3}}{18}$, $c_2 = \dfrac{-1}{18}$. Thus we obtain the solution of (2) in the form

$$x = e^{-4t}\,\frac{-\sqrt{3}\sin 4\sqrt{3}t - \cos 4\sqrt{3}t}{18} + \frac{\sqrt{2}\sin 4\sqrt{2}t + \cos 4\sqrt{2}t}{18}.$$

(b) In this part m, k, and F(t) are as in (1), but a = 0, so we have the D.E.

$$\frac{3}{8}x'' + 24x = 2\cos \omega t. \tag{3}$$

The auxiliary equation of the corresponding homogeneous equation is $\left(\frac{3}{8}\right)r^2 + 24 = 0$ with roots $r = \pm 8i$. Thus the complementary function is

$$x_c = c_1 \sin 8t + c_2 \cos 8t. \tag{4}$$

Undamped resonance occurs when the frequency $\frac{\omega}{2\pi}$ of the impressed force equals the natural frequency $\frac{4}{\pi}$ and hence when $\omega = 8$. Letting $\omega = 8$ in (3), we have the D.E.

$$\frac{3}{8}x'' + 24x = 2\cos 8t. \tag{5}$$

The complementary function is given by (4). We find a particular integral using undetermined coefficients. We modify the UC set $\{\sin 8t, \cos 8t\}$ of $\cos 8t$ by multiplying each member by t, obtaining $\{t \sin 8t, t \cos 8t\}$. Thus we assume $x_p = At \sin 8t + Bt \cos 8t$. Differentiating twice and substituting into (5), we find $A = \frac{1}{3}$, $B = 0$. Thus we obtain the general solution of (5) in the form

$$x = c_1 \sin 8t + c_2 \cos 8t + t \sin \frac{8t}{3}.$$

Differentiating this, we find

$$x' = 8c_1 \cos 8t - 8c_2 \sin 8t + 8t \cos \frac{8t}{3} + \sin \frac{8t}{3}.$$

The I.C.'s are the same as in part (a). Applying them to these expressions for x and x′, we readily find $c_1 = 0$, $c_2 = 0$. Thus we obtain the solution of (5) in the form $x = t \sin \frac{8t}{3}$.

2. The 20 lb. weight stretches the spring 6 inches; so by Hooke's Law, $F = ks$, we have $20 = k\left(\frac{1}{2}\right)$, from which $k = 40$ lb./ft. Also, $m = \frac{w}{g} = \frac{20}{32} = \frac{5}{8}$ (slugs). Thus the D.E. is

$$\frac{5}{8} x'' + ax' + 40x = \cos \omega t$$

or

$$x'' + \frac{8}{5} ax' + 64x = \frac{8}{5} \cos \omega t . \qquad (1)$$

The auxiliary equation of the corresponding homogeneous D.E. is $r^2 + (8a/5)r + 64 = 0$, with roots $r = -\frac{4}{5} a \pm \sqrt{\frac{16}{25} a^2 - 64}$. For resonance the system must be underdamped, so $\frac{16}{25} a^2 - 64 < 0$. Then the complementary function can be written

$$x_c = ce^{-(4a/5)t} \cos\left(\sqrt{64 - \frac{16a^2}{25}}\, t + \phi\right). \qquad (2)$$

The D.E. (1) is of the form (5.51) on page 212 of the text, and the complementary function (2) is of the form (5.52), where $2b = \frac{8}{5}a$, $\lambda^2 = 64$.

The resonance frequency is given as 0.5 (cycles/sec.). From the text, pages 219-220, this is $\omega_1/2\pi$, where $\omega_1 = \sqrt{\lambda^2 - 2b^2}$. With the values of 2b and $\lambda^2$ just noted in this problem, as have $\lambda^2 - 2b^2 = 64 - \frac{32}{25}a^2$. Thus we obtain

$$\frac{\sqrt{64 - \frac{32}{25}a^2}}{2\pi} = 0.5.$$

Thus $\sqrt{64 - \frac{32}{25}a^2} = \pi$, and from this $a^2 = \frac{25}{32}(64 - \pi^2)$.

Thus recalling that $a > 0$, we find $a = 5\sqrt{2 - \frac{\pi^2}{32}}$.

**Section 5.6, Page 232**

1. Let i denote the current in amperes at time t. The total electromotive force is 40 V. Using the voltage drop laws 1 and 2 of the text (page 225) we find the following voltage drops:

   1. across the resistor: $E_R = Ri = 10i$.
   2. across the inductor: $E_L = Li' = 0.2i'$.

   Applying Kirchhoff's Law, we have the D.E.

$$0.2i' + 10i = 40. \qquad (1)$$

Since the initial current is 0, the I.C. is

$$i(0) = 0.$$

The D.E. (1) is a first order linear D.E. In standard form it is

$$i' + 50\,i = 200; \qquad (2)$$

an I.F. is $e^{\int 50t\,d} = e^{50t}$. Multiplying (2) through by this, we obtain

$$e^{50t}i' + 50e^{50t}i = 200e^{50t}$$

or

$$[e^{50t}i]' = 200e^{50t}.$$

Integrating and simplifying, we find

$$i = 4 + ce^{-50t}.$$

Applying the I.C. $i = 0$ at $t = 0$ to this, we find $c = -4$. Thus we obtain the solution

$$i = 4(1 - e^{-50t}).$$

3. Let i denote the current and let q denote the charge on the capacitor at time t. The electromotive force is 100 V. Using the voltage drop laws 1 and 3, we find the following voltage drops:

   1. across the resistor: $E_R = Ri = 10i$.
   2. across the capacitor: $E_C = \frac{1}{c}\,q = (10)^4\,\frac{q}{2}$.

Applying Kirchhoff's Law, we have the equation

$$10i + (10)^4 \frac{q}{2} = 100.$$

Since $i = q'$, this reduces to

$$10q' + (10)^4 \frac{q}{2} = 100. \qquad (1)$$

Since the charge is initially zero, we have the I.C.

$$q(0) = 0.$$

The D.E. (1) is a first order linear D.E. In standard form it is

$$q' + 500q = 10, \qquad (2)$$

and an I.F. is $e^{\int 500\,dt} = e^{500t}$. Multiplying (2) through by this, we obtain

$$e^{500t}q' + 500e^{500t}q = 10e^{500t}$$

or

$$[e^{500t}q]' = 10e^{500t}.$$

Integrating and simplifying, we find

$$q = \frac{1}{50} + ce^{-500t}.$$

Applying the I.C. $q = 0$ at $t = 0$ to this, we find $c = \frac{-1}{50}$. Thus we obtain the solution

$$q = \frac{1 - e^{-500t}}{50}.$$

This is the charge. To find the current, we return to $10i + (10)^4 \frac{q}{2} = 100$, substitute the expression for q just found, and solve for i. We find $i = -500q + 10$ and hence $i = 10e^{-500t}$.

6. Let i denote the current in amperes at time t. The total electromotive force is $200e^{-100t}$. Using the voltage drop laws 1, 2, and 3, we find the following voltage drops:

1. across the resistor: $E_R = Ri = 80i$.
2. across the inductor: $E_L = Li' = 0.2i'$
3. across the capacitor: $E_c = \frac{1}{c} q = 10^6 \frac{q}{5}$.

Applying Kirchhoff's Law, we have the D.E.

$$0.2i' + 80i + 10^6 \frac{q}{5} = 200e^{-100t}.$$

Since $i = q'$, this reduces to

$$0.2q'' + 80q' + 10^6 \frac{q}{5} = 200e^{-100t}. \qquad (1)$$

Since the charge q is initially zero, we have the first I.C.

$$q(0) = 0.$$

Since the current i is initially zero and $i = q'$, we have the second I.C.

$$q'(0) = 0.$$

The homogeneous D.E. corresponding to D.E. (1) has the auxiliary equation

$$\frac{1}{5} r^2 + 80r + 200{,}000 = 0,$$

and the roots of this are $-200 \pm 979.8i$. Thus the complementary function of D.E. (1) is

$$q_c = e^{-200t}(c_1 \sin 979.8t + c_2 \cos 979.8t).$$

We use undetermined coefficients to find a particular integral. We write

$$q_p = Ae^{-100t}.$$

Differentiating twice and substituting into (1), we find $A = \frac{1}{970} \approx 0.0010$. Thus the general solution of D.E. (1) is

$$q = e^{-200t}(c_1 \sin 979.8t + c_2 \cos 979.8t) + \frac{e^{-100t}}{970}.$$

Differentiating this, we obtain

$$q' = e^{-200t}[(-200c_1 - 979.8c_2) \sin 979.8t + (979.8c_1 - 200c_2) \cos 979.8t] - \frac{10e^{-100t}}{97}. \quad (2)$$

Applying the I.C.'s to these expressions for q and q', we have

$$c_2 + \frac{1}{970} = 0, \quad 979.8c_1 - 200c_2 - \frac{10}{97} = 0,$$

from these we find that $c_1 = \dfrac{-10}{(97)(979.8)} \approx -0.0001$ and $c_2 = \dfrac{-1}{970} \approx -0.0010$. Thus we obtain

$$q = e^{-200t}(-0.0001 \sin 979.8t - 0.0010 \cos 979.8t) + 0.0010\, e^{-100t}.$$

Since $i = q'$, using formula (2) for $q'$ and the values of $c_1$ and $c_2$ determined above, we find

$$i = e^{-200t}(1.0311 \sin 979.8t + 0.1031 \cos 979.8t) - 0.1031e^{-100t}.$$

# Chapter 6

**Section 6.1, Page 249**

2. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Substituting into the D.E., we obtain

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 8x \sum_{n=1}^{\infty} n c_n x^{n-1} - 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 8 \sum_{n=1}^{\infty} n c_n x^n - 4 \sum_{n=0}^{\infty} c_n x^n = 0.$$

We rewrite the first summation so that x has the exponent n. Doing so, we have:

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + 8 \sum_{n=1}^{\infty} n c_n x^n - 4 \sum_{n=0}^{\infty} c_n x^n = 0.$$

The common range of these three summations is from 1 to $\infty$. We write out the individual terms in each that do *not* belong to this common range. Thus we have

$$(2c_2 - 4c_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} + 4(2n-1) c_n] x^n = 0.$$

Equating to zero the coefficient of each power of x, we obtain

$$2c_2 - 4c_0 = 0, \tag{1}$$

$$(n + 2)(n + 1)c_{n+2} + 4(2n - 1)c_n = 0, \quad n \geq 1. \tag{2}$$

From (1) we have $c_2 = 2c_0$. From (2), we obtain

$$c_{n+2} = -\frac{4(2n - 1)c_n}{(n + 1)(n + 2)}, \quad n \geq 1.$$

Using this, we find $c_3 = -\frac{2c_1}{3}$, $c_4 = -c_2 = -2c_0$, $c_5 = -c_3 = \frac{2c_1}{3}$. Substituting these values into the assumed solution, we have

$$y = c_0 + c_1x + 2c_0x^2 - \frac{2c_1}{3}x^3 - 2c_0x^4 + \frac{2c_1}{3}x^5 + \cdots$$

or

$$y = c_0(1 + 2x^2 - 2x^4 + \cdots) + c_1\left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5 + \cdots\right).$$

4. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2}$. Substituting into the D.E., we obtain

$$\sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^{n-1} + 3x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2} = 0.$$

We rewrite the summations so that x has the same exponent in each. We choose n - 2 as this common exponent, leave the first summation alone, and rewrite the second and third accordingly. We have

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} (n-1)c_{n-1} x^{n-2} + 3\sum_{n=4}^{\infty} c_{n-4} x^{n-2} = 0.$$

The common range of these three summations is from 4 to $\infty$. We write out the individual terms in each that do *not* belong to this common range. Thus we have

$$(2c_2 + c_1) + (6c_3 + 2c_2)x + \sum_{n=4}^{\infty} [n(n-1)c_n + (n-1)c_{n-1} + 3c_{n-4}]x^{n-2} = 0.$$

Equating to zero the coefficient of each power of x, we obtain

$$2c_2 + c_1 = 0,\ 6c_3 + 2c_2 = 0, \tag{1}$$

$$n(n-1)c_n + (n-1)c_{n-1} + 3c_{n-4} = 0,\ n \geq 4. \tag{2}$$

From (1), we have $c_2 = -\frac{c_1}{2}$, $c_3 = -\frac{c_2}{3} = \frac{c_1}{6}$. From (2), we obtain

$$c_n = -\frac{(n-1)c_{n-1} + 3c_{n-4}}{n(n-1)}, \quad n \geq 4.$$

Using this, we find $c_4 = -\frac{c_3 + c_0}{4} = -\frac{c_0}{4} - \frac{c_1}{24}$,

$c_5 = -\frac{4c_4 + 3c_1}{20} = -\frac{1}{5}\left(-\frac{c_0}{4} - \frac{c_1}{24}\right) - \frac{3}{20}c_1 = \frac{c_0}{20} - \frac{17c_1}{120}$.

Substituting these values into the assumed solution, we have

$$y = c_0 + c_1x - \frac{c_1}{2}x^2 + \frac{c_1}{6}x^3 - \left(\frac{c_0}{4} + \frac{c_1}{24}\right)x^4 + \left(\frac{c_0}{20} - \frac{17c_1}{120}\right)x^5 + \cdots$$

or

$$y = c_0\left(1 - \frac{1}{4}x^4 + \frac{1}{20}x^5 + \cdots\right) + c_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{17}{120}x^5 + \cdots\right).$$

5. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting into the D.E., we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + 2x^2\sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^{n+2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We rewrite the first and third summations so that x has the exponent n in each. Thus we have

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=2}^{\infty} 2c_{n-2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

The common range of these four summations is from 2 to $\infty$. We write out the individual terms in each that do *not* belong to this range. Thus we have:

$$(c_0 + 2c_2) + (2c_1 + 6c_3)x + \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n + 2c_{n-2}]x^n = 0.$$

Equating to zero the coefficient of each power of x, we obtain

$$c_0 + 2c_2 = 0, \quad 2c_1 + 6c_3 = 0, \tag{1}$$

$$(n+2)(n+1)c_{n+2} + (n+1)c_n + 2c_{n-2} = 0, \quad n \geq 2. \tag{2}$$

From (1), we find $c_2 = \frac{-c_0}{2}$, $c_3 = \frac{-c_3}{3} = \frac{-c_1}{3}$. From (2), we find

$$c_{n+2} = -\frac{(n+1)c_n + 2c_{n-2}}{(n+1)(n+2)}, \quad n \geq 2.$$

Using this, we find $c_4 = -\frac{3c_2 + 2c_0}{12} = \frac{-c_0}{24}$;

$c_5 = -\frac{4c_3 + 2c_1}{20} = \frac{-c_1}{30}$. Substituting these values into the assumed solution, we have

$$y = c_0 + c_1x - \left(\frac{c_0}{2}\right)x^2 - \left(\frac{c_1}{3}\right)x^3 - \left(\frac{c_0}{24}\right)x^4 - \left(\frac{c_1}{30}\right)x^5 + \cdots$$

or

$$y = c_0\left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + c_1\left(x - \frac{x^3}{3} - \frac{x^5}{30} + \cdots\right).$$

9. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting into the D.E., we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x^3 \sum_{n=1}^{\infty} nc_n x^{n-1} - 2\sum_{n=1}^{\infty} nc_n x^{n-1} - 6x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^{n+2} - 2\sum_{n=1}^{\infty} nc_n x^{n-1} - 6\sum_{n=0}^{\infty} c_n x^{n+2} = 0.$$

We write the first and third summations so that x has the exponent n + 2 in each, as it already has in the second and fourth summations. We have

$$\sum_{n=-2}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2} - \sum_{n=1}^{\infty} nc_n x^{n+2} - 2\sum_{n=-2}^{\infty} (n+3)c_{n+3}x^{n+2} - 6\sum_{n=0}^{\infty} c_n x^{n+2} = 0.$$

The common range of these four summations is from 1 to $\infty$. We write out the individual terms in each that do *not* belong to this common range. We have

$$(2c_2 - 2c_1) + (6c_3 - 4c_2)x + (12c_4 - 6c_3 - 6c_0)x^2 + \sum_{n=1}^{\infty} [(n+4)(n+3)c_{n+4} - 2(n+3)c_{n+3} - (n+6)c_n]x^{n+2} = 0.$$

Equating to zero the coefficient of each power of x, we obtain

$$2c_2 - 2c_1 = 0,\ 6c_3 - 4c_2 = 0,$$
$$12c_4 - 6c_3 - 6c_0 = 0, \tag{1}$$
$$(n+4)(n+3)c_{n+4} - 2(n+3)c_{n+3} - (n+6)c_n = 0,\ n \geq 1. \tag{2}$$

From (1), we find $c_2 = c_1$, $c_3 = \frac{2c_2}{3} = \frac{2c_1}{3}$, $c_4 = \frac{c_0 + c_3}{2} = \frac{c_0}{2} + \frac{c_1}{3}$. From (2), we obtain

$$c_{n+4} = \frac{(n+6)c_n + 2(n+3)c_{n+3}}{(n+3)(n+4)}, \quad n \geq 1.$$

Using this, we find $c_5 = \frac{7c_1 + 8c_4}{20} = \frac{c_0}{5} + \frac{29c_1}{60}$.

Substituting these values into the assumed solution, we have

$$y = c_0 + c_1x + c_1x^2 + \frac{2c_1}{3}x^3 + \left(\frac{c_0}{2} + \frac{c_1}{3}\right)x^4 + \left(\frac{c_0}{5} + \frac{29c_1}{60}\right)x^5 + \cdots$$

or

$$y = c_0\left(1 + \frac{4}{2}x^2 + \frac{x^5}{5} + \cdots\right) + c_1\left(x + x^2 + \frac{2x^3}{3} + \frac{x^4}{3} + \frac{29x^5}{60} + \cdots\right).$$

10. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting into the D.E., we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x^2 \sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$- x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^{n+1} - \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We rewrite the summations so that x has the common exponent n in each. We have

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=2}^{\infty} (n-1)c_{n-1} x^n$$

$$- \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

The common range of these four summations is from 2 to $\infty$. We write out the individual terms in each that do *not* belong to this common range. Thus we have

$$(2c_2 + c_0) + (6c_3 - c_1 + c_1)x$$

$$+ \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} - (n-1)c_{n-1}$$

$$- (n-1)c_n] x^n = 0.$$

Equating to zero the coefficients of each power of x, we obtain

$$2c_2 + c_0 = 0,\ 6c_3 = 0, \tag{1}$$

$$(n+1)(n+2)c_{n+2} - (n-1)c_{n-1} - (n-1)c_n = 0,\ n \geq 2. \tag{2}$$

From (1), we have $c_2 = -\frac{c_0}{2}$ and $c_3 = 0$. From (2), we obtain

$$c_{n+2} = \frac{(n-1)(c_{n-1} + c_n)}{(n+1)(n+2)}, \quad n \geq 2.$$

Using this, we find $c_4 = \frac{c_1 + c_2}{12} = -\frac{c_0}{24} + \frac{c_1}{12}$,

$c_5 = \frac{c_2 + c_3}{10} = -\frac{c_0}{20}$, $c_6 = \frac{c_3 + c_4}{10} = -\frac{c_0}{240} + \frac{c_1}{120}$.

Substituting these values into the assumed solution, we have

$$y = c_0 + c_1 x - \frac{c_0}{2}x^2 + \left(-\frac{c_0}{24} + \frac{c_1}{12}\right)x^4 - \frac{c_0}{20}x^5 + \left(-\frac{c_0}{240} + \frac{c_1}{120}\right)x^6 + \cdots$$

or

$$y = c_0\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{20}x^5 - \frac{1}{240}x^6 \cdots\right) + c_1\left(x + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \cdots\right).$$

11. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting into the D.E., we obtain

$$x^2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$+ x\sum_{n=1}^{\infty} nc_n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

We rewrite the second and fourth summations so that x has the exponent n in each. We have:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$+ \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0.$$

The common range of these four summations is from 2 to $\infty$. We write out the individual terms in each that do *not* belong to this range. Thus we have:

$$2c_2 + (c_0 + c_1 + 6c_3)x$$

$$+ \sum_{n=2}^{\infty} \{(n+2)(n+1)c_{n+2} + [n(n-1) + n]c_n$$

$$+ c_{n-1}\} x^n = 0.$$

Equating to zero the coefficients of each power of x, we obtain

$$2c_2 = 0,\ c_0 + c_1 + 6c_3 = 0, \tag{1}$$

$$(n+2)(n+1)c_{n+2} + [n(n-1) + n]c_n + c_{n-1} = 0,\ n \geq 2 \tag{2}$$

From (1), we find $c_2 = 0$, $c_3 = -\dfrac{c_0 + c_1}{6}$. From (2), we find

$$c_{n+2} = -\frac{n^2 c_n + c_{n-1}}{(n+1)(n+2)}, \quad n \geq 2.$$

Using this, we find $c_4 = -\dfrac{4c_2 + c_1}{12} = -\dfrac{c_1}{12}$;

$c_5 = -\dfrac{9c_3 + c_2}{20} = \left(\dfrac{3}{40}\right)(c_0 + c_1)$. Substituting these values into the assumed solution, we have

$$y = c_0 + c_1 x - \frac{(c_0 + c_1)x^3}{6} - \frac{c_1 x^4}{12} + \frac{3(c_0 + c_1)x^5}{40} + \cdots$$

or

$$y = c_0\left(1 - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots\right) + c_1\left(x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{3x^5}{40} + \cdots\right).$$

13. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting into the D.E., we obtain

$$x^3\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$+ x^2\sum_{n=1}^{\infty} nc_n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n+1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$+ \sum_{n=1}^{\infty} nc_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

We rewrite the second summation so that x has the exponent n + 1 in it (since this is the exponent of x in all the other summations). Thus we have:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n+1} - \sum_{n=-1}^{\infty} (n+3)(n+2)c_{n+3} x^{n+1}$$

$$+ \sum_{n=1}^{\infty} nc_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

The common range of these four summations is from 2 to $\infty$. We write out the individual terms in each that do *not* belong to this range. Thus we have:

$$-2c_2 + (c_0 - 6c_3)x + (2c_1 - 12c_4)x^2$$

$$+ \sum_{n=2}^{\infty} \{-(n + 3)(n + 2)c_{n+3}$$

$$+ [n(n - 1) + n + 1]c_n\}x^n = 0.$$

Equating to zero the coefficients of each power of x, we obtain

$$-2c_2 = 0,\ c_0 - 6c_3 = 0,\ 2c_1 - 12c_4 = 0, \tag{1}$$

$$-(n + 3)(n + 2)c_{n+3} + [n(n - 1) + n + 1]c_n = 0,\ n \geq 2. \tag{2}$$

From (1), we find $c_2 = 0$, $c_3 = \frac{c_0}{6}$, $c_4 = \frac{c_1}{6}$. From (2), we find

$$c_{n+3} = \frac{(n^2 + 1)c_n}{(n + 2)(n + 3)}, \quad n \geq 2.$$

Using this, we find $c_5 = \frac{5c_2}{20} = 0$; $c_6 = \frac{10c_3}{30} = \frac{c_3}{3} = \frac{c_0}{18}$; and $c_7 = \frac{17c_4}{42} = \frac{17c_1}{252}$. Substituting these values into the assumed solution, we have

$$y = c_0 + c_1x + \frac{c_0x^3}{6} + \frac{c_1x^4}{6} + \frac{c_0x^6}{18} + \frac{17c_1x^7}{252} + \cdots$$

or

$$y = c_0\left(1 + \frac{x^3}{6} + \frac{x^6}{18} + \cdots\right) + c_1\left(x + \frac{x^4}{6} + \frac{17x^7}{252} + \cdots\right).$$

15. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Substituting into the D.E., we obtain

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} (-n c_n) x^n + \sum_{n=0}^{\infty} (-c_n) x^n = 0.$$

We rewrite the first summation so that x has an exponent n. Thus we have:

$$\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n + \sum_{n=1}^{\infty} (-n c_n) x^n + \sum_{n=0}^{\infty} (-c_n) x^n = 0.$$

The common range of these three summations is from 1 to $\infty$. We write out the terms from the 1st and 3rd that do *not* belong in this range. Thus we have:

$$(2c_2 - c_0) + \sum_{n=1}^{\infty} [(n+1)(n+2) c_{n+2} + (-n-1) c_n] x^n = 0.$$

Equating the coefficients of each power of x to zero we obtain

$$2c_2 - c_0 = 0, \tag{1}$$

$$(n + 1)(n + 2)c_{n+2} + (-n - 1)c_n = 0, \; n \geq 1. \tag{2}$$

From (1), we find $c_2 = \frac{1}{2} c_0$. From (2), we find

$$c_{n+2} = \frac{1}{n + 2} c_n, \quad n \geq 1.$$

Using this, we find $c_3 = \frac{1}{3} c_1$; $c_4 = \frac{1}{4} c_2 = \frac{1}{8} c_0$;

$c_5 = \frac{1}{5} c_3 = \frac{1}{15} c_1$; $c_6 = \frac{1}{6} c_4 = \frac{1}{48} c_0$. Thus the G.S. of the D.E. is

$$y = c_0\left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \cdots\right) + c_1\left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots\right).$$

We now apply the I.C. $y(0) = 1$ to this, obtaining $c_0 = 1$. Differentiating we find

$$y' = x + \frac{1}{2} x^3 + \frac{1}{8} x^5 + \cdots + c_1\left(1 + x^2 + \frac{1}{3} x^4 + \cdots\right).$$

Applying the I.C. $y'(0) = 0$ to this, we have $c_1 = 0$. Thus we have the solution

$$y = 1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \cdots.$$

Writing this as

$$y = 1 + \frac{1}{2^1 1!} x^2 + \frac{1}{2^2 2!} x^4 + \frac{1}{2^3 3!} x^6 + \cdots$$

and observing that for $n \geq 1$,

$$c_{2n} = \frac{1}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{1}{2^n n!},$$

we express the solution simply as

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}.$$

18. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Substituting into the D.E., we obtain

$$x^2 \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} + 2x \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^{n+1} = 0.$$

We rewrite the second and fourth summations so that x has the exponent n in each. Thus we have:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$+ \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=1}^{\infty} 2c_{n-1} x^n = 0.$$

The common range of these four summations is from 2 to $\infty$. We write out the individual terms in each that do *not* belong to this range. Thus we have:

$$2c_2 + (6c_3 + c_1 + 2c_0)x + \sum_{n=2}^{\infty} \{[n(n-1) + n]c_n$$

$$+ (n+2)(n+1)c_{n+2} + 2c_{n-1}\} x^n = 0.$$

Equating to zero the coefficients of each power of x, we obtain

$$2c_2 = 0,\ 2c_0 + c_1 + 6c_3 = 0, \tag{1}$$

$$[n(n-1) + n]c_n + (n+2)(n+1)c_{n+2} + 2c_{n-1} = 0,\ n \geq 2. \tag{2}$$

From (1), we find $c_2 = 0$, $c_3 = -\dfrac{2c_0 + c_1}{6}$. From (2), we find

$$c_{n+2} = -\frac{n^2 c_n + 2c_{n-1}}{(n+1)(n+2)}, \quad n \geq 2.$$

Using this, we find $c_4 = -\dfrac{4c_2 + 2c_1}{12} = -\dfrac{c_1}{6}$,

$c_5 = -\dfrac{9c_3 + 2c_2}{20} = \dfrac{3(2c_0 + c_1)}{40}$. Substituting these values into the assumed solution, we have

$$y = c_0 + c_1x - \frac{(2c_0 + c_1)x^3}{6} - \frac{c_1x^4}{6} + \frac{3(2c_0 + c_1)x^5}{40} + \cdots$$

or

$$y = c_0\left(1 - \frac{x^3}{3} + \frac{3x^5}{20} + \cdots\right) + c_1\left(x - \frac{x^3}{6} - \frac{x^4}{6} + \frac{3x^5}{40} + \cdots\right).$$

We now apply the I.C. $y(0) = 2$ to this, obtaining $c_0 = 2$. Differentiating we find

$$y' = c_0\left(-x^2 + \frac{3x^4}{4} + \cdots\right) + c_1\left(1 - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{3x^4}{8} + \cdots\right).$$

Applying the I.C. $y'(0) = 3$ to this, we have $c_1 = 3$. Thus we have the solution

$$y = 2\left(1 - \frac{x^3}{3} + \frac{3x^5}{20} + \cdots\right) + 3\left(x - \frac{x^3}{6} - \frac{x^4}{6} + \frac{3x^5}{40} + \cdots\right)$$

or

$$y = 2 + 3x - \frac{7x^3}{6} - \frac{x^4}{2} + \frac{21x^5}{40} + \cdots.$$

20. We assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Substituting into the D.E., we obtain

$$x^2 \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 4x \sum_{n=1}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 4 \sum_{n=1}^{\infty} n c_n x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0.$$

We rewrite the second summation so that x has the exponent n, as in the other three summations. We have:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + 4 \sum_{n=1}^{\infty} n c_n x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0.$$

The common range of these summations is from 2 to $\infty$. We write out the individual terms that do *not* belong to this common range. We have:

$$(-2c_2 + 2c_0) + (-6c_3 + 6c_1)x$$

$$+ \sum_{n=2}^{\infty} \{[n(n-1) + 4n + 2]c_n$$

$$- (n+2)(n+1)c_{n+2}\}x^n = 0.$$

Noting that $n(n-1) + 4n + 2 = (n+1)(n+2)$, we equate to zero the coefficients of each power of x, to obtain

$$-2c_2 + 2c_0 = 0, \quad -6c_3 + 6c_1 = 0, \tag{1}$$

$$(n+1)(n+2)c_n - (n+1)(n+2)c_{n+2} = 0, \quad n \geq 2. \tag{2}$$

From (1), we have $c_2 = c_0$ and $c_3 = c_1$. From (2), we obtain $c_{n+2} = c_n$, $n \geq 2$. From this we observe that $c_{2n} = c_0$ and $c_{2n+1} = c_1$ for all $n \geq 1$. Thus we obtain the G.S.

$$y = c_0\left(1 + x^2 + x^4 + \cdots\right) + c_1\left(x + x^3 + x^5 + \cdots\right)$$

$$= c_0 \sum_{n=0}^{\infty} x^{2n} + c_1 \sum_{n=0}^{\infty} x^{2n+1}.$$

We now apply the I.C. $y(0) = 1$ to this, obtaining $c_0 = 1$. Then differentiating we have

$$y' = (2x + 4x^3 + \cdots) + c_1(1 + 3x^2 + 5x^4 + \cdots).$$

Applying the I.C. $y'(0) = -1$ to this, we find $c_1 = -1$. Thus we obtain the desired solution

$$y = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n.$$

22. We assume $y = \sum_{n=0}^{\infty} c_n(x - 1)^n$. Then $y' = \sum_{n=1}^{\infty} nc_n(x - 1)^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n - 1)c_n(x - 1)^{n-2}$. Substituting into the D.E., we obtain

$$x^2\sum_{n=2}^{\infty} n(n - 1)c_n(x - 1)^{n-2} + 3x\sum_{n=1}^{\infty} nc_n(x - 1)^{n-1}$$

$$- \sum_{n=0}^{\infty} c_n(x - 1)^n = 0.$$

Since the summations involve powers of $x - 1$, we must express the respective "coefficients" $x^2$ and $3x$ of the first two summations in powers of $x - 1$. Thus we have

$$[(x - 1)^2 + 2(x - 1) + 1]\sum_{n=2}^{\infty} n(n - 1)c_n(x - 1)^{n-2}$$

$$+ [3(x - 1) + 3]\sum_{n=1}^{\infty} nc_n(x - 1)^{n-1}$$

$$- \sum_{n=0}^{\infty} c_n(x - 1)^n = 0.$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^n + \sum_{n=2}^{\infty} 2n(n-1)c_n(x-1)^{n-1}$$
$$+ \sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} + \sum_{n=1}^{\infty} 3nc_n(x-1)^n$$
$$+ \sum_{n=1}^{\infty} 3nc_n(x-1)^{n-1} - \sum_{n=0}^{\infty} c_n(x-1)^n = 0.$$

We rewrite the second, third, and fifth summations so that $x - 1$ has the exponent $n$ in each. Thus we have:

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^n + \sum_{n=1}^{\infty} 2(n+1)nc_{n+1}(x-1)^n$$
$$+ \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}(x-1)^n + \sum_{n=1}^{\infty} 3nc_n(x-1)^n$$
$$+ \sum_{n=0}^{\infty} 3(n+1)c_{n+1}(x-1)^n - \sum_{n=0}^{\infty} c_n(x-1)^n = 0.$$

The common range of these six summations is from 2 to $\infty$. We write out the individual terms in each that do *not* belong to this range. Thus we have:

$$(-c_0 + 3c_1 + 2c_2) + (2c_1 + 10c_2 + 6c_3)(x-1)$$
$$+ \sum_{n=2}^{\infty} \{[n(n-1) + 3n - 1]c_n$$
$$+ [2(n+1)n + 3(n+1)]c_{n+1}$$
$$+ (n+2)(n+1)c_{n+2}\}(x-1)^n = 0.$$

Equating to zero the coefficients of each power of x, we obtain

$$-c_0 + 3c_1 + 2c_2 = 0,\quad 2c_1 + 10c_2 + 6c_3 = 0, \tag{1}$$

$$(n^2 + 2n - 1)c_n + (2n + 3)(n + 1)c_{n+1} + (n + 2)(n + 1)c_{n+2} = 0,\ n \geq 2. \tag{2}$$

From (1), we find $c_2 = \dfrac{c_0 - 3c_1}{2}$, $c_3 = -\dfrac{c_1 + 5c_2}{3}$ $= \dfrac{-5c_0 + 13c_1}{6}$. From (2), we find

$$c_{n+2} = -\frac{(n^2 + 2n - 1)c_n + (2n + 3)(n + 1)c_{n+1}}{(n + 1)(n + 2)},\ n \geq 2.$$

Using this, we find $c_4 = -\dfrac{7c_2 + 21c_3}{12} = \dfrac{7(2c_0 - 5c_1)}{12}$.

Substituting these values into the assumed solution, we have

$$y = c_0 + c_1(x - 1) + \frac{(c_0 - 3c_1)(x - 1)^2}{2} + \frac{(-5c_0 + 13c_1)(x - 1)^3}{6} + \frac{7(2c_0 - 5c_1)(x - 1)^4}{12} + \cdots$$

or

$$y = c_0\left[1 + \frac{(x - 1)^2}{2} - \frac{5(x - 1)^3}{6} + \frac{7(x - 1)^4}{6} + \cdots\right] + c_1\left[(x - 1) - \frac{3(x - 1)^2}{2} + \frac{13(x - 1)^3}{6} - \frac{35(x - 1)^4}{12} + \cdots\right].$$

23. Since the initial value is 1, we assume that $y = \sum_{n=0}^{\infty} c_n(x - 1)^n$. Then $y' = \sum_{n=1}^{\infty} nc_n(x - 1)^{n-1}$, and $y'' = \sum_{n=2}^{\infty} (n - 1)(n)c_n(x - 1)^{n-2}$. In order to obtain multiples of $(x - 1)$ we rewrite the original D.E. as follows:

$$(x - 1)y'' + 1y'' + y' + 2y = 0$$

Substituting into this D.E. we obtain:

$$(x - 1)\sum_{n=2}^{\infty} (n - 1)(n)c_n(x - 1)^{n-2}$$

$$+ \sum_{n=2}^{\infty} (n - 1)(n)c_n(x - 1)^{n-2}$$

$$+ \sum_{n=1}^{\infty} nc_n(x - 1)^{n-1} + 2\sum_{n=0}^{\infty} c_n(x - 1)^n = 0$$

or

$$\sum_{n=2}^{\infty} (n - 1)(n)c_n(x - 1)^{n-1}$$

$$+ \sum_{n=2}^{\infty} (n - 1)(n)c_n(x - 1)^{n-2}$$

$$+ \sum_{n=1}^{\infty} nc_n(x - 1)^{n-1} + \sum_{n=0}^{\infty} 2c_n(x - 1)^n = 0.$$

We rewrite the first three summations so that $(x - 1)$ has an exponent of n in each one. Thus we have:

$$\sum_{n=1}^{\infty}(n)(n+1)c_{n+1}(x-1)^n + \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}(x-1)^n + \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-1)^n + \sum_{n=0}^{\infty}2c_n(x-1)^n = 0.$$

The common range of these summations is from 1 to $\infty$. We write out the terms that are not in this range separately. Thus we have:

$$(2c_2 + c_1 + 2c_0) + \sum_{n=1}^{\infty}[(n+1)(n+2)c_{n+2} + (n+1)(n+1)c_{n+1} + 2c_n](x-1)^n = 0.$$

Equating the coefficients of each power of $(x - 1)$ to 0 we obtain

$$2c_2 + c_1 + 2c_0 = 0, \tag{1}$$

$$(n+1)(n+2)c_{n+2} + (n+1)^2c_{n+1} + 2c_n = 0. \tag{2}$$

From (1), we find $c_2 = -c_0 - \frac{c_1}{2}$. From (2), we find

$$c_{n+2} = -\frac{2c_n + (n+1)^2c_{n+1}}{(n+1)(n+2)}, \quad n \geq 1.$$

Using this, $c_3 = -\frac{c_1 + 2c_2}{3} = \frac{2c_0}{3}$, $c_4 = -\frac{c_2}{6} - \frac{3c_3}{4}$ $= -\frac{c_0}{3} + \frac{c_1}{12}$, $c_5 = -\frac{c_3}{10} - \frac{4c_4}{5} = \frac{c_0}{5} - \frac{c_1}{15}$. Thus the G.S. of the D.E. is

$$y = c_0\left[1 - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \frac{1}{5}(x-1)^5 + \cdots\right] + c_1\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^4 - \frac{1}{15}(x-1)^5 + \cdots\right].$$

Applying the I.C. $y(1) = 2$ to this, we find $c_0 = 2$. Differentiating, we find

$$y' = [-2(x-1) + 2(x-1)^2 - \cdots] + c_1[1 - (x-1) + \cdots].$$

Applying the I.C. $y'(1) = 4$ to this, we find $c_1 = 4$. Thus we obtain the solution

$$y = 2\left[1 - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \frac{1}{5}(x-1)^5 + \cdots\right] + 4\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^4 - \frac{1}{15}(x-1)^5 + \cdots\right]$$

or

$$y = 2 + 4(x-1) - 4(x-1)^2 + \frac{4}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \frac{2}{15}(x-1)^5 + \cdots.$$

**Section 6.2, Page 269**

1. We write the D.E. in the normalized form (6.3) of the text. This is:

$$y'' + \frac{x + 2}{x^2 - 3x} y' + \frac{1}{x^2 - 3x} y = 0.$$

Here $P_1(x) = \frac{x + 2}{x(x - 3)}$ and $P_2(x) = \frac{1}{x(x - 3)}$. We see from this that the singular points are $x = 0$ and $x = 3$.

We consider $x = 0$, and the functions formed by the products

$$xP_1(x) = \frac{x + 2}{x - 3} \quad \text{and} \quad x^2P_2(x) = \frac{x}{x - 3}.$$

Both of the product functions are defined at $x = 0$, so $x = 0$ is a *regular singular point.*

We now consider $x = 3$, and form the product functions

$$(x - 3)P_1(x) = \frac{x + 2}{x} \quad \text{and} \quad (x - 3)^2P_2(x) = \frac{x - 3}{x}.$$

Both of these product functions are defined at $x = 3$, so $x = 3$ is a *regular singular point.*

2. We write the D.E. in the normalized form (6.3) of the text. This is:

$$y'' + \frac{x - 2}{x(x + 1)} y' + \frac{4}{x^2(x + 1)} y = 0.$$

Here $P_1(x) = \frac{x - 2}{x(x + 1)}$ and $P_2(x) = \frac{4}{x^2(x + 1)}$. We see from this that the singular points are $x = 0$ and $x = -1$.

We consider $x = 0$ and form the functions defined by the products

$$xP_1(x) = \frac{x - 2}{x + 1} \quad \text{and} \quad x^2P_2(x) = \frac{4}{x + 1}.$$

Both of the product functions thus defined are analytic at $x = 0$; so $x = 0$ is a *regular singular point.*

We now consider $x = -1$ and form the functions defined by the products

$$(x + 1)P_1(x) = \frac{x - 2}{x} \quad \text{and} \quad (x + 1)^2P_2(x) = \frac{4(x + 1)}{x^2}.$$

Both of these product functions thus defined are analytic at $x = -1$; so $x = -1$ is a *regular singular point.*

4. We write the D.E. in the normalized form (6.3) of the text. This is:

$$y'' + \frac{1}{x(x + 3)(x - 2)}\, y' + \frac{1}{x^3(x + 3)}\, y = 0.$$

Here $P_1(x) = \frac{1}{x(x + 3)(x - 2)}$ and $P_2(x) = \frac{1}{x^3(x + 3)}$. From this, we see that the singular points are $x = 0$, $x = -3$ and $x = 2$.

We consider $x = 0$, and form the product functions

$$xP_1(x) = \frac{1}{(x + 3)(x - 2)} \quad \text{and} \quad x^2P_2(x) = \frac{1}{x(x + 3)}.$$

Although $xP_1(x)$ is defined at $x = 0$, $x^2P_2(x)$ is *not* defined at $x = 0$. Therefore $x = 0$ is an *irregular singular point.*

Consider $x = -3$, and form the product functions:

$$(x + 3)P_1(x) = \frac{1}{x(x - 2)} \quad \text{and} \quad (x + 3)^2P_2(x) = \frac{x + 3}{x^3}.$$

Both are defined at $x = -3$, so $x = -3$ is a *regular singular point.*

Consider $x = 2$, and form the product functions:

$$(x - 2)P_1(x) = \frac{1}{x(x + 3)} \quad \text{and} \quad (x - 2)^2P_2(x) = \frac{(x - 2)^2}{x^3(x + 3)}.$$

Both are defined at $x = 2$, so $x = 2$ is a *regular singular point.*

5. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$ and

$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2}$. Substituting these into the D.E., we obtain

$$2x^2 \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2}$$

$$+ x \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$$

$$+ (x^2 - 1) \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Simplifying, we obtain

$$\sum_{n=0}^{\infty}[(n + r)(n + r - 1)c_n + (n + r)c_n - c_n]x^{n+r}$$

$$+ \sum_{n=2}^{\infty}c_{n-2}x^{n+r} = 0$$

or

$$[2r(r - 1) + r - 1]c_0x^r + [2(r + 1)(r)$$

$$+ (r + 1) - 1]c_1x^{r+1}$$

$$+ \sum_{n=2}^{\infty}[(2n + 2r + 1)(n + r - 1)c_n$$

$$+ c_n - 2]x^{n+r} = 0. \qquad (1)$$

Equating the coefficient of the lowest power of x to zero yields the *indicial equation* $2r(r - 1) + r - 1 = 0$ or $2r^2 - r - 1 = 0$ with roots $r_1 = 1$ and $r_2 = -1/2$. Since the difference between these roots is not zero or a positive integer, Conclusion 1 of Theorem 6.3 tells us that the D.E. has two linearly independent solutions of the assumed form, one corresponding to each of the roots $r_1$ and $r_2$. Equating the coefficients of the higher powers of x in (1) to zero we obtain

$$[2(r + 1)r + r]c_1 = 0 \qquad (2)$$

and

$$(2n + 2r + 1)(n + r - 1)c_n + c_{n-2} = 0, \; n \geq 2. \qquad (3)$$

Letting $r = r_1 = 1$ in (2) we obtain $5c_1 = 0$ so $c_1 = 0$.

Letting $r = r_1 = 1$ in (3) we obtain

$$(2n + 3)(n)c_n + c_{n-2} = 0, \quad n \geq 2$$

or

$$c_n = -\frac{c_{n-2}}{(2n + 3)n}, \quad n \geq 2.$$

From this $c_2 = \frac{-1}{14} c_0$, $c_3 = \frac{-1}{27} c_1 = 0$, $c_4 = \frac{-1}{44} c_2 = \frac{1}{616} c_0$ $c_5 = \frac{-1}{65} c_3 = 0$. Using these values, we obtain the solution corresponding to the larger root $r_1 = 1$:

$$y_1(x) = c_0 x\left(1 - \frac{1}{14} x^2 + \frac{1}{616} x^4 + \cdots\right).$$

Letting $r = r_2 = -1/2$ in (2), we obtain $-c_1 = 0$ so $c_1 = 0$.

Letting $r = r_2 = -1/2$ in (3) we obtain

$$(n - 3/2)(2n)c_n + c_{n-2} = 0$$

or

$$c_n = -\frac{c_{n-2}}{(2n - 3)n}, \quad n \geq 2.$$

From this, $c_2 = -\frac{1}{2} c_0$, $c_3 = -\frac{1}{9} c_1 = 0$,

$c_4 = -\frac{1}{20} c_2 = \frac{1}{40} c_0$, $c_5 = 0$. Using these values, we obtain the solution corresponding to the smaller root $r_2 = -1/2$

$$y_2(x) = c_0 x^{-1/2}\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + \cdots\right).$$

The G.S. of the D.E. is $y = C_1 y_1(x) + C_2 y_2(x)$.

7. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1}$ and

$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-2}$. Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r}$$

$$- x \sum_{n=0}^{\infty} (n + r) c_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r+2} + \sum_{n=0}^{\infty} \frac{8}{9} c_n x^{n+r} = 0.$$

Simplifying as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty} \left[ (n + r)(n + r - 1) - (n + r) + \frac{8}{9} \right] c_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 0$$

or

$$\left[r(r-1) - r + \frac{8}{9}\right]c_0 x^r + \left[(r+1)r - (r+1) + \frac{8}{9}\right]c_1 x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[(n+r)(n+r-1) - (n+r) + \frac{8}{9}\right]c_n + c_{n-2}\right\}x^{n+r} = 0. \tag{1}$$

Equating to zero the coefficient of the lowest power of x we have the *indicial equation* $r(r-1) - r + \left(\frac{8}{9}\right) = 0$ or $r^2 - 2r + \left(\frac{8}{9}\right) = 0$ with roots $r_1 = \frac{4}{3}$ and $r_2 = \frac{2}{3}$. Since the difference between these roots is *not* zero or a positive integer, Conclusion 1 of Theorem 6.3 tells us that the D.E. has *two* linearly independent solutions of the assumed form, one corresponding to each of the roots $r_1$ and $r_2$.

Equating to zero the coefficients of the higher powers of x in (1), we have

$$\left[(r+1)r - (r+1) + \left(\frac{8}{9}\right)\right]c_1 = 0 \tag{2}$$

and

$$\left[(n+r)(n+r-1) - (n+r) + \frac{8}{9}\right]c_n + c_{n-2} = 0, \quad n \geq 2. \tag{3}$$

Letting $r = r_1 = \frac{4}{3}$ in (2) we obtain $\left(\frac{5}{3}\right)c_1 = 0$ so $c_1 = 0$.

Letting $r = r_1 = \frac{4}{3}$ in (3) we obtain $n\left(n + \frac{2}{3}\right)c_n + c_{n-2} = 0$, or

$$c_n = -\frac{3c_{n-2}}{n(3n+2)}, \quad n \geq 2.$$

From this $c_2 = -\frac{3c_0}{16}$, $c_3 = -\frac{c_1}{11} = 0$, $c_4 = -\frac{3c_2}{56} = \frac{9c_0}{896}$.

Using these values, we obtain the solution corresponding to the larger root $r_1 = \frac{4}{3}$:

$$y_1(x) = c_0 x^{4/3}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} - \cdots\right).$$

Letting $r = r_2 = \frac{2}{3}$ in (2), we obtain $\frac{c_1}{3} = 0$; so $c_1 = 0$.
Letting $r = r_2 = \frac{2}{3}$ in (3) we obtain $n\left(n - \frac{2}{3}\right)c_n + c_{n-2} = 0$

or

$$c_n = -\frac{3c_{n-2}}{n(3n - 2)}, \quad n \geq 2.$$

From this, $c_2 = -\frac{3c_0}{8}$, $c_3 = -\frac{c_1}{7} = 0$, $c_4 = -\frac{3c_2}{40} = \frac{9c_0}{320}$.

Using these values, we obtain the solution corresponding to the smaller root $r_2 = \frac{2}{3}$:

$$y_2(x) = c_0 x^{2/3}\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} - \cdots\right).$$

The G.S. of the D.E. is $y = C_1 y_1(x) + C_2 y_2(x)$.

9. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$ and

$y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-2}$. Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n + r)c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r+2} - \frac{1}{9}\sum_{n=0}^{\infty}c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty}\left[(n + r)(n + r - 1) + (n + r) - \frac{1}{9}\right]c_n x^{n+r} + \sum_{n=2}^{\infty}c_{n-2}x^{n+r} = 0$$

or

$$\left[r(r - 1) + r - \frac{1}{9}\right]c_0 x^r + \left[(r + 1)r + (r + 1) - \frac{1}{9}\right]c_1 x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[(n + r)(n + r - 1) + (n + r) - \frac{1}{9}\right]c_n + c_{n-2}\right\}x^{n+r} = 0. \quad (1)$$

Equating to zero the coefficient of the lowest power of x we have the *indicial equation* $r^2 - \frac{1}{9} = 0$ with roots $r_1 = \frac{1}{3}$, $r_2 = -\frac{1}{3}$. Since the difference between these roots is *not* zero or a positive integer, Conclusion 1 of Theorem 6.3 tells us that the D.E. has *two* linearly

independent solutions of the assumed form, one corresponding to each of the roots $r_1$ and $r_2$. Equating to zero the coefficients of the higher powers of x in (1) we have

$$\left[(r + 1)r + (r + 1) - \frac{1}{9}\right]c_1 = 0 \tag{2}$$

and

$$\left[(n + r)(n + r - 1) + (n + r) - \frac{1}{9}\right]c_n + c_{n-2} = 0, \quad n \geq 2. \tag{3}$$

Letting $r = r_1 = \frac{1}{3}$ in (2) we obtain $\frac{5}{3}\, c_1 = 0$; so $c_1 = 0$. Letting $r = r_1 = \frac{1}{3}$ in (3) we obtain

$$n\left(n + \frac{2}{3}\right)c_n + c_{n-2} = 0$$

or

$$c_n = -\frac{3c_{n-2}}{n(3n + 2)}, \quad n \geq 2.$$

From this, $c_2 = -\frac{3c_0}{16}$, $c_3 = -\frac{c_1}{11} = 0$, $c_4 = -\frac{3c_2}{56} = \frac{9c_0}{896}$, $c_{ODD} = 0$ for $n \geq 1$. Using these values, we obtain the solution corresponding to the larger root $r_1 = \frac{1}{3}$:

$$y_1(x) = c_0 x^{1/3}\left[1 - \frac{3x^2}{16} + \frac{9x^4}{896} - \cdots\right].$$

Letting $r = r_2 = -\frac{1}{3}$ in (2), we obtain $\frac{1}{3}\, c_1 = 0$; so $c_1 = 0$. Letting $r = r_2 = -\frac{1}{3}$ in (3) we obtain

$$n\left(n - \frac{2}{3}\right)c_n + c_{n-2} = 0$$

or

$$c_n = -\frac{3c_{n-2}}{n(3n-2)}, \quad n \geq 2.$$

From this, $c_2 = -\frac{3c_0}{8}$, $c_3 = -\frac{c_1}{7}$, $c_4 = -\frac{3c_2}{40} = \frac{9c_0}{320}$,

$c_{ODD} = 0$ for $n \geq 1$. Using these values, we obtain the solution corresponding to the smaller root $r_2 = -\frac{1}{3}$:

$$y_2(x) = c_0 x^{-1/3}\left[1 - \frac{3x^2}{8} + \frac{9x^4}{320} - \cdots\right].$$

The G.S. of the D.E. is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

12. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty} c_n (n + r) x^{n+r-1}$ and

$y'' = \sum_{n=0}^{\infty} c_n (n + r)(n + r - 1) x^{n+r-2}$. Substituting these into the D.E., we obtain

$$2\sum_{n=0}^{\infty} c_n(n + r)(n + r - 1)x^{n+r}$$

$$+ 5\sum_{n=0}^{\infty} c_n(n + r)x^{n+r}$$

$$+ 2\sum_{n=0}^{\infty} c_n x^{n+r+1} - 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty} [2(n + r)(n + r - 1) + 5(n + r) - 2]c_n x^{n+r}$$

$$+ 2\sum_{n=1}^{\infty} c_{n-1} x^{n+r} = 0$$

or

$$[2r(r - 1) + 5r - 2]c_0 x^r$$

$$+ \sum_{n=1}^{\infty} \{[2(n + r)(n + r - 1) + 5(n + r) - 2]c_n$$

$$+ 2c_{n-1}\}x^{n+r} = 0. \qquad (1)$$

Equating to zero the coefficient of the lowest power of $x$, we have the *indicial equation* $2r^2 + 3r - 2 = 0$, that is $(2r - 1)(r + 2) = 0$, with roots $r_1 = \frac{1}{2}$ and $r_2 = -2$. Since the difference between these roots is *not* zero or a positive integer, Conclusion 1 of Theorem 6.3 tells us that the D.E. has *two* linearly independent solutions of

the assumed form, one corresponding to each of the roots $r_1$ and $r_2$. Equating to zero the coefficients of the higher powers of x in (1), we have

$$[2(n + r)(n + r - 1) + 5(n + r) - 2]c_n + 2c_{n-1} = 0, \quad n \geq 1. \tag{2}$$

Letting $r = r_1 = \frac{1}{2}$ in (2) we obtain $(2n^2 + 5n)c_n + 2c_{n-1} = 0$ or

$$c_n = -\frac{2c_{n-1}}{n(2n + 5)}, \quad n \geq 1.$$

From this $c_1 = -\frac{2c_0}{7}$, $c_2 = -\frac{2c_1}{2(9)} = -\frac{c_1}{9} = \frac{2c_0}{63}$, $c_3 = -\frac{2c_2}{33} = -\frac{4c_0}{2079}$. Using these values, we obtain the solution corresponding to the larger root $r_1 = \frac{1}{2}$:

$$y_1(x) = c_0 x^{1/2}\left[1 - \frac{2x}{7} + \frac{2x^2}{63} - \frac{4x^3}{2079} + \cdots\right].$$

Letting $r = r_2 = -2$ in (2), we obtain $(2n^2 - 5n)c_n + 2c_{n-1} = 0$ so $c_1 = 0$ or

$$c_n = -\frac{2c_{n-1}}{n(2n - 5)}, \quad n \geq 1.$$

From this, $c_1 = -\frac{2c_0}{(-3)} = \frac{2c_0}{3}$, $c_2 = -\frac{2c_1}{2(-1)} = c_1 = \frac{2c_0}{3}$, $c_3 = -\frac{2c_2}{3} = -\frac{4c_0}{9}$. Using these values, we obtain the solution corresponding to the smaller root $r_2 = -2$:

$$y_2(x) = c_0 x^{-2}\left[1 + \frac{2}{3}x + \frac{2}{3}x^2 - \frac{4}{9}x^3 + \cdots\right].$$

The G.S. of the D.E. is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

14. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1}$ and

$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-2}$. Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r+1}$$

$$+ 2\sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty} (n + r) c_n x^{n+r+1} + \sum_{n=0}^{\infty} (n + r) c_n x^{n+r}$$

$$- 10\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty}[2(n+r)(n+r-1)+(n+r)-10]c_n x^{n+r}$$

$$+\sum_{n=1}^{\infty}[(n+r-1)(n+r-2)+$$

$$(n+r-1)]c_{n-1}x^{n+r}=0$$

or

$$[2r(r-1)+r-10]c_0x^r$$

$$+\sum_{n=1}^{\infty}\{[2(n+r)(n+r-1)+(n+r)-10]c_n$$

$$+(n+r-1)^2c_{n-1}\}x^{n+r}=0. \qquad (1)$$

Equating to zero the coefficient of the lowest power of x, we have the *indicial equation* $2r^2 - r - 10 = 0$ or $(2r-5)(r+2)=0$, with roots $r_1 = 5/2$ and $r_2 = -2$. Since the difference between these roots is *not* zero or a positive integer, Conclusion 1 of Theorem 6.3 tells us that the D.E. has *two* linearly independent solutions of the assumed form, one corresponding to each of the roots $r_1$ and $r_2$. Equating to zero the coefficients of the higher powers of x in (1), we have

$$[2(n+r)(n+r-1)+(n+r)-10]c_n$$

$$+(n+r-1)^2c_{n-1}=0,\ n>1. \qquad (2)$$

Letting $r = r_1 = 5/2$ in (2) we obtain $n(2n+9)c_n$ $+\left(n+\frac{3}{2}\right)^2 c_{n-1}=0$ or

$$c_n = -\frac{(2n+3)^2 c_{n-1}}{4n(2n+9)}, \quad n \geq 1.$$

From this $c_1 = -\frac{25c_0}{44}$, $c_2 = -\frac{49c_1}{104} = \frac{1225c_0}{4576}$. Using these values, we obtain the solution corresponding to the larger root $r_1 = 5/2$:

$$y = c_0 x^{5/2}\left[1 - \frac{25x}{44} + \frac{1225x^2}{4576} - \cdots\right].$$

Letting $r = r_2 = -2$ in (2), we obtain $n(2n-9)c_n + (n-3)^2 c_{n-1} = 0$ or

$$c_n = -\frac{(n-3)^2 c_{n-1}}{n(2n-9)}, \quad n \geq 1.$$

From this, $c_1 = \frac{4c_0}{7}$, $c_2 = \frac{c_1}{10} = \frac{2c_0}{35}$, $c_3 = \frac{0c_2}{9} = 0$; and hence $c_n = 0$ for all $n \geq 3$. The solution corresponding to the smaller root $r_2 = -2$ is the finite sum

$$y(x) = c_0 x^{-2}\left(1 + \frac{4x}{7} + \frac{2x^2}{35}\right).$$

The G.S. of the D.E. is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

15. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1},$$

$y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-2}$. Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n + r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}c_n x^{n+r+1} = 0,$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty}[(n + r)(n + r - 1) + 2(n + r)]c_n x^{n+r-1} + \sum_{n=2}^{\infty}c_{n-2}x^{n+r-1} = 0$$

or

$$[r(r - 1) + 2r]c_0 x^{r-1} + [(r + 1)r + 2(r + 1)]c_1 x^r + \sum_{n=2}^{\infty}\{[(n + r)(n + r - 1) + 2(n + r)]c_n + c_{n-2}\}x^{n+r-1} = 0. \quad (1)$$

Equating to zero the coefficient of the lowest power of x, we have the indicial equation $r^2 - r + 2r = 0$ or $r^2 + r = 0$, with roots $r_1 = 0$, $r_2 = -1$. Note that the difference between these roots is a positive integer. By Conclusion 2 of Theorem 6.3, the D.E. has a solution of

the assumed form corresponding to the larger root $r_1 = 0$. Equating to zero the coefficients of the higher powers of x in (1), we have

$$[(r + 1)r + 2(r + 1)]c_1 = 0, \tag{2}$$

$$[(n + r)(n + r - 1) + 2(n + r)]c_n + c_{n-2} = 0, \quad n \geq 2. \tag{3}$$

Letting $r = r_1 = 0$ in (2), we obtain $2c_1 = 0$; so $c_1 = 0$. Letting $r = r_1 = 0$ in (3), we obtain

$$n(n + 1)c_n + c_{n-2} = 0 \quad \text{or}$$

$$c_n = -\frac{c_{n-2}}{n(n + 1)}, \quad n \geq 2.$$

From this, $c_2 = -\frac{c_0}{3!}$, $c_3 = -\frac{c_1}{4!} = 0$, $c_4 = -\frac{c_2}{(4)(5)} = \frac{c_0}{5!}$, $\cdots$. Using these values, we obtain the solution corresponding to the larger root $r_1 = 0$.

$$y_1(x) = c_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)$$

$$= c_0 x^{-1}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

or

$$y_1(x) = c_0 x^{-1} \sin x. \tag{4}$$

Letting $r = r_2 = -1$ in (2), we obtain $0c_1 = 0$; so $c_1$ is arbitrary. Letting $r = r_2 = -1$ in (3), we obtain $n(n - 1)c_n + c_{n-2} = 0$ or

$$c_n = -\frac{c_{n-2}}{n(n-1)}, \quad n \geq 2.$$

From this, $c_2 = -\frac{c_0}{2!}$, $c_3 = -\frac{c_1}{3!}$, $c_4 = -\frac{c_2}{(4)(3)} = \frac{c_0}{4!}$, $c_5 = -\frac{c_3}{(5)(4)} = \frac{c_1}{5!}$, $\cdots$. Using these values, we obtain the solution corresponding to the smaller root $r_2 = -1$:

$$y_2(x) = c_0 x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + c_1 x^{-1}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

or

$$y_2(x) = c_0 x^{-1}\cos x + c_1 x^{-1}\sin x. \tag{5}$$

The situation here is analogous to that of Example 6.13 of the text (in particular, see text, page 261). In like manner to the case of that example, we see from (4) and (5) that the G.S. of the given D.E. is of the form

$$y = C_1 x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + C_2 x^{-1}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

or

$$y = x^{-1}(C_1\cos x + C_2\sin x).$$

16. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-2}.$$ Substituting these into the D.E., we obtain

$$x^2 \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-2}$$

$$+ x \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1}$$

$$+ (x^2 - 1/4) \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Simplifying, we obtain

$$\sum_{n=0}^{\infty} [(n + r)(n + r - 1) + (n + r) - 1/4] c_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 0$$

or

$$[r(r-1)+r-1/4]c_0x^r + [(1+r)(r) + (1+r) - 1/4]c_1x^{1+r} + \sum_{n=2}^{\infty}\{[(n+r)(n+r-1) + (n+r) - 1/4]c_n + c_{n-2}\}x^{n+r} = 0. \qquad (1)$$

Equating the coefficient of the lowest power of x to zero yields the indicial equation $r(r-1)+r-1/4 = 0$ or $r^2 - 1/4 = 0$ with roots $r_1 = 1/2$ and $r_2 = -1/2$. Note that the difference between these roots is a positive integer. By Conclusion 2 of Theorem 6.3, the D.E. has a solution of the assumed form corresponding to the larger root $r_1 = 1/2$. However, as in Example 6.13 of the text, if the smaller root $r_2 = -1/2$ has a solution, it will include that of the larger root. Therefore let us try the smaller root first.

Letting $r = r_2 = -1/2$ in (1), we obtain $[-1/4 + 1/2 - 1/4]c_1 = 0$ so $c_1$ is arbitrary. Next we obtain $[(n-1/2)(n-3/2) + (n-3/4)]c_n + c_{n-2} = 0$ or $c_n = -\frac{1}{n^2 - n}c_{n-2}$; $n \geq 2$. Since $n \geq 2$, we have no undefined terms and we will obtain two linearly independent solutions from this one root.

Using successive values of n we obtain $c_2 = -\frac{1}{2}c_0$, $c_3 = -\frac{1}{6}c_1$, $c_4 = -\frac{1}{12}c_2 = \frac{1}{24}c_0$, $c_5 = -\frac{1}{20}c_3 = \frac{1}{120}c_1$,

$c_6 = -\frac{1}{30}c_4 = -\frac{1}{720}c_0$, $c_7 = -\frac{1}{42}c_5 = -\frac{1}{5040}c_1$. Thus

$$y = c_0 x^{-1/2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$+ c_1 x^{-1/2}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

or

$$y = c_0 x^{-1/2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$

$$+ c_1 x^{1/2}\left(1 - \frac{x^2}{6} + \frac{x^5}{120} - \cdots\right).$$

The second part of this solution is the same expression that $r = r_1 = 1/2$ will yield.

17. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty}(n + r)c_n x^{n+r-1}$,

$y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-2}$. Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n + r)c_n x^{n+r+3}$$

$$+ \sum_{n=0}^{\infty}(n + r)c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty}[(n + r)(n + r - 1) + (n + r) - 1]c_n x^{n+r} + \sum_{n=3}^{\infty}(n + r - 3)c_{n-3}x^{n+r} = 0$$

or

$$\begin{aligned}&[r(r - 1) + r - 1]c_0x^r + [(r + 1)r + r]c_1x^{r+1} \\ &\quad + [(r + 2)(r + 1) + r + 1]c_2x^{r+2} \\ &\quad + \sum_{n=3}^{\infty}[(n + r + 1)(n + r - 1)c_n \\ &\quad + (n + r - 3)c_{n-3}]x^{n+r} = 0. \qquad (1)\end{aligned}$$

Equating to zero the coefficient of the lowest power of x, we have the indicial equation $r(r - 1) + r - 1 = 0$ or $r^2 - 1 = 0$, with roots $r_1 = 1$, $r_2 = -1$. Equating to zero the coefficients of the higher powers of x in (1), we have

$$(r^2 + 2r)c_1 = 0, \qquad (2)$$

$$(r^2 + 4r + 3)c_2 = 0, \qquad (3)$$

$$(n + r + 1)(n + r - 1)c_n + (n + r - 3)c_{n-3} = 0, \quad n \geq 3. \qquad (4)$$

Letting $r = r_1 = 1$ in (2), we obtain $3c_1 = 0$, so $c_1 = 0$. Letting $r = r_1 = 1$ in (3), we obtain $8c_2 = 0$, so $c_2 = 0$. Letting $r = r_1 = 1$ in (4), we obtain

$$(n + 2)nc_n + (n - 2)c_{n-3} = 0 \qquad \text{or}$$

$$c_n = -\frac{(n - 2)c_{n-3}}{n(n + 2)}, \qquad n \geq 3.$$

From this, $c_3 = -\frac{c_0}{15}$, $c_4 = -\frac{c_1}{12} = 0$, $c_5 = -\frac{3c_2}{35} = 0$, $c_6 = -\frac{c_3}{12} = \frac{c_0}{180}$, $\cdots$. Using these values we obtain the solution corresponding to the larger root $r_1 = 1$:

$$y_1(x) = c_0 x\left(1 - \frac{x^3}{15} + \frac{x^6}{180} - \cdots\right). \tag{5}$$

Letting $r = r_2 = -1$ in (2), we obtain $-c_1 = 0$, so $c_1 = 0$. Letting $r = r_2 = -1$ in (3), we obtain $0c_2 = 0$, so $c_2$ is arbitrary. Letting $r = r_2 = -1$ in (4), we obtain

$$n(n - 2)c_n + (n - 4)c_{n-3} = 0 \qquad \text{or}$$

$$c_n = -\frac{(n - 4)c_{n-3}}{n(n - 2)}, \qquad n \geq 3.$$

From this, $c_3 = \frac{c_0}{3}$, $c_4 = 0$, $c_5 = -\frac{c_2}{15}$, $c_6 = -\frac{c_3}{12} = -\frac{c_0}{36}$, $c_7 = -\frac{3c_4}{35} = 0$, $c_8 = -\frac{c_5}{12} = \frac{c_2}{180}$, $\cdots$. Using these values we obtain the solution corresponding to the smaller root $r_2 = -1$:

$$y_2(x) = c_0 x^{-1}\left(1 + \frac{x^3}{3} - \frac{x^6}{36} + \cdots\right)$$

$$+ c_2 x^{-1}\left(x^2 - \frac{x^5}{15} + \frac{x^8}{180} - \cdots\right). \tag{6}$$

The situation here is analogous to that of Example 6.13 of the text. In like manner to the case of that example, we see from (5) and (6) that the G.S. of the D.E. is of the form

$$y = C_1 x\left(1 - \frac{x^3}{15} + \frac{x^6}{180} - \cdots\right) + C_2 x^{-1}\left(1 + \frac{x^3}{3} - \frac{x^6}{36} + \cdots\right).$$

22. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$,

$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2}$. Substituting into the D.E., we obtain

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n + r)c_n x^{n+r+1} + 5\sum_{n=0}^{\infty} (n + r)c_n x^{n+r} + 2\sum_{n=0}^{\infty} c_n x^{n+r+1} + 3\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we obtain

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r} + \sum_{n=1}^{\infty}(n + r - 1)c_{n-1}x^{n+r}$$

$$+ 5\sum_{n=0}^{\infty}(n + r)c_n x^{n+r} + 2\sum_{n=1}^{\infty}c_{n-1}x^{n+r} + 3\sum_{n=0}^{\infty}c_n x^{n+r} = 0$$

or

$$[r(r - 1) + 5r + 3]c_0 x^r + \sum_{n=1}^{\infty}\{[(n + r)(n + r - 1) + 5(n + r) + 3]c_n + [(n + r - 1) + 2]c_{n-1}\}x^{n+r} = 0. \quad (1)$$

Equating to zero the coefficient of the lowest power of x in (1), we have the indicial equation $r^2 + 4r + 3 = 0$, with roots $r_1 = -1$, $r_2 = -3$. Equating to zero the coefficients of the higher powers of x in (1), we obtain

$$(n + r + 1)(n + r + 3)c_n + (n + r + 1)c_{n-1} = 0$$

or

$$(n + r + 1)[(n + r + 3)c_n + c_{n-1}] = 0, \quad n \geq 1. \quad (2)$$

Letting $r = r_1 = -1$ in (2), we obtain $n[(n + 2)c_n + c_{n-1}] = 0$ and so $c_n = -\dfrac{c_{n-1}}{(n + 2)}$, $n \geq 1$. From this, $c_1 = -\dfrac{c_0}{3} = -\dfrac{2c_0}{3!}$, $c_2 = -\dfrac{c_1}{4} = \dfrac{2c_0}{4!}$, and in general, $c_n = \dfrac{(-1)^n 2c_0}{(n + 2)!}$. From this we obtain the solution corresponding to the larger root $r_1 = -1$:

$$y_1(x) = 2c_0 x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+2)!}.$$

Letting $r = r_2 = -3$ in (2), we obtain $(n-2)(nc_n + c_{n-1}) = 0$, $n \geq 1$, and so

$$c_n = -\frac{c_{n-1}}{n}, \quad n \geq 1, \; n \neq 2. \tag{3}$$

From (3) with $n = 1$, we find $c_1 = -c_0$. For $n = 2$, we go back to $(n-2)(nc_n + c_{n-1}) = 0$. When $n = 2$, this becomes $0(2c_2 + c_1) = 0$. Thus $c_2$ is independent of $c_1$ and is a second arbitrary constant. Then from (3) with $n = 3$, $c_3 = -c_2/3 = -\frac{2c_2}{3!}$; with $n = 4$, $c_4 = -c_3/4 = \frac{2c_2}{4!}$; and in general $c_n = -c_{n-1}/n = \frac{(-1)^n 2c_2}{n!}$. From this we obtain the solution corresponding to the smaller root $r_2 = -3$:

$$y_2(x) = c_0 x^{-3}(1 - x) + x^{-3} \sum_{n=2}^{\infty} \frac{(-1)^n 2c_2 x^n}{n!}. \tag{4}$$

The situation here is analogous to that of Example 6.13 of the text. In like manner to the case of that example, we see from (3) and (4) that the G.S. of the D.E. is of the form

$$y = C_1 x^{-3}(1 - x) + C_2 x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+2)!}.$$

24. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-2}.$$ Substituting into the D.E., we obtain

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r} + 2 \sum_{n=0}^{\infty} (n + r) c_n x^{n+r+2}$$

$$- \sum_{n=0}^{\infty} c_n x^{n+r+2} - \frac{15}{4} \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we obtain

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r} + 2 \sum_{n=2}^{\infty} (n + r - 2) c_{n-2} x^{n+r}$$

$$- \sum_{n=2}^{\infty} c_{n-2} x^{n+r} - \frac{15}{4} \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

or

$$\left[r(r - 1) - \frac{15}{4}\right] c_0 x^r + \left[(r + 1)r - \frac{15}{4}\right] c_1 x^{r+1}$$

$$+ \sum_{n=2}^{\infty} \left\{ \left[ (n + r)(n + r - 1) - \frac{15}{4} \right] c_n \right.$$

$$\left. + [2(n + r - 2) - 1] c_{n-2} \right\} x^{n+r} = 0. \qquad (1)$$

Equating to zero the coefficient of the lowest power of x in (1), we have the indicial equation $r^2 - r - \frac{15}{4} = 0$, with roots $r_1 = 5/2$, $r_2 = -3/2$. Equating to zero the coefficients of the higher powers of x in (1), we obtain

$$\left(r^2 + r - \frac{15}{4}\right)c_1 = 0, \tag{2}$$

$$\left[(n + r)(n + r - 1) - \frac{15}{4}\right]c_n + [2(n + r - 2) - 1]c_{n-2} = 0, \quad n \geq 2. \tag{3}$$

Letting $r = r_1 = \frac{5}{2}$ in (2), we obtain $5c_1 = 0$, so $c_1 = 0$. Letting $r = r_1 = \frac{5}{2}$ in (3), we obtain $n(n + 4)c_n + 2nc_{n-2} = 0$ and hence $c_n = -\frac{2c_{n-2}}{n + 4}$, $n \geq 2$. From this, $c_2 = -\frac{c_0}{3}$, $c_4 = -\frac{c_2}{4} = \frac{c_0}{12}$, and $c_{2n+1} = 0$ for $n \geq 0$. Using these values we obtain the solution corresponding to the larger root $r_1 = \frac{5}{2}$:

$$y_1(x) = c_0 x^{5/2}\left(1 - \frac{x^2}{3} + \frac{x^4}{12} - \cdots\right). \tag{4}$$

Letting $r = r_2 = -\frac{3}{2}$ in (2), we obtain $-3c_1 = 0$, since $c_1 = 0$. Letting $r = r_2 = -\frac{3}{2}$ in (3), we obtain $n(n - 4)c_n + 2(n - 4)c_{n-2} = 0$, $n \geq 2$, and hence

$$c_n = -\frac{2c_{n-2}}{n}, \quad n \geq 2, \quad n \neq 4. \tag{5}$$

From (5) with $n = 2$, we find $c_2 = -c_0$; and with $n = 3$, $c_3 = -2c_1/3$. For $n = 4$, we return to $n(n - 4)c_n$

$+ 2(n - 4)c_{n-2} = 0$. With $n = 4$, this becomes $0c_4 + 0c_2 = 0$. Thus $c_4$ is independent of $c_2$ and so is a second arbitrary constant. Then from (5) with $n = 6$, $c_6 = -c_4/3$; and with $n = 8$, $c_8 = -c_6/4 = c_4/12$. Also note that $c_{2n+1} = 0$ for $n \geq 0$. Using these values we obtain the solution corresponding to the smaller root $r_2 = -\frac{3}{2}$:

$$y_2(x) = c_0 x^{-3/2}(1 - x^2) + c_4 x^{-3/2}\left(x^4 - \frac{x^6}{3} + \frac{x^8}{12} - \cdots\right). \tag{6}$$

The situation here is analogous to that of Example 6.13 of the text. In like manner to the case in that example, we see from (4) and (6) that the G.S. of the D.E. is of the form

$$y = C_1 x^{-3/2}(1 - x^2) + C_2 x^{5/2}\left(1 - \frac{x^2}{3} + \frac{x^4}{12} - \cdots\right).$$

25. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2}.$$ Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n + r)c_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty}c_n x^{n+r+1} - \sum_{n=0}^{\infty}c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty}[(n + r)(n + r - 1) + (n + r) - 1]c_n x^{n+r}$$

$$+ \sum_{n=1}^{\infty}c_{n-1}x^{n+r} = 0$$

and hence as

$$[r(r - 1) + r - 1]c_0 + \sum_{n=1}^{\infty}[(n + r + 1)(n + r - 1)c_n$$

$$+ c_{n-1}]x^{n+r} = 0.$$

Equating to zero the coefficient of the lowest power of x, we have the indicial equation $r(r - 1) + r - 1 = 0$ or $(r + 1)(r - 1) = 0$, with roots $r_1 = 1$, $r_2 = -1$.

Equating to zero the coefficients of the higher powers of x, we have

$$(n + r + 1)(n + r - 1)c_n + c_{n-1} = 0, \quad n \geq 1. \qquad (1)$$

Letting $r = r_1 = 1$ in (1), we find $(n + 2)nc_n + c_{n-1} = 0$ and hence

$$c_n = -\frac{c_{n-1}}{n(n+2)}, \quad n \geq 1.$$

From this, $c_1 = -\frac{c_0}{(1)(3)} = -\frac{2c_0}{1!3!}$, $c_2 = -\frac{c_1}{(2)(4)} = \frac{2c_0}{2!4!}$, $c_3 = -\frac{c_2}{(3)(5)} = -\frac{2c_0}{3!5!}$, and, in general, $\cdots$, $c_n = (-1)^n \frac{2c_0}{n!(n+2)!}$. Using these values, we obtain the solution corresponding to the larger root $r_1 = 1$ and given by $y = c_0 x\left[1 - \frac{2x}{1!3!} + \frac{2x^2}{2!4!} - \frac{2x^3}{3!5!} + \cdots\right]$. Choosing $c_0 = 1$, we obtain the particular solution denoted by $y_1(x)$ and defined by

$$y_1(x) = x\left[1 - \frac{2x}{1!3!} + \frac{2x^2}{2!4!} - \frac{2x^3}{3!5!} + \cdots\right]$$

$$= x\left[1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(n+2)!}\right]. \tag{2}$$

Letting $r = r_2 = -1$ in (1), we find

$$n(n-2)c_n + c_{n-1} = 0 \tag{3}$$

and hence

$$c_n = -\frac{c_{n-1}}{n(n-2)}, \quad n \geq 1, \quad n \neq 2. \tag{4}$$

For $n = 1$, (4) gives $c_1 = c_0$. For $n = 2$, (4) does not apply and we must use (3). For $n = 2$, (3) is $0c_2 + c_1 = 0$, and hence $c_1 = 0$. But then, since $c_1 = c_0$, we must have $c_0 = 0$. However, we assumed $c_0 \neq 0$ at the start. This contradiction shows there is no solution of the

assumed form with $c_0 \neq 0$ corresponding to the smaller root $r_2 = -1$. Moreover, use of (4) for $n \geq 3$ will only lead us to the solution $y_1(x)$ already obtained and given by (2) (the student should check that this is true).

We now seek a solution that is linearly independent of the solution $y_1(x)$. From Theorem 6.3, we know that it is of the form $\sum_{n=0}^{\infty} c_n^* x^{n-1} + Cy_1(x)\ln x$, where $c_0^* \neq 0$ and $C \neq 0$. We shall use reduction of order to find it. We let

$$y = y_1(x)v.$$

From this we obtain

$$y' = y_1(x)v' + y_1'(x)v \quad \text{and}$$
$$y'' = y_1(x)v'' + 2y_1'(x)v' + y_1''(x)v.$$

Substituting these into the given D.E., after some simplifications we obtain

$$xy_1(x)v'' + [2xy_1'(x) + y_1(x)]v' = 0. \tag{5}$$

Letting $w = v'$, this reduces to

$$xy_1(x)\frac{dw}{dx} + [2xy_1'(x) + y_1(x)]w = 0.$$

From this, we have

$$\frac{dw}{w} = -\left[2\frac{y_1'(x)}{y_1(x)} + \frac{1}{x}\right]dx,$$

and integrating we obtain the particular solution

$$w = \frac{1}{x[y_1(x)]^2}.$$

Writing out the first three terms of $y_1(x)$ defined by (2), we have

$$y_1(x) = x - \frac{x^2}{3} + \frac{x^3}{24} + \cdots. \tag{6}$$

From this, using basic multiplication and division, we find

$$[y_1(x)]^2 = x^2 - \frac{2x^3}{3} + \frac{7x^4}{36} + \cdots$$

and $\frac{1}{[y_1(x)]^2} = \frac{1}{x^2} + \frac{2}{3x} + \frac{1}{4} + \cdots$. Hence we obtain

$$w = \frac{1}{x[y_1(x)]^2} = \frac{1}{x^3} + \frac{2}{3x^2} + \frac{1}{4x} + \cdots.$$

Integrating we obtain the particular solution of (5) given by

$$v = -\frac{1}{2x^2} - \frac{2}{3x} + \frac{\ln|x|}{4} + \cdots. \tag{7}$$

Multiplying (6) by (7), we obtain the desired linearly independent solution of the given D.E. We thus find

$$y = x^{-1}\left(-\frac{1}{2} - \frac{x}{2} + \frac{29x^2}{144} + \cdots\right) + \frac{y_1(x)\ln|x|}{4}. \tag{8}$$

The general solution is a linear combination of (2) and (8).

28. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} (n + r) c_n x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r-2}.$$ Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n x^{n+r} + \sum_{n=0}^{\infty} (n + r) c_n x^{n+r+1}$$

$$- \sum_{n=0}^{\infty} \frac{3}{4} c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty} \left[ (n + r)(n + r - 1) - \frac{3}{4} \right] c_n x^{n+r}$$

$$+ \sum_{n=1}^{\infty} (n + r - 1) c_{n-1} x^{n+r} = 0$$

and hence as

$$\left[ r(r - 1) - \frac{3}{4} \right] c_0 x^r + \sum_{n=1}^{\infty} \left\{ \left[ (n + r)(n + r - 1) - \frac{3}{4} \right] c_n \right.$$

$$\left. + (n + r - 1) c_{n-1} \right\} x^{n+r} = 0.$$

Equating to zero the coefficient of the lowest power of x, we have the indicial equation $r^2 - r - \frac{3}{4} = 0$, with roots, $r_1 = \frac{3}{2}$, $r_2 = -\frac{1}{2}$. Equating to zero the coefficients of the higher powers of x, we have

$$\left[(n + r)(n + r - 1) - \frac{3}{4}\right]c_n + (n + r - 1)c_{n-1} = 0, \quad n \geq 1. \tag{1}$$

Letting $r = r_1 = \frac{3}{2}$ in (1), we obtain $(n^2 + 2n)c_n + \left(n + \frac{1}{2}\right)c_{n-1} = 0$ and hence

$$c_n = -\frac{(2n + 1)c_{n-1}}{2n(n + 2)}, \quad n \geq 1.$$

From this, $c_1 = -\frac{c_0}{2}$, $c_2 = -\frac{5c_1}{16} = \frac{5c_0}{32}$, $c_3 = -\frac{7c_2}{30} = -\frac{7c_0}{192}$, $\cdots$. Observe that we can also write

$$c_1 = -\frac{3c_0}{2^0 1!3!}, \quad c_2 = \frac{(3)(5)c_0}{2^1 2!4!}, \quad c_3 = -\frac{(3)(5)(7)c_0}{2^2 3!5!}, \quad \cdots,$$

and in general,

$$c_n = (-1)^n \frac{(3)(5)(7)\cdots(2n + 1)c_0}{2^{n-1} n!(n + 2)!}, \quad n \geq 1.$$

Using these values we obtain the solution corresponding to the larger root $r_1 = \frac{3}{2}$ and given by $y = c_0 x^{3/2}\left(1 - \frac{x}{2} + \frac{5x^2}{32} - \frac{7x^3}{192} + \cdots\right)$. Choosing $c_0 = 1$, we obtain the particular solution denoted by $y_1(x)$ and defined by

$$y_1(x) = x^{3/2}\left(1 - \frac{x}{2} + \frac{5x^2}{32} - \frac{7x^3}{192} + \cdots\right)$$

$$= x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n[(3)(5)(7)\cdots(2n+1)]}{2^{n-1}n!(n+2)!}\, x^n\right] \quad (2)$$

Letting $r = r_2 = -\frac{1}{2}$ in (1), we obtain $(n^2 - 2n)c_n + \left(n - \frac{3}{2}\right)c_{n-1} = 0$, or

$$n(n-2)c_n + \frac{1}{2}(2n-3)c_{n-1} = 0 \quad (3)$$

and hence

$$c_n = -\frac{(2n-3)c_{n-1}}{2n(n-2)}, \quad n \geq 1, \ n \neq 2. \quad (4)$$

For $n = 1$, (4) gives $c_1 = -\frac{c_0}{2}$. For $n = 2$, (4) does not apply and we must use (3). For $n = 2$, (3) is $0c_2 + \frac{c_1}{2} = 0$, and hence $c_1 = 0$. But then, since $c_1 = -\frac{c_0}{2}$, we must have $c_0 = 0$. However, we assumed $c_0 \neq 0$ at the start. This contradiction shows there is no solution of the assumed form with $c_0 \neq 0$ corresponding to the smaller root $r_2 = -\frac{1}{2}$. Moreover, use of (4) for $n \geq 3$ will only lead us to the solution $y_1(x)$ already obtained and given by (2) (the student should check that this is true).

We now seek a solution that is linearly independent of the solution $y_1(x)$. From Theorem 6.3, we know that it is of the form

$$\sum_{n=0}^{\infty} c_n^* x^{n-1/2} + Cy_1(x)\ln x,$$

where $c_0^* \neq 0$ and $C \neq 0$.

We shall use reduction of order to find it. We let $y = y_1(x)v$. From this we obtain $y' = y_1(x)v' + y_1'(x)v$ and $y'' = y_1(x)v'' + 2y_1'(x)v' + y_1''(x)v$. Substituting these into the given D.E., after some simplifications, we obtain

$$y_1(x)v'' + [2y_1'(x) + y_1(x)]v' = 0. \tag{5}$$

Letting $w = v'$, this reduces to

$$y_1(x)\frac{dw}{dx} + [2y_1'(x) + y_1(x)]w = 0.$$

From this, we have $\frac{dw}{w} = -\left[\frac{2y_1'(x)}{y_1(x)} + 1\right]dx$, and integrating we obtain the particular solution

$$w = \frac{e^{-x}}{[y_1(x)]^2}.$$

From (2), using basic multiplication and division, we find

$$[y_1(x)]^2 = x^3 - x^4 + \frac{9x^5}{16} - \frac{11x^6}{48} + \cdots$$

and

$$\frac{1}{[y_1(x)]^2} = x^{-3} + x^{-2} + \frac{7x^{-1}}{16} + \frac{5}{48} + \cdots.$$

Thus using $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$, we obtain

$$w = \frac{e^{-x}}{[y_1(x)]^2} = x^{-3} - \frac{x^{-1}}{16} + \cdots.$$

Integrating we obtain the particular solution of (5) given by

$$v = -\frac{x^{-2}}{2} - \frac{\ln|x|}{16} + \cdots, \tag{6}$$

where the next nonzero term in this expansion is the term in $x^2$. Multiplying (2) by (6), we obtain the desired linearly independent solution of the given D.E. We thus find

$$y = x^{-1/2}\left(-\frac{1}{2} + \frac{x}{4} - \frac{5x^2}{64} + \cdots\right) - \frac{y_1(x)\ln|x|}{16}. \tag{7}$$

The general solution is a linear combination of (2) and (7).

31. We assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, where $c_0 \neq 0$. Then

$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$,

$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2}$. Substituting these into the D.E., we obtain

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n + r)c_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty}c_n x^{n+r+2} + \sum_{n=0}^{\infty}c_n x^{n+r} = 0.$$

Simplifying, as in the solutions of Section 6.1, we write this as

$$\sum_{n=0}^{\infty}[(n + r)(n + r - 1) - (n + r) + 1]c_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty}c_{n-2}x^{n+r} = 0$$

and hence as

$$[r(r - 1) - r + 1]c_0 x^r + [(r + 1)r - (r + 1) + 1]c_1 x^{r+1} + \sum_{n=2}^{\infty}\{[(n + r)(n + r - 1) - (n + r) + 1]c_n + c_{n-2}\}x^{n+r} = 0.$$

Equating to zero the coefficient of the lowest power of x, we have the indicial equation $r^2 - 2r + 1 = 0$, with the double root $r = 1$. Equating to zero the coefficients of the higher powers of x, we have

$$[(r + 1)r - (r + 1) + 1]c_1 = 0, \tag{1}$$

$$[(n + r)(n + r - 1) - (n + r) + 1]c_n + c_{n-2} = 0, \quad n \geq 2. \tag{2}$$

Letting $r = 1$ in (1), we have $c_1 = 0$. Letting $r = 1$ in (2), we obtain $n^2 c_n + c_{n-2} = 0$ and hence

$$c_n = -\frac{c_{n-2}}{n^2}, \quad n \geq 2.$$

From this, $c_2 = -\frac{c_0}{4}$, $c_3 = -\frac{c_1}{9} = 0$, $c_4 = -\frac{c_2}{16} = \frac{c_0}{64}$, $\cdots$. Note that all odd coefficients are zero and we can write the general even coefficients as

$$c_{2n} = \frac{(-1)^n}{[(2)(4)(6)\cdots(2n)]^2}, \quad n \geq 1.$$

Using these values we obtain the solution of the assumed form corresponding to the double root $r = 1$ and given by

$$y = c_0 x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right).$$

Choosing $c_0 = 1$, we obtain the particular solution denoted by $y_1(x)$ and defined by

$$y_1(x) = x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right)$$

$$= x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{[(2)(4)(6)\cdots(2n)]^2}\right]. \tag{3}$$

Since the indicial equation has the double root $r = 1$, by Conclusion 3 of Theorem 6.3, a linearly independent solution is of the form

$$\sum_{n=0}^{\infty} c_n^* x^{n+2} + y_1(x) \ln x.$$

We shall use reduction of order to find it. We let $y = y_1(x)v$. From this we obtain

$$y' = y_1(x)v' + y_1'(x)v \quad \text{and}$$

$$y'' = y_1(x)v'' + 2y_1'(x)v' + y_1''(x)v.$$

Substituting these into the given D.E., after some simplifications, we obtain

$$xy_1(x)v'' + [2xy_1'(x) - y_1(x)]v' = 0. \tag{4}$$

Letting $w = \frac{dv}{dx}$, this reduces to

$$xy_1(x)\frac{dw}{dx} + [2xy_1'(x) - y_1(x)]w = 0.$$

From this, we have

$$\frac{dw}{w} = \left[\frac{1}{x} - \frac{2y_1'(x)}{y_1(x)}\right]dx$$

and integrating, we obtain the particular solution

$$w = \frac{x}{[y_1(x)]^2}.$$

From (3), using basic multiplication and division, we find

$$[y_1(x)]^2 = x^2 - \frac{x^4}{2} + \frac{3x^6}{32} - \frac{5x^8}{576} + \cdots \quad \text{and}$$

$$\frac{1}{[y_1(x)]^2} = x^{-2} + \frac{1}{2} + \frac{5x^2}{32} + \frac{23x^4}{576} + \cdots.$$

Thus we obtain

$$w = x^{-1} + \frac{x}{2} + \frac{5x^3}{32} + \frac{23x^5}{576} + \cdots.$$

Integrating we obtain the particular solution of (4) given by

$$v = \ln|x| + \frac{x^2}{4} + \frac{5x^4}{128} + \frac{23x^6}{3456} + \cdots. \tag{5}$$

Multiplying (3) by (5) we obtain the desired linearly independent solution of the given D.E. We thus find

$$y = \frac{x^3}{4} - \frac{3x^5}{128} + \frac{11x^7}{13824} + \cdots + y_1(x)\ln|x|. \tag{6}$$

The general solution is a linear combination of (3) and (6).

**Section 6.3, Page 280**

2. Differentiating $y = \dfrac{u}{\sqrt{x}}$ twice and simplifying, we obtain

$$y' = \frac{2xu' - u}{2x^{3/2}}, \qquad y'' = \frac{4x^2u'' - 4xu' + 3u}{4x^{5/2}}.$$

Substituting these into $x^2y'' + xy' + (x^2 - p^2)y = 0$, we have

$$\frac{4x^2u'' - 4xu' + 3u}{4\sqrt{x}} + \frac{2xu' - u}{2\sqrt{x}} + \frac{(x^2 - p^2)u}{\sqrt{x}} = 0$$

which reduces to

$$\frac{4x^2u'' + u + 4(x^2 - p^2)u}{4\sqrt{x}} = 0.$$

Now dividing through by $x^{3/2}$, this quickly reduces to

$$u'' + \left[1 + \left(\frac{1}{4} - p^2\right)\frac{1}{x^2}\right]u = 0.$$

3. The Bessel equation of order $p = \frac{1}{2}$ is

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0. \qquad (1)$$

To use the result of Exercise 2, we let $y = \frac{u}{\sqrt{x}}$ and reduce (1) to

$$u'' + \left[1 + \left(\frac{1}{4} - \frac{1}{4}\right)\frac{1}{x^2}\right]u = 0$$

or simply $u'' + u = 0$. The auxiliary equation of this is $m^2 + 1 = 0$, with roots $m = \pm i$, and the G.S. is $u = c_1 \sin x + c_2 \cos x$. Thus, since $y = u/\sqrt{x}$, the G.S. of (1) is

$$y = \frac{c_1 \sin x + c_2 \cos x}{\sqrt{x}}.$$

4. From the series definition (6.124), we at once have

$$x^p J_p(kx) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!}\left(\frac{k}{2}\right)^{2n+p} x^{2n+2p}.$$

We may differentiate this series term-by-term to obtain

$$\frac{d}{dx}\left[x^p J_p(kx)\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^n}{n!(n+p)!}\left(\frac{k}{2}\right)^{2n+p} x^{2n+2p}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+p)}{n!(n+p)!}\left(\frac{k}{2}\right)^{2n+p} x^{2n+2p-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2x^p k}{n!(n+p-1)!2} \left(\frac{k}{2}\right)^{2n+p-1} x^{2n+p-1}$$

$$= kx^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p-1)!} \left(\frac{k}{2}\right)^{2n+p-1} x^{2n+p-1}$$

$$= kx^p J_{p-1}(kx).$$

The other identity is verified in a similar manner.

# Chapter 7

**Section 7.1, Page 296**

1. We introduce operator notation and write the system in the form

$$\begin{cases} (D-2)x + (D-4)y = e^{t}, \\ Dx + (D-1)y = e^{4t}. \end{cases} \tag{1}$$

We apply the operator $(D-1)$ to the first equation and the operator $(D-4)$ to the second, obtaining

$$\begin{cases} (D-1)(D-2)x + (D-1)(D-4)y = (D-1)e^{t}, \\ (D-4)Dx + (D-4)(D-1)y = (D-4)e^{4t}. \end{cases}$$

Subtracting the second equation from the first, we obtain $[(D-1)(D-2) - (D-4)D]x = (D-1)e^{t} - (D-4)e^{4t}$ or $(D+2)x = 0$. The G.S. of this D.E. is

$$x = c\,e^{-2t}. \tag{2}$$

Now there are two ways to obtain y, and we give both here. First, we proceed by returning to the system (1), applying the operator D to the first operator and the operator $(D-2)$ to the second equation, obtaining

$$\begin{cases} D(D-2)x + D(D-4)y = De^{t}, \\ (D-2)Dx + (D-2)(D-1)y = (D-2)e^{4t}. \end{cases}$$

Subtracting the first equation from the second, we obtain $[(D - 2)(D - 1) - D(D - 4)]y = (D - 2)e^{4t} - De^{t}$ or $(D + 2)y = 2e^{4t} = e^{t}$. Using undetermined coefficients, we find the G.S. of this D.E. to be

$$y = ke^{-2t} + \frac{e^{4t}}{3} - \frac{e^{t}}{3}. \tag{3}$$

Now the determinant of the operator "coefficients" of x and y in (1) is

$$\begin{vmatrix} D - 2 & D - 4 \\ D & D - 1 \end{vmatrix} = D + 2.$$

Since this is of order 1, only one of the two constants c and k in (2) and (3) can be independent. To determine the relation which must thus exist between c and k, we substitute x given by (2) and y given by (3) into system (1). Substituting into the second equation of (1), we have

$$-2ce^{-2t} + \left[-2ke^{-2t} + \frac{4e^{4t}}{3} - \frac{e^{t}}{3}\right]$$

$$- \left[ke^{-2t} + \frac{e^{4t}}{3} - \frac{e^{t}}{3}\right] = e^{4t}.$$

or $(-2c - 3k)e^{-2t} = 0$. Thus $k = -\frac{2c}{3}$. Hence we obtain the G.S. of (1) in the form

$$\begin{cases} x = ce^{-2t}, \\ y = -\dfrac{2ce^{-2t}}{3} + \dfrac{e^{4t}}{3} - \dfrac{e^{t}}{3} \end{cases} \tag{4}$$

We now obtain y by the alternative procedure of the text, pages 293-295. We return to system (1) and subtract the second equation from the first, thereby eliminating Dy but not y, and thus obtaining $-2x - 3y = e^{t} - e^{4t}$. From this, $y = -\frac{2x}{3} + \frac{e^{4t}}{3} - \frac{e^{t}}{3}$. Substituting x into this from (2), we at once obtain $y = -\frac{2ce^{-2t}}{3} + \frac{e^{4t}}{3} - \frac{e^{t}}{3}$. Using this and (2), we again have the G.S. (4) of the original system.

2. We introduce operator notation and write the system in the form

$$\begin{cases} (D - 1)x + \quad Dy = -2t \\ (D - 3)x + (D - 1)y = t^{2}. \end{cases} \tag{1}$$

We apply the operator (D - 1) to the first equation and the operator D to the second to obtain

$$\begin{cases} (D - 1)^{2}x + (D - 1)Dy = (D - 1)(-2t), \\ D(D - 3)x + D(D - 1)y = Dt^{2}. \end{cases}$$

Subtracting the second equation from the first, we obtain $[(D - 1)^{2} - D(D - 3)]x = (D-1)(-2t) - Dt^{2}$ or $(D + 1)x = -2$. The G.S. of this D.E. is

$$x = c\,e^{-t} - 2 \tag{2}$$

There are two ways to obtain y, and we give both. First, we return to (1) and apply the operator (D - 3) to the first equation and the operator (D - 1) to the second equation. We obtain

$$\begin{cases}(D - 3)(D - 1)x + (D - 3)Dy = (D - 3)(-2t) \\ (D - 1)(D - 3)x + (D - 1)^2 y = (D - 1)t^2.\end{cases}$$

Subtract the first equation from the second, to obtain $[(D - 1)^2 - (D - 3)D]y = (D - 1)t^2 - (D - 3)(-2t)$, or $(D + 1)y = -t^2 - 4t + 2$.

The G.S. of this D.E. is

$$y = k e^{-t} - t^2 - 2t + 4. \qquad (3)$$

The determinant of the operator "coefficients" of x and y in (1) is

$$\begin{vmatrix} D - 1 & D \\ D - 3 & D - 1 \end{vmatrix} = D + 1.$$

Since this is of order 1, only one of the two constants c and k in (2) and (3) can be independent. To determine the relation which must thus exist between c and k, we substitute x given by (2) and y given by (3) into system (1). Substituting into the first equation of (1), we have $-c e^{-t} - c e^{-t} + 2 - k e^{-t} - 2t - 2 = -2t$ or $(2c + k)e^{-t} = 0$. Thus $k = -2c$. Hence we obtain the G.S. of (1) in the form

$$\begin{cases} x = c\,e^{-t} - 2, \\ y = -2c\,e^{-t} - t^2 - 2t + 4. \end{cases} \tag{4}$$

We now obtain y by the alternative procedure of the text, pages 293-295. We return to the system (1) and subtract the second equation from the first, thereby eliminating Dy but not y, and thus obtaining

$$2x + y = -t^2 - 2t.$$

Substituting x into this from (2) and solving for y, we at once find

$$y = -2c\,e^{-t} - t^2 - 2t + 4.$$

Using this and (2), we again have the G.S. (4) of the original system.

4. We introduce operator notation and write the system in the form

$$\begin{cases} (2D - 1)x + (D + 2)y = 0, \\ (3D - 2)x + (2D + 1)y = 4e^{2t}. \end{cases} \tag{1}$$

We apply the operator (2D + 1) to the first equation and the operator (D + 2) to the second, obtaining

$$\begin{cases} (2D + 1)(2D - 1)x + (2D + 1)(D + 2)y = (2D + 1)0, \\ (D + 2)(3D - 2)x + (D + 2)(2D + 1)y = (D + 2)e^{2t}. \end{cases}$$

Subtracting the second equation from the first, we obtain $(D^2 - 4D + 3)x = -(D + 2)e^{2t}$ or $(D^2 - 4D + 3)x = -4e^{2t}$.

The G.S. of this is

$$x = c_1e^t + c_2e^{3t} + 4e^{2t}. \qquad (2)$$

We find y using the alternative procedure of the text, pages 293-295. We subtract the second equation of (1) from 2 times the first, thereby eliminating Dy but not y. We have

$$Dx + 3y = -e^{2t}. \qquad (3)$$

Differentiating (2), we find

$$dx = c_1e^t + 3c_2e^{3t} + 8e^{2t}.$$

We substitute this into (3) and solve for y. We find

$$y = \frac{1}{3}\left[-c_1e^t - 3c_2e^{3t} - 9e^{2t}\right]$$

or

$$y = -\frac{c_1}{3}e^t - c_2e^{3t} - 3e^{2t}.$$

The G.S. of (1) is this

$$\begin{cases} x = c_1e^t + c_2e^{3t} + 4e^{2t}, \\ y = -\dfrac{c_1}{3}e^t - c_2e^{3t} - 3e^{2t}. \end{cases}$$

6. We introduce operator notation and write the system in the form

$$\begin{cases} Dx + (D + 2)y = \sin t, \\ (D - 1)x + (D - 1)y = 0 \end{cases} \qquad (1)$$

We apply the operator $(D - 1)$ to the first equation and the operator $(D + 2)$ to the second, obtaining

$$\begin{cases} (D - 1)Dx + (D - 1)(D + 2)y = (D - 1)\sin t, \\ (D + 2)(D - 1)x + (D + 2)(D - 1)y = (D + 2)0. \end{cases}$$

Subtracting the first equation from the second, we obtain $[(D + 2)(D - 1) - (D - 1)D]x = -(D - 1)\sin t$ or $(2D - 2)x = -(D - 1)\sin t$ or finally $(D - 1)x = \frac{1}{2}\sin t - \frac{1}{2}\cos t$. The G.S. of this is

$$x = c e^{t} - \frac{1}{2}\sin t \qquad (2).$$

We find y using the alternative procedure of the text, pages 293 - 295. We subtract the second equation of (1) from the first, thereby eliminating Dy but not y. We have $x + 3y = \sin t$. We substitute x given by (2) into this and solve for y, obtaining $y = -\frac{1}{3}c e^{t} + \frac{1}{2}\sin t$. The G.S. of (1) is thus

$$\begin{cases} x = c e^{t} - \frac{1}{2}\sin t, \\ y = -\frac{1}{3} c e^{t} + \frac{1}{2}\sin t. \end{cases}$$

7. We introduce operator notation and write the system in the form

$$\begin{cases} (5D - 5)x + (D - 1)y = 0, \\ (4D - 3)x + \quad D\,y = t. \end{cases} \tag{1}$$

We apply the operator D to the first equation and the operator (D - 1) to the second, thereby obtaining

$$\begin{cases} \quad D(5D - 5)x + D(D - 1)y = D0, \\ (D - 1)(4D - 3)x + (D - 1)Dy = (D - 1)t. \end{cases}$$

We subtract the second equation from the first, obtaining $(D^2 + 2D - 3)x = -(D - 1)t$, that is, $(D^2 + 2D - 3)x = t - 1$. The G.S. of this is

$$x = c_1e^t + c_2e^{-3t} - \frac{1}{3}t + \frac{1}{9}. \tag{2}$$

We find y using the alternative procedure of the text, pages 293-295. We subtract the second equation of (1) from the first, obtaining

$$Dx - 2x - y = -t,$$

which involves y but does not involve Dy. From this,

$$y = Dx - 2x + t. \tag{3}$$

Differentiating (2), we have

$$Dx = c_1e^t - 3c_2e^{-3t} - \frac{1}{3}. \tag{4}$$

Now substituting x from (2) and Dx from (4) into (3), we find

$$y = c_1e^t - 3c_2e^{-3t} - \frac{1}{3} - 2c_1e^t - 2c_2e^{-3t} + \frac{2}{3}t - \frac{2}{9} + t$$

or

$$y = -c_1e^t - 5c_2e^{-3t} + \frac{5}{3}t - \frac{5}{9}.$$

The G.S. of system (1) is thus

$$\begin{cases} x = c_1e^t + c_2e^{-3t} - \frac{1}{3}t + \frac{1}{9}, \\ y = -c_1e^t - 5c_2e^{-3t} + \frac{5}{3}t - \frac{5}{9}. \end{cases}$$

9. We introduce operator notation and write the system in the form

$$\begin{cases} (D - 1)x + (D - 6)y = e^{3t}, \\ (D - 2)x + (2D - 6)y = t. \end{cases} \qquad (1)$$

We apply the operator (2D - 6) to the first equation and the operator (D - 6) to the second equation, obtaining

$$\begin{cases} (2D - 6)(D - 1)x + (2D - 6)(D - 6)y = (2D - 6)e^{3t}, \\ (D - 6)(D - 2)x + (D - 6)(2D - 6)y = (D - 6)t. \end{cases}$$

Subtracting the second equation from the first, we obtain $[(2D - 6)(D - 1) - (D - 6)(D - 2)]x = (2D - 6)e^{3t} - (D - 6)t$ or $(D^2 - 6)x = 6t - 1$. Using the undetermined coefficients, the G.S. of this D.E. is found to be

$$x = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} - t + \frac{1}{6}. \qquad (2)$$

We find y using the alternate procedure of the text, pages 293-295. We multiply the first equation of (1) by 2, obtaining $(2D - 2)x + (2D - 12)y = 2e^{3t}$. We now subtract the second equation of (1) from this, obtaining $Dx - 6y = 2e^{3t} - t$, which involves y but not Dy. From this,

$$y = \frac{Dx - 2e^{3t} + t}{6}. \qquad (3)$$

From (2), $Dx = \sqrt{6}c_1 e^{\sqrt{6}t} - \sqrt{6}c_2 e^{-\sqrt{6}t} - 1$.

Substituting this into (3), we get

$$y = \frac{\sqrt{6}c_1 e^{\sqrt{6}t}}{6} - \frac{\sqrt{6}c_2 e^{-\sqrt{6}t}}{6} + \frac{t}{6} - \frac{1}{6} - \frac{e^{3t}}{3}. \qquad (4)$$

The pair (2) and (4) together constitute the G.S. of system (1).

11. We introduce operator notation and write the system in the form

$$\begin{cases} (2D - 1)x + (D - 1)y = e^{-t}, \\ (D + 2)x + (D + 1)y = e^{t}. \end{cases} \qquad (1)$$

We apply the operator $(D + 1)$ to the first equation and the operator $(D - 1)$ to the second, obtaining

$$\begin{cases}(D + 1)(2D - 1)x + (D + 1)(D - 1)y = (D + 1)e^{-t} \\ (D - 1)(D + 2)x + (D - 1)(D + 1)y = (D - 1)e^{t}.\end{cases}$$

Subtracting the second from the first, we obtain $[(D + 1)(2D - 1) - (D - 1)(D + 2)]x = (D + 1)e^{-t} - (D - 1)e^{t}$ or $(D^2 + 1)x = 0$. The G.S. of this D.E. is

$$x = c_1 \sin t + c_2 \cos t. \qquad (2)$$

We now find y using the alternate procedure of the text, pages 293-295. We subtract the second equation of (1) from the first, thereby eliminating Dy but not y, and thus obtaining $(D - 3)x - 2y = e^{-t} - e^{t}$. From this, $y = \frac{Dx - 3x - e^{-t} + e^{t}}{2}$. From (2), $Dx = c_1 \cos t - c_2 \sin t$. Substituting into the preceding expression for y, we get

$$y = \frac{c_1 \cos t - c_2 \sin t - 3c_1 \sin t - 3c_2 \cos t + e^{t} - e^{t}}{2} \text{ or}$$

$$y = -\frac{[(3c_1 + c_2)\sin t]}{2} + \frac{[c_1 - 3c_2)\cos t]}{2} + \frac{e^{t}}{2} - \frac{e^{-t}}{2}.$$

The G.S. of system (1) is thus:

$$\begin{cases}x = c_1 \sin t + c_2 \cos t, \\ y = -\frac{3c_1 + c_2}{2}\sin t + \frac{c_1 - 3c_2}{2}\cos t + \frac{e^{t}}{2} - \frac{e^{-t}}{2}.\end{cases}$$

12. We introduce operator notation and write the system in the form

$$\begin{cases}(3D - 1)x + (2D + 1)y = t - 1, \\ (D - 1)x + Dy = t + 2.\end{cases} \qquad (1)$$

We apply the operator D to the first equation and the operator (2D + 1) to the second, thereby obtaining

$$\begin{cases} D(3D - 1)x + D(2D + 1)y = D(t - 1), \\ (2D + 1)(D - 1)x + (2D + 1)Dy = (2D + 1)(t + 2). \end{cases}$$

We subtract the second equation from the first, obtaining $[D(3D - 1) - (2D + 1)(D - 1)]x = D(t - 1) - (2D + 1)(t + 2)$ or $(D^2 + 1)x = -t - 3$. The auxiliary equation of this D.E. is $m^2 + 1 = 0$ with roots $\pm i$, and the complementary function is $x_c = c_1 \sin t + c_2 \cos t$. A particular integral is $-t - 3$, and so the G.S. is

$$x = c_1 \sin t + c_2 \cos t - t - 3. \qquad (2)$$

We find y using the alternative procedure of the text, pages 293-295. We multiply the second equation of (1) by 2 to obtain $(2D - 2)x + 2Dy = 2t + 4$. Then we subtract this from the first equation of (1) to obtain $Dx + x + y = -t - 5$, which involves y but does not involve Dy. From this,

$$y = -x - Dx - t - 5. \qquad (3)$$

Differentiating (2), we have

$$Dx = c_1 \cos t - c_2 \sin t - 1. \qquad (4)$$

Now substituting x from (2) and Dx from (4) into (3), we find $y = -c_1 \sin t - c_2 \cos t + t + 3 - c_1 \cos t + c_2 \sin t + 1 - t - 5$ or $y = (c_2 - c_1)\sin t - (c_1 + c_2)\cos t - 1$.

The G.S. of system (1) is thus

$$\begin{cases} x = c_1 \sin t + c_2 \cos t - t - 3, \\ y = (c_2 - c_1)\sin t - (c_1 + c_2)\cos t - 1. \end{cases}$$

15. We introduce operator notation and write the system in the form

$$\begin{cases} (2D + 1)x + (D + 5)y = 4t, \\ (D + 2)x + (D + 2)y = 2. \end{cases} \qquad (1)$$

We apply the operator $(D + 2)$ to the first equation and the operator $(D + 5)$ to the second, thereby obtaining

$$\begin{cases} (D + 2)(2D + 1)x + (D + 2)(D + 5)y = (D + 2)4t, \\ (D + 5)(D + 2)x + (D + 5)(D + 2)y = (D + 5)2. \end{cases}$$

We subtract the second equation from the first, obtaining $(D^2 - 2D - 8)x = 8t - 6$. The auxiliary equation of this D.E. is $m^2 - 2m - 8 = 0$, that is, $(m - 4)(m + 2) = 0$, with roots $m = 4, -2$; and the complementary function is $x_c = c_1e^{4t} + c_2e^{-2t}$. A particular integral is $x_p = -t + 1$, so the G.S. is

$$x = c_1e^{4t} + c_2e^{-2t} - t + 1. \qquad (2)$$

We find y using the alternative procedure of the text, pages 293-295. We subtract the second equation of (1) from the first to obtain $Dx - x + 3y = 4t - 2$, which involves y but does not involve Dy. From this,

$$y = \frac{1}{3}\,[x - Dx + 4t - 2]. \qquad (3)$$

Differentiating (2), we have

$$Dx = 4c_1e^{4t} - 2c_2e^{-2t} - 1. \qquad (4)$$

Now substituting x from (2) and Dx from (4) into (3), we find $y = \frac{1}{3}\left[c_1e^{4t} + c_2e^{-2t} - t + 1 - 4c_1e^{4t} + 2c_2e^{-2t} + 1 + 4t - 2\right]$ or $y = -c_1e^{4t} + c_2e^{-2t} + t$. The G.S. of system (1) is thus

$$\begin{cases} x = c_1e^{4t} + c_2e^{-2t} - t + 1, \\ y = -c_1e^{4t} + c_2e^{-2t} + t. \end{cases}$$

17. We introduce operator notation and write the system in the form

$$\begin{cases} (2D + 1)x + (4D - 1)y = 3e^t, \\ (D + 2)x + (D + 2)y = e^t. \end{cases} \qquad (1)$$

We apply the operator (D + 2) to the first equation and the operator (4D - 1) to the second equation, obtaining

$$\begin{cases} (D + 2)(2D + 1)x + (D + 2)(4D - 1)y = (D + 2)3e^t, \\ (4D - 1)(D + 2)x + (4D - 1)(D + 2)y = (4D - 1)e^t. \end{cases}$$

We subtract the first equation from the second, obtaining $(2D^2 + 2D - 4)x = -6e^t$, that is, $(D^2 + D - 2)x = -3e^t$.

The auxiliary equation of this D.E. is $m^2 + m - 2 = 0$, with roots $m = 1, -2$; and the complementary function is $x_c = c_1 e^t + c_2 e^{-2t}$. Using the method of undetermined coefficients, we assume a particular integral of the form $x_p = A\,t\,e^t$. We quickly find $A = -1$, so $x_p = -t\,e^t$. Thus, the G.S. of the D.E. is

$$x = c_1 e^t + c_2 e^{-2t} - t\,e^t. \qquad (2)$$

We find y using the alternative procedure of the text, pages 293-295. We multiply the second equation of (1) by 4 to obtain

$$(4D + 8)x + (4D + 8)y = 4\,e^t.$$

We then subtract the first equation of (1) from this, obtaining $2\,Dx + 7x + 9y = e^t$, which involves y but does not involve Dy. From this

$$y = \frac{1}{9}\left[-7x - 2\,Dx + e^t\right]. \qquad (3)$$

Differentiating (2), we have

$$Dx = c_1 e^t - 2c_2 e^{-2t} - t\,e^t - e^t. \qquad (4)$$

Now substituting x from (2) and Dx from (4) into (3), we find

$$y = \frac{1}{9}\left[-7c_1 e^t - 7c_2 e^{-2t} + 7t\,e^t - 2c_1 e^t + 4c_2 e^{-2t} + 2t\,e^t + 2e^t + e^t\right]$$

or $y = -c_1 e^t - \frac{c_2}{3} e^{-2t} + t\,e^t + \frac{1}{3} e^t$. The G.S. of system (1) is thus

$$\begin{cases} x = c_1 e^t + c_2 e^{-2t} - t\,e^t, \\ y = (1/3 - c_1)e^t - (1/3)\,c_2 e^{-2t} + t\,e^t. \end{cases}$$

20. We introduce operator notation and write the system in the form

$$\begin{cases} (D - 5)x + Dy = 2t - 8, \\ -8x + (D - 1)y = -t^2. \end{cases} \qquad (1)$$

We apply the operator $(D - 1)$ to the first equation and the operator $D$ to the second, obtaining

$$\begin{cases} (D + 2)(D - 5)x + (D - 1)Dy = (D - 1)(2t - 8), \\ -8Dx + D(D - 1)y = -Dt^2. \end{cases}$$

We subtract the second equation from the first, obtaining $(D^2 + 2D + 5)x = 10$.

The auxiliary equation of this D.E. is $m^2 + 2m + 5 = 0$, with roots $m = -1 \pm 2i$; and the complementary function is $x_c = e^{-t}(c_1 \sin 2t + c_2 \cos 2t)$. A particular integral is $x_p = 2$; and the G.S. is

$$x = e^{-t}(c_1 \sin 2t + c_2 \cos 2t) + 2. \qquad (2)$$

We find y using the alternative procedure of the text, pages 293-295. We subtract the second equation of (1) from the first to obtain $Dx + 3x + y = t^2 + 2t - 8$, which involves y but does not involve Dy. From this,

$$y = -3x - Dx + t^2 + 2t - 8. \qquad (3)$$

Differentiating (2), we find

$$Dx = e^{-t}\left[(-c_1 - 2c_2)\sin 2t + (2c_1 - c_2)\cos 2t\right]. \qquad (4)$$

Now substituting x given by (2) and Dx given by (4) into (3), we obtain

$$\begin{aligned} y = &- 3e^{-t}(c_1 \sin 2t + c_2 \cos 2t) - 6 \\ &+ e^{-t}\left[(c_1 + 2c_2)\sin 2t + (c_2 - 2c_1)\cos 2t\right] \\ &+ t^2 + 2t - 8 \end{aligned}$$

or

$$\begin{aligned} y = e^{-t}&\left[2(c_2 - c_1)\sin 2t - 2(c_1 + c_2)\cos 2t\right] \\ &+ t^2 + 2t - 14. \end{aligned}$$

The G.S. of the system (1) is thus

$$\begin{aligned} y &= e^{-t}(c_1 \sin 2t + c_2 \cos 2t) + 2, \\ y &= e^{-t}\left[2(c_2 - c_1)\sin 2t - 2(c_1 + c_2)\cos 2t\right] \\ &\quad + t^2 + 2t - 14. \end{aligned}$$

21. We introduce operator notation and write the system in the form

$$\begin{cases} (2D - 1)x + (D - 1)y = 1, \\ (D + 2)x + (D - 1)y = t. \end{cases} \tag{1}$$

For this system we merely have to subtract the second equation from the first to eliminate both y and Dy. We have

$$[(2D - 1) - (D + 2)]x = 1 - t \text{ or } (D - 3)x = 1 - t.$$

The G.S. of this is

$$x = c_1 e^{3t} + \frac{t}{3} - \frac{2}{9}. \tag{2}$$

We cannot apply the alternate procedure of pages 293-295 to find y here; for, as noted above, elimination of Dy from (1) also results in elimination of y itself. Thus we proceed to eliminate x and Dx from (1). We apply the operator (D + 2) to the first equation of (1) and the operator (2D - 1) to the second equation, obtaining

$$\begin{cases} (D + 2)(2D - 1)x + (D + 2)(D - 1)y = (D + 2)1, \\ (2D - 1)(D + 2)x + (2D - 1)(D - 1)y = (2D - 1)t. \end{cases}$$

Subtracting the first from the second, we obtain

$$[(2D - 1)(D - 1) - (D + 2)(D - 1)]y = (2D - 1)t - (D + 2)1$$

or $(D^2 - 4D + 3)y = -t$. Using undetermined coefficients, the G.S. of this D.E. is found to be

$$y = k_1 e^t + k_2 e^{3t} - \frac{t}{3} - \frac{4}{9}. \qquad (3)$$

The determinant of the operator "coefficients" of x and y in (1) is

$$\begin{vmatrix} 2D - 1 & D - 1 \\ D + 2 & D - 1 \end{vmatrix} = D^2 - 4D + 3.$$

Since this is of order 2, only two of the three constants $c_1$, $k_1$, $k_2$ in (2) and (3) can be independent. To determine the relations which must thus exist among these three constants, we substitute x given by (2) and y given by (3) into system (1). Substituting into the first equation of (1), we have

$$2\left(3c_1 e^{3t} + \frac{1}{3}\right) + \left(k_1 e^t + 3k_2 e^{3t} - \frac{1}{3}\right)$$

$$- \left(c_1 e^{3t} + \frac{t}{3} - \frac{2}{9}\right) - \left(k_1 e^t + k_2 e^{3t} - \frac{t}{3} - \frac{4}{9}\right) = 1$$

or

$$(5c_1 + 2k_2)e^{3t} + 0k_1 e^t = 0.$$

Thus $k_2 = -\frac{5c_1}{2}$ and $k_1$ is arbitrary. Thus $k_1$ is the "second" arbitrary constant of the G.S., and we now write it as $c_2$. Hence we have the G.S. in the form

$$\begin{cases} x = c_1 e^{3t} + \dfrac{t}{3} - \dfrac{2}{9}, \\ \\ y = c_2 e^{t} - \dfrac{5c_1 e^{3t}}{2} - \dfrac{t}{3} - \dfrac{4}{9}. \end{cases}$$

22. We introduce operator notation and write the system in the form

$$\begin{cases} D^2x + Dy = e^{2t}, \\ \\ (D - 1)x + (D - 1)y = 0. \end{cases} \tag{1}$$

We apply the operator $(D - 1)$ to the first equation and the operator $D$ to the second equation, obtaining

$$\begin{cases} (D - 1)D^2x + (D - 1)Dy = (D - 1)e^{2t}, \\ \\ D(D - 1)x + D(D - 1)y = D0. \end{cases}$$

Subtracting the second equation from the first, we obtain

$$[(D - 1)D^2 - D(D - 1)]x = (D - 1)e^{2t} - 0$$

or

$$(D^3 - 2D^2 + D)x = e^{2t}.$$

The auxiliary equation of this D.E. is $m^3 - 2m^2 + m = 0$ with roots $m = 0, 1, 1$ (double root); and the G.S., found by using undetermined coefficients, is

$$x = c_1 + (c_2 + c_3 t)e^{t} + \frac{e^{2t}}{2}. \tag{2}$$

We find using the alternate procedure of pages 293-295 of the text. We subtract the second equation of (1) from

the first, thereby eliminating Dy but not y, and thus obtaining $D^2x - Dx + x + y = e^{2t}$. From this,

$$y = -x + Dx - D^2x + e^{2t}. \tag{3}$$

From (2), we find $Dx = (c_2 + c_3 + c_3t)e^t + e^{2t}$, $D^2x = (c_2 + 2c_3 + c_3t)e^t + 2e^{2t}$. Substituting x from (2) and these derivatives into (3), we get

$$\begin{aligned} y = &- \left[c_1 + (c_2 + c_3t)e^t + \frac{e^{2t}}{2}\right] \\ &+ [(c_2 + c_3 + c_3t)e^t + e^{2t}] \\ &- [(c_2 + 2c_3 + c_3t)\, e^t + 2e^{2t}] + e^{2t} \end{aligned}$$

or

$$y = -c_1 - (c_2 + c_3 + c_3t)e^t - \frac{e^{2t}}{2}. \tag{4}$$

The pair (2) and (4) together constitute the G.S. of system (1).

26. We introduce operator notation and write the system in the form

$$\begin{cases} (D^2 + 1)x + (4D - 4)y = 0, \\ (D - 1)x + (D + 9)y = e^{2t}. \end{cases} \tag{1}$$

We apply the operator $(D + 9)$ to the first equation and the operator $(4D - 4)$ to the second equation, obtaining

$$\begin{cases} (D + 9)(D^2 + 1)x + (D + 9)(4D - 4)y = (D + 9)0, \\ (4D - 4)(D - 1)x + (4D - 4)(D + 9)y = (4D - 4)e^{2t}. \end{cases}$$

Subtracting the second equation from the first, we obtain

$$[(D + 9)(D^2 + 1) - (4D - 4)(D - 1)]x = 0 - (4D - 4)e^{2t}$$

or

$$(D^3 + 5D^2 + 9D + 5)x = -4e^{2t}.$$

The auxiliary equation of this D.E. is $m^3 + 5m^2 + 9m + 5 = 0$ or $(m + 1)(m^2 + 4m + 5) = 0$, with roots $m = -1$, $m = -2 \pm i$. Using this information and undetermined coefficients, the G.S. of this D.E. is found to be

$$x = c_1e^{-t} + e^{-2t}(c_2 \sin t + c_3 \cos t) - \frac{4e^{2t}}{51}. \qquad (2)$$

We find y using the alternate procedure of pages 293-295 of the text. We multiply the second equation of (1) by 4, obtaining $(4D - 4)x + (4D + 36)y = 4e^{2t}$. We now subtract this from the first equation of (1), obtaining $D^2x - 4Dx + 5x - 40y = -4e^{2t}$. From this,

$$y = \frac{D^2x - 4Dx + 5x + 4e^{2t}}{40}. \qquad (3)$$

From (2),

$$Dx = -c_1e^{-t} + e^{-2t}(-2c_2 - c_3)\sin t + e^{-2t}(c_2 - 2c_3)\cos t - \frac{8e^{2t}}{51},$$

$$D^2x = c_1e^{-t} + e^{-2t}(3c_2 + 4c_3)\sin t + e^{-2t}(-4c_2 + 3c_3)\cos t - \frac{16e^{2t}}{51}.$$

Substituting x from (2) and these derivatives into (3), we get

$$y = \frac{\left[c_1e^{-t} + e^{-2t}(3c_2 + 4c_3)\sin t + e^{-2t}(-4c_2 + 3c_3)\cos t - \frac{16e^{2t}}{51}\right]}{40} - \frac{4\left[-c_1e^{-t} - e^{-2t}(-2c_2 - c_3)\sin t + e^{-2t}(c_2 - 2c_3)\cos t - \frac{8e^{2t}}{51}\right]}{40} + \frac{5\left[c_1e^{-t} + e^{-2t}(c_2\sin t + c_3\cos t) - \frac{4e^{2t}}{51}\right]}{40} + \frac{4e^{2t}}{40}$$

or

$$y = \frac{(c_1 + 4c_1 + 5c_1)e^{-t}}{40}$$

$$+ \frac{e^{-2t}(3c_2 + 4c_3 + 8c_2 + 4c_3 + 5c_2)(\sin t)}{40}$$

$$+ \frac{e^{-2t}(-4c_2 + 3c_3 - 4c_2 + 8c_3 + 5c_3)(\cos t)}{40}$$

$$+ \frac{(-16 + 32 - 20 + 204)e^{2t}}{(40)(51)}$$

or finally

$$y = \frac{c_1 e^{-t}}{4} + e^{-2t}\left[\left(\frac{2c_2}{5} + \frac{c_3}{5}\right)\sin t\right.$$

$$\left. + \left(-\frac{c_2}{5} + \frac{2c_3}{5}\right)\cos t\right] + \frac{5e^{2t}}{51}. \qquad (4)$$

The pair (2) and (4) constitute the G.S. of system (1).

30. By text, page 286, equations (7.7), we let $x_1 = x_1$, $x_2 = x'$, $x_3 = x''$, $x_4 = x'''$. From these and the given fourth order D.E., we have

$$x_1' = x' = x_2,\ x_2' = x'' = x_3,\ x_3' = x''' = x_4,$$

$$x_4' = x^{iv} = -2tx + t^2x'' + \cos t = -2tx_1 + t^2x_3 + \cos t\,.$$

Thus we have the system

$$\begin{cases} x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4, \\ x_4' = -2tx_1 + t^2 x_3 + \cos t\,. \end{cases}$$

**Section 7.2, Page 307.**

3. We apply Kirchhoff's voltage law (text, Section 5.6) to each of the three loops indicated in Figure 7.6. For the left hand loop, the voltage drops are as follows:

   1. across the resistor $R_1$: $20i_1$.
   2. across the inductor $L_1$: $0.01\,i'$.

   Thus applying the voltage law to this loop, we have the D.E.

$$0.01\,i' + 20i_1 = 100. \qquad (1)$$

   For the right hand loop, the voltage drops are as follows:

   1. across the resistor $R_1$: $-20i_1$.
   2. across the resistor $R_2$: $40i_2$.
   3. across the inductor $L_2$: $0.02\,i_2'$.

   Thus applying the voltage law to this loop, we have the D.E.

$$0.02\,i_2' - 20i_1 + 40i_2 = 0. \qquad (2)$$

   For the outside loop, the voltage drops are

   1. across the resistor $R_2$: $40i_2$.
   2. across the inductor $L_1$: $0.01\,i'$.
   3. across the inductor $L_2$: $0.02\,i_2'$.

Thus applying the voltage law to this loop, we have the D.E.

$$0.02\,i_2' + 0.01\,i' + 40i_2 = 100. \qquad (3)$$

The three equations are not all independent. Equation (3) may be obtained by adding equations (1) and (2). Hence we retain only equations (1) and (2).

We now apply Kirchhoff's current law to the upper junction point in the figure. We have $i = i_1 + i_2$. Hence we replace i by $i_1 + i_2$ in (1), retain (2) as is, and obtain the linear system

$$\begin{cases} 0.01\,i_1' + 0.01\,i_2' + 20i_1 = 100, \\ 0.02\,i_2' - 20i_1 + 40i_2 = 0. \end{cases} \qquad (4)$$

The initially zero currents give the I.C.'s

$$i_1(0) = 0, \quad i_2(0) = 0.$$

We introduce operator notation and write (4) in the form

$$\begin{cases} (0.01D + 20)i_1 + 0.01Di_2 = 100, \\ -20i_1 + (0.02D + 40)i_2 = 0. \end{cases} \qquad (5)$$

We apply the operator 0.02D + 40 to the first equation of (5) and the operator 0.01D to the second and subtract to obtain

$$[(0.01D + 20)(0.02D + 40) + 0.2D]i_1 = (0.02D + 40)100$$

or

$$(.0002D^2 + D + 800)i_1 = 4000$$

or finally

$$(D^2 + 5000D + 4{,}000{,}000)i_1 = 20{,}000{,}000. \quad (6)$$

The auxiliary equation of the corresponding homogeneous D.E. is $m^2 + 5000m + 4{,}000{,}000 = 0$, with roots $m = -1000, -4000$. Using this information and the method of undetermined coefficients, we see that the solution of (6) is

$$i_1 = c_1e^{-1000t} + c_2e^{-4000t} + 5. \quad (7)$$

We use the alternative procedure of page 293 of the text to find $i_2$. Returning to (5), we multiply the first equation by 2 and then subtract the second equation from this, obtaining

$$.02Di_1 + 60i_1 - 40i_2 = 200.$$

Note that this contains $i_2$ but not $Di_2$. From it, we have

$$i_2 = \frac{3i_1}{2} + .0005Di_1 - 5. \quad (8)$$

From (7), we find

$$Di_1 = -1000c_1e^{-1000t} - 4000c_2e^{-4000t}.$$

Now substituting $i_1$ from (7) and $Di_1$ just obtained into (8) and simplifying, we obtain

$$i_2 = c_1e^{-1000t} - \frac{c_2e^{-4000t}}{2} + \frac{5}{2}. \qquad (9)$$

Finally, applying the I.C.'s to (7) and (9), we have $c_1 + c_2 + 5 = 0$ and $c_1 - \frac{c_2}{2} + \frac{5}{2} = 0$, from which we find $c_1 = -\frac{10}{3}$, $c_2 = -\frac{5}{3}$. Thus we obtain

$$\begin{cases} i_1 = -\dfrac{10e^{-1000t}}{3} - \dfrac{5e^{-4000t}}{3} + 5, \\ i_2 = -\dfrac{10e^{-1000t}}{3} + \dfrac{5e^{-4000t}}{6} + \dfrac{5}{2}. \end{cases}$$

6. Let x = the amount of salt in tank X at time t, and let y = the amount of salt in tank Y at time t, each measured in kilograms. Each tank always contains 30 liters of fluid, so the concentration of salt in tank X is $\frac{x}{30}$ (kg/liter) and that in tank Y is $\frac{y}{30}$ (kg/liter) . Salt enters tank X two ways: (a) 1 kg. of salt per liter of brine enters at the rate of 2 liters/min. from outside the system, and (b) salt in the brine pumped from tank Y back to tank X at the rate of 1 liter/min. By (a), salt enters tank X at the

rate of 2 kg./min.; and by (b), it enters at the rate of $\frac{y}{30}$ kg./min. Salt only leaves tank X in the brine flowing from tank X into tank Y at the rate of 4 liters/min. Thus salt leaves tank X at the rate of $\frac{4x}{30}$ kg./min. Hence we obtain the D.E.

$$x' = 2 + \frac{y}{30} - \frac{4x}{30} \tag{1}$$

for the amount of salt in tank X at time t. The D.E. for the amount in tank Y is obtained similarly. It is

$$y' = \frac{4x}{30} - \frac{4y}{30}. \tag{2}$$

Since initially there was 30 kg. of salt in tank X and pure water in tank Y, the I.C.'s are $x(0) = 30$, $y(0) = 0$.

We introduce operator notation and write the D.E.'s (1) and (2) in the forms

$$\begin{cases} \left(D + \frac{2}{15}\right)x - \frac{1}{30}y = 2, \\ -\frac{2}{15}x + \left(D + \frac{2}{15}\right)y = 0. \end{cases} \tag{3}$$

We apply the operator $\left(D + \frac{2}{15}\right)$ to the first equation (3), multiply the second by $\frac{1}{30}$, and add to obtain

$$\left[\left(D + \frac{2}{15}\right)^2 - \frac{1}{225}\right]x = \frac{4}{15}$$

or

$$\left(D^2 + \frac{4}{15}D + \frac{3}{225}\right)x = \frac{4}{15}. \qquad (4)$$

The auxiliary equation of the corresponding homogeneous D.E. is

$$m^2 + \frac{4m}{15} + \frac{3}{225} = 0$$

with roots $m = -\frac{1}{5}, -\frac{1}{15}$. Using this information and the method of undetermined coefficients, we see that the solution of (4) is

$$x = c_1e^{-t/5} + c_2e^{-t/15} + 20. \qquad (5)$$

To find y, we return to (3), multiply the first equation by $\frac{2}{15}$, apply the operator $\left(D + \frac{2}{15}\right)$ to the second, and add. After simplification, we obtain

$$\left(D^2 + \frac{4}{15}D + \frac{3}{225}\right)y = \frac{4}{15}.$$

Comparing this with (4) and its solution (5), we see that its solution is

$$y = c_3e^{-t/5} + c_4e^{-t/15} + 20. \qquad (6)$$

Only two of the four constants $c_1$, $c_2$, $c_3$, $c_4$ in (5) and (6) are independent. To determine the relations which

must exist among these, we substitute (5) and (6) into the second equation of (3). We find

$$\frac{(-2c_1 - c_3)e^{-t/5}}{15} + \frac{(-2c_2 + c_4)e^{-t/15}}{15} = 0.$$

Hence we must have $c_3 = -2c_1$, $c_4 = 2c_2$. Thus we have

$$\begin{cases} x = c_1e^{-t/5} + c_2e^{-t/15} + 20, \\ y = -2c_1e^{-t/5} + 2c_2e^{-t/15} + 20. \end{cases}$$

Finally we apply the I.C.'s. We have $c_1 + c_2 + 20 = 30$, $-2c_1 + 2c_2 + 20 = 0$, from which we find $c_1 = 10$, $c_2 = 0$. Thus we obtain the solution

$$\begin{cases} x = 10e^{-t/5} + 20, \\ y = -20e^{-t/5} + 20. \end{cases}$$

**Section 7.3, Page 317.**

2. (a) Let $\begin{cases} x = 3e^{7t}, \\ y = 2e^{7t}. \end{cases}$ and $\begin{cases} x = e^{-t}, \\ y = -2e^{-t}. \end{cases}$

**Consider the first pair:** $\frac{dx}{dt} = 21e^{7t}$, and $5x + 3y = 5(3e^{7t}) + 3(2e^{7t}) = 15e^{7t} + 6e^{7t} = 21e^{7t}$. So $\frac{dx}{dt} = 5x + 3y$. Now, $\frac{dy}{dt} = 14e^{7t}$, and $4x + y = 4(3e^{7t}) + 2e^{7t} = 12e^{7t} + 2e^{7t} = 14e^{7t}$. So $\frac{dy}{dt} = 4x + y$. Thus the first pair satisfies the linear system.

**Consider the second pair:** $\frac{dx}{dt} = -e^{-t}$, and $5x + 3y = 5(e^{-t}) + 3(-2e^{-t}) = 5e^{-t} - 6e^{-t} = -e^{-t}$. So $\frac{dx}{dt} = 5x + 3y$. Now, $\frac{dy}{dt} = 2e^{-t}$, and $4x + y = 4(e^{-t}) + (-2e^{-t}) = 4e^{-t} - 2e^{-t} = 2e^{-t}$. So $\frac{dy}{dt} = 4x + y$. Thus the second pair satisfies the linear system. Hence each pair is a solution of the system.

(b) Consider $\begin{vmatrix} 3e^{7t} & e^{-t} \\ 2e^{7t} & -2e^{-t} \end{vmatrix} = (3e^{7t})(-2e^{-t}) - (2e^{7t})(e^{-t})$

$$= -6e^{6t} - 2e^{6t} = -8e^{6t} \neq 0.$$

Thus these two solutions are linearly independent for all $a \leq t \leq b$.

(c) The general solution of the system is

$$\begin{cases} x = 3c_1e^{7t} + c_2e^{-t}, \\ y = 2c_1e^{7t} - 2c_2e^{-t}. \end{cases}$$

The I.C. $f(0) = 0$ gives $3c_1 + c_2 = 0$; and the I.C. $g(0) = 8$ gives $2c_1 - 2c_2 = 8$. The unique solution of this system in $c_1$ and $c_2$ is $c_1 = 1$, $c_2 = -3$. Thus

$$\begin{cases} x = 3e^{7t} - 3e^{-t}, \\ y = 2e^{7t} + 6e^{-t}, \end{cases}$$

is the solution of the given linear system that satisfies the two initial conditions.

4. The Wronskian determinant of $f_1$ and $f_2$ is

$$W(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix} = f_1(t)g_2(t) - f_2(t)g_1(t).$$

Differentiating, we find

$$\begin{aligned} W'(t) &= f_1(t)g_2'(t) + f_1'(t)g_2(t) \\ &\quad - f_2(t)g_1'(t) - f_2'(t)g_1(t). \end{aligned} \tag{1}$$

Since $x = f_1(t)$, $y = g_1(t)$ is a solution of system (7.67), we have $f_1'(t) = a_{11}(t)f_1(t) + a_{12}(t)g_1(t)$ and $g_1'(t) = a_{21}(t)f_1(t) + a_{22}(t)g_1(t)$. Since $x = f_2(t)$, $y = g_2(t)$ is a solution of system (7.67), we have $f_2'(t) = a_{11}(t)f_2(t) + a_{12}(t)g_2(t)$ and $g_2'(t) = a_{21}(t)f_2(t) + a_{22}(t)g_2(t)$. Substituting these derivatives into (1), we find

$$\begin{aligned} W'(t) &= f_1(t)[a_{21}(t)f_2(t) + a_{22}(t)g_2(t)] \\ &\quad + g_2(t)[a_{11}(t)f_1(t) + a_{12}(t)g_1(t)] \\ &\quad - f_2(t)[a_{21}(t)f_1(t) + a_{22}(t)g_1(t)] \\ &\quad - g_1(t)[a_{11}(t)f_2(t) + a_{12}(t)g_2(t)] \\ &= a_{11}(t)[f_1(t)g_2(t) - f_2(t)g_1(t)] \\ &\quad + a_{22}(t)[f_1(t)g_2(t) - f_2(t)g_1(t)] \\ &= [a_{11}(t) + a_{22}(t)][f_1(t)g_2(t) - f_2(t)g_1(t)]. \end{aligned}$$

But

$$f_1(t)g_2(t) - f_2(t)g_1(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix} = W(t),$$

so we have proved $W'(t) = [a_{11}(t) + a_{22}(t)]\, W(t)$, as required.

**Section 7.4, Page 328.**

1. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 5Ae^{\lambda t} - 2Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 4Ae^{\lambda t} - Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (5 - \lambda)A - 2B = 0, \\ 4A + (-1 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 5 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 4\lambda + 3 = 0$, whose roots are $\lambda_1 = 1$, $\lambda_2 = 3$.

Setting $\lambda = \lambda_1 = 1$ in (1), we obtain $4A - 2B = 0$ (twice), a nontrivial solution of which is $A = 1$, $B = 2$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^t, \quad y = 2e^t. \tag{2}$$

Setting $\lambda = \lambda_2 = 3$ in (1), we obtain $2A - 2B = 0$ and $4A - 4B = 0$, a nontrivial solution of which is $A = B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{3t}, \quad y = e^{3t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find $x = c_1e^t + c_2e^{3t}$, $y = 2c_1e^t + c_2e^{3t}$.

2. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$, Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 5Ae^{\lambda t} - Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 3Ae^{\lambda t} + Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (5 - \lambda)A - B = 0, \\ 3A + (1 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 5 - \lambda & -1 \\ 3 & 1 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 6\lambda + 8 = 0$, whose roots are $\lambda_1 = 4$, $\lambda_2 = 2$.

Setting $\lambda = \lambda_1 = 4$ in (1), we obtain $A - B = 0$, $3A - 3B = 0$, a nontrivial solution of which is $A = B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{4t}, \quad y = e^{4t}. \tag{2}$$

Setting $\lambda = \lambda_2 = 2$ in (1), we obtain $3A - B = 0$ (twice), a nontrivial solution of which is $A = 1$, $B = 3$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{2t}, \quad y = 3e^{2t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$x = c_1e^{4t} + c_2e^{2t}, \quad y = c_1e^{4t} + 3c_2e^{2t}.$$

5. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 3Ae^{\lambda t} + Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 4Ae^{\lambda t} + 3Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (3 - \lambda)A + B = 0, \\ 4A + (3 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 3 - \lambda & -1 \\ 4 & 3 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 6\lambda + 5 = 0$, whose roots are $\lambda_1 = 1$, $\lambda_2 = 5$.

Setting $\lambda = \lambda_1 = 1$ in (1), we obtain $2A + B = 0$, $4A + 2B = 0$, a nontrivial solution of which is $A = 1$, $B = -2$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^t, \quad y = -2e^t. \tag{2}$$

Setting $\lambda = \lambda_2 = 5$ in (1), we obtain $-2A + B = 0$, $4A - 2B = 0$, a nontrivial solution of which is $A = 1$, $B = 2$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{5t}, \quad y = 2e^{5t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$x = c_1e^t + c_2e^{5t}, \quad y = -2c_1e^t + 2c_2e^{5t}.$$

6. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 6Ae^{\lambda t} - Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 3Ae^{\lambda t} + 2Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (6 - \lambda)A - B = 0, \\ 3A + (2 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 6 - \lambda & -1 \\ 3 & 2 - \lambda \end{vmatrix} = 0.$$

This leads at once to the characteristic equation $\lambda^2 - 8\lambda + 15 = 0$, whose roots are $\lambda_1 = 5$, $\lambda_2 = 3$.

Setting $\lambda = \lambda_1 = 5$ in (1), we obtain $A - B = 0$, $3A - 3B = 0$, a nontrivial solution of which is $A = 1$, $B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{5t}, \quad y = e^{5t}. \tag{2}$$

Setting $\lambda = \lambda_2 = 3$ in (1), we obtain $3A - B = 0$ (twice), a nontrivial solution of which is $A = 1$, $B = 3$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{3t}, \quad y = 3e^{3t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$x = c_1e^{5t} + c_2e^{3t}, \quad y = c_1e^{5t} + 3c_2e^{3t}.$$

7. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 3Ae^{\lambda t} - 4Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 2Ae^{\lambda t} - 3Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (3 - \lambda)A - 4B = 0, \\ 2A + (-3 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 3 - \lambda & -4 \\ 2 & -3 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 1 = 0$, whose roots are $\lambda_1 = 1$, $\lambda_2 = -1$.

Setting $\lambda = \lambda_1 = 1$ in (1), we obtain $2A - 4B = 0$ (twice), a nontrivial solution of which is $A = 2$, $B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = 2e^t, \quad y = e^t. \tag{2}$$

Setting $\lambda = \lambda_2 = -1$ in (1), we obtain $4A - 4B = 0$, $2A - 2B = 0$, a nontrivial solution of which is $A = B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{-t}, \quad y = e^{-t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$x = 2c_1e^t + c_2e^{-t}, \quad y = c_1e^t + c_2e^{-t}.$$

8. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 2Ae^{\lambda t} - Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 9Ae^{\lambda t} + 2Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (2 - \lambda)A - B = 0, \\ 9A + (2 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 2 - \lambda & -1 \\ 9 & 2 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 4\lambda + 13 = 0$, whose roots are $\lambda = 2 \pm 3i$.

Letting $\lambda = 2 + 3i$ in (1), we obtain $-3iA - B = 0$, $9A - 3iB = 0$, a nontrivial solution of which is $A = 1$, $B = -3i$. With these values of A, B, and $\lambda$, we find the complex solution

$$\begin{cases} x = e^{(2+3i)t}, \\ y = -3ie^{(2+3i)t}. \end{cases}$$

Using Euler's formula this takes the form

$$\begin{cases} x = e^{2t}(\cos 3t + i \sin 3t), \\ y = e^{2t}(3 \sin 3t - 3i \cos 3t). \end{cases}$$

Both the real and imaginary parts of this are solutions. Thus we obtain the two linearly independent solutions

$$\begin{cases} x = e^{2t}\cos 3t \\ y = 3e^{2t}\sin 3t \end{cases} \quad \text{and} \quad \begin{cases} x = e^{2t}\sin 3t \\ y = -3e^{2t}\cos 3t. \end{cases}$$

The G.S. of the system is thus

$$\begin{cases} x = e^{2t}(c_1 \cos 3t + c_2 \sin 3t), \\ y = e^{2t}(3c_1 \sin 3t - 3c_2 \cos 3t). \end{cases}$$

11. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = Ae^{\lambda t} - 4Be^{\lambda t}, \\ B\lambda e^{\lambda t} = Ae^{\lambda t} + Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (1 - \lambda)A - 4B = 0, \\ A + (1 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 1 - \lambda & -4 \\ 1 & 1 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 2\lambda + 5 = 0$, whose roots are $1 \pm 2i$.

Letting $\lambda = 1 + 2i$ in (1), we obtain $-2iA - 4B = 0$, $A - 2iB = 0$, a nontrivial solution of which is $A = 2i$, $B = 1$. With these values of A, B, and $\lambda$, we find the complex solution

$$x = 2ie^{(1+2i)t},$$

$$y = e^{(1+2i)t}.$$

Using Euler's formula this takes the form

$$x = 2ie^{t}(\cos 2t + i \sin 2t),$$

$$y = e^{t}(\cos 2t + i \sin 2t).$$

Both the real and imaginary parts of this are solutions. Thus we obtain the two linearly independent solutions

$$\begin{cases} x = -2e^{t}\sin 2t \\ y = e^{t}\cos 2t . \end{cases} \quad \text{and} \quad \begin{cases} x = 2e^{2t}\cos 2t \\ y = e^{t}\sin 2t \end{cases}.$$

The G.S. of the system is thus

$$\begin{cases} x = 2e^{t}(-c_1 \sin 2t + c_2 \cos 2t), \\ y = e^{t}(c_1 \cos 2t + c_2 \sin 2t). \end{cases}$$

15. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 4Ae^{\lambda t} - 2Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 5Ae^{\lambda t} + 2Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (4 - \lambda)A - 2B = 0, \\ 5A + (2 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 4 - \lambda & -2 \\ 5 & 2 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 6\lambda + 18 = 0$, whose roots are $\lambda = 3 \pm 3i$.

Letting $\lambda = 3 + 3i$ in (1), we obtain $(1 - 3i)A - 2B = 0$, $5A - (1 + 3i)B = 0$, a nontrivial solution of which is $A = 2$, $B = 1 - 3i$. With these values of A, B, and $\lambda$, we find the complex solution

$$\begin{cases} x = 2e^{(3+3i)t}, \\ y = (1 - 3i)e^{(3+3i)t}. \end{cases}$$

Using Euler's formula this takes the form

$$\begin{cases} x = 2e^{3t}(\cos 3t + i \sin 3t), \\ y = (1 - 3i)e^{3t}(\cos 3t + i \sin 3t). \end{cases}$$

Both the real and imaginary parts of this are solutions. Thus we obtain the two linearly independent solutions

$$\begin{cases} x = 2e^{3t}\cos 3t \\ y = e^{3t}(\cos 3t + 3 \sin 3t), \end{cases}$$

and

$$\begin{cases} x = 2e^{3t}\sin 3t \\ y = e^{3t}(\sin 3t - 3 \cos 3t). \end{cases}$$

The G.S. of the system is thus

$$\begin{cases} x = 2e^{3t}(c_1 \cos 3t + c_2 \sin 3t), \\ y = e^{3t}[c_1(\cos 3t + 3 \sin 3t) + c_2(\sin 3t - 3 \cos 3t)]. \end{cases}$$

17. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 3Ae^{\lambda t} - 2Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 2Ae^{\lambda t} + 3Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (3 - \lambda)A - 2B = 0, \\ 2A + (3 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions, we must have

$$\begin{vmatrix} 3 - \lambda & -2 \\ 2 & 3 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 6\lambda + 13 = 0$, whose roots are $\lambda = 3 \pm 2i$.

Letting $\lambda = 3 + 2i$ in (1), we obtain $-2iA - 2B = 0$, $2A - 2iB = 0$, a nontrivial solution of which is $A = i$, $B = 1$. With these values of A, B, and $\lambda$, we find the complex solution

$$\begin{cases} x = ie^{(3+2i)t}, \\ y = e^{(3+2i)t}. \end{cases}$$

Using Euler's formula this takes the form

$$\begin{cases} x = ie^{3t}(\cos 2t + i \sin 2t), \\ y = e^{3t}(\cos 2t + i \sin 2t). \end{cases}$$

Both the real and imaginary parts of this are solutions. Thus we obtain the two linearly independent solutions

$$\begin{cases} x = -e^{3t}\sin 2t, \\ y = e^{3t}\cos 2t, \end{cases} \quad \text{and} \quad \begin{cases} x = e^{3t}\cos 2t \\ y = e^{3t}\sin 2t. \end{cases}$$

The G.S. of the system is thus

$$\begin{cases} x = e^{3t}(-c_1 \sin 2t + c_2 \cos 2t), \\ y = e^{3t}(c_1 \cos 2t + c_2 \sin 2t). \end{cases}$$

18. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 6Ae^{\lambda t} - 5Be^{\lambda t}, \\ B\lambda e^{\lambda t} = Ae^{\lambda t} + 2Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (6 - \lambda)A - 5B = 0, \\ A + (2 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solution of this, we must have

$$\begin{vmatrix} 6 - \lambda & -5 \\ 1 & 2 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 8\lambda + 17 = 0$, whose roots are $\lambda = 4 \pm i$.

Letting $\lambda = 4 + i$ in (1), we obtain $(2 - i)A - 5B = 0$, $A - (2 + i)B = 0$, a nontrivial solution of which is $A = -5$, $B = i - 2$. With these values of A, B, and $\lambda$, we find the complex solution

$$\begin{cases} x = -5e^{(4+i)t}, \\ y = (i - 2)e^{(4+i)t}. \end{cases}$$

Using Euler's formula this takes the form

$$\begin{cases} x = -5e^{4t}(\cos t + i \sin t), \\ y = (i - 2)e^{4t}(\cos t + i \sin t). \end{cases}$$

Both the real and imaginary parts of this are solutions. Thus we obtain the two linearly independent solutions

$$\begin{cases} x = -5e^{4t}\cos t, \\ y = e^{4t}(-2\cos t - \sin t) \end{cases} \quad \text{and} \quad \begin{cases} x = -5e^{4t}\sin t \\ y = e^{4t}(\cos t - 2\sin t). \end{cases}$$

The G.S. of the system is thus

$$\begin{cases} x = -5e^{4t}(c_1 \cos t + c_2 \sin t), \\ y = e^{4t}[(c_2 - 2c_1)\cos t - (c_1 + 2c_2)\sin t]. \end{cases}$$

19. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 3Ae^{\lambda t} - Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 4Ae^{\lambda t} - Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (3 - \lambda)A - B = 0 \\ 4A + (-1 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 3 - \lambda & -1 \\ 4 & -1 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 2\lambda + 1 = 0$, whose roots are 1, 1 (real and equal).

Setting $\lambda = 1$ in (1), we obtain $2A - B = 0$, $4A - 2B = 0$, a nontrivial solution of which is $A = 1$, $B = 2$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^t, \quad y = 2e^t. \tag{2}$$

We now seek a second solution of the form $x = (A_1t + A_2)e^t$, $y = (B_1t + B_2)e^t$. Substituting these into the given system, we obtain

$$\begin{cases}(A_1t + A_2 + A_1)e^t = 3(A_1t + A_2)e^t - (B_1t + B_2)e^t, \\ (B_1t + B_2 + B_1)e^t = 4(A_1t + A_2)e^t - (B_1t + B_2)e^t.\end{cases}$$

These equations reduce to

$$\begin{cases}(2A_1 - B_1)t + (2A_2 - A_1 - B_2) = 0, \\ (4A_1 - 2B_1)t + (4A_2 - B_1 - 2B_2) = 0.\end{cases}$$

Thus we must have

$$\begin{cases}2A_1 - B_1 = 0, \quad 2A_2 - A_1 - B_2 = 0, \\ 4A_1 - 2B_1 = 0, \quad 4A_2 - B_1 - 2B_2 = 0.\end{cases}$$

A simple nontrivial solution of these is $A_1 = 1$, $B_1 = 2$, $A_2 = 0$, $B_2 = -1$. This leads to the solution

$$x = te^t, \quad y = (2t - 1)e^t. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$x = c_1e^t + c_2te^t, \quad y = 2c_1e^t + c_2(2t - 1)e^t.$$

20. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 7Ae^{\lambda t} + 4Be^{\lambda t}, \\ B\lambda e^{\lambda t} = -Ae^{\lambda t} + 3Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (7 - \lambda)A + 4B = 0, \\ -A + (3 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 7 - \lambda & 4 \\ -1 & 3 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 10\lambda + 25 = 0$, whose roots are $\lambda = 5, 5$ (real and equal).

Setting $\lambda = 5$ in (1), we obtain $2A + 4B = 0$, $-A - 2B = 0$, a nontrivial solution of which is $A = 2$, $B = -1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = 2e^{5t}, \quad y = -e^{5t}. \tag{2}$$

We now seek a second solution of the form $x = (A_1 t + A_2)e^{5t}$, $y = (B_1 t + B_2)e^{5t}$. Substituting these into the given system, we obtain

$$\begin{cases}(5A_1t + 5A_2 + A_1)e^{5t} = 7(A_1t + A_2)e^{5t} + 4(B_1t + B_2)e^{5t}, \\ (5B_1t + 5B_2 + B_1)e^{5t} = -(A_1t + A_2)e^{5t} + 3(B_1t + B_2)e^{5t}. \end{cases}$$

These equations reduce to

$$\begin{cases}(2A_1 + 4B_1)t + (2A_2 + 4B_2 - A_1) = 0, \\ (A_1 + 2B_1)t + (A_2 + 2B_2 + B_1) = 0. \end{cases}$$

Thus we must have

$$\begin{cases}2A_1 + 4B_1 = 0, \\ A_1 + 2B_1 = 0, \end{cases} \qquad \begin{cases}2A_2 + 4B_2 - A_1 = 0, \\ A_2 + 2B_2 + B_1 = 0. \end{cases}$$

A simple nontrivial solution of this is $A_1 = 2$, $B_1 = -1$, $A_2 = 1$, $B_2 = 0$. This leads to the solution

$$x = (2t + 1)e^{5t}, \quad y = -t\,e^{5t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find the G.S. in the form

$$\begin{cases}x = 2c_1e^{5t} + c_2(2t + 1)e^{5t}, \\ y = -c_1e^{5t} - c_2t\,e^{5t}. \end{cases}$$

22. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = Ae^{\lambda t} - 2Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 2Ae^{\lambda t} - 3Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (1 - \lambda)A - 2B = 0, \\ 2A + (-3 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 1 - \lambda & -2 \\ 2 & -3 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$, whose roots are $-1$, $-1$ (real and equal).

Setting $\lambda = -1$ in (1), we obtain $2A - 2B = 0$ (twice), a nontrivial solution of which is $A = 1$, $B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{-t}, \quad y = e^{-t}. \tag{2}$$

We now seek a second solution of the form $x = (A_1 t + A_2)e^{-t}$, $y = (B_1 t + B_2)e^{-t}$. Substituting these into the given system, we obtain

$$\begin{cases}(-A_1t - A_2 + A_1)e^{-t} = (A_1t + A_2)e^{-t} - 2(B_1t + B_2)e^{-t}, \\ (-B_1t - B_2 + B_1)e^{-t} = 2(A_1t + A_2)e^{-t} - 3(B_1t + B_2)e^{-t}. \end{cases}$$

These equations reduce to

$$\begin{cases}(2A_1 - 2B_1)t + (2A_2 - 2B_2 - A_1) = 0, \\ (2A_1 - 2B_1)t + (2A_2 - 2B_2 - B_1) = 0. \end{cases}$$

Thus we must have

$$\begin{cases}2A_1 - 2B_1 = 0, \quad 2A_2 - 2B_2 - A_1 = 0, \\ 2A_1 - 2B_1 = 0, \quad 2A_2 - 2B_2 - B_1 = 0. \end{cases}$$

A simple nontrivial solution of these is $A_1 = 2$, $B_1 = 2$, $A_2 = 1$, $B_2 = 0$. This leads to the solution

$$x = (2t + 1)e^{-t}, \; y = 2te^{-t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$x = c_1e^{-t} + c_2(2t + 1)e^{-t}, \; y = c_1e^{-t} + 2c_2te^{-t}.$$

23. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases}A\lambda e^{\lambda t} = 3Ae^{\lambda t} - Be^{\lambda t}, \\ B\lambda e^{\lambda t} = Ae^{\lambda t} + Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases}(3 - \lambda)A - B = 0, \\ A + (1 - \lambda)B = 0.\end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 3 - \lambda & -1 \\ 1 & 1 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 4\lambda + 4 = 0$, whose roots are 2, 2 (real and equal).

Setting $\lambda = 2$ in (1), we obtain $A - B = 0$, (twice), a nontrivial solution of which is $A = B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{2t}, \quad y = e^{2t}. \tag{2}$$

We now seek a second solution of the form $x = (A_1t + A_2)e^{2t}$, $y = (B_1t + B_2)e^{2t}$. Substituting these into the given system, we obtain

$$\begin{cases}(2A_1t + 2A_2 + A_1)e^{2t} = 3(A_1t + A_2)e^{2t} - (B_1t + B_2)e^{2t}, \\ (2B_1t + 2B_2 + B_1)e^{2t} = (A_1t + A_2)e^{2t} + (B_1t + B_2)e^{2t}.\end{cases}$$

These equations reduce to

$$\begin{cases}(A_1 - B_1)t + (A_2 - B_2 - A_1) = 0, \\ (A_1 - B_1)t + (A_2 - B_2 - B_1) = 0.\end{cases}$$

Thus we must have

$$\begin{cases} A_1 - B_1 = 0, \\ A_1 - B_1 = 0, \end{cases} \qquad \begin{cases} A_2 - B_2 - A_1 = 0, \\ A_2 - B_2 - B_1 = 0. \end{cases}$$

A simple nontrivial solution of this is $A_1 = B_1 = A_2 = 1$, $B_2 = 0$. This leads to the solution

$$x = (t + 1)e^{2t}, \quad y = te^{2t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find the G.S. in the form

$$\begin{cases} x = c_1 e^{2t} + c_2(t + 1)e^{2t}, \\ y = c_1 e^{2t} + c_2 te^{2t}. \end{cases}$$

26. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 2Ae^{\lambda t} - 4Be^{\lambda t}, \\ B\lambda e^{\lambda t} = Ae^{\lambda t} - 2Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (2 - \lambda)A - 4B = 0, \\ A + (-2 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 2 - \lambda & -4 \\ 1 & -2 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 = 0$, whose roots are $\lambda = 0, 0$ (real and equal).

Setting $\lambda = 0$ in (1), we obtain $2A - 4B = 0$, $A - 2B = 0$, a nontrivial solution of which is $A = 2$, $B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = 2e^{0t} = 2, \quad y = e^{0t} = 1. \tag{2}$$

We now seek a second solution of the form $x = (A_1t + A_2)e^{0t} = A_1t + A_2$, $y = (B_1t + B_2)e^{0t} = B_1t + B_2$. Substituting these into the given system, we obtain

$$\begin{cases} A_1 = 2(A_1t + A_2) - 4(B_1t + B_2), \\ B_1 = (A_1t + A_1) - 2(B_1t + B_1). \end{cases}$$

These equations reduce to

$$\begin{cases} (2A_1 - 4B_1)t + (2A_2 - 4B_2 - A_1) = 0, \\ (A_1 - 2B_1)t + (A_2 - 2B_2 - B_1) = 0. \end{cases}$$

Thus we must have

$$\begin{cases} 2A_1 - 4B_1 = 0, \\ A_1 - 2B_1 = 0, \end{cases} \qquad \begin{cases} 2A_2 - 4B_2 - A_1 = 0, \\ A_2 - 2B_2 - B_1 = 0. \end{cases}$$

A simple nontrivial solution of this is $A_1 = 2$, $B_1 = 1$, $A_2 = 1$, $B_2 = 0$. This leads to the solution

$$x = 2t + 1, \; y = t. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$\begin{cases} x = 2c_1 + c_2(2t + 1), \\ y = c_1 + c_2 t. \end{cases} \tag{4}$$

Here is an alternative way to solve the given system. From the given D.E.'s themselves, we see that $x' = 2y'$. Then integrating, $x = 2y + k_1$, where $k_1$ is an arbitrary constant. Now substitute $x = 2y + k_1$ into the second given D.E. $y' = x - 2y$. We obtain $y' = 2y + k_1 - 2y = k_1$. Integrating this, we obtain $y = k_1 t + k_2$, where $k_2$ is a second arbitrary constant. Then since $x = 2y + k_1$, we have $x = 2k_1 t + 2k_2 + k_1$. Thus we have found

$$\begin{cases} x = 2k_1 t + 2k_2 + k_1, \\ y = k_1 t + k_2, \end{cases}$$

which is G.S. (4).

27. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = -2Ae^{\lambda t} + 7Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 3Ae^{\lambda t} + 2Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (-2 - \lambda)A + 7B = 0, \\ 3A + (2 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} -2 - \lambda & 7 \\ 3 & 2 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 = 25$, whose roots are $\lambda_1 = 5$, $\lambda_2 = -5$.

Setting $\lambda = \lambda_1 = 5$ in (1), we obtain $-7A + 7B = 0$, $3A - 3B = 0$, a nontrivial solution of which is $A = 1$, $B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{5t}, \quad y = e^{5t}. \tag{2}$$

Setting $\lambda = \lambda_2 = -5$ in (1), we obtain $3A + 7B = 0$ (twice), a nontrivial solution of which is $A = 7$, $B = -3$.

With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = 7e^{-5t}, \quad y = -3e^{-5t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find

$$x = c_1e^{5t} + 7c_2e^{-5t}, \quad y = c_1e^{5t} - 3c_2e^{-5t}. \tag{4}$$

We apply the I.C.'s $x(0) = 9$, $y(0) = -1$ to (4), obtaining $c_1 + 7c_2 = 9$, $c_1 - 3c_2 = -1$. The solution of this is $c_1 = 2$, $c_2 = 1$. Substituting these values back into (4), we get the desired particular solution

$$x = 2e^{5t} + 7e^{-5t}, \quad y = 2e^{5t} - 3e^{-5t}.$$

29. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 2Ae^{\lambda t} - 8Be^{\lambda t}, \\ B\lambda e^{\lambda t} = Ae^{\lambda t} + 6Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (2 - \lambda)A - 8B = 0, \\ A + (6 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 2 - \lambda & -8 \\ 1 & 6 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 8\lambda + 20 = 0$, whose roots are $4 \pm 2i$.

Letting $\lambda = 4 + 2i$ in (1), we obtain $(2 - 2i)A - 8B = 0$, $A + (2 - 2i)B = 0$, a nontrivial solution of which is $A = 2(i - 1)$, $B = 1$. With these values of A, B, and $\lambda$, we find the complex solution

$$\begin{cases} x = 2(i - 1)e^{(4+2i)t} \\ y = e^{(4+2i)t}. \end{cases}$$

Using Euler's formula this takes the form

$$\begin{cases} x = 2(i - 1)e^{4t}(\cos 2t + i \sin 2t), \\ y = e^{4t}(\cos 2t + i \sin 2t). \end{cases}$$

Both the real and imaginary parts of this are solutions. Thus we obtain the two linearly independent solutions

$$\begin{cases} x = -2e^{4t}(\cos 2t + \sin 2t) \\ y = e^{4t}\cos 2t, \end{cases}$$

and

$$\begin{cases} x = 2e^{4t}(\cos 2t - \sin 2t), \\ y = e^{4t}\sin 2t. \end{cases}$$

The G.S. of the system is thus

$$\begin{cases} x = 2e^{4t}[(-c_1 + c_2)\cos 2t - (c_1 + c_2)\sin 2t], \\ y = e^{4t}(c_1 \cos 2t + c_2 \sin 2t). \end{cases} \tag{2}$$

We apply the I.C.'s $x(0) = 4$, $y(0) = 1$ to (2), obtaining $2(-c_1 + c_2) = 4$, $c_1 = 1$. Thus $c_1 = 1$, $c_2 = 3$. Substituting back into (2), we get the desired particular solution

$$\begin{cases} x = 4e^{4t}(\cos 2t - 2\sin 2t), \\ y = e^{4t}(\cos 2t + 3\sin 2t). \end{cases}$$

31. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = 6Ae^{\lambda t} - 4Be^{\lambda t}, \\ B\lambda e^{\lambda t} = Ae^{\lambda t} + 2Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (6 - \lambda)A - 4B = 0, \\ A + (2 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 6 - \lambda & -4 \\ 1 & 2 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 8\lambda + 16 = 0$, whose roots are 4, 4 (real and equal).

Setting $\lambda = 4$ in (1), we obtain $2A - 4B = 0$, $A - 2B = 0$, a nontrivial solution of which is $A = 2$, $B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = 2e^{4t}, \quad y = e^{4t}. \tag{2}$$

We now seek a second solution of the form $x = (A_1t + A_2)e^{4t}$, $y = (B_1t + B_2)e^{4t}$. Substituting these into the given system, we obtain

$$\begin{cases} (4A_1t + 4A_2 + A_1)e^{4t} = 6(A_1t + A_2)e^{4t} - 4(B_1t + B_2)e^{4t}, \\ (4B_1t + 4B_2 + B_1)e^{4t} = (A_1t + A_2)e^{4t} + 2(B_1t + B_2)e^{4t}. \end{cases}$$

These equations reduce to

$$\begin{cases} (2A_1 - 4B_1)t + (2A_2 - 4B_2 - A_1) = 0, \\ (A_1 - 2B_1)t + (A_2 - 2B_2 - B_1) = 0. \end{cases}$$

Thus we must have

$$\begin{cases} 2A_1 - 4B_1 = 0, \\ A_1 - 2B_1 = 0, \end{cases} \qquad \begin{cases} 2A_2 - 4B_2 - A_1 = 0, \\ A_2 - 2B_2 - B_1 = 0. \end{cases}$$

A simple nontrivial solution of this is $A_1 = 2$, $B_1 = 1$, $A_2 = 1$, $B_2 = 0$. This leads to the solution

$$x = (2t + 1)e^{4t}, \quad y = te^{4t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find the G.S. in the form

$$x = 2c_1e^{4t} + c_2(2t + 1)e^{4t}, \quad y = c_1e^{4t} + c_2te^{4t}. \tag{4}$$

We apply the I.C.'s $x(0) = 2$, $y(0) = 3$ to (4), obtaining $2c_1 + c_2 = 2$, $c_1 + 0 = 3$. Thus $c_1 = 3$, $c_2 = -4$. Substituting these values back into (4), we get the desired particular solution

$$x = 6e^{4t} - 4(2t + 1)e^{4t}, \quad y = 3e^{4t} - 4te^{4t};$$

or

$$x = 2e^{4t} - 8te^{4t}, \quad y = 3e^{4t} - 4te^{4t}.$$

34. We assume a solution of the form $x = Ae^{\lambda t}$, $y = Be^{\lambda t}$. Substituting into the given system, we obtain

$$\begin{cases} A\lambda e^{\lambda t} = Ae^{\lambda t} - 2Be^{\lambda t}, \\ B\lambda e^{\lambda t} = 8Ae^{\lambda t} - 7Be^{\lambda t}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (1 - \lambda)A - 2B = 0, \\ 8A + (-7 - \lambda)B = 0. \end{cases} \tag{1}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 1 - \lambda & -2 \\ 8 & -7 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 + 6\lambda + 9 = 0$, that is, $(\lambda + 3)^2 = 0$, with roots $\lambda = -3, -3$ (real and equal).

Setting $\lambda = -3$ in (1), we obtain $4A - 2B = 0$, $8A - 4B = 0$, a nontrivial solution of which is $A = 1$, $B = 2$. With these values of A, B, and $\lambda$, we find the nontrivial solution

$$x = e^{-3t}, \quad y = 2e^{-3t}. \tag{2}$$

We now seek a second solution of the form $x = (A_1t + A_2)e^{-3t}$, $y = (B_1t + B_2)e^{-3t}$. Substituting these into the given system, we obtain

$$\begin{cases} (-3A_1t - 3A_2 + A_1)e^{-3t} = \\ (A_1t + A_2)e^{-3t} - 2(B_1t + B_2)e^{-3t}, \end{cases}$$

$$\begin{cases} (-3B_1t - 3B_2 + B_1)e^{-3t} = \\ 8(A_1t + A_2)e^{-3t} - 7(B_1t + B_2)e^{-3t}. \end{cases}$$

These equations reduce to

$$\begin{cases}(4A_1 - 2B_1)t + (4A_2 - 2B_2 - A_1) = 0,\\ (8A_1 - 4B_1)t + (8A_2 - 4B_2 - B_1) = 0.\end{cases}$$

Thus we must have

$$\begin{cases}4A_1 - 2B_1 = 0,\\ 8A_1 - 4B_1 = 0,\end{cases} \qquad \begin{cases}4A_2 - 2B_2 - A_1 = 0,\\ 8A_2 - 4B_2 - B_1 = 0.\end{cases}$$

A simple nontrivial solution of this is $A_1 = 1$, $B_1 = 2$, $A_2 = 1/4$, $B_2 = 0$. This leads to the solution

$$x = (t + 1/4)e^{-3t},\ y = 2te^{-3t}. \tag{3}$$

The G.S. of the given system is a linear combination of (2) and (3). Thus we find the G.S.

$$\begin{cases}x = c_1e^{-3t} + c_2(t + 1/4)e^{-3t},\\ y = 2c_1e^{-3t} + 2c_2te^{-3t}.\end{cases} \tag{4}$$

We apply the I.C.'s $x(0) = 6$, $y(0) = 8$ to (4), obtaining $c_1 + \frac{1}{4}c_2 = 6$, $2c_1 = 8$. Thus $c_1 = 4$, $c_2 = 8$. Substituting these values back into (4), we get the desired particular solution

$$x = 4e^{-3t} + 8(t + 1/4)e^{-3t},\ y = 8e^{-3t} + 16te^{-3t}$$

that is,

$$\begin{cases} x = (8t + 6)e^{-3t}, \\ y = (16t + 8)e^{-3t}. \end{cases}$$

36. We let $t = e^w$, where we assume $t > 0$. Then $w = \ln t$, and $\frac{dx}{dt} = \left(\frac{dx}{dw}\right)\left(\frac{dw}{dt}\right) = \left(\frac{1}{t}\right)\frac{dx}{dw}$. Thus $t\frac{dx}{dt} = \frac{dx}{dw}$; and similarily $t\frac{dy}{dt} = \frac{dy}{dw}$. Substituting into the original system, we obtain the constant coefficient system

$$\begin{cases} \frac{dx}{dw} = x + y, \\ \frac{dy}{dw} = -3x + 5y. \end{cases} \tag{1}$$

We assume a solution of (1) of the form $x = Ae^{\lambda w}$, $y = Be^{\lambda w}$. Substituting this into (1), we obtain

$$\begin{cases} A\lambda e^{\lambda w} = Ae^{\lambda w} + Be^{\lambda w}, \\ B\lambda e^{\lambda w} = -3Ae^{\lambda w} + 5Be^{\lambda w}, \end{cases}$$

which leads at once to the algebraic system

$$\begin{cases} (1 - \lambda)A + B = 0, \\ -3A + (5 - \lambda)B = 0. \end{cases} \tag{2}$$

For nontrivial solutions of this, we must have

$$\begin{vmatrix} 1 - \lambda & 1 \\ -3 & 5 - \lambda \end{vmatrix} = 0.$$

This leads to the characteristic equation $\lambda^2 - 6\lambda + 8 = 0$, whose roots are $\lambda_1 = 2$, $\lambda_2 = 4$.

Setting $\lambda = \lambda_1 = 2$ in (2), we obtain $-A + B = 0$, $-3A + 3B = 0$, a nontrivial solution of which is $A = 1$, $B = 1$. With these values of A, B, and $\lambda$, we find the nontrivial solution of (1),

$$x = e^{2w}, \quad y = e^{2w}.$$

Setting $\lambda = \lambda_2 = 4$ in (2), we obtain $-3A + B = 0$ (twice), a nontrivial solution of which is $A = 1$, $B = 3$. With these values of A, B, and $\lambda$, we find the nontrivial solution of (1),

$$x = e^{4w}, \quad y = 3e^{4w}.$$

The G.S. of (1) is thus $x = c_1e^{2w} + c_2e^{4w}$, $y = c_1e^{2w} + 3c_2e^{4w}$. We must now return to the original variable t. Since $t = e^w$, we see that the G.S. of the original system is

$$x = c_1t^2 + c_2t^4, \quad y = c_1t^2 + 3c_2t^4.$$

38. The characteristic equation of the given linear system (S) is $\lambda^2 - (a_1 + b_2)\lambda + (a_1b_2 - a_2b_1) = 0$, and its roots are given by $\lambda = \dfrac{a_1 + b_2 \pm \sqrt{(a_1 - b_2)^2 + 4a_2b_1}}{2}$.

Observe that if $(a_1 - b_2)^2 + 4a_2b_1 > 0$, then $\sqrt{(a_1 - b_2)^2 + 4a_2b_1}$ is real and positive, so the roots $\lambda_1$ and $\lambda_2$ of the characteristic equation are real and distant. Then by Theorem 7.7, the system has two real linearly independent solutions of the stated form.

Now note that if $a_2b_1 > 0$, then $(a_1 - b_2)^2 + 4a_2b_1 > 0$ and the reasoning of the previous paragraph follows. Thus $a_2b_1 > 0$ is sufficient for the system to have two solutions of the stated type and form.

But now observe that there exist systems for which $a_2b_1 \leq 0$, but $(a_1 - b_2)^2 + 4a_2b_1 > 0$. For example, consider the system (S) in which $a_1 = 3$, $b_1 = -1$, $a_2 = 1$, $b_2 = 0$. Then $a_2b_1 = -1 < 0$, but $(a_1 - b_2)^2 + 4a_2b_1 = 5 > 0$. Since $(a_1 - b_2)^2 + 4a_2b_1 > 0$, by the reasoning of the first paragraph, the system (S) has two solutions of the stated type and form. But for the system (S), $a_2b_1 > 0$ does <u>not</u> hold. Thus the condition $a_2b_1 > 0$ is not necessary for a system to have two solutions of the stated type and form.

40. <u>Suppose</u> there exists a <u>nontrivial</u> solution of the form

$$x = A\,t\,e^{\lambda t},\quad y = B\,t\,e^{\lambda t}. \tag{1}$$

If indeed (1) is a solution of the given system, then it must satisfy the equations of the system. So,

differentiating (1) and substituting into the system, we must have

$$\begin{cases} (A\lambda t + A)e^{\lambda t} = a_1 A t e^{\lambda t} + b_1 B t e^{\lambda t}, \\ (B\lambda t + B)e^{\lambda t} = a_2 A t e^{\lambda t} + b_2 B t e^{\lambda t}. \end{cases}$$

These equations quickly reduce to

$$\begin{cases} [(a_1 - \lambda)A + b_1 B]t - A = 0, \\ [a_2 A + (b_2 - \lambda)B]t - B = 0. \end{cases}$$

Thus we must have

$$\begin{cases} (a_1 - \lambda)A + b_1 B = 0, \\ a_2 A + (b_2 - \lambda)B = 0, \end{cases} \qquad \begin{cases} A = 0, \\ B = 0. \end{cases}$$

To satisfy these we must have $A = B = 0$, which means that (1) must be the <u>trivial</u> solution. This is a contradiction. Thus our supposition is invalid, so there exists <u>no</u> nontrivial solution of the stated form.

**Section 7.5 A, Page 340.**

Here and following, we shall indicate vectors and matrices by placing a bar over the corresponding letter. Thus, for example, we write $\bar{A}$, $\bar{v}$, etc.

1. (b) $\bar{A} + \bar{B} = \begin{pmatrix} 2+7 & 1-1 & 3+6 \\ -1+2 & 0+4 & 5-3 \\ -4+5 & 3-5 & -2+1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 9 \\ 1 & 4 & 2 \\ 1 & -2 & -1 \end{pmatrix}.$

$$\text{(c) } \bar{A} + \bar{B} = \begin{pmatrix} -5+7 & 0-2 & 4-3 \\ -2+6 & -1-3 & -3+1 \\ 6-2 & 2+1 & 5-3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 4 & -4 & -2 \\ 4 & 3 & 2 \end{pmatrix}.$$

2. $$\text{(b) } -4\bar{A} = \begin{pmatrix} -4(1) & -4(-3) & -4(5) \\ -4(6) & -4(-2) & -4(0) \\ -4(3) & -4(1) & -4(2) \end{pmatrix} = \begin{pmatrix} -4 & 12 & -20 \\ -24 & 8 & 0 \\ 12 & -4 & -8 \end{pmatrix}.$$

$$\text{(c) } -3\bar{A} = \begin{pmatrix} -3(5) & -3(-1) & -3(2) \\ -3(4) & -3(-3) & -3(-2) \\ -3(0) & -3(3) & -3(-6) \end{pmatrix} = \begin{pmatrix} -15 & 3 & -6 \\ -12 & 9 & 6 \\ 0 & -9 & 18 \end{pmatrix}.$$

3. (b) We have

$$3\bar{x}_1 - 2\bar{x}_2 + 4\bar{x}_4 = 3\begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 3 \\ 5 \\ -2 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 2 \\ -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 6 \\ -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \\ -10 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 8 \\ -12 \\ 20 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ -25 \\ 36 \end{bmatrix}.$$

(c) We have

$$-\bar{x}_1 + 5\bar{x}_2 - 2\bar{x}_3 + 3\bar{x}_4 = -\begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 3 \\ 5 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -2 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} -5 \\ 15 \\ 25 \\ -10 \end{bmatrix} + \begin{bmatrix} -4 \\ -8 \\ 0 \\ -12 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \\ -9 \\ 15 \end{bmatrix} = \begin{bmatrix} -15 \\ 11 \\ 17 \\ -11 \end{bmatrix}.$$

4. (b) We have

$$\bar{A}\bar{x} = \begin{pmatrix} -3 & -5 & 7 \\ 0 & 4 & 1 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -3(2) - 5(3) + 7(-2) \\ 0(2) + 4(3) + 1(-2) \\ -2(2) + 1(3) + 3(-2) \end{pmatrix} = \begin{pmatrix} -35 \\ 10 \\ -7 \end{pmatrix}.$$

(c) We have

$$\bar{A}\bar{x} = \begin{pmatrix} 1 & 0 & -3 \\ 2 & -5 & 4 \\ -3 & 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_2 - x_3 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 + x_2) + 0 - 3(x_2 - x_3) \\ 2(x_1 + x_2) - 5(x_1 + 2x_2) + 4(x_2 - x_3) \\ -3(x_1 + x_2) + (x_1 + 2x_2) + 2(x_2 - x_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - 2x_2 + 3x_3 \\ -3x_1 - 4x_2 - 4x_3 \\ -2x_1 + x_2 - 2x_3 \end{bmatrix}.$$

5. We have

$$\bar{A} = \begin{pmatrix} 3 & -1 & 2 \\ 5 & 4 & -3 \\ -5 & 1 & 2 \end{pmatrix}, \ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \ c = 4. \quad \text{Then}$$

$$\bar{A}(\bar{x} + \bar{y}) = \begin{pmatrix} 3 & -1 & 2 \\ 5 & 4 & -3 \\ -5 & 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(x_1 + y_1) + (-1)(x_2 + y_2) + 2(x_3 + y_3) \\ 5(x_1 + y_1) + 4(x_2 + y_2) + (-3)(x_3 + y_3) \\ (-5)(x_1 + y_1) + 1(x_2 + y_2) + 2(x_3 + y_3) \end{bmatrix}$$

$$\begin{bmatrix} (3x_1 - x_2 + 2x_3) + (3y_1 - y_2 + 2y_3) \\ (5x_1 + 4x_2 - 3x_3) + (5y_1 + 4y_2 - 3y_3) \\ (-5x_1 + x_2 + 2x_3) + (-5y_1 + y_2 + 2y_3) \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 - x_2 + 2x_3 \\ 5x_1 + 4x_2 - 3x_3 \\ -5x_1 + x_2 + 2x_3 \end{bmatrix} + \begin{bmatrix} 3y_1 - y_2 + 2y_3 \\ 5y_1 + 4y_2 - 3y_3 \\ -5y_1 + y_2 + 2y_3 \end{bmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 2 \\ 5 & 4 & -3 \\ -5 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 2 \\ 5 & 4 & -3 \\ -5 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \bar{A}\bar{x} + \bar{A}\bar{y}.$$

Also,

$$\bar{A}(4\bar{x}) = \begin{pmatrix} 3 & -1 & 2 \\ 5 & 4 & -3 \\ -5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix} = \begin{pmatrix} 12x_1 - 4x_2 + 8x_3 \\ 20x_1 + 16x_2 - 12x_3 \\ -20x_1 + 4x_2 + 8x_3 \end{pmatrix}$$

$$= 4\begin{pmatrix} 3x_1 - x_2 + 2x_3 \\ 5x_1 + 4x_2 - 3x_3 \\ -5x_1 + x_2 + 2x_3 \end{pmatrix} = 4(\bar{A}\bar{x}).$$

**Section 7.5B, Page 351.**

1. $$\bar{A}\bar{B} = \begin{pmatrix} 3 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (3)(4) + (5)(2) & (3)(6) + (5)(1) \\ (1)(4) + (7)(2) & (1)(6) + (7)(1) \end{pmatrix} = \begin{pmatrix} 22 & 23 \\ 18 & 13 \end{pmatrix}.$$

$$\bar{B}\bar{A} = \begin{pmatrix} 4 & 6 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (4)(3) + (6)(1) & (4)(5) + (6)(7) \\ (2)(3) + (1)(1) & (2)(5) + (1)(7) \end{pmatrix} = \begin{pmatrix} 18 & 62 \\ 7 & 17 \end{pmatrix}.$$

2. $\bar{A}\,\bar{B} = \begin{pmatrix} 5 & -2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -1 & 8 \\ 2 & -5 \end{pmatrix}$

$$= \begin{pmatrix} (5)(-1) + (-2)(2) & (5)(8) + (-2)(-5) \\ (4)(-1) + (3)(2) & (4)(8) + (3)(-5) \end{pmatrix} = \begin{pmatrix} -9 & 50 \\ 2 & 17 \end{pmatrix}.$$

$\bar{B}\,\bar{A} = \begin{pmatrix} -1 & 8 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 4 & 3 \end{pmatrix}$

$$= \begin{pmatrix} (-1)(5) + (8)(4) & (-1)(-2) + (8)(3) \\ (2)(5) + (-5)(4) & (2)(-2) + (-5)(3) \end{pmatrix}$$

$$= \begin{pmatrix} 27 & 26 \\ -10 & -19 \end{pmatrix}.$$

4. $\bar{A}\,\bar{B} = \begin{pmatrix} 6 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 2 & -1 & -3 \end{pmatrix}$

$$= \begin{pmatrix} (6)(1) + (1)(0) & (6)(3) + (1)(2) \\ (5)(1) + (-2)(0) & (5)(3) + (-2)(2) \end{pmatrix}$$

$$\begin{pmatrix} (6)(2) + (1)(-1) & (6)(4) + (1)(-3) \\ (5)(2) + (-2)(-1) & (5)(4) + (-2)(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 20 & 11 & 21 \\ 5 & 11 & 12 & 26 \end{pmatrix}.$$

$\bar{B}\,\bar{A}$ is not defined.

6. $$\bar{A}\bar{B} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & -3 \\ -6 & 0 & 1 \\ 1 & -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} (3)(2) + (2)(-6) + (1)(1) & (3)(-1) + (2)(0) + (1)(-3) \\ (0)(2) + (1)(-6) + (2)(1) & (0)(-1) + (1)(0) + (2)(-3) \\ (5)(2) + (4)(-6) + (3)(1) & (5)(-1) + (4)(0) + (3)(-3) \end{pmatrix}$$

$$\begin{pmatrix} (3)(-3) + (2)(1) + (1)(4) \\ (0)(-3) + (1)(1) + (2)(4) \\ (5)(-3) + (4)(1) + (3)(4) \end{pmatrix} = \begin{pmatrix} -5 & -6 & -3 \\ -4 & -6 & 9 \\ -11 & -14 & 1 \end{pmatrix}.$$

$$\bar{B}\bar{A} = \begin{pmatrix} 2 & -1 & -3 \\ -6 & 0 & 1 \\ 1 & -3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} (2)(3)+(-1)(0)+(-3)(5) & (2)(3)+(-1)(1)+(-3)(4) \\ (-6)(3)+(0)(0)+(1)(5) & (-6)(2)+(0)(1)+(1)(4) \\ (1)(3)+(-3)(0)+(4)(5) & (1)(2)+(-3)(1)+(4)(4) \end{pmatrix}$$

$$\begin{pmatrix} (2)(1)+(-1)(2)+(-3)(3) \\ (-6)(1)+(0)(2)+(1)(3) \\ (1)(1)+(-3)(2)+(4)(3) \end{pmatrix} = \begin{pmatrix} -9 & -9 & -9 \\ -13 & -8 & -3 \\ 23 & 15 & 7 \end{pmatrix}.$$

7. $$\bar{A}\bar{B} = \begin{pmatrix} -2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & -2 & -1 \\ 5 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (-2)(0)+(4)(3)+(6)(5) & (-2)(1)+(4)(-2)+(6)(4) \\ (1)(0)+(3)(3)+(5)(5) & (1)(1)+(3)(-2)+(5)(4) \\ (0)(0)+(2)(3)+(0)(5) & (0)(1)+(2)(-2)+(0)(4) \end{pmatrix}$$

$$\begin{pmatrix} (-2)(2)+(4)(-1)+(6)(2) \\ (1)(2)+(3)(-1)+(5)(2) \\ (0)(2)+(2)(-1)+(0)(2) \end{pmatrix} = \begin{pmatrix} 42 & 14 & 4 \\ 34 & 15 & 9 \\ 6 & -4 & -2 \end{pmatrix}.$$

$$\bar{B}\,\bar{A} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & -2 & -1 \\ 5 & 4 & 2 \end{pmatrix} \begin{pmatrix} -2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (0)(-2)+(1)(1)+(2)(0) & (0)(4)+(1)(3)+(2)(2) \\ (3)(-2)+(-2)(1)+(-1)(0) & (3)(4)+(-2)(3)+(-1)(2) \\ (5)(-2)+(4)(1)+(2)(0) & (5)(4)+(4)(3)+(2)(2) \end{pmatrix}$$

$$\begin{pmatrix} (0)(6)+(1)(5)+(2)(0) \\ (3)(6)+(-2)(5)+(-1)(0) \\ (5)(6)+(4)(5)+(2)(0) \end{pmatrix} = \begin{pmatrix} 1 & 7 & 5 \\ -8 & 4 & 8 \\ -6 & 36 & 50 \end{pmatrix}.$$

9. $$\bar{A}\,\bar{B} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 0 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (2)(1)+(1)(0)+(0)(-1)+(1)(1) \\ (0)(1)+(2)(0)+(3)(-1)+(-1)(1) \\ (1)(1)+(-2)(0)+(1)(-1)+(0)(1) \end{pmatrix}$$

$$\begin{pmatrix} (2)(2)+(1)(3)+(0)(0)+(1)(-2) \\ (0)(2)+(2)(3)+(3)(0)+(-1)(-2) \\ (1)(2)+(-2)(3)+(1)(0)+(0)(-2) \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -4 & 8 \\ 0 & -4 \end{pmatrix}.$$

Since the number of columns in $\bar{B}$,2, is unequal to the number of rows in $\bar{A}$,3, the product $\bar{B}\,\bar{A}$ is not defined.

11. $\bar{A}^2 = \bar{A}\,\bar{A} = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} (1)(1)+(2)(-2)+(-1)(2) & (1)(2)+(2)(1)+(-1)(0) \\ (-2)(1)+(1)(-2)+(3)(2) & (-2)(2)+(1)(1)+(3)(0) \\ (2)(1)+(0)(-2)+(1)(2) & (2)(2)+(0)(1)+(1)(0) \end{pmatrix}$$

$$\begin{pmatrix} (1)(-1)+(2)(3)+(-1)(1) \\ (-2)(-1)+(1)(3)+(3)(1) \\ (2)(-1)+(0)(3)+(1)(1) \end{pmatrix} = \begin{pmatrix} -5 & 4 & 4 \\ 2 & -3 & 8 \\ 4 & 4 & -1 \end{pmatrix},$$

$$\bar{A}^3 = \bar{A}\,\bar{A}\,\bar{A} = \bar{A}^2\bar{A} = \begin{pmatrix} -5 & 4 & 4 \\ 2 & -3 & 8 \\ 4 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-5)(1)+(4)(-2)+(4)(2) & (-5)(2)+(4)(1)+(4)(0) \\ (2)(1)+(-3)(-2)+(8)(2) & (2)(2)+(-3)(1)+(8)(0) \\ (4)(1)+(4)(-2)+(-1)(2) & (4)(2)+(4)(1)+(-1)(0) \end{pmatrix}$$

$$\begin{pmatrix} (4)(1)+(4)(-2)+(-1)(2) \\ (4)(2)+(4)(1)+(-1)(0) \\ (4)(-1)+(4)(3)+(-1)(0) \end{pmatrix} = \begin{pmatrix} -5 & -6 & -6 \\ 24 & 1 & 12 \\ -6 & 12 & 8 \end{pmatrix}.$$

12. We have $\bar{A}^2 = \bar{A}\,\bar{A} = \begin{pmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ -3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ -3 & -1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} (2)(2)+(3)(1)+(3)(-3) & (2)(3)+(3)(-2)+(3)(-1) \\ (1)(2)+(-2)(1)+(1)(-3) & (1)(3)+(-2)(-2)+(1)(-1) \\ (-3)(2)+(-1)(1)+(0)(-3) & (-3)(3)+(-1)(-2)+(0)(-1) \end{pmatrix}$$

$$\begin{pmatrix} (2)(3)+(3)(1)+(3)(0) \\ (1)(3)+(-2)(1)+(1)(0) \\ (-3)(3)+(-1)(1)+(0)(0) \end{pmatrix} = \begin{pmatrix} -2 & -3 & 9 \\ -3 & 6 & 1 \\ -7 & -7 & -10 \end{pmatrix}.$$

Then

$$\bar{A}^2 + 3\bar{A} + 2\bar{I} = \begin{pmatrix} -2 & -3 & 9 \\ -3 & 6 & 1 \\ -7 & -7 & -10 \end{pmatrix} + \begin{pmatrix} 6 & 9 & 9 \\ 3 & -6 & 3 \\ -9 & -3 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 18 \\ 0 & 2 & 4 \\ -16 & -10 & -8 \end{pmatrix}.$$

13. $\bar{A} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$. We have cof $\bar{A} = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$; adj $\bar{A}$ = $(\text{cof } \bar{A})^T$ = $\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}$; and $|\bar{A}| = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} = -1$. Thus $\bar{A}^{-1} = \frac{1}{|\bar{A}|}(\text{adj } \bar{A}) = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$.

14. $\bar{A} = \begin{pmatrix} -1 & 5 \\ -2 & 8 \end{pmatrix}$. We have cof $\bar{A} = \begin{pmatrix} 8 & 2 \\ -5 & -1 \end{pmatrix}$; adj $\bar{A}$ = $(\text{cof } \bar{A})^T$ = $\begin{pmatrix} 8 & -5 \\ 2 & -1 \end{pmatrix}$; and $|\bar{A}| = \begin{pmatrix} -1 & 5 \\ -2 & 8 \end{pmatrix} = 2$. Thus $\bar{A}^{-1} = \frac{1}{|\bar{A}|}(\text{adj } \bar{A}) = \begin{pmatrix} 4 & -5/2 \\ 1 & -1/2 \end{pmatrix}$.

17. We find

$$\text{cof } \bar{A} = \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & - \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ -1 & 1 \end{vmatrix} \\ - \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ -1 & 1 \end{vmatrix} & - \begin{vmatrix} 4 & 3 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & - \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{pmatrix} 1 & -5 & 6 \\ -2 & 5 & -7 \\ 3 & -5 & 3 \end{pmatrix}; \text{ adj } \bar{A} = (\text{cof } \bar{A})^T$$

$$= \begin{pmatrix} 1 & -2 & 3 \\ -5 & 5 & -5 \\ 6 & -7 & 3 \end{pmatrix}; \text{ and } |\bar{A}| = \begin{vmatrix} 4 & 3 & 1 \\ 3 & 3 & 2 \\ -1 & 1 & 1 \end{vmatrix} = -5.$$

Thus

$$\bar{A}^{-1} = \frac{1}{|\bar{A}|}(\text{adj } \bar{A}) = \begin{pmatrix} -1/5 & 2/5 & -3/5 \\ 1 & -1 & 1 \\ -6/5 & 7/5 & -3/5 \end{pmatrix}.$$

18. We find

$$\text{cof } \bar{A} = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} \\ - \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} & - \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} & - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{pmatrix} -2 & -6 & 4 \\ 7 & 7 & -5 \\ -3 & -3 & 3 \end{pmatrix}.$$

$$\text{adj } \bar{A} = (\text{cof } \bar{A})^T = \begin{pmatrix} -2 & 7 & -3 \\ -6 & 9 & -3 \\ 4 & -5 & 3 \end{pmatrix}. \quad \bar{A} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ -1 & 3 & 4 \end{vmatrix} = 6.$$

Thus

$$\bar{A}^{-1} = \frac{1}{|\bar{A}|}(\text{adj } \bar{A})$$

$$= \frac{1}{6} \begin{pmatrix} -2 & 7 & -3 \\ -6 & 9 & -3 \\ 4 & -5 & 3 \end{pmatrix} = \begin{pmatrix} -1/3 & 7/6 & -1/2 \\ -1 & 3/2 & -1/2 \\ 2/3 & -5/6 & 1/2 \end{pmatrix}.$$

21. We find

$$\text{cof } \bar{A} = \begin{bmatrix} \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} \\ - \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} & - \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} & - \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \end{bmatrix} = \begin{pmatrix} 8 & 1 & -6 \\ 2 & 1 & -2 \\ -12 & -2 & -10 \end{pmatrix};$$

$$\text{adj } \bar{A} = (\text{cof } \bar{A})^T = \begin{pmatrix} 8 & 2 & -12 \\ 1 & 1 & -2 \\ -6 & -2 & 10 \end{pmatrix}; \text{ and } \bar{A} = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 4 & 2 \\ 2 & 2 & 3 \end{vmatrix} = 2.$$

$$\text{Thus} \quad \bar{A}^{-1} = \frac{1}{|\bar{A}|}(\text{adj } \bar{A}) = \begin{pmatrix} 4 & 1 & -6 \\ 1/2 & 1/2 & -1 \\ -3 & -1 & 5 \end{pmatrix}$$

22. We have

$$\text{cof } \bar{A} = \begin{pmatrix} \begin{vmatrix} 7 & 1 \\ 3 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 7 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 7 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 18 & -8 & 2 \\ -6 & 3 & -1 \\ 2 & -1 & 1 \end{pmatrix};$$

$$\text{adj } \bar{A} = (\text{cof } \bar{A})^T = \begin{pmatrix} 18 & -6 & 2 \\ -6 & 3 & -1 \\ 2 & -1 & 1 \end{pmatrix}; \text{ and } \bar{A} = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 1 & 3 & 3 \end{vmatrix} = 2.$$

$$\text{Thus} \quad \bar{A}^{-1} = \frac{1}{|\bar{A}|}(\text{adj } \bar{A}) = \begin{pmatrix} 9 & -3 & 1 \\ -4 & 3/2 & -1/2 \\ 1 & -1/2 & 1/2 \end{pmatrix}.$$

**Section 7.5C, Page 356.**

1. a. We must show there exist numbers $c_1$, $c_2$, $c_3$, not all zero, such that $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$. This is

$$c_1\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 13 \\ 5 \\ -4 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the homogeneous linear system

$$\begin{cases} 3c_1 + 13c_2 + 2c_3 = 0, \\ -c_1 + 5c_2 + 4c_3 = 0, \\ 2c_1 - 4c_2 - 5c_3 = 0, \end{cases} \qquad (1)$$

the determinant of coefficients of which is

$$\begin{vmatrix} 3 & 13 & 2 \\ -1 & 5 & 4 \\ 2 & -4 & -5 \end{vmatrix} = 0.$$

Hence, by Theorem A (text, page 353), the system (1) has a nontrivial solution for $c_1$, $c_2$, $c_3$. For example, one solution is $c_1 = 3$, $c_2 = -1$, $c_3 = 2$. Thus $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ are linearly dependent.

2. a. Here we must show that if $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$, then $c_1 = c_2 = c_3 = 0$. Thus, we suppose

$$c_1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the homogeneous linear system

$$\begin{cases} 2c_1 + c_2 = 0, \\ c_1 - c_3 = 0, \\ 3c_2 + c_3 = 0, \end{cases} \qquad (1)$$

the determinant of which is

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 3 & 1 \end{vmatrix} = 5 \neq 0.$$

Thus by Theorem A (text, page 353), the system (1) has only the trivial solution $c_1 = c_2 = c_3 = 0$, and so $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ are linearly independent.

3. a. The given vectors are lineraly dependent if and only if there exist numbers $c_1$, $c_2$, $c_3$, not all zero, such that

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}. \qquad (1)$$

This is

$$c_1\begin{pmatrix} k \\ 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 3 \\ 7 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the homogeneous linear system

$$\begin{cases} kc_1 + c_2 + 3c_3 = 0, \\ 2c_1 + c_2 + 7c_3 = 0, \\ -c_1 + 2c_2 - 8c_3 = 0, \end{cases} \qquad (2)$$

the determinant of coefficient of which is

$$\begin{vmatrix} k & 1 & 3 \\ 2 & -1 & 7 \\ -1 & 2 & -8 \end{vmatrix} = -6k + 18. \qquad (3)$$

By Theorem A (text, page 353), the system (2) has a nontrivial solution for $c_1$, $c_2$, $c_3$, and hence there exist $c_1$, $c_2$, $c_3$, not all zero, such that (1) holds, if and only if the determinant (3) is zero. Thus we have $-6k + 18 = 0$, and hence $k = 3$.

4. b. Note that

$$4\bar{\phi}_1(t) - 2\bar{\phi}_2(t) - \bar{\phi}_3(t)$$

$$= \begin{pmatrix} 4\sin t + 4\cos t \\ 8\sin t \\ -4\cos t \end{pmatrix} + \begin{pmatrix} -4\sin t \\ -8\sin t + 2\cos t \\ 2\sin t \end{pmatrix}$$

$$+ \begin{pmatrix} -4\cos t \\ -2\cos t \\ -2\sin t + 4\cos t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus there exists the set of three numbers 4, -2, -1, none of which are zero, such that $4\bar{\phi}_1(t) + (-2)\bar{\phi}_2(t) + (-1)\bar{\phi}_3(t) = 0$ for all t such that $a \leq t \leq b$, for any a and b. Therefore $\bar{\phi}_1$, $\bar{\phi}_2$, and $\bar{\phi}_3$ are linearly dependent on $a \leq t \leq b$.

5. b. Suppose there exist numbers $c_1$ and $c_2$ such that $c_1\bar{\phi}_1(t) + c_2\bar{\phi}_2(t) = 0$ for all t on $a \leq t \leq b$. Then

$$\begin{cases} 2c_1e^{2t} + c_2e^{-t} = 0, \\ -c_1e^{2t} + 3c_2e^{-t} = 0, \text{ for all t on } a \leq t \leq b. \end{cases}$$

From this, we have both

$$\begin{cases} 2c_1 + c_2e^{-3t} = 0, \\ -2c_1 + 6c_2e^{-3t} = 0, \end{cases} \text{ and } \begin{cases} 6c_1e^{3t} + 3c_2 = 0, \\ c_1e^{3t} - 3c_2 = 0. \end{cases} \quad (1)$$

From the former of (1), $7c_2e^{-3t} = 0$ on $a \leq t \leq b$, so $c_2 = 0$; and from the latter of (1), $7c_1e^{3t} = 0$ on $a \leq t \leq b$, so $c_1 = 0$. Thus if $c_1\bar{\phi}_1(t) + c_2\bar{\phi}_2(t) = \bar{0}$ for all t on $a \leq t \leq b$, we must have $c_1 = c_2 = 0$, and hence $\phi_1$ and $\phi_2$ are linearly independent on $a \leq t \leq b$, for any a and b.

**Section 7.5D, Page 367.**

1. The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = 0,$$

that is, $\lambda^2 - 3\lambda - 4 = 0$ or $(\lambda + 1)(\lambda - 4) = 0$. Thus, the characteristic values are $\lambda = -1$ and 4.

The characteristic vectors corresponding to $\lambda = -1$ have components $x_1$ and $x_2$ such that

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $x_1$ and $x_2$ must satisfy the system

$$\begin{cases} x_1 + 2x_2 = -x_1 \\ 3x_1 + 2x_2 = -x_2 \end{cases} \quad \text{or} \quad \begin{cases} 2x_1 + 2x_2 = 0 \\ 3x_1 + 3x_2 = 0. \end{cases}$$

We find $x_1 = k$, $x_2 = -k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = -1$ are $\begin{pmatrix} k \\ -k \end{pmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = 4$ have components $x_1$ and $x_2$ such that

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $x_1$ and $x_2$ must satisfy the system

$$\begin{cases} x_1 + 2x_2 = 4x_1 \\ 3x_1 + 2x_2 = 4x_2 \end{cases} \quad \text{or} \quad \begin{cases} -3x_1 + 2x_2 = 0 \\ 3x_1 - 3x_2 = 0. \end{cases}$$

We find $x_1 = 2k$, $x_2 = 3k$ for every real k. Hence the characteristic vectors corresponding to $\lambda = 4$ are $\begin{pmatrix} 2k \\ 3k \end{pmatrix}$ for every nonzero real k.

2. The characteristic equation is

$$\begin{vmatrix} 3 - \lambda & 2 \\ 6 & -1 - \lambda \end{vmatrix} = 0,$$

that is, $\lambda^2 - 2\lambda - 15 = 0$ or $(\lambda - 5)(\lambda + 3) = 0$. Thus, the characteristic values are $\lambda = 5$ and $\lambda = -3$.

The characteristic vectors corresponding to $\lambda = 5$ have components $x_1$ and $x_2$ such that

$$\begin{pmatrix} 3 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $x_1$ and $x_2$ must satisfy the system

$$\begin{cases} 3x_1 + 2x_2 = 5x_1 \\ 6x_1 - x_2 = 5x_2 \end{cases} \quad \text{or} \quad \begin{cases} -2x_1 + 2x_2 = 0, \\ x_1 - x_2 = 0. \end{cases}$$

We find $x_1 = x_2 = k$ for every real k. Hence the characteristic vectors corresponding to $\lambda = 5$ are $\begin{pmatrix} k \\ k \end{pmatrix}$ for every nonzero real k.

The characteristic vectors corresponding to $\lambda = -3$ have components $x_1$ and $x_2$ such that

$$\begin{pmatrix} 3 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-3) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $x_1$ and $x_2$ must satisfy the system

$$\begin{cases} 3x_1 + 2x_2 = -3x_1, \\ 6x_1 - x_2 = -3x_2. \end{cases}$$

From these, $6x_1 + 2x_2 = 0$, so $x_2 = -3x_1$. Thus we find $x_1 = k$, $x_2 = -3k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = -3$ are $\begin{pmatrix} k \\ -3k \end{pmatrix}$ for every nonzero real $k$.

7. The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 1 & -1 \\ 2 & 3 - \lambda & -4 \\ 4 & 1 & -4 - \lambda \end{vmatrix} = 0,$$

which reduces to

$$(1 - \lambda) \begin{vmatrix} 3 - \lambda & -4 \\ 1 & 4 - \lambda \end{vmatrix} - \begin{vmatrix} 2 & -4 \\ 4 & -4 - \lambda \end{vmatrix} - \begin{vmatrix} 2 & 3 - \lambda \\ 4 & 1 \end{vmatrix} = 0$$

or $\lambda^3 - 7\lambda + 6 = 0$ or $(\lambda - 1)(\lambda - 2)(\lambda + 3) = 0$. Thus, the characteristic values are $\lambda = 1, 2, -3$.

The characteristic vectors corresponding to $\lambda = 1$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + x_2 - x_3 = x_1, \\ 2x_1 + 3x_2 - 4x_3 = x_2, \\ 4x_1 + x_2 - 4x_3 = x_3, \end{cases} \quad \text{or} \quad \begin{cases} x_2 - x_3 = 0, \\ 2x_1 + 2x_2 - 4x_3 = 0, \\ 4x_1 + x_2 - 5x_3 = 0. \end{cases}$$

From the first of these, $x_3 = x_2$; and then the second and third become

$$\begin{cases} 2x_1 - 2x_2 = 0, \\ 4x_1 - 4x_2 = 0, \end{cases}$$

a solution of which is $x_1 = x_2 = k$. Thus we find $x_1 = k$, $x_2 = k$, $x_3 = k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 1$ are $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = 2$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + x_2 - x_3 = 2x_1, \\ 2x_1 + 3x_2 - 4x_3 = 2x_2, \\ 4x_1 + x_2 - 4x_3 = 2x_3, \end{cases} \quad \text{or} \quad \begin{cases} -x_1 + x_2 - x_3 = 0, \\ 2x_1 + x_2 - 4x_3 = 0, \\ 4x_1 + x_2 - 6x_3 = 0. \end{cases}$$

From the first and second, $3x_1 - 3x_3 = 0$, so $x_3 = x_1$, and $3x_2 - 6x_3 = 0$, so $x_2 = 2x_3$. Thus we find $x_1 = k$, $x_2 = 2k$, $x_3 = k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 2$ are $\begin{bmatrix} k \\ 2k \\ k \end{bmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = -3$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + x_2 - x_3 = -3x_1, \\ 2x_1 + 3x_2 - 4x_3 = -3x_2, \\ 4x_1 + x_2 - 4x_3 = -3x_3, \end{cases} \quad \text{or} \quad \begin{cases} 4x_1 + x_2 - x_3 = 0, \\ 2x_1 + 6x_2 - 4x_3 = 0, \\ 4x_1 + x_2 - x_3 = 0. \end{cases}$$

Regarding the first and second equations as two equations in $x_2$ and $x_3$, we find $x_2 = 7x_1$, $x_3 = 11x_1$. Thus we find $x_1 = k$, $x_2 = 7k$, $x_3 = 11k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = -3$ are $\begin{bmatrix} k \\ 7k \\ 11k \end{bmatrix}$ for every nonzero real $k$.

8. The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & -1 & -1 \\ 1 & 3 - \lambda & 1 \\ -3 & -6 & 6 - \lambda \end{vmatrix} = 0,$$

which reduces to

$$(1 - \lambda)\begin{vmatrix} 3 - \lambda & 1 \\ -6 & 6 - \lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -3 & 6 - \lambda \end{vmatrix} - \begin{vmatrix} 1 & 3 - \lambda \\ -3 & -6 \end{vmatrix} = 0$$

or $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$ or $(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$.

The characteristic vectors corresponding to $\lambda = 2$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 - x_2 - x_3 = 2x_1, \\ x_1 + 3x_2 + x_3 = 2x_2, \\ -3x_1 - 6x_2 + 6x_3 = 2x_3, \end{cases} \quad \text{or} \quad \begin{cases} -x_1 - x_2 - x_3 = 0, \\ x_1 + x_2 + x_3 = 0, \\ -3x_1 + 6x_2 + 4x_3 = 0. \end{cases}$$

The first two of these are essentially the same. We write the second and third as

$$\begin{cases} x_1 + x_2 = -x_3, \\ 3x_1 + 6x_2 = 4x_3, \end{cases}$$

and solve for $x_1$ and $x_2$ in terms of $x_3$. We find $x_1 = -\frac{10}{3}x_3$, $x_2 = \frac{7}{3}x_3$. Letting $x_3 = -3k$, we find $x_1 = 10k$, $x_2 = -7k$, $x_3 = -3k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 2$ are $\begin{bmatrix} 10k \\ -7k \\ -3k \end{bmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = 3$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy

$$\begin{cases} x_1 - x_2 - x_3 = 3x_1, \\ x_1 + 3x_2 + x_3 = 3x_2, \\ -3x_1 - 6x_2 + 6x_3 = 3x_3, \end{cases} \quad \text{or} \quad \begin{cases} -2x_1 - x_2 - x_3 = 0, \\ x_1 + x_3 = 0, \\ -3x_1 - 6x_2 + 3x_3 = 0. \end{cases}$$

From the second equation, $x_3 = -x_1$; and then the first and third become

$$\begin{cases} -x_1 - x_2 = 0, \\ -6x_1 - 6x_2 = 0. \end{cases}$$

From these $x_2 = -x_1$. Thus we find $x_1 = k$, $x_2 = -k$, $x_3 = -k$, for every real $k$. Hence the characteristic vectors to $\lambda = 3$ are $\begin{bmatrix} k \\ -k \\ -k \end{bmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = 5$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy

$$\begin{cases} x_1 - x_2 - x_3 = 5x_1, \\ x_1 + 3x_2 + x_3 = 5x_2, \\ -3x_1 - 6x_2 + 6x_3 = 5x_3, \end{cases} \quad \text{or} \quad \begin{cases} -4x_1 - x_2 - x_3 = 0, \\ x_1 - 2x_2 + x_3 = 0, \\ -3x_1 - 6x_2 + x_3 = 0. \end{cases}$$

Adding the first two of these, we obtain $-3x_1 - 3x_2 = 0$, from which $x_2 = -x_1$. Then the third equation becomes $3x_1 + x_3 = 0$, from which $x_3 = -3x_1$. Thus we find $x_1 = k$, $x_2 = -k$, $x_3 = -3k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 5$ are $\begin{bmatrix} k \\ -k \\ -3k \end{bmatrix}$ for every nonzero real $k$.

10. The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0,$$

which reduces to

$$(1 - \lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 - \lambda \end{vmatrix} = 0$$

or $\lambda^3 - 2\lambda - \lambda + 2 = 0$ or $(\lambda - 1)(\lambda - 2)(\lambda + 1) = 0$. Thus, the characteristic values are $\lambda = 1, 2, -1$.

The characteristic vectors corresponding to $\lambda = 1$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + x_2 = x_1, \\ x_1 + x_3 = x_2, \\ x_2 + x_3 = x_3, \end{cases} \quad \text{or} \quad \begin{cases} x_2 = 0, \\ x_1 - x_2 + x_3 = 0, \\ x_2 = 0. \end{cases}$$

From these, we see at once that $x_2 = 0$ and then $x_3 = -x_1$. Thus we find $x_1 = k$, $x_2 = 0$, $x_3 = -k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 1$ are $\begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = 2$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + x_2 = 2x_1, \\ x_1 + x_3 = 2x_2, \\ x_2 + x_3 = 2x_3, \end{cases} \quad \text{or} \quad \begin{cases} x_1 + x_2 = 0 \\ x_1 - 2x_2 + x_3 = 0, \\ x_2 - x_3 = 0. \end{cases}$$

We find $x_1 = k$, $x_2 = k$, $x_3 = k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 2$ are $\begin{bmatrix} k \\ k \\ k \end{bmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = -1$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (-1) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + x_2 = -x_1, \\ x_1 + x_3 = -x_2, \\ x_2 + x_3 = -x_3, \end{cases} \quad \text{or} \quad \begin{cases} 2x_1 + x_2 = 0, \\ x_1 + x_2 + x_3 = 0, \\ x_2 + 2x_3 = 0. \end{cases}$$

We find $x_1 = k$, $x_2 = -2k$, $x_3 = k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = -1$ are

$$\begin{bmatrix} k \\ -2k \\ k \end{bmatrix}$$ for every nonzero real k.

11. The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 3 & -6 \\ 0 & 2 - \lambda & 2 \\ 0 & -1 & 5 - \lambda \end{vmatrix} = 0,$$

which reduces to

$$(1 - \lambda)\begin{vmatrix} 2 - \lambda & 2 \\ -1 & 5 - \lambda \end{vmatrix} = 0$$

or $\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$ or $(\lambda - 1)(\lambda - 3)(\lambda - 4) = 0$. Thus, the characteristic values are $\lambda = 1, 3, 4$.

The characteristic vectors corresponding to $\lambda = 1$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (1)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy

$$\begin{cases} x_1 + 3x_2 - 6x_3 = x_1, \\ 2x_2 + 2x_3 = x_2, \\ -x_2 + 5x_3 = x_3, \end{cases} \quad \text{or} \quad \begin{cases} 3x_2 - 6x_3 = 0, \\ x_2 + 2x_3 = 0, \\ -x_2 + 4x_3 = 0. \end{cases}$$

From these we see that $x_1$ is arbitrary and $x_2 = x_3 = 0$. Thus we find $x_1 = k$, $x_2 = x_3 = 0$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 1$ are $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = 3$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + 3x_2 - 6x_3 = 3x_1, \\ 2x_2 + 2x_3 = 3x_2, \\ -x_2 + 5x_3 = 3x_3, \end{cases} \quad \text{or} \quad \begin{cases} -2x_1 + 3x_2 - 6x_3 = 0, \\ -x_2 + 2x_3 = 0, \\ -x_2 + 2x_3 = 0. \end{cases}$$

The last two equations are identical. From the second, $x_2 = 2x_3$. Then the first equation becomes $-2x_1 + 6x_3 - 6x_3 = 0$, from which $x_1 = 0$. Thus we find $x_1 = 0$, $x_2 = 2k$, $x_3 = k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 3$ are $\begin{bmatrix} 0 \\ 2k \\ k \end{bmatrix}$ for every nonzero real $k$.

The characteristic vectors corresponding to $\lambda = 4$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} x_1 + 3x_2 - 6x_3 = 4x_1, \\ 2x_2 + 2x_3 = 4x_2, \\ -x_2 + 5x_3 = 4x_3, \end{cases} \quad \text{or} \quad \begin{cases} -3x_1 + 3x_2 - 6x_3 = 0, \\ -2x_2 + 2x_3 = 0, \\ -x_2 + x_3 = 0. \end{cases}$$

The last two equations are equivalent, and from them, $x_2 = x_3$. Then from the first equation, $x_1 = x_2 - 2x_3 = -x_3$. Thus we find $x_1 = -k$, $x_2 = k$, $x_3 = k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 4$ are $\begin{bmatrix} -k \\ k \\ k \end{bmatrix}$ for every nonzero real $k$.

14. The characteristic equation is

$$\begin{vmatrix} -2 - \lambda & 6 & -18 \\ 12 & -23 - \lambda & 66 \\ 5 & -10 & 29 - \lambda \end{vmatrix} = 0,$$

which reduces to

$$(-2 - \lambda)\begin{vmatrix} -23 - \lambda & 66 \\ -10 & 29 - \lambda \end{vmatrix} - 6\begin{vmatrix} 12 & 66 \\ 5 & 29 - \lambda \end{vmatrix} - \begin{vmatrix} 12 & -23 - \lambda \\ 5 & -10 \end{vmatrix} = 0$$

or $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$ or $(\lambda - 1)(\lambda + 1)(\lambda - 4) = 0$. Thus the characteristic values are $\lambda = 1, -1, 4$.

The characteristic vectors corresponding to $\lambda = 1$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} -2x_1 + 6x_2 - 18x_3 = x_1, \\ 12x_1 - 23x_2 + 66x_3 = x_2, \\ 5x_1 - 10x_2 + 29x_3 = x_3, \end{cases} \quad \text{or} \quad \begin{cases} -3x_1 + 6x_2 - 18x_3 = 0, \\ 12x_1 - 24x_2 + 66x_3 = 0, \\ 5x_1 - 10x_2 + 28x_3 = 0. \end{cases}$$

Adding four times the first to the second, we find $-6x_3 = 0$, and so $x_3 = 0$. Then the first reduces to $-3x_1 + 6x_2 = 0$, from which $x_1 = 2x_2$. Thus we find $x_1 = 2k$, $x_2 = k$, $x_3 = 0$ for every real k. Hence the characteristic vectors

corresponding to $\lambda = 1$ are $\begin{bmatrix} 2k \\ k \\ 0 \end{bmatrix}$ for every nonzero real k.

The characteristic vectors corresponding to $\lambda = -1$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (-1) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} -2x_1 + 6x_2 - 18x_3 = -x_1, \\ 12x_1 - 23x_2 + 66x_3 = -x_2, \\ 5x_1 - 10x_2 + 29x_3 = -x_3, \end{cases} \quad \text{or} \quad \begin{cases} -x_1 + 6x_2 - 18x_3 = 0, \\ 12x_1 - 22x_2 + 66x_3 = 0, \\ 5x_1 - 10x_2 + 30x_3 = 0. \end{cases}$$

Adding 3/5 the third equation to the first gives $2x_1 = 0$, so $x_1 = 0$. Then the first equation becomes $6x_2 - 18x_3 = 0$, from which $x_2 = 3x_3$. Thus we find $x_1 = 0$, $x_2 = 3k$, $x_3 = k$, for every real k. Hence the characteristic vectors corresponding to $\lambda = -1$ are $\begin{bmatrix} 0 \\ 3k \\ k \end{bmatrix}$ for every nonzero real k.

The characteristic vectors corresponding to $\lambda = 4$ have components $x_1$, $x_2$, $x_3$ such that

$$\begin{pmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus $x_1$, $x_2$, $x_3$ must satisfy the system

$$\begin{cases} -2x_1 + 6x_2 - 18x_3 = 4x_1, \\ 12x_1 - 23x_2 + 66x_3 = 4x_2, \\ 5x_1 - 10x_2 + 29x_3 = 4x_3, \end{cases} \quad \text{or} \quad \begin{cases} -6x_1 + 6x_2 - 18x_3 = 0, \\ 12x_1 - 27x_2 + 66x_3 = 0, \\ 5x_1 - 10x_2 + 25x_3 = 0. \end{cases}$$

The first and third reduce to

$$\begin{cases} -x_1 + x_2 - 3x_3 = 0, \\ x_1 - 2x_2 + 5x_3 = 0. \end{cases}$$

Regarding these as two equations in $x_1$ and $x_2$, we find $x_1 = -x_3$, $x_2 = 2x_3$. Letting $x_3 = -k$, we find $x_1 = k$, $x_2 = -2k$, $x_3 = -k$ for every real $k$. Hence the characteristic vectors corresponding to $\lambda = 4$ are $\begin{bmatrix} k \\ -2k \\ -k \end{bmatrix}$ for every nonzero real $k$.

**Section 7.6, Page 377.**

1. The characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \text{ is}$$

$$|\bar{A} - \lambda \bar{I}| = \begin{vmatrix} 5 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 4\lambda + 3 = 0$ with roots $\lambda_1 = 1$, $\lambda_2 = 3$. These are the characteristic values of A. They are distinct (and real), and so Theorem 7.10 of the text applies. We use equation (7.118) of the text to find corresponding characteristic vectors.

With $\lambda = \lambda_1 = 1$ and $\bar{a} = \bar{a}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 5a_1 - 2a_2 &= a_1, \\ 4a_1 - a_2 &= a_2, \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2a_1 &= a_2, \\ 2a_1 &= a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = 1$, $a_2 = 2$, and thus a characteristic vectors corresponding to $\lambda = 1$ is

$\bar{a}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t}$, that is,

$$\bar{x} = \begin{pmatrix} e^{t} \\ 2e^{t} \end{pmatrix}. \qquad (*)$$

With $\lambda = \lambda_2 = 3$ and $\bar{a} = \bar{a}^{(2)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 3 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 5a_1 - 2a_2 &= 3a_1, & a_1 &= a_2, \\ & \text{or} & & \\ 4a_1 - a_2 &= 3a_2, & a_1 &= a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = a_2 = 1$, and thus a characteristic vectors corresponding to $\lambda_2 = 3$ is $\bar{a}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$, that is,

$$\bar{x} = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}. \qquad (**)$$

By Theorem 7.10 the solutions (*) and (**) are linearly independent, and a general solution is

$$\bar{x} = c_1 \begin{pmatrix} e^t \\ 2e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix},$$

where $c_1$ and $c_2$ are arbitrary constants. In scalar language,

$$\begin{aligned} x_1 &= c_1 e^t + c_2 e^{3t}, \\ x_2 &= 2c_1 e^t + c_2 e^{3t}. \end{aligned}$$

5. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$ is

$$|\bar{A} - \lambda I| = \begin{vmatrix} 3 - \lambda & 1 \\ 4 & 3 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 6\lambda + 5 = 0$ with roots $\lambda_1 = 1$, $\lambda_2 = 5$. These are the characteristic values of $\bar{A}$. They are distinct (and real), and so Theorem 7.10 of the text applies. We use equation (7.118) of the text to find corresponding characteristic vectors.

With $\lambda = \lambda_1 = 1$ and $\bar{a} = \bar{a}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{cases} 3a_1 + a_2 = a_1, \\ 4a_1 + 3a_2 = a_2, \end{cases} \quad \text{or} \quad \begin{cases} 2a_1 = -a_2, \\ 4a_1 = -2a_2. \end{cases}$$

A simple nontrivial solution is $a_1 = 1$, $a_2 = -2$, and thus a characteristic vector corresponding to $\lambda_1 = 1$ is $\bar{a}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^t$, that is,

$$\bar{x} = \begin{pmatrix} e^t \\ -2e^t \end{pmatrix}.$$

With $\lambda = \lambda_2 = 5$ and $\bar{a} = \bar{a}^{(2)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 5 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{cases} 3a_1 + a_2 = 5a_1, \\ 4a_1 + 3a_2 = 5a_2, \end{cases} \quad \text{or} \quad \begin{cases} 2a_1 = a_2, \\ 4a_1 = 2a_2. \end{cases}$$

A simple nontrivial solution is $a_1 = 1$, $a_2 = 2$, and thus a characteristic vector corresponding to $\lambda_2 = 5$ is $\bar{a}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$, that is,

$$\bar{x} = \begin{pmatrix} e^{5t} \\ 2e^{5t} \end{pmatrix}. \qquad (**)$$

By Theorem 7.10 the solutions (*) and (**) are linearly independent, and a general solution is

$$\bar{x} = c_1 \begin{pmatrix} e^{t} \\ -2e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ 2e^{5t} \end{pmatrix}$$

where $c_1$ and $c_2$ are arbitrary constants. In scalar language,

$$\begin{cases} x_1 = c_1 e^{t} + c_2 e^{5t}, \\ x_2 = -2c_1 e^{t} + 2c_2 e^{5t}. \end{cases}$$

6. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 6 & -1 \\ 3 & 2 \end{pmatrix}$ is

$$|\bar{A} - \lambda\bar{I}| = \begin{vmatrix} 6 - \lambda & -1 \\ 3 & 2 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 8\lambda + 15 = 0$ with roots $\lambda_1 = 3$, $\lambda_2 = 5$. These are the characteristic values of $\bar{A}$. They are distinct (and real), and so Theorem 7.10 of the text applies. We use equation (7.118) of the text to find corresponding characteristic vectors.

With $\lambda = \lambda_1 = 3$ and $\bar{a} = \bar{a}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 6 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 3\begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 6a_1 - a_2 &= 3a_1, & 3a_1 &= a_2, \\ & \text{or} & & \\ 3a_1 + 2a_2 &= 3a_2, & 3a_1 &= a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = 1$, $a_2 = 3$, and thus a characteristic vector corresponding to $\lambda_1 = 3$ is $\bar{a}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}$, that is,

$$\bar{x} = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}. \qquad (*)$$

With $\lambda = \lambda_2 = 5$ and $\bar{a} = \bar{a}^{(2)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 6 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 5 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 6a_1 - a_2 &= 5a_1, & a_1 &= a_2, \\ & \text{or} & & \\ 3a_1 + 2a_2 &= 5a_2, & a_1 &= a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = a_2 = 1$, and thus a characteristic vector corresponding to $\lambda_2 = 5$ is $\bar{a}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$, that is,

$$\bar{x} = \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}. \qquad (**)$$

By Theorem 7.10 the solutions (*) and (**) are linearly independent, and a general solution is

$$\bar{x} = c_1 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix},$$

where $c_1$ and $c_2$ are arbitrary constants. In scalar language,

$$\begin{aligned} x_1 &= c_1 e^{3t} + c_2 e^{5t}, \\ x_2 &= 3c_1 e^{3t} + c_2 e^{5t}. \end{aligned}$$

9. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$ is

$$|\bar{A} - \lambda \bar{I}| = \begin{vmatrix} 1 - \lambda & -4 \\ 1 & 1 - \lambda \end{vmatrix} = 0,$$

Expanding this determinant and simplifying, this takes the form $\lambda^2 - 2\lambda + 5 = 0$ with roots $1 \pm 2i$. These are the characteristic values of $\bar{A}$. They are distinct conjugate complex numbers, and Theorem 7.10 applies. We use equation (7.118) of the text.

With $\lambda = \lambda_1 = 1 + 2i$ and $\bar{a} = \bar{a}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118)

becomes

$$\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (1 + 2i) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} a_1 - 4a_2 &= (1 + 2i)a_1, & 2i\,a_1 &= -4a_2, \\ a_1 + a_2 &= (1 + 2i)a_2, & a_1 &= 2i\,a_2. \end{aligned} \qquad \text{or}$$

A simple nontrivial solution is $a_1 = 2i$, $a_2 = 1$, and thus a characteristic vectors corresponding to $\lambda_1 = 1 + 2i$ is $\bar{a}^{(1)} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{(1+2i)t}$, that is,

$$\bar{x} = \begin{pmatrix} 2i\, e^{(1+2i)t} \\ e^{(1+2i)t} \end{pmatrix}. \qquad (*)$$

We could now let $\lambda = \lambda_2 = 1 - 2i$ in (7118) and find a corresponding characteristic vector $\bar{a}^{(2)}$ and a linearly independent complex solution $\bar{a}^{(2)} e^{(1-2i)t}$. Instead we proceed as in Example 7.18 of Section 7.4C and apply Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$ to each component of (*). We have

$$\begin{cases} x_1 = 2i\, e^{(1+2i)t} \\ x_2 = e^{(1+2i)t} \end{cases}$$

and applying the formula and simplifying, this takes the form

$$\begin{aligned} x_1 &= 2e^t(-\sin 2t + i\cos 2t), \\ x_2 &= e^t(\cos 2t + i\sin 2t). \end{aligned}$$

The real and imaginary parts of this solution are themselves solutions, so we obtain

$$x_1 = -2e^t\sin 2t\,, \qquad x_1 = 2e^t\cos 2t,$$

and

$$x_2 = e^t\cos 2t, \qquad x_2 = e^t\sin 2t.$$

These two solutions are linearly independent, so a G.S. may be written

$$\begin{aligned} x_1 &= 2e^t(-c_1\sin 2t + c_2\cos 2t), \\ x_2 &= e^t(c_1\cos 2t + c_2\sin 2t). \end{aligned}$$

where $c_1$ and $c_2$ are arbitrary constants.

10. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ is

$$|\bar{A} - \lambda \bar{I}| = \begin{vmatrix} 2 - \lambda & -3 \\ 3 & 2 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 4\lambda + 13 = 0$ with roots $\lambda = 2 \pm 3i$. These are the characteristic values of $\bar{A}$. They are distinct conjugate complex numbers, and Theorem 7.10 applies. We use equation (7.118) of the text.

$$\text{With } \lambda = \lambda_1 = 2 + 3i \text{ and } \bar{a} = \bar{a}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (7.118)$$

becomes

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (2 + 3i) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 2a_1 - 3a_2 &= (2 + 3i)a_1, & -a_2 &= i\,a_1, \\ & \text{or} & & \\ 3a_1 + 2a_2 &= (2 + 3i)a_2, & a_1 &= i\,a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = i$, $a_2 = 1$, and thus a characteristic vector corresponding to $\lambda_1 = 2 + 3i$ is

$\bar{a}^{(1)} = \begin{pmatrix} i \\ 1 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+3i)t}$, that is,

$$\bar{x} = \begin{pmatrix} i\, e^{(2+3i)t} \\ e^{(2+3i)t} \end{pmatrix}. \qquad (*)$$

As in our solution to Exercise 9 just preceding this solution, we now proceed as in Example 7.18 of Section 7.4C. We apply Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$ to each component of (*). We have

$$\begin{cases} x_1 = i\, e^{(2+3i)t}, \\ x_2 = e^{(2+3i)t}, \end{cases}$$

and applying the formula and simplifying, this takes the form

$$\begin{aligned} x_1 &= e^{2t}(-\sin 3t + i \cos 3t), \\ x_2 &= e^{2t}(\cos 3t + i \sin 3t). \end{aligned}$$

The real and imaginary parts of this solution are themselves solutions, so we obtain

$$\begin{cases} x_1 = -e^{2t}\sin 3t, \\ x_2 = e^{2t}\cos 3t, \end{cases} \quad \text{and} \quad \begin{cases} x_1 = e^{2t}\cos 3t, \\ x_2 = e^{2t}\sin 3t. \end{cases}$$

These two solutions are linearly independent, so a G.S. may be written

$$\begin{cases} x_1 = e^{2t}(-c_1 \sin 3t + c_2 \cos 3t), \\ x_2 = e^{2t}(c_1 \cos 3t + c_2 \sin 3t). \end{cases}$$

12. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}$ is

$$|\bar{A} - \lambda \bar{I}| = \begin{vmatrix} 5 - \lambda & -4 \\ 2 & 1 - \lambda \end{vmatrix} = 0,$$

Expanding this determinant and simplifying, this takes the form $\lambda^2 - 6\lambda + 13 = 0$ with roots $3 \pm 2i$. These are the characteristic values of $\bar{A}$. They are distinct conjugate complex numbers, and Theorem 7.10 applies. We use equation (7.118) of the text.

With $\lambda = \lambda_1 = 3 + 2i$ and $\bar{\alpha} = \bar{\alpha}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118)

becomes

$$\begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (3 + 2i) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 5a_1 - 4a_2 &= (3 + 2i)a_1, \qquad (1 - i)\,a_1 = 2a_2, \\ &\text{or} \\ 2a_1 + a_2 &= (3 + 2i)a_2, \qquad a_1 = (1 + i)a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = 2$, $a_2 = 1 - i$, and thus a characteristic vector corresponding to $\lambda_1 = 3 + 2i$ is $\bar{a}^{(1)} = \begin{pmatrix} 2 \\ 1 - i \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 2 \\ 1 - i \end{pmatrix} e^{(3+2i)t}$, that is,

$$\bar{x} = \begin{pmatrix} 2e^{(3+2i)t} \\ (1 - i)e^{(3+2i)t} \end{pmatrix}. \qquad (*)$$

As in our solutions to Exercises 9 and 10, we apply Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$ to each component of (*). We have

$$x_1 = 2e^{(3+2i)t},$$

$$x_2 = (1 - i)e^{(3+2i)t};$$

and applying the formula and simplifying, this takes the form

$$\begin{aligned} x_1 &= 2e^{3t}(\cos 2t + i \sin 2t), \\ x_2 &= e^{3t}[(\cos 2t + \sin 2t) + i(\sin 2t - \cos 2t)]. \end{aligned}$$

The real and imaginary parts of this solution are themselves solutions, so we obtain

$$x_1 = 2e^{3t}\cos 2t\,, \qquad x_1 = 2e^{3t}\sin 2t,$$

and

$$x_2 = e^{3t}(\cos 2t + \sin 2t)\,, \qquad x_2 = e^{3t}(\sin 2t - \cos 2t).$$

These two solutions are linearly independent, so a G.S. may be written

$$x_1 = 2e^{3t}(c_1 \cos 2t + c_2 \sin 2t),$$
$$x_2 = e^{3t}[c_1(\cos 2t + \sin 2t) + c_2(\sin 2t - \cos 2t),$$

where $c_1$ and $c_2$ are arbitrary constants.

14. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 4 & -5 \\ 1 & 6 \end{pmatrix}$ is

$$|\bar{A} - \lambda\bar{I}| = \begin{vmatrix} 4 - \lambda & -5 \\ 1 & 6 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 10\lambda + 29 = 0$ with roots $5 \pm 2i$. These are the characteristic values of $\bar{A}$. They are distinct conjugate complex numbers, and Theorem 7.10 applies. We use equation (7.118) of the text.

$$\text{With } \lambda = \lambda_1 = 5 + 2i \text{ and } \bar{a} = \bar{a}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (7.118)$$

becomes

$$\begin{pmatrix} 4 & -5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (5 + 2i) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{cases} 4a_1 - 5a_2 = (5 + 2i)a_1, \\ a_1 + 6a_2 = (5 + 2i)a_2, \end{cases} \quad \text{or} \quad \begin{cases} (1 + 2i)a_1 + 5a_2 = 0, \\ a_1 + (1 - 2i)a_2 = 0. \end{cases}$$

A simple nontrivial solution is $a_1 = 5$, $a_2 = -(1 + 2i)$, and thus a characteristic vector corresponding to $\lambda_1 = 5 + 2i$ is $\bar{a}^{(1)} = \begin{pmatrix} 5 \\ -(1 + 2i) \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 5 \\ -(1 + 2i) \end{pmatrix} e^{(5+2i)t}$, that is,

$$\bar{x} = \begin{pmatrix} 5\, e^{(5+2i)t} \\ -(1+2i)e^{(5+2i)t} \end{pmatrix}. \qquad (*)$$

As in our solution to Exercises 9, 10, and 12 we apply Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$ to each component of (*). We have

$$\begin{cases} x_1 = 5\,e^{(5+2i)t}, \\ x_2 = -(1+2i)e^{(5+2i)t}, \end{cases}$$

and applying the formula and simplifying, this takes the form

$$\begin{cases} x_1 = 5e^{5t}(\cos 2t + i \sin 2t), \\ x_2 = -e^{5t}[(\cos 2t - 2 \sin 2t) + i(2 \cos 2t) + \sin 2t)]. \end{cases}$$

The real and imaginary parts of this solution are themselves solutions, so we obtain

$$\begin{cases} x_1 = 5e^{5t}\cos 2t, \\ x_2 = -e^{5t}(\cos 2t - 2 \sin 2t), \end{cases}$$

and

$$\begin{cases} x_1 = 5e^{5t}\sin 2t, \\ x_2 = -e^{5t}(2 \cos 2t + \sin 2t). \end{cases}$$

These two solutions are linearly independent, so a G.S. may be written

$$\begin{cases} x_1 = 5e^{5t}(c_1 \cos 2t + c_2 \sin 2t), \\ x_2 = -e^{5t}[c_1(\cos 2t - 2 \sin 2t) + c_2(2 \cos 2t + \sin 2t)]. \end{cases}$$

15. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$ is

$$|\bar{A} - \lambda I| = \begin{vmatrix} 3 - \lambda & -1 \\ 4 & -1 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 2\lambda + 1 = 0$ with double root $\lambda_1 = 1$. That is the characteristic values of A are real and equal and so Theorem 7.11 applies. We use equation (7.118) to find a characteristic vector $\bar{a}$.

With $\lambda = 1$ and $\bar{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 3a_1 - a_2 &= a_1, \\ 4a_1 - a_2 &= a_2, \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2a_1 &= a_2, \\ 2a_1 &= a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = 1$, $a_2 = 2$, and thus a characteristic vector corresponding to $\lambda = 1$ is $\bar{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

A solution is $\bar{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t$, that is,

$$\bar{x} = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}. \qquad (*)$$

By Theorem 7.11, a linearly independent solution is of the form $(\bar{\alpha}\, t + \bar{\beta})e^{\lambda t}$, where $\bar{\alpha} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda = 1$, and $\bar{\beta}$ satisfies $(\bar{A} - \lambda\bar{I})\bar{\beta} = \bar{\alpha}$. Thus $\bar{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ satisfies

$$\left[ \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

From this we find that $\beta_1$ and $\beta_2$ must satisfy

$$2\beta_1 - \beta_2 = 1,$$
$$4\beta_1 - 2\beta_2 = 2.$$

A simple nontrivial solution is $\beta_1 = 0$, $\beta_2 = -1$. Thus we find $\bar{\beta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and the "second" solution is

$$\bar{x} = \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] e^t, \text{ that is,}$$

$$\bar{x} = \begin{pmatrix} te^t \\ (2t - 1)e^t \end{pmatrix}. \qquad (**)$$

By Theorem 7.11 the solutions (*)(**) are linearly independent, and a general solution is

$$\bar{x} = c_1 \begin{pmatrix} e^t \\ 2e^t \end{pmatrix} + c_2 \begin{pmatrix} te^t \\ (2t - 1)e^t \end{pmatrix},$$

where $c_1$ and $c_2$ are arbitrary constants. In scalar language,

$$x_1 = c_1 e^t + c_2 te^t,$$
$$x_2 = 2c_1 e^t + c_2(2t - 1)e^t.$$

16. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 7 & 4 \\ -1 & 3 \end{pmatrix}$ is

$$|\bar{A} - \lambda I| = \begin{vmatrix} 7 - \lambda & 4 \\ -1 & 3 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 10\lambda + 25 = 0$ with double root $\lambda = 5$. That is the characteristic values of $\bar{A}$ are real and equal and so Theorem 7.11 applies. We first use equation (7.118) to find a characteristic vector $\bar{a}$.

With $\lambda = 5$ and $\bar{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 7 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 5 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{aligned} 7a_1 + 4a_2 &= 5a_1, & a_1 &= -2a_2, \\ & \text{or} & & \\ -a_1 + 3a_2 &= 5a_2, & a_1 &= -2a_2. \end{aligned}$$

A simple nontrivial solution is $a_1 = -2$, $a_2 = 1$, and thus a characteristic vector corresponding to $\lambda = 5$ is

$\bar{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{5t}$, that is,

$$\bar{x} = \begin{pmatrix} -2e^{5t} \\ e^{5t} \end{pmatrix}. \qquad (*)$$

By Theorem 7.11, a linearly independent solution is of the form $(\bar{\alpha}\, t + \bar{\beta})e^{\lambda t}$, where $\bar{\alpha} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\lambda = 5$, and $\bar{\beta}$ satisfies $(\bar{A} - \lambda \bar{I})\bar{\beta} = \bar{\alpha}$. Thus $\bar{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ satisfies

$$\left[ \begin{pmatrix} 7 & 4 \\ -1 & 3 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

From this we find that $\beta_1$ and $\beta_2$ must satisfy

$$\begin{aligned} 2\beta_1 + 4\beta_2 &= -2, \\ -\beta_1 - 2\beta_2 &= 1. \end{aligned}$$

A simple nontrivial solution is $\beta_1 = -1$, $\beta_2 = 0$. Thus we find $\bar{\beta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and the "second" solution is

$$\bar{x} = \left[ \begin{pmatrix} -2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{5t}, \text{ that is,}$$

$$\bar{x} = \begin{pmatrix} -(2t + 1)e^{5t} \\ te^{5t} \end{pmatrix}. \qquad (**)$$

By Theorem 7.11 the solutions (*) and (**) are linearly independent, and a general solution is

$$\bar{x} = c_1 \begin{pmatrix} -2e^{5t} \\ e^{5t} \end{pmatrix} + c_2 \begin{pmatrix} -(2t + 1)e^{5t} \\ te^{5t} \end{pmatrix},$$

where $c_1$ and $c_2$ are arbitrary constants. In scalar language,

$$\begin{aligned} x_1 &= -2c_1e^{5t} - c_2(2t + 1)e^{5t}, \\ x_2 &= c_1e^{5t} + c_2te^{5t}. \end{aligned}$$

19. The characteristic equation of the coefficient matrix $\bar{A} = \begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix}$ is

$$|\bar{A} - \lambda\bar{I}| = \begin{vmatrix} 6 - \lambda & -4 \\ 1 & 2 - \lambda \end{vmatrix} = 0,$$

Expanding the determinant and simplifying, this takes the form $\lambda^2 - 8\lambda + 16 = 0$ with double root $\lambda = 4$. That is, the characteristic values of $\bar{A}$ are real and equal and so Theorem 7.11 applies. We first use equation (7.118) to find a characteristic vector $\bar{a}$.

With $\lambda = 4$ and $\bar{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, (7.118) becomes

$$\begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 4 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

from which we find that $a_1$ and $a_2$ must satisfy

$$\begin{cases} 6a_1 - 4a_2 = 4a_1, \\ a_1 + 2a_2 = 4a_2, \end{cases} \quad \text{or} \quad \begin{cases} 2a_1 = 4a_2, \\ a_1 = 2a_2. \end{cases}$$

A simple nontrivial solution is $a_1 = 2$, $a_2 = 1$, and thus a characteristic vector corresponding to $\lambda = 4$ is $\bar{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. A solution is $\bar{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t}$, that is,

$$\bar{x} = \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}. \qquad (*)$$

By Theorem 7.11, a linearly independent solution is of the form $(\bar{a}\,t + \bar{\beta})e^{\lambda t}$, where $\bar{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\lambda = 4$, and $\bar{\beta}$

satisfies $(\bar{A} - \lambda\bar{I})\bar{\beta} = \bar{\alpha}$. Thus $\bar{\beta} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$ satisfies

$$\left[ \begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}.$$

From this we find that $\beta_1$ and $\beta_2$ must satisfy

$$\begin{aligned} 2\beta_1 - 4\beta_2 &= 2, \\ \beta_1 - 2\beta_2 &= 1. \end{aligned}$$

A simple nontrivial solution is $\beta_1 = 1$, $\beta_2 = 0$. Thus we find $\bar{\beta} = \begin{pmatrix}1\\ 0\end{pmatrix}$, and the "second" solution is

$$\bar{x} = \left[ \begin{pmatrix}2\\ 1\end{pmatrix} t + \begin{pmatrix}1\\ 0\end{pmatrix} \right] e^{4t}, \text{ that is,}$$

$$\bar{x} = \begin{pmatrix} (2t + 1)e^{4t} \\ te^{4t} \end{pmatrix}. \qquad (**)$$

By Theorem 7.11 the solutions (*) and (**) are linearly independent, and a general solution is

$$\bar{x} = c_1 \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix} + c_2 \begin{pmatrix} (2t + 1)e^{4t} \\ te^{4t} \end{pmatrix},$$

where $c_1$ and $c_2$ are arbitrary constants. In scalar language,

$$\begin{aligned} x_1 &= 2c_1e^{4t} + c_2(2t + 1)e^{4t}, \\ x_2 &= c_1e^{4t} + c_2te^{4t}. \end{aligned}$$

**Section 7.7, Page 400.**

1. We assume a solution of the form $\bar{x} = \bar{a}\, e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 1 & -1 \\ 2 & 3 - \lambda & -4 \\ 4 & 1 & -4 - \lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, it reduces to $\lambda^3 - 7\lambda + 6 = 0$ or $(\lambda - 1)(\lambda - 2)(\lambda + 3) = 0$ (see solution

of Exercise 7 of Section 7.5D). Its roots are the characteristic values $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -3$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 1$ this is

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} a_1 - a_3 &= 0, \\ 2a_1 + 2a_2 - 4a_3 &= 0, \\ 4a_1 + a_2 - 5a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 1$, $a_3 = 1$ (see solution of Exercise 7 of Section 7.5D, where x's there are used in place of $a$'s here). Thus a characteristic vector corresponding to $\lambda = 1$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t, \quad \text{that is,} \quad \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}. \qquad (*)$$

With $\lambda = 2$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -a_1 + a_2 - a_3 &= 0, \\ 2a_1 + a_2 - 4a_3 &= 0, \\ 4a_1 + a_2 - 6a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 2$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 2$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t}, \quad \text{that is,} \quad \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ e^{2t} \end{bmatrix}. \qquad (**)$$

With $\lambda = -3$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = -3 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} 4a_1 + a_2 - a_3 &= 0, \\ 2a_1 + 6a_2 - 4a_3 &= 0, \\ 4a_1 + a_2 - a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 7$, $a_3 = 11$. Thus a characteristic vector corresponding to $\lambda = -3$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 7 \\ 11 \end{pmatrix} e^{-3t}, \quad \text{that is,} \quad \begin{bmatrix} e^{-3t} \\ 7e^{-3t} \\ 11e^{-3t} \end{bmatrix}. \qquad (***)$$

By Theorem 7.16, the solutions (*), (**), and (***) are linearly independent, and a G.S. is

$$\bar{x} = c_1 \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{-3t} \\ 7e^{-3t} \\ 11e^{-3t} \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{aligned} x_1 &= c_1e^{t} + c_2e^{2t} + c_3e^{-3t}, \\ x_2 &= c_1e^{t} + 2c_2e^{2t} + 7c_3e^{-3t}, \\ x_3 &= c_1e^{t} + c_2e^{2t} + 11c_3e^{-3t}. \end{aligned}$$

2. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & -6 & 6-\lambda \end{vmatrix} = 0. \qquad (*)$$

We evaluate the determinant in (*) by expanding by cofactors along its first row. We obtain

$$(1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -6 & 6-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ -3 & 6-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & 3-\lambda \\ -3 & -6 \end{vmatrix} = 0.$$

This reduces to

$$(1 - \lambda)[(3 - \lambda)(6 - \lambda) + 6] + [6 - \lambda + 3] - [-6 + 9 - 3\lambda] = 0$$

which simplifies to

$$(\lambda - 1)(\lambda^2 - 9\lambda + 24) - (-\lambda + 9) + (-3\lambda + 3) = 0$$

and hence to

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0.$$

Thus the characteristic equation of $\bar{A}$ reduces to the cubic equation $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$. We see by inspection that $\lambda = 2$ is a root of this, and then by synthetic division we obtain $(\lambda - 2)(\lambda^2 - 8\lambda + 15) = 0$. Finally, then, we obtain the characteristic equation in the factored form $(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$. The roots of this are the characteristic values $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$.

We have gone through the preceeding evaluation and solution in some detail. In third-order problems it is essential that we carry out the evaluation and solution of the characteristic equation carefully and correctly. Making an error here can be very time-consuming and frustrating. We shall omit the corresponding details from future solutions in this manual. We refer the reader in need of help to Appendices 1 and 2 of the text.

Going on with the present solution, we use $\bar{A}\,\bar{a} = \lambda\,\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 2$, this is

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -a_1 - a_2 - a_3 &= 0, \\ a_1 + a_2 + a_3 &= 0, \\ -3a_1 - 6a_2 + 4a_3 &= 0. \end{aligned}$$

The first two equations are equivalent. Solving the last two for $a_1$ and $a_2$ in terms of $a_3$, we find $a_1 = -(10/3)a_3$, $a_2 = (7/3)a_3$. Thus a nontrivial solution of the algebraic system is $a_1 = 10$, $a_2 = -7$, $a_3 = -3$. Thus a characteristic vector corresponding to $\lambda = 2$ is

$$\bar{a} = \begin{bmatrix} 10 \\ -7 \\ -3 \end{bmatrix},$$

and a corresponding solution is

$$\begin{bmatrix} 10 \\ -7 \\ -3 \end{bmatrix} e^{2t}, \quad \text{that is,} \quad \begin{bmatrix} 10e^{2t} \\ -7e^{2t} \\ -3e^{2t} \end{bmatrix}. \qquad (**)$$

With $\lambda = 3$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 3\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -2a_1 - a_2 - a_3 &= 0, \\ a_1 + a_3 &= 0, \\ -3a_1 - 6a_2 + 3a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = -1$, $a_3 = -1$. Thus a characteristic vector corresponding to $\lambda = 3$ is

$$\bar{a} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}e^{3t}, \quad \text{that is,} \quad \begin{bmatrix} e^{3t} \\ -e^{3t} \\ -e^{3t} \end{bmatrix}. \qquad (***)$$

With $\lambda = 5$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 5 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -4a_1 - a_2 - a_3 &= 0, \\ a_1 - 2a_2 + a_3 &= 0, \\ -3a_1 - 6a_2 + a_3 &= 0. \end{aligned}$$

Adding the first two equations, we obtain $-3a_1 - 3a_2 = 0$, from which $a_2 = -a_1$. Then the third equation gives $a_3 = 3a_1 + 6a_2 = -3a_1$. So a nontrivial solution of the algebraic system is $a_1 = 1$, $a_2 = -1$, $a_3 = -3$. Thus a characteristic vector corresponding to $\lambda = 5$ is

$$\bar{a} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} e^{5t}, \quad \text{that is,} \quad \begin{bmatrix} e^{5t} \\ -e^{5t} \\ -3e^{5t} \end{bmatrix}. \qquad (****)$$

By Theorem 7.16, the solutions (**), (***), and (****) are linearly independent, and a G.S. is

$$\bar{x} = c_1\begin{bmatrix} 10e^{2t} \\ -7e^{2t} \\ -3e^{2t} \end{bmatrix} + c_2\begin{bmatrix} e^{3t} \\ -e^{3t} \\ -e^{3t} \end{bmatrix} + c_3\begin{bmatrix} e^{5t} \\ -e^{5t} \\ -3e^{5t} \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{aligned} x_1 &= 10c_1e^{2t} + c_2e^{3t} + c_3e^{5t}, \\ x_2 &= -7c_1e^{2t} - c_2e^{3t} - c_3e^{5t}, \\ x_3 &= -3c_1e^{2t} - c_2e^{3t} - 3c_3e^{5t}. \end{aligned}$$

6. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, it reduces to $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$ or $(\lambda - 1)(\lambda - 2)(\lambda + 1) = 0$. Its roots are the characteristic values $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -1$.

We use $\bar{A}\,\bar{a} = \lambda a$ to find the corresponding characteristic vectors. With $\lambda = 1$ this is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} a_1 &= 0, \\ a_1 - a_2 + a_3 &= 0, \\ a_1 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 0$, $a_3 = -1$. Thus a characteristic vector corresponding to $\lambda = 1$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^t, \quad \text{that is,} \quad \begin{bmatrix} e^t \\ 0 \\ -e^t \end{bmatrix}. \qquad (*)$$

With $\lambda = 2$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -a_1 + a_2 &= 0, \\ a_1 - 2a_2 + a_3 &= 0, \\ a_1 - a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 1$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 2$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \quad \text{that is,} \quad \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}. \qquad (**)$$

With $\lambda = -1$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = -1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} 2a_1 + a_2 &= 0, \\ a_1 + a_2 + a_3 &= 0, \\ a_2 + 2a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = -2$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = -1$ is

$$\bar{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t}, \quad \text{that is,} \quad \begin{bmatrix} e^{-t} \\ -2e^{-t} \\ e^{-t} \end{bmatrix}. \qquad (***)$$

By Theorem 7.16, the solutions (*), (**), and (***) are linearly independent, and a G.S. is

$$\bar{x} = c_1\begin{bmatrix} e^{t} \\ 0 \\ -e^{t} \end{bmatrix} + c_2\begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_3\begin{bmatrix} e^{-t} \\ -2e^{-t} \\ e^{-t} \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{aligned} x_1 &= c_1e^{t} + c_2e^{2t} + c_3e^{-t}, \\ x_2 &= c_2e^{2t} - 2c_3e^{-t}, \\ x_3 &= c_1e^{t} + c_2e^{2t} + c_3e^{-t}. \end{aligned}$$

7. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, it reduces to $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$ or $(\lambda - 2)(\lambda^2 - 4\lambda - 1) = 0$. Its roots are the characteristic values $\lambda_1 = 2 + \sqrt{5}$, $\lambda_2 = 2 - \sqrt{5}$, $\lambda_3 = 2$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 2 + \sqrt{5}$ this is

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (2 + \sqrt{5}) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} (-1 - \sqrt{5})a_1 - 2a_2 &= 0, \\ -2a_1 + (1 - \sqrt{5})a_2 &= 0, \\ -\sqrt{5}\,a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = -2$, $a_2 = 1 + \sqrt{5}$, $a_3 = 0$. Thus a characteristic vector corresponding to $\lambda = 2 + \sqrt{5}$ is

$$\bar{a} = \begin{pmatrix} -2 \\ 1 + \sqrt{5} \\ 0 \end{pmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} -2 \\ 1 + \sqrt{5} \\ 0 \end{pmatrix} e^{(2+\sqrt{5})t}, \text{ that is, } \begin{bmatrix} -2e^{(2+\sqrt{5})t} \\ (1 + \sqrt{5})e^{(2+\sqrt{5})t} \\ 0 \end{bmatrix} \quad (*)$$

With $\lambda = 2 - \sqrt{5}$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (2 - \sqrt{5}) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} (-1 + \sqrt{5})a_1 - 2a_2 &= 0, \\ -2a_1 + (1 + \sqrt{5})a_2 &= 0, \\ \sqrt{5}\,a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 2$, $a_2 = -1 + \sqrt{5}$, $a_3 = 0$. Thus a characteristic vector corresponding to $\lambda = 2 - \sqrt{5}$ is

$$\begin{pmatrix} 2 \\ -1 + \sqrt{5} \\ 0 \end{pmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 2 \\ 1 + \sqrt{5} \\ 0 \end{pmatrix} e^{(2-\sqrt{5})t}, \text{ that is, } \begin{bmatrix} 2e^{(2-\sqrt{5})t} \\ (-1 + \sqrt{5})e^{(2-\sqrt{5})t} \\ 0 \end{bmatrix} \quad (**)$$

With $\lambda = 2$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -a_1 - 2a_2 &= 0, \\ -2a_1 + a_2 &= 0, \\ 0a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 0$, $a_2 = 0$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 2$ is

$$\bar{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}, \quad \text{that is,} \quad \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}. \qquad (***)$$

By Theorem 7.16, the solutions (*), (**), and (***) are linearly independent, and a G.S. is

$$\bar{x} = c_1\begin{bmatrix} -2e^{(2+\sqrt{5})t} \\ (1+\sqrt{5})e^{(2+\sqrt{5})t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2e^{(2-\sqrt{5})t} \\ (-1+\sqrt{5})e^{(2-\sqrt{5})t} \\ 0 \end{bmatrix}$$

$$+ c_3\begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}.$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$x_1 = -2c_1e^{(2+\sqrt{5})t} + 2c_2e^{(2-\sqrt{5})t},$$
$$x_2 = (1+\sqrt{5})c_1e^{(2+\sqrt{5})t} + (-1+\sqrt{5})c_2e^{(2-\sqrt{5})t},$$
$$x_3 = c_3e^{2t}.$$

10. We assume a solution of the form $\bar{x} = \bar{a}e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 3 & 7 & -3 \\ 1 & 2 & -2 \\ 1 & 6 & -2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 3-\lambda & 7 & -3 \\ 1 & 2-\lambda & -2 \\ 1 & 6 & -2-\lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, it reduces to $\lambda^3 - 3\lambda^2 + 4\lambda - 12 = 0$, or $(\lambda - 3)(\lambda^2 + 4) = 0$. Its roots are the characteristic values $\lambda_1 = 3$, $\lambda_2 = 2i$, $\lambda_3 = -2i$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 3$ this is

$$\begin{bmatrix} 3 & 7 & -3 \\ 1 & 2 & -2 \\ 1 & 6 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 3 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} 7a_2 - 3a_3 &= 0, \\ a_1 - a_2 - 2a_3 &= 0, \\ a_1 + 6a_2 - 5a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 17$, $a_2 = 3$, $a_3 = 7$. Thus a characteristic vector corresponding to $\lambda = 3$ is

$$\bar{a} = \begin{bmatrix} 17 \\ 3 \\ 7 \end{bmatrix},$$

and a corresponding solution is

$$\begin{bmatrix} 17 \\ 3 \\ 7 \end{bmatrix} e^{3t}, \quad \text{that is,} \quad \begin{bmatrix} 17e^{3t} \\ 3e^{3t} \\ 7e^{3t} \end{bmatrix}. \qquad (*)$$

With $\lambda = 2i$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 3 & 7 & -3 \\ 1 & 2 & -2 \\ 1 & 6 & -2 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2i \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} (3 - 2i)a_1 + 7a_2 - 3a_3 &= 0, \\ a_1 + (2 - 2i)a_2 - 2a_3 &= 0, \\ a_1 + 6a_2 + (-2 - 2i)a_3 &= 0. \end{aligned}$$

Subtracting the second equation from the third, we obtain

$$(4 + 2i)a_2 - 2i\,a_3 = 0$$

and hence $(2 + i)a_2 = i\,a_3$. A nontrivial solution of this is $a_2 = 1$, $a_3 = 1 - 2i$. Then $a_1 = (-2 + 2i)a_2 + 2a_3 = (-2 + 2i)(1) + 2(1 - 2i) = -2i$. Thus a characteristic vector corresponding to $\lambda = 2i$ is

$$\bar{a} = \begin{bmatrix} -2i \\ 1 \\ 1 - 2i \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} -2i \\ 1 \\ 1 - 2i \end{pmatrix} e^{2i\,t}, \quad \text{that is,} \quad \begin{bmatrix} -2ie^{2i\,t} \\ e^{2i\,t} \\ (1-2)e^{2i\,t} \end{bmatrix}.$$

Proceeding as in Section 7.6, we apply Euler's formula $e^{i\,\theta} = \cos\theta + i\sin\theta$ to each component. We obtain the solution

$$\begin{pmatrix} -2i(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t \\ (1 - 2i)(\cos 2t + i\sin 2t) \end{pmatrix},$$

which simplifies to

$$\begin{pmatrix} 2\sin 2t + i(-2\cos 2t) \\ \cos 2t + i\sin 2t \\ \cos 2t + 2\sin 2t + i(-2\cos 2t + \sin 2t) \end{pmatrix}.$$

The real and imaginary parts of this solution are themselves solutions, so we obtain the solutions

$$\begin{pmatrix} 2\sin 2t \\ \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2\cos 2t \\ \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix}. \qquad (**)$$

The solution (*) and the two solutions (**) are linearly independent, so a G.S.is

$$\bar{x} = c_1\begin{pmatrix} 17e^{3t} \\ 3e^{3t} \\ 7e^{3t} \end{pmatrix} + c_2\begin{pmatrix} 2\sin 2t \\ \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + c_3\begin{pmatrix} -2\cos 2t \\ \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix}.$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{cases} x_1 = 17c_1e^{3t} + 2c_2\sin 2t - 2c_3\cos 2t, \\ x_2 = 3c_1e^{3t} + c_2\cos 2t + c_3\sin 2t, \\ x_3 = 7c_1e^{3t} + c_2(\cos 2t + 2\sin 2t) \\ \qquad\quad + c_3(\sin 2t - 2\cos 2t). \end{cases}$$

13. We assume a solution of the form $\bar{x} = \bar{a}e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & -3 & 9 \\ 0 & -5 & 18 \\ 0 & -3 & 10 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -3 & 9 \\ 0 & -5-\lambda & 18 \\ 0 & -3 & 10-\lambda \end{vmatrix} = 0.$$

Expanding this determinant and simplifying, it reduces to $(\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0$ or $(\lambda - 1)^2(\lambda - 4) = 0$. Its roots are the characteristic vectors $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 1$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 4$ this is

$$\begin{bmatrix} 1 & -3 & 9 \\ 0 & -5 & 18 \\ 0 & -3 & 10 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 4 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -3a_1 - 3a_2 + 9a_3 &= 0, \\ -9a_2 + 18a_3 &= 0, \\ -3a_2 + 6a_3 &= 0. \end{aligned}$$

From either the second or third equation $a_2 = 2a_3$. Then the first equation gives $a_1 = -a_2 + 3a_3 = a_3$. A non-trivial solution of this is $a_1 = 1$, $a_2 = 2$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 4$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix}1\\2\\1\end{pmatrix}e^{4t}, \quad \text{that is,} \quad \begin{bmatrix}e^{4t}\\2e^{4t}\\e^{4t}\end{bmatrix}. \qquad (*)$$

With $\lambda = 1$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix}1 & -3 & 9\\0 & -5 & 18\\0 & -3 & 10\end{pmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = (1)\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned}-3a_2 + 9a_3 &= 0,\\-6a_2 + 18a_3 &= 0,\\-3a_2 + 9a_3 &= 0.\end{aligned}$$

Each of these three relationships is equivalent to the one relation

$$a_2 = 3a_3, \qquad (**)$$

and this the only relationship which $a_1$, $a_2$, $a_3$ must satisfy. In particular, observe that $a_1$ is arbitrary. Two linearly independent solutions of (**) are $a_1 = 0$, $a_2 = 3$, $a_3 = 1$ and $a_1 = 1$, $a_2 = 3$, $a_3 = 1$. That is,

corresponding to the double characteristic value $\lambda = 1$, we have the two linearly independent characteristic vectors

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Respective corresponding solutions are

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} e^t, \quad \text{and} \quad \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} e^t,$$

that is,

$$\begin{bmatrix} 0 \\ 3e^t \\ e^t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e^t \\ 3e^t \\ e^t \end{bmatrix}. \qquad (***)$$

The three solutions given by (*) and (***) are linearly independent, and a G.S. is

$$\bar{x} = c_1 \begin{bmatrix} e^{4t} \\ 2e^{4t} \\ e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ 3e^t \\ e^t \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3e^t \\ e^t \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$x_1 = c_1e^{4t} + c_2e^{t},$$
$$x_2 = 2c_1e^{4t} + 3c_2e^{t} + 3c_3e^{t},$$
$$x_3 = c_1e^{4t} + c_2e^{t} + c_3e^{t}.$$

15. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 11 & 6 & 18 \\ 9 & 8 & 18 \\ -9 & -6 & -16 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 11 - \lambda & 6 & 18 \\ 9 & 8 - \lambda & 18 \\ -9 & -6 & -16 - \lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, it reduces to $\lambda^3 - 3\lambda^2 + 4 = 0$ or $(\lambda + 1)(\lambda - 2)^2 = 0$. Its roots are the characteristic vectors $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 2$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = -1$ this is

$$\begin{pmatrix} 11 & 6 & 18 \\ 9 & 8 & 18 \\ -9 & -6 & -16 \end{pmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (-1)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} 12a_1 + 6a_2 + 18a_3 &= 0, \\ 9a_1 + 9a_2 + 18a_3 &= 0, \\ -9a_1 - 6a_2 - 15a_3 &= 0. \end{aligned}$$

These simplify to

$$\begin{aligned} 2a_1 + a_2 + 3a_3 &= 0, \\ a_1 + a_2 + 2a_3 &= 0, \\ 3a_1 + 2a_2 + 5a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 1$, $a_3 = -1$. Thus a characteristic vector corresponding to $\lambda = -1$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-t}, \quad \text{that is,} \quad \begin{bmatrix} e^{-t} \\ e^{-t} \\ -e^{-t} \end{bmatrix}. \qquad (*)$$

With $\lambda = 2$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 11 & 6 & 18 \\ 9 & 8 & 18 \\ -9 & -6 & -16 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} 9a_1 + 6a_2 + 18a_3 &= 0, \\ 9a_1 + 6a_2 + 18a_3 &= 0, \\ -9a_1 - 6a_2 - 18a_3 &= 0. \end{aligned}$$

Each of these three relationships is equivalent to the one relation

$$3a_1 + 2a_2 + 6a_3 = 0,$$

and this the only relationship which $a_1$, $a_2$, $a_3$ must satisfy. Two linearly independent solutions of it are $a_1 = 2$, $a_2 = 0$, $a_3 = -1$ and $a_1 = 0$, $a_2 = 3$, $a_3 = -1$. That is, corresponding to the double characteristic value $\lambda = 2$, we have the two linearly independent characteristic vectors

$$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}.$$

Respective corresponding solutions are

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{2t}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} e^{2t},$$

that is,

$$\begin{bmatrix} 2e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 3e^{2t} \\ -e^{2t} \end{bmatrix}. \qquad (**)$$

The three solutions given by (*) and (**) are linearly independent, and a G.S. is

$$\bar{x} = c_1 \begin{bmatrix} e^{-t} \\ e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3e^{2t} \\ -e^{2t} \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{aligned} x_1 &= c_1 e^{-t} + 2c_2 e^{2t}, \\ x_2 &= c_1 e^{-t} + 3c_3 e^{2t}, \\ x_3 &= -c_1 e^{-t} + c_2 e^{2t} - c_3 e^{2t}. \end{aligned}$$

16. We assume a solution of the form $\bar{x} = \bar{a} e^{\lambda t}$, that is, $x_1 = a_1 e^{\lambda t}$, $x_2 = a_2 e^{\lambda t}$, $x_3 = a_3 e^{\lambda t}$. We know that $\lambda$ must

be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & 9 & 8 \\ 0 & 19 & 18 \\ 0 & 9 & 10 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 9 & 9 \\ 0 & 19 - \lambda & 18 \\ 0 & 9 & 10 - \lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, it reduces to $\lambda^3 - 30\lambda^2 + 57\lambda - 28 = 0$ or $(\lambda - 1)^2(\lambda - 28) = 0$. Its roots are the characteristic vectors $\lambda_1 = 28$, $\lambda_2 = \lambda_3 = 1$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 28$ this is

$$\begin{pmatrix} 1 & 9 & 9 \\ 0 & 19 & 18 \\ 0 & 9 & 10 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 28 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} -27a_1 + 9a_2 + 9a_3 &= 0, \\ -9a_2 + 18a_3 &= 0, \\ 9a_2 - 18a_3 &= 0. \end{aligned}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 2$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 28$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{28t}, \quad \text{that is,} \quad \begin{bmatrix} e^{28t} \\ 2e^{28t} \\ e^{28t} \end{bmatrix}. \qquad (*)$$

With $\lambda = 1$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} 1 & 9 & 9 \\ 0 & 19 & 18 \\ 0 & 9 & 10 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} 9a_2 + 9a_3 &= 0, \\ 18a_2 + 18a_3 &= 0, \\ 9a_2 + 9a_3 &= 0. \end{aligned}$$

Each of these three relationships is equivalent to the one relation

$$a_2 + a_3 = 0,$$

and this the only relationship which $a_1$, $a_2$, $a_3$ must satisfy. Note that it allows $a_1$ to be arbitrary. Two linearly independent solutions of it are $a_1 = 1$, $a_2 = a_3 = 0$, and $a_1 = 0$, $a_2 = 1$, $a_3 = -1$. That is, corresponding to the double characteristic value $\lambda = 1$, we have the two linearly independent characteristic vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Respective corresponding solutions are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t, \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^t,$$

that is,

$$\begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ e^t \\ -e^t \end{bmatrix}. \qquad (**)$$

The three solutions given by (*) and (**) are linearly independent, and a G.S. is

$$\bar{x} = c_1 \begin{bmatrix} e^{28t} \\ 2e^{28t} \\ e^{28t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^t \\ -e^t \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{aligned} x_1 &= c_1e^{28t} + c_2e^{t}, \\ x_2 &= 2c_1e^{28t} + c_3e^{t}, \\ x_3 &= c_1e^{28t} - c_3e^{t}. \end{aligned}$$

17. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} -5 - \lambda & -12 & 6 \\ 1 & 5 - \lambda & -1 \\ -7 & -10 & 8 - \lambda \end{vmatrix} = 0.$$

This reduces to $\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$, that is, $(\lambda - 1)(\lambda - 3)(\lambda - 4) = 0$. Its roots are $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 1$ this is

$$\begin{pmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (1) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} -6a_1 - 12a_2 + 6a_3 = 0, \\ a_1 + 4a_2 - a_3 = 0, \\ -7a_1 - 10a_2 + 7a_3 = 0. \end{cases}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 0$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 1$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^t, \quad \text{that is,} \quad \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}. \qquad (*)$$

With $\lambda = 3$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 3 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} -8a_1 - 12a_2 + 6a_3 = 0, \\ a_1 + 2a_2 - a_3 = 0, \\ -7a_1 - 10a_2 + 5a_3 = 0. \end{cases}$$

A nontrivial solution of this is $a_1 = 0$, $a_2 = 1$, $a_3 = 2$. Thus a characteristic vector corresponding to $\lambda = 3$ is

$$\bar{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

and a corresponding solution is

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} e^{3t}, \quad \text{that is,} \quad \begin{bmatrix} 0 \\ e^{3t} \\ 2e^{3t} \end{bmatrix}. \qquad (**)$$

With $\lambda = 4$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 4 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} -9a_1 - 12a_2 + 6a_3 = 0, \\ a_1 + a_2 - a_3 = 0, \\ -7a_1 - 10a_2 + 4a_3 = 0. \end{cases}$$

A nontrivial solution of this is $a_1 = 2$, $a_2 = -1$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 4$ is

$$\bar{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

and a corresponding solution is

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} e^{4t}, \quad \text{that is,} \quad \begin{bmatrix} 2e^{4t} \\ -e^{4t} \\ e^{4t} \end{bmatrix}. \qquad (***)$$

By Theorem 7.16, the solutions (*), (**), and (**) are linearly independent, and a G.S. is

$$\bar{x} = c_1 \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 2e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 2e^{4t} \\ -e^{4t} \\ e^{4t} \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$x_1 = c_1e^{t} + 2c_3e^{4t},$$
$$x_2 = c_2e^{3t} - c_3e^{4t},$$
$$x_3 = c_1e^{t} + 2c_2e^{3t} + c_3e^{4t}.$$

19. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} -5 & -3 & -3 \\ 8 & 5 & 7 \\ -2 & -1 & -3 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} -5 - \lambda & -3 & -3 \\ 8 & 5 - \lambda & 7 \\ -2 & -1 & -3 - \lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, it reduces to $\lambda^3 + 3\lambda^2 - 4 = 0$, or $(\lambda - 1)(\lambda + 2)^2 = 0$. Its roots are the characteristic values $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -2$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 1$ this is

$$\begin{pmatrix} -5 & -3 & -3 \\ 8 & 5 & 7 \\ -2 & -1 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = (1)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} -6a_1 - 3a_2 - 3a_3 = 0, \\ 8a_1 + 4a_2 + 7a_3 = 0, \\ -2a_1 - a_2 - 4a_3 = 0. \end{cases}$$

Adding the second equation to four times the third equation, we find $-9a_3 = 0$, from which $a_3 = 0$. With $a_3 = 0$, each equation is equivalent to $a_3 = -2a_1$. Hence a nontrivial solution of the algebraic system is $a_1 = 1$, $a_2 = -2$, $a_3 = 0$. Thus a characteristic vector corresponding to $\lambda = 1$ is

$$\bar{a} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^t, \quad \text{that is,} \quad \begin{bmatrix} e^t \\ -2e^t \\ 0 \end{bmatrix}. \qquad (*)$$

With $\lambda = -2$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ is

$$\begin{pmatrix} -5 & -3 & -3 \\ 8 & 5 & 7 \\ -2 & -1 & -3 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = -2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} -3a_1 - 3a_2 - 3a_3 = 0, \\ 8a_1 + 7a_2 + 7a_3 = 0, \\ -2a_1 - a_2 - a_3 = 0. \end{cases}$$

Adding the first equation to (-3) times the third, we find $3a_1 = 0$, from which $a_1 = 0$. With $a_1 = 0$, each equation is equivalent to $a_3 = -a_2$. Hence a nontrivial solution of the algebraic system is $a_1 = 0$, $a_2 = 1$, $a_3 = -1$. Thus a characteristic vector corresponding to $\lambda = -2$ is

$$\bar{a} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and a corresponding solution is

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-2t}, \quad \text{that is,} \quad \begin{bmatrix} 0 \\ e^{-2t} \\ -e^{-2t} \end{bmatrix}. \qquad (**)$$

A second solution corresponding to $\lambda = -2$ is of the form $(\bar{a}t + \bar{\beta})e^{-2t}$, where $\bar{\beta}$ satisfies $(\bar{A} + 2\bar{I})\bar{\beta} = \bar{a}$. This is

$$\begin{pmatrix} -3 & -3 & -3 \\ 8 & 7 & 7 \\ -2 & -1 & -1 \end{pmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Multiplying this out yields

$$\begin{cases} -3\beta_1 - 3\beta_2 - 3\beta_3 = 0, \\ 8\beta_1 + 7\beta_2 + 7\beta_3 = 1, \\ -2\beta_1 - \beta_2 - \beta_3 = -1. \end{cases}$$

a solution of which is $\beta_1 = 1$, $\beta_2 = -1$, $\beta_3 = 0$. Thus we obtain

$$\bar{\beta} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and the second solution $(\bar{a}t + \bar{\beta})e^{-2t}$ is

$$\left[\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right] e^{-2t}, \quad \text{that is,} \quad \begin{bmatrix} e^{-2t} \\ (t-1)e^{-2t} \\ -te^{-2t} \end{bmatrix}. \qquad (***)$$

The three solutions (*), (**), and (**) are linearly independent. Thus a G.S. is

$$\bar{x} = c_1 \begin{bmatrix} e^{t} \\ -2e^{t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-2t} \\ -2e^{-2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{-2t} \\ (t-1)e^{-2t} \\ -te^{-2t} \end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{cases} x_1 = c_1e^{t} + c_3e^{-2t}, \\ x_2 = -2c_1e^{t} + c_2e^{-2t} + c_3(t - 1)e^{-2t}, \\ x_3 = -2c_2e^{-2t} - c_3te^{-2t}. \end{cases}$$

22. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 3 & -2 & -1 \\ -4 & 2 & 4 \\ 5 & -3 & -3 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 3 - \lambda & -2 & -1 \\ -4 & 2 - \lambda & 4 \\ 5 & -3 & -3 - \lambda \end{vmatrix} = 0.$$

It reduces to $\lambda^3 - 2\lambda^2 = 0$. Its roots are the characteristic values $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 0$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 2$ this is

$$\begin{pmatrix} 3 & -2 & -1 \\ -4 & 2 & 4 \\ 5 & -3 & -3 \end{pmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} a_1 - 2a_2 - a_3 = 0, \\ -4a_1 + 4a_3 = 0, \\ 5a_1 - 3a_2 - 5a_3 = 0. \end{cases}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = 0$, $a_3 = 1$. Thus a characteristic vector corresponding to $\lambda = 2$ is

$$\bar{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

and a corresponding solution is

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}, \quad \text{that is,} \quad \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}. \qquad (*)$$

We now consider the repeated characteristic value $\lambda_2 = \lambda_3 = 0$. With $\lambda = 0$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ becomes

$$\begin{pmatrix} 3 & -2 & -1 \\ -4 & 2 & 4 \\ 5 & -3 & -3 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} 3a_1 - 2a_2 - a_3 = 0, \\ -4a_1 + 2a_2 + 4a_3 = 0, \\ 5a_1 - 3a_2 - 3a_3 = 0. \end{cases}$$

Adding the first two equations, we obtain $-a_1 + 3a_3 = 0$. From this $a_1 = 3a_3$. Then $2a_2 = 3a_1 - a_3 = 8a_3$, so $a_2 = 4a_3$. Thus a simple nontrivial solution is $a_1 = 3$, $a_2 = 4$, $a_3 = 1$. A characteristic vector corresponding to $\lambda = 0$ is

$$\bar{a} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

and the corresponding solution is

$$\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} e^{0t}, \quad \text{that is,} \quad \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}. \qquad (**)$$

A second solution corresponding to $\lambda = 0$ is of the form $(\bar{a}t + \bar{\beta})e^{0t} = \bar{a}t + \bar{\beta}$, where $\bar{\beta}$ satisfies $(\bar{A} - \lambda\bar{I})\bar{\beta} = \bar{a}$, that is $\bar{A}\,\bar{\beta} = \bar{a}$. This is

$$\begin{pmatrix} 3 & -2 & -1 \\ -4 & 2 & 4 \\ 5 & -3 & -3 \end{pmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}.$$

Multiplying this out yields

$$\begin{cases} 3\beta_1 - 2\beta_2 - \beta_3 = 3, \\ -4\beta_1 + 2\beta_2 + 4\beta_3 = 4, \\ 5\beta_1 - 3\beta_2 - 3\beta_3 = 1. \end{cases}$$

Multiplying the first equation by 4, the second by 3, and adding, we find $-2\beta_2 + 8\beta_3 = 24$ and so $\beta_2 = 4\beta_3 - 12$. The first equation gives $\beta_1 = \frac{2}{3}\beta_2 + \frac{1}{3}\beta_3 + 1$. Then substituting $\beta_2 = 4\beta_3 - 12$ into this, we have $\beta_1 = 3\beta_3 - 7$. One quickly checks that the pair $\beta_1 = 3\beta_3 - 7$, $\beta_2 = 4\beta_3 - 12$ satisfies the third equation for arbitrary $\beta_3$. Choosing $\beta_3 = 3$, we have the simple nontrivial solution $\beta_1 = 2$, $\beta_2 = 0$, $\beta_3 = 3$. Thus we obtain

$$\bar{\beta} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix},$$

and the second solution $\bar{\alpha}t + \bar{\beta}$ is

$$\left[\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}\right], \quad \text{that is,} \quad \begin{bmatrix} 3t + 2 \\ 4t \\ t + 3 \end{bmatrix}. \qquad (***)$$

The three solutions (*), (**), and (**) are linearly independent. Thus a G.S. is

$$\bar{x} = c_1 \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3t + 2 \\ 4t \\ t + 3 \end{bmatrix}$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{cases} x_1 = c_1 e^{2t} + 3c_2 + c_3(3t + 2), \\ x_2 = 4c_2 + 4c_2 t, \\ x_3 = c_1 e^{2t} + c_2 + c_3(t + 3). \end{cases}$$

23. We assume a solution of the form $\bar{x} = \bar{a}\, e^{\lambda t}$, that is, $x_1 = a_1 e^{\lambda t}$, $x_2 = a_2 e^{\lambda t}$, $x_3 = a_3 e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 7 & 4 & 4 \\ -6 & -4 & -7 \\ -2 & -1 & 2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 7 - \lambda & 4 & 4 \\ -6 & -4 - \lambda & -7 \\ -2 & -1 & 2 - \lambda \end{vmatrix} = 0.$$

It reduces to $\lambda^3 - 5\lambda^2 + 3\lambda + 9 = 0$, that is, $(\lambda + 1)(\lambda - 3)^2 = 0$. Its roots are the characteristic values $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 3$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = -1$ this is

$$\begin{pmatrix} 7 & 4 & 4 \\ -6 & -4 & -7 \\ -2 & -1 & 2 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = -1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{cases} 8a_1 + 4a_2 + 4a_3 = 0, \\ -6a_1 - 3a_2 - 7a_3 = 0, \\ -2a_1 - a_2 + 3a_3 = 0. \end{cases}$$

A nontrivial solution of this is $a_1 = 1$, $a_2 = -2$, $a_3 = 0$. Thus a characteristic vector corresponding to $\lambda = -1$ is

$$\bar{a} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$

and the corresponding solution is

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t}, \quad \text{that is,} \quad \begin{bmatrix} e^{-t} \\ -2e^{-t} \\ 0 \end{bmatrix}. \qquad (*)$$

We now consider the repeated characteristic value $\lambda_2 = \lambda_3 = 3$. With $\lambda = 3$, $\bar{A}\,\bar{a} = \lambda\bar{a}$ becomes

$$\begin{pmatrix} 7 & 4 & 4 \\ -6 & -4 & -7 \\ -2 & -1 & 2 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 3 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Thus $a_1$, $a_2$, $a_3$ must satisfy

$$\begin{aligned} 4a_1 + 4a_2 + 4a_3 &= 0, \\ -6a_1 - 7a_2 - 7a_3 &= 0, \\ -2a_1 - a_2 - a_3 &= 0. \end{aligned}$$

A solution of this is $a_1 = 0$, $a_2 = 1$, $a_3 = -1$. Thus a characteristic vector corresponding to $\lambda = 3$ is

$$\bar{a} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

and the corresponding solution is

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{3t}, \quad \text{that is,} \quad \begin{bmatrix} 0 \\ e^{3t} \\ -e^{3t} \end{bmatrix}. \qquad (**)$$

A second solution corresponding to $\lambda = 3$ is of the form $(\bar{a}t + \bar{\beta})e^{3t}$, where $\bar{\beta}$ satisfies $(\bar{A} - 3\bar{I})\bar{\beta} = \bar{a}$. This is

$$\begin{pmatrix} 4 & 4 & 4 \\ -6 & -7 & -7 \\ -2 & -1 & -1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Multiplying this out yields

$$\begin{aligned} 4\beta_1 + 4\beta_2 + 4\beta_3 &= 0, \\ -6\beta_1 - 7\beta_2 - 7\beta_3 &= 1, \\ -2\beta_1 - \beta_2 - \beta_3 &= -1. \end{aligned}$$

a solution of which is $\beta_1 = 1$, $\beta_2 = -1$, $\beta_3 = 0$. Thus we obtain

$$\bar{\beta} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

and the second solution $(\bar{\alpha}t + \bar{\beta})e^{3t}$ is

$$\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right] e^{3t}, \quad \text{that is,} \quad \begin{pmatrix} e^{3t} \\ (t-1)e^{3t} \\ -te^{3t} \end{pmatrix}. \qquad (***)$$

The three solutions (*), (**), and (***) are linearly independent. Thus a G.S. is

$$\bar{x} = c_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{3t} \\ -e^{3t} \end{pmatrix} + c_3 \begin{pmatrix} e^{3t} \\ (t-1)e^{3t} \\ -te^{3t} \end{pmatrix},$$

where $c_1$, $c_2$, and $c_3$ are arbitrary constants. In component form, this is

$$x_1 = c_1e^{-t} + c_3e^{3t},$$
$$x_2 = -2c_1e^{-t} + c_2e^{3t} + c_3(t - 1)e^{3t},$$
$$x_3 = -c_2e^{3t} - c_3te^{3t}.$$

26. We assume a solution of the form $\bar{x} = \bar{a}\,e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 4 - \lambda & -1 & -1 \\ 2 & 1 - \lambda & -1 \\ 2 & -1 & 1 - \lambda \end{vmatrix} = 0.$$

It reduces to $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$, that is, $(\lambda - 2)^3 = 0$. Its roots are the characteristic values $\lambda_1 = \lambda_2 = \lambda_3 = 2$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 2$ this is

$$\begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Each of the three resulting relationships in $a_1$, $a_2$, $a_3$ is $2a_1 - a_2 - a_3 = 0$. Both $a_1 = 1$, $a_2 = 2$, $a_3 = 0$ and $a_1 = 1$, $a_2 = 0$, $a_3 = 2$ are distinct solutions of this. Thus we obtain the characteristic vectors

$$\bar{a}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{a}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

and the corresponding solutions $\bar{a}^{(1)} e^{2t}$ and $\bar{a}^{(2)} e^{2t}$, that is

$$\begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{2t} \\ 0 \\ 2e^{2t} \end{pmatrix}. \qquad (*)$$

A third solution corresponding to $\lambda = 2$ is of the form $(\bar{a}t + \bar{\beta})e^{2t}$, where $\bar{a}$ satisfies $(\bar{A} - 2\bar{I})\bar{a} = \bar{0}$ and $\bar{\beta}$ satisfies $(\bar{A} - 2\bar{I})\bar{\beta} = \bar{a}$. We apply this last equation with

$$\bar{a} = k_1\bar{a}^{(1)} + k_2\bar{a}^{(2)} = k_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 \\ 2k_1 \\ 2k_2 \end{pmatrix}.$$

Thus we have

$$\begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{pmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ 2k_1 \\ 2k_2 \end{bmatrix}$$

Multiplying this out yields

$$\begin{aligned} 2\beta_1 - \beta_2 - \beta_3 &= k_1 + k_2, \\ 2\beta_1 - \beta_2 - \beta_3 &= 2k_1, \qquad (**) \\ 2\beta_1 - \beta_2 - \beta_3 &= 2k_2. \end{aligned}$$

From these $k_1 + k_2 = 2k_1 = 2k_2$, a nontrivial solution of which is $k_1 = k_2 = 1$. Thus

$$\bar{\alpha} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

and relations (**) each become $2\beta_1 - \beta_2 - \beta_3 = 2$. A solution of this is $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$; and we obtain

$$\bar{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the third solution $(\bar{\alpha}t + \bar{\beta})e^{2t}$ is

$$\left(\begin{bmatrix}2\\2\\2\end{bmatrix}t + \begin{bmatrix}1\\0\\0\end{bmatrix}\right)e^{2t}, \quad \text{that is,} \quad \begin{bmatrix}(2t+1)e^{2t}\\ 2te^{2t}\\ 2te^{2t}\end{bmatrix}.$$

This solution and the two solutions (*) are linearly independent, and a G.S. is the linear combination

$$\bar{x} = c_1\begin{bmatrix}e^{2t}\\2e^{2t}\\0\end{bmatrix} + c_2\begin{bmatrix}e^{2t}\\0\\2e^{2t}\end{bmatrix} + c_3\begin{bmatrix}(2t+1)e^{2t}\\ 2te^{2t}\\ 2te^{2t}\end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{aligned} x_1 &= c_1e^{2t} + c_2e^{2t} + c_3(2t+1)e^{2t},\\ x_2 &= 2c_1e^{2t} + 2c_3te^{2t},\\ x_3 &= 2c_2e^{2t} + 2c_3te^{2t}. \end{aligned}$$

28. We assume a solution of the form $\bar{x} = \bar{a}e^{\lambda t}$, that is, $x_1 = a_1e^{\lambda t}$, $x_2 = a_2e^{\lambda t}$, $x_3 = a_3e^{\lambda t}$. We know that $\lambda$ must be a solution of the characteristic equation of the coefficient matrix

$$\bar{A} = \begin{pmatrix}4 & 6 & -1\\ -1 & -2 & 1\\ -2 & -8 & 4\end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 4-\lambda & 6 & -1 \\ -1 & -2-\lambda & 1 \\ -2 & -8 & 4-\lambda \end{vmatrix} = 0.$$

It reduces to $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$, that is, $(\lambda - 2)^3 = 0$. Its roots are the characteristic values $\lambda_1 = \lambda_2 = \lambda_3 = 2$.

We use $\bar{A}\,\bar{a} = \lambda\bar{a}$ to find the corresponding characteristic vectors. With $\lambda = 2$ this is

$$\begin{pmatrix} 4 & 6 & -1 \\ -1 & -2 & 1 \\ -2 & -8 & 4 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

The resulting relationships in $a_1$, $a_2$, $a_3$ are

$$\begin{aligned} 2a_1 + 6a_2 - a_3 &= 0, \\ -a_1 - 4a_2 + a_3 &= 0, \\ -2a_1 - 8a_2 + 2a_3 &= 0. \end{aligned}$$

We find that $a_1 = 2$, $a_2 = -1$, $a_3 = -2$ is a nontrivial solution of these. Thus we obtain the characteristic vector

$$\bar{a} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

and the corresponding solution $\bar{\alpha}e^{2t}$, that is,

$$\begin{bmatrix} 2e^{2t} \\ -e^{2t} \\ -2e^{2t} \end{bmatrix}. \qquad (*)$$

A second solution corresponding to $\lambda = 2$ is of the form $(\bar{\alpha}t + \bar{\beta})e^{2t}$, where $\bar{\beta}$ satisfies $(\bar{A} - 2\bar{I})\beta = \bar{\alpha}$. This is

$$\begin{pmatrix} 2 & 6 & -1 \\ -1 & -4 & 1 \\ -2 & -8 & 2 \end{pmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.$$

Multiplying this out yields

$$\begin{aligned} 2\beta_1 + 6\beta_2 - \beta_3 &= 2, \\ -\beta_1 - 4\beta_2 + \beta_3 &= -1, \\ -2\beta_1 - 8\beta_2 + 2\beta_3 &= -2, \end{aligned}$$

a solution of which is $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$. Thus we obtain

$$\bar{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and the second solution $(\bar{\alpha}t + \bar{\beta})e^{2t}$ is

$$\left[\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right] e^{2t}, \quad \text{that is,} \quad \begin{bmatrix} (2t+1)e^{2t} \\ -t\,e^{2t} \\ -2t\,e^{2t} \end{bmatrix}. \qquad (**)$$

A third solution if of the form $\left(\frac{\bar{\alpha}t^2}{2} + \bar{\beta}t + \bar{\gamma}\right)e^{2t}$, where $\bar{\gamma}$ satisfies $(\bar{A} - 2\bar{I})\bar{\gamma} = \bar{\beta}$. This is

$$\begin{pmatrix} 2 & 6 & -1 \\ -1 & -4 & 1 \\ -2 & -8 & 2 \end{pmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Multiplying this out yields

$$\begin{aligned} 2\gamma_1 + 6\gamma_2 - \gamma_3 &= 1, \\ -\gamma_1 - 4\gamma_2 + \gamma_3 &= 0, \\ -2\gamma_1 - 8\gamma_2 + 2\gamma_3 &= 0. \end{aligned}$$

A solution of this is $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 1$; and we obtain

$$\bar{\gamma} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus the third solution $\left(\frac{\bar{\alpha}t^2}{2} + \bar{\beta}t + \bar{\gamma}\right)e^{2t}$ is

$$\left(\begin{bmatrix}2\\-1\\-2\end{bmatrix}\frac{t^2}{2} + \begin{bmatrix}1\\0\\0\end{bmatrix}t + \begin{bmatrix}1\\0\\1\end{bmatrix}\right)e^{2t}, \quad \text{that is,} \quad \begin{bmatrix}(t^2 + t + 1)e^{2t}\\(-1/2)t^2e^{2t}\\(-t^2 + 1)e^{2t}\end{bmatrix}.$$

This and the two solutions (*) and (**) are linearly independent, and a general solution is the linear combination

$$\bar{x} = c_1\begin{bmatrix}2e^{2t}\\-e^{2t}\\-2e^{2t}\end{bmatrix} + c_2\begin{bmatrix}(2t + 1)e^{2t}\\-te^{2t}\\-2te^{2t}\end{bmatrix} + c_3\begin{bmatrix}(t^2 + t + 1)e^{2t}\\(-1/2)t^2e^{2t}\\(-t^2 + 1)e^{2t}\end{bmatrix},$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants. In component form, this is

$$\begin{aligned}x_1 &= 2c_1e^{2t} + c_2(2t + 1)e^{2t} + c_3(t^2 + t + 1)e^{2t},\\x_2 &= -c_1e^{2t} - c_2te^{2t} - (c_3/2)t^2e^{2t},\\x_3 &= -2c_1e^{2t} - 2c_2te^{2t} + c_3(-t^2 + 1)e^{2t}.\end{aligned}$$

# Chapter 8

**Section 8.1B, Page 427**

**Exercise 2.**

We separate the solution into four graphs. The differential equation $y' = xy$ determines the family of isoclines $c = xy$; these are the hyperbolas $y = c/x$. Our first graph is the result of drawing these hyperbolas for $c = \pm 1/10, \pm 3/10, \pm 1/2, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8$, and $\pm 10$. We note that if $c = 0$, the corresponding isoclines are the x and y axes. See figure 8.1B-2A.

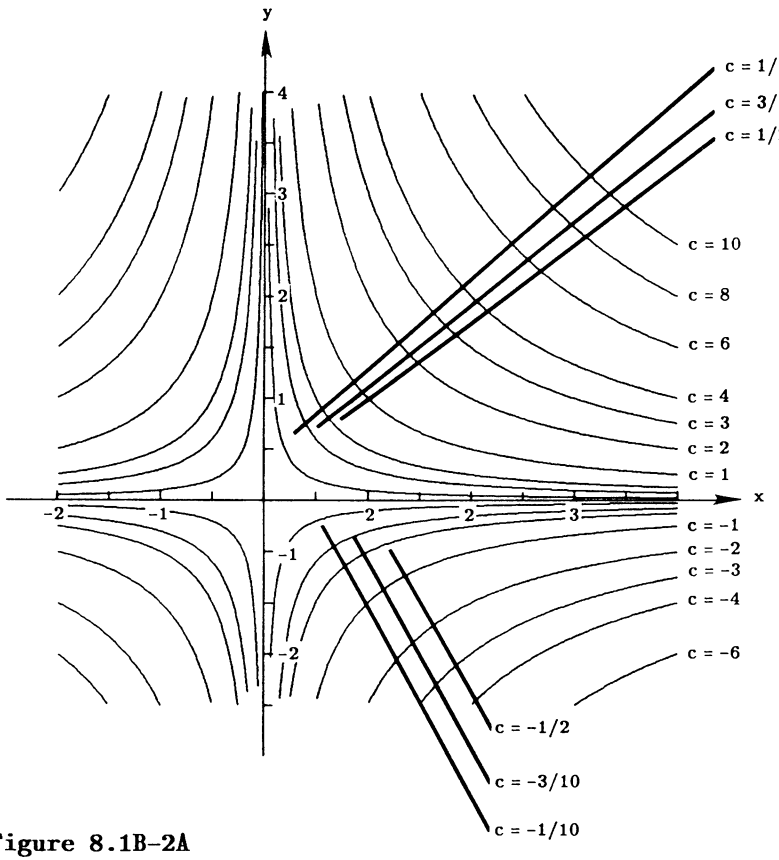


**Figure 8.1B-2A**

Now, on each of the isoclines drawn in our first graph, we draw several line elements having the appropriate slope, or, angle of inclination. The slope of each line element on the fixed isocline $y = c_0/x$ is $c_0$; the corresponding angle of inclination is then $\alpha_0 = \arctan c_0$. We obtain the graph found in Figure 8.1B-2B.

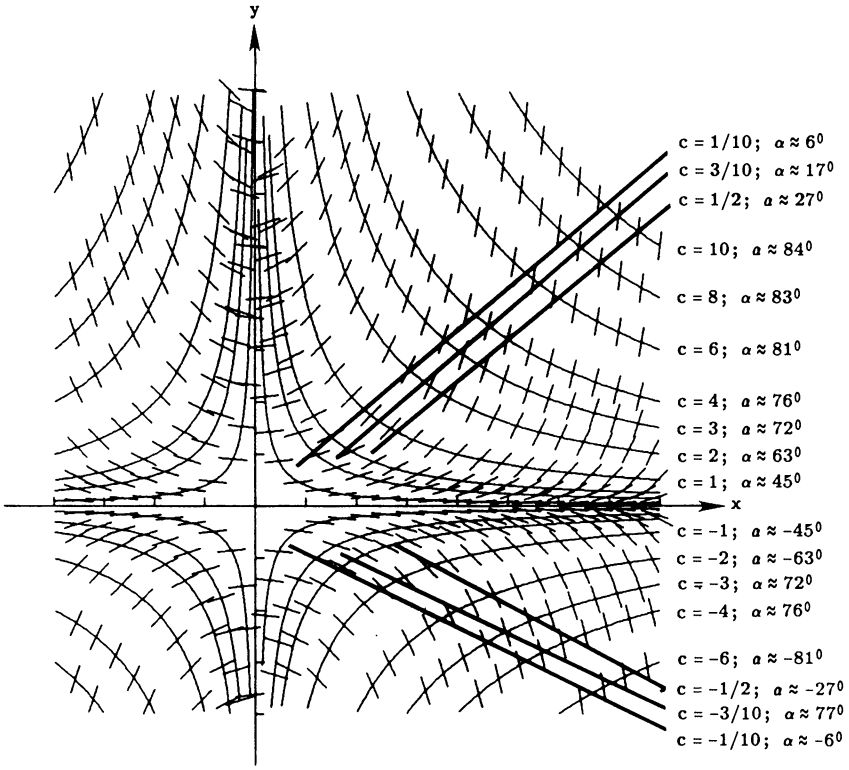


**Figure 8.1B-2B**

The line element configuration seen in the second graph indicates how to draw the approximate integral curves. We sketch several. In particular, we show those curves which have y intercepts -1, -0.1, -0.01, -0.001, 1/16, 1/8, 1/4, 1/2, 1, and 2, respectively. See Figure 8.1B-2C.

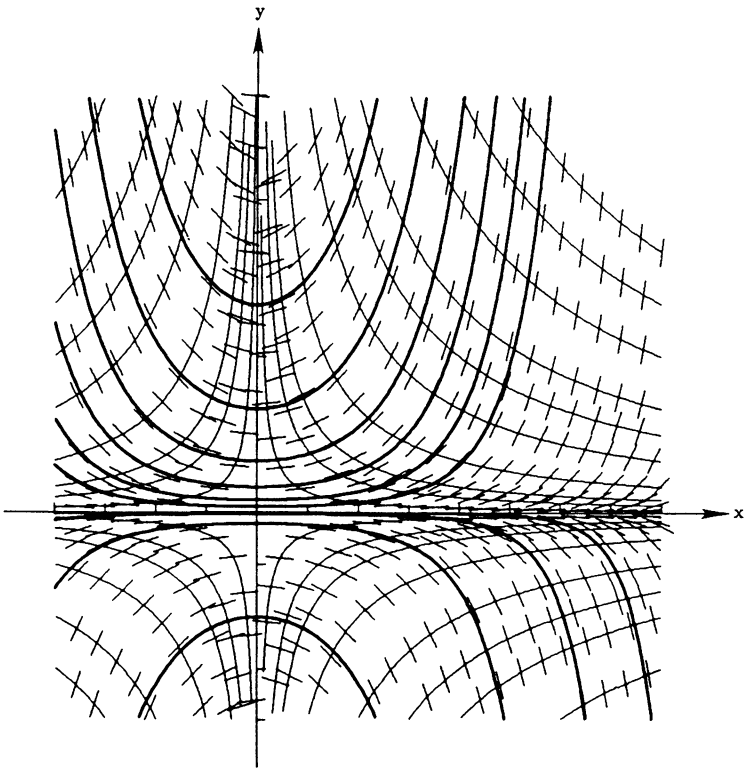


**Figure 8.1B-2C**

Our last graph just shows how the integral curves appear without the background of isoclines and the corresponding line elements. See Figure 8.1B-2D.

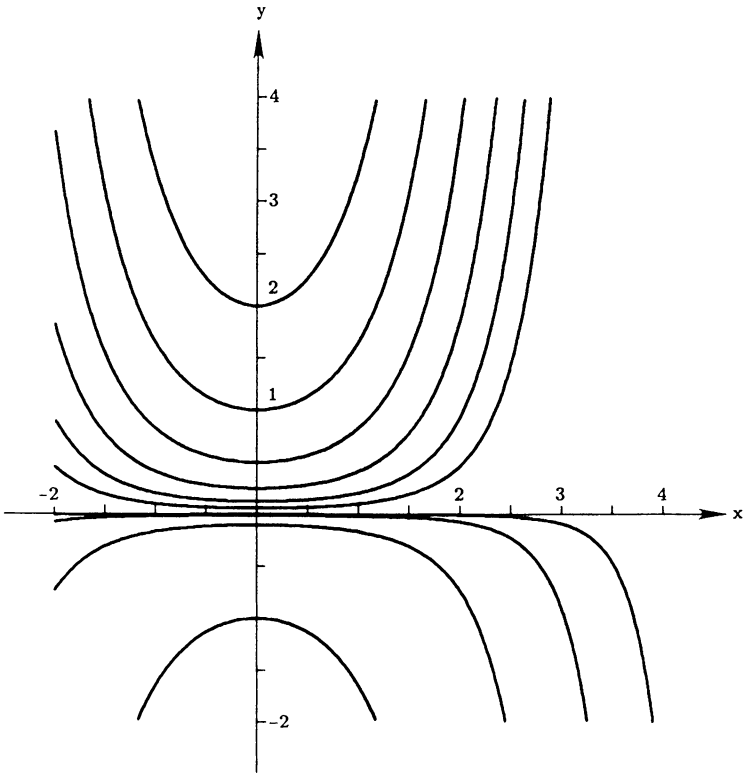


Figure 8.1B-2D

**Exercise 11.**

We separate the solution into four graphs. The differential equation $y' = y \sin x$ determines the family of isoclines $c = y \sin x$, or equivalently, $y = c \csc x$. Our first graph is the result of drawing these isoclines for $c = \pm 1/10, \pm 1/2, \pm 1, \pm 2, \pm 3, \pm 4$, and $-5$. See Figure 8.1B-11A.

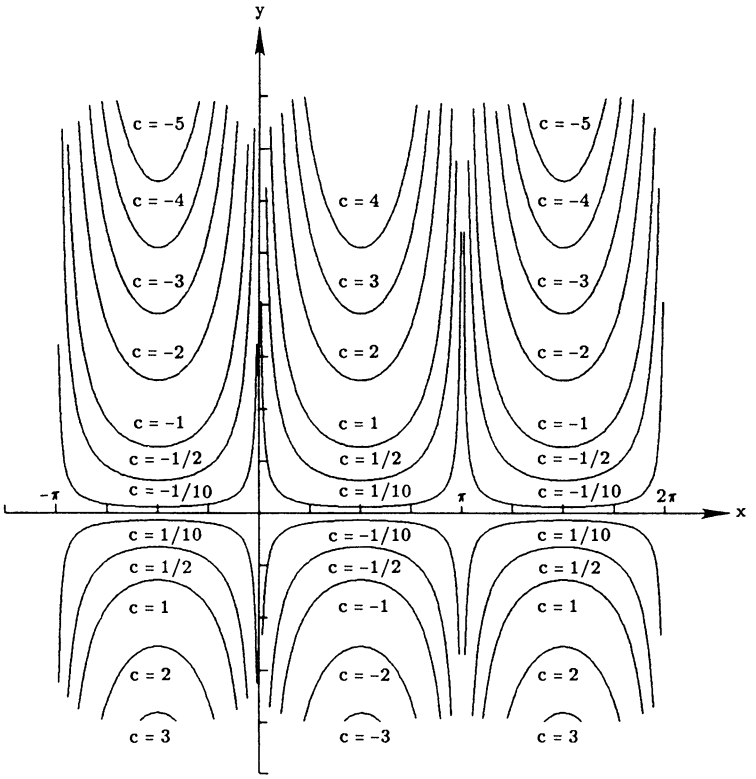


**Figure 8.1B-11A**

Now, on each of the isoclines drawn in our first graph, we draw several line elements having the appropriate slope, or, angle of inclination. The slope of each line element on the fixed isocline $y = c_0 \csc x$ is $c_0$; the corresponding angle of inclination is then $\alpha_0 = \arctan c_0$. We obtain the graph found in Figure 8.1B-11B.

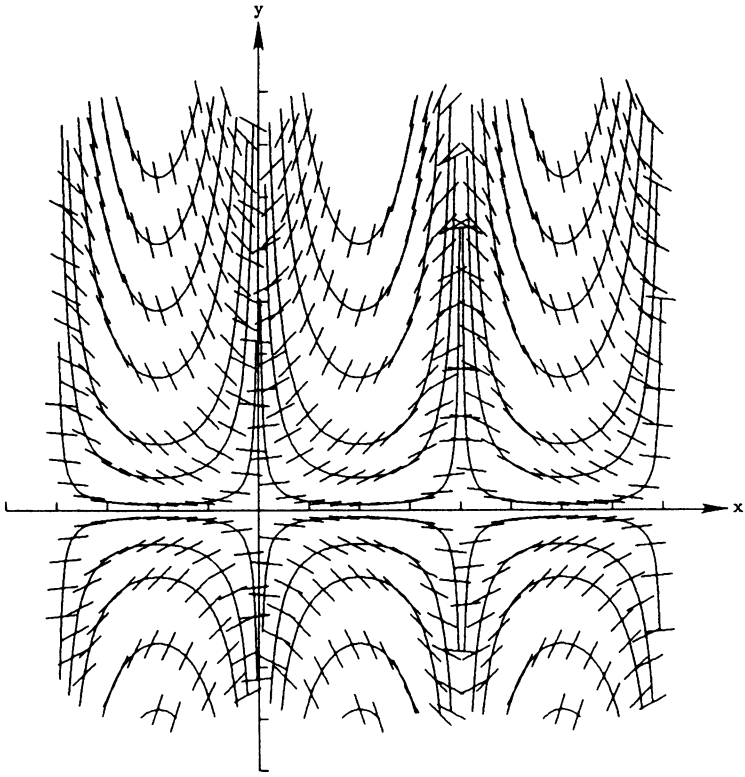


Figure 8.1B-11B

The line element configuration seen in the second graph indicates how to draw the approximate integral curves. We sketch several, including one whose y intercept is $\pi$. See Figure 8.1B-11C.

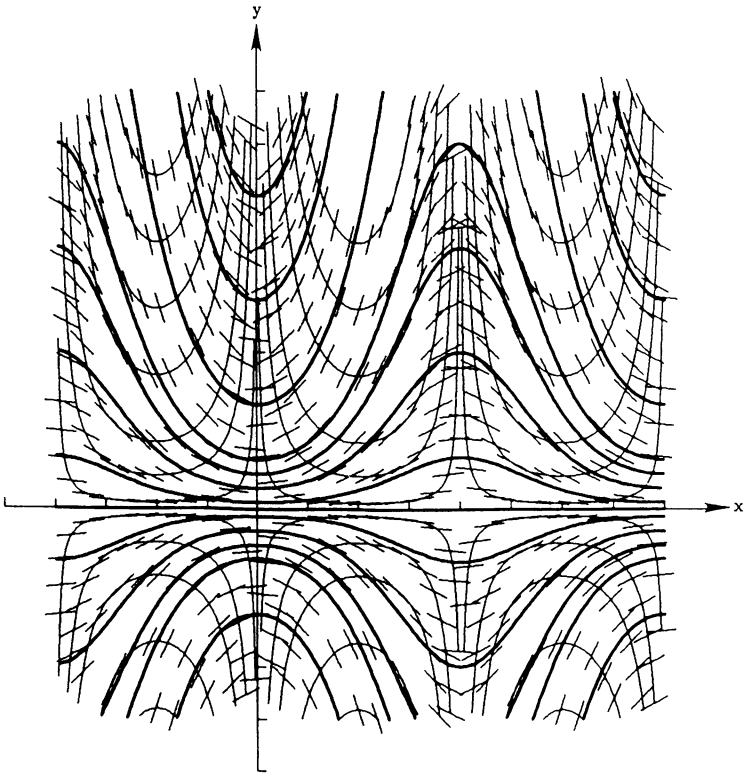


**Figure 8.1B-11C**

Our last graph just shows how the integral curves appear without the background of isoclines and the corresponding line elements. See Figure 8.1B-11D.

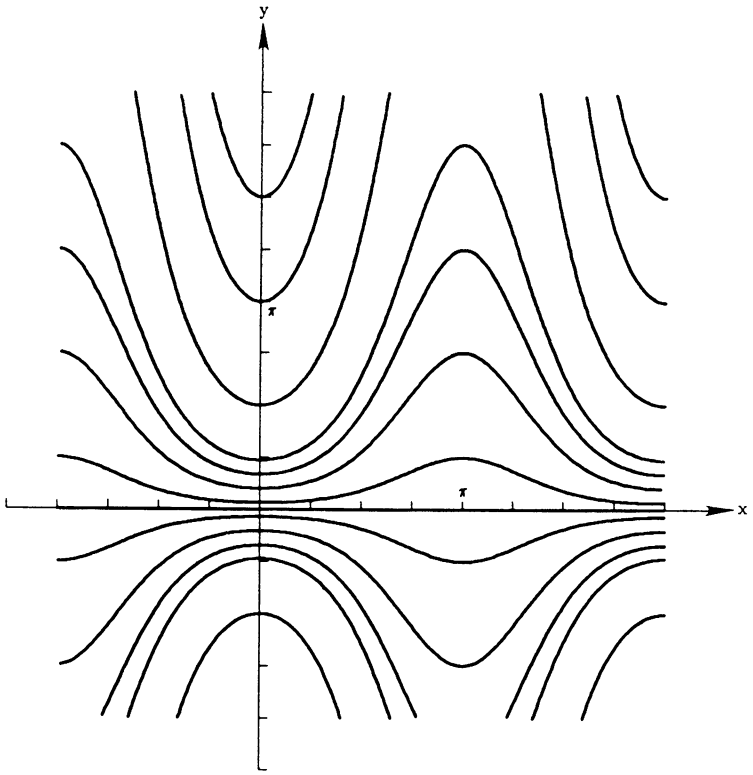


Figure 8.1B-11D

**Section 8.2, Page 434.**

3. a. We have

$$y(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y'''(0)\frac{x^3}{3!} + y^{iv}(0)\frac{x^4}{4!} + \cdots. \qquad (1)$$

The I.C. states $y(0) = 2$, and the D.E. gives $y'(0) = 1 + (0)(2)^2 = 1$. Differentiating the D.E., we obtain

$$y'' = 2xyy' + y^2, \qquad (2)$$

$$y''' = 2xyy'' + 2x(y')^2 + 4yy', \qquad (3)$$

$$y^{iv} = 2xyy''' + 6xy'y'' + 6yy'' + 6(y')^2. \qquad (4)$$

Substituting $x = 0$, $y = 2$, $y' = 1$ into (2), we obtain

$$y''(0) = 2(0)(2)(1) + (2)^2 = 4.$$

Substituting $x = 0$, $y = 2$, $y' = 1$, $y'' = 4$ into (3), we obtain

$$y'''(0) = (2)(0)(2)(4) + (2)(0)(1)^2 + (4)(2)(1) = 8.$$

Substituting $x = 0$, $y = 2$, $y' = 1$, $y'' = 4$, $y''' = 8$ into (4), we obtain

$$y^{iv}(0) = (2)(0)(1)(8) + (6)(0)(1)(4) + (6)(2)(4) + (6)(1)^2 = 54.$$

Now substituting the values of $y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$, and $y^{iv}(0)$ so determined into (1), we obtain

$$y = 2 + 1x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{54x^4}{4!} + \cdots$$

$$= 2 + x + 2x^2 + \frac{4x^3}{3} + \frac{9x^4}{4} + \cdots.$$

b. We assume

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots.$$

To satisfy the I.C. $y(0) = 2$, we must have $c_0 = 2$, and hence

$$y = 2 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots. \qquad (5)$$

Differentiating this, we find

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots. \qquad (6)$$

Since the initial y value is 2, we must express $y^2$ in the D.E. in powers of $y - 2$. Then the D.E. takes the form

$$y' = 1 + x[(y - 2)^2 + 4(y - 2) + 4].$$

Now substituting (5) and (6) into this, we obtain

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots$$
$$= 1 + x[(c_1x + c_2x^2 + \cdots)^2$$
$$+ 4(c_1x + c_2x^2 + c_3x^3 + \cdots) + 4]$$

or

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots$$
$$= 1 + 4x + 4c_1x^2 + (c_1^2 + 4c_2)x^3 + \cdots$$

From this, $c_1 = 1$, $2c_2 = 4$, $3c_3 = 4c_1$, $4c_4 = c_1^2 + 4c_2$; and from these, $c_1 = 1$, $c_2 = 2$, $c_3 = \frac{4c_1}{3} = \frac{4}{3}$, $c_4 = \frac{c_1^2 + 4c_2}{4} = \frac{9}{4}$. Substituting these into (5) we again obtain

$$y = 2 + x + 2x^2 + \frac{4x^3}{3} + \frac{9x^4}{4} + \cdots.$$

4. a. We have

$$y(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y'''(0)\frac{x^3}{3!}$$
$$+ y^{iv}(0)\frac{x^4}{4!} + \cdots. \qquad (1)$$

The I.C. states that $y(0) = 3$, and the D.E. gives $y'(0) = (0)^3 + (3)^3 = 27$. Differentiating the D.E., we obtain

$$y'' = 3x^2 + 3y^2y', \tag{2}$$

$$y''' = 6x + 3y^2y'' + 6y(y')^2, \tag{3}$$

$$y^{iv} = 6 + 3y^2y''' + 18yy'y'' + 6(y')^3 \tag{4}$$

Substituting $x = 0$, $y = 3$, $y' = 27$ into (1), we obtain

$$y''(0) = 3(0)^2 + 3(3)^2(27) = 729.$$

Substituting $x = 0$, $y = 3$, $y' = 27$, $y'' = 729$ into (3), we obtain

$$y'''(0) = 6(0) + 3(3)^2(729) + 6(3)(27)^2 = 32,805.$$

Substituting $y = 3$, $y' = 27$, $y'' = 729$, $y''' = 32,805$ into (4), we obtain

$$y^{iv}(0) = 6 + 3(3)^2(32,805) + (18)(3)(27)(729) + 6(27)^3 = 2,066,721.$$

Now substituting the values of $y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$ and $y^{iv}(0)$ so determined into (1), we obtain

$$y = 3 + 27x + \frac{729x^2}{2} + \frac{32,805x^3}{6} + \frac{2,066,721x^4}{24} + \cdots$$

$$= 3 + 27x + \frac{729x^2}{2} + \frac{10,935x^3}{2} + \frac{688,907x^4}{8} + \cdots$$

b. We assume $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$. To satisfy the I.C. $y(0) = 3$, we must have $c_0 = 3$, and hence

$$y = 3 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots \qquad (5)$$

Differentiating this, we find

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots. \qquad (6)$$

Since the initial y value is 3, we must express $y^3$ in the D.E. in powers of $y - 3$. Then the D.E. takes the form

$$y' = x^3 + (y - 3)^3 + 9(y - 3)^2 + 27(y - 3) + 27.$$

Now substituting (5) and (6) into this, we obtain

$$\begin{aligned} &c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots \\ &\quad = x^3 + (c_1x + c_2x^2 + \cdots)^3 + 9(c_1x + c_2x^2 + \cdots)^2 \\ &\quad + 27(c_1x + c_2x^2 + c_3x^3 + \cdots) + 27 \end{aligned}$$

or

$$\begin{aligned} &c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots \\ &\quad = 27 + 27c_1x + (9c_1^2 + 27c_2)x^2 \\ &\quad + (1 + c_1^3 + 18c_1c_2 + 27c_3)x^3 + \cdots. \end{aligned}$$

From this, $c_1 = 27$, $2c_2 = 27c_1$, $3c_3 = 9c_1^2 + 27c_2$,

$4c_4 = 1 + c_1^3 + 18c_1c_2 + 27c_3$; and from these, $c_1 = 27$,

$$c_2 = \frac{27c_1}{2} = \frac{729}{2},\quad c_3 = 3c_1^2 + 9c_2 = 3(27)^2 + 9\,\frac{729}{2} = \frac{10,935}{2},$$

$$c_4 = \frac{1 + c_1^3 + 18c_1c_2 + 27c_3}{4}$$

$$= \frac{1 + 19,683 + 177,147 + \dfrac{295,245}{2}}{4} = \frac{688,907}{8}.$$

Substituting these into (5), we again obtain

$$y = 3 + 27x + \frac{729x^2}{2} + \frac{10,935x^3}{2} + \frac{688,907x^4}{8} + \cdots.$$

5. a. We have

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \frac{y^{v}(0)}{5!}x^5 + \cdots.$$

The I.C. states $y(0) = 0$, and the D.E. gives $y'(0) = 0 + \sin 0 = 0$. Differentiating the D.E., we obtain

$$y'' = 1 + (\cos y)y',$$

$$y''' = (\cos y)y'' - (\sin y)(y')^2.$$

$$y^{iv} = (\cos y)y''' - 3(\sin y)y'y'' - (\cos y)(y')^3,$$

$$y^{v} = (\cos y)y^{iv} - 4(\sin y)y'y''' - 3(\sin y)(y'')^2 - 6(\cos y)(y')^2y'' + (\sin y)(y')^4.$$

Substituting $x = 0$, $y = 0$, $y' = 0$ into these, we find

$$y''(0) = 1, \quad y'''(0) = 1, \quad y^{iv}(0) = 1, \quad y^{v}(0) = 1.$$

Thus the assumed Taylor Series solution is

$$y = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots.$$

b. We assume

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots.$$

The I.C. $y(0) = 0$ gives $c_0 = 0$, so

$$y = c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots. \qquad (1)$$

Differentiating, we find

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots. \qquad (2)$$

Also, the Maclaurin Series for $\sin y$ is

$$\sin y = y - \frac{1}{6}y^3 + \cdots.$$

Thus the D.E. takes the form

$$y' = x + y - \frac{1}{6}y^3 + \cdots.$$

Now substituting (1) and (2) into this, we obtain

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$$

$$= x + [c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots]$$

$$- \frac{1}{6}[c_1^3x^3 + 3c_1^2c_2x^4 + \cdots] + \cdots.$$

or

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$$

$$= (1 + c_1)x + c_2x^2 + (c_3 - \frac{1}{6}c_1^3)x^3$$

$$+ (c_4 - \frac{1}{2}c_1^2c_2)x^4 + \cdots.$$

From this, $c_1 = 0$, $2c_2 = 1 + c_1$, $3c_3 = c_2$,

$4c_4 = c_3 - \frac{1}{6}c_1^3$, $5c_5 = c_4 - \frac{1}{2}c_1^2c_2$, $\cdots$.

and from these, $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{3}c_2 = \frac{1}{6}$,

$c_4 = \frac{1}{4}c_3 = \frac{1}{24}$; $c_5 = \frac{1}{5}c_4 = \frac{1}{120}$.

Substituting this into (1), we again obtain

$$y = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots.$$

8. a. We have

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \cdots.$$

The I.C. states that $y(0) = 1$, and the D.E. gives $y'(0) = 1$. Differentiating the D.E., we obtain

$$y'' = 4x^3 + 4y^3y',$$

$$y''' = 12x^2 + 4y^3y'' + 12y^2(y')^2,$$

$$y^{iv} = 24x + 4y^3y''' + 36y^2y'y'' + 24y(y')^3.$$

Substituting $x = 0$, $y = 1$, $y' = 1$ into these, we find

$$y''(0) = 4, \qquad y'''(0) = 28, \qquad y^{iv}(0) = 280.$$

Thus the assumed Taylor Series is

$$y = 1 + x + 2x^2 + \frac{14}{3}x^3 + \frac{35}{3}x^4 + \cdots.$$

b. We assume

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots.$$

The I.C. $y(0) = 1$ gives $c_0 = 1$, so

$$y = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots. \qquad (1)$$

Differentiating we find

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots \qquad (2)$$

Since the initial y value is 1, we express $y^4$ in the D.E. in powers of y - 1. Then the D.E. takes the form

$$y' = x^4 + 1 + 4(y - 1) + 6(y - 1)^2 + 4(y - 1)^3 + (y - 1)^4.$$

Substituting (1) and (2) into this, we obtain

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$$

$$= x^4 + 1 + 4(c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots)$$

$$+ 6[c_1^2x^2 + 2c_1c_2x^3 + (2c_1c_3 + c_2^2)x^4 + \cdots]$$

$$+ 4[c_1^3x^3 + 3c_1^2c_2x^4 + \cdots] + [c_1^4x^4 + \cdots]$$

or

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$$

$$= 1 + 4c_1x + (4c_2 + 6c_1^2)x^2 + (4c_3 + 12c_1c_2 + 4c_1^3)x^3$$

$$+ (1 + 4c_4 + 12c_1c_3 + 6c_2^2 + 12c_1^2c_2 + c_1^4)x^4 + \cdots.$$

From these, $c_1 = 1$, $2c_2 = 4c_1$, $3c_3 = 4c_2 + 6c_1^2$,

$4c_4 = 4c_3 + 12c_1c_2 + 4c_1^3$, $\cdots$;

and from these, $c_1 = 1$, $c_2 = 2c_1 = 2$,

$$c_3 = \frac{4}{3}c_2 + 2c_1^2 = \frac{14}{3},\quad c_4 = c_3 + 3c_1c_2 + c_1^3 + \frac{35}{3},\ \ldots.$$

Substituting these into (1), we again obtain

$$y = 1 + x + 2x^2 + \frac{14}{3}x^3 + \frac{35}{3}x^4 + \cdots.$$

12. a. We have

$$\begin{aligned} y(x) &= y(1) + y'(1)(x - 1) + y''(1)\frac{(x - 1)^2}{2!} \\ &\quad + y'''(1)\frac{(x - 1)^3}{3!} + \cdots \\ &= \sum_{n = 0}^{\infty} y^{(n)}(1)\frac{(x - 1)^n}{n!}. \end{aligned} \tag{1}$$

The I.C. states that $y(1) = \pi$, and the D.E. gives $y'(1) = 1 + \cos\pi = 0$. Differentiating the D.E., we obtain

$$y'' = 1 - (\sin y)y', \tag{2}$$

$$y''' = -(\sin y)y'' - (\cos y)(y')^2, \tag{3}$$

$$\begin{aligned} y^{iv} &= -(\sin y)y''' - 3(\cos y)y''y' \\ &\quad + (\sin y)(y')^2, \end{aligned} \tag{4}$$

$$y^{v} = -(\sin y)y^{iv} - 4(\cos y)y'''y' - 3(\cos y)(y'')^{2} + 3(\sin y)y''(y')^{2} + 2(\sin y)y'y'' + (\cos y)(y')^{3}. \tag{5}$$

Substituting $x = 1$, $y = \pi$, $y' = 0$ into (2), we obtain $y''(1) = 1 - (\sin \pi)(0) = 1$.
Substituting $y = \pi$, $y' = 0$, $y'' = 1$ into (3), we obtain $y'''(1) = -(\sin \pi)(1) - (\cos \pi)(0)^{2} = 0$.
Substituting $y = \pi$, $y' = 0$, $y'' = 1$, $y''' = 0$ into (4), we obtain

$$y^{iv}(1) = -(\sin \pi)(0) - (3 \cos \pi)(1)(0) + (\sin \pi)(0)^{2} = 0.$$

Substituting $y = \pi$, $y' = 0$, $y'' = 1$, $y''' = 0$, $y^{iv} = 0$ into (5), we obtain

$$y^{v}(1) = -(\sin \pi)(0) - 4(\cos \pi)(0)(0) - 3(\cos \pi)(1)^{2} + 3(\sin \pi)(1)(0)^{2} + 2(\sin \pi)(0)(1) + (\cos \pi)(0)^{3} = 3.$$

Now substituting the values of $y(1)$, $y'(1)$, $\cdots$, $y^{v}(1)$ so determined into (1), we obtain

$$y = \pi + \frac{(x - 1)^{2}}{2!} + 3\,\frac{(x - 1)^{5}}{5!} + \cdots$$

$$= \pi + \frac{(x - 1)^{2}}{2} + \frac{(x - 1)^{5}}{40} + \cdots$$

(b) We assume

$$y = c_0 + c_1(x - 1) + c_2(x - 1)^2 + c_3(x - 1)^3 + c_4(x - 1)^4 + c_5(x - 1)^5 + \cdots$$

To satisfy the I.C. $y(1) = \pi$, we must have $c_0 = \pi$, and hence

$$y = \pi + c_1(x - 1) + c_2(x - 1)^2 + c_3(x - 1)^3 + c_4(x - 1)^4 + c_5(x - 1)^5 + \cdots. \tag{6}$$

Differentiating this, we find

$$y' = c_1 + 2c_2(x - 1) + 3c_3(x - 1)^2 + 4c_4(x - 1)^3 + 5c_5(x - 1)^4 + \cdots. \tag{7}$$

Since the initial y value is $\pi$, we must express $\cos y$ in the D.E. in powers of $y - \pi$. By Taylor's Theorem,

$$\phi(y) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(\pi)}{n!} (y - \pi)^n$$

with $\phi(y) = \cos y$, we obtain

$$\cos y = -1 + \frac{(y - \pi)^2}{2!} - \frac{(y - \pi)^4}{4!} + \cdots$$

We also express x in powers of x - 1, and then the D.E. takes the form

$$y' = [(x - 1) + 1] + \left[-1 + \frac{(y - \pi)^2}{2!} - \frac{(y - \pi)^4}{4!} + \cdots\right]$$

or

$$y' = x - 1 + \frac{(y - \pi)^2}{2} - \frac{(y - \pi)^4}{24} + \cdots.$$

Now substituting (6) and (7) into this, we obtain

$$\begin{aligned} &c_1 + 2c_2(x - 1) + 3c_3(x - 1)^2 + 4c_4(x - 1)^3 \\ &\qquad + 5c_5(x - 1)^4 + \cdots \\ &= x - 1 + \frac{[c_1(x - 1) + c_2(x - 1)^2 + \cdots]^2}{2} \\ &\qquad - \frac{[c_1(x - 1) + c_2(x - 1)^2 + \cdots]^4}{24} + \cdots \\ &= (x - 1) + \left(\frac{c_1^2}{2}\right)(x - 1)^2 + c_1c_2(x - 1)^3 \\ &\qquad + \left(c_1c_3 + \frac{c_2^2}{2} + \frac{c_1^4}{24}\right)(x - 1)^4 + \cdots. \end{aligned}$$

From this, $c_1 = 0$, $2c_2 = 1$, $3c_3 = \frac{c_1^2}{2}$, $4c_4 = c_1c_2$, $5c_5 = c_1c_3 + \frac{c_2^2}{2} + \frac{c_1^2}{24}$; and from these $c_1 = 0$, $c_2 = \frac{1}{2}$,

$c_3 = 0$, $c_4 = 0$, $c_5 = \dfrac{0 + \frac{1}{8} + 0}{5} = \dfrac{1}{40}$. Substituting these into (6), we again obtain

$$y = \pi + \frac{(x - 1)^2}{2} + \frac{(x - 1)^5}{40} + \cdots.$$

14. (a) We have

$$y(x) = y(1) + y'(1)(x - 1) + \frac{y''(1)}{2!}(x - 1)^2 + \frac{y'''(1)}{3!}(x - 1)^3 + \cdots.$$

The I.C. states that $y(1) = 2$, and the D.E. gives $y'(1) = e + 2$. Differentiating the D.E., we obtain

$$y'' = y' + (x + 1)e^x,$$

$$y''' = y'' + (x + 2)e^x,$$

$$y^{iv} = y''' + (x + 3)e^x.$$

Substituting $x = 1$, $y = 2$, $y' = e + 2$ into these, we find

$$y''(1) = 3e + 2, \quad y'''(1) = 6e + 2, \quad y^{iv}(1) = 10e + 2.$$

Thus the assumed Taylor Series is

$$y = 2 + (e + 2)(x - 1) + \left(\frac{3}{2}e + 1\right)(x - 1)^2 + \left(e + \frac{1}{3}\right)(x - 1)^3 + \left(\frac{5}{12}e + \frac{1}{12}\right)(x - 1)^4 + \cdots.$$

(b) We assume

$$y = c_0 + c_1(x - 1) + c_2(x - 1)^2 + c_3(x - 1)^3 + c_4(x - 1)^4 + \cdots.$$

The I.C. $y(1) = 2$ gives $c_0 = 2$, so

$$y = 2 + c_1(x - 1) + c_2(x - 1)^2 + c_3(x - 1)^3 + c_4(x - 1)^4 + \cdots. \tag{1}$$

Differentiating, we find

$$y' = c_1 + 2c_2(x - 1) + 3c_3(x - 1)^2 + 4c_4(x - 1)^3 + \cdots. \tag{2}$$

We also need to expand $xe^x$ in powers of $x - 1$. We have

$$\begin{aligned} xe^x &= e[1 + (x - 1)]e^{(x - 1)} \\ &= e[1 + (x - 1)]\left[1 + (x - 1) + \frac{1}{2!}(x - 1)^2 + \frac{1}{3!}(x - 1)^3 + \cdots\right]. \end{aligned}$$

Thus the D.E. takes the form

$$y' = y + e[1 + (x - 1)]\left[1 + (x - 1) + \frac{1}{2}(x - 1)^2 + \frac{1}{6}(x - 1)^3 + \cdots\right]$$

Now substituting (1) and (2) into this and multiplying the bracketed x factors together, we obtain

$$c_1 + 2c_2(x - 1) + 3c_3(x - 1)^2 + 4c_4(x - 1)^3 + \cdots$$

$$= 2 + c_1(x - 1) + c_2(x - 1)^2 + c_3(x - 1)^3 + \cdots$$

$$+e + 2e(x - 1) + \frac{3}{2}\,e(x - 1)^2 + \frac{2}{3}\,e(x - 1)^3 + \cdots$$

$$= (e + 2) + (2e + c_1)(x - 1) + \left(\frac{3}{2}\,e + c_2\right)(x - 1)^2$$

$$+ \left(\frac{2}{3}\,e + c_3\right)(x - 1)^3 + \cdots.$$

From this, $c_1 = e + 2$, $2c_2 = 2e + c_1$,

$3c_3 = \frac{3}{2}\,e + c_2$, $4c_4 = \frac{2}{3}\,e + c_3$;

and from these, $c_1 = e + 2$, $c_2 = e + \frac{1}{2}\,c_1 = \frac{3}{2}\,e + 1$,

$c_3 = \frac{1}{2}\,e + \frac{1}{3}\,c_2 = e + \frac{1}{3}$, $c_4 = \frac{1}{6}\,e + \frac{1}{4}\,c_3 = \frac{5}{12}\,e + \frac{1}{12}$.

Substituting these into (1), we again obtain

$$y = 2 + (e + 2)(x - 1) + \left(\frac{3}{2}\,e + 1\right)(x - 1)^2$$

$$+ \left(e + \frac{1}{3}\right)(x - 1)^3 + \left(\frac{5}{12}\,e + \frac{1}{12}\right)(x - 1)^4 + \cdots.$$

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4. Since the initial y value is 0, we choose the zeroth approximation $\phi_0$ to be the function defined by $\phi_0(x) = 0$ for all x. The $n^{th}$ approximation $\phi_n$ for $n \geq 1$ is given by

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f[t, \ \phi_{n-1}(t)] \ dt$$

$$= 0 + \int_0^x \{1 + t[\phi_{n-1}(t)^2\} \ dt, \ n \geq 1.$$

Using this formula for n = 1, 2, 3, $\cdots$, we obtain successively

$$\phi_1(x) = \int_0^x \{1 + t[\phi_0(t)]^2\} \ dt = \int_0^x (1 + 0) \ dt = x,$$

$$\phi_2(x) = \int_0^x \{1 + t[\phi_1(t)]^2\} \ dt = \int_0^x (1 + t^3) \ dt$$

$$= x + \frac{x^4}{4},$$

$$\phi_3(x) = \int_0^x \{1 + t[\phi_2(t)]^2\} \ dt$$

$$= \int_0^x \left[1 + t\left(t + \frac{t^4}{4}\right)^2\right] dt$$

$$= \int_0^x \left(1 + t^3 + \frac{t^6}{2} + \frac{t^9}{16}\right) dt$$

$$= x + \frac{x^4}{4} + \frac{x^7}{14} + \frac{x^{10}}{160}.$$

5. Since the initial y value is 0, we choose the zeroth approximation $\phi_0$ to be the function defined by $\phi_0(x) = 0$ for all x. The $n^{th}$ approximation $\phi_n$ for $n \geq 1$ is given by

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f[t, \phi_{n-1}(t)]\, dt$$

$$= 0 + \int_0^x \{e^t + [\phi_{n-1}(t)]^2\}\, dt, \quad n \geq 1.$$

Using this formula for $n = 1, 2, 3, \cdots$, we obtain successively

$$\phi_1(x) = \int_0^x (e^t + 0)\, dt = e^t \Big|_0^x = e^x - 1.$$

$$\phi_2(x) = \int_0^x [e^t + (e^t - 1)^2]\, dt$$

$$= \int_0^x (e^{2t} - e^t + 1)\, dt$$

$$= \frac{e^{2t}}{2} - e^t + t \Big|_0^x = \frac{e^{2x}}{2} - e^x + x + \frac{1}{2},$$

$$\phi_3(x) = \int_0^x \left[e^t + \left(\frac{e^{2t}}{2} - e^t + t + \frac{1}{2}\right)^2\right] dt$$

$$= \int_0^x \left( \frac{e^{4t}}{4} - e^{3t} + te^{2t} + \frac{3e^{2t}}{2} \right.$$

$$\left. - 2te^t + t^2 + t + \frac{1}{4} \right) dt$$

$$= \left[ \frac{e^{4t}}{16} - \frac{e^{3t}}{3} + e^{2t}\left(\frac{2t - 1}{4}\right) + \frac{3e^{2t}}{4} \right.$$

$$\left. - 2e^t(t - 1) + \frac{t^3}{3} + \frac{t^2}{2} + \frac{t}{4} \right]_0^x$$

$$= \frac{e^{4x}}{16} - \frac{e^{3x}}{3} + \frac{xe^{2x}}{2} + \frac{e^{2x}}{2}$$

$$- 2xe^x + 2e^x + \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4} - \frac{107}{48}.$$

7. As in the solutions of Exercises 4 and 5, we choose $\phi_0(x) = 0$ for all $x$. For $n \geq 1$, $\phi_n(x)$ is given by

$$\phi_n(x) = y_0 + \int_{x_0}^x f[t, \phi_{n-1}(t)]\, dt$$

$$= 0 + \int_0^x \{2t + [\phi_{n-1}(t)]^2\}\, dt.$$

Using this formula for $n = 1, 2, 3, \cdots$, we obtain successively

$$\phi_1(x) = \int_0^x [2t + (0)^3]\, dt = x^2,$$

$$\phi_2(x) = \int_0^x [2t + (t^2)^3]\, dt = x^2 + \frac{x^7}{7},$$

$$\phi_3(x) = \int_0^x \left[2t + \left(t^2 + \frac{t^7}{7}\right)^3\right] dt$$

$$= \int_0^x \left(2t + t^6 + \frac{3t^{11}}{7} + \frac{3t^{16}}{49} + \frac{t^{21}}{343}\right) dt$$

$$= x^2 + \frac{x^7}{7} + \frac{x^{12}}{28} + \frac{3x^{17}}{833} + \frac{x^{22}}{7546}.$$

**NOTE** In the solutions of Sections 8.4 through 8.7, the exact solutions of the problems in Exercises 1, 4, 5, 8, 9, and 12 are required. The D.E.'s in Exercises 1 and 4 are linear (see Section 2.3A), the ones in Exercises 9 and 12 are separable (see Section 2.2A), and the ones in Exercises 5 and 8 are both linear and separable.

We list the exact solution of each of these problems here:

1. $y = \frac{1}{4}(2x - 1 + 5e^{-2x})$.

4. $y = \frac{1}{4}(-2x - 1 + 3e^2e^{2x})$.

5. $y = e^{x^2/2 - 2x + 2}$.

8. $y = \frac{1}{2} e^{1 - \cos x}$.

9. $y = \frac{1}{5} \sqrt{25x^2 + 1}$.

12. $y = \sqrt{3 - 2 \cos x}$.

**Section 8.4, Page 447.**

General Information: For each problem below, we show the first three calculations in detail, while summarizing calculations at each value of $x_n$ in a table. The problems were done maintaining 11 to 12 digit accuracy from step to step; however, we've rounded the results to fewer places (usually four to six) as we tabulate them below.

1. Let $h = 0.2$ and $f(x,y) = x - 2y$ in (8.53); we have $x_0 = 0$ and $y_0 = 1$.

(a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

To find $y_1$, we use (8.53) with $n = 0$: $y_1 = y_0 + hf(x_0, y_0) = 1.0000 + (0.2)f(0.0, 1.0000) = 1.0000 + (0.2)(0.0 - 2(1.0000)) = 0.6000$.

(b) $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$.

To find $y_2$, we use (8.53) with $n = 1$: $y_2 = y_1 + hf(x_1,y_1) = 0.6000 + (0.2)f(0.2, 0.6000) = 0.6000 + (0.2)(0.2 - 2(0.6000)) = 0.4000$.

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$.

To find $y_3$, we use (8.53) with n = 2: $y_3 = y_2 + hf(x_2,y_2) = 0.4000 + (0.2)f(0.4, 0.4000) = 0.4000 + (0.2)(0.4 - 2(0.4000)) = 0.3200$.

Proceeding in this manner, using (8.53) with n = 3 and n = 4, we successively obtain $y_4 = 0.3120$ corresponding to $x_4 = 0.8$, and $y_5 = 0.3472$ corresponding to $x_5 = 1.0$. All results and errors are summarized in Table 8.4.1. The exact solution of the differential equation is given in the pages immediately preceding this section of this manual.

**TABLE 8.4.1** Euler Method for $y' = x - 2y$, $y(0) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | Euler Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 0.687900 | 0.600000 | 0.087900 | 12.778027 |
| 0.4 | 0.511661 | 0.400000 | 0.111661 | 21.823270 |
| 0.6 | 0.426493 | 0.320000 | 0.106493 | 24.969419 |
| 0.8 | 0.402371 | 0.312000 | 0.090371 | 22.459553 |
| 1.0 | 0.419169 | 0.347200 | 0.071969 | 17.169468 |

4. Let h = 0.2 and $f(x,y) = x + 2y$ in (8.53); we have $x_0 = -1$ and $y_0 = 1$.

(a) $x_1 = x_0 + h = -1.0 + 0.2 = -0.8$.

To find $y_1$, we use (8.53) with n = 0: $y_1 = y_0 + hf(x_0,y_0) = 1.0000 + (0.2)f(-1.0, 1.0000) = 1.0000 + (0.2)(-1.0 + 2(1.0000)) = 1.2000$.

(b) $x_2 = x_1 + h = -0.8 + 0.2 = -0.6$.

To find $y_2$, we use (8.53) with n = 1: $y_2 = y_1 + hf(x_1,y_1) = 1.2000 + (0.2)f(-0.8, 1.2000) = 1.2000 + (0.2)(-0.8 + 2(1.2000)) = 1.5200$.

(c) $x_3 = x_2 + h = -0.6 + 0.2 = -0.4$.

To find $y_3$, we use (8.53) with n = 2: $y_3 = y_2 + hf(x_2,y_2) = 1.5200 + (0.2)f(-0.6, 1.5200) = 1.5200 + (0.2)(-0.6 + 2(1.5200)) = 2.0080$.

Proceeding in this manner, using (8.53) with n = 3 we successively obtain $y_4 = 2.7312$ corresponding to $x_4 = -0.2$, and $y_5 = 3.7837$ corresponding to $x_5 = 0.0$. All results and errors are summarized in Table 8.4.4. The exact solution of the differential equation is given in the pages immediately preceding this section of this manual.

**TABLE 8.4.4** Euler Method for $y' = x + 2y$, $y(-1) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | Euler Method | Error | % Rel Error |
|---|---|---|---|---|
| -0.8 | 1.268869 | 1.200000 | 0.068869 | 5.427554 |
| -0.6 | 1.719156 | 1.520000 | 0.199156 | 11.584506 |
| -0.4 | 2.440088 | 2.008000 | 0.432088 | 17.707876 |
| -0.2 | 3.564774 | 2.731200 | 0.833574 | 23.383649 |
| 0.0 | 5.291792 | 3.783680 | 1.508112 | 28.499080 |

5. Let $h = 0.1$ and $f(x,y) = xy - 2y$ in (8.53); we have $x_0 = 2$ and $y_0 = 1$.

(a) $x_1 = x_0 + h = 2.0 + 0.1 = 2.1$.

To find $y_1$, we use (8.53) with $n = 0$: $y_1 = y_0 + hf(x_0,y_0) = 1.0000 + (0.1)f(2.0, 1.0000) = 1.0000 + (0.1)((2.0)(1.0000) - 2(1.0000)) = 1.0000 + (0.1)(0.0000) = 1.0000$.

(b) $x_2 = x_1 + h = 2.1 + 0.1 = 2.2$.

To find $y_2$, we use (8.53) with $n = 1$: $y_2 = y_1 + hf(x_1,y_1) = 1.0000 + (0.1)f(2.1,1.0000) = 1.0000 + (0.1)((2.1)(1.0000) - 2(1.0000)) = 1.0000 + (0.1)(0.1000) = 1.0100$.

(c) $x_3 = x_2 + h = 2.2 + 0.1 = 2.3$.

To find $y_3$, we use (8.53) with $n = 2$: $y_3 = y_2 + hf(x_2,y_2) = 1.0100 + (0.1)f(2.2, 1.0100) = 1.0100 + (0.1)((2.2)(1.0100) - 2(1.0100)) = 1.0100 + (0.1)(0.2020) = 1.0302$.

Proceeding in this manner, using (8.53) with $n = 3$ and $n = 4$, we successively obtain $y_4 = 1.0611$ corresponding to $x_4 = 2.4$, and $y_5 = 1.1036$ corresponding to $x_5 = 2.5$. All results and errors are summarized in Table 8.4.5. The exact solution of the differential equation is given in the pages immediately preceding this section of this manual.

TABLE 8.4.5 Euler Method for $y' = xy - 2y$, $y(2) = 1$, with $h = 0.1$

| $x_n$ | Exact Solution | Euler Method | Error | % Rel Error |
|---|---|---|---|---|
| 2.1 | 1.005013 | 1.000000 | 0.005013 | 0.498752 |
| 2.2 | 1.020201 | 1.010000 | 0.010201 | 0.999934 |
| 2.3 | 1.046028 | 1.030200 | 0.015828 | 1.513139 |
| 2.4 | 1.083287 | 1.061106 | 0.022181 | 2.047571 |
| 2.5 | 1.133148 | 1.103550 | 0.029598 | 2.612033 |

8. Let $h = 0.2$ and $f(x,y) = y \sin x$ in (8.53); we have $x_0 = 0$ and $y_0 = 0.5$.

(a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

To find $y_1$, we use (8.53) with $n = 0$: $y_1 = y_0 + hf(x_0, y_0) = 0.5000 + (0.2)f(0.0, 0.5000) = 0.5000 + (0.2)(0.5000 \sin(0.0)) = 0.5000 + (0.2)(0.0000) = 0.5000$.

(b) $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$.

To find $y_2$, we use (8.53) with $n = 1$: $y_2 = y_1 + hf(x_1,y_1) = 0.5000 + (0.2)f(0.2, 0.5000) = 0.5000 + (0.2)(0.5000 \sin(0.2)) = 0.5000 + (0.2)(0.0993) = 0.5199$.

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$.

To find $y_3$, we use (8.53) with $n = 2$: $y_3 = y_2 + hf(x_2,y_2) = 0.5199 + (0.2)f(0.4, 0.5199) = 0.5199 + (0.2)(0.5199 \sin(0.4)) = 0.5199 + (0.2)(0.2024) = 0.5604$.

Proceeding in this manner, using (8.53) with n = 3 and n = 4, we successively obtain $y_4 = 0.6236$ corresponding to $x_4 = 0.8$, and $y_5 = 0.7131$ corresponding to $x_5 = 1.0$. All results and errors are summarized in Table 8.4.8. The exact solution of the differential equation is given in the pages immediately preceding this section of this manual.

**TABLE 8.4.8** Euler Method for $y' = y \sin x$, $y(0) = 0.5$, with $h = 0.2$.

| $x_n$ | Exact Solution | Euler Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 0.510067 | 0.500000 | 0.010067 | 1.973607 |
| 0.4 | 0.541069 | 0.519867 | 0.021202 | 3.918579 |
| 0.6 | 0.595423 | 0.560356 | 0.035067 | 5.889452 |
| 0.8 | 0.677156 | 0.623636 | 0.053520 | 7.903582 |
| 1.0 | 0.791798 | 0.713110 | 0.078687 | 9.937829 |

9. Let $h = 0.2$ and $f(x,y) = x/y$ in (8.53); we have $x_0 = 0$ and $y_0 = 0.2$.

(a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

To find $y_1$, we use (8.53) with n = 0: $y_1 = y_0 + hf(x_0,y_0) = 0.2000 + (0.2)f(0.0, 0.2000) = 0.2000 + (0.2)(0.0/0.2000) = 0.2000$.

(b) $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$.

To find $y_2$, we use (8.53) with n = 1: $y_2 = y_1 + hf(x_1,y_1) = 0.2000 + (0.2)f(0.2, 0.2000) = 0.2000 + (0.2)(0.2/0.2000) = 0.4000$.

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$.

To find $y_3$, we use (8.53) with $n = 2$: $y_3 = y_2 + hf(x_2,y_2) = 0.4000 + (0.2)f(0.4,0.4000) = 0.4000 + (0.2)(0.4/0.4000) = 0.6000$.

Proceeding in this manner, using (8.53) with $n = 3$, $n = 4$, ..., $n = 9$, we successively obtain $y_4 = 0.8000$ corresponding to $x_4 = 0.8$, $y_5 = 1.000$ corresponding to $x_5 = 1.0$, ..., and finally $y_{10} = 2.0000$ corresponding to $x_{10} = 2.0$. Results for all $x_n$ and the errors are summarized in Table 8.4.9. The exact solution of the differential equation is given in the pages immediately preceding this section of this manual.

**TABLE 8.4.9** Euler Method for $y' = x/y$, $y(0) = 0.2$, with $h = 0.2$.

| $x_n$ | Exact Solution | Euler Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 0.282843 | 0.200000 | 0.082843 | 29.289322 |
| 0.4 | 0.447214 | 0.400000 | 0.047214 | 10.557281 |
| 0.6 | 0.632456 | 0.600000 | 0.032456 | 5.131670 |
| 0.8 | 0.824621 | 0.800000 | 0.024621 | 2.985750 |
| 1.0 | 1.019804 | 1.000000 | 0.019804 | 1.941932 |
| 1.2 | 1.216553 | 1.200000 | 0.016553 | 1.360608 |
| 1.4 | 1.414214 | 1.400000 | 0.014214 | 1.005051 |
| 1.6 | 1.612452 | 1.600000 | 0.012452 | 0.772212 |
| 1.8 | 1.811077 | 1.800000 | 0.011077 | 0.611627 |
| 2.0 | 2.009975 | 2.000000 | 0.009975 | 0.496281 |

12. Let $h = 0.2$ and $f(x,y) = (\sin x)/y$ in (8.53); we have $x_0 = 0$ and $y_0 = 1$.

(a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

To find $y_1$, we use (8.53) with $n = 0$: $y_1 = y_0 + hf(x_0,y_0) = 1.0000 + (0.2)f(0.0, 1.0000) = 1.0000 + (0.2)(\sin(0.0)/1.0000) = 1.0000 + (0.2)(0.0000) = 1.0000$.

(b) $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$.

To find $y_2$, we use (8.53) with $n = 1$: $y_2 = y_1 + hf(x_1,y_1) = 1.0000 + (0.2)f(0.2, 1.0000) = 1.0000 + (0.2)(\sin(0.2)/1.0000) = 1.0000 + (0.2)(0.1987) = 1.0397$.

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$.

To find $y_3$, we use (8.53) with $n = 2$: $y_3 = y_2 + hf(x_2,y_2) = 1.0397 + (0.2)f(0.4, 1.0397) = 1.0397 + (0.2)(\sin(0.4)/1.0397) = 1.0397 + (0.2)(0.3745) = 1.1146$.

Proceeding in this manner, using (8.53) with $n = 3$, $n = 4$, ..., $n = 9$, we successively obtain $y_4 = 1.2160$ corresponding to $x_4 = 0.8$, $y_5 = 1.3339$ corresponding to $x_5 = 1.0$, ..., and $y_{10} = 1.9352$ corresponding to $x_{10} = 2.0$. Results for all $x_n$ and corresponding errors

are summarized in Table 8.4.12. The exact solution of the differential equation is given in the pages immediately preceding this section of this manual.

**TABLE 8.4.12** Euler Method for $y' = (\sin x)/y$, $y(0) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | Euler Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 1.019739 | 1.000000 | 0.019739 | 1.935654 |
| 0.4 | 1.076047 | 1.039734 | 0.036314 | 3.374715 |
| 0.6 | 1.161606 | 1.114641 | 0.046965 | 4.043104 |
| 0.8 | 1.267512 | 1.215955 | 0.051557 | 4.067577 |
| 1.0 | 1.385422 | 1.333946 | 0.051477 | 3.715614 |
| 1.2 | 1.508405 | 1.460108 | 0.048296 | 3.201819 |
| 1.4 | 1.630971 | 1.587775 | 0.043195 | 2.648449 |
| 1.6 | 1.748828 | 1.711905 | 0.036923 | 2.111296 |
| 1.8 | 1.858603 | 1.828684 | 0.029919 | 1.609740 |
| 2.0 | 1.957624 | 1.935192 | 0.022432 | 1.145899 |

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General Information: For each problem below, we show the first three calculations in detail, while summarizing calculations at each value of $x_n$ in a table. The problems were done maintaining 11 to 12 digit accuracy from step to step; however, we've rounded the results to fewer places (usually four to six) as we tabulate them below.

1. Let $h = 0.2$ and $f(x,y) = x - 2y$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 1$.

   (a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

   To find $y_1$, we use (8.59) and (8.60) with $n = 0$.

   First using (8.59), we find

$$\hat{y}_1 = y_0 + hf(x_0,y_0) = 1.0000 + (0.2)f(0.0,\ 1.0000)$$
$$= 1.0000 + (0.2)(-2.0000) = 0.60000.$$

Now using (8.60) with this value of $\hat{y}_1$ we obtain

$$y_1 = y_0 + h\,\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}$$
$$= 1.0000 + (0.2)\,\frac{f(0.0,\ 1.0000) + f(0.2,\ 0.6000)}{2}$$
$$= 1.0000 + (0.2)\,\frac{-2.0000 - 1.0000}{2} = 0.7000.$$

(b) $x_2 = x_2 + h = 0.2 + 0.2 = 0.4.$

To find $y_2$, we use (8.59) and (8.60) with n = 1.

First using (8.59), we find

$$\hat{y}_2 = y_1 + hf(x_1,y_1) = 0.7000 + (0.2)f(0.2,\ 0.7000)$$
$$= 0.7000 + (0.2)(-1.2000) = 0.4600.$$

Now using (8.60) with this value of $\hat{y}_2$ we obtain

$$y_2 = y_1 + h\,\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}$$
$$= 0.7000 + (0.2)\,\frac{f(0.2,\ 0.7000) + f(0.4,\ 0.4600)}{2}$$
$$= 0.7000 + (0.2)\,\frac{-1.2000 - 0.5200}{2} = 0.5280.$$

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6.$

To find $y_3$, we use (8.59) and (8.60) with n = 2.

First using (8.59), we find

$$\hat{y}_3 = y_2 + hf(x_2,y_2) = 0.5280 + (0.2)f(0.4,0.5280)$$
$$= 0.5280 + (0.2)(-0.6560) = 0.3968.$$

Now using (8.60) with this value of $\hat{y}_3$ we obtain

$$y_3 = y_2 + h\,\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}$$

$$= 0.5280 + (0.2)\,\frac{f(0.4,\ 0.5280) + f(0.6,\ 0.3968)}{2}$$

$$= 0.5280 + (0.2)\,\frac{-0.6560 - 0.1936}{2} = 0.4430.$$

These results and those for the remaining $x_n$'s, as well as the errors are summarized in Table 8.5.1. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.5.1** Improved Euler Method for $y' = x - 2y$, $y(0) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | Improved Euler | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 0.687900 | 0.700000 | 0.012100 | 1.7590 |
| 0.4 | 0.511661 | 0.528000 | 0.016339 | 3.1933 |
| 0.6 | 0.426493 | 0.443040 | 0.016547 | 3.8798 |
| 0.8 | 0.402371 | 0.417267 | 0.014897 | 3.7022 |
| 1.0 | 0.419169 | 0.431742 | 0.012573 | 2.9994 |

4. Let $h = 0.2$ and $f(x,y) = x + 2y$ in (8.59) and (8.60); we have $x_0 = -1$ and $y_0 = 1$.

(a) $x_1 = x_0 + h = -1.0 + 0.2 = -0.8$.

To find $y_1$, we use (8.59) and (8.60) with $n = 0$.

First using (8.59), we find

$$\hat{y}_1 = y_0 + hf(x_0,y_0) = 1.0000 + (0.2)f(-1.0, 1.0000)$$
$$= 1.0000 + (0.2)(1.0000) = 1.2000.$$

Now using (8.60) with this value of $\hat{y}_1$ we obtain

$$y_1 = y_0 + h\,\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}$$

$$= 1.0000 + (0.2)\,\frac{f(-1.0, 1.0000) + f(-0.8, 1.2000)}{2}$$

$$= 1.0000 + (0.2)\,\frac{1.0000 + 1.6000}{2} = 1.2600.$$

(b) $x_2 = x_1 + h = -0.8 + 0.2 = -0.6$.

To find $y_2$, we use (8.59) and (8.60) with $n = 1$.

First using (8.59), we find

$$\hat{y}_2 = y_1 + hf(x_1,y_1) = 1.2600 + (0.2)f(-0.8, 1.2600)$$
$$= 1.2600 + (0.2)(1.7200) = 1.6040.$$

Now using (8.60) with this value of $\hat{y}_2$ we obtain

$$y_2 = y_1 + h\,\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}$$

$$= 1.2600 + (0.2)\,\frac{f(-0.8,\ 1.2600) + f(-0.6,\ 1.6040)}{2}$$

$$= 1.2600 + (0.2)\,\frac{1.7200 + 2.6080}{2} = 1.6928.$$

(c) $x_3 = x_2 + h = -0.6 + 0.2 = -0.4.$

To find $y_3$, we use (8.59) and (8.60) with n = 2.

First using (8.59), we find

$$\hat{y}_3 = y_2 + hf(x_2,y_2) = 1.6928 + (0.2)f(-0.6,\ 1.6928)$$

$$= 1.6928 + (0.2)(2.7856) = 2.2499.$$

Now using (8.60) with this value of $\hat{y}_3$ we obtain

$$y_3 = y_2 + h\,\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}$$

$$= 1.6928 + (0.2)\,\frac{f(-0.6,\ 1.6928) + f(-0.4,\ 2.2499)}{2}$$

$$= 1.6928 + (0.2)\,\frac{2.7856 + 4.0998}{2} = 2.3813.$$

These results and those for the remaining $x_n$'s, as well as the errors are summarized in Table 8.5.4. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

TABLE 8.5.4 Improved Euler Method for $y' = x + 2y$, $y(-1) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | Improved Euler | Error | % Rel Error |
|---|---|---|---|---|
| -0.8 | 1.268869 | 1.260000 | 0.008869 | 0.6989 |
| -0.6 | 1.719156 | 1.692800 | 0.026356 | 1.5331 |
| -0.4 | 2.440088 | 2.381344 | 0.058744 | 2.4074 |
| -0.2 | 3.564774 | 3.448389 | 0.116385 | 3.2649 |
| 0.0 | 5.291792 | 5.075616 | 0.216176 | 4.0851 |

5. Let $h = 0.1$ and $f(x,y) = xy - 2y$ in (8.59) and (8.60); we have $x_0 = 2$ and $y_0 = 1$.

(a) $x_1 = x_0 + h = 2.0 + 0.1 = 2.1$.

To find $y_1$, we use (8.59) and (8.60) with $n = 0$.

First using (8.59), we find

$$\hat{y}_1 = y_0 + hf(x_0,y_0) = 1.0000 + (0.1)f(2.0,\ 1.0000)$$
$$= 1.0000 + (0.1)(0.0000) = 1.0000.$$

Now using (8.60) with this value of $\hat{y}_1$ we obtain

$$y_1 = y_0 + h\,\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}$$

$$= 1.0000 + (0.1)\,\frac{f(2.0,\ 1.0000) + f(2.1,\ 1.0000)}{2}$$

$$= 1.0000 + (0.1)\,\frac{0.0000 + 0.1000}{2} = 1.0050.$$

(b) $x_2 = x_1 + h = 2.1 + 0.1 = 2.2$.

To find $y_2$, we use (8.59) and (8.60) with $n = 1$.

First using (8.59), we find

$$\hat{y}_2 = y_1 + hf(x_1,y_1) = 1.0050 + (0.1)f(2.1, 1.0050)$$
$$= 1.0050 + (0.1)(1.1005) = 1.0150.$$

Now using (8.60) with this value of $\hat{y}_2$ we obtain

$$y_2 = y_1 + h\,\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}$$
$$= 1.0050 + (0.1)\,\frac{f(2.1, 1.0050) + f(2.2, 1.0150)}{2}$$
$$= 1.0050 + (0.1)\,\frac{0.1005 + 0.2030}{2} = 1.0202.$$

(c) $x_3 = x_2 + h = 2.2 + 0.1 = 2.3.$

To find $y_3$, we use (8.59) and (8.60) with n = 2.

First using (8.59), we find

$$\hat{y}_3 = y_2 + hf(x_2,y_2) = 1.0202 + (0.1)f(2.2, 1.0202)$$
$$= 1.0202 + (0.1)(0.2040) = 1.0406.$$

Now using (8.60) with this value of $\hat{y}_3$ we obtain

$$y_3 = y_2 + h\,\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}$$
$$= 1.0202 + (0.1)\,\frac{f(2.2, 1.0202) + f(2.3, 1.0406)}{2}$$
$$= 1.0202 + (0.1)\,\frac{0.2040 + 0.3122}{2} = 1.0460.$$

These results and those for the remaining $x_n$'s, as well as the errors are summarized in Table 8.5.5. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

TABLE 8.5.5 Improved Euler Method for $y' = xy - 2y$, $y(2) = 1$, with $h = 0.1$.

| $x_n$ | Exact Solution | Improved Euler | Error | % Rel Error |
|---|---|---|---|---|
| 2.1 | 1.005013 | 1.005000 | 0.000013 | 0.0012 |
| 2.2 | 1.020201 | 1.020175 | 0.000026 | 0.0025 |
| 2.3 | 1.046028 | 1.045986 | 0.000042 | 0.0040 |
| 2.4 | 1.083287 | 1.083223 | 0.000064 | 0.0059 |
| 2.5 | 1.133148 | 1.133051 | 0.000097 | 0.0086 |

8. Let $h = 0.2$ and $f(x,y) = y \sin x$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 0.5$.

(a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

To find $y_1$, we use (8.59) and (8.60) with $n = 0$.

First using (8.59), we find

$$\hat{y}_1 = y_0 + hf(x_0,y_0) = 0.5000 + (0.2)f(0.0, 0.5000)$$
$$= 0.5000 + (0.2)(0.0000) = 0.5000.$$

Now using (8.60) with this value of $\hat{y}_1$ we obtain

$$y_1 = y_0 + h\,\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}$$

$$= 0.5000 + (0.2)\,\frac{f(0.0,\ 0.5000) + f(0.2,\ 0.5000)}{2}$$

$$= 0.5000 + (0.2)\,\frac{0.0000 + 0.0993}{2} = 0.5099.$$

(b) $x_2 = x_1 + h = 0.2 + 0.2 = 0.4.$

To find $y_2$, we use (8.59) and (8.60) with n = 1.

First using (8.59), we find

$$\hat{y}_2 = y_1 + hf(x_1,y_1) = 0.5099 + (0.2)f(0.2,\ 0.5099)$$

$$= 0.5099 + (0.2)(0.1013) = 0.5302.$$

Now using (8.60) with this value of $\hat{y}_2$ we obtain

$$y_2 = y_1 + h\,\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}$$

$$= 0.5099 + (0.2)\,\frac{f(0.2,\ 0.5099) + f(0.4,\ 0.5302)}{2}$$

$$= 0.5099 + (0.2)\,\frac{0.1013 + 0.2065}{2} = 0.5407.$$

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6.$

To find $y_3$, we use (8.59) and (8.60) with n = 2.

First using (8.59), we find

$$\hat{y}_3 = y_2 + hf(x_2,y_2) = 0.5407 + (0.2)f(0.4,\ 0.5407)$$

$$= 0.5407 + (0.2)(0.2106) = 0.5828.$$

Now using (8.60) with this value of $\hat{y}_3$ we obtain

$$y_3 = y_2 + h\,\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}$$

$$= 0.5407 + (0.2)\,\frac{f(0.4,\ 0.5407) + f(0.6,\ 0.5828)}{2}$$

$$= 0.5407 + (0.2)\,\frac{0.2106 + 0.3291}{2} = 0.5947.$$

These results and those for the remaining $x_n$'s, as well as the errors are summarized in Table 8.5.8. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.5.8** Improved Euler Method for $y' = y \sin x$, $y(0) = 0.5$, with $h = 0.2$.

| $x_n$ | Exact Solution | Improved Euler | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 0.510067 | 0.509933 | 0.000133 | 0.0261 |
| 0.4 | 0.541069 | 0.540711 | 0.000358 | 0.0662 |
| 0.6 | 0.595423 | 0.594676 | 0.000747 | 0.1255 |
| 0.8 | 0.677156 | 0.675731 | 0.001425 | 0.2104 |
| 1.0 | 0.791798 | 0.789224 | 0.002574 | 0.3251 |

9. Let $h = 0.2$ and $f(x,y) = x/y$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 0.2$.

(a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

To find $y_1$, we use (8.59) and (8.60) with $n = 0$.

First using (8.59), we find

$$\hat{y}_1 = y_0 + hf(x_0,y_0) = 0.2000 + (0.2)f(0.0,\ 0.2000)$$

$$= 0.2000 + (0.2)(0.0000) = 0.2000.$$

Now using (8.60) with this value of $\hat{y}_1$ we obtain

$$y_1 = y_0 + h\,\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}$$

$$= 0.2000 + (0.2)\,\frac{f(0.0,\ 0.2000) + f(0.2,\ 0.2000)}{2}$$

$$= 0.2000 + (0.2)\,\frac{0.0000 + 1.0000}{2} = 0.3000.$$

(b) $x_2 = x_1 + h = 0.2 + 0.2 = 0.4.$

To find $y_2$, we use (8.59) and (8.60) with n = 1.

First using (8.59), we find

$$\hat{y}_2 = y_1 + hf(x_1,y_1) = 0.3000 + (0.2)f(0.2,\ 0.3000)$$

$$= 0.3000 + (0.2)(0.6667) = 0.4333.$$

Now using (8.60) with this value of $\hat{y}_2$ we obtain

$$y_2 = y_1 + h\,\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}$$

$$= 0.3000 + (0.2)\,\frac{f(0.2,\ 0.3000) + f(0.4,\ 0.4333)}{2}$$

$$= 0.3000 + (0.2)\,\frac{0.6667 + 0.9231}{2} = 0.4590.$$

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6.$

To find $y_3$, we use (8.59) and (8.60) with n = 2.

First using (8.59), we find

$$\hat{y}_3 = y_2 + hf(x_2,y_2) = 0.4590 + (0.2)f(0.4,\ 0.4590)$$

$$= 0.4590 + (0.2)(0.8715) = 0.6333.$$

Now using (8.60) with this value of $\hat{y}_3$ we obtain

$$y_3 = y_2 + h\,\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}$$

$$= 0.4590 + (0.2)\,\frac{f(0.4,\ 0.4590) + f(0.6,\ 0.6333)}{2}$$

$$= 0.4590 + (0.2)\,\frac{0.8715 + 0.9475}{2} = 0.6409.$$

These results and those for the remaining $x_n$'s, as well as the errors are summarized in Table 8.5.9. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.5.9** Improved Euler Method for $y' = x/y$, $y(0) = 0.2$, with $h = 0.2$.

| $x_n$ | Exact Solution | Improved Euler | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 0.282843 | 0.300000 | 0.017157 | 6.0660 |
| 0.4 | 0.447214 | 0.458974 | 0.011761 | 2.6298 |
| 0.6 | 0.632456 | 0.640871 | 0.008415 | 1.3305 |
| 0.8 | 0.824621 | 0.831098 | 0.006477 | 0.7854 |
| 1.0 | 1.019804 | 1.025049 | 0.005245 | 0.5144 |
| 1.2 | 1.216553 | 1.220953 | 0.004401 | 0.3617 |
| 1.4 | 1.414214 | 1.418001 | 0.003787 | 0.2678 |
| 1.6 | 1.612452 | 1.615774 | 0.003323 | 0.2061 |
| 1.8 | 1.811077 | 1.814036 | 0.002959 | 0.1634 |
| 2.0 | 2.009975 | 2.012642 | 0.002667 | 0.1327 |

12. Let $h = 0.2$ and $f(x,y) = (\sin x)/y$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 1$.

(a) $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$.

To find $y_1$, we use (8.59) and (8.60) with $n = 0$.

First using (8.59), we find

$$\hat{y}_1 = y_0 + hf(x_0,y_0) = 1.0000 + (0.2)f(0.0,\ 1.0000)$$

$$= 1.0000 + (0.2)(0.0000) = 1.0000.$$

Now using (8.60) with this value of $\hat{y}_1$ we obtain

$$y_1 = y_0 + h\,\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}$$

$$= 1.0000 + (0.2)\,\frac{f(0.0,\ 1.0000) + f(0.2,\ 1.0000)}{2}$$

$$= 1.0000 + (0.2)\,\frac{0.0000 + 0.1987}{2} = 1.0199.$$

(b) $x_2 = x_1 + h = 0.2 + 0.2 = 0.4.$

To find $y_2$, we use (8.59) and (8.60) with n = 1.

First using (8.59), we find

$$\hat{y}_2 = y_1 + hf(x_1,y_1) = 1.0199 + (0.2)f(0.2,\ 1.0199)$$

$$= 1.0199 + (0.2)(0.1948) = 1.0588.$$

Now using (8.60) with this value of $\hat{y}_2$ we obtain

$$y_2 = y_1 + h\,\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}$$

$$= 1.0199 + (0.2)\,\frac{f(0.2,\ 1.0199) + f(0.4,\ 1.0588)}{2}$$

$$= 1.0199 + (0.2)\,\frac{0.1948 + 0.3678}{2} = 1.0761.$$

(c) $x_3 = x_2 + h = 0.4 + 0.2 = 0.6.$

To find $y_3$, we use (8.59) and (8.60) with n = 2.

First using (8.59), we find

$$\hat{y}_3 = y_2 + hf(x_2,y_2) = 1.0761 + (0.2)f(0.4,\ 1.0761)$$

$$= 1.0761 + (0.2)(0.3619) = 1.1485.$$

Now using (8.60) with this value of $\hat{y}_3$ we obtain

$$y_3 = y_2 + h\,\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}$$

$$= 1.0761 + (0.2)\,\frac{f(0.4,\ 1.0761) + f(0.6,\ 1.1485)}{2}$$

$$= 1.0761 + (0.2)\,\frac{0.3619 + 0.4916}{2} = 1.1615.$$

These results and those for the remaining $x_n$'s, as well as the errors are summarized in Table 8.5.12. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.5.12** Improved Euler Method for $y' = (\sin x)/y$, $y(0) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | Improved Euler | Error | % Rel Error |
|---|---|---|---|---|
| 0.2 | 1.019739 | 1.019867 | 0.000128 | 0.0126 |
| 0.4 | 1.076047 | 1.076125 | 0.000078 | 0.0072 |
| 0.6 | 1.161606 | 1.161476 | 0.000130 | 0.0112 |
| 0.8 | 1.267512 | 1.267082 | 0.000430 | 0.0340 |
| 1.0 | 1.385422 | 1.384659 | 0.000764 | 0.0551 |
| 1.2 | 1.508405 | 1.507310 | 0.001095 | 0.0726 |
| 1.4 | 1.630971 | 1.629565 | 0.001405 | 0.0862 |
| 1.6 | 1.748828 | 1.747140 | 0.001688 | 0.0965 |
| 1.8 | 1.858603 | 1.856666 | 0.001937 | 0.1042 |
| 2.0 | 1.957624 | 1.955473 | 0.002152 | 0.1099 |

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General Information: For each problem below, we show the first two calculations in detail, while summarizing calculations at each value of $x_n$ in a table. The problems were done maintaining 11 to 12 digit accuracy from step to

step; however, we've rounded the results to fewer places as we tabulate them below.

1. Let $h = 0.2$ and $f(x,y) = x - 2y$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 1$.

$$k_1 = hf(x_0,y_0) = 0.2f(0.0,\ 1.0000000) = 0.2(-2.0000000)$$
$$= -0.4000000,$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2) = 0.2f(0.1,\ 0.8000000)$$
$$= 0.2(-1.5000000) = -0.3000000,$$

$$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2) = 0.2f(0.1,\ 0.8500000)$$
$$= 0.2(-1.6000000) = -0.3200000,$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.2f(0.2,\ 0.6800000)$$
$$= 0.2(-1.1600000) = -0.2320000.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (-0.4000000 - 0.6000000 - 0.6400000 - 0.2320000)/6$$
$$= -0.3120000,$$

and so the approximate value of the solution at $x_1 = 0.2$ is

$$y_1 = 1.0000000 - 0.3120000 = 0.6880000.$$

Now using the above value $y_1$, we calculate successively *new* $k_1$, $k_2$, $k_3$, $k_4$, and then K. We first find

$$k_1 = hf(x_1,y_1) = 0.2f(0.2,\ 0.6880000) = 0.2(-1.1760000)$$
$$= -0.2352000,$$

$$k_2 = hf(x_1 + h/2,\ y_1 + k_1/2) = 0.2f(0.3,0.5704000)$$
$$= 0.2(-0.8408000) = -0.1681600,$$

$$k_3 = hf(x_1 + h/2,\ y_1 + k_2/2) = 0.2f(0.3,0.6039200)$$
$$= 0.2(-0.9078400) = -0.1815680,$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3) = 0.2f(0.4,\ 0.5064320)$$
$$= 0.2(-0.6128640) = -0.1225728.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (-0.2352000 - 0.3363200 - 0.3631360 - 0.1225728)/6$$
$$= -0.1762048,$$

and so the approximate value of the solution at $x_2 = 0.4$ is

$$y_2 = 0.6880000 - 0.1762048 = 0.5117952.$$

We summarize the calculations for these and the remaining $y_n$'s in Table 8.6.1. The exact solution of the

differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.6.1** Runge-Kutta Method for x - 2y, y(0) = 1, with h = 0.2.

| $x_n$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $K$ | Exact Solution | Runge-Kutta | Error | % Rel Error |
|---|---|---|---|---|---|---|---|---|---|
| 0.2 | -0.4000 | -0.3000 | -0.3200 | -0.2320 | -0.3120 | 0.687900 | 0.688000 | 0.000100 | 0.014529 |
| 0.4 | -0.2352 | -0.1682 | -0.1816 | -0.1226 | -0.1762 | 0.511661 | 0.511795 | 0.000134 | 0.026188 |
| 0.6 | -0.1247 | -0.0798 | -0.0888 | -0.0492 | -0.0852 | 0.426493 | 0.426628 | 0.000135 | 0.031592 |
| 0.8 | -0.0507 | -0.0205 | -0.0265 | 0.0000 | -0.0241 | 0.402371 | 0.402491 | 0.000120 | 0.029930 |
| 1.0 | -0.0010 | 0.0192 | 0.0152 | 0.0329 | 0.0168 | 0.419169 | 0.419270 | 0.000101 | 0.024075 |

4. Let h = 0.2 and f(x,y) = x + 2y in (8.59) and (8.60); we have $x_0 = -1$ and $y_0 = 1$.

$$k_1 = hf(x_0,y_0) = 0.2f(-1.0,\ 1.0000000)$$
$$= 0.2(1.0000000) = 0.2000000,$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2) = 0.2f(-0.9,\ 1.1000000)$$
$$= 0.2(1.3000000) = 0.2600000,$$

$$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2) = 0.2f(-0.9,\ 1.1300000)$$
$$= 0.2(1.3600000) = 0.2720000,$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.2f(-0.8,\ 1.2720000)$$
$$= 0.2(1.7440000) = 0.3488000.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.2000000 + 0.5200000 + 0.5440000 + 0.3488000)/6$$
$$= 0.2688000,$$

and so the approximate value of the solution at $x_1 = -0.8$ is

$$y_1 = 1.0000000 + 0.2688000 = 1.2688000.$$

Now using the above value $y_1$, we calculate successively *new* $k_1$, $k_2$, $k_3$, $k_4$, and then K. We first find

$$k_1 = hf(x_1, y_1) = 0.2f(-0.8, 1.2688000) = 0.2(1.7376000)$$
$$= 0.3475200,$$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2f(-0.7, 1.4425600)$$
$$= 0.2(2.1851200) = 0.4370240,$$

$$k_3 = hf(x_1 + h/2, y_1 + k_2/2) = 0.2f(-0.7, 1.4873120)$$
$$= 0.2(2.2746240) = 0.4549248,$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(-0.6, 1.7237248)$$
$$= 0.2(2.8474496) = 0.5694899.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.3475200 + 0.8740480 + 0.9098496 + 0.5694899)/6$$
$$= 0.4501513,$$

and so the approximate value of the solution at $x_2 = -0.6$ is

$$y_2 = 1.2688000 + 0.4501513 = 1.7189513.$$

We summarize the calculations for these and the remaining $y_n$'s in Table 8.6.4. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.6.4** Runge-Kutta Method for $y' = x + 2y$, $y(-1) = 1$, with $h = 0.2$.

| $x_n$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $K$ | Exact Solution | Runge-Kutta | Error | % Rel Error |
|---|---|---|---|---|---|---|---|---|---|
| -0.8 | 0.2000 | 0.2600 | 0.2720 | 0.3488 | 0.2688 | 1.268869 | 1.268800 | 0.000069 | 0.005400 |
| -0.6 | 0.3475 | 0.4370 | 0.4549 | 0.5695 | 0.4502 | 1.719156 | 1.718951 | 0.000204 | 0.011892 |
| -0.4 | 0.5676 | 0.7011 | 0.7278 | 0.8987 | 0.7207 | 2.440088 | 2.439630 | 0.000457 | 0.018748 |
| -0.2 | 0.8959 | 1.0950 | 1.1349 | 1.3898 | 1.1242 | 3.564774 | 3.563864 | 0.000910 | 0.025526 |
| 0.0 | 1.3855 | 1.6827 | 1.7421 | 2.1224 | 1.7262 | 5.291792 | 5.290095 | 0.001697 | 0.032064 |

5. Let $h = 0.1$ and $f(x,y) = xy - 2y$ in (8.59) and (8.60); we have $x_0 = 2$ and $y_0 = 1$.

$$k_1 = hf(x_0,y_0) = 0.1f(2.0,\ 1.0000000) = 0.1(0.0000000)$$
$$= 0.0000000,$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2) = 0.1f(2.05,\ 1.00000000)$$
$$= 0.1(0.0500000) = 0.0050000,$$

$$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2) = 0.1f(2.05,\ 1.0025000)$$
$$= 0.1(0.0501250) = 0.0050125,$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.1f(2.10,\ 1.0050125)$$
$$= 0.1(0.1005012) = 0.0100501.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.0000000 + 0.0100000 + 0.0100250 + 0.0100501)/6$$
$$= 0.0050125,$$

and so the approximate value of the solution at $x_1 = 2.1$ is

$$y_1 = 1.0000000 + 0.0050125 = 1.0050125.$$

Now using the above value $y_1$, we calculate successively *new* $k_1$, $k_2$, $k_3$, $k_4$, and then K. We first find

$$k_1 = hf(x_1,y_1) = 0.1f(2.1,\ 1.0050125 = 0.1(0.1005013)$$
$$= 0.0100501,$$

$$k_2 = hf(x_1 + h/2,\ y_1 + k_1/2) = 0.1f(2.15,\ 1.0100376)$$
$$= 0.1(0.1515056) = 0.0151506,$$

$$k_3 = hf(x_1 + h/2,\ y_1 + k_2/2) = 0.1f(2.15,\ 1.0125878)$$
$$= 0.1(0.1518882) = 0.0151888,$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3) = 0.1f(2.20,\ 1.0202013)$$
$$= 0.1(0.2040403) = 0.0204040.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.0100501 + 0.0303011 + 0.0303776 + 0.0204040)/6$$
$$= 0.0151888,$$

and so the approximate value of the solution at $x_2 = 2.2$ is

$$y_2 = 1.0050125 + 0.0151888 = 1.0202013.$$

We summarize the calculations for these and the remaining $y_n$'s in Table 8.6.5. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.6.5** Runge-Kutta Method for $y' = xy - 2y$, $y(2) = 1$, with $h = 0.1$.

| $x_n$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $K$ | Exact Solution | Runge-Kutta | Error | % Rel Error |
|---|---|---|---|---|---|---|---|---|---|
| 2.1 | 0.0000 | 0.0050 | 0.0050 | 0.0101 | 0.0050 | 1.005013 | 1.005013 | 0.000000 | 0.000000 |
| 2.2 | 0.0101 | 0.0152 | 0.0152 | 0.0204 | 0.0152 | 1.020201 | 1.020201 | 0.000000 | 0.000000 |
| 2.3 | 0.0204 | 0.0258 | 0.0258 | 0.0314 | 0.0258 | 1.046028 | 1.046028 | 0.000000 | 0.000000 |
| 2.4 | 0.0314 | 0.0372 | 0.0373 | 0.0433 | 0.0373 | 1.083287 | 1.083287 | 0.000000 | 0.000000 |
| 2.5 | 0.0433 | 0.0497 | 0.0499 | 0.0567 | 0.0499 | 1.133148 | 1.133148 | 0.000000 | 0.000001 |

8. Let $h = 0.2$ and $f(x,y) = y \sin x$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 0.5$.

$$k_1 = hf(x_0,y_0) = 0.2f(0.0,\ 0.5000000) = 0.2(0.0000000)$$
$$= 0.0000000,$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2) = 0.2f(0.1,\ 0.5000000)$$
$$= 0.2(0.0499167) = 0.0099833,$$

$$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2) = 0.2f(0.1,\ 0.5049917)$$
$$= 0.2(0.0504150) = 0.0100830,$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.2f(0.2,\ 0.5100830)$$
$$= 0.2(0.1013379) = 0.0202676.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.0000000 + 0.0199667 + 0.0201660 + 0.0202676)/6$$
$$= 0.0100667,$$

and so the approximate value of the solution at $x_1 = 0.2$ is

$$y_1 = 0.5000000 + 0.0100667 = 0.5100667.$$

Now using the above value $y_1$, we calculate successively *new* $k_1$, $k_2$, $k_3$, $k_4$, and then K. We first find

$$k_1 = hf(x_1,y_1) = 0.2f(0.2, 0.5100667) = 0.2(0.1013346)$$
$$= 0.0202669,$$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2f(0.3, 0.5202002)$$
$$= 0.2(0.1537297) = 0.0307459,$$

$$k_3 = hf(x_1 + h/2, y_1 + k_2/2) = 0.2f(0.3, 0.5254397)$$
$$= 0.2(0.1552780) = 0.0310556,$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 0.5411223)$$
$$= 0.2(0.2107230) = 0.0421446.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.0202669 + 0.0614919 + 0.0621112 + 0.0421446)/6$$
$$= 0.0310024,$$

and so the approximate value of the solution at $x_2 = 0.4$ is

$$y_2 = 0.5100667 + 0.0310024 = 0.5410691.$$

We summarize the calculations for these and the remaining $y_n$'s in Table 8.6.8. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.6.8** Runge-Kutta Method for $y' = y \sin x$, $y(0) = 0.5$, with $h = 0.2$.

| $x_n$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $K$ | Exact Solution | Runge-Kutta | Error | % Rel Error |
|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.0000 | 0.0100 | 0.0101 | 0.0203 | 0.0101 | 0.510067 | 0.510067 | 0.000000 | 0.000000 |
| 0.4 | 0.0203 | 0.0307 | 0.0311 | 0.0421 | 0.0310 | 0.541069 | 0.541069 | 0.000000 | 0.000003 |
| 0.6 | 0.0421 | 0.0539 | 0.0545 | 0.0673 | 0.0544 | 0.595423 | 0.595423 | 0.000000 | 0.000016 |
| 0.8 | 0.0672 | 0.0810 | 0.0819 | 0.0972 | 0.0817 | 0.677156 | 0.677155 | 0.000000 | 0.000055 |
| 1.0 | 0.0972 | 0.1137 | 0.1150 | 0.1333 | 0.1146 | 0.791798 | 0.791796 | 0.000001 | 0.000139 |

9. Let $h = 0.2$ and $f(x,y) = x/y$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 0.2$.

$$k_1 = hf(x_0,y_0) = 0.2f(0.0,\ 0.2000000) = 0.2(0.0000000)$$
$$= 0.0000000,$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2) = 0.2f(0.1,\ 0.2000000)$$
$$= 0.2(0.5000000) = 0.1000000,$$

$$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2) = 0.2f(0.1,\ 0.2500000)$$
$$= 0.2(0.4000000) = 0.0800000,$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.2f(0.2,\ 0.2800000)$$
$$= 0.2(0.7142857) = 0.1428571.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.0000000 + 0.2000000 + 0.1600000 + 0.1428571)/6$$
$$= 0.0838095,$$

and so the approximate value of the solution at $x_1 = 0.2$ is

$$y_2 = 0.2000000 + 0.0838095 = 0.2838095.$$

Now using the above value $y_1$, we calculate successively *new* $k_1$, $k_2$, $k_3$, $k_4$, and then K. We first find

$$k_1 = hf(x_1, y_1) = 0.2f(0.2,\ 0.2838095) = 0.2(0.7046980)$$
$$= 0.1409396,$$

$$k_2 = hf(x_1 + h/2,\ y_1 + k_1/2) = 0.2f(0.3,\ 0.3542793)$$
$$= 0.2(0.8467895) = 0.1693579,$$

$$k_3 = hf(x_1 + h/2,\ y_1 + k_2/2) = 0.2f(0.3,\ 0.3684885)$$
$$= 0.2(0.8141367) = 0.1628273,$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3) = 0.2f(0.4,\ 0.4466369)$$
$$= 0.2(0.8955821) = 0.1791164.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.1409396 + 0.3387158 + 0.3256547 + 0.1791164)/6$$
$$= 0.1640711,$$

and so the approximate value of the solution at $x_2 = 0.4$ is

$$y_2 = 0.2838095 + 0.1640711 = 0.4478806.$$

We summarize the calculations for these and the remaining $y_n$'s in Table 8.6.9. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.6.9** Runge-Kutta Method for $y' = x/y$, $y(0) = 0.2$, with $h = 0.2$.

| $x_n$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $K$ | Exact Solution | Runge-Kutta | Error | % Rel Error |
|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.0000 | 0.1000 | 0.0800 | 0.1429 | 0.0838 | 0.282843 | 0.283810 | 0.000967 | 0.341819 |
| 0.4 | 0.1409 | 0.1694 | 0.1628 | 0.1791 | 0.1641 | 0.447214 | 0.447881 | 0.000667 | 0.149149 |
| 0.6 | 0.1786 | 0.1862 | 0.1849 | 0.1897 | 0.1850 | 0.632456 | 0.632930 | 0.000474 | 0.074943 |
| 0.8 | 0.1896 | 0.1924 | 0.1920 | 0.1940 | 0.1921 | 0.824621 | 0.824985 | 0.000364 | 0.044113 |
| 1.0 | 0.1939 | 0.1952 | 0.1951 | 0.1961 | 0.1951 | 1.019804 | 1.020098 | 0.000294 | 0.028848 |
| 1.2 | 0.1961 | 0.1968 | 0.1967 | 0.1972 | 0.1967 | 1.216553 | 1.216799 | 0.000247 | 0.020273 |
| 1.4 | 0.1972 | 0.1977 | 0.1976 | 0.1980 | 0.1976 | 1.414214 | 1.414426 | 0.000212 | 0.015002 |
| 1.6 | 0.1980 | 0.1982 | 0.1982 | 0.1984 | 0.1982 | 1.612452 | 1.612638 | 0.000186 | 0.011541 |
| 1.8 | 0.1984 | 0.1986 | 0.1986 | 0.1988 | 0.1986 | 1.811077 | 1.811243 | 0.000166 | 0.009148 |
| 2.0 | 0.1988 | 0.1989 | 0.1989 | 0.1990 | 0.1989 | 2.009975 | 2.010124 | 0.000149 | 0.007427 |

12. Let $h = 0.2$ and $f(x,y) = (\sin x)/y$ in (8.59) and (8.60); we have $x_0 = 0$ and $y_0 = 1$.

$$k_1 = hf(x_0,y_0) = 0.2f(0.0,\ 1.0000000) = 0.2(0.0000000)$$
$$= 0.0000000,$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2) = 0.2f(0.1,\ 1.0000000)$$
$$= 0.2(0.0998334) = 0.0199667,$$

$$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2) = 0.2f(0.1,\ 1.0099833)$$
$$= 0.2(0.0988466) = 0.0197693,$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.2f(0.2,\ 1.0197693)$$
$$= 0.2(0.1948179) = 0.0389636.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.0000000 + 0.0399334 + 0.0395386 + 0.0389636)/6$$
$$= 0.0197393,$$

and so the approximate value of the solution at $x_1 = 0.2$ is

$$y_1 = 1.0000000 + 0.0197393 = 1.0197393.$$

Now using the above value $y_1$, we calculate successively *new* $k_1$, $k_2$, $k_3$, $k_4$, and then K. We first find

$$k_1 = hf(x_1,y_1) = 0.2f(0.2, 1.0197393) = 0.2(0.1948237)$$
$$= 0.0389647,$$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2f(0.3, 1.0392216)$$
$$= 0.2(0.2843669) = 0.0568734,$$

$$k_3 = hf(x_1 + h/2, y_1 + k_2/2) = 0.2f(0.3, 1.0481760)$$
$$= 0.2(0.2819376) = 0.0563875,$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.0761268)$$
$$= 0.2(0.3618703) = 0.0723741.$$

Therefore

$$K = (k_1 + 2k_2 + 2k_3 + k_4)/6$$
$$= (0.0389647 + 0.1137467 + 0.1127750 + 0.0723741)/6$$
$$= 0.0563101,$$

and so the approximate value of the solution at $x_2 = 0.4$ is

$$y_2 = 1.0197393 + 0.0563101 = 1.0760494.$$

We summarize the calculations for these and the remaining $y_n$'s in Table 8.6.12. The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.6.12** Runge-Kutta Method for $y' = (\sin x)/y$, $y(0) = 1$, with $h = 0.2$.

| $x_n$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $K$ | Exact Solution | Runge-Kutta | Error | % Rel Error |
|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.0000 | 0.0200 | 0.0198 | 0.0390 | 0.0197 | 1.019739 | 1.019739 | 0.000001 | 0.000064 |
| 0.4 | 0.0390 | 0.0569 | 0.0564 | 0.0724 | 0.0563 | 1.076047 | 1.076049 | 0.000002 | 0.000182 |
| 0.6 | 0.0724 | 0.0862 | 0.0857 | 0.0972 | 0.0856 | 1.161606 | 1.161609 | 0.000003 | 0.000248 |
| 0.8 | 0.0972 | 0.1065 | 0.1061 | 0.1132 | 0.1059 | 1.267512 | 1.267515 | 0.000003 | 0.000251 |
| 1.0 | 0.1132 | 0.1183 | 0.1181 | 0.1215 | 0.1179 | 1.385422 | 1.385426 | 0.000003 | 0.000225 |
| 1.2 | 0.1215 | 0.1233 | 0.1232 | 0.1236 | 0.1230 | 1.508405 | 1.508408 | 0.000003 | 0.000196 |
| 1.4 | 0.1236 | 0.1227 | 0.1228 | 0.1208 | 0.1226 | 1.630971 | 1.630974 | 0.000003 | 0.000173 |
| 1.6 | 0.1208 | 0.1179 | 0.1181 | 0.1143 | 0.1179 | 1.748828 | 1.748831 | 0.000003 | 0.000156 |
| 1.8 | 0.1143 | 0.1098 | 0.1100 | 0.1048 | 0.1098 | 1.858603 | 1.858605 | 0.000003 | 0.000144 |
| 2.0 | 0.1048 | 0.0990 | 0.0992 | 0.0929 | 0.0990 | 1.957624 | 1.957627 | 0.000003 | 0.000137 |

**Section 8.7, Page 468.**

General Information: For each problem below, we show the first two calculations in detail, while summarizing calculations at each value of $x_n$ in a table. The problems were done maintaining 11 to 12 digit accuracy from step to step; however, we've rounded the results to fewer places as we tabulate them below.

1. Let h = 0.2 and f(x,y) = x - 2y in (8.67), (8.68) and (8.69); we have $x_0 = 0$ and $y_0 = 1$.

Before we can begin using the ABAM method, we need to have values for $y_0$, $y_1$, $y_2$, and $y_3$. The first of these is given by the initial condition y(0) = 1.0, while values for the other three are supplied by the text and were found using the Runge-Kutta method. We have

$$\begin{aligned} x_0 &= 0.0, & y_0 &= 1.0000000000, \\ x_1 &= 0.2, & y_1 &= 0.6880000000, \\ x_2 &= 0.4, & y_2 &= 0.5117952000, \\ x_3 &= 0.6, & y_3 &= 0.4266275021, \end{aligned}$$

and we set $x_4 = 0.8$. Now we find

$$\begin{aligned} y_0' &= f(x_0,y_0) = f(0.0,\ 1.0000000) = -2.0000000, \\ y_1' &= f(x_1,y_1) = f(0.2,\ 0.6880000) = -1.1760000, \\ y_2' &= f(x_2,y_2) = f(0.4,\ 0.5117952) = -0.6235904, \\ y_3' &= f(x_3,y_3) = f(0.6,\ 0.4266275) = -0.2532550. \end{aligned}$$

We now use (8.67) with n = 3 and h = 0.2 to determine $\hat{y}_4$. We have

$$\hat{y}_4 = y_3 + \frac{0.2}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

$$= 0.4266275 + \frac{(-13.9290252 + 36.7918336 - 43.5120000 + 18.0000000)}{120.0}$$

$$= 0.4045509.$$

Having thus determined $\hat{y}_4$, we use (8.68) with n = 3 to find $\hat{y}_4'$. We obtain

$$\hat{y}_4' = f(x_4, \hat{y}_4) = f(0.8,\ 0.4045509) = -0.0091018.$$

We use this value of $\hat{y}_4'$ in (8.69) with n = 3 and h = 0.2 to finally obtain $y_4$ as follows:

$$y_4 = y_3 + \frac{0.2}{24}(9\hat{y}_4' + 19y_3' - 5y_2' + y_1')$$

$$= 0.4266275 + \frac{(-0.0819163 - 4.8118451 + 3.1179520 - 1.1760000)}{120.0}$$

$$= 0.4020291.$$

Now we set n = 4 in order to calculate $y_{n+1} = y_5$. Using the value we just found for $y_4$, we first calculate

$$y_4' = f(x_4, y_4) = f(0.80,\ 0.4020291) = -0.0040582.$$

We next use (8.67) with n = 4 and h = 0.20 to determine $\hat{y}_5$. We find

$$\hat{y}_5 = y_4 + \frac{0.2}{24}(55y'_4 - 59y'_3 + 37y'_2 - 9y'_1)$$

$$= 0.4020291 + \frac{(-0.2232000 + 14.9420452 - 23.0728448 + 10.5840000)}{120.0}$$

$$= 0.4206124.$$

Having thus determined $\hat{y}_5$, we use (8.68) with n = 4 to find $\hat{y}'_5$. We obtain

$$\hat{y}'_5 = f(x_5, \hat{y}_5) = f(1.0, 0.4206124) = 0.1587751.$$

We use this value of $\hat{y}'_5$ in (8.69) with n = 4 and h = 0.2 to finally obtain $y_5$ as follows:

$$y_5 = y_4 + \frac{0.2}{24}(9\hat{y}'_5 + 19y'_4 - 5y'_3 + y'_2)$$

$$= 0.4020291 + \frac{(1.4289763 - 0.0771054 + 1.2662750 - 0.6235904)}{120.0}$$

$$= 0.4186504.$$

The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.7.1A** ABAM Method (Calculations Summary) for $y' = x - 2y$, $y(0) = 1$, with $h = 0.2$.

| $n$ | $x_n$ | $y_n$ | $y'_n$ | $x_{n+1}$ | $\hat{y}_{n+1}$ | $\hat{y}'_{n+1}$ | $y_{n+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 1.000000 | -2.000000 | - | — | — | — |
| 1 | 0.2 | 0.688000 | -1.176000 | - | — | — | — |
| 2 | 0.4 | 0.511795 | -0.623590 | - | — | — | — |
| 3 | 0.6 | 0.426628 | -0.253255 | 0.8 | 0.404551 | -0.009102 | 0.402029 |
| 4 | 0.8 | 0.402029 | -0.004058 | 1.0 | 0.420612 | 0.158775 | 0.418650 |
| 5 | 1.0 | 0.418650 | | | | | |

**TABLE 8.7.1B** ABAM Method (Comparisons and Errors) for $y' = x - 2y$, $y(0) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | ABAM Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.8 | 0.402371 | 0.402029 | 0.000342 | 0.084886 |
| 1.0 | 0.419169 | 0.418650 | 0.000519 | 0.123749 |

4. Let $h = 0.2$ and $f(x,y) = x + 2y$ in (8.67), (8.68) and (8.69); we have $x_0 = -1$ and $y_0 = 1$.

Before we can begin using the ABAM method, we need to have values for $y_0$, $y_1$, $y_2$, and $y_3$. The first of these is given by the initial condition $y(-1) = 1.0$, while values for the other three are supplied by the text and were found using the Runge-Kutta method. We have

$$x_0 = -1.0, \qquad y_0 = 1.0000000000,$$
$$x_1 = -0.8, \qquad y_1 = 1.2688000000,$$
$$x_2 = -0.6, \qquad y_2 = 1.7189512533,$$
$$x_3 = -0.4, \qquad y_3 = 2.4396302163,$$

and we set $x_4 = -0.2$. Now we find

$$y_0' = f(x_0,y_0) = f(-1.0,\ 1.0000000) = 1.0000000,$$
$$y_1' = f(x_1,y_1) = f(-0.8,\ 1.2688000) = 1.7376000,$$
$$y_2' = f(x_2,y_2) = f(-0.6,\ 1.7189513) = 2.8379025$$
$$y_3' = f(x_3,y_3) = f(-0.4,\ 2.4396302) = 4.4792604.$$

We now use (8.67) with n = 3 and h = 0.2 to determine $\hat{y}_4$. We have

$$\hat{y}_4 = y_3 + \frac{0.2}{24}\,(55y_3' - 59y_2' + 37y_1' - 9y_0')$$
$$= 2.4396302$$
$$+ \frac{(246.3593238 - 167.4362479 + 64.2912000 - 9.0000000)}{120.0}$$
$$= 3.5580825.$$

Having thus determined $\hat{y}_4$, we use (8.68) with n = 3 to find $\hat{y}_4'$. We obtain

$$\hat{y}_4' = f(x_4,\hat{y}_4) = f(-0.2,\ 3.5580825) = 6.9161650.$$

We use this value of $\hat{y}_4'$ in (8.69) with n = 3 and h = 0.2 to finally obtain $y_4$ as follows:

$$y_4 = y_3 + \frac{0.2}{24}\,(9\hat{y}'_4 + 19y'_3 - 5y'_2 + y'_1)$$

$$= 2.4396302 + \frac{(62.2454853 + 85.1059482 - 14.1895125 + 1.7376000)}{120.0}$$

$$= 3.5637929.$$

Now we set n = 4 in order to calculate $y_{n+1} = y_5$. Using the value we just found for $y_4$, we first calculate

$$y'_4 = f(x_4, y_4) = f(-0.20,\ 3.5637929) = 6.9275858.$$

We next use (8.67) with n = 4 and h = 0.20 to determine $\hat{y}_5$. We find

$$\hat{y}_5 = y_4 + \frac{0.2}{24}\,(55y'_4 - 59y'_3 + 37y'_2 - 9y'_1)$$

$$= 3.5637929 + \frac{(381.0172180 - 264.2763655 + 105.0023927 - 15.6384000)}{120.0}$$

$$= 5.2813333,$$

Having thus determined $\hat{y}_5$ we use (8.68) with n = 4 to find $\hat{y}'_5$. We obtain

$$\hat{y}'_5 = f(x_5, \hat{y}_5) = f(0.0,\ 5.2813333) = 10.5626665.$$

We use this value of $\hat{y}'_5$ in (8.69) with n = 4 and h = 0.2 to finally obtain $y_5$ as follows:

$$y_5 = y_4 + \frac{0.2}{24}\,(9\hat{y}'_5 + 19y'_4 - 5y'_3 + y'_2)$$

$$= 3.5637929 + \frac{(95.0639988 + 131.6241299 - 22.3963022 + 2.8379025)}{120.0}$$

$$= 5.2898740.$$

The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.7.4A** ABAM Method (Calculations Summary) for $y' = x + 2y$, $y(-1) = 1$, with $h = 0.2$.

| $n$ | $x_n$ | $y_n$ | $y'_n$ | $x_{n+1}$ | $\hat{y}_{n+1}$ | $\hat{y}'_{n+1}$ | $y_{n+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | -1.0 | 1.000000 | 1.000000 | - | — | — | — |
| 1 | -0.8 | 1.268800 | 1.737600 | - | — | — | — |
| 2 | -0.6 | 1.718951 | 2.837903 | - | — | — | — |
| 3 | -0.4 | 2.439630 | 4.479260 | -0.2 | 3.558083 | 6.916165 | 3.563793 |
| 4 | -0.2 | 3.563793 | 6.927586 | 0.0 | 5.281333 | 10.562667 | 5.289874 |
| 5 | 0.0 | 5.289874 | | | | | |

**TABLE 8.7.4B** ABAM Method (Comparisons and Errors) for $y' = x + 2y$, $y(-1) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | ABAM Method | Error | % Rel Error |
|---|---|---|---|---|
| -0.2 | 3.564774 | 3.563793 | 0.000981 | 0.027531 |
| 0.0 | 5.291792 | 5.289874 | 0.001918 | 0.036247 |

5. Let $h = 0.1$ and $f(x,y) = xy - 2y$ in (8.67), (8.68) and (8.69); we have $x_0 = 2$ and $y_0 = 1$.

Before we can begin using the ABAM method, we need to have values for $y_0$, $y_1$, $y_2$, and $y_3$. The first of these is

given by the initial condition $y(2) = 1.0$, while values for the other three are supplied by the text and were found using the Runge-Kutta method. We have

$$\begin{aligned} x_0 &= 2.0, & y_0 &= 1.0000000000, \\ x_1 &= 2.1, & y_1 &= 1.0050125208, \\ x_2 &= 2.2, & y_2 &= 1.0202013398, \\ x_3 &= 2.3, & y_3 &= 1.0460278589, \end{aligned}$$

and we set $x_4 = 2.4$. Now we find

$$\begin{aligned} y_0' &= f(x_0,y_0) = f(2.0,\ 1.0000000) = 0.0000000, \\ y_1' &= f(x_1,y_1) = f(2.1,\ 1.0050125) = 0.1005013, \\ y_2' &= f(x_2,y_2) = f(2.2,\ 1.0202013) = 0.2040403, \\ y_3' &= f(x_3,y_3) = f(2.3,\ 1.0460279) = 0.3138084. \end{aligned}$$

We now use (8.67) with $n = 3$ and $h = 0.1$ to determine $\hat{y}_4$. We have

$$\begin{aligned} \hat{y}_4 &= y_3 + \frac{0.1}{24}\,(55y_3' - 59y_2' + 37y_1' - 9y_0') \\ &= 1.0460279 \\ &\quad + \frac{(17.2594597 - 12.0383758 + 3.7185463 - 0.0000000)}{240.0} \\ &= 1.0832763. \end{aligned}$$

Having thus determined $\hat{y}_4$, we use (8.68) with $n = 3$ to find $\hat{y}_4'$. We obtain

$$\hat{y}_4' = f(x_4, \hat{y}_4) = f(2.4,\ 1.0832763) = 0.4333105.$$

We use this value of $\hat{y}_4'$ in (8.69) with n = 3 and h = 0.1 to finally obtain $y_4$ as follows:

$$\begin{aligned} y_4 &= y_3 + \frac{0.1}{24}\,(9\hat{y}_4' + 19y_3' - 5y_2' + y_1') \\ &= 1.0460279 \\ &\quad + \frac{(3.8997947 + 5.9623588 - 1.0202013 + 0.1005013)}{240.0} \\ &= 1.0832881. \end{aligned}$$

Now we set n = 4 in order to calculate $y_{n+1} = y_5$. Using the value we just found for $y_4$, we first calculate

$$y_4' = f(x_4, y_4) = f(2.40,\ 1.0832881) = 0.4333152.$$

We next use (8.67) with n = 4 and h = 0.10 to determine $\hat{y}_5$. We find

$$\begin{aligned} \hat{y}_5 &= y_4 + \frac{0.1}{24}\,(55y_4' - 59y_3' + 37y_2' - 9y_1') \\ &= 1.0832881 \\ &\quad + \frac{(23.8323378 - 18.5146931 + 7.5494899 - 0.9045113)}{240.0} \\ &= 1.1331323. \end{aligned}$$

Having thus determined $\hat{y}_5$, we use (8.68) with n = 4 to find $\hat{y}_5'$. We obtain

$$\hat{y}'_5 = f(x_5, \hat{y}_5) = f(2.5,\ 1.1331323) = 0.5665662.$$

We use this value of $\hat{y}'_5$ in (8.69) with n = 4 and h = 0.1 to finally obtain $y_5$ as follows:

$$y_5 = y_4 + \frac{0.1}{24}\,(9\hat{y}'_5 + 19y'_4 - 5y'_3 + y'_2)$$

$$= 1.0832881 + \frac{(5.0990956 + 8.2329894 - 1.5690418 + 0.2040403)}{240.0}$$

$$= 1.1331509.$$

The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.7.5A** ABAM Method (Calculations Summary) for $y' = xy - 2y$, $y(2) = 1$, with $h = 0.1$.

| $n$ | $x_n$ | $y_n$ | $y'_n$ | $x_{n+1}$ | $\hat{y}_{n+1}$ | $\hat{y}'_{n+1}$ | $y_{n+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | 2.0 | 1.000000 | 0.000000 | - | — | — | — |
| 1 | 2.1 | 1.005013 | 0.100501 | - | — | — | — |
| 2 | 2.2 | 1.020201 | 0.204040 | - | — | — | — |
| 3 | 2.3 | 1.046028 | 0.313808 | 2.4 | 1.083276 | 0.433311 | 1.083288 |
| 4 | 2.4 | 1.083288 | 0.433315 | 2.5 | 1.133132 | 0.566566 | 1.133151 |
| 5 | 2.5 | 1.133151 | | | | | |

**TABLE 8.7.5B** ABAM Method (Comparisons and Errors) for $y' = xy - 2y$, $y(2) = 1$, with $h = 0.1$.

| $x_n$ | Exact Solution | ABAM Method | Error | % Rel Error |
|---|---|---|---|---|
| 2.4 | 1.083287 | 1.083288 | 0.000001 | 0.000094 |
| 2.5 | 1.133148 | 1.133151 | 0.000002 | 0.000219 |

8. Let $h = 0.2$ and $f(x,y) = y \sin x$ in (8.67), (8.68) and (8.69); we have $x_0 = 0$ and $y_0 = 0.5$.

Before we can begin using the ABAM method, we need to have values for $y_0$, $y_1$, $y_2$, and $y_3$. The first of these is given by the initial condition $y(0) = 0.5$, while values for the other three are supplied by the text and were found using the Runge-Kutta Method. We have

$$\begin{aligned} x_0 &= 0.0, & y_0 &= 0.5000000000, \\ x_1 &= 0.2, & y_1 &= 0.5100667118, \\ x_2 &= 0.4, & y_2 &= 0.5410691444, \\ x_3 &= 0.6, & y_3 &= 0.5954231442, \end{aligned}$$

and we set $x_4 = 0.8$. Now we find

$$\begin{aligned} y_0' &= f(x_0,y_0) = f(0.0,\ 0.5000000) = 0.0000000, \\ y_1' &= f(x_1,y_1) = f(0.2,\ 0.5100667) = 0.1013346, \\ y_2' &= f(x_2,y_2) = f(0.4,\ 0.5410691) = 0.2107022, \\ y_3' &= f(x_3,y_3) = f(0.6,\ 0.5954231) = 0.3362012. \end{aligned}$$

We now use (8.67) with $n = 3$ and $h = 0.2$ to determine $\hat{y}_4$. We have

$$\hat{y}_4 = y_3 + \frac{0.2}{24}\,(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

$$= 0.5954231 + \frac{(18.4910658 - 12.4314327 + 3.7493807 - 0.0000000)}{120.0}$$

$$= 0.6771649.$$

Having thus determined $\hat{y}_4$, we use (8.68) with n = 3 to find $\hat{y}_4'$. We obtain

$$\hat{y}_4' = f(x_4, \hat{y}_4) = f(0.8,\ 0.6771649) = 0.4857684.$$

We use this value of $\hat{y}_4'$ in (8.69) with n = 3 and h = 0.2 to finally obtain $y_4$ as follows:

$$y_4 = y_3 + \frac{0.2}{24}\,(9\hat{y}_4' + 19y_3' - 5y_2' + y_1')$$

$$= 0.5954231 + \frac{(4.3719155 + 6.3878227 - 1.0535112 + 0.1013346)}{120.0}$$

$$= 0.6771528.$$

Now we set n = 4 in order to calculate $y_{n+1} = y_5$. Using the value we just found for $y_4$, we first calculate

$$y_4' = f(x_4, y_4) = f(0.80,\ 0.6771528) = 0.4857597.$$

We next use (8.67) with n = 4 and h = 0.20 to determine $\hat{y}_5$. We find

$$\hat{y}_5 = y_4 + \frac{0.2}{24}\,(55y'_4 - 59y'_3 + 37y'_2 - 9y'_1)$$

$$= 0.6771528 + \frac{(26.7167836 - 19.8358706 + 7.7959832 - 0.9120115)}{120.0}$$

$$= 0.7918602.$$

Having thus determined $\hat{y}_5$, we use (8.68) with n = 4 to find $\hat{y}'_5$. We obtain

$$\hat{y}'_5 = f(x_5, \hat{y}_5) = f(1.0,\ 0.7918602) = 0.6663274.$$

We use this value of $\hat{y}'_5$ in (8.69) with n = 4 and h = 0.2 to finally obtain $y_5$ as follows:

$$y_5 = y_4 + \frac{0.2}{24}\,(9\hat{y}'_5 + 19y'_4 - 5y'_3 + y'_2)$$

$$= 0.6771528 + \frac{(5.9969464 + 9.2294344 - 1.6810060 + 0.2107022)}{120.0}$$

$$= 0.7917868.$$

The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.7.8A** ABAM Method (Calculations Summary) for $y' = y \sin x$, $y(0) = 0.5$, with $h = 0.2$.

| $n$ | $x_n$ | $y_n$ | $y'_n$ | $x_{n+1}$ | $\hat{y}_{n+1}$ | $\hat{y}'_{n+1}$ | $y_{n+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 0.500000 | 0.000000 | – | — | — | — |
| 1 | 0.2 | 0.510067 | 0.101335 | – | — | — | — |
| 2 | 0.4 | 0.541069 | 0.210702 | – | — | — | — |
| 3 | 0.6 | 0.595423 | 0.336201 | 0.8 | 0.677165 | 0.485768 | 0.677153 |
| 4 | 0.8 | 0.677153 | 0.485760 | 1.0 | 0.791860 | 0.666327 | 0.791787 |
| 5 | 1.0 | 0.791787 | | | | | |

**TABLE 8.7.8B** ABAM Method (Comparisons and Errors) for $y' = y \sin x$, $y(0) = 0.5$, with $h = 0.2$.

| $x_n$ | Exact Solution | ABAM Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.8 | 0.677156 | 0.677153 | 0.000003 | 0.000440 |
| 1.0 | 0.791798 | 0.791787 | 0.000011 | 0.001363 |

9. Let $h = 0.2$ and $f(x,y) = x/y$ in (8.67), (8.68) and (8.69); we have $x_0 = 0$ and $y_0 = 0.2$.

Before we can begin using the ABAM method, we need to have values for $y_0$, $y_1$, $y_2$, and $y_3$. The first of these is given by the initial condition $y(0) = 0.2$, while values for the other three are supplied by the text and were found using the Runge-Kutta method. We have

$$x_0 = 0.0, \qquad y_0 = 0.2000000000,$$
$$x_1 = 0.2, \qquad y_1 = 0.2838095238,$$
$$x_2 = 0.4, \qquad y_2 = 0.4478806080,$$
$$x_3 = 0.6, \qquad y_3 = 0.6329295155,$$

and we set $x_4 = 0.8$. Now we find

$$y_0' = f(x_0,y_0) = f(0.0,\ 0.2000000) = 0.0000000,$$
$$y_1' = f(x_1,y_1) = f(0.2,\ 0.2838095) = 0.7046980,$$
$$y_2' = f(x_2,y_2) = f(0.4,\ 0.4478806) = 0.8930952,$$
$$y_3' = f(x_3,y_3) = f(0.6,\ 0.6329295) = 0.9479729.$$

We now use (8.67) with n = 3 and h = 0.2 to determine $\hat{y}_4$. We have

$$\hat{y}_4 = y_3 + \frac{0.2}{24}\,(55y_3' - 59y_2' + 37y_1' - 9y_0')$$
$$= 0.6329295$$
$$+ \frac{(52.1385070 - 52.6926140 + 26.0738255 - 0.0000000)}{120.0}$$
$$= 0.8455938.$$

Having thus determined $\hat{y}_4$, we use (8.68) with n = 3 to find $\hat{y}_4'$. We obtain

$$\hat{y}_4' = f(x_4,\hat{y}_4) = f(0.8,\ 0.8455938) = 0.9460807.$$

We use this value of $\hat{y}_4'$ in (8.69) with n = 3 and h = 0.2 to finally obtain $y_4$ as follows:

$$y_4 = y_3 + \frac{0.2}{24}\,(9\hat{y}'_4 + 19y'_3 - 5y'_2 + y'_1)$$

$$= 0.6329295 + \frac{(8.5147262 + 18.0114843 - 4.4654758 + 0.7046980)}{120.0}$$

$$= 0.8226415.$$

Now we set n = 4 in order to calculate $y_{n+1} = y_5$. Using the value we just found for $y_4$, we first calculate

$$y'_4 = f(x_4, y_4) = f(0.80,\ 0.8226415) = 0.9724771.$$

We next use (8.67) with n = 4 and h = 0.20 to determine $\hat{y}_5$. We find

$$\hat{y}_5 = y_4 + \frac{0.2}{24}\,(55y'_4 - 59y'_3 + 37y'_2 - 9y'_1)$$

$$= 0.8226415 + \frac{(53.4862421 - 55.9303985 + 33.0445207 - 6.3422819)}{120.0}$$

$$= 1.0247921.$$

Having thus determined $\hat{y}_5$, we use (8.68) with n = 4 to find $\hat{y}'_5$. We obtain

$$\hat{y}'_5 = f(x_5, \hat{y}_5) = f(1.0,\ 1.0247921) = 0.9758076.$$

We use this value of $\hat{y}'_5$ in (8.69) with n = 4 and h = 0.2 to finally obtain $y_5$ as follows:

$$y_5 = y_4 + \frac{0.2}{24}\,(9\hat{y}'_5 + 19y'_4 - 5y'_3 + y'_2)$$

$$= 0.8226415 + \frac{(8.7822688 + 18.4770655 - 4.7398643 + 0.8930952)}{120.0}$$

$$= 1.0177462.$$

The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.7.9A** ABAM Method (Calculations Summary) for $y' = x/y$, $y(0) = 0.2$, with $h = 0.2$.

| $n$ | $x_n$ | $y_n$ | $y'_n$ | $x_{n+1}$ | $\hat{y}_{n+1}$ | $\hat{y}'_{n+1}$ | $y_{n+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 0.200000 | 0.000000 | – | — | — | — |
| 1 | 0.2 | 0.283810 | 0.704698 | – | — | — | — |
| 2 | 0.4 | 0.447881 | 0.893095 | – | — | — | — |
| 3 | 0.6 | 0.632930 | 0.947973 | 0.8 | 0.845594 | 0.946081 | 0.822641 |
| 4 | 0.8 | 0.822641 | 0.972477 | 1.0 | 1.024792 | 0.975808 | 1.017746 |
| 5 | 1.0 | 1.017746 | 0.982563 | 1.2 | 1.215263 | 0.987441 | 1.214757 |
| 6 | 1.2 | 1.214757 | 0.987852 | 1.4 | 1.413178 | 0.990675 | 1.412631 |
| 7 | 1.4 | 1.412631 | 0.991058 | 1.6 | 1.611193 | 0.993053 | 1.611055 |
| 8 | 1.6 | 1.611055 | 0.993138 | 1.8 | 1.809868 | 0.994547 | 1.809831 |
| 9 | 1.8 | 1.809831 | 0.994568 | 2.0 | 2.008869 | 0.995585 | 2.008851 |
| 10 | 2.0 | 2.008851 | | | | | |

**TABLE 8.7.9B** ABAM Method (Comparisons and Errors) for $y' = x/y$, $y(0) = 0.2$, with $h = 0.2$.

| $x_n$ | Exact Solution | ABAM Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.8 | 0.824621 | 0.822641 | 0.001980 | 0.240070 |
| 1.0 | 1.019804 | 1.017746 | 0.002058 | 0.201778 |
| 1.2 | 1.216553 | 1.214757 | 0.001796 | 0.147619 |
| 1.4 | 1.414214 | 1.412631 | 0.001583 | 0.111901 |
| 1.6 | 1.612452 | 1.611055 | 0.001396 | 0.086603 |
| 1.8 | 1.811077 | 1.809831 | 0.001246 | 0.068800 |
| 2.0 | 2.009975 | 2.008851 | 0.001124 | 0.055916 |

12. Let $h = 0.2$ and $f(x,y) = (\sin x)/y$ in (8.67), (8.68) and (8.69); we have $x_0 = 0$ and $y_0 = 1$.

Before we can begin using the ABAM method, we need to have values for $y_0$, $y_1$, $y_2$, and $y_3$. The first of these is given by the initial condition $y(0) = 1.0$, while values for the other three are supplied by the text and were found using the Runge-Kutta method. We have

$$\begin{aligned} x_0 &= 0.0, & y_0 &= 1.0000000000, \\ x_1 &= 0.2, & y_1 &= 1.0197392647, \\ x_2 &= 0.4, & y_2 &= 1.0760493618, \\ x_3 &= 0.6, & y_3 &= 1.1616089997, \end{aligned}$$

and we set $x_4 = 0.8$. Now we find

$$\begin{aligned} y_0' &= f(x_0,y_0) = f(0.0,\ 1.0000000) = 0.0000000, \\ y_1' &= f(x_1,y_1) = f(0.2,\ 1.0197393) = 0.1948237, \\ y_2' &= f(x_2,y_2) = f(0.4,\ 1.0760494) = 0.3618964, \\ y_3' &= f(x_3,y_3) = f(0.6,\ 1.1616090) = 0.4860865. \end{aligned}$$

We now use (8.67) with $n = 3$ and $h = 0.2$ to determine $\hat{y}_4$. We have

$$\hat{y}_4 = y_3 + \frac{0.2}{24}\,(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$$

$$= 1.1616090 + \frac{(26.7347585 - 21.3518850 + 7.2084752 - 0.0000000)}{120.0}$$

$$= 1.2665369.$$

Having thus determined $\hat{y}_4$, we use (8.68) with n = 3 to find $\hat{y}'_4$. We obtain

$$\hat{y}'_4 = f(x_4,\hat{y}_4) = f(0.8,\ 1.2665369) = 0.5663918.$$

We use this value of $\hat{y}'_4$ in (8.69) with n = 3 and h = 0.2 to finally obtain $y_4$ as follows:

$$y_4 = y_3 + \frac{0.2}{24}\,(9\hat{y}'_4 + 19y'_3 - 5y'_2 + y'_1)$$

$$= 1.1616090 + \frac{(5.0975260 + 9.2356438 - 1.8094818 + 0.1948237)}{120.0}$$

$$= 1.2675966.$$

Now we set n = 4 in order to calculate $y_{n+1} = y_5$. Using the value we just found for $y_4$, we first calculate

$$y'_4 = f(x_4,y_4) = f(0.80,\ 1.2675966) = 0.5659183.$$

We next use (8.67) with n = 4 and h = 0.20 to determine $\hat{y}_5$. We find

$$\hat{y}_5 = y_4 + \frac{0.2}{24}\,(55y'_4 - 59y'_3 + 37y'_2 - 9y'_1)$$

$$= 1.2675966 + \frac{(31.1255056 - 28.6791045 + 13.3901652 - 1.7534129)}{120.0}$$

$$= 1.3849562.$$

Having thus determined $\hat{y}_5$, we use (8.68) with n = 4 to find $\hat{y}_5$. We obtain

$$\hat{y}'_5 = f(x_5, \hat{y}_5) = f(1.0,\ 1.3849562) = 0.6075795.$$

We use this value of $\hat{y}'_5$ in (8.69) with n = 4 and h = 0.2 to finally obtain $y_5$ as follows:

$$y_5 = y_4 + \frac{0.2}{24}\,(9\hat{y}'_5 + 19y'_4 - 5y'_3 + y'_2)$$

$$= 1.2675966 + \frac{(5.4682154 + 10.7524474 - 2.4304326 + 0.3618964)}{120.0}$$

$$= 1.3855310.$$

The exact solution of the differential equation is given in the pages immediately preceding Section 8.4 of this manual.

**TABLE 8.7.12A** ABAM Method (Calculations Summary) for $y' = (\sin x)/y$, $y(0) = 1$, with $h = 0.2$.

| $n$ | $x_n$ | $y_n$ | $y'_n$ | $x_{n+1}$ | $\hat{y}_{n+1}$ | $\hat{y}'_{n+1}$ | $y_{n+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 1.000000 | 0.000000 | - | — | — | — |
| 1 | 0.2 | 1.019739 | 0.194824 | - | — | — | — |
| 2 | 0.4 | 1.076049 | 0.361896 | - | — | — | — |
| 3 | 0.6 | 1.161609 | 0.486087 | 0.8 | 1.266537 | 0.566392 | 1.267597 |
| 4 | 0.8 | 1.267597 | 0.565918 | 1.0 | 1.384956 | 0.607579 | 1.385531 |
| 5 | 1.0 | 1.385531 | 0.607327 | 1.2 | 1.508381 | 0.617907 | 1.508505 |
| 6 | 1.2 | 1.508505 | 0.617856 | 1.4 | 1.631121 | 0.604155 | 1.631054 |
| 7 | 1.4 | 1.631054 | 0.604180 | 1.6 | 1.749006 | 0.571509 | 1.748897 |
| 8 | 1.6 | 1.748897 | 0.571545 | 1.8 | 1.858756 | 0.523924 | 1.858660 |
| 9 | 1.8 | 1.858660 | 0.523951 | 2.0 | 1.957744 | 0.464462 | 1.957674 |
| 10 | 2.0 | 1.957674 | | | | | |

**TABLE 8.7.12B** ABAM Method (Comparisons and Errors) for $y' = (\sin x)/y$, $y(0) = 1$, with $h = 0.2$.

| $x_n$ | Exact Solution | ABAM Method | Error | % Rel Error |
|---|---|---|---|---|
| 0.8 | 1.267512 | 1.267597 | 0.000085 | 0.006677 |
| 1.0 | 1.385422 | 1.385531 | 0.000109 | 0.007833 |
| 1.2 | 1.508405 | 1.508505 | 0.000100 | 0.006654 |
| 1.4 | 1.630971 | 1.631054 | 0.000084 | 0.005132 |
| 1.6 | 1.748828 | 1.748897 | 0.000069 | 0.003923 |
| 1.8 | 1.858603 | 1.858660 | 0.000057 | 0.003090 |
| 2.0 | 1.957624 | 1.957674 | 0.000050 | 0.002540 |

**Section 8.8, Page 477.**

General Information: For each problem below, we show in detail the first two steps for the Euler Method, and the first for the Runge-Kutta method. All calculations for each value of $t_n$ are summarized in tables. The problems were done maintaining 11 to 12 digit accuracy from step to step; however, we've rounded the results to fewer places (usually four to seven) as we tabulate them below.

1. The exact solution is $x = 2e^t$, $y = 4e^t$. First we find the solution by the Euler method. We have $f(t,x,y) = 5x - 2y$, $g(t,x,y) = 4x - y$, $t_0 = 0.0$, $x_0 = 2.0000$, $y_0 = 4.0000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.74). We have

$$\begin{aligned} x_1 &= x_0 + hf(t_0,x_0,y_0) \\ &= 2.0000 + 0.1f(0.0, 2.0000, 4.0000) \\ &= 2.0000 + (0.1)2.0000 = 2.2000, \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + hg(t_0,x_0,y_0) \\ &= 4.0000 + 0.1g(0.0, 2.0000, 4.0000) \\ &= 4.0000 + (0.1)4.0000 = 4.4000. \end{aligned}$$

Now to find the approximate values at $t_2 = 0.1 + 0.1 = 0.2$, we use Formulas 8.75 with $n = 1$. We have

$$\begin{aligned} x_2 &= x_1 + hf(t_1,x_1,y_1) \\ &= 2.2000 + 0.1f(0.1,2.2000, 4.4000) \\ &= 2.2000 + (0.1)2.2000 = 2.4200, \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hg(t_1,x_1,y_1) \\ &= 4.4000 + 0.1g(0.1, 2.2000, 4.4000) \\ &= 4.4000 + (0.1)4.4000 = 4.8400. \end{aligned}$$

The results for all $t_n$ and the corresponding errors have been summarized in Table 8.8.1-Euler.

Now we solve the problem by the Runge-Kutta method. Again, $f(t,x,y) = 5x - 2y$, $g(t,x,y) = 4x - y$, $t_0 = 0.0$, $x_0 = 2.0000000$, $y_0 = 4.0000000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.78-80) with $n = 0$. By (8.80) we have

$$\begin{aligned} k_1 &= hf(t_0,x_0,y_0) \\ &= (0.1)f(0.0,\ 2.0000000,\ 4.0000000) \\ &= (0.1)2.0000000 = 0.2000000, \end{aligned}$$

$$\begin{aligned} m_1 &= hg(t_0,x_0,y_0) \\ &= (0.1)g(0.0,\ 2.0000000,\ 4.0000000) \\ &= (0.1)4.0000000 = 0.4000000. \end{aligned}$$

Using these values for $k_1$ and $m_1$ we now find

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)f(0.05,\ 2.1000000,\ 4.2000000) \\ &= (0.1)2.1000000 = 0.2100000, \end{aligned}$$

$$\begin{aligned} m_2 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)g(0.05,\ 2.1000000,\ 4.2000000) \\ &= (0.1)4.2000000 = 0.4200000. \end{aligned}$$

Next, the above values for $k_2$ and $m_2$ are used to obtain

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)f(0.05,\ 2.1050000,\ 4.2100000) \\ &= (0.1)2.1050000 = 0.2105000, \end{aligned}$$

$$m_3 = hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right)$$
$$= (0.1)g(0.05,\ 2.1050000,\ 4.2100000)$$
$$= (0.1)4.2100000 = 0.4210000.$$

With these values for $k_3$ and $m_3$ we find

$$k_4 = hf(t_0 + h,\ x_0 + k_3,\ y_0 + m_3)$$
$$= (0.1)f(0.1,\ 2.2105000,\ 4.4210000)$$
$$= (0.1)2.2105000 = 0.2210500,$$

$$m_4 = hg(t_0 + h,\ x_0 + k_3,\ y_0 + m_3)$$
$$= (0.1)g(0.1,\ 2.2105000,\ 4.4210000)$$
$$= (0.1)4.4210000 = 0.4421000.$$

Then from (8.79) we set

$$K = \frac{1}{6}\ (0.2000000 + 2(0.2100000) + 2(0.2105000) + 0.2210500)$$
$$= \frac{1}{6}\ (1.2620500) = 0.2103417,$$

$$M = \frac{1}{6}\ (0.4000000 + 2(0.4200000) + 2(0.4210000) + 0.4421000)$$
$$= \frac{1}{6}\ (2.5241000) = 0.4206833.$$

Finally from (8.78) we obtain

$$x_1 = 2.0000000 + 0.2103417 = 2.2103417,$$
$$y_1 = 4{,}0000000 + 0.4206833 = 4.4206833.$$

These calculations and those for the remaining $t_n$ are summarized in Table 8.8.1-RK.

**TABLE 8.8.1-Euler** Summary: Euler Method Solution (with h = 0.1) of the System

$$x' = 5x - 2y,\ y' = 4x - y,\ x(0.0) = 2.0,\ y(0.0) = 4.0.$$

| $t_n$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | Euler $\begin{cases} x_n \\ y_n \end{cases}$ | Errors | % Rel Errors |
|---|---|---|---|---|
| 0.1 | 2.21034 | 2.20000 | 0.01034 | 0.4679 |
| | 4.42068 | 4.40000 | 0.02068 | 0.4679 |
| 0.2 | 2.44281 | 2.42000 | 0.02281 | 0.9336 |
| | 4.88561 | 4.84000 | 0.04561 | 0.9336 |
| 0.3 | 2.69972 | 2.66200 | 0.03772 | 1.3971 |
| | 5.39944 | 5.32400 | 0.07544 | 1.3971 |
| 0.4 | 2.98365 | 2.92820 | 0.05545 | 1.8584 |
| | 5.96730 | 5.85640 | 0.11090 | 1.8584 |
| 0.5 | 3.29744 | 3.22102 | 0.07642 | 2.3176 |
| | 6.59489 | 6.44204 | 0.15285 | 2.3176 |

**TABLE 8.8.1-R-K** Summary: Runge-Kutta Solution (with h = 0.1) of the System

$x' = 5x - 2y$, $y' = 4x - y$, $x(0.0) = 2.0$, $y(0.0) = 4.0$.

| $t_n$ | $k_1$, $m_1$ | $k_2$, $m_2$ | $k_3$, $m_3$ | $k_4$, $m_4$ | $K$, $M$ | Exact $\begin{Bmatrix} x(t_n) \\ y(t_n) \end{Bmatrix}$ | R-K $\begin{Bmatrix} x_n \\ y_n \end{Bmatrix}$ | Errors | % Rel Errors |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.2000 | 0.2100 | 0.2105 | 0.2210 | 0.2103 | 2.2103418 | 2.2103417 | 0.0000002 | 0.00001 |
| | 0.4000 | 0.4200 | 0.4210 | 0.4421 | 0.4207 | 4.4206837 | 4.4206833 | 0.0000003 | 0.00001 |
| 0.2 | 0.2210 | 0.2321 | 0.2326 | 0.2443 | 0.2325 | 2.4428055 | 2.4428051 | 0.0000004 | 0.00002 |
| | 0.4421 | 0.4642 | 0.4653 | 0.4886 | 0.4649 | 4.8856110 | 4.8856103 | 0.0000007 | 0.00002 |
| 0.3 | 0.2443 | 0.2565 | 0.2571 | 0.2700 | 0.2569 | 2.6997176 | 2.6997170 | 0.0000006 | 0.00002 |
| | 0.4886 | 0.5130 | 0.5142 | 0.5400 | 0.5138 | 5.3994352 | 5.3994340 | 0.0000012 | 0.00002 |
| 0.4 | 0.2700 | 0.2835 | 0.2841 | 0.2984 | 0.2839 | 2.9836494 | 2.9836485 | 0.0000009 | 0.00003 |
| | 0.5399 | 0.5669 | 0.5683 | 0.5968 | 0.5679 | 5.9672988 | 5.9672970 | 0.0000018 | 0.00003 |
| 0.5 | 0.2984 | 0.3133 | 0.3140 | 0.3298 | 0.3138 | 3.2974425 | 3.2974413 | 0.0000013 | 0.00004 |
| | 0.5967 | 0.6266 | 0.6281 | 0.6595 | 0.6276 | 6.5948851 | 6.5948826 | 0.0000025 | 0.00004 |

4. The exact solution is $x = e^{-t}$, $y = -e^{-t}$. First we find the solution by the Euler method. We have $f(t,x,y) = x + 2y$, $g(t,x,y) = 3x + 2y$, $t_0 = 0.0$, $x_0 = 1.0000$, $y_0 = -1.0000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.74). We have

$$\begin{aligned} x_1 &= x_0 + hf(t_0,x_0,y_0) \\ &= 1.0000 + 0.1f(0.0,\ 1.0000,\ -1.0000) \\ &= 1.0000 + (0.1)(-1.0000) = 0.9000, \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + hg(t_0,x_0,y_0) \\ &= -1.0000 + 0.1g(0.0,\ 1.0000,\ -1.0000) \\ &= -1.0000 + (0.1)1.0000 = -0.9000. \end{aligned}$$

Now to find the approximate values at $t_2 = 0.1 + 0.1 = 0.2$, we use Formulas 8.75 with $n = 1$. We have

$$\begin{aligned} x_2 &= x_1 + hf(t_1,x_1,y_1) \\ &= 0.9000 + 0.1f(0.1,\ 0.9000,\ -0.9000) \\ &= 0.9000 + (0.1)(-0.9000) = 0.8100, \end{aligned}$$

$$\begin{aligned} y_2 &+ y_1 + hg(t_1,x_1,y_1) \\ &= -0.9000 + 0.1g(0.1,\ 0.9000,\ -0.9000) \\ &= -0.9000 + (0.1)0.9000 = -0.8100. \end{aligned}$$

The results for all $t_n$ and the corresponding errors have been summarized in Table 8.8.4-Euler.

Now we solve the problem by the Runge-Kutta method. Again, $f(t,x,y) = x + 2y$, $g(t,x,y) = 3x + 2y$, $t_0 = 0.0$, $x_0 = 1.0000000$, $y_0 = -1.0000000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.78-80) with $n = 0$. By (8.80) we have

$$\begin{aligned} k_1 &= hf(t_0,x_0,y_0) \\ &= (0.1)f(0.0,\ 1.0000000,\ -1.0000000) \\ &= (0.1)(-1.0000000) = -0.1000000, \end{aligned}$$

$$\begin{aligned} m_1 &= hg(t_0,x_0,y_0) \\ &= (0.1)g(0.0,\ 1.0000000,\ -1.0000000) \\ &= (0.1)1.0000000 = 0.1000000. \end{aligned}$$

Using these values for $k_1$ and $m_1$ we now find

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)f(0.05,\ 0.9500000,\ -0.9500000) \\ &= (0.1)(-0.9500000) = -0.0950000, \end{aligned}$$

$$\begin{aligned} m_2 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)g(0.05,\ 0.9500000,\ -0.9500000) \\ &= (0.1)0.9500000 = 0.0950000. \end{aligned}$$

Next, the above values for $k_2$ and $m_2$ are used to obtain

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)f(0.05,\ 0.9525000,\ -0.9525000) \\ &= (0.1)(-0.9525000) = -0.0952500, \end{aligned}$$

$$\begin{aligned} m_3 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)g(0.05,\ 0.9525000,\ -0.9525000) \\ &= (0.1)0.9525000 = 0.0952500. \end{aligned}$$

With these values for $k_3$ and $m_3$ we find

$$\begin{aligned} k_4 &= hf(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)f(0.1,\ 0.9047500,\ -0.9047500) \\ &= (0.1)(-0.9047500) = -0.0904750, \end{aligned}$$

$$\begin{aligned} m_4 &= hg(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)g(0.1,\ 0.9047500,\ -0.9047500) \\ &= (0.1)0.9047500 = 0.0904750. \end{aligned}$$

Then from (8.79) we set

$$\begin{aligned} K &= \frac{1}{6}\ (-0.1000000 + 2(0.0950000) + 2(-0.0952500) \\ &\qquad + -0.0904750) \\ &= \frac{1}{6}\ (-0.5709750) = -0.0951625, \end{aligned}$$

$$M = \frac{1}{6}\,(0.1000000 + 2(0.0950000) + 2(0.0952500) + 0.0904750)$$

$$= \frac{1}{6}\,(0.5709750) = 0.0951625.$$

Finally from (8.78) we obtain

$$x_1 = 1.0000000 + -0.0951625 = 0.9048375,$$

$$y_1 = -1.0000000 + 0.0951625 = -0.9048375.$$

These calculations and those for the remaining $t_n$ are summarized in Table 8.8.4-RK.

**TABLE 8.8.4-Euler** Summary: Euler Method Solution (with h = 0.1) of the System

$x' = x + 2y,\ y' = 3x + 2y,\ x(0.0) = 1.0,\ y(0.0) = -1.0.$

| $t_n$ | Exact $\begin{Bmatrix} x(t_n) \\ y(t_n) \end{Bmatrix}$ | Euler $\begin{Bmatrix} x_n \\ y_n \end{Bmatrix}$ | Errors | % Rel Errors |
|---|---|---|---|---|
| 0.1 | 0.90484<br>-0.90484 | 0.90000<br>-0.90000 | 0.00484<br>0.00484 | 0.5346<br>0.5346 |
| 0.2 | 0.81873<br>-0.81873 | 0.81000<br>-0.81000 | 0.00873<br>0.00873 | 1.0664<br>1.0664 |
| 0.3 | 0.74082<br>-0.74082 | 0.72900<br>-0.72900 | 0.01182<br>0.01182 | 1.5953<br>1.5953 |
| 0.4 | 0.67032<br>-0.67032 | 0.65610<br>-0.65610 | 0.01422<br>0.01422 | 2.1214<br>2.1214 |
| 0.5 | 0.60653<br>-0.60653 | 0.59049<br>-0.59049 | 0.01604<br>0.01604 | 2.6447<br>2.6447 |

**TABLE 8.8.4-R-K** Summary: Runge-Kutta Solution (with h = 0.1) of the System

$x' = x + 2y$, $y' = 3x + 2y$, $x(0.0) = 1.0$, $y(0.0) = -1.0$.

| $t_n$ k | $k_1$, $m_1$ | $k_2$, $m_2$ | $k_3$, $m_3$ | $k_4$, $m_4$ | $K$, $M$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | R-K $\begin{cases} x_n \\ y_n \end{cases}$ | Errors | % Rel Errors |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | -0.1000 | -0.0950 | -0.0952 | -0.0905 | -0.0952 | 0.9048374 | 0.9048375 | 0.0000001 | 0.00001 |
| | 0.1000 | 0.0950 | 0.0952 | 0.0905 | 0.0952 | -0.9048374 | -0.9048375 | 0.0000001 | 0.00001 |
| 0.2 | -0.0905 | -0.0860 | -0.0862 | -0.0819 | -0.0861 | 0.8187308 | 0.8187309 | 0.0000001 | 0.00002 |
| | 0.0905 | 0.0860 | 0.0862 | 0.0819 | 0.0861 | -0.8187308 | -0.8187309 | 0.0000001 | 0.00002 |
| 0.3 | -0.0819 | -0.0778 | -0.0780 | -0.0741 | -0.0779 | 0.7408182 | 0.7408184 | 0.0000002 | 0.00003 |
| | 0.0819 | 0.0778 | 0.0780 | 0.0741 | 0.0779 | -0.7408182 | -0.7408184 | 0.0000002 | 0.00003 |
| 0.4 | -0.0741 | -0.0704 | -0.0706 | -0.0670 | -0.0705 | 0.6703200 | 0.6703203 | 0.0000002 | 0.00004 |
| | 0.0741 | 0.0704 | 0.0706 | 0.0670 | 0.0705 | -0.6703200 | -0.6703203 | 0.0000002 | 0.00004 |
| 0.5 | -0.0670 | -0.0637 | -0.0638 | -0.0606 | -0.0638 | 0.6065307 | 0.6065309 | 0.0000003 | 0.00005 |
| | 0.0670 | 0.0637 | 0.0638 | 0.0606 | 0.0638 | -0.6065307 | -0.6065309 | 0.0000003 | 0.00005 |

7. The exact solution is $x = -e^{3t} + 3e^{4t} + 2e^{2t}$, $y = -e^{3t} + 2e^{4t} + 3e^{2t}$. First we find the solution by the Euler method. We have $f(t,x,y) = 6x - 3y + e^{2t}$, $g(t,x,y) = 2x + y - e^{2t}$, $t_0 = 0.0$, $x_0 = 4.0000$, $y_0 = 4.0000$ and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.74). We have

$$\begin{aligned} x_1 &= x_0 + hf(t_0,x_0,y_0) \\ &= 4.0000 + 0.1f(0.0, 4.0000, 4.0000) \\ &= 4.0000 + (0.1)13.0000 = 5.3000, \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + hg(t_0, x_0, y_0) \\ &= 4.0000 + 0.1g(0.0, 4.0000, 4.0000) \\ &= 4.0000 + (0.1)11.0000 = 5.1000. \end{aligned}$$

Now to find the approximate values at $t_2 = 0.1 + 0.1 = 0.2$, we use Formulas 8.75 with $n = 1$. We have

$$\begin{aligned} x_2 &= x_1 + hf(t_1,x_1,y_1) \\ &= 5.3000 + 0.1f(0.1, 5.3000, 5.1000) \\ &= 5.3000 + (0.1)17.7214 = 7.0721, \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hg(t_1,x_1,y_1) \\ &= 5.1000 + 0.1g(0.1, 5.3000, 5.1000) \\ &= 5.1000 + (0.1)14.4786 = 6.5479. \end{aligned}$$

The results for all $t_n$ and the corresponding errors have been summarized in Table 8.8.7-Euler.

Now we solve the problem by the Runge-Kutta method. Again, $f(t,x,y) = 6x - 3y + e^{2t}$, $g(t,x,y) = 2x + y - e^{2t}$, $t_0 = 0.0$, $x_0 = 4.0000000$, $y_0 = 4.0000000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.78-80) with $n = 0$. By (8.80) we have

$$\begin{aligned} k_1 &= hf(t_0,x_0,y_0) \\ &= (0.1)f(0.0,\ 4.0000000,\ 4.0000000) \\ &= (0.1)13.0000000 = 1.3000000, \end{aligned}$$

$$\begin{aligned} m_1 &= hg(t_0,x_0,y_0) \\ &= (0.1)g(0.0,\ 4.0000000,\ 4.0000000) \\ &= (0.1)11.0000000 = 1.1000000. \end{aligned}$$

Using these values for $k_1$ and $m_1$ we now find

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)f(0.05,\ 4.6500000,\ 4.5500000) \\ &= (0.1)15.3551709 = 1.5355171, \end{aligned}$$

$$\begin{aligned} m_2 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)g(0.05,\ 4.6500000,\ 4.5500000) \\ &= (0.1)12.7448291 = 1.2744829. \end{aligned}$$

Next, the above values for $k_2$ and $m_2$ are used to obtain

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)f(0.05,\ 4.7677585,\ 4.6372415) \\ &= (0.1)15.7999978 = 1.5799998, \end{aligned}$$

$$\begin{aligned} m_3 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)g(0.05,\ 4.7677585,\ 4.6372415) \\ &= (0.1)13.0675876 = 1.3067588. \end{aligned}$$

With these values for $k_3$ and $m_3$ we find

$$\begin{aligned} k_4 &= hf(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)f(0.1,\ 5.5799998,\ 5.3067588) \\ &= (0.1)18.7811252 = 1.8781125, \end{aligned}$$

$$\begin{aligned} m_4 &= hg(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)g(0.1,\ 5.5799998,\ 5.3067588) \\ &= (0.1)15.2453556 = 1.5245356. \end{aligned}$$

Then from (8.79) we set

$$\begin{aligned} K &= \frac{1}{6}\,(1.3000000 + 2(1.5355171) + 2(1.5799998) \\ &\qquad + 1.8781125) \\ &= \frac{1}{6}\,(9.4091463) = 1.5681910, \end{aligned}$$

$$M = \frac{1}{6}\,(1.1000000 + 2(1.2744829) + 2(1.3067588) + 1.5245356)$$

$$= \frac{1}{6}\,(7.7870189) = 1.2978365.$$

Finally from (8.78) we obtain

$$x_1 = 4.0000000 + 1.5681910 = 5.5681910,$$

$$y_1 = 4.0000000 + 1.2978365 = 5.2978365.$$

These calculations and those for the remaining $t_n$ are summarized in Table 8.8.7-RK.

**TABLE 8.8.7-Euler** Summary: Euler Method Solution (with h = 0.1) of the System $x' = 6x - 3y + e^{2t}$, $y' = 2x + y - e^{2t}$, $x(0.0) = 4.0$, $y(0.0) = 4.0$.

| $t_n$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | Euler $\begin{cases} x_n \\ y_n \end{cases}$ | Errors | % Rel Errors |
|---|---|---|---|---|
| 0.1 | 5.56842<br>5.29800 | 5.30000<br>5.10000 | 0.26842<br>0.19800 | 4.8204<br>3.7372 |
| 0.2 | 7.83815<br>7.10444 | 7.07214<br>6.54786 | 0.76601<br>0.55658 | 9.7729<br>7.8342 |
| 0.3 | 11.14499<br>9.64699 | 9.50025<br>8.46789 | 1.64474<br>1.17910 | 14.758<br>12.222 |
| 0.4 | 15.99006<br>13.26257 | 12.84224<br>11.03252 | 3.14782<br>2.23005 | 19.686<br>16.815 |
| 0.5 | 23.12204<br>18.45127 | 17.46039<br>14.48166 | 5.66166<br>3.96960 | 24.486<br>21.514 |

**TABLE 8.8.7-R-K** Summary: Runge-Kutta Solution (with h = 0.1) of the System

$x' = 6x - 3y + e^{2t}$, $y' = 2x + y - e^{2t}$, $x(0.0) = 4.0$, $y(0.0) = 4.0$.

| $t_n$ k | $k_1$, $m_1$ | $k_2$, $m_2$ | $k_3$, $m_3$ | $k_4$, $m_4$ | $K$, $M$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | R-K $\begin{cases} x_n \\ y_n \end{cases}$ | Errors | % Rel Errors |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 1.3000 | 1.5355 | 1.5800 | 1.8781 | 1.5682 | 5.5684208 | 5.5681910 | 0.0002298 | 0.00413 |
| | 1.1000 | 1.2745 | 1.3068 | 1.5245 | 1.2978 | 5.2979989 | 5.2978365 | 0.0001624 | 0.00306 |
| 0.2 | 1.8737 | 2.2205 | 2.2869 | 2.7272 | 2.2693 | 7.8381534 | 7.8374626 | 0.0006908 | 0.00881 |
| | 1.5213 | 1.7719 | 1.8191 | 2.1335 | 1.8061 | 7.1044371 | 7.1039538 | 0.0004833 | 0.00680 |
| 0.3 | 2.7205 | 3.2330 | 3.3323 | 3.9846 | 3.3060 | 11.1449853 | 11.1434276 | 0.0015577 | 0.01398 |
| | 2.1287 | 2.4915 | 2.5609 | 3.0182 | 2.5420 | 9.6469871 | 9.6459066 | 0.0010806 | 0.01120 |
| 0.4 | 3.9745 | 4.7343 | 4.8830 | 5.8519 | 4.8435 | 15.9900622 | 15.9869409 | 0.0031213 | 0.01952 |
| | 3.0111 | 3.5399 | 3.6423 | 4.3116 | 3.6145 | 13.2625707 | 13.2604207 | 0.0021500 | 0.01621 |
| 0.5 | 5.8366 | 6.9658 | 7.1883 | 8.6305 | 7.1292 | 23.1220429 | 23.1161805 | 0.0058624 | 0.02535 |
| | 4.3009 | 5.0762 | 5.2279 | 6.2121 | 5.1868 | 18.4512686 | 18.4472544 | 0.0040143 | 0.02176 |

9. Before we begin using the Euler method, we convert the given second order D.E. into a system, following the procedure described in section 7.1A. We have $f(t,x,y) = y$, $g(t,x,y) = -6x + 5y$, $t_0 = 0.0$, $x_0 = 1.0000$, $y_0 = 2.5000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.74). We have

$$\begin{aligned} x_1 &= x_0 + hf(t_0,x_0,y_0) \\ &= 1.0000 + 0.1f(0.0,\ 1.0000,\ 2.5000) \\ &= 1.0000 + (0.1)2.5000 = 1.2500, \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + hg(t_0,x_0,y_0) \\ &= 2.5000 + 0.1g(0.0,\ 1.0000,\ 2.5000) \\ &= 2.5000 + (0.1)6.5000 = 3.1500. \end{aligned}$$

Now to find the approximate values at $t_2 = 0.1 + 0.1 = 0.2$, we use Formulas 8.75 with $n = 1$. We have

$$\begin{aligned} x_2 &= x_1 + hf(t_1,x_1,y_1) \\ &= 1.2500 + 0.1f(0.1,\ 1.2500,\ 3.1500) \\ &= 1.2500 + (0.1)3.1500 = 1.5650, \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hg(t_1,x_1,y_1) \\ &= 3.1500 + 0.1g(0.1,\ 1.2500,\ 3.1500) \\ &= 3.1500 + (0.1)8.2500 = 3.9750. \end{aligned}$$

The results for all $t_n$ and the corresponding errors have been summarized in Table 8.8.9-Euler.

Now we solve the problem by the Runge-Kutta method. Again, $f(t,x,y) = y$, $g(t,x,y) = -6x + 5y$, $t_0 = 0.0$, $x_0 = 1.0000000$, $y_0 = 2.50000000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.78-80) with $n = 0$. By (8.80) we have

$$\begin{aligned} k_1 &= hf(t_0,x_0,y_0) \\ &= (0.1)f(0.0,\ 1.0000000,\ 2.5000000) \\ &= (0.1)2.5000000 = 0.2500000, \end{aligned}$$

$$\begin{aligned} m_1 &= hg(t_0,x_0,y_0) \\ &= (0.1)g(0.0,\ 1.0000000,\ 2.5000000) \\ &= (0.1)6.5000000 = 0.6500000. \end{aligned}$$

Using these values for $k_1$ and $m_1$ we now find

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)f(0.05,\ 1.1250000,\ 2.8250000) \\ &= (0.1)2.8250000 = 0.2825000, \end{aligned}$$

$$\begin{aligned} m_2 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)g(0.05,\ 1.1250000,\ 2.8250000) \\ &= (0.1)7.3750000 = 0.7375000. \end{aligned}$$

Next, the above values for $k_2$ and $m_2$ are used to obtain

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)f(0.05,\ 1.1412500,\ 2.8687500) \\ &= (0.1)2.8687500 = 0.2868750, \end{aligned}$$

$$\begin{aligned} m_3 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)g(0.05,\ 1.1412500,\ 2.8687500) \\ &= (0.1)7.4962500 = 0.7496250. \end{aligned}$$

With these values for $k_3$ and $m_3$ we find

$$\begin{aligned} k_4 &= hf(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)f(0.1,\ 1.2868750,\ 3.2496250) \\ &= (0.1)3.2496250 = 0.3249625, \end{aligned}$$

$$\begin{aligned} m_4 &= hg(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)g(0.1,\ 1.2868750,\ 3.2496250 \\ &= (0.1)8.5268750 = 0.8526875. \end{aligned}$$

Then from (8.79) we set

$$\begin{aligned} K &= \frac{1}{6}\ (0.2500000 + 2(0.2825000) + 2(0.2868750) \\ &\qquad + 0.3249625) \\ &= \frac{1}{6}\ (1.7137125) = 0.2856187, \end{aligned}$$

$$M = \frac{1}{6}\,(0.6500000 + 2(0.7375000) + 2(0.7496250) + 0.8526875)$$

$$= \frac{1}{6}\,(4.4769375) = 0.7461562.$$

Finally from (8.78) we obtain

$$x_1 = 1.0000000 + 0.2856187 = 1.2856187,$$

$$y_1 = 2.5000000 + 0.7461562 = 3.2461562.$$

These calculations and those for the remaining $t_n$ are summarized in Table 8.8.9-R.K. The exact solution to the stated problem is $x = \frac{1}{2}\,(e^{2t} + e^{3t})$; we have also $y = x' = \frac{1}{2}\,(2e^{2t} + 3e^{3t})$.

**TABLE 8.8.9-Euler** Summary: Euler Method Solution (with h = 0.1) of the System

$x' = y,\ y' = -6x + 5y,\ x(0.0) = 1.0,\ y(0.0) = 2.5.$

| $t_n$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | Euler $\begin{cases} x_n \\ y_n \end{cases}$ | Errors | % Rel Errors |
|---|---|---|---|---|
| 0.1 | 1.28563 | 1.25000 | 0.03563 | 2.7715 |
| | 3.24619 | 3.15000 | 0.09619 | 2.9632 |
| 0.2 | 1.65697 | 1.56500 | 0.09197 | 5.5506 |
| | 4.22500 | 3.97500 | 0.25000 | 5.9172 |
| 0.3 | 2.14086 | 1.96250 | 0.17836 | 8.3313 |
| | 5.51152 | 5.02350 | 0.48802 | 8.8546 |
| 0.4 | 2.77283 | 2.46485 | 0.30798 | 11.107 |
| | 7.20572 | 6.35775 | 0.84797 | 11.768 |
| 0.5 | 3.59999 | 3.10062 | 0.49936 | 13.871 |
| | 9.44082 | 8.05771 | 1.38310 | 14.650 |

**TABLE 8.8.9-R-K** Summary: Runge-Kutta Solution (with h = 0.1) of the System

$$x' = y,\ y' = -6x + 5y,\ x(0.0) = 1.0,\ y(0.0) = 2.5.$$

| $t_n$ k | $k_1$, $m_1$ | $k_2$, $m_2$ | $k_3$, $m_3$ | $k_4$, $m_4$ | $K$, $M$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | R-K $\begin{cases} x_n \\ y_n \end{cases}$ | Errors | % Rel Errors |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.2500 | 0.2825 | 0.2869 | 0.3250 | 0.2856 | 1.2856308 | 1.2856187 | 0.0000120 | 0.00094 |
| | 0.6500 | 0.7375 | 0.7496 | 0.8527 | 0.7462 | 3.2461910 | 3.2461562 | 0.0000347 | 0.00107 |
| 0.2 | 0.3246 | 0.3672 | 0.3730 | 0.4230 | 0.3713 | 1.6569717 | 1.6569396 | 0.0000321 | 0.00194 |
| | 0.8517 | 0.9672 | 0.9834 | 1.1196 | 0.9788 | 4.2250029 | 4.2249099 | 0.0000930 | 0.00220 |
| 0.3 | 0.4225 | 0.4784 | 0.4860 | 0.5517 | 0.4839 | 2.1408610 | 2.1407965 | 0.0000644 | 0.00301 |
| | 1.1183 | 1.2711 | 1.2925 | 1.4729 | 1.2864 | 5.5115235 | 5.5113364 | 0.0001871 | 0.00339 |
| 0.4 | 0.5511 | 0.6247 | 0.6348 | 0.7214 | 0.6319 | 2.7728289 | 2.7727141 | 0.0001149 | 0.00414 |
| | 1.4712 | 1.6736 | 1.7022 | 1.9414 | 1.6940 | 7.2057163 | 7.2053818 | 0.0003345 | 0.00464 |
| 0.5 | 0.7205 | 0.8175 | 0.8309 | 0.9451 | 0.8271 | 3.5999854 | 3.5997932 | 0.0001922 | 0.00534 |
| | 1.9391 | 2.2077 | 2.2457 | 2.5634 | 2.2349 | 9.4408154 | 9.4402542 | 0.0005613 | 0.00594 |

12. Before we begin using the Euler method, we convert the given second order D.E. into a system, following the procedure described in section 7.1A. We have $f(t,x,y) = y$, $g(t,x,y) = 3y - 2x + 4t^2$, $t_0 = 0.0$, $x_0 = 8.0000$, $y_0 = 8.0000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.74). We have

$$\begin{aligned} x_1 &= x_0 + hf(t_0,x_0,y_0) \\ &= 8.0000 + 0.1f(0.0, 8.0000, 8.0000) \\ &= 8.0000 + (0.1)8.0000 = 8.8000, \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + hg(t_0,x_0,y_0) \\ &= 8.0000 + 0.1g(0.0, 8.0000, 8.0000) \\ &= 8.0000 + (0.1)8.0000 = 8.8000. \end{aligned}$$

Now to find the approximate values at $t_2 = 0.1 + 0.1 = 0.2$, we use Formulas 8.75 with $n = 1$. We have

$$\begin{aligned} x_2 &= x_1 + hf(t_1,x_1,y_1) \\ &= 8.8000 + 0.1f(0.1, 8.8000, 8.8000) \\ &= 8.8000 + (0.1)8.8000 = 9.6800, \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hg(t_1,x_1,y_1) \\ &= 8.8000 + 0.1g(0.1, 8.8000, 8.8000) \\ &= 8.8000 + (0.1)8.8400 = 9.6840. \end{aligned}$$

The results for all $t_n$ and the corresponding errors have been summarized in Table 8.8.12-Euler.

Now we solve the problem by the Runge-Kutta method. Again, $f(t,x,y) = y$, $g(t,x,y) = 3y - 2x + 4t^2$, $t_0 = 0.0$, $x_0 = 8.0000000$, $y_0 = 8.0000000$, and $h = 0.1$. To find the approximate values at $t_1 = 0.0 + 0.1 = 0.1$, we use Formulas (8.78-80) with $n = 0$. By (8.80) we have

$$\begin{aligned} k_1 &= hf(t_0,x_0,y_0) \\ &= (0.1)f(0.0,\ 8.0000000,\ 8.0000000) \\ &= (0.1)8.0000000 = 0.8000000, \end{aligned}$$

$$\begin{aligned} m_1 &= hg(t_0,\ x_0,\ y_0) \\ &= (0.1)g(0.0,\ 8.0000000,\ 8.0000000) \\ &= (0.1)8.0000000 = 0.8000000. \end{aligned}$$

Using these values for $k_1$ and $m_1$ we now find

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)f(0.05,\ 8.4000000,\ 8.4000000) \\ &= (0.1)8.4000000 = 0.8400000, \end{aligned}$$

$$\begin{aligned} m_2 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{m_1}{2}\right) \\ &= (0.1)g(0.05,\ 8.4000000,\ 8.4000000) \\ &= (0.1)8.4100000 = 0.8410000. \end{aligned}$$

Next, the above values for $k_2$ and $m_2$ are used to obtain

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)f(0.05,\ 8.4200000,\ 8.4205000) \\ &= (0.2)8.4205000 = 0.8420500, \end{aligned}$$

$$\begin{aligned} m_3 &= hg\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{m_2}{2}\right) \\ &= (0.1)g(0.05,\ 8.4200000,\ 8.4205000) \\ &= (0.1)8.4315000 = 0.8431500. \end{aligned}$$

With these values for $k_3$ and $m_3$ we find

$$\begin{aligned} k_4 &= hf(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)f(0.1,\ 8.8420500,\ 8.8431500) \\ &= (0.1)8.8431500 = 0.8843150, \end{aligned}$$

$$\begin{aligned} m_4 &= hg(t_0 + h,\ x_0 + k_3,\ y_0 + m_3) \\ &= (0.1)g(0.1,\ 8.8420500,\ 8.8431500) \\ &= (0.1)8.8853500 = 0.8885350. \end{aligned}$$

Then from (8.79) we set

$$\begin{aligned} K &= \frac{1}{6}\ (0.8000000 + 2(0.8400000) + 2(0.8420500) \\ &\qquad + 0.8843150) \\ &= \frac{1}{6}\ (5.0484150) = 0.8414025, \end{aligned}$$

$$M = \frac{1}{6}\,(0.8000000 + 2(0.8410000) + 2(0.8431500) + 0.8885350)$$

$$= \frac{1}{6}\,(5.0568350) = 0.8428058.$$

Finally from (8.78) we obtain

$$x_1 = 8.0000000 + 0.8414025 = 8.8414025,$$

$$y_1 = 8.0000000 + 0.8428058 = 8.8428058.$$

These calculations and those for the remaining $t_n$ are summarized in Table 8.8.12-R-K.

**TABLE 8.8.12-Euler** Summary: Euler Method Solution (with h = 0.1) of the System

$x' = y,\ y' = 3y - 2x + 4t^2,\ x(0.0) = 8.0,\ y(0.0) = 8.0.$

| $t_n$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | Euler $\begin{cases} x_n \\ y_n \end{cases}$ | Errors | % Rel Errors |
|---|---|---|---|---|
| 0.1 | 8.84140 | 8.80000 | 0.04140 | 0.4683 |
| | 8.84281 | 8.80000 | 0.04281 | 0.4841 |
| 0.2 | 9.77182 | 9.68000 | 0.09182 | 0.9397 |
| | 9.78365 | 9.68400 | 0.09965 | 1.0185 |
| 0.3 | 10.80212 | 10.64840 | 0.15372 | 1.4230 |
| | 10.84424 | 10.66920 | 0.17504 | 1.6141 |
| 0.4 | 11.94554 | 11.71532 | 0.23022 | 1.9273 |
| | 12.05108 | 11.77628 | 0.27480 | 2.2803 |
| 0.5 | 13.21828 | 12.89295 | 0.32533 | 2.4612 |
| | 13.43656 | 13.03010 | 0.40646 | 3.0251 |

**TABLE 8.8.12-R-K** Summary: Runge-Kutta Solution (with h = 0.1) of the System

$$x' = y,\ y' = 3y - 2x + 4t^2,\ x(0.0) = 8.0,\ y(0.0) = 8.0.$$

| $t_n$ k | $k_1$, $m_1$ | $k_2$, $m_2$ | $k_3$, $m_3$ | $k_4$, $m_4$ | $K$, $M$ | Exact $\begin{Bmatrix} x(t_n) \\ y(t_n) \end{Bmatrix}$ | R-K $\begin{Bmatrix} x_n \\ y_n \end{Bmatrix}$ | Errors | % Rel Errors |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.8000 | 0.8400 | 0.8420 | 0.8843 | 0.8414 | 8.8414028 | 8.8414025 | 0.0000003 | 0.00000 |
| | 0.8000 | 0.8410 | 0.8431 | 0.8885 | 0.8428 | 8.8428055 | 8.8428058 | 0.0000003 | 0.00000 |
| 0.2 | 0.8843 | 0.9287 | 0.9312 | 0.9784 | 0.9304 | 9.7718247 | 9.7718236 | 0.0000011 | 0.00001 |
| | 0.8886 | 0.9384 | 0.9415 | 0.9968 | 0.9408 | 9.7836494 | 9.7836490 | 0.0000004 | 0.00000 |
| 0.3 | 0.9784 | 1.0282 | 1.0312 | 1.0845 | 1.0303 | 10.8021188 | 10.8021161 | 0.0000027 | 0.00003 |
| | 0.9967 | 1.0574 | 1.0615 | 1.1289 | 1.0606 | 10.8442376 | 10.8442349 | 0.0000027 | 0.00002 |
| 0.4 | 1.0844 | 1.1409 | 1.1446 | 1.2052 | 1.1434 | 11.9455409 | 11.9455354 | 0.0000055 | 0.00005 |
| | 1.1288 | 1.2027 | 1.2082 | 1.2904 | 1.2068 | 12.0510819 | 12.0510747 | 0.0000072 | 0.00006 |
| 0.5 | 1.2051 | 1.2696 | 1.2741 | 1.3438 | 1.2727 | 13.2182818 | 13.2182719 | 0.0000100 | 0.00008 |
| | 1.2902 | 1.3802 | 1.3873 | 1.4876 | 1.3855 | 13.4365637 | 13.4365489 | 0.0000148 | 0.00011 |

The exact solution to the stated problem is $x = e^{2t} + 2t^2 + 6t + 7$; we also have $y = x' = 2e^{2t} + 4t + 6$.

# Chapter 9

**Section 9.1A, Page 488**

1. We have

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty t^2 e^{-st} dt$$

$$= \lim_{R\to\infty} \left[ -\frac{e^{-st}}{s^3} (s^2t^2 + 2st + 2) \right]\Bigg|_0^R$$

$$= \lim_{R\to\infty} \left[ -\frac{e^{-sR}}{s^3} (s^2R^2 + 2sR + 2) \right] + \frac{2}{s^3}$$

$$= 0 + \frac{2}{s^3} = \frac{2}{s^3}, \text{ where } s > 0.$$

4. We have

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^3 4e^{-st} dt + \int_3^\infty 2e^{-st} dt$$

$$= \frac{-4e^{-st}}{s}\Bigg|_0^3 + \lim_{R\to\infty} \left[ \frac{-2e^{-st}}{s} \right]\Bigg|_3^R$$

$$= \frac{-4e^{-3s}}{s} + \frac{4}{s} + \lim_{R\to\infty} \left[ \frac{-2e^{-sR}}{s} + \frac{2e^{-3s}}{s} \right]$$

$$= \frac{-4e^{-3s}}{s} + \frac{4}{s} + 0 + \frac{2e^{-3s}}{s}$$

$$= \frac{2(2 - e^{-3s})}{s}, \text{ where } s > 0.$$

5. We have

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^2 te^{-st}\,dt + \int_2^{\infty} 3e^{-st}\,dt$$

$$= -\frac{e^{-st}(st+1)}{s^2}\Bigg|_0^2 + \lim_{R\to\infty}\left[\frac{-3e^{-st}}{s}\right]\Bigg|_2^R$$

$$= -\frac{e^{-2s}(2s+1)}{s^2} + \frac{1}{s^2} + \lim_{R\to\infty}\left[\frac{-3e^{-sR}}{s} + \frac{3e^{-2s}}{s}\right]$$

$$= \frac{-e^{-2s}(2s+1)}{s^2} + \frac{1}{s^2} + 0 + \frac{3e^{-2s}}{s}$$

$$= \frac{1}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}, \text{ where } s > 0.$$

8. We have

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^1 2te^{-st}\,dt + \int_1^3 2e^{-st}\,dt$$

$$+ \int_3^{\infty} (8-2t)e^{-st}\,dt$$

$$= \frac{2e^{-st}}{s^2}(-st-1)\Bigg|_0^1 - \frac{2e^{-st}}{s}\Bigg|_1^3$$

$$+ \lim_{R\to\infty}\left[-\frac{8e^{-st}}{s} - \frac{2e^{-st}}{s^2}(-st-1)\right]\Bigg|_3^R$$

$$= -\frac{2e^{-s}}{s^2}(s+1) + \frac{2}{s^2} - \frac{2e^{-3s}}{s} + \frac{2e^{-s}}{s}$$

$$+ \lim_{R\to\infty}\left[-\frac{8e^{-sR}}{s} + \frac{2e^{-sR}}{s^2}(sR+1)\right]$$

$$+ \left[\frac{8e^{-3s}}{s} - \frac{2e^{-3s}}{s^2}(3s + 1)\right]$$

$$= -\frac{2e^{-s}}{s} - \frac{2e^{-s}}{s^2} + \frac{2}{s^2} - \frac{2e^{-3s}}{s} + \frac{2e^{-s}}{s}$$

$$+ 0 + \frac{8e^{-3s}}{s} - \frac{6e^{-3s}}{s} - \frac{2e^{-3s}}{s^2}$$

$$= \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{2e^{-3s}}{s^2}$$

$$= \frac{2}{s^2}(1 - e^{-s} - e^{-3s}), \text{ where } s > 0.$$

**Section 9.1B, Page 496.**

2. We use the identity

$$\sin at \sin bt = \frac{1}{2}\cos(a - b)t - \frac{1}{2}\cos(a + b)t.$$

Applying Theorem 9.2, we have

$$\mathcal{L}\{\sin at \sin bt\} = \frac{1}{2}\mathcal{L}\{\cos(a - b)t\} - \frac{1}{2}\mathcal{L}\{\cos(a + b)t\}.$$

By (9.6), $\mathcal{L}\{\cos ct\} = \dfrac{s}{s^2 + c^2}$, $(s > 0)$, where c is a constant. Applying this with $c = a - b$ and $c = a + b$, we have

$$\mathcal{L}\{\sin at \sin bt\} = \frac{1}{2} \cdot \frac{s}{s^2 + (a - b)^2} - \frac{1}{2} \cdot \frac{s}{s^2 + (a + b)^2}$$

$$= \frac{s}{2} \cdot \frac{[s^2 + (a + b)^2] - [s^2 + (a - b)^2]}{[s^2 + (a - b)^2][s^2 + (a + b)^2]}$$

$$= \frac{2abs}{[s^2 + (a - b)^2][s^2 + (a + b)^2]}.$$

3. We use the identity $\sin^3 at = \frac{3 \sin at - \sin 3at}{4}$. Applying Theorem 9.2, we have

$$\mathcal{L}\{\sin^3 at\} = \frac{3\mathcal{L}\{\sin at\}}{4} - \frac{\mathcal{L}\{\sin 3at\}}{4}.$$

By (9.5), $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$ and $\mathcal{L}\{\sin 3at\} = \frac{3a}{s^2 + 9a^2}$. Thus

$$\mathcal{L}\{\sin^3 at\} = \frac{3a}{4(s^2 + a^2)} - \frac{3a}{4(s^2 + 9a^2)}$$

$$= \frac{6a^3}{(s^2 + a^2)(s^2 + 9a^2)}.$$

Now note that $(\sin^3 at)' = 3a \sin^2 at \cos at$. Then by Theorem 9.3, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$, so with $f(t) = \sin^3 at$ we have, $\mathcal{L}\{3a \sin^2 at \cos at\} = s\mathcal{L}\{\sin^3 at\} - 0$. Thus $\mathcal{L}\{\sin^2 at \cos at\} = \frac{s\mathcal{L}\{\sin^3 at\}}{3a}$

$$= \frac{2a^2 s}{(s^2 + a^2)(s^2 + 9a^2)}.$$

6. We apply Theorem 9.4 with $n = 2$. We have $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$. Letting $f(t) = t^4$,

$f'(t) = 4t^3$, $f''(t) = 12t^2$, this becomes $\mathcal{L}\{12t^2\} = s^2\mathcal{L}\{t^4\} - 0 - 0$, so $\mathcal{L}\{t^4\} = \dfrac{\mathcal{L}\{12t^2\}}{s^2} = \dfrac{24}{s^5}$.

7. We apply Theorem 9.2 to obtain $\mathcal{L}\{f''(t)\} + 3\mathcal{L}\{f'(t)\} + 2\mathcal{L}\{f(t)\} = \mathcal{L}\{0\}$. Then, using (9.11) and (9.13), we have $[s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)] + 3[s\mathcal{L}\{f(t)\} - f(0)] + 2\mathcal{L}\{f(t)\} = 0$. Now applying the conditions $f(0) = 1$, $f'(0) = 2$, this becomes

$$s^2\mathcal{L}\{f(t)\} - s - 2 + 3s\mathcal{L}\{f(t)\} - 3 + 2\mathcal{L}\{f(t)\} = 0,$$

or $(s^2 + 3s + 2)\mathcal{L}\{f(t)\} = s + 5$. Hence

$$\mathcal{L}\{f(t)\} = \frac{s + 5}{s^2 + 3s + 2}.$$

9. We take the Laplace Transform of both sides of $f'''(t) = f'(t)$ to obtain

$$\mathcal{L}\{f'''(t)\} = \mathcal{L}\{f'(t)\}. \tag{1}$$

By (9.17) with $n = 3$, we have

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0).$$

By (9.11), $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$. Using these expressions, (1) becomes

$$s^3\mathcal{L}\{ft)\} - s^2f(0) - sf'(0) - f''(0) = s\mathcal{L}\{f(t)\} - f(0).$$

Now apply the conditions $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$ to obtain

$$s^3\mathcal{L}\{f(t)\} - s - 2 = s\mathcal{L}\{f(t)\}.$$

From this,

$$(s^3 - s)\mathcal{L}\{f(t)\} = s + 2$$

and hence

$$\mathcal{L}\{f(t)\} = \frac{s + 2}{s^3 - s}.$$

10. We take the Laplace Transform of both sides of $f^{iv}(t) = f''(t)$, to obtain

$$\mathcal{L}\{f^{iv}(t)\} = \mathcal{L}\{f''(t)\}. \tag{1}$$

By (9.17), with n = 4, we have

$$\mathcal{L}\{f^{iv}(t)\} = s^4\mathcal{L}\{f(t)\} - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0).$$

By (9.18), $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$. Using these expressions, (1) becomes

$$s^4\mathcal{L}\{f(t)\} - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Now apply the conditions $f(0) = -1$, $f'(0) = 0$, $f''(0) = 0$, $f'''(0) = 1$ to obtain

$$s^4\mathcal{L}\{f(t)\} + s^3 - 1 = s^2\mathcal{L}\{f(t)\} + s.$$

From this, $(s^4 - s^2)\mathcal{L}\{f(t)\} = -s^3 + s + 1$, and hence

$$\mathcal{L}\{f(t)\} = \frac{-s^3 + s + 1}{s^4 - s^2}.$$

12. We apply Theorem 9.2 to obtain

$$3\mathcal{L}\{f''(t)\} - 5\mathcal{L}\{f'(t)\} + 7\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\}. \tag{1}$$

Then by formula (9.18), $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$; and by formula (9.11), $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$. By the given conditions, these become $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - 4s - 6$ and $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - 4$. Substituting these expressions into the left member of (1), the left member becomes $3s^2\mathcal{L}\{f(t)\} - 12s - 18 - 5s\mathcal{L}\{f(t)\} + 20 + 7\mathcal{L}\{f(t)\}$ or $(3s^2 - 5s + 7)\mathcal{L}\{f(t)\} - 12s + 2$. By Example 9.4, $\mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2 + 4}$; and so the right member of (1) becomes $\dfrac{2}{s^2 + 4}$. Thus (1) becomes

$$(3s^2 - 5s + 7)\mathcal{L}\{f(t)\} - 12s + 2 = \frac{2}{s^2 + 4}.$$

From this,

$$(3s^2 - 5s + 7)\mathcal{L}\{f(t)\} = 12s - 2 + \frac{2}{s^2 + 4}$$

and hence

$$\mathcal{L}\{f(t)\} = \frac{12s - 2}{3s^2 - 5s + 7} + \frac{2}{(3s^2 - 5s + 7)(s^2 + 4)}.$$

13. We let $f(t) = t^2$. Then, using information provided in Exercises 5 or 6, $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} = \dfrac{2}{s^3}$. Then by Theorem 9.5, we have

$$\mathcal{L}\{e^{at}t^2\} = \mathcal{L}\{e^{at}f(t)\} = F(s - a) = \frac{2}{(s - a)^3}.$$

16. We let $f(t) = \sin bt$. Then $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin bt\} = \dfrac{b}{s^2 + b^2}$. Then by Theorem 9.6,

$$\mathcal{L}\{t^3 \sin bt\} = (-1)^3 \frac{d^3}{ds^3}[F(s)]. \qquad (1)$$

As in Example 9.18, $\frac{d^2}{ds^2}[F(s)] = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}$. From this, $\frac{d^3}{ds^3}[F(s)] = \frac{24bs(b^2 - s^2)}{(s^2 + b^2)^4}$. Hence by (1),

$$\mathcal{L}\{t^3 \sin bt\} = \frac{24bs(s^2 - b^2)}{(s^2 + b^2)^4}.$$

**Section 9.2A, Page 504.**

1. $\mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{3}{s - 5}\right\} = 2 + 3e^{5t}$, where we used Table 9.1, number 1, and number 2 with a = 5, respectively.

4. $\mathcal{L}^{-1}\left\{\frac{2s}{s^2 + 9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 3^2}\right\} = 2\cos 3t$, when we used Table 9.1, number 4, with b = 3.

6. $\mathcal{L}^{-1}\left\{\frac{5s + 6}{s^3}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = 5t + 3t^2$, where we used Table 9.1, number 7, with n = 1 and n = 2, respectively.

7. We write

$$F(s) = \frac{s + 2}{s^2 + 4s + 7} = \frac{s + 2}{(s + 2)^2 + (\sqrt{3})^2}.$$

Then using Table 9.1, number 12, with $a = 2$, $b = \sqrt{3}$, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s + 2}{(s + 2)^2 + (\sqrt{3})^2}\right\} = e^{-2t}\cos\sqrt{3}t.$$

10. We write $\frac{2s + 3}{s^2 - 4} = 2\left(\frac{s}{s^2 - 4}\right) + \frac{3}{2}\left(\frac{2}{s^2 - 4}\right)$. By Table 9.1, number 6, $\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 4}\right\} = \cosh 2t$; and by number 5, $\mathcal{L}^{-1}\left\{\frac{2}{s^2 - 4}\right\} = \sinh 2t$. Thus $\mathcal{L}^{-1}\left\{\frac{2s + 3}{s^2 - 4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 4}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2 - 4}\right\} = 2\mathcal{L}^{-1}\cosh 2t + \frac{3}{2}\sinh 2t$.

12. We employ partial fractions. We have

$$\frac{2s + 6}{8s^2 - 2s - 3} = \frac{A}{4s - 3} + \frac{B}{2s + 1}$$

and hence $2s + 6 = A(2s + 1) + B(4s - 3)$. Letting $s = -\frac{1}{2}$, we find $B = -1$; and letting $s = \frac{3}{4}$. we find $A = 3$. Thus we have the partial fractions decomposition

$$\frac{2s + 6}{8s^2 - 2s - 3} = \frac{3}{4s - 3} - \frac{1}{2s + 1} = \frac{3}{4\left(s - \frac{3}{4}\right)} - \frac{1}{2\left(s + \frac{1}{2}\right)}.$$

Then, using Table 9.1, number 2, we find

$$\mathcal{L}^{-1}\{F(s)\} = \frac{3e^{3t/4}}{4} - \frac{e^{-t/2}}{2}.$$

13. We express F(s) as follows:

$$F(s) = \frac{5s}{s^2 + 4s + 4} = \frac{5(s + 2) - 10}{(s + 2)^2} = \frac{5}{s + 2} - \frac{10}{(s + 2)^2}.$$

Now, using Table 9.1, number 2 with a = -2, and number 8 with a = -2 and n = 1, we find

$$\mathcal{L}^{-1}\{F(s)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\} - 10\mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2}\right\}$$

$$= 5e^{-2t} - 10te^{-2t} = 5e^{-2t}(1 - 2t).$$

14. We first employ partial fractions. We have $\frac{s + 1}{s^3 + 2s}$ $= \frac{A}{s} + \frac{Bs + c}{s^2 + 2}$ and hence $s + 1 = (A + B)s^2 + Cs + 2A$. Thus $A + B = 0$, $C = 1$, $2A = 1$. Hence $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = 1$; and we have the partial fractions decomposition

$$\frac{s + 1}{s^3 + 2s} = \frac{1}{2s} - \frac{s}{2(s^2 + 2)} + \frac{1}{s^2 + 2};$$

so

$$\mathcal{L}^{-1}\left\{\frac{s + 1}{s^3 + 2s}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2}\right\}$$

$$+ \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2 + 2}\right\}.$$

Then using Table 9.1, numbers 1, 4, and 3, respectively, we find

$$\mathcal{L}^{-1}\left\{\frac{s + 1}{s^2 + 2s}\right\} = \frac{1}{2} - \frac{1}{2}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t.$$

16. We write

$$F(s) = \frac{2s + 7}{(s + 3)^4} = \frac{2(s + 3) + 1}{(s + 3)^4} = \frac{2}{(s + 3)^3} + \frac{1}{(s + 3)^4}.$$

Then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2!}{(s + 3)^3}\right\} + \frac{1}{6}\,\mathcal{L}^{-1}\left\{\frac{3!}{(s + 3)^4}\right\}.$$

Using Table 9.1, number 8, with a = -3 and n = 2 and 3, respectively, we find

$$\mathcal{L}^{-1}\{F(s)\} = t^2e^{-3t} + \frac{t^3e^{-3t}}{6} = t^2e^{-3t}\left(1 + \frac{1}{6}\right).$$

18. We write $\dfrac{8(s + 1)}{(2s + 1)^3} = \dfrac{8\left(s + \frac{1}{2}\right) + 4}{\left[2\left(s + \frac{1}{2}\right)\right]^3} = \dfrac{1}{\left(s + \frac{1}{2}\right)^2}$ $+ \dfrac{1}{4} \cdot \dfrac{2}{\left(s + \frac{1}{2}\right)^3}$. By Table 9.1, number 8, with n = 1, $\mathcal{L}^{-1}\left\{\dfrac{1}{\left(s + \frac{1}{2}\right)^2}\right\} = te^{-t/2}$; and by number 8, with n = 2, $\mathcal{L}^{-1}\left\{\dfrac{2}{\left(s + \frac{1}{2}\right)^3}\right\} = t^2e^{-t/2}$. Thus $\mathcal{L}^{-1}\left\{\dfrac{8(s + 1)}{(2s - 1)^3}\right\} = te^{-t/2}$ $+ \dfrac{1}{4}\,t^2e^{-t/2}$.

20. Write

$$\frac{s^2 - 4s - 4}{(s^2 + 4)^2} = \frac{s^2 - 4}{(s^2 + 4)^2} - \frac{4s}{(s^2 + 4)^2}.$$

By Table 9.1, number 10, with b = 2,

$$\mathcal{L}^{-1}\left\{\frac{s^2 - 4}{(s^2 + 4)^2}\right\} = t \cos 2t.$$

By number 9, with b = 2,

$$\mathcal{L}^{-1}\left\{\frac{4s}{(s^2 + 4)^2}\right\} = t \sin 2t.$$

Thus $\mathcal{L}^{-1}\left\{\frac{s^2 - 4s - 4}{(s^2 + 4)^2}\right\} = t \cos 2t - t \sin 2t$

$= t(\cos 2t - \sin 2t).$

22. Write

$$\frac{5s + 17}{s^2 + 4s + 13} = \frac{5(s + 2) + \frac{7}{3}(3)}{(s + 2)^2 + (3)^2}$$

$$= 5\left[\frac{s + 2}{(s + 2)^2 + (3)^2}\right] + \frac{7}{3}\left[\frac{3}{(s + 2)^2 + (3)^2}\right].$$

By Table 9.1, number 12, with a = 2, b = 3,

$$\mathcal{L}^{-1}\left\{\frac{s + 2}{(s + 2)^2 + (3)^2}\right\} = e^{-2t}\cos 3t.$$

By number 11, with a = 2, b = 3,

$$\mathcal{L}^{-1}\left\{\frac{3}{(s + 2)^2 + (3)^2}\right\} = e^{-2t}\sin 3t.$$

Thus $\mathcal{L}^{-1}\left\{\frac{5s + 17}{s^2 + 4s + 13}\right\} = 5e^{-2t}\cos 3t + \frac{7}{3}e^{-2t}\sin 3t$

$= e^{-2t}\left[5 \cos 3t + \frac{7}{3} \sin 3t\right].$

23. We first employ partial fractions,

$$\frac{10s + 23}{s^2 + 7s + 12} = \frac{A}{s + 3} + \frac{B}{s + 4}.$$

and hence $10s + 23 = A(s + 4) + B(s + 3)$. Letting $s = -3$, we find $A = -7$; and letting $s = -4$, we find $B = 17$. Thus

$$\mathcal{L}^{-1}\left\{\frac{10s + 23}{s^2 + 7s + 12}\right\} = -7\mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\} + 17\mathcal{L}^{-1}\left\{\frac{1}{s + 4}\right\}$$

$$= -7e^{-3t} + 17e^{-4t},$$

when we used Table 9.1, number 2, with $a = -3$ and $a = -4$, respectively.

25. We first employ partial fractions

$$\frac{1}{s^3 + 4s^2 + 3s} = \frac{1}{s(s + 1)(s + 3)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 3}.$$

Then, clearing fractions,

$$1 = A(s + 1)(s + 3) + Bs(s + 3) + Cs(s + 1).$$

Letting $s = 0$, we find $A = 1/3$; letting $s = -1$, we find $B = -1/2$; and letting $s = -3$, we find $C = 1/6$. Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3 + 4s^2 + 3s}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\}$$

$$+ \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\}.$$

By Table 9.1, number 1, $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$. By number 2, $\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} = e^{-t}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\} = e^{-3t}$. Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3 + 4s^2 + 3s}\right\} = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}.$$

26. We first employ partial fractions. We have

$$\frac{s+5}{s^4+3s^3+2s^2} = \frac{s+5}{s^2(s+1)(s+2)}$$
$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}.$$

and hence $s + 5 = As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$. Letting $s = 0$, we find $2B = 5$ so $B = \frac{5}{2}$; letting $s = -1$, we find $C = 4$; letting $s = -2$, we find $-4D = 3$, so $D = -\frac{3}{4}$; and letting $s = 1$, we have $6A + 6B + 3C + 2D = 6$ so $A = 1 - B - \frac{C}{2} - \frac{D}{3} = -\frac{13}{4}$. Thus we have the partial fractions decomposition

$$\frac{s+5}{s^4+3s^3+2s^2} = -\frac{13}{4}\left(\frac{1}{s}\right) + \frac{5}{2}\left(\frac{1}{s^2}\right) + 4\left(\frac{1}{s+1}\right) - \frac{3}{4}\left(\frac{1}{s+2}\right).$$

Then using Table 9.1, numbers 1, 7, 2, and 2 respectively, we find

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{13}{4} + \frac{5t}{2} + 4e^{-t} - \frac{3e^{-2t}}{4}.$$

27. We first employ partial fractions,

$$\frac{7s^2+8s+8}{s^3+4s} = \frac{7s^2+8s+8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

Clearing fractions, we have

$$7s^2 + 8s + 8 = A(s^2+4) + s(Bs+C)$$
$$= (A+B)s^2 + Cs + 4A.$$

Equating coefficients of like powers of s, we have $A + B = 7$, $C = 8$, $4A = 8$, and hence $A = 2$, $B = 5$, $C = 8$. Thus

$$\mathcal{L}^{-1}\left\{\frac{7s^2 + 8s + 8}{s^3 + 4s}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{5s + 8}{s^2 + 4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + 4\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}.$$

By Table 9.1, number 1, $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$. By number 4, with $b = 2$, $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} = \cos 2t$. By number 3, with $b = 2$, $\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} = \sin 2t$. Thus

$$\mathcal{L}^{-1}\left\{\frac{7s^2 + 8s + 8}{s^2 + 4s}\right\} = 2 + 5\cos 2t + 4\sin 2t.$$

28. We can first employ partial fractions,

$$\frac{3s^3 + 4s^2 - 16s + 16}{s^3(s - 2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s - 2} + \frac{E}{(s - 2)^2}.$$

Clearing fractions, expanding and multiplying, and then rearranging terms, we find

$$\begin{aligned} 3s^3 + 4s^2 - 16s + 16 &= As^2(s - 2)^2 + Bs(s - 2)^2 \\ &\quad + C(s - 2)^2 + Ds^3(s - 2) + Es^3 \\ &= (A + D)s^4 + (-4A + B - 2D + E)s^3 \\ &\quad + (4A - 4B + C)s^2 + (4B - 4C)s + (4C). \qquad (*) \end{aligned}$$

Letting $s = 0$, one quickly obtains $16 = 4C$, so $C = 4$. Letting $s = 2$, one easily finds $24 = 8E$, so $E = 3$. Then equating coefficients of s in the extreme members of (*),

one obtains $4B - 4C = -16$, from which $B = C - 4$. But $C = 4$, so $B = 0$. Next equating coefficients of $s^2$ in (*), one has $4A - 4B + C = 4$; but $B = 0$ and $C = 4$, so this reduces to $4A + 4 = 4$, so $A = 0$. Finally, equating coefficients of $s^4$ in (*) gives $A + D = 0$; and since $A = 0$, $D = 0$ also. Thus

$$\frac{3s^3 + 4s^2 - 16s + 16}{s^3(s - 2)^2} = \frac{4}{s^3} + \frac{3}{(s - 2)^2}.$$

Alternatively, we could have done this by careful observation. For, we have

$$\frac{3s^3 + 4s^2 - 16s + 16}{s^3(s - 2)^2} = \frac{3s^3}{s^3(s - 2)^2} + \frac{4(s^2 - 4s + 4)}{s^3(s - 2)^2}$$

$$= \frac{3}{(s - 2)^2} + \frac{4}{s^3}.$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{3s^3 + 4s^2 - 16s + 16}{s^3(s - 2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s - 2)^2}\right\}.$$

By Table 9.1, number 7, with $n = 2$, $\mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$ $= 2t^2$. By number 8, with $n = 1$ and $a = 2$, $\mathcal{L}^{-1}\left\{\frac{3}{(s - 2)^2}\right\}$ $= 3\mathcal{L}^{-1}\left\{\frac{1}{(s - 2)^2}\right\} = 3te^{2t}$. Thus, finally,

$$\mathcal{L}^{-1}\left\{\frac{3s^3 + 4s^2 - 16s + 16}{s^3(s - 2)^2}\right\} = 2t^2 + 3te^{2t}.$$

**Section 9.2B, Page 509.**

4. We write $\dfrac{1}{s(s^2 + 4s + 13)}$ as the product $F(s)G(s)$, where $F(s) = \dfrac{1}{s}$ and $G(s) = \dfrac{1}{s^2 + 4s + 13}$. By Table 9.1, number 1, $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$, and by number 11,

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4s + 13}\right\} = \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{(s + 2)^2 + 9}\right\}$$

$$= \frac{1}{3}\,e^{-2t}\sin 3t.$$ Then

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} = f(t)*g(t)$$

$$= \int_0^t f(\tau)g(t - \tau)\,d\tau$$

$$= \int_0^t [1]\left[\frac{1}{3}\,e^{-2(t-\tau)}\sin 3(t - \tau)\right]d\tau$$

or

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{G(s)F(s)\} = g(t)*f(t)$$

$$= \int_0^t g(\tau)f(t - \tau)\,d\tau$$

$$= \int_0^t \left[\frac{1}{3}\,e^{-2\tau}\sin 3\tau\right][1]\,d\tau.$$

We evaluate the latter of these two integral expressions.

$$\mathcal{L}^{-1}\{H(s)\} = \frac{1}{3}\,\frac{e^{-2\tau}}{13}\,(-2\sin 3\tau - 3\cos 3\tau)\Big|_0^t$$

$$= \frac{1}{39}\,[3 - e^{-2t}(2\sin 3t + 3\cos 3t)].$$

5. We write $H(s) = \dfrac{1}{s^2(s+3)}$ as the product $F(s)G(s)$, where $F(s) = \dfrac{1}{s^2}$ and $G(s) = \dfrac{1}{s+3}$. By Table 9.1, number 7, $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$, and by number 2, $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+3}\right\} = e^{-3t}$. Then

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= \mathcal{L}^{-1}\{F(s)G(s)\} = f(t)*g(t)\\ &= \int_0^t f(\tau)g(t-\tau)d\tau\\ &= \int_0^t \tau e^{-3(t-\tau)}d\tau.\end{aligned}$$

or

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= \mathcal{L}^{-1}\{G(s)F(s)\} = g(t)*f(t)\\ &= \int_0^t g(\tau)f(t-\tau)d\tau\\ &= \int_0^t e^{-3\tau}(t-\tau)d\tau.\end{aligned}$$

We evaluate the former of these two integral expressions. We find

$$\mathcal{L}^{-1}\{H(s)\} = e^{-3t}\left[e^{3\tau}\,\frac{(3\tau-1)}{9}\right]_0^t = \frac{-1+3t+e^{-3t}}{9}.$$

6. We write $H(s) = \dfrac{1}{(s+2)(s^2+1)}$ as the product $F(s)G(s)$, where $F(s) = \dfrac{1}{s+2}$ and $G(s) = \dfrac{1}{s^2+1}$. By Table 9.1,

numbers 2 and 3, respectively, $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$. Then

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= \mathcal{L}^{-1}\{F(s)G(s)\} = f(t)*g(t)\\ &= \int_0^t f(\tau)g(t-\tau)d\tau\\ &= \int_0^t e^{-2\tau}\sin(t-\tau)d\tau\end{aligned}$$

or

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= \mathcal{L}^{-1}\{G(s)F(s)\} = g(t)*f(t)\\ &= \int_0^t g(\tau)f(t-\tau)d\tau\\ &= \int_0^t [\sin\tau]e^{-2(t-\tau)}d\tau.\end{aligned}$$

We evaluate the latter of these two integral expressions. We find

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= e^{-2t}\cdot\frac{e^{2\tau}}{5}[2\sin\tau - \cos\tau]\Big|_0^t\\ &= e^{-2t}\left[\frac{e^{2t}}{5}(2\sin t - \cos t) + \frac{1}{5}\right]\\ &= \frac{2\sin t - \cos t + e^{-2t}}{5}.\end{aligned}$$

**Section 9.3, Page 519.**

3. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 6\mathcal{L}\{e^{-t}\}. \qquad (1)$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.3, we have

$$\mathcal{L}\{y'\} = sY(s) - y(0).$$

Applying the I.C., this becomes

$$\mathcal{L}\{y'\} = sY(s) - 5.$$

Substituting this into the left member of (1) it becomes $sY(s) - 5 + 4Y(s)$. By Table 9.1, number 2, the right member becomes $\frac{6}{s+1}$. Thus (1) becomes

$$(s + 4)Y(s) - 5 = \frac{6}{s+1}.$$

Step 2. Solving this for $Y(s)$, we obtain

$$Y(s) = \frac{5}{s+4} + \frac{6}{(s+1)(s+4)},$$

that is

$$Y(s) = \frac{5s + 11}{(s+1)(s+4)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{5s + 11}{(s+1)(s+4)}\right\}.$$

We employ partial fractions. We have

$$\frac{5s + 11}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}.$$

Then $5s + 11 = A(s + 4) + B(s + 1)$. Letting $s = -1$, we find $A = 2$; and letting $s = -4$, we find $B = 3$. Thus

$$\frac{5s + 11}{(s + 1)(s + 4)} = \frac{2}{s + 1} + \frac{3}{s + 4}.$$

Thus

$$y = 2\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s + 4}\right\}.$$

Then using Table 9.1, number 2, with $a = -1$ and $a = -4$, respectively, we obtain

$$y = 2e^{-t} + 3e^{-4t}.$$

4. Step 2. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 16\mathcal{L}\{t^2\}. \qquad (1)$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.3, we have $\mathcal{L}\{y'\} = sY(s) - y(0)$. Applying the I.C., this becomes $\mathcal{L}\{y'\} = sY(s) - 7$. Substituting this into the left member of (1) it becomes $sY(s) - 7 + 2Y(s)$. By Table 9.1, number 7, the right member becomes $\frac{32}{s^3}$. Thus (1) becomes

$$(s + 2)Y(s) - 7 = \frac{32}{s^3}.$$

Step 2. Solving this for $Y(s)$, we obtain

$$Y(s) = \frac{7}{s + 2} + \frac{32}{s^3(s + 2)} = \frac{7s^3 + 32}{s^3(s + 2)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{7s^3 + 32}{s^3(s + 2)}\right\}.$$

We employ partial fractions. We have

$$\frac{7s^3 + 32}{s^3(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s + 2}.$$

Then $7s^3 + 32 = As^2(s + 2) + Bs(s + 2) + C(s + 2) + Ds^3$ $= (A + D)s^3 + (2A + B)s^2 + (2B + C)s + 2C$. Letting $s = 0$ in this, we obtain $2C = 32$, so $C = 16$; letting $s = -2$, we have $-8D = -24$, so $D = 3$. Then equating coefficients of $s^3$, we have $A + D = 7$, so $A = 4$; and equating coefficients of $s^2$, we have $2A + B = 0$, so $B = -8$. Thus we find

$$\frac{7s^3 + 32}{s^3(s + 2)} = \frac{4}{s} - \frac{8}{s^2} + \frac{16}{s^3} + \frac{3}{s + 2}.$$

Then

$$y = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 8\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 8\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\}.$$

Finally, using Table 9.1, numbers 1, 7 (with $n = 1$), 7 (with $n = 2$), and 2, respectively, we find

$$y = 4 - 8t + 8t^2 + 3e^{-2t}.$$

5. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{0\}. \qquad (1)$$

Denoting $\mathcal{L}\{y(t)\}$ by $Y(s)$ and applying Theorem 9.4, we have the following expressions for $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0),$$
$$\mathcal{L}\{y'\} = sY(s) - y(0).$$

Applying the I.C.'s to these, they become

$$\mathcal{L}\{y''\} = s^2Y(s) - s - 2, \quad \mathcal{L}\{y'\} = sY(s) - 1.$$

Substituting these expressions into the left member of (1) and using $\mathcal{L}(0) = 0$, (1) becomes $[s^2Y(s) - s - 2] - 5[sY(s) - 1] + 6Y(s) = 0$ or $(s^2 - 5s + 6)Y(s) - s + 3 = 0$.

Step 2. Solving the preceding for $Y(s)$, we have

$$Y(s) = \frac{s - 3}{s^2 - 5s + 6} = \frac{1}{s - 2}.$$

Step 3. We must now determine

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\}.$$

By Table 9.1, number 2, we immediately find

$$y = e^{2t}.$$

7. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}(0). \qquad (1)$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.4, we have $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$, $\mathcal{L}(y'\} = sY(s) - y(0)$. Applying the I.C.'s, these become $\mathcal{L}\{y''\} = s^2Y(s) - 2s - 9$, $\mathcal{L}\{y'\} = sY(s) - 2$. Substituting into the left member of (1) and using $\mathcal{L}(0) = 0$, (1) becomes

$$s^2Y(s) - 2s - 9 - 6sY(s) + 12 + 9Y(s) = 0$$

or

$$(s^2 - 6s + 9)Y(s) - 2s + 3 = 0.$$

Step 2. Solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s - 3}{s^2 - 6s + 9} = \frac{2s - 3}{(s - 3)^2}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{2s - 3}{(s - 3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s - 3) + 3}{(s - 3)^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s - 3}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s - 3)^2}\right\}.$$

By Table 9.1, numbers 2 and 8, respectively, $\mathcal{L}^{-1}\left\{\frac{1}{s - 3}\right\} = e^{3t}$ and $\mathcal{L}^{-1}\left\{\frac{1}{(s - 3)^2}\right\} = te^{3t}$. Thus

$$y = 2e^{3t} + 3te^{3t} = (3t + 2)e^{3t}.$$

10. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = 36\mathcal{L}\{e^{-3t}\}. \qquad (1)$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.4, we have the following expression for $\mathcal{L}\{y''\}$:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0).$$

Applying the I.C.'s to this, it becomes

$$\mathcal{L}\{y''\} = s^2Y(s) - 2s - 3.$$

Thus the left member of (1) becomes $s^2Y(s) - 2s - 3 + 9Y(s)$. By Table 9.1, number 2, the right member of (1) becomes $\frac{36}{s+3}$. Thus equation (1) reduces to

$$(s^2 + 9)Y(s) - 2s - 3 = \frac{36}{s+3}.$$

Step 2. Solving this for $Y(s)$, we obtain

$$Y(s) = \frac{2s+3}{s^2+9} + \frac{36}{(s+3)(s^2+9)},$$

that is,

$$Y(s) = \frac{2s^2 + 9s + 45}{(s+3)(s^2+9)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{2s^2 + 9s + 45}{(s+3)(s^2+9)}\right\}.$$

We employ partial fractions. We have

$$\frac{2s^2 + 9s + 45}{(s+3)(s^2+9)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+9}.$$

Then $2s^2 + 9s + 45 = A(s^2 + 9) + (Bs + C)(s + 3)$ $= (A + B)s^2 + (3B + C)s + (9A + 3C)$. From this, $A + B = 2$, $3B + C = 9$, $9A + 3C = 45$; and hence $A = 2$, $B = 0$, $C = 9$. Thus

$$\frac{2s^2 + 9s + 45}{(s + 3)(s^2 + 9)} = \frac{2}{s + 3} + \frac{9}{s^2 + 9}.$$

Therefore,

$$y = 2\mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\} + 3\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\}.$$

Then by Table 9.1, numbers 2 and 3, respectively, we find

$$y = 2e^{-3t} + 3 \sin 3t.$$

12. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} = 5\mathcal{L}\{e^{2t}\}. \qquad (1)$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.4, we have

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0),$$
$$\mathcal{L}\{y'\} = sY(s) - y(0).$$

Applying the I.C. to these, they become

$$\mathcal{L}\{y''\} = s^2Y(s) - 2s,$$
$$\mathcal{L}\{y'\} = sY(s) - 2.$$

Thus the left member of (1) becomes $2s^2Y(s) - 4s + sY(s) - 2$. By Table 9.1, number 2, the right member of (1) becomes $\frac{5}{s - 2}$. Thus equation (1) reduces to

$$(2s^2 + s)Y(s) - 4s - 2 = \frac{5}{s - 2}.$$

Step 2. Solving for $Y(s)$, we obtain

$$Y(s) = \frac{4s + 2}{2s^2 + s} + \frac{5}{(s - 2)(2s^2 + s)},$$

that is,

$$Y(s) = \frac{2}{s} + \frac{5}{s(s - 2)(2s + 1)},$$

or

$$Y(s) = \frac{4s^2 - 6s + 1}{s(s - 2)(2s + 1)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{4s^2 - 6s + 1}{s(s - 2)(2s + 1)}\right\}. \tag{2}$$

We first employ partial fractions. We have

$$\frac{4s^2 - 6s + 1}{s(s - 2)(2s + 1)} = \frac{A}{s} + \frac{B}{s - 2} + \frac{C}{2s + 1}.$$

Then $4s^2 - 6s + 1 = A(s - 2)(2s + 1) + Bs(2s + 1) + Cs(s - 2)$. Letting $s = 0$, we find $A = -1/2$; letting $s = 2$, we find $B = 1/2$; and letting $s = -1/2$, we find $C = 4$. Thus

$$\frac{4s^2 - 6s + 1}{s(s - 2)(2s + 1)} = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s - 2}\right) + \frac{4}{2s + 1}.$$

Thus

$$y = -\frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s + 1/2}\right\}.$$

Then by Table 9.1, numbers 1, 2, and 2, respectively, we find

$$y = -\frac{1}{2} + \frac{1}{2}e^{2t} + 2e^{-t/2}.$$

14. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y^{iv}\} - 2\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}. \tag{1}$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.4 with $n = 4$ and $n = 2$, respectively, we obtain

$$\mathcal{L}\{y^{iv}\} = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0),$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0).$$

Applying the I.C.'s to these, they become

$$\mathcal{L}\{y^{iv}\} = s^4Y(s) - 4s^2 - 8,$$

$$\mathcal{L}\{y''\} = s^2Y(s) - 4.$$

Substituting into the left member of (1) and using $\mathcal{L}\{0\} = 0$, (1) becomes $s^4Y(s) - 4s^2 - 8 - 2s^2Y(s) + 8 + Y(s) = 0$ or

$$(s^4 - 2s^2 + 1)Y(s) - 4s^2 = 0.$$

Step 2. Solving for $Y(s)$, we find

$$Y(s) = \frac{4s^2}{s^4 - 2s^2 + 1} = \frac{4s^2}{(s^2 - 1)^2}$$

$$= \frac{4s^2}{(s - 1)^2(s + 1)^2}.$$

Step 3. We must now find

$$\mathcal{L}^{-1}\left\{\frac{4s^2}{(s - 1)^2(s + 1)^2}\right\}.$$

We employ partial fractions

$$\frac{4s^2}{(s-1)^2(s+1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}.$$

Then $4s^2 = A(s-1)(s+1)^2 + B(s+1)^2 + C(s+1)(s-1)^2 + D(s-1)^2 = (A+C)s^3 + (A+B-C+D)s^2 + (-A+2B-C-2D)s + (-A+B+C+D) = 0$. First, letting $s = 1$ in this gives $4B = 4$, so $B = 1$. Then letting $s = -1$, we get $4D = 4$, so $D = 1$. Next, equating coefficients of $s^3$ gives $A + C = 0$, so $C = -A$. Equating coefficients of $s^2$ then gives $A + B - C + D = 4$. Since $B = D = 1$ and $C = -A$, this reduces to $2A + 2 = 4$, from which $A = 1$, and then $C = -1$. Thus we find

$$\frac{4s^2}{(s-1)^2(s+1)^2} = \frac{1}{s-1} + \frac{1}{(s-1)^2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}.$$

So

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}.$$

Using Table 9.1, numbers 2, 8, 2, and 8, respectively, we find

$$\begin{aligned} y &= e^t + te^t - e^{-t} + te^{-t} \\ &= (t+1)e^t + (t-1)e^{-t}. \end{aligned}$$

15. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{18e^{-t}\sin 3t\}. \qquad (1)$$

Denoting $\mathcal{L}(y(t)\}$ by $Y(s)$ and applying Theorem 9.4, we have the following expressions for $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0),$$
$$\mathcal{L}\{y'\} = sY(s) - y(0).$$

Applying the I.C.'s to these, they become

$$\mathcal{L}\{y''\} = s^2Y(s) - 3,$$
$$\mathcal{L}\{y'\} = sY(s).$$

Substituting these expressions into the left member of (1), this left member becomes $[s^2Y(s) - 3] - sY(s) - 2Y(s)$ or $(s^2 - s - 2)Y(s) - 3$. By Table 9.1, number 11, the right member of (1) becomes,

$$\frac{54}{(s+1)^2 + 9}.$$

Thus (1) reduces to

$$(s^2 - s - 2)Y(s) - 3 = \frac{54}{(s+1)^2 + 9}.$$

Step 2. Solving the preceding for $Y(s)$, we have

$$Y(s) = \frac{3s^2 + 6s + 84}{(s+1)(s-2)[(s+1)^2 + 9]}.$$

Step 3. We must now determine

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3s^2 + 6s + 84}{(s+1)(s-2)(s^2 + 2s + 10)}\right\}.$$

We employ partial fractions. We have

$$\frac{3s^2 + 6s + 84}{(s + 1)(s - 2)(s^2 + 2s + 10)} = \frac{A}{s + 1} + \frac{B}{s - 2} + \frac{Cs + D}{s^2 + 2s + 10}$$

or

$$\begin{aligned} 3s^2 + 6s + 84 &= A(s - 2)(s^2 + 2s + 10) \\ &\quad + B(s + 1)(s^2 + 2s + 10) \\ &\quad + (Cs + D)(s + 1)(s - 2) \end{aligned} \tag{2}$$

or

$$\begin{aligned} 3s^2 + 6s + 84 = (A + B + C)s^3 &+ (3B - C + D)s^2 \\ &+ (6A + 12B - 2C - D)s \\ &+ (-20A + 10B - 2D). \end{aligned}$$

From this, we obtain

$$\begin{cases} A + B + C = 0, \quad 3B - C + D = 3, \\ 6A + 12B - 2C - D = 6, \quad -20A + 10B - 2D = 84. \end{cases} \tag{3}$$

Letting $s = -1$ in (2), we find $-27A = 81$, so $A = -3$; and letting $s = 2$ in (2), we find $54B = 108$, so $B = 2$. Using the values for A and B, we find from (3), that $C = 1$, $D = -2$. Thus we have

$$\begin{aligned} &\mathcal{L}^{-1}\left\{\frac{3s^2 + 6s + 84}{(s + 1)(s - 2)(s^2 + 2s + 10)}\right\} \\ &\quad = -3\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} \end{aligned}$$

$$+ \mathcal{L}^{-1}\left\{\frac{s - 2}{s^2 + 2s + 10}\right\}$$

$$= -3\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\}$$

$$+ \mathcal{L}^{-1}\left\{\frac{s + 1}{(s + 1)^2 + 9}\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{3}{(s + 1)^2 + 9}\right\}.$$

Then by Table 9.1, numbers 2, 2, 12, and 11, respectively, we find

$$y = -3e^{-t} + 2e^{2t} + e^{-t}\cos 3t - e^{-t}\sin 3t.$$

19. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 10\mathcal{L}\{\cos t\}. \qquad (1)$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.4 we have

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0),$$
$$\mathcal{L}\{y'\} = sY(s) - y(0).$$

Applying the I.C.'s to these, they become

$$\mathcal{L}\{y''\} = s^2Y(s) - 7,$$
$$\mathcal{L}\{y'\} = sY(s).$$

Substituting into the left member of (1), it becomes $s^2Y(s) - 7 + 3sY(s) + 2Y(s)$. By Table 9.1, number 4, the right member becomes $\frac{10s}{s^2 + 1}$. Thus (1) becomes

$$(s^2 + 3s + 2)Y(s) - 7 = \frac{10s}{s^2 + 1}.$$

Step 2. Solving for Y(s), we obtain

$$Y(s) = \frac{7}{s^2 + 3s + 2} + \frac{10s}{(s^2 + 3s + 2)(s^2 + 1)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{7}{s^2 + 3s + 2} + \frac{10s}{(s^2 + 3s + 2)(s^2 + 1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{7s^2 + 10s + 7}{(s + 1)(s + 2)(s^2 + 1)}\right\}.$$

We first employ partial fractions. We have

$$\frac{7s^2 + 10s + 7}{(s + 1)(s + 2)(s^2 + 1)} = \frac{A}{s + 1} + \frac{B}{s + 2} + \frac{Cs + D}{s^2 + 1}.$$

Then $7s^2 + 10s + 7 = A(s + 2)(s^2 + 1) + B(s + 1)(s^2 + 1) + (Cs + D)(s + 1)(s + 2)$. Letting $s = -1$, we have $2A = 4$, so $A = 2$; and letting $s = -2$, gives $-5B = 15$, so $B = -3$. Letting $s = 0$, we obtain $2A + B + 2D = 7$. With $A = 2$, $B = -3$, this gives $D = 3$. Finally, letting $s = 1$, we find $6A + 4B + 6(C + D) = 24$. With $A = 2$, $B = -3$, $D = 3$, this gives $C = 1$. Thus we have

$$\frac{7s^2 + 10s + 7}{(s + 1)(s + 2)(s^2 + 1)} = \frac{2}{s + 1} - \frac{3}{s + 2} + \frac{s + 3}{s^2 + 1}.$$

Thus

$$y = 2\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}.$$

Then using Table 9.1, numbers 2, 2, 4, and 3, we have

$$y = 2e^{-t} - 3e^{-2t} + \cos t + 3 \sin t.$$

20. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 6\mathcal{L}\{te^{-t}\} + 8\mathcal{L}\{e^{-t}\}. \tag{1}$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and applying Theorem 9.4 we have

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0),$$
$$\mathcal{L}\{y'\} = sY(s) - y(0).$$

Applying the I.C.'s, these become

$$\mathcal{L}\{y''\} = s^2Y(s) - s - 1,$$
$$\mathcal{L}\{y'\} = sY(s) - 1.$$

Substituting into the left member of (1), it becomes $s^2Y(s) - s - 1 + 5sY(s) - 5 + 4Y(s)$ or $(s^2 + 5s + 4)Y(s) - s - 6$. Using Table 9.1, numbers 8 and 2, the right member of (1) becomes $\frac{6}{(s + 1)^2} + \frac{8}{s + 1}$. Thus (1) becomes

$$(s^2 + 5s + 4)Y(s) - s - 6 = \frac{6}{(s + 1)^2} + \frac{8}{s + 1}.$$

Step 2. Solving for $Y(s)$, we obtain

$$Y(s) = \frac{s + 6}{s^2 + 5s + 4} + \frac{8}{(s^2 + 5s + 4)(s + 1)} + \frac{6}{(s^2 + 5s + 4)(s + 1)^2}$$

Adding fractions and observing that $s^2 + 5s + 4 = (s + 1)(s + 4)$, we find

$$Y(s) = \frac{s^3 + 8s^2 + 21s + 20}{(s + 1)^3(s + 4)}.$$

Step 3. Thus we must determine

$$y = \mathcal{L}^{-1}\left\{\frac{s^3 + 8s^2 + 21s + 20}{(s + 1)^3(s + 4)}\right\}. \tag{2}$$

We employ partial fractions, writing

$$\frac{s^3 + 8s^2 + 21s + 20}{(s + 1)^3(s + 4)} = \frac{A}{s + 1} + \frac{B}{(s + 1)^2} + \frac{C}{(s + 1)^3} + \frac{D}{s + 4}.$$

Then $s^3 + 8s^2 + 21s + 20 = A(s + 1)^2(s + 4) + B(s + 1)(s + 4) + C(s + 4) + D(s + 1)^3$. Letting $s = -1$, we find $3C = 6$, so $C = 2$; and letting $s = -4$, we find $D = 0$. Then letting $s = 0$, we obtain $4A + 4B + 4C + D = 20$. With $C = 2$, $D = 0$, this gives $A + B = 3$. Finally, letting $s = 1$, we obtain $20A + 10B + 5C + 8D = 50$, which, with $C = 2$ and $D = 0$, reduces to $2A + B = 4$. From the two equations in $A$ and $B$, we find $A = 1$, $B = 2$. Thus

$$\frac{s^3 + 8s^2 + 21s + 20}{(s+1)^3(s+4)} = \frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}. \tag{3}$$

Alternatively, an astute observer might have noticed that $s^3 + 8s^2 + 21s + 20 = (s^2 + 4s + 5)(s + 4)$ and reduced the fraction in (2) to

$$\frac{s^2 + 4s + 5}{(s+1)^3}.$$

Applying partial fractions to this, one writes

$$\frac{s^2 + 4s + 5}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}.$$

Then $s^2 + 4s + 5 = A(s+1)^2 + B(s+1) + C$ $= As^2 + (2A + B)s + (A + B + C)$. From this, $A = 1$, $2A + B = 4$, $A + B + C = 5$, and we find $A = 1$, $B = 2$, $C = 2$. So we again obtain (3), and this time with an easier use of partial fractions. Therefore

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \mathcal{L}\left\{\frac{2}{(s+1)^3}\right\}.$$

By Table 9.1, numbers 2, 8, and 8, respectively, we find

$$y = e^{-t} + 2te^{-t} + t^2e^{-t} = (t+1)^2e^{-t}.$$

21. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y'''\} - 5\mathcal{L}\{y''\} + 7\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{20 \sin t\}. \tag{1}$$

Denoting $\mathcal{L}\{y(t)\}$ by $Y(s)$ and applying Theorem 9.4, we have the following expressions for $\mathcal{L}\{y'''\}$, $\mathcal{L}\{y''\}$, and $\mathcal{L}\{y'\}$:

$$\begin{cases} \mathcal{L}\{y'''\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0), \\ \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0), \\ \mathcal{L}\{y'\} = sY(s) - y(0). \end{cases}$$

Applying the I.C.'s to these, they become

$$\begin{aligned} \mathcal{L}\{y'''\} &= s^3Y(s) + 2, \\ \mathcal{L}\{y''\} &= s^2Y(s), \\ \mathcal{L}\{y'\} &= sY(s). \end{aligned}$$

Substituting these expressions into the left member of (1), this left member becomes

$$s^3Y(s) + 2 - 5s^2Y(s) + 7sY(s) - 3Y(s)$$

or

$$(s^3 - 5s^2 + 7s - 3)Y(s) + 2.$$

By Table 9.1, number 3, the right member of (1) becomes $\frac{20}{s^2 + 1}$. Thus (1) reduces to

$$(s^3 - 5s^2 + 7s - 3)Y(s) + 2 = \frac{20}{s^2 + 1}.$$

Step 2. Solving the preceding for $Y(s)$, we have

$$Y(s) = \frac{18 - 2s^2}{(s^3 - 5s^2 + 7s - 3)(s^2 + 1)}$$

or

$$Y(s) = \frac{-2(s^2 - 9)}{(s - 1)^2(s - 3)(s^2 + 1)}$$

or finally

$$Y(s) = \frac{-2(s + 3)}{(s - 1)^2(s^2 + 1)}.$$

Step 3. We must now determine

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-2s - 6}{(s - 1)^2(s^2 + 1)}\right\}.$$

We employ partial fractions. We have

$$\frac{-2s - 6}{(s - 1)^2(s^2 + 1)} = \frac{A}{s - 1} + \frac{B}{(s - 1)^2} + \frac{Cs + D}{s^2 + 1}$$

or $-2s - 6 = A(s - 1)(s^2 + 1) + B(s^2 + 1)$
$+ (Cs + D)(s - 1)^2$ or $-2s - 6 = (A + C)s^3$
$+ (-A + B - 2C + D)s^2 + (A + C - 2D)s + (-A + B + D)$.
From this, we obtain

$$\begin{cases} A + C = 0, & -A + B - 2C + D = 0, \\ A + C - 2D = -2, & -A + B + D = -6. \end{cases}$$

The first and third of these give $C = -A$, $D = 1$. Then the second and fourth reduce to $A + B = -1$, $-A + B = -7$, respectively, from which $A = 3$, $B = -4$. Hence $A = 3$, $B = -4$, $C = -3$, $D = 1$; and we have

$$\mathcal{L}^{-1}\left\{\frac{-2s - 6}{(s - 1)^2(s^2 + 1)}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\}$$
$$- 4\mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\}$$
$$+ \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}.$$

Then by Table 9.1, numbers 2, 8, 4, and 3, respectively, we find

$$y = 3e^{t} - 4te^{t} - 3\cos t + \sin t.$$

**Section 9.4A, Page 527.**

1. The given function is $5u_6(t)$. Using formula (9.75), we have $\mathcal{L}\{5u_6(t)\} = \dfrac{5e^{-6s}}{s}$.

4. We may express the values of f in the form

$$f(t) = \begin{cases} 2 - 0, & 0 < t < 5, \\ 2 - 2, & t > 5. \end{cases}$$

Thus f(t) can be expressed as $2 - 2u_5(t)$. Then using formulas (9.2) and (9.75) we find $\mathcal{L}\{f(t)\}$

$= 2\mathcal{L}\{1\} - 2\mathcal{L}\{u_5(t)\} = \dfrac{2}{s} - \dfrac{2e^{-5s}}{s} = \dfrac{2(1 - e^{-5s})}{s}$.

6. We may express the values of f in the form

$$f(t) = \begin{cases} 0 + 0, & 0 < t < 3, \\ -6 + 0, & 3 < t < 9, \\ -6 + 6, & t > 9. \end{cases}$$

Thus f can be expressed as

$$-6u_3(t) + 6u_9(t).$$

Then using formula (9.75), we find

$$\mathcal{L}\{f(t)\} = -6\mathcal{L}\{u_3(t)\} + 6\mathcal{L}\{u_9(t)\}$$

$$= \frac{-6e^{-3s}}{s} + \frac{6e^{-9s}}{s}$$

$$= \frac{6}{s}\left(e^{-9s} - e^{-3s}\right).$$

7. We may express the values of f in the form

$$f(t) = \begin{cases} 1 + 0 + 0 - 0, & 0 < t < 2, \\ 1 + 1 + 0 - 0, & 2 < t < 4, \\ 1 + 1 + 1 - 0, & 4 < t < 6, \\ 1 + 1 + 1 - 3, & t > 6. \end{cases}$$

Thus $f(t)$ can be expressed as

$$1 + u_2(t) + u_4(t) - 3u_6(t).$$

Then using formulas (9.2) and (9.75) we find

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{u_2(t)\} + \mathcal{L}\{u_4(t)\} - 3\mathcal{L}\{u_6(t)\}$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s} + \frac{e^{-4s}}{s} - \frac{3e^{-6s}}{s}$$

$$= \frac{1 + e^{-2s} + e^{-4s} - 3e^{-6s}}{s}.$$

9. We may express the values of f in the form

$$f(t) = \begin{cases} 2 - 0 + 0, & 0 < t < 3, \\ 2 - 2 + 0, & 3 < t < 6, \\ 2 - 2 + 2, & t > 6. \end{cases}$$

Thus f can be expressed as

$$2 - 2u_3(t) + 2u_6(t).$$

Then using formulas (9.2) and (9.75), we find

$$L\{f(t)\} = 2\mathcal{L}\{1\} - 2\mathcal{L}\{u_3(t)\} + 2\mathcal{L}\{u_6(t)\}$$

$$= \frac{2}{s} - \frac{2e^{-3s}}{s} + \frac{2e^{-6s}}{s}$$

$$= \frac{2}{s}(1 - e^{-3s} + e^{-6s}).$$

11. We must first express $f(t)$ for $t > 2$ in terms of $t - 2$. That is, we express $t$ as $(t - 2) + 2$ and write

$$f(t) = \begin{cases} 0, & 0 < t < 2, \\ (t - 2) + 2, & t > 2. \end{cases}$$

This is the translated function defined by

$$u_2(t)\phi(t - 2) = \begin{cases} 0, & 0 < t < 2, \\ \phi(t - 2), & t > 2, \end{cases}$$

where $\phi(t) = t + 2$. By Theorem 9.9,
$\mathcal{L}\{u_2(t)\phi(t - 2)\} = e^{-2s}\mathcal{L}\{\phi(t)\} = e^{-2s}\mathcal{L}\{t + 2\}$

$= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$, where we have used (9.3) and (9.2).

14. We express the values of $f$ in the form

$$f(t) = \begin{cases} 2t - 0, & 0 < t < 5, \\ 2t - 2(t - 5), & t > 5. \end{cases}$$

Thus $f(t)$ can be expressed as $f(t) = 2t - u_5(t)\phi(t - 5)$, where $\phi(t) = 2t$. By Formula (9.3), $\mathcal{L}\{2t\} = \frac{2}{s^2}$. By Theorem 9.9, $\mathcal{L}\{u_5(t)\phi(t - 5)\} = e^{-5s}\mathcal{L}\{\phi(t)\} = e^{-5s}\mathcal{L}\{2t\}$

$$= \frac{2e^{-5s}}{s^2}. \quad \text{Thus } \mathcal{L}\{f(t)\} = \mathcal{L}\{2t\} - \mathcal{L}\{u_5(t)\phi(t-5)\}$$

$$= \frac{2}{s^2} - \frac{2e^{-5s}}{s^2} = \frac{2(1 - e^{-5s})}{s^2}.$$

16. We express the values of f in the form

$$f(t) = \begin{cases} 0, & 0 < t < 2, \\ e^{-2}e^{-(t-2)}, & t > 2. \end{cases}$$

This is the translated function defined by

$$u_2(t)\phi(t-2) = \begin{cases} 0, & 0 < t < 2, \\ \phi(t-2), & t > 2, \end{cases}$$

where $\phi(t) = e^{-2}e^{-t}$. By Theorem 9.9,

$$\mathcal{L}\{u_2(t)\phi(t-2)\} = e^{-2s}\mathcal{L}\{\phi(t)\}$$

$$= e^{-2s}\mathcal{L}\{e^{-2}e^{-t}\} = e^{-2s}e^{-2}\left(\frac{1}{s+1}\right)$$

$$= \frac{e^{-2(s+1)}}{s+1},$$

where we have used (9.4).

17. We express the values of f in the form

$$f(t) = \begin{cases} 0 - 0, & 0 < t < 4, \\ (t-4) - 0, & 4 < t < 7, \\ (t-4) - (t-7), & t > 7. \end{cases}$$

Thus f(t) can be expressed as

$$f(t) = u_4(t)\phi(t-4) - u_7(t)\phi(t-7),$$

where $\phi(t) = t$ and $\psi(t) = t$. By formula (9.3), $\mathcal{L}\{\phi(t)\} = \mathcal{L}\{\psi(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$. Then by Theorem 9.9,

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{u_4(t)\phi(t-4)\} - \mathcal{L}\{u_7(t)\psi(t-7)\} \\ &= e^{-4s}\mathcal{L}\{\phi(t)\} - e^{-7s}\mathcal{L}\{\psi(t)\} \\ &= \frac{e^{-4s}}{s^2} - \frac{e^{-7s}}{s^2} \\ &= \frac{e^{-4s} - e^{-7s}}{s^2}.\end{aligned}$$

19. We express the values of f in the form

$$f(t) = \begin{cases} 0 - 0, & 0 < t < 2\pi, \\ \sin(t - 2\pi) - 0, & 2\pi < t < 4\pi, \\ \sin(t - 2\pi) - \sin(t - 4\pi), & t > 4\pi. \end{cases}$$

Thus f(t) can be expressed as

$$u_{2\pi}(t)\sin(t - 2\pi) - u_{4\pi}(t)\sin(t - 4\pi)$$

or

$$u_{2\pi}(t)\phi(t - 2\pi) - u_{4\pi}(t)\psi(t - 4\pi),$$

where $\phi(t) = \sin t$, $\psi(t) = \sin t$. By Theorem 9.9,

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{u_{2\pi}(t)\phi(t - 2\pi)\} - \mathcal{L}\{u_{4\pi}(t)\psi(t - 4\pi)\} \\ &= e^{-2\pi s}\mathcal{L}\{\phi(t)\} - e^{-4\pi s}\mathcal{L}\{\psi(t)\} \\ &= e^{-2\pi s}\mathcal{L}\{\sin t\} - e^{-4\pi s}\mathcal{L}\{\sin t\}\end{aligned}$$

$$= e^{-2\pi s}\left[\frac{1}{s^2 + 1}\right] - e^{-4\pi s}\left[\frac{1}{s^2 + 1}\right]$$

$$= \frac{e^{-2\pi s} - e^{-4\pi s}}{s^2 + 1},$$

where we have used formula (9.5).

22. The period P = 2. By Theorem 9.10, formula (9.80),

$$\mathcal{L}\{f(t)\} = \frac{\int_0^2 e^{-st} f(t)\,dt}{1 - e^{-2s}}$$

$$= \frac{1}{1 - e^{-2s}}\left[\int_0^1 te^{-st}\,dt + \int_1^2 e^{-st}\,dt\right]$$

$$= \frac{1}{1 - e^{-2s}}\left[\frac{e^{-st}}{s^2}(-st - 1)\Big|_0^1 - \frac{1}{s}e^{-st}\Big|_1^2\right]$$

$$= \frac{1}{1 - e^{-2s}}\left[\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}\right].$$

24. The period P = $\pi$. By Theorem 9.10, formula (9.80),

$$\mathcal{L}\{f(t)\} = \frac{\int_0^\pi e^{-st} f(t)\,dt}{1 - e^{-\pi s}}$$

$$= \frac{1}{1 - e^{-\pi s}}\left[\int_0^\pi e^{-st}\sin t\,dt\right]$$

$$= \frac{1}{1 - e^{-\pi s}}\left[\frac{e^{-st}}{s^2 + 1}(-s\sin t - \cos t)\right]\Bigg|_0^\pi$$

$$= \frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(s^2 + 1)}.$$

**Section 9.4B, Page 530.**

2. $F(s)$ is of the form $e^{-as}\Phi(s)$, where $a = 5$ and $\Phi(s) = \dfrac{3s + 1}{(s - 2)^2}$. By formula (9.86) $\mathcal{L}^{-1}\{e^{-as}\Phi(s)\}$ $= u_a(t)\phi(t - a)$, where $u_a$ is defined by (9.73) and $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$. [See Theorem 9.9.] We must find $\phi(t)$. Using Table 9.1, numbers 2 and 8, respectively,

$$\phi(t) = \mathcal{L}^{-1}\left\{\frac{3s + 1}{(s - 2)^2}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{(s - 2)}\right\} + 7\mathcal{L}^{-1}\left\{\frac{1}{(s - 2)^2}\right\}$$

$= 3e^{2t} + 7te^{2t} = e^{2t}(7t + 3)$. Thus $\phi(t - 5)$ $= e^{2(t-5)}[7(t - 5) + 3] = e^{2(t-5)}(7t - 32)$. Then by formula (9.86) [or Table 9.1, number 16],

$$\mathcal{L}^{-1}\left\{\frac{3s + 1}{(s - 2)^2}\, e^{-5s}\right\} = u_5(t)\phi(t - 5)$$

$$= \begin{cases} 0, & 0 < t < 5, \\ e^{2(t-5)}(7t - 32), & t > 5. \end{cases}$$

4. $F(s)$ is of the form $e^{-as}\Phi(s)$, where $a = 4$ and $\Phi(s) = \dfrac{12}{s^2 + s - 2}$. By formula (9.86) $\mathcal{L}^{-1}\{e^{-as}\Phi(s)\}$ $= u_a(t)\phi(t - a)$, where $u_a$ is defined by (9.73) and $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ [see Theorem 9.9]. We must find $\phi(t)$. We first employ partial fractions. We write $\dfrac{12}{s^2 + s - 12} = \dfrac{A}{s - 1} + \dfrac{B}{s + 2}$. Clearing fractions, we at once find $A = 4$, $B = -4$. Thus, using Table 9.1, number 2,

twice, we have $\phi(t) = \mathcal{L}^{-1}\left\{\frac{12}{s^2 + s - 12}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\}$ $- 4\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\} = 4e^{t} - 4e^{-2t}$. Thus $\phi(t - 4) = 4e^{t-4}$ $- 4e^{-2(t-4)}$. Then by formula (9.86) [or Table 9.1, number 16],

$$\mathcal{L}^{-1}\left\{\frac{12}{s^2 + s - 2}\, e^{-4s}\right\} = u_4(t)\phi(t - 4)$$

$$= \begin{cases} 0, & 0 < t < 4, \\ 4e^{t-4} - 4e^{-2(t-4)}, & t > 4. \end{cases}$$

5. $F(s)$ is of the form $e^{-as}\Phi(s)$, where $a = \pi$ and $\Phi(s) = \frac{5s + 6}{s^2 + 9}$. By formula (9.86), $\mathcal{L}^{-1}\{e^{-as}\Phi(s)\}$ $= u_a(t)\phi(t - a)$, where $u_a$ is defined by (9.73) and $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$. [See Theorem 9.9.] We must find $\phi(t)$. Using Table 9.1, numbers 4 and 3, respectively, we find

$$\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\left\{\frac{5s + 6}{s^2 + 9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 9}\right\}$$

$$+ 2\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\} = 5\cos 3t + 2\sin 3t. \quad \text{Thus}$$

$\phi(t - a) = \phi(t - \pi) = 5\cos 3(t - \pi) + 2\sin 3(t - \pi)$ $= -5\cos 3t - 2\sin 3t$. Then by formula (9.86) [or Table 9.1, number 16], $\mathcal{L}^{-1}\{e^{-\pi s}\Phi(s)\} = u_{\pi}(t)\phi(t - \pi)$, that is,

$$\mathcal{L}^{-1}\{F(s)\} = u_{\pi}(t)(-5\cos 3t - 2\sin 3t)$$

$$= \begin{cases} 0, & 0 < t < \pi, \\ -5\cos 3t - 2\sin 3t, & t > \pi. \end{cases}$$

7. $F(s)$ is of the form $e^{-as}\Phi(s)$, where $a = \pi/2$ and $\Phi(s) = \dfrac{s + 8}{s^2 + 4s + 13}$. By formula (9.86), $\mathcal{L}^{-1}\{e^{-as}\Phi(s)\} = u_a(t)\phi(t - a)$, where $u_a$ is defined by (9.73) and $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ [see theorem 9.9]. We must find $\phi(t)$. We write

$$\frac{s + 8}{s^2 + 4s + 13} = \frac{s + 2}{(s + 2)^2 + 9} + \frac{6}{(s + 2)^2 + 9}$$

and proceed to use Table 9.1, numbers 12 and 11. By these respective entries, we find

$$\begin{aligned}
\phi(t) &= \mathcal{L}^{-1}\left\{\frac{s + 8}{s^2 + 4s + 13}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s + 2}{(s + 2)^2 + 3^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{3}{(s + 2)^2 + 3^2}\right\} \\
&= e^{-2t}\cos 3t + 2e^{-2t}\sin 3t \\
&= e^{-2t}(\cos 3t + 2\sin 3t).
\end{aligned}$$

Thus $\phi(t - \pi/2) = e^{-2(t-\pi/2)}[\cos 3(t - \pi/2) + 2\sin 3(t - \pi/2)] = e^{-2t+\pi}(2\cos 3t - \sin 3t)$. Then by formula (9.86) [or Table 9.1, number 16],

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s + 8}{s^2 + 4s + 13}\, e^{-(\pi s)/2}\right\} &= u_{(\pi/2)}(t)\phi(t - \pi/2) \\
&= \begin{cases} 0, & 0 < t < \pi/2, \\ e^{-2t+\pi}(2\cos 3t - \sin 3t), & t > \pi/2. \end{cases}
\end{aligned}$$

8. $F(s)$ is of the form $e^{-as}\Phi(s)$, where $a = 3$ and $\Phi(s) = \dfrac{2s + 9}{s^2 + 4s + 13}$. By formula (9.86), $\mathcal{L}^{-1}\{e^{-as}\Phi(s)\} = u_a(t)\phi(t - a)$, where $u_a$ is defined by (9.73) and $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$. We must find $\phi(t)$. We have

$$\frac{2s + 9}{s^2 + 4s + 13} = \frac{2(s + 2)}{(s + 2)^2 + (3)^2} + \frac{5}{(s + 2)^2 + (3)^2}.$$

Using number 12 and 11, respectively, we find

$$\begin{aligned}\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} &= \mathcal{L}^{-1}\left\{\frac{2s + 9}{s^2 + 4s + 13}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{s + 2}{(s + 2)^2 + (3)^2}\right\} + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s + 2)^2 + (3)^2}\right\} \\ &= 2e^{-2t}\cos 3t + \left(\frac{5}{3}\right)e^{-2t}\sin 3t.\end{aligned}$$

Thus

$$\begin{aligned}\phi(t - a) &= \phi(t - 3) \\ &= e^{-2(t-3)}\left[2\cos 3(t - 3) + \left(\frac{5}{3}\right)\sin 3(t - 3)\right]\end{aligned}$$

Then by formula (9.86) [or Table 9.1, number 16], $\mathcal{L}^{-1}\{e^{-3s}\Phi(s)\} = u_3(t)\phi(t - 3)$, that is,

$$\mathcal{L}^{-1}\{F(s)\} = u_3(t)e^{-2(t-3)}[2\cos 3(t - 3)] + \left(\frac{5}{3}\right)\sin 3(t - 3)$$

$$= \begin{cases} 0, & 0 < t < 3, \\ e^{-2(t-3)}\left[2\cos 3(t - 3) + \left(\frac{5}{3}\right)\sin 3(t - 3)\right], & t > 3. \end{cases}$$

10. We first write

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s} - e^{-8s}}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-8s}}{s^3}\right\}.$$

Each of the two inverse transforms on the right is of the form $\mathcal{L}^{-1}\{e^{-as}\Phi(s)\}$. By formula (9.86), $\mathcal{L}^{-1}\{e^{-as}\Phi(s)\} = u_a(t)\phi(t - a)$, where $u_a$ is defined by (9.73) and $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$.

First consider $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\}$. Here $a = 3$, $\Phi(s) = \frac{1}{s^3}$; and from Table 9.1, number 7, $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ $= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}$. Thus $\phi(t - 3) = \frac{(t - 3)^2}{2}$. By formula (9.86),

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\} = u_3(t)\phi(t - 3) = \begin{cases} 0, & 0 < t < 3, \\ \frac{(t - 3)^2}{2}, & t > 3. \end{cases}$$

Now consider $\mathcal{L}^{-1}\left\{\frac{e^{-8s}}{s^3}\right\}$. This is similar to the preceding, with $a = 8$, $\Phi(s) = \frac{1}{s^3}$, and $\phi(t) = \frac{t^2}{2}$. Thus $\phi(t - 8) = \frac{(t - 8)^2}{2}$, and

$$\mathcal{L}^{-1}\left\{\frac{e^{-8s}}{s^3}\right\} = u_8(t)\phi(t - 8) = \begin{cases} 0, & 0 < t < 8, \\ \frac{(t - 8)^2}{2}, & t > 8. \end{cases}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s} - e^{-8s}}{s^3}\right\} = u_3(t)\phi(t - 3) - u_8(t)\phi(t - 8)$$

$$= \begin{cases} 0 - 0, & 0 < t < 3, \\ \dfrac{(t - 3)^2}{2} - 0, & 3 < t < 8, \\ \dfrac{(t - 3)^2}{2} - \dfrac{(t - 8)^2}{2}, & t > 8, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < 3, \\ \dfrac{1}{2}(t^2 - 6t + 9), & 3 < t < 8, \\ 5t - \dfrac{55}{2}, & t > 8. \end{cases}$$

13. We write

$$F(s) = \frac{2}{s^2 - 2s + 5} + \frac{2e^{-(\pi s)/2}}{s^2 - 2s + 5}, \tag{1}$$

and first determine $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2 - 2s + 5}\right\}$. By Table 9.1, number 11, we have

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2 - 2s + 5}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 4}\right\}$$

$$= e^t \sin 2t. \tag{2}$$

Now letting $\Phi(s) = \dfrac{2}{s^2 - 2s + 5}$ and $\phi(t) = e^t \sin 2t$, we see that (2) is $\mathcal{L}^{-1}\{\Phi(s)\} = \phi(t)$. Then by formula (9.86) [or Table 9.1, number 16],

$$\mathcal{L}^{-1}\left\{\frac{2e^{-(\pi s)/2}}{s^2 - 2s + 5}\right\} = \mathcal{L}^{-1}\{e^{-(\pi s)/2}\Phi(s)\}$$

$$= u_{\pi/2}(t)\phi\left(t - \frac{\pi}{2}\right) = \begin{cases} 0, & 0 < t < \frac{\pi}{2}, \\ \phi\left(t - \frac{\pi}{2}\right), & t > \frac{\pi}{2}, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < \frac{\pi}{2}, \\ e^{t-\pi/2}\sin 2\left(t - \frac{\pi}{2}\right), & t > \frac{\pi}{2}. \end{cases}$$

That is,

$$\mathcal{L}^{-1}\left\{\frac{2e^{-(\pi s)/2}}{s^2 - 2s + 5}\right\} = \begin{cases} 0, & 0 < t < \frac{\pi}{2}, \\ -e^{t-\pi/2}\sin 2t, & t > \frac{\pi}{2}. \end{cases} \tag{3}$$

Then, using (1), (2), and (3), we find

$$\mathcal{L}^{-1}\{F(s)\} = \begin{cases} e^t\sin 2t - 0, & 0 < t < \frac{\pi}{2}, \\ e^t\sin 2t - e^{t-\pi/2}\sin 2t. & t > \frac{\pi}{2}, \end{cases}$$

$$= \begin{cases} e^t\sin 2t, & 0 < t < \frac{\pi}{2}, \\ (1 - e^{-\pi/2})e^t\sin 2t, & t > \frac{\pi}{2}. \end{cases}$$

**Section 9.4C, Page 533.**

2. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$3\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{h(t)\}. \tag{1}$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$, applying Theorem 9.3, and then applying the I.C., we find that

$$\mathcal{L}\{y'\} = sY(s) - 4.$$

Substituting this into the left member of (1), it becomes $3[sY(s) - 4] - 5Y(s)$ or $(3s - 5)Y(s) - 12$. By the definition of the Laplace Transform

$$\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st}h(t)dt = \int_6^\infty 10e^{-st}dt$$

$$= \lim_{R\to\infty}\left[-\frac{10}{s}e^{-st}\Big|_6^R\right] = \frac{10e^{-6s}}{s}.$$

Alternatively, we note that $h(t) = 10u_6(t)$; and then using Table 9.1, number 15, we have

$$\mathcal{L}\{h(t)\} = 10\mathcal{L}\{u_6(t)\} = \frac{10e^{-6s}}{s}.$$

Thus, (1) reduces to

$$(3s - 5)Y(s) - 12 = \frac{10e^{-6s}}{s}.$$

Step 2. Solving for $Y(s)$, we obtain

$$Y(s) = \frac{12}{3s - 5} + \frac{10e^{-6s}}{s(3s - 5)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{12}{3s - 5} + \frac{10e^{-6s}}{s(3s - 5)}\right\}. \tag{2}$$

We first determine

$$\mathcal{L}^{-1}\left\{\frac{12}{3s - 5}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s - 5/3}\right\} = 4e^{(5/3)t}, \tag{3}$$

where we have used Table 9.1, number 2. Now consider $\mathcal{L}^{-1}\left\{\frac{10e^{-6s}}{s(3s - 5)}\right\}$. We first employ partial fractions to $\frac{1}{s(s - 5/3)}$, and write $\frac{1}{s(s - 5/3)} = \frac{A}{s} + \frac{B}{s - 5/3}$. Clearing fractions, we at once find $A = -3/5$, $B = 3/5$. Then $\frac{10}{s(3s - 5)} = \frac{10}{3}\left[\frac{-3/5}{s} + \frac{3/5}{s - 5/3}\right] = -\frac{2}{s} + \frac{2}{s - 5/3}$. Then by Table 9.1, numbers 1 and 2, respectively,

$$\mathcal{L}^{-1}\left\{\frac{10}{s(3s - 5)}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s - 5/3}\right\}$$
$$= -2 + 2e^{(5/3)t}.$$

Now letting $F(s) = \frac{10}{s(3s - 5)}$ and $f(t) = -2 + 2e^{(5/3)t}$ we thus have $\mathcal{L}^{-1}\{F(s)\} = f(t)$. Then by Theorem 9.9,

$$\mathcal{L}^{-1}\{F(s)e^{-6s}\} = \mathcal{L}^{-1}\left\{\frac{10e^{-6s}}{s(3s - 5)}\right\}$$
$$= u_6(t)f(t - 6) = \begin{cases} 0, & 0 < t < 6, \\ f(t - 6), & t > 6, \end{cases}$$
$$= \begin{cases} 0, & 0 < t < 6, \\ -2 + 2e^{(5/3)(t-6)}, & t > 6. \end{cases} \qquad (4)$$

Then from (2), (3), and (4), we obtain

$$y = \begin{cases} 4e^{(5/3)t}, & 0 < t < 6, \\ 4e^{(5/3)t} - 2 + 2e^{(5/3)(t-6)}, & t > 6, \end{cases}$$
$$= \begin{cases} 4e^{(5/3)t}, & 0 < t < 6, \\ -2 + (4 + 2e^{-10})e^{(5/3)t}, & t > 6. \end{cases}$$

3. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{h(t)\}. \qquad (1)$$

Denoting $\mathcal{L}\{y(t)\}$ by $Y(s)$, applying Theorem 9.4 as in the previous exercises, and then applying the I.C.'s we find that

$$\mathcal{L}\{y''\} = s^2Y(s) \quad \text{and} \quad \mathcal{L}\{y'\} = sY(s).$$

Substituting these expressions into the left member of (1), this left member becomes

$$s^2Y(s) - 3sY(s) + 2Y(s)$$

or $(s^2 - 3s + 2)Y(s)$. By the definition of the Laplace Transform,

$$\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st}h(t)dt = \int_0^4 2e^{-st}dt$$

$$= \frac{2}{s}(1 - e^{-4s}).$$

Alternatively, writing

$$h(t) = \begin{cases} 2 - 0, & 0 < t < 4 \\ 2 - 2, & t > 4, \end{cases}$$

we have $h(t) = 2 - 2u_4(t)$. Then using Table 9.1, numbers 1 and 15, respectively, we have

$$\mathcal{L}\{h(t)\} = \frac{2}{s} - \frac{2e^{-4s}}{s} = \frac{2}{s}(1 - e^{-4s}).$$

Thus (1) reduces to $(s^2 - 3s + 2)Y(s) = \frac{2}{s}(1 - e^{-4s})$.

Step 2. Solving the preceding for $Y(s)$, we obtain

$$Y(s) = \frac{2(1 - e^{-4s})}{s(s - 1)(s - 2)}.$$

Step 3. We must now determine

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s(s - 1)(s - 2)}\right\} - \mathcal{L}^{-1}\left\{\frac{2e^{-4s}}{s(s - 1)(s - 2)}\right\}. \tag{2}$$

We first apply partial fractions to the first term on the right of (2). We have

$$\frac{2}{s(s - 1)(s - 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s - 2},$$

and from this we readily find $A = 1$, $B = -2$, $C = 1$. Thus

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s - 1)(s - 2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\}.$$

By Table 9.1, numbers 1, 2, and 2, respectively, we find

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s - 1)(s - 2)}\right\} = 1 - 2e^{t} + e^{2t}.$$

Letting $F(s) = \dfrac{2}{s(s - 1)(s - 2)}$ and $f(t) = 1 - 2e^{t} + e^{2t}$, we thus have $\mathcal{L}^{-1}\{F(s)\} = f(t)$. We now consider $\mathcal{L}^{-1}\left\{\dfrac{2e^{-4s}}{s(s - 1)(s - 2)}\right\} = \mathcal{L}^{-1}\{F(s)e^{-4s}\}$. By Theorem 9.9,

$$\mathcal{L}^{-1}\{F(s)e^{-4s}\} = u_4(t)f(t-4),$$

$$= \begin{cases} 0, & 0 < t < 4, \\ f(t-4), & t > 4, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < 4, \\ 1 - 2e^{t-4} + 2e^{2(t-4)}, & t > 4. \end{cases}$$

Thus from (2), we find $y = f(t) - u_4(t)f(t-4)$

$$= \begin{cases} 1 - 2e^{t} + e^{2t}, & 0 < t < 4, \\ 1 - 2e^{t} + e^{2t} - [1 - 2e^{t-4} + e^{2(t-4)}], & t > 4. \end{cases}$$

Hence $y = \begin{cases} 1 - 2e^{t} + e^{2t}, & 0 < t < 4, \\ 2(e^{-4} - 1)e^{t} + (1 - e^{-8})e^{2t}, & t > 4. \end{cases}$

6. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 8\mathcal{L}\{y\} = \mathcal{L}\{h(t)\}. \qquad (1)$$

Denoting $\mathcal{L}\{y(t)\}$ by $Y(s)$, applying Theorem 9.4 as in previous exercises, and then applying the I.C.'s, we find that

$$\mathcal{L}\{y''\} = s^2Y(s) - s + 1, \ \mathcal{L}\{y'\} = sY(s) - 1.$$

Substituting these expressions into the left member of (1), this left member becomes $[s^2Y(s) - s + 1] + 6[sY(s) - 1] + 8Y(s)$ or $(s^2 + 6s + 8)Y(s) - s - 5$. By the definition of the Laplace Transform,

$$\mathcal{L}\{h(t)\} = \int_0^{\infty} e^{-st} h(t)\,dt$$

$$= \int_0^{2\pi} 3e^{-st}\,dt = \frac{3}{s}\,(1 - e^{-2\pi s}).$$

Alternatively, writing

$$h(t) = \begin{cases} 3 - 0, & 0 < t < 2\pi, \\ 3 - 3, & t > 2\pi, \end{cases}$$

we have $h(t) = 3 - 3u_{2\pi}(t)$. Then using Table 9.1, numbers 1 and 15, respectively, we have

$$\mathcal{L}\{h(t)\} = \frac{3}{s} - \frac{3e^{-2\pi s}}{s} = \frac{3}{s}\,(1 - e^{-2\pi s}).$$

Thus (1) reduces to $(s^2 + 6s + 8)Y(s) - s - 5$
$= \frac{3}{s}\,(1 - e^{-2\pi s})$.

Step 2. Solving the preceding for $Y(s)$, we obtain

$$Y(s) = \frac{s + 5}{s^2 + 6s + 8} + \frac{3(1 - e^{-2\pi s})}{s(s^2 + 6s + 8)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{s + 5}{(s + 2)(s + 4)}\right\} + \mathcal{L}^{-1}\left\{\frac{3(1 - e^{-2\pi s})}{s(s + 2)(s + 4)}\right\}. \tag{2}$$

We first apply partial fractions to the first term on the right of (2). We have

$$\frac{s + 5}{(s + 2)(s + 4)} = \frac{A}{s + 2} + \frac{B}{s + 4},$$

and from this we readily find A = 3/2, B = -1/2. Thus

$$\mathcal{L}^{-1}\left\{\frac{s+5}{(s+2)(s+4)}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}.$$

By Table 9.1, number 2, we find

$$\mathcal{L}^{-1}\left\{\frac{s+5}{(s+2)(s+4)}\right\} = \frac{3}{2}e^{-2t} - \frac{1}{2}e^{-4t}. \tag{3}$$

We now consider $\mathcal{L}^{-1}\left\{\frac{3(1-e^{-2\pi s})}{s(s+2)(s+4)}\right\}$. We first apply partial fractions to $\frac{3}{s(s+2)(s+4)}$. We have

$$\frac{3}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4},$$

and from this we readily find A = $\frac{3}{8}$, B = $-\frac{3}{4}$, C = $\frac{3}{8}$. Thus

$$\mathcal{L}^{-1}\left\{\frac{3}{s(s+2)(s+4)}\right\} = \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}.$$

By Table 9.1, numbers 1, 2, and 2, respectively, we find

$$\mathcal{L}^{-1}\left\{\frac{3}{s(s+2)(s+4)}\right\} = \frac{3}{8} - \frac{3}{4}e^{-2t} + \frac{3}{8}e^{-4t}.$$

Letting $F(s) = \frac{3}{s(s+2)(s+4)}$ and $f(t) = \frac{3}{8} - \frac{3}{4}e^{-2t} + \frac{3}{8}e^{-4t}$, we thus have $\mathcal{L}^{-1}\{F(s)\} = f(t)$. We now consider

$\mathcal{L}^{-1}\left\{\frac{3e^{-2\pi s}}{s(s+2)(s+4)}\right\} = \mathcal{L}^{-1}\{F(s)e^{-2\pi s}\}$. By Theorem 9.9,

$$\mathcal{L}^{-1}\{F(s)e^{-2\pi s}\} = u_{2\pi}(t)f(t - 2\pi)$$

$$= \begin{cases} 0, & 0 < t < 2\pi \\ f(t - 2\pi), & t > 2\pi, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < 2\pi, \\ \frac{3}{8} - \frac{3}{4}e^{-2(t-2\pi)} + \frac{3}{8}e^{-4(t-2\pi)}, & t > 2\pi. \end{cases}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{3(1 - e^{-2\pi s})}{s(s + 2)(s + 4)}\right\}$$

$$= \begin{cases} \frac{3}{8} - \frac{3}{4}e^{-2t} + \frac{3}{8}e^{-4t}, & 0 < t < 2\pi, \\ \frac{3}{8} - \frac{3}{4}e^{-2t} + \frac{3}{8}e^{-4t} - \frac{3}{8} + \frac{3}{4}e^{-2(t-2\pi)} - \frac{3}{8}e^{-4(t-2\pi)}, & \\ & t > 2\pi, \end{cases}$$

$$= \begin{cases} \frac{3}{8} - \frac{3}{4}e^{-2t} + \frac{3}{8}e^{-4t}, & 0 < t < 2\pi, \\ \frac{3}{4}(e^{4\pi} - 1)e^{-2t} + \frac{3}{8}(1 - e^{8\pi})e^{-4t}, & t > 2\pi. \end{cases} \tag{4}$$

Hence, using (3) and (4), (2) becomes

$$y = \begin{cases} \frac{3}{8} + \frac{3}{4}e^{-2t} - \frac{1}{8}e^{-4t}, & 0 < t < 2\pi, \\ \frac{3}{4}(e^{4\pi} + 1)e^{-2t} - \frac{1}{8}(1 + 3e^{8\pi})e^{-4t}, & t > 2\pi. \end{cases}$$

8. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{h(t)\}. \tag{1}$$

Denoting $\mathcal{L}\{y(t)\}$ by $Y(s)$, applying Theorem 9.4 as in previous exercises, and then applying the I.C.'s, we find that

$$\mathcal{L}\{y''\} = s^2Y(s) - 2s - 3.$$

Substituting this expression into the left member of (1), this left member becomes $(s^2 + 1)Y(s) - 2s - 3$. We now find $\mathcal{L}\{h(t)\}$. We have

$$h(t) = \begin{cases} t, & 0 < t < \pi, \\ \pi, & t > \pi, \end{cases}$$

$$= \begin{cases} t - 0, & 0 < t < \pi, \\ t - (t - \pi), & t > \pi, \end{cases}$$

and hence $h(t) = t - u_\pi(t)f(t - \pi)$, where $f(t) = t$. Thus, using Theorem 9.9 and Table 9.1, number 7, we find

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{u_\pi(t)f(t - \pi)\} = \frac{1}{s^2} - e^{-\pi s}\left(\frac{1}{s^2}\right)$$

$$= \frac{1 - e^{-\pi s}}{s^2}. \quad \text{Thus (1) reduces to } (s^2 + 1)Y(s) - 2s - 3$$

$$= \frac{1 - e^{-\pi s}}{s^2}.$$

Step 2. Solving the preceding for $Y(s)$, we obtain

$$Y(s) = \frac{2s + 3}{s^2 + 1} + \frac{1 - e^{-\pi s}}{s^2(s^2 + 1)}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{2s + 3}{s^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 1)}\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2(s^2 + 1)}\right\}. \qquad (2)$$

Using Table 9.1, numbers 4 and 3, we at once find

$$\mathcal{L}^{-1}\left\{\frac{2s + 3}{s^2 + 1}\right\} = 2\cos t + 3\sin t. \tag{3}$$

We now apply partial fractions to the second term in the right member of (2). We have

$$\frac{1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1}$$

and from this we readily find $A = 0$, $B = 1$, $C = 0$, $D = -1$. Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}.$$

Then using Table 9.1, numbers 7 and 3, we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 1)}\right\} = t - \sin t. \tag{4}$$

Now letting $F(s) = \dfrac{1}{s^2(s^2 + 1)}$ and $f(t) = t - \sin t$,

(4) states that $\mathcal{L}^{-1}\{F(s)\} = f(t)$. Using Theorem 9.9, we now find

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2(s^2 + 1)}\right\} &= \mathcal{L}^{-1}\{F(s)e^{-\pi s}\} \\
&= u_{\pi}(t)f(t - \pi) \\
&= \begin{cases} 0, & 0 < t < \pi, \\ f(t - \pi), & t > \pi, \end{cases} \\
&= \begin{cases} 0, & 0 < t < \pi, \\ t - \pi - \sin(t - \pi), & t > \pi. \end{cases}
\end{aligned}$$

That is,

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2(s^2+1)}\right\} = \begin{cases} 0, & 0 < t < \pi, \\ t - \pi + \sin t, & t > \pi, \end{cases} \tag{5}$$

Thus, using (3), (4), and (5), (2) becomes

$$y = \begin{cases} (2\cos t + 3\sin t) + (t - \sin t) - 0, & 0 < t < \pi, \\ (2\cos t + 3\sin t) + (t - \sin t) - (t - \pi + \sin t), & t > \pi \end{cases}$$

$$= \begin{cases} 2\sin t + 2\cos t + t, & 0 < t < \pi, \\ \sin t + 2\cos t + \pi, & t > \pi. \end{cases}$$

10. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 25\mathcal{L}\{y\} = 25[\mathcal{L}\{u_2(t)\} - \mathcal{L}\{u_4(t)\}]. \tag{1}$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$, applying Theorem 9.4, and then applying the I.C.'s, we have

$$\mathcal{L}\{y''\} = s^2Y(s) - 3s + 9, \quad \mathcal{L}\{y'\} = sY(s) - 3.$$

Substituting these into the left member of (1), it becomes $s^2Y(s) - 3s + 9 + 6sY(s) - 18 + 25Y(s)$ or $(s^2 + 6s + 25)Y(s) - 3s - 9$. By Table 9.1, number 15, twice, the right member becomes $25\left[\frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}\right]$. Thus (1) becomes

$$(s^2 + 6s + 25)Y(s) - 3s - 9 = \frac{25}{s}\left[e^{-2s} - e^{-4s}\right].$$

Step 2. We solve this for $Y(s)$ to obtain

$$Y(s) = \frac{3s + 9}{s^2 + 6s + 25} + \frac{25[e^{-2s} - e^{-4s}]}{s(s^2 + 6s + 25)}.$$

Step 3. We must find

$$y = \mathcal{L}^{-1}\left\{\frac{3s + 9}{s^2 + 6s + 25} + \frac{25[e^{-2s} - e^{-4s}]}{s(s^2 + 6s + 25)}\right\} \tag{2}$$

Using Table 9.1, number 12, we first find

$$\mathcal{L}^{-1}\left\{\frac{3s + 9}{s^2 + 6s + 25}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s + 3}{(s + 3)^2 + 4^2}\right\}$$

$$= 3e^{-3t}\cos 4t. \tag{3}$$

Now consider $\mathcal{L}^{-1}\left\{\frac{25}{s(s^2 + 6s + 25)}\right\}$ and employ partial fractions. We have

$$\frac{25}{s(s^2 + 6s + 25)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 6s + 25}$$

Clearing fractions, we find $25 = A(s^2 + 6s + 25) + s(Bs + C) = (A + B)s^2 + (6A + C)s + 25A$. From this, $A = 1$, $B = -1$, $C = -6$. Thus

$$\mathcal{L}^{-1}\left\{\frac{25}{s(s^2 + 6s + 25)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s + 6}{s^2 + 6s + 25}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s + 3}{(s + 3)^2 + 4^2}\right\}$$

$$- \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{4}{(s + 3)^2 + 4^2}\right\}$$

$$= 1 - e^{-3t}\cos 4t - \frac{3}{4}e^{-3t}\sin 4t,$$

where we have used Table 9.1, numbers 1, 12, and 11, respectively.

Letting $F(s) = \dfrac{25}{s(s^2 + 6s + 25)}$ and $f(t) = 1 - e^{-3t}\cos 4t - \dfrac{3}{4}e^{-3t}\sin 4t$, we thus have $\mathcal{L}^{-1}\{F(s)\} = f(t)$. Then by Table 9.1, number 16, we find

$$\mathcal{L}^{-1}\left\{\frac{25e^{-2s}}{s(s^2 + 6s + 25)}\right\} = \mathcal{L}^{-1}\{F(s)e^{-2s}\}$$

$$= u_2(t)f(t - 2) = \begin{cases} 0, & 0 < t < 2, \\ f(t - 2), & t > 2, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < 2, \\ 1-e^{-3(t-2)}\cos 4(t - 2) - \dfrac{3}{4}e^{-3(t-2)}\sin 4(t - 2), \\ \quad t > 2. \end{cases} \tag{4}$$

Similarly, we find

$$\mathcal{L}^{-1}\left\{\frac{25e^{-4s}}{s(s^2 + 6s + 25)}\right\} = \begin{cases} 0, & 0 < t < 4, \\ f(t - 4), & t > 4, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < 4, \\ 1-e^{-3(t-4)}\cos 4(t-4) - \dfrac{3}{4}e^{-3(t-4)}\sin 4(t-4), & t>4. \end{cases} \tag{5}$$

Finally, using (2), (3), (4), and (5), we find

$$y = \begin{cases} 3e^{-3t}\cos 4t, & 0 < t < 2, \\ 3e^{-3t}\cos 4t+1-e^{-3(t-2)}\cos 4(t-2) - \dfrac{3}{4}e^{-3(t-2)}\sin 4(t-2), \\ \quad 2 < t < 4, \\ 3e^{-3t}\cos 4t-e^{-3(t-2)}\cos 4(t-2) - \dfrac{3}{4}e^{-3(t-2)}\sin 4(t-2) \\ \quad + e^{-3(t-4)}\cos 4(t-4) + \dfrac{3}{4}e^{-3(t-4)}\sin 4(t-4), & t > 4. \end{cases}$$

12. Step 1. Taking the Laplace Transform of both sides of the D.E., we have

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} = \mathcal{L}\{h(t)\}. \tag{1}$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$, applying Theorem 9.4, and then applying the I.C.'s, we have

$$\mathcal{L}\{y''\} = s^2Y(s) - s + 2, \quad \mathcal{L}\{y'\} = sY(s) - 1.$$

Substituting these into the left member of (1) and simplifying, it becomes $(s^2 - s)Y(s) - s + 3$. Now observe that

$$\begin{aligned} h(t) &= \begin{cases} 4 - 2t, & 0 \le t \le 2, \\ 0, & t > 2, \end{cases} \\ &= \begin{cases} 4 - 2t + 0, & 0 \le t \le 2, \\ 4 - 2t + 2(t - 2), & t > 2, \end{cases} \\ &= 4 - 2t + 2u_2(t)(t - 2) \\ &= 4 - 2t + 2u_2(t)f(t - 2), \end{aligned}$$

where $f(t - 2) = t - 2$. Then by Table 9.1, numbers 1, 7, and 16, respectively, we have

$$\mathcal{L}\{h(t)\} = \frac{4}{s} - \frac{2}{s^2} + 2e^{-2s}F(s),$$

where $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \dfrac{1}{s^2}$, using number 7 again. Thus the right member of (1) becomes $\dfrac{4}{s} - \dfrac{2}{s^2} + \dfrac{2e^{-2s}}{s^2}$, and (1) becomes

$$(s^2 - s)Y(s) - s + 3 = \frac{4}{s} - \frac{2}{s^2} + \frac{2e^{-2s}}{s^2}.$$

Step 2. We solve this for Y(s), obtaining

$$Y(s) = \frac{s - 3}{s(s - 1)} + \frac{4}{s^2(s - 1)} - \frac{2}{s^3(s - 1)} + \frac{2e^{-2s}}{s^3(s - 1)}.$$

Step 3. We must find

$$y = \mathcal{L}^{-1}\left\{\frac{s - 3}{s(s - 1)} + \frac{4}{s^2(s - 1)} - \frac{2}{s^3(s - 1)}\right\} + \mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^3(s - 1)}\right\}. \tag{2}$$

We first find the first of these two inverse transforms. We write the expression in the first pair of braces over a lowest common denominator and then employ partial fractions. We have

$$\frac{s^3 - 3s^2 + 4s - 2}{s^3(s - 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s - 1}.$$

Clearing fractions, we obtain

$$s^3 - 3s^2 + 4s - 2 = As^2(s - 1) + Bs(s - 1) + C(s - 1) + Ds^3 = (A + D)s^3 + (B - A)s^2 + (C - B)s - C.$$

From this we easily find $A = 1$, $B = -2$, $C = 2$, $D = 0$. Hence we seek

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\};$$

and using Table 9.1, numbers 1, 7, and 7, respectively, we obtain $1 - 2t + t^2$. Thus

$$\mathcal{L}^{-1}\left\{\frac{s-3}{s(s-1)}+\frac{4}{s^2(s-1)}-\frac{2}{s^3(s-1)}\right\}$$

$$=1-2t+t^2. \tag{3}$$

We now return to (2) and proceed to find $\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^3(s-1)}\right\}$. We consider $\frac{2}{s^3(s-1)}$, and again employ partial fractions. We have

$$\frac{2}{s^3(s-1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-1}.$$

Clearing fractions and simplifying, we obtain

$$2=(A+D)s^3+(B-A)s^2+(C-B)s-C,$$

from which $A=B=C=-2$ and $D=2$. Thus

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3(s-1)}\right\}=-2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$-\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}.$$

Then by Table 9.1, numbers 1, 7, 7, and 2, respectively, we obtain

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3(s-1)}\right\}=-2-2t-t^2+2e^t.$$

Letting $F(s)=\frac{2}{s^3(s-1)}$ and $f(t)=-2-2t-t^2+2e^t$, we thus have $\mathcal{L}^{-1}\{F(s)\}=f(t)$. Then by Table 9.1, number 16, we find

$$\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^3(s-1)}\right\} = \mathcal{L}^{-1}\{F(s)e^{-2s}\}$$

$$= u_2(t)f(t-2)$$

$$= \begin{cases} 0, & 0 < t < 2, \\ f(t-2), & t > 2, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < 2, \\ -2 - 2(t-2) - (t-2)^2 + 2e^{t-2}, & t > 2, \end{cases}$$

$$= \begin{cases} 0, & 0 < t < 2, \\ -2 + 2t - t^2 + 2e^{t-2}, & t > 2. \end{cases} \qquad (4)$$

Then, using (2), (3), and (4), we obtain

$$y = \begin{cases} (t-1)^2, & 0 < t < 2, \\ -1 + 2e^{t-2}, & t > 2. \end{cases}$$

**Section 9.4D, Page 539.**

1. Step 1. We take the Laplace Transform of both sides of the D.E. to obtain

$$\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\}. \qquad (1)$$

We denote $\mathcal{L}\{y\}$ by $Y(s)$ and apply Theorem 9.3 and then the I.C., to obtain

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 3.$$

Then the left member of equation (1) becomes $(s-4)Y(s) - 3$. By formula (9.103), $\mathcal{L}\{\delta(t-2)\} = e^{-2s}$. Thus (1) reduces to

$$(s-4)Y(s) - 3 = e^{-2s}.$$

Step 2. We solve this for Y(s), obtaining

$$Y(s) = \frac{3}{s - 4} + \frac{e^{-2s}}{s - 4}.$$

Step 3. We must now determine

$$y = 3\mathcal{L}^{-1}\left\{\frac{1}{s - 4}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s - 4}\right\}.$$

Using Table 9.1, numbers 2 and 16, respectively, we find $y = 3e^{4t} + u_2(t)f(t - 2)$, where $f(t) = e^{4t}$. Thus

$$y = 3e^{4t} + u_2(t)e^{4(t-2)}$$

$$= \begin{cases} 3e^{4t}, & 0 < t < 2, \\ 3e^{4t} + e^{4(t-2)}, & t > 2. \end{cases}$$

3. Step 1. We take the Laplace Transform of both sides of the D.E. to obtain

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t - \pi)\}. \qquad (1)$$

We denote $\mathcal{L}\{y\}$ by Y(s) and apply Theorem 9.3 and then the I.C., to obtain

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

$$= s^2Y(s) - 1.$$

Then the left member of equation (1) becomes $(s^2 + 1)Y(s) - 1$. By formula (9.103), $\mathcal{L}\{\delta(t - \pi)\} = e^{-\pi s}$. Thus (1) reduces to

$$(s^2 + 1)Y(s) - 1 = e^{-\pi s}.$$

Step 2. We solve this for Y(s), obtaining

$$Y(s) = \frac{1}{s^2 + 1} + \frac{e^{-\pi s}}{s^2 + 1}.$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2 + 1}\right\}.$$

Using Table 9.1, numbers 3 and 16, respectively, we find $y = \sin t + u_\pi(t)f(t - \pi)$, where $f(t) = \sin t$. Thus

$$y = \sin t + u_\pi(t)\sin(t - \pi)$$

$$= \begin{cases} \sin t, & 0 < t < \pi, \\ \sin t - \sin t, & t > \pi, \end{cases}$$

$$= \begin{cases} \sin t, & 0 < t < \pi, \\ 0, & t > \pi. \end{cases}$$

4. Step 1. We take the Laplace Transform of both sides of the D.E. to obtain

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \delta(t - 4). \qquad (1)$$

We denote $\mathcal{L}\{y\}$ by Y(s) and apply Theorem 9.4 and then the I.C.'s, to obtain

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s + 6,$$
$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2.$$

Substituting these into the left member of (1) that left member becomes

$$s^2Y(s) - 2s + 6 + 3sY(s) - 6 + 2Y(s)$$

or

$$(s^2 + 3s + 2)Y(s) - 2s.$$

By formula (9.103), $\mathcal{L}\{\delta(t - 4)\} = e^{-4s}$. Thus (1) reduces to

$$(s^2 + 3s + 2)Y(s) - 2s = e^{-4s}.$$

Step 2. We solve this for Y(s), obtaining

$$Y(s) = \frac{2s}{s^2 + 3s + 2} + \frac{e^{-4s}}{s^2 + 3s + 2}$$

Step 3. We must now determine

$$y = \mathcal{L}^{-1}\left\{\frac{2s}{(s + 1)(s + 2)}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{(s + 1)(s + 2)}\right\}. \tag{2}$$

We employ partial fractions to find the first of the two inverse transforms needed. We write $\frac{2s}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}$, and upon clearing fractions, at once find A = -2, B = 4. Thus

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s + 1)(s + 2)}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\}.$$

Then using Table 9.1, number 2, twice, we find

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s + 1)(s + 2)}\right\} = -2e^{-t} + 4e^{-2t}. \tag{3}$$

Now we proceed to determine $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{(s + 1)(s + 2)}\right\}$.

Again using partial fractions, we find

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Thus by Table 9.1, number 2, $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$ $= e^{-t} - e^{-2t}$. Then by Table 9.1, number 16,

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{(s+1)(s+2)}\right\} = u_4(t)f(t-4), \tag{4}$$

where $f(t) = e^{-t} - e^{-2t}$. Hence using (2), (3), and (4), we find

$$y = \begin{cases} -2e^{-t} + 4e^{-2t}, & 0 < t < 4, \\ -2e^{-t} + 4e^{-2t} + e^{-(t-4)} - e^{-2(t-4)}, & t > 4, \end{cases}$$

$$= \begin{cases} -2e^{-t} + 4e^{-2t}, & 0 < t < 4, \\ (-2 + e^4)e^{-t} + (4 - e^8)e^{-2t}, & t > 4. \end{cases}$$

**Section 9.5, Page 542.**

3. Step 1. Taking the Laplace Transform of both sides of each D.E. of the system, we have

$$\begin{cases} \mathcal{L}\{x'\} - 5\mathcal{L}\{x\} + 2\mathcal{L}\{y\} = 3\mathcal{L}\{e^{4t}\}, \\ \mathcal{L}\{y'\} - 4\mathcal{L}\{x\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}. \end{cases} \tag{1}$$

We denote $\mathcal{L}\{x\}$ by $X(s)$ and $\mathcal{L}\{y\}$ by $Y(s)$.

Applying Theorem 9.3 and the given I.C.'s, we express $\mathcal{L}\{x'\}$ and $\mathcal{L}\{y'\}$ in terms of $X(s)$ and $Y(s)$, respectively, as follows

$$\begin{cases} \mathcal{L}\{x'\} = sX(s) - x(0) = sX(s) - 3, \\ \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s). \end{cases} \tag{2}$$

By Table 9.1, number 2, $\mathcal{L}\{e^{4t}\} = \dfrac{1}{s - 4}$. Thus, from this and (2), we see that (1) becomes

$$\begin{cases} sX(s) - 3 - 5X(s) + 2Y(s) = \dfrac{3}{s - 4}, \\ sY(s) - 4X(s) + Y(s) = 0, \end{cases}$$

or

$$\begin{cases} (s - 5)X(s) + 2Y(s) = \dfrac{3s - 9}{s - 4}, \\ -4X(s) + (s + 1)Y(s) = 0. \end{cases}$$

Step 2. We solve this system for the two unknowns X(s) and Y(s). We have

$$\begin{cases} (s + 1)(s - 5)X(s) + 2(s + 1)Y(s) = \dfrac{(3s - 9)(s + 1)}{s - 4}, \\ -8X(s) + 2(s + 1)Y(s) = 0. \end{cases}$$

Subtracting, we obtain

$$(s^2 - 4s + 3)X(s) = \frac{(3s - 9)(s + 1)}{s - 4},$$

from which we find

$$X(s) = \frac{3(s - 3)(s + 1)}{(s - 1)(s - 3)(s - 4)} = \frac{3(s + 1)}{(s - 1)(s - 4)}.$$

In like manner, we find

$$Y(s) = \frac{12}{(s - 1)(s - 4)}.$$

Step 3. We must now determine

$$x = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3(s+1)}{(s-1)(s-4)}\right\}.$$

and

$$y = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{12}{(s-1)(s-4)}\right\}.$$

We first find x. We employ partial fractions. We have $\frac{3(s+1)}{(s-1)(s-4)} = \frac{A}{s-1} + \frac{B}{s-4}$, from which we readily find $A = -2$, $B = 5$. Thus

$$x = -2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\},$$

and using Table 9.1, number 2, we obtain

$$x = -2e^{t} + 5e^{4t}.$$

Similarly, we find y. Again employing partial fractions, we obtain

$$y = -4\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}.$$

Again using Table 9.1, number 2, we obtain

$$y = -4e^{t} + 4e^{4t}.$$

4. Step 1. Taking the Laplace Transform of both sides of each D.E. of the system, we have

$$\begin{cases} \mathcal{L}\{x'\} - 2\mathcal{L}\{x\} - 3\mathcal{L}\{y\} = 0, \\ \mathcal{L}\{y'\} + \mathcal{L}\{x\} + 2\mathcal{L}\{y\} = \mathcal{L}\{t\}. \end{cases} \tag{1}$$

Denote $\mathcal{L}\{x\}$ by $X(s)$ and $\mathcal{L}\{y\}$ by $Y(s)$. Applying Theorem 9.3 and the given I.C.'s, we express $\mathcal{L}\{x\}$ in terms of $X(s)$ and $\mathcal{L}\{y\}$ in terms of $Y(s)$ as follows:

$$\begin{cases} \mathcal{L}\{x'\} = sX(s) - x(0) = sX(s) + 1, \\ \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s). \end{cases} \tag{2}$$

By Table 9.1, number 7, $\mathcal{L}\{t\} = \dfrac{1}{s^2}$. Substituting this and the expressions for $\mathcal{L}\{x'\}$ and $\mathcal{L}\{y'\}$ given by (2) into (1), we see that (1) becomes

$$\begin{cases} sX(s) + 1 - 2X(s) - 3Y(s) = 0, \\ sY(s) + X(s) + 2Y(s) = 1/s^2. \end{cases}$$

or

$$\begin{cases} (s - 2)X(s) - 3Y(s) = -1, \\ X(s) + (s + 2)Y(s) = 1/s^2. \end{cases}$$

Step 2. We solve this system for the two unknowns $X(s)$ and $Y(s)$. We have

$$\begin{cases} (s - 2)(s + 2)X(s) - 3(s + 2)Y(s) = -(s + 2), \\ 3X(s) + 3(s + 2)Y(s) = 3/s^2. \end{cases}$$

Adding, we obtain

$$(s^2 - 1)X(s) = -(s + 2) + 3/s^2,$$

from which we find

$$X(s) = -\frac{s + 2}{s^2 - 1} + \frac{3}{s^2(s^2 - 1)} = \frac{-s^3 - 2s^2 + 3}{s^2(s^2 - 1)}.$$

In like manner, we find

$$Y(s) = \frac{1}{s^2 - 1} + \frac{s - 2}{s^2(s^2 - 1)} = \frac{s^2 + s - 2}{s^2(s^2 - 1)}.$$

Step 3. We must now determine

$$x = \mathcal{L}^{-1}\left\{\frac{-s^3 - 2s^2 + 3}{s^2(s^2 - 1)}\right\}.$$

$$y = \mathcal{L}^{-1}\left\{\frac{s^2 + s - 2}{s^2(s^2 - 1)}\right\}.$$

We first determine x. Using partial fractions, we write $\frac{-s^3 - 2s^2 + 3}{s^2(s - 1)(s + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 1} + \frac{D}{s + 1}$. Clearing fractions, we obtain $-s^3 - 2s^2 + 3 = As(s - 1)(s + 1) + B(s - 1)(s + 1) + Cs^2(s + 1) + Ds^2(s - 1) = (A + C + D)s^3 + (B + C - D)s^2 - As - B$. From this, we find $A = 0$, $B = -3$, $C = 0$, $D = -1$. Thus

$$x = -3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\};$$

and using Table 9.1, numbers 7 and 2, respectively, we find

$$x = -3t + e^{-t}.$$

In like manner, using partial fractions, we obtain

$$y = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\};$$

and using Table 9.1, numbers 1, 7, and 2, respectively, we find

$$y = -1 + 2t + e^{-t}.$$

7. Step 1. Taking the Laplace Transform of both sides of each D.E. of the system, we have

$$\begin{cases} 2\mathcal{L}\{x'\} + \mathcal{L}\{y'\} - \mathcal{L}\{x\} - \mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\} \\ \mathcal{L}\{x'\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{x\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{t}\}. \end{cases} \tag{1}$$

We denote $\mathcal{L}\{x\}$ by $X(s)$ and $\mathcal{L}\{y\}$ by $Y(s)$. Applying Theorem 9.3 and the given I.C.'s, we express $\mathcal{L}\{x'\}$ and $\mathcal{L}\{y'\}$ in terms of $X(s)$ and $Y(s)$, respectively, as follows:

$$\begin{aligned} \mathcal{L}\{x'\} &= sX(s) - x(0) = sX(s) - 2, \\ \mathcal{L}\{y'\} &= sY(s) - y(0) = sY(s) - 1. \end{aligned} \tag{2}$$

By Table 9.1, number 2, $\mathcal{L}\{e^{-t}\} = \dfrac{1}{s+1}$, $\mathcal{L}\{e^{t}\} = \dfrac{1}{s-1}$. Thus, from this and (2), we see that (1) becomes

$$\begin{cases} 2sX(s) - 4 + sY(s) - 1 - X(s) - Y(s) = \dfrac{1}{s+1}, \\ sX(s) - 2 + sY(s) - 1 + 2X(s) + Y(s) = \dfrac{1}{s-1}, \end{cases}$$

or

$$\begin{cases} (2s-1)X(s) + (s-1)Y(s) = \dfrac{5s+6}{s+1}, \\ (s+2)X(s) + (s+1)Y(s) = \dfrac{3s-2}{s-1}. \end{cases}$$

Step 2. We solve this system for the two unknowns $X(s)$ and $Y(s)$. We have

$$\begin{cases} (s+1)(2s-1)X(s) + (s+1)(s-1)Y(s) = 5s+6, \\ (s-1)(s+2)X(s) + (s+1)(s-1)Y(s) = 3s-2. \end{cases}$$

Subtracting, we obtain $(s^2+1)X(s) = 2s + 8$, from which we find

$$X(s) = \frac{2s + 8}{s^2 + 1}.$$

In like manner, we find

$$Y(s) = \frac{s^3 - 12s^2 - s + 14}{(s + 1)(s - 1)(s^2 + 1)}.$$

Step 3. We must now determine

$$x = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2s + 8}{s^2 + 1}\right\}$$

and

$$y = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^3 - 12s^2 - s + 14}{(s + 1)(s - 1)(s^2 + 1)}\right\}.$$

We first obtain x. Using Table 9.1, numbers 4 and 3, we find

$$\begin{aligned} x &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} \\ &= 2\cos t + 8\sin t. \end{aligned}$$

We proceed to find y. We first employ partial fractions. We have $\dfrac{s^3 - 12s^2 - s + 14}{(s + 1)(s - 1)(s^2 + 1)} = \dfrac{A}{s - 1} + \dfrac{B}{s + 1} + \dfrac{Cs + D}{s^2 + 1}$ or

$$\begin{aligned} s^3 - 12s^2 - s + 14 &= A(s + 1)(s^2 + 1) \\ &\quad + B(s - 1)(s^2 + 1) + (Cs + D)(s - 1)(s + 1). \quad (3) \end{aligned}$$

or

$$s^3 - 12s^2 - s + 14 = (A + B + C)s^3 + (A - B + D)s^2 + (A + B - C)s + (A - B - D).$$

From this, we have

$$\begin{cases} A + B + C = 1, & A - B + D = -12, \\ A + B - C = -1, & A - B - D = 14. \end{cases} \tag{4}$$

Letting $s = 1$ in (3), we find $A = \frac{1}{2}$; and letting $s = -1$ in (3), we find $B = -\frac{1}{2}$. Using these values and (4), we find $C = 1$, $D = -13$. Thus we have

$$y = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} - 13\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}.$$

Using Table 9.1, numbers 2, 2, 4, and 3, respectively, we find

$$y = \frac{1}{2}e^t - \frac{1}{2}e^{-t} + \cos t - 13 \sin t.$$

8. Step 1. Taking the Laplace Transform of both sides of each D.E. of the system, we have

$$\begin{cases} 2\mathcal{L}\{x'\} + \mathcal{L}\{y'\} + \mathcal{L}\{x\} + 5\mathcal{L}\{y\} = 4\mathcal{L}\{t\}, \\ \mathcal{L}\{x'\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{x\} + 2\mathcal{L}\{y\} = 2\mathcal{L}\{1\}. \end{cases} \tag{1}$$

Denote $\mathcal{L}\{x\}$ by $X(s)$ and $\mathcal{L}\{y\}$ by $Y(s)$. Applying Theorem 9.3 and the given I.C.'s, we express $\mathcal{L}\{x\}$ in terms of $X(s)$ and $\mathcal{L}\{y\}$ in terms of $Y(s)$ as follows:

$$\begin{cases} \mathcal{L}\{x'\} = sX(s) - x(0) = sX(s) - 3, \\ \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) + 4. \end{cases} \tag{2}$$

By Table 9.1, numbers 7 and 2, respectively, we find $\mathcal{L}\{t\} = 1/s^2$ and $\mathcal{L}\{1\} = 1/s$. Substituting these and the expressions for $\mathcal{L}\{x'\}$ and $\mathcal{L}\{y'\}$ given by (2) into (1), we see that (1) becomes

$$\begin{cases} 2sX(s) - 6 + sY(s) + 4 + X(s) + 5Y(s) = 4/s^2, \\ sX(s) - 3 + sY(s) + 4 + 2X(s) + 2Y(s) = 2/s. \end{cases}$$

or

$$\begin{cases} (2s + 1)X(s) + (s + 5)Y(s) = 2 + 4/s^2, \\ (s + 2)X(s) + (s + 2)Y(s) = -1 + 2/s. \end{cases}$$

Step 2. We solve this system for the two unknowns $X(s)$ and $Y(s)$. We have

$$\begin{cases} (s+2)(2s+1)X(s) + (s+2)(s+5)Y(s) = (s+2)(2+4/s^2), \\ (s+2)(s+5)X(s) + (s+2)(s+5)Y(s) = (s+5)(-1+2/s). \end{cases}$$

Subtracting, we obtain

$$(s^2 - 2s - 8)X(s) = 3s + 7 - \frac{6}{s} + \frac{8}{s^2},$$

from which we find

$$\begin{aligned} X(s) &= \frac{3s + 7}{s^2 - 2s - 8} - \frac{6}{s(s^2 - 2s - 8)} \\ &\quad + \frac{8}{s^2(s^2 - 2s - 8)} \\ &= \frac{3s^3 + 7s^2 - 6s + 8}{s^2(s - 4)(s + 2)}. \end{aligned}$$

In like manner, we find

$$Y(s) = \frac{-4s - 1}{s^2 - 2s - 8} - \frac{2}{s(s^2 - 2s - 8)} - \frac{8}{s^2(s^2 - 2s - 8)} = \frac{-4s^3 - s^2 - 2s - 8}{s^2(s - 4)(s + 2)}.$$

Step 3. We must now determine

$$x = \mathcal{L}^{-1}\left\{\frac{3s^3 + 7s^2 - 6s + 8}{s^2(s - 4)(s + 2)}\right\},$$

$$y = \mathcal{L}^{-1}\left\{\frac{-4s^3 - s^2 - 2s - 8}{s^2(s - 4)(s + 2)}\right\}.$$

We first determine x. Using partial fractions, we write $\frac{3s^3 + 7s^2 - 6s + 8}{s^2(s - 4)(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 4} + \frac{D}{s + 2}$. Clearing fractions, we have $3s^3 + 7s^2 - 6s + 8 = As(s - 4)(s + 2) + B(s - 4)(s + 2) + Cs^2(s + 2) + Ds^2(s - 4) = (A + C + D)s^3 + (-2A + B + 2C - 4D)s^2 + (-8A - 2B)s - 8B$. From this, we find $A = 1$, $B = -1$, $C = 3$, $D = -1$. Thus

$$x = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s - 4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\}.$$

Then, using Table 9.1, numbers 1, 7, 2, and 2, respectively, we find

$$x = 1 - t + 3e^{4t} - e^{-2t}.$$

In like manner, using partial fractions, we obtain

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

Then, using Table 9.1, numbers 7, 2, and 2, respectively, we find

$$y = t - 3e^{4t} - e^{-2t}.$$

10. Step 1. Taking the Laplace Transform of both sides of each D.E. of the system, we have

$$\begin{cases} \mathcal{L}\{x''\} - 3\mathcal{L}\{x'\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{x\} - \mathcal{L}\{y\} = 0, \\ \mathcal{L}\{x'\} + \mathcal{L}\{y'\} - 2\mathcal{L}\{x\} + \mathcal{L}\{y\} = 0. \end{cases} \tag{1}$$

We denote $\mathcal{L}\{x\}$ by $X(s)$ and $\mathcal{L}\{y\}$ by $Y(s)$. Applying Theorem 9.4 and the given I.C.'s, we express $\mathcal{L}\{x''\}$ and $\mathcal{L}\{x'\}$ in terms of $X(s)$ and $\mathcal{L}\{y'\}$ in terms of $Y(s)$ as follows:

$$\mathcal{L}\{x''\} = s^2X(s), \quad \mathcal{L}\{x'\} = sX(s),$$
$$\mathcal{L}\{y'\} = sY(s) + 1.$$

Substituting these expressions into (1), we see that (1) becomes

$$\begin{cases} s^2X(s) - 3sX(s) + sY(s) + 1 + 2X(s) - Y(s) = 0, \\ sX(s) + sY(s) + 1 - 2X(s) + Y(s) = 0, \end{cases}$$

or

$$\begin{cases} (s^2 - 3s + 2)X(s) + (s - 1)Y(s) = -1, \\ (s - 2)X(s) + (s + 1)Y(s) = -1. \end{cases}$$

Step 2. We solve this system for the two unknowns $X(s)$ and $Y(s)$. We have

$$\begin{cases}(s^2 - 3s + 2)(s + 1)X(s) + (s - 1)(s + 1)Y(s) = -(s + 1), \\ (s - 2)(s - 1)X(s) + (s + 1)(s - 1)Y(s) = -(s - 1).\end{cases}$$

Subtracting, we obtain $(s^3 - 3s^2 + 2s)X(s) = -2$ from which we find

$$X(s) = -\frac{2}{s(s - 1)(s - 2)}.$$

In like manner, we find

$$Y(s) = -\frac{s - 2}{s(s - 1)}.$$

Step 3. We must now determine

$$x = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s(s - 1)(s - 2)}\right\}, \quad \text{and}$$

$$y = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-s + 2}{s(s - 1)}\right\}.$$

We first find x using partial fractions. We have $\frac{-2}{s(s - 1)(s - 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s - 2}$ or $-2 = A(s - 1)(s - 2) + Bs(s - 2) + Cs(s - 1)$. Letting $s = 0$, we find $A = -1$; letting $s = 1$, we find $B = 2$; and letting $s = 2$, we find $C = -1$. Thus we have

$$x = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\}.$$

Now using Table 9.1, numbers 1, 2, and 2, respectively, we find $x(t) = -1 + 2e^t - e^{2t}$.

We now find y, again using partial fractions. We have $\frac{-s + 2}{s(s - 1)} = \frac{A}{s} + \frac{B}{s - 1}$. We readily find $A = -2$, $B = 1$; and hence

$$y = -2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\}.$$

Then from Table 9.1, numbers 1 and 2, we find

$$y = -2 + e^t.$$

# Answers to Even-Numbered Exercises

**Section 1.1, Page 5.**

2. ordinary; third; linear.

4. ordinary; first; nonlinear.

6. partial; fourth; linear.

8. ordinary; second; nonlinear.

10. ordinary; second; nonlinear.

**Section 1.3, Page 21.**

2. (a) $y = (x^2 + 2)e^{-x}$; (b) $y = (x^2 + 3e^{-1})e^{-x}$.

**Section 2.1, Page 36.**

2. $xy^2 + 3x - 4y = c$.

4. not exact.

6. $(\theta^2 + 1)\sin r = c$.

8. not exact.

10. not exact.

12. $x^3y^2 - y^3x + x^2 + y + 1 = 0$.

14. $e^x y + xy^2 + 2e^x = 8$.

16. $2x^{1/3}y^{-1/3} + 4x^{4/3}y^{1/3} = 9$.

18. (a) $A = 3$; $x^3y + 2xy^2 = c$.
(b) $A = -2$; $x^{-2}y - x^{-1}y = c$.

20. (a) $xy^4 + 2x^3y^2 + \phi(x)$; (b) $e^x y^2 + e^{3x}y^3 + \phi(x)$.

22. (b) $n = -2$; (c) $x + x^2y^{-1} = c$.

24. $4 \arctan \frac{x}{y} + (x^2 + y^2)^2 = c$.

**Section 2.2, Page 46.**

2. $x(x + 2)(y + 2)^2 = c$.

4. $\sin x - \cos y = c$.

6. $(\sin u + 1)(e^v + 1) = c$.

8. $y = x \ln|cx|$.

10. $v^2 = u^2(\ln v^2 + c)$.

12. $t^3 - 3t^2s - 3ts^2 - 2s^3 = c$.

14. $x\left(\sqrt{\frac{x^2 - y^2}{x^2}} + 1\right) = c$.

16. $4x - 2 \sin 2x + \tan y = \frac{\pi}{3}$.

18. $x^2 + y^2 = 5x^3$.

20. $2(3x^2 + 3xy + y^2)^2 = 9x^5$.

22. (a) $x^2 + 4xy - y^2 = c$;
(b) $3x^2 - 2xy - y^2 = c$.

26. (a) $y = x \ln|cx|$;
(b) $y^2 + xy = cx^3$.

**Section 2.3, Page 56.**

2. $x^2y + \frac{1}{x} = c.$

4. $y = 2 + ce^{-2x^2}.$

6. $(u^2 + 1)^2 v = \frac{3u^4}{4} + \frac{3u^4}{2} + c.$

8. $3(x + 2)y = x - 1 + c(x - 1)^{-2}.$

10. $xy = 1 + ce^{-y^2/2}.$

12. $2r = (\theta + \sin\theta\cos\theta + c)\cos\theta$

14. $(1 + \sin^2 x)y = \sin x + c.$

16. $x^3y^3(2x^3 + c) = 1.$

18. $x^2 = 2 + ct^{-1}e^{-t}.$

20. $3y = 1 + 5e^{-x^3}.$

22. $y = -2x^2 - 3.$

24. $x = \frac{1}{5}(2e^t - \sin 2t - 2\cos 2t).$

26. $\frac{1}{\sqrt{xy}} = -\frac{1}{2}x + 1.$

28. $\begin{cases} y = 5 + e^{-x} & , \ 0 \le x < 10; \\ y = 1 + (4e^{10} + 1)e^{-x} & , \ x \ge 10. \end{cases}$

30. $\begin{cases} y = \frac{x^2 + 1}{2(x + 1)} & , \ 0 \le x < 3, \\ y = \frac{3x - 4}{x + 1} & , \ x \ge 3. \end{cases}$

36. (b) $y = ce^{-x} + \sum_{k=1}^{5} \frac{\sin kx - k \cos kx}{1 + k^2}$

40. $\frac{e^{-x^2/2}}{y - x} = \int e^{-x^2/2}\,dx + c.$

**Miscellaneous Review Exercises, Page 59.**

2. $x^2y^3 - xy = c.$

4. $y = \frac{x^2}{4} + \frac{c}{x^2}.$

6. $(e^{2x} - 2)y^2 = c.$

8. $\left(\frac{y + x}{y + 2x}\right)^2 = \left|\frac{c}{x}\right|.$

10. $y = e^{-x}[-1 + c(x + 1)].$

12. $y^2 = \frac{1}{(1 + cx^2)}.$

14. $(y - 2x)(y + x) = cx.$

16. $(x + 1)\sqrt{y^2 + 4} = 4(x - 1).$

18. $x^3 + x^2y^2 + 2y^3 = 21.$

20. $5(2x + y)^2 = 16(x + 2y).$

22. $y = \begin{cases} 1 - e^{-x}, & 0 \leq x < 2, \\ (e^2 - 1)e^{-x}, & x > 2. \end{cases}$

24. $y = \frac{\sqrt{2}x}{\sqrt{x^4 + 1}}$

**Section 2.4, Page 67.**

2. $x^2 \cos y + x \sin y = c.$

4. $x^2 + xy^{-1} + y^2 = c.$

6. $x^{2/3}y^{5/3}(x^2 - y) = c.$

8. $x - 2y + \ln|3x - y - 2| = c.$

10. $(2x + y + 3)^3(x - y + 1)^2 = c.$

12. $\ln[3(x - 1)^2 + (y + 3)^2] + \frac{2}{\sqrt{3}} \arctan \frac{y + 3}{\sqrt{3}(x - 1)} = \ln 4 + \frac{\pi}{3\sqrt{3}}.$

14. $\ln|2x + y - 1| + \frac{x - 2}{2x + y - 1} = 1.$

**Section 3.1, Page 78.**

2. $y^2 + 2x^2 = k^2.$

4. $y^2\left(\ln|y| - \frac{1}{2}\right) = -x^2 + k.$

6. $x^2 + y^2 - 2x + \ln(x + 2)^4 = k.$

8. $x^2y = \frac{x^4}{4} + k.$

10. $y = k(x^2 + 3y^2).$

12. $x^2 + y^2 = ky.$

14. $n = 3.$

16. $\ln(x^2 + y^2) + 2 \arctan(y/x) = k.$

18. $\ln|3x^2 + 3xy + 4y^2| - (2/\sqrt{39}) \arctan[(3x + 8y)/\sqrt{39}x] = k.$

**Section 3.2, Page 88.**

2. (a) $v = 9$ ft./sec.; $x = 539.16$ ft.;
   (b) $v = 9$ ft./sec.

4. (a) $v = \frac{25}{2}(1 - e^{-t/250})$; (b) 12.5 ft./sec.;
   (c) 402 sec.

6. (a) 33.99 cm/sec.;
   (b) 88.82 cm/sec.

8. (a) 7.16 ft./sec.;
   (b) 4.95 sec.

10. (a) $v = 100 \tan(\arctan 10 - 0.32t)$;
    (b) 4.60 sec.

12. 3.06 ft./sec.

14. 17.5 ft./sec.

16. 0.25

18. $v = \left[\frac{2gR^2}{x} + v_0^2 - 2gR\right]^{1/2}$.

**Section 3.3, Page 102.**

2. (a) 34.5%;
   (b) $t = 43.4$ min.

4. (a) 19.8%;
   (b) 9 hrs., 58 min.

6. (a) 10.22 oz.;
   (b) 23 hrs., 34 min.

8. 18.91 min.

10. (a) $151.41^0$;
    (b) Between 22 min., 31 sec., after 10 A.M., and 30 min., 12 sec., after.

12. 40,833.

14. (a) $x = (4219)(10^6)e^{0.02(t-1978)}$.
(b) 2410 million. (c) 6551 million.
(d) 886 million. (e) 48,405 million.

16. (a) $x = \dfrac{kx_0}{\lambda x_0 + (k - \lambda x_0)e^{-k(t-t_0)}}$

18. 94,742

20. (a) 34.66%; (b) 0.85 years

22. (a) 7.81 lbs; concentration is 0.0539 lb./gal.
(b) 0.0217 lb./gal.

24. (a) 466.12 lb.; (b) 199.99 lb.

26. 4,119.65 gm. 28. 292.96 $(\text{ft.})^3/\text{min.}$

30. 2016. 32. 7.14 grams.

**Section 4.1B, Page 122.**

2. $y = 0$ for all real $x$.

4. (b) and (c). Theorem 4.2.

8. (b) $y = c_1e^x + c_2xe^x$.
(c) $y = e^x + 3xe^x$; $-\infty < x < \infty$.

10. (b) $y = c_1 x^2 + \dfrac{c_2}{x^2}$.

(c) $y = \dfrac{1}{4} x^2 + \dfrac{8}{x^2}$; Theorem 4.1; $0 < x < \infty$.

12. $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{4x}$.

**Section 4.1D, Page 132.**

2. $y = c_1(x + 1) + c_2(x + 1)^3$.

4. $y = c_1 x + c_2(x - 1)e^x$.

6. $y = xe^x$; $y = c_1 x^2 + c_2 x e^x$.

8. $y = c_1 e^x + c_2 x^{-1}$.

10. (b) $y = c_1 e^x + c_2 e^{2x}$.

(d) $y = c_1 e^x + c_2 e^{2x} + 2x^2 + 6x + 7$.

**Section 4.2, Page 143.**

2. $y = c_2 e^{3x} + c_2 e^{-x}$.

4. $y = c_1 e^{5x} + c_2 e^{-x/3}$.

6. $y = c_1 e^{x/2} + c_2 e^{-2x}$.

8. $y = (c_1 + c_2 x)e^{2x}$.

10. $y = e^{-x}\left(c_1 \sin \dfrac{3x}{4} + c_2 \cos \dfrac{3x}{4}\right)$.

12. $y = c_1e^{-x} + c_2e^{3x} + c_3e^{4x}$.

14. $y = c_1e^{-2x} + (c_2 + c_3x)e^{x/2}$.

16. $y = c_1e^{-3x} + e^{-x/2}\left(c_2 \sin \frac{\sqrt{7}}{2} x + c_3 \cos \frac{\sqrt{7}}{x} x\right)$.

18. $y = (c_1 + c_2x)e^{-x/2}$.

20. $y = e^{-3x}(c_2 \sin 4x + c_2 \cos 4x)$.

22. $7 = c_1 \sin \frac{x}{2} + c_2 \cos \frac{x}{2}$.

24. $y = (c_1 + c_2x + c_3x^2)e^{-x/2}$.

26. $y = c_1e^{x} + c_2e^{-x} + c_3 \sin x + c_4 \cos x$.

28. $y = c_1e^{x} + c_2e^{2x} + (c_3 + c_4x)e^{-x}$.

30. $y = (c_1 + c_2x)e^{-2x} + e^{-x}(c_3 \sin \sqrt{2}x + c_4 \cos \sqrt{2}x)$.

32. $y = (c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4)e^{-x}$.

34. $y = (c_1 + c_2x)e^{x}$

$$+ e^{-x/2}\left[(c_3 + c_4x)\sin \frac{\sqrt{3}x}{2} + (c_5 + c_6x)\cos \frac{\sqrt{3}x}{2}\right]$$

36. $y = c_1 \sin 2x + c_2 \cos 2x + e^{\sqrt{3}x}(c_3 \sin x + c_4 \cos x) + e^{-\sqrt{3}x}(c_5 \sin x + c_6 \cos x)$.

38. $y = -6e^{-2x} + 2e^{-5x}$.

40. $y = 2e^{-2x}$.

42. $y = 3xe^{3x/2} + 4e^{3x/2}$.

44. $y = (3 - 2x)e^{x/3}$.

46. $y = e^{-3x}\left(\frac{2}{7} \sin 7x - \cos 7x\right)$.

48. $y = e^{-x}(4 \sin 2x + 2 \cos 2x)$.

50. $y = e^{-x/2}(-\sin 3x + 2 \cos 3x)$.

52. $y = e^{2x} - \sin 2x + \cos 2x$.

54. $y = e^{x} + e^{2x}(2 \sin x - \cos x)$.

56. $y = e^{-2x} - \sin 2x$.

58. $y = e^{-x/2}\left[(c_1 + c_2 x)\sin \frac{\sqrt{7}x}{2} + (c_3 + c_4 x)\cos \frac{\sqrt{7}x}{2}\right]$.

60. $y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5)e^{2x} + e^{3x}[(c_7 + c_8 x + c_9 x^2)\sin 4x + (c_{10} + c_{11} x + c_{12} x^2)\cos 4x]$.

62. $y = c_1 e^{-2x} + c_2 e^{-3x} + e^{x}(c_3 \sin 2x + c_4 \cos 2x)$.

**Section 4.3, Page 159.**

2. $y = c_1e^{4x} + c_2e^{-2x} - \frac{1}{2}e^{2x} - 3e^{-3x}$.

4. $y = e^{-x}(c_1\sin x + c_2\cos x) - \frac{7}{13}\sin 4x - \frac{4}{13}\cos 4x$.

6. $y = c_1e^{4x} + c_2e^{-x} + 2e^{2x} - 4x + 3$.

8. $y = e^{-x}(c_1\sin 3x + c_2\cos 3x) + \frac{1}{2}xe^{-2x} + \frac{1}{10}e^{-2x}$.

10. $y = c_1e^{-2x} + c_2e^{-4x} + c_2e^{-4x} + \frac{1}{4}xe^{2x} - \frac{5}{48}e^{2x}$
$+ x^2 - \frac{3}{2}x + \frac{7}{8}$.

12. $y = c_1e^{2x} + c_2e^{-2x} + 2x^2e^{2x} - xe^{2x}$.

14. $y = (c_1 + c_2x)e^{3x} + 3x^2e^{3x} + 5xe^{4x} - 10e^{4x}$.

16. $y = c_1e^{2x} + e^{-2x}(c_2\sin x + c_3\cos x) - 2xe^{-2x} - \frac{1}{2}e^{-2x}$.

18. $y = c_1e^{-x} + c_2e^{x/2} + c_3e^{3x/2} + x^3 + 5x^2 + 22x + 42$.

20. $y = c_1e^{x} + c_2e^{-2x} + xe^{x} - 2xe^{-2x} + 2x^2 + 2x + 3$.

22. $y = e^{2x}(c_1\sin x + c_2\cos x) + 3xe^{2x}\sin x$.

24. $y = c_1e^{x} + c_2e^{-x} + c_3e^{2x} + 3xe^{2x} - e^{3x}$.

26. $y = c_1 + c_2x + c_3e^x + c_4e^{2x} + \frac{1}{2}e^{-x} + \frac{3}{2}xe^{2x} - \frac{1}{2}x^3 - \frac{9}{4}x^2.$

28. $y = (c_1 + c_2x)e^x + c_3e^{2x} - \frac{1}{4}x^4e^x + \frac{1}{2}x^2e^x.$

30. $y = c_1 + c_2e^{2x} + c_3e^{-2x} + 2x^2e^{2x} - 3xe^{2x} + 2x^3 + 3x.$

32. $y = c_1\sin 2x + c_2\cos 2x + 3x^2 - \frac{3}{2} - 2x^2\sin 2x - x\cos 2x.$

34. $y = c_1e^{2x} + c_2e^{3x} + c_3\sin x + c_4\cos x + \frac{1}{13}\sin 2x + \frac{5}{13}\cos 2x + \frac{1}{4}x\sin x - \frac{1}{4}x\cos x.$

36. $y = 3e^{-x} + 2e^x + 4x - 5.$

38. $y = e^{-2x} - 2xe^{-3x} - e^{-3x}.$

40. $y = (3x - 5)e^{-3x} + 3e^{-6x}.$

42. $y = 2e^{5x}(2\sin 2x - \cos 2x + 1).$

44. $y = e^{-2x} + 4e^{3x} - 2e^{2x} - xe^{3x}.$

46. $y = \frac{9e^x}{8} - \frac{e^{-x}}{8} + \frac{x^3e^x}{2} - \frac{3x^2e^x}{4} + \frac{3xe^x}{4}.$

48. $y = 5\sin 2x + 6\cos 2x - 2x\cos 2x.$

50. $y = \left(\frac{122}{9} - \frac{4}{3}x\right)e^x - \frac{5}{9}e^{4x} - 2x^2 - 9x - 15 + 3e^{2x}.$

52. $y_p = Ae^{3x} + Be^{-3x} + Ce^{3x}\sin 3x + De^{3x}\cos 3x.$

54. $y_p = Ax^4e^x + Bx^3e^x + Cx^2e^x + Dxe^x + Ee^x + Fx^3e^{2x}$
$+ Gx^2e^{2x} + Hxe^{2x} + Ie^{2x} + Jx^4e^{3x} + Kx^3e^{3x} + Lx^2e^{3x}.$

56. $y_p = Ax^3e^x + Bx^2e^x + Cxe^x + Dx^2e^{2x} + Exe^{2x} + Fx^3$
$+ Gx^2 + Hx.$

58. $y_p = Ax^4e^{-x} + Bx^3e^{-x} + Cx^2e^{-x} + Dxe^{-x/2}\sin \frac{\sqrt{3}}{2}x$
$+ Exe^{-x/2}\cos \frac{\sqrt{3}}{2}x.$

60. $y_p = Ax^7 + Bx^6 + Cx^5 + Dx^4 + Ex^2e^{-x} + Fxe^{-x} + Ge^{-x}$
$+ Hxe^{-x}\sin 2x + Ixe^{-x}\cos 2x.$

62. $y_p = Ax^2e^{\sqrt{2}x}\sin\sqrt{2}x + Bx^2e^{\sqrt{2}x}\cos\sqrt{2}x + Cxe^{\sqrt{2}x}\sin\sqrt{2}x$
$+ Dxe^{\sqrt{2}x}\cos\sqrt{2}x + Exe^{-\sqrt{2}x}\sin\sqrt{2}x + Fxe^{-\sqrt{2}x}\cos\sqrt{2}x.$

64. $y_p = Ax\sin x\sin 2x + Bx\sin x\cos 2x + Cx\cos x\sin 2x$
$+ Dx\cos x\cos 2x.$

**Section 4.4, Page 169.**

2. $y = c_1\sin x + c_2\cos x + (\sin x)[\ln|\sec x + \tan x|] - 2.$

4. $y = c_1\sin x + c_2\cos x + \sin x\tan x - \frac{\sec x}{2}.$

OR $y = c_1 \sin x + c_2 \cos x + \frac{(\sin x \tan x)}{2}$.

6. $y = c_1 \sin x + c_2 \cos x - \sin x \ln|\cos x| + x \cos x$.

8. $y = e^x(c_1 \sin 2x + c_2 \cos 2x)$

$$- \frac{1}{4} e^x \cos 2x[\ln|\sec 2x + \tan x|].$$

10. $y = c_1 e^x + c_2 x e^x - \frac{5x^3 e^x}{36} + \frac{x^3 e^x \ln|x|}{6}$.

12. $y = c_1 \sin x + c_2 \cos x + \frac{\tan x}{2} + \frac{3}{2} \cos x[\ln|\sec x + \tan x|]$.

14. $y = c_1 e^{-x} + c_2 e^{-2x} + e^{-x} \arctan e^x - \frac{1}{2} e^{-2x} \ln(1 + e^{2x})$.

16. $y = c_1 e^x + c_2 x e^x + \frac{e^x \sin^{-1} x}{4} + \frac{x^2 e^x \sin^{-1} x}{2} + \frac{3xe^x\sqrt{1 - x^2}}{4}$.

18. $y = (c_1 + c_2 x)e^x - e^x \left[\int \frac{x^2 \ln x}{e^x} dx\right] + xe^x \left[\int \frac{x \ln x}{e^x} dx\right]$.

20. $y = c_1(x + 1) + c_2(x + 1)^2 + \frac{1}{2}$.

22. $y = c_1 x + c_2 x e^x - x^2 - x$.

24. $y = c_1 x + c_2(x + 1)^{-1} + x^2 - \frac{(2x^3 + 3x^2)(x + 1)^{-1}}{6}$.

26. $y = c_1 e^x + c_2 x^2 e^x + \frac{2}{3} e^x x^{-1}$.

28. $y = c_1e^x + c_2e^{-x} + c_3e^{3x} - \frac{x^3e^x}{12} - \frac{xe^x}{8}$.

**Section 4.5, Page 176.**

2. $y = c_1x^2 + \frac{c_2}{x^2}$.

4. $y = c_1x^2 + c_2x^2\ln x$.

6. $y = x^2[c_1\sin(\ln x^3) + c_2\cos(\ln x^3)]$.

8. $y = c_1\sin(\ln x^3) + c_2\cos(\ln x^3)$.

10. $y = x^3[c_1\sin(\ln x) + c_2\cos(\ln x)]$.

12. $y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3x^4$.

14. $y = c_1x + c_2x^{-1} + c_3x^2 + c_4x^4$.

16. $y = c_1x^2 + c_2x^4 - 2x^3$.

18. $y = c_1\sin(\ln x^2) + c_2\cos(\ln x^2) + \frac{x\ln x^2}{5} - \frac{4x}{25}$.

20. $y = x^2[c_1\sin(\ln x) + c_2\cos(\ln x)] + 5x^2$.

22. $y = (c_1 + c_2\ln x)x + c_3x^2 + \frac{x^3}{4}$.

24. $y = -2x^2 + x^3$

26. $y = x^{-1} + x^2 - 2x + 4$.

28. $y = -x^2 + 2x^{-3} + 2x^2 \ln x$.

30. $y = \frac{1}{6}\left(\frac{x^{-2}}{2} + \frac{x^3}{3} - \ln x + \frac{1}{6}\right)$.

32. $y = c_1(2x - 3) + c_2(2x - 3)^3$.

**Section 4.6, Page 184.**

1. (a) $f_1(x) = 2e^x - e^{2x}$, $f_2(x) = -e^x + e^{2x}$.

   (b) $5f_1(x) + 7f_2(x)$.

**Section 5.2, Page 197.**

2. (a) $x = \frac{1}{4}\sin 8t + \frac{1}{3}\cos 8t$.

   (b) $x = -\frac{1}{4}\sin 8t + \frac{1}{3}\cos 8t$.

   (c) $x = \frac{1}{4}\sin 8t - \frac{1}{3}\cos 8t$.

4. $\sqrt{442}/7$ (cm); $\pi/7$ (sec.); $7/\pi$ (cycles/sec.).

6. (a) $\frac{1}{2}$ (ft.); $2\sqrt{3}$ ft./sec.

   (b) $\frac{\pi}{3} + \frac{2n\pi}{4}$ sec.; $2\sqrt{3}$ ft./sec.

8. $k = 4$; $A = \frac{\sqrt{71}}{4}$.

10. (a) $\theta = c_1 \sin\sqrt{\frac{g}{\ell}}\,t + c_2 \cos\sqrt{\frac{g}{\ell}}\,t$ ; $A = \theta_0$; period $= 2\pi\sqrt{\frac{\ell}{g}}$.

(b) $\frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{\ell}}\sqrt{\cos\theta - \cos\theta_0}$.

**Section 5.3, Page 208.**

2. $x = \frac{\sqrt{3}}{3} e^{-6t} \sin 2\sqrt{3}t$.

4. $x = -\frac{1}{12} e^{-16t} + \frac{1}{3} e^{-4t}$.

6. $x = \frac{e^{-4t}}{100} (\sqrt{6} \sin 2\sqrt{6}t + 2 \cos 2\sqrt{6}t)$.

8. (a) $x = e^{-4t}(\sin 4t + \cos 4t)$; $x = \sqrt{2}e^{-4t}\cos\left(4t - \frac{\pi}{4}\right)$;

(b) $\pi/2$; $\sqrt{2}e^{-4t}$;

(c) $\pi/4$; 0.04(ft.)

10. (a) $x = \left(\frac{1}{4} + 2t\right)e^{-8t}$.

(b) $x = e^{-4t}\left(\frac{\sqrt{3}}{12} \sin 4\sqrt{3}t + \frac{1}{4} \cos 4\sqrt{3}t\right)$.

(c) $x = \left(\frac{3 + 2\sqrt{3}}{24}\right)e^{(-16+8\sqrt{3})t} + \left(\frac{3 - 2\sqrt{3}}{24}\right)e^{(-16-8\sqrt{3})t}$

12. (a) $8\sqrt{6}$;

(b) $8\sqrt{2}$

**Section 5.4, Page 217.**

2. (a) $x = e^{-4t}\left(-\frac{21}{74} \sin 8t + \frac{11}{37} \cos 8t\right)$

$+ \frac{8}{37} \sin 16t - \frac{11}{37} \cos 16t$.

4. (a) $x = e^{-8t}\left(\frac{11}{20}\sin 8t + \frac{9}{10}\cos 8t\right) + \frac{7}{10}\sin 4t - \frac{2}{5}\cos 4t.$

   (b) amplitude: $\frac{\sqrt{65}}{10}$.

6. $x = e^{-t}(c_1 \sin 3t + c_2 \cos 3t) + \frac{9}{85}\sin t - \frac{2}{85}\cos t$

   $+ \frac{3}{104}\sin 2t - \frac{1}{52}\cos 2t + \frac{1}{333}\sin 3t - \frac{6}{333}\cos 3t,$

   where $c_1 = -54{,}854/1{,}471{,}860$, $c_2 = 29{,}819/490{,}620$.

10. (b) $x = \frac{\sqrt{37}}{8}e^{-2t}[\cos(4t - \phi)] + \frac{\sqrt{5}}{4}[\cos(2t - \theta)]$,

    where $\phi = \cos^{-1}\frac{6}{\sqrt{37}}$ and $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

    or $x = e^{-2t}\left[\frac{3}{4}\cos 4t + \frac{1}{8}\sin 4t\right] + \frac{1}{2}\sin 2t - \frac{1}{4}\cos 2t.$

**Section 5.5, Page 224.**

2. $a = 5\sqrt{2 - \frac{\pi^2}{32}}.$

**Section 5.6, Page 232.**

2. $i = \frac{15}{17}(\cos 200t + 4\sin 200t - e^{-50t}).$

4. $q = \left(\frac{1}{200} + \frac{t}{2}\right)e^{-100t} - \frac{1}{200}\cos 100t.$

6. $i = e^{-200t}(1.0311 \sin 979.8t + 0.1031 \cos 979.8t) - 0.1031\, e^{-100t}$.

8. (d) $\omega = 50$; ampl.: $\dfrac{100}{\sqrt{(-187.5)^2 + (20)^2}} \approx \dfrac{100}{190}$.

**Section 6.1, Page 249.**

2. $y = c_0(1 + 2x^2 - 2x^4 + \cdots) + c_1\left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5 + \cdots\right)$.

4. $y = c_0\left(1 - \frac{1}{4}x^4 + \frac{1}{20}x^5 - \cdots\right) + c_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{17}{120}x^5 + \cdots\right)$.

6. $y = c_0\left(1 + 2x^2 + \frac{1}{4}x^4 + \cdots\right) + c_1\left(x + \frac{1}{2}x^3 - \frac{1}{20}x^5 + \cdots\right)$.

8. $y = c_0\left(1 + x^2 - \frac{1}{2}x^3 + \cdots\right) + c_1\left(x + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \cdots\right)$.

10. $y = c_0\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{20}x^5 - \frac{1}{240}x^6 + \cdots\right) + c_1\left(x + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \cdots\right)$.

12. $y = c_0\left(1 + \frac{1}{3}x^3 + \frac{1}{3^5}x^4 + \cdots\right) + c_1\left(x + x^2 + \frac{1}{2}x^3 + \cdots\right)$.

14. $y = c_0\left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \cdots\right) + c_1\left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 + \cdots\right)$.

16. $y = x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \cdots$.

18. $y = 2 + 3x - \frac{7x^3}{6} - \frac{x^4}{2} + \frac{21x^5}{40} + \cdots$.

20. $y = \sum_{n=0}^{\infty}(-1)^n x^n$.

22. $y = c_0\left[1 + \frac{1}{2}(x-1)^2 - \frac{5}{6}(x-1)^3 + \cdots\right]$
$+ c_1\left[(x-1) - \frac{3}{2}(x-1)^2 + \frac{13}{6}(x-1)^3 - \cdots\right]$.

24. (a) $y = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right]$
$+ c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right]$.

**Section 6.2, Page 269.**

2. $x = 0$ and $x = -1$ are regular singular pts.

4. $x = 0$ is an irregular singular pt.;
$x = -3$ and $x = 2$ are regular singular pts.

6. $y = C_1x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \cdots\right)$
$+ C_2x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \cdots\right)$.

8. $y = C_1 x^{5/3}\left(1 - \frac{3}{10}x^2 + \frac{9}{320}x^4 + \cdots\right)$
$+ C_2 x^{1/3}\left(1 - \frac{3}{2}x^2 + \frac{9}{32}x^4 + \cdots\right).$

10. $y = C_1 x^{1/2}\left(1 - \frac{2}{3}x + \frac{2}{15}x^2 - \cdots\right)$
$+ C_2\left(1 - 2x + \frac{2}{3}x^2 - \frac{4}{45}x^3 + \cdots\right).$

12. $y = C_1 x^{1/2}\left(1 - \frac{2x}{7} + \frac{2x^2}{63} - \frac{4x^3}{2079} + \cdots\right)$
$+ C_2 x^{-2}\left(1 + \frac{2x}{3} + \frac{2x^2}{3} - \frac{4x^3}{9} + \cdots\right).$

14. $y = C_1 x^{5/2}\left(1 - \frac{25x}{44} + \frac{1225x^2}{4576} - \cdots\right) + C_2 x^{-2}\left(1 + \frac{4x}{7} + \frac{2x^2}{35}\right).$

16. $y = C_1 x^{1/2}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right)$
$+ C_2 x^{-1/2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right).$

18. $y = C_1\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!}\right]$
$+ C_2 x^3\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{[5 \cdot 7 \cdot 9 \cdots (2n+3)]}\right]$
$= C_1\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right) + C_2 x^3\left(1 + \frac{x^2}{5} + \frac{x^4}{35} + \cdots\right).$

20. $y = C_1x^{1/2} + C_2x^{3/2}$.

22. $y = C_1x^{-1}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+2)!} + C_2x^{-3}(1 - x)$.

24. $y = C_1x^{5/2}\left(1 - \frac{x^2}{3} + \frac{x^4}{12} - \cdots\right) + C_2x^{-3/2}(1 - x^2)$.

26. $y = C_1x^3\left(1 - \frac{1}{4}x^2 + \frac{5}{128}x^4 - \cdots\right)$
$+ C_2\left[x^{-1}\left(1 - \frac{1}{4}x + \frac{1}{192}x^5 - \cdots\right) + \frac{1}{16}y_1(x)\ln|x|\right]$,
where $y_1(x)$ denotes the solution of which $C_1$ is the coefficient.

28. $y = C_1x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n[3 \cdot 5 \cdot 7 \cdots (2n+1)]}{2^{n-1}n!(n+2)!}x^n\right]$
$+ C_2\left[x^{-1/2}\left(-\frac{1}{2} + \frac{1}{4}x - \frac{5}{64}x^2 + \cdots\right)\right.$
$\left. - \frac{1}{16}y_1(x)\ln|x|\right]$, where $y_1(x)$ denotes the solution of which $C_1$ is the coefficient.

30. $y = C_1\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^{n-1}n!(n+2)!}\right]$
$+ C_2\left[x^{-2}\left(-\frac{1}{2} - \frac{1}{4}x + \frac{29}{576}x^2 + \cdots\right)\right.$
$\left. + \frac{1}{16}y_1(x)\ln|x|\right]$, where $y_1(x)$ denotes the solution of which $C_1$ is the coefficient.

32. $y = C_1 x^3 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{n-1} n!(n+2)!}\right]$

$+ C_2 \left[x^{-1}\left(-\frac{1}{4} - \frac{1}{16}x^2 + \frac{29}{4608}x^4 + \cdots\right)\right.$

$\left. + \frac{1}{64} y_1(x) \ln|x|\right]$, where $y_1(x)$ denotes the solution of which $C_1$ is the coefficient.

**Section 7.1, Page 296.**

2. $\begin{cases} x = ce^{-t} - 2, \\ y = -2ce^{-t} - t^2 - 2t + 4. \end{cases}$

4. $\begin{cases} x = c_1 e^t + c_2 e^{3t}, \\ y = -\frac{c_1}{3} e^t - c_2 e^{3t} - 3e^{2t}. \end{cases}$

6. $x = c_1 e^t - \frac{\sin t}{2}, \quad y = -\frac{c_1 e^t}{3} + \frac{\sin t}{2}.$

8. $\begin{cases} x = c_1 e^{-t} + t - 1 - \frac{1}{2} e^t, \\ y = c_2 e^t - \frac{5}{2} c_1 e^{-t} - \frac{1}{2} te^t - 4t + 1. \end{cases}$

10. $\begin{cases} x = c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t} + 3t - 3, \\ y = \frac{\sqrt{3}}{3} c_1 e^{\sqrt{3}t} - \frac{\sqrt{3}}{3} c_2 e^{-\sqrt{3}t} - 2t + \frac{4}{3}. \end{cases}$

12. $\begin{cases} x = c_1 \sin t + c_2 \cos t - t - 3, \\ y = (c_2 - c_1)\sin t - (c_1 + c_2)\cos t - 1. \end{cases}$

14. $\begin{cases} x = 3e^{t}, \\ y = -2e^{t} - \frac{1}{2} e^{2t}. \end{cases}$

16. $\begin{cases} x = c_1 + c_2 e^{-2t} + 2t^2 + t, \\ y = (1 - c_1) - 3c_2 e^{-2t} - t^2 - 3t. \end{cases}$

18. $\begin{cases} x = c_1 e^{2t} + \frac{1}{2} t^2 + \frac{3}{2} t + \frac{3}{4}, \\ y = -3c_1 e^{2t} + c_2 e^{t} - \frac{1}{2} t^2 + \frac{3}{2} t + \frac{15}{4}. \end{cases}$

20. $\begin{cases} x = e^{-t}(c_1 \sin 2t + c_2 \cos 2t) + 2, \\ y = e^{-t}[2(-c_1 + c_2)\sin 2t - 2(c_1 + c_2)\cos 2t] \\ \qquad + t^2 + 2t - 14. \end{cases}$

22. $\begin{cases} x = c_1 + c_2 e^{t} + c_3 t e^{t} + \frac{1}{2} e^{2t}, \\ y = -c_1 - c_2 e^{t} - c_3(t + 1)e^{t} - \frac{1}{2} e^{2t}. \end{cases}$

24. $\begin{cases} x = c_1 + c_2 e^{2t} + c_3 e^{-2t} - \frac{2}{3} e^{t}, \\ y = -4c_1 + 2c_2 e^{2t} - 2c_3 e^{-2t} - \frac{5}{3} e^{t}. \end{cases}$

26.
$$\begin{cases} x = c_1 e^{-t} + e^{-2t}(c_2 \sin t + c_3 \cos t) - \frac{4}{51} e^{2t}, \\ y = \frac{1}{4} c_1 e^{-t} + e^{-2t}\left(\frac{2}{5} c_2 + \frac{1}{5} c_3\right)\sin t \\ \qquad + \left(-\frac{1}{5} c_2 + \frac{2}{5} c_3\right)\cos t + \frac{5}{51} e^{2t}. \end{cases}$$

28.
$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = x_3, \\ \frac{dx_3}{dt} = 2x_1 + x_2 - 2x_3 + e^{3t}. \end{cases}$$

30.
$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = x_3, \\ \frac{dx_3}{dt} = x_4, \\ \frac{dx_4}{dt} = -2tx_1 + t^2 x_3 + \cos t. \end{cases}$$

**Section 7.2, Page 307.**

2. (b)
$$\begin{cases} x = c_1 + c_2 e^{-kt/m}, \\ y = c_3 + c_4 e^{-kt/m} - \frac{mg}{k} t. \end{cases}$$

6.
$$\begin{cases} x = 10e^{-t/5} + 20, \\ y = -20e^{-t/5} + 20. \end{cases}$$

## Section 7.3, Page 317.

2. (b)
$$\begin{cases} x = 3c_1e^{7t} + c_2e^{-t}, \\ y = 2c_1e^{7t} - 2c_2e^{-t}. \end{cases}$$

(c)
$$\begin{cases} x = 3e^{7t} - 3e^{-t}, \\ y = 2e^{7t} + 6e^{-t}. \end{cases}$$

## Section 7.4, Page 328.

2.
$$\begin{cases} x = c_1e^{4t} + c_2e^{2t}, \\ y = c_1e^{4t} + 3c_2e^{2t}. \end{cases}$$

4.
$$\begin{cases} x = c_1e^{t} + c_2e^{-t}, \\ y = -3c_1e^{t} - c_2e^{-t}. \end{cases}$$

6.
$$\begin{cases} x = c_1e^{5t} + c_2e^{3t}, \\ y = c_1e^{5t} + 3c_2e^{3t}. \end{cases}$$

8.
$$\begin{cases} x = e^{2t}(c_1\cos 3t + c_2\sin 3t), \\ y = e^{2t}(3c_1\sin 3t - 3c_2\cos 3t). \end{cases}$$

10. $\begin{cases} x = c_1 e^{5t} + c_2 e^{-3t}, \\ y = c_1 e^{5t} - 3c_2 e^{-3t}. \end{cases}$

12. $\begin{cases} x = e^{2t}(c_1 \cos 3t - c_2 \sin 3t), \\ y = e^{2t}(c_1 \cos 3t + c_2 \sin 3t). \end{cases}$

14. $\begin{cases} x = 2e^{3t}(c_1 \cos 2t + c_2 \sin 2t), \\ y = e^{3t}[c_1(\cos 2t + \sin 2t) + c_2(\sin 2t - \cos 2t)]. \end{cases}$

16. $\begin{cases} x = 5(c_1 \cos 3t + c_2 \sin 3t), \\ y = -c_1(\cos 3t + 15 \sin 3t) + c_2(3 \cos 3t - 5 \sin 3t). \end{cases}$

18. $\begin{cases} x = -5e^{4t}(c_1 \cos t + c_2 \sin t), \\ y = e^{4t}[(c_2 - 2c_1)\cos t - (c_1 + 2c_2 \sin t]. \end{cases}$

20. $\begin{cases} x = 2c_1 e^{5t} + c_2(2t + 1)e^{5t}, \\ y = -c_1 e^{5t} - c_2 t e^{5t}. \end{cases}$

22. $\begin{cases} x = c_1 e^{-t} + c_2(2t + 1)e^{-t}, \\ y = c_1 e^{-t} + 2c_2 t e^{-t}. \end{cases}$

24. $\begin{cases} x = 2c_1e^{6t} + c_2(2t + 1)e^{6t}, \\ y = c_1e^{6t} + c_2te^{6t}. \end{cases}$

26. $\begin{cases} x = 2c_1 + c_2(2t + 1), \\ y = c_1 + c_2t. \end{cases}$

28. $\begin{cases} x = e^{5t} + 5e^{-3t}, \\ y = 7e^{5t} - 5e^{-3t}. \end{cases}$

30. $\begin{cases} x = e^{4t}\left(5\cos 3t - \frac{10}{3}\sin 3t\right), \\ y = -e^{4t}\left(\cos 3t + \frac{11}{3}\sin 3t\right). \end{cases}$

32. $\begin{cases} x = e^{5t} - te^{5t}, \\ y = 3e^{5t} - 2te^{5t}. \end{cases}$

34. $\begin{cases} x = (8t + 6)e^{-3t}, \\ y = (16t + 8)e^{-3t}. \end{cases}$

36. $\begin{cases} x = c_1t^2 + c_2t^4, \\ y = c_1t^2 + 3c_2t^4. \end{cases}$

**Section 7.5A, Page 340.**

2. (a) $\begin{pmatrix} 3 & 6 \\ 21 & -9 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & 12 & -20 \\ -24 & 8 & 0 \\ 12 & -4 & -8 \end{pmatrix}$

(c) $\begin{pmatrix} -15 & 3 & -6 \\ -12 & 9 & 6 \\ 0 & -9 & 18 \end{pmatrix}$

4. (a) $\begin{pmatrix} 2x_1 + x_2 - 4x_3 \\ 5x_1 - 2x_2 + 3x_3 \\ x_1 - 3x_2 + 2x_3 \end{pmatrix}$ (b) $\begin{pmatrix} -35 \\ 10 \\ -7 \end{pmatrix}$

(c) $\begin{pmatrix} x_1 - 2x_2 + 3x_3 \\ -3x_1 - 4x_2 - 4x_3 \\ -2x_1 + x_2 - 2x_3 \end{pmatrix}$

6. (a) (i.) $\begin{pmatrix} 10t \\ -18t^2 + 2t \\ 4t - 5 \end{pmatrix}$ (ii.) $\begin{pmatrix} \frac{5t^3}{3} \\ -\frac{3t^4}{2} + \frac{t^3}{3} \\ \frac{2t^3}{3} - \frac{5t^2}{2} \end{pmatrix}$

(b) (i.) $\begin{pmatrix} 3e^{3t} \\ (6t + 11)e^{3t} \\ (3t^2 + 2t)e^{3t} \end{pmatrix}$

(ii.) $\begin{pmatrix} \frac{(e^{3t} - 1)}{3} \\ \frac{[(6t + 7)e^{3t} - 7]}{9} \\ \frac{[(9t^2 - 6t + 2)e^{3t} - 2]}{27} \end{pmatrix}$

(c) (i.) $\begin{pmatrix} 3\cos 3t \\ -3\sin 3t \\ 3t\cos 3t + \sin 3t \\ -3t\sin 3t + \cos 3t \end{pmatrix}$

(ii.) $\begin{pmatrix} \frac{(1 - \cos 3t)}{3} \\ \frac{1}{3}\sin 3t \\ \frac{1}{9}\sin 3t - \frac{t}{3}\cos 3t \\ \frac{1}{9}\cos 3t + \frac{t}{3}\sin 3t - \frac{1}{9} \end{pmatrix}$

**Section 7.5B, Page 351.**

2. $AB = \begin{pmatrix} -9 & 50 \\ 2 & 17 \end{pmatrix}$; $BA = \begin{pmatrix} 27 & 26 \\ -10 & -19 \end{pmatrix}$.

4. $AB = \begin{pmatrix} 6 & 20 & 11 & 21 \\ 5 & 11 & 12 & 26 \end{pmatrix}$; BA not defined.

6. $AB = \begin{pmatrix} -5 & -6 & -3 \\ -4 & -6 & 9 \\ -11 & -14 & 1 \end{pmatrix}$; $BA = \begin{pmatrix} -9 & -9 & -9 \\ -13 & -8 & -3 \\ 23 & 15 & 7 \end{pmatrix}$.

8. $AB = \begin{pmatrix} 7 & 14 & 21 \\ 5 & -4 & 6 \\ 5 & 17 & 20 \end{pmatrix}$; $BA = \begin{pmatrix} 11 & 7 & -3 \\ 29 & 19 & -8 \\ -3 & -4 & -7 \end{pmatrix}$.

10. $AB = \begin{pmatrix} 12 & 8 & -16 \\ 21 & 13 & 19 \\ -6 & 6 & -22 \end{pmatrix}$; $BA = \begin{pmatrix} 20 & 8 & 20 \\ 2 & 7 & -4 \\ 12 & -2 & -24 \end{pmatrix}$.

12. $\begin{pmatrix} 6 & 6 & 18 \\ 0 & 2 & 4 \\ -16 & -10 & -8 \end{pmatrix}$.

14. $\begin{pmatrix} 4 & -\frac{5}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$.

16. $\begin{pmatrix} \frac{5}{3} & 2 \\ \frac{2}{3} & 1 \end{pmatrix}$.

18. $\begin{pmatrix} -\frac{1}{3} & \frac{7}{6} & -\frac{1}{2} \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{5}{6} & \frac{1}{2} \end{pmatrix}$.

20. $\begin{pmatrix} \frac{3}{5} & -\frac{2}{5} & 1 \\ -\frac{2}{5} & \frac{3}{5} & -1 \\ -\frac{6}{5} & \frac{4}{5} & -1 \end{pmatrix}$.

22. $\begin{pmatrix} 9 & -3 & 1 \\ -4 & \frac{3}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

24. $\begin{pmatrix} \frac{5}{2} & 3 & -\frac{3}{2} \\ -\frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{25}{12} & \frac{17}{6} & -\frac{13}{12} \end{pmatrix}$.

**Section 7.5D, Page 367.**

2. Characteristic values: 5 and -3; Respective corresponding characteristic vectors:

$$\begin{pmatrix} k \\ k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k \\ -3k \end{pmatrix},$$

where in each vector k is an arbitrary nonzero real number.

4. Characteristic values: 5 and -5; Respective corresponding characteristic vectors:

$$\begin{pmatrix} k \\ k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 7k \\ -3k \end{pmatrix},$$

where in each vector k is an arbitrary nonzero real number.

6. Characteristic values: 7 and -2; Respective corresponding characteristic vectors:

$$\begin{pmatrix} 5k \\ -4k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k \\ k \end{pmatrix},$$

where in each vector k is an arbitrary nonzero real number.

8. Characteristic values: 2, 3, and 5; Respective corresponding characteristic vectors:

$$\begin{pmatrix} 10k \\ -7k \\ -3k \end{pmatrix}, \quad \begin{pmatrix} k \\ -k \\ -k \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} k \\ -k \\ -3k \end{pmatrix},$$

where in each case k is an arbitrary nonzero real number.

10. Characteristic values: 1, 2, and -1; Respective corresponding characteristic vectors:

$$\begin{pmatrix} k \\ 0 \\ -k \end{pmatrix}, \quad \begin{pmatrix} k \\ k \\ k \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} k \\ -2k \\ k \end{pmatrix},$$

where in each case k is an arbitrary nonzero real number.

12. Characteristic values: 1, 3, and 4; Respective corresponding characteristic vectors:

$$\begin{pmatrix} k \\ 0 \\ k \end{pmatrix}, \quad \begin{pmatrix} 0 \\ k \\ 2k \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2k \\ -k \\ k \end{pmatrix},$$

where in each vector k is an arbitrary nonzero real number.

14. Characteristic values: 1, -1, and 4; Respective corresponding characteristic vectors:

$$\begin{pmatrix} 2k \\ k \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 3k \\ k \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} k \\ -2k \\ -k \end{pmatrix},$$

where in each vector k is an arbitrary nonzero real number.

**Section 7.6, Page 377.**

2. $\begin{cases} x_1 = c_1e^{4t} + c_2e^{2t}, \\ x_2 = c_1e^{4t} + 3c_2e^{2t}. \end{cases}$

4. $\begin{cases} x_1 = c_1e^{t} + c_2e^{-t}, \\ x_2 = -3c_1e^{t} - c_2e^{-t}. \end{cases}$

6. $\begin{cases} x_1 = c_1e^{5t} + c_2e^{3t}, \\ x_2 = c_1e^{5t} + 3c_2e^{3t}. \end{cases}$

8. $\begin{cases} x_1 = c_1e^{5t} + c_2e^{-3t}, \\ x_2 = c_1e^{5t} - 3c_2e^{-3t}. \end{cases}$

10. $$\begin{cases} x_1 = e^{2t}(c_1 \cos 3t - c_2 \sin 3t), \\ x_2 = e^{2t}(c_1 \cos 3t + c_2 \sin 3t). \end{cases}$$

12. $$\begin{cases} x_1 = 2e^{3t}(c_1 \cos 2t + c_2 \sin 2t), \\ x_2 = e^{3t}[c_1(\cos 2t + \sin 2t) + c_2(\sin 2t - \cos 2t)]. \end{cases}$$

14. $$\begin{cases} x_1 = 5e^{5t}(c_1 \cos 2t + c_2 \sin 2t), \\ x_2 = -e^{5t}[c_1(\cos 2t - 2 \sin 2t) + c_2(2 \cos 2t + \sin 2t)] \end{cases}$$

16. $$\begin{cases} x_1 = -2c_1 e^{5t} - c_2(2t + 1)e^{5t}, \\ x_2 = c_1 e^{5t} + c_2 t\, e^{5t}. \end{cases}$$

18. $$\begin{cases} x_1 = c_1 e^{-t} + c_2(2t + 1)e^{-t}, \\ x_2 = c_1 e^{-t} + 2c_2 t e^{-t}. \end{cases}$$

20. $$\begin{cases} x_1 = c_1 e^{5t} + c_2(t + 1)e^{5t}, \\ x_2 = 2c_1 e^{5t} + c_2(2t + 1)e^{5t}. \end{cases}$$

Section 7.7, Page 400.

2. $$\begin{cases} x_1 = 10c_1e^{2t} + c_2e^{3t} + c_3e^{5t}, \\ x_2 = -7c_1e^{2t} - c_2e^{3t} - c_3e^{5t}, \\ x_3 = -3c_1e^{2t} - c_2e^{3t} - 3c_3e^{5t}. \end{cases}$$

4. $$\begin{cases} x_1 = c_1e^{t} + c_3e^{4t}, \\ x_2 = 2c_2e^{3t} - c_3e^{4t}, \\ x_3 = c_2e^{3t} - c_3e^{4t}. \end{cases}$$

6. $$\begin{cases} x_1 = c_1e^{t} + c_2e^{2t} + c_3e^{-t}, \\ x_2 = c_2e^{2t} - 2c_3e^{-t}, \\ x_3 = -c_1e^{t} + c_2e^{2t} + c_3e^{-t}. \end{cases}$$

8. $$\begin{cases} x_1 = c_1e^{2t} + c_2e^{-t} + c_3e^{-4t}, \\ x_2 = 2c_1e^{2t} + c_2e^{-t} + c_3e^{-4t}, \\ x_3 = c_1e^{2t} + c_2e^{-t} + 2c_3e^{-4t}. \end{cases}$$

10. $$\begin{cases} x_1 = 17c_1e^{3t} + 2c_2\sin 2t - 2c_3\cos 2t, \\ x_2 = 3c_1e^{3t} + c_2\cos 2t + c_3\sin 2t, \\ x_3 = 7c_1e^{3t} + c_2(\cos 2t + 2\sin 2t) + c_3(\sin 2t - 2\cos 2t). \end{cases}$$

12. $$\begin{cases} x_1 = c_1e^{2t} + c_2e^{t}, \\ x_2 = 2c_1e^{2t} - c_2e^{t} + c_3e^{t}, \\ x_3 = c_1e^{2t} + 2c_3e^{t}. \end{cases}$$

14. $\begin{cases} x_1 = c_1e^{t} + c_2e^{3t}, \\ x_2 = -c_1e^{t} + 2c_3e^{3t}, \\ x_3 = 3c_1e^{t} + c_3e^{3t}. \end{cases}$

16. $\begin{cases} x_1 = c_1e^{28t} + c_2e^{t}, \\ x_2 = 2c_1e^{28t} + c_3e^{t}, \\ x_3 = c_1e^{28t} - c_3e^{t}. \end{cases}$

18. $\begin{cases} x_1 = 5c_1e^{-t} + 5c_2e^{2t} + c_3e^{3t}, \\ x_2 = 3c_1e^{-t} + 5c_2e^{2t} + c_3e^{3t}, \\ x_3 = -2c_1e^{-t} - c_2e^{2t}. \end{cases}$

20. $\begin{cases} x_1 = 3c_1e^{-t} + c_2e^{3t} + c_3te^{3t}, \\ x_2 = -8c_1e^{-t} + 2c_3e^{3t}, \\ x_3 = c_1e^{-t} - c_2e^{3t} - c_3(t + 3)e^{3t}. \end{cases}$

22. $\begin{cases} x_1 = c_1e^{2t} + 3c_2 + c_3(3t + 2), \\ x_2 = 4c_2 + 4c_2t, \\ x_3 = c_1e^{2t} + c_2 + c_3(t + 3). \end{cases}$

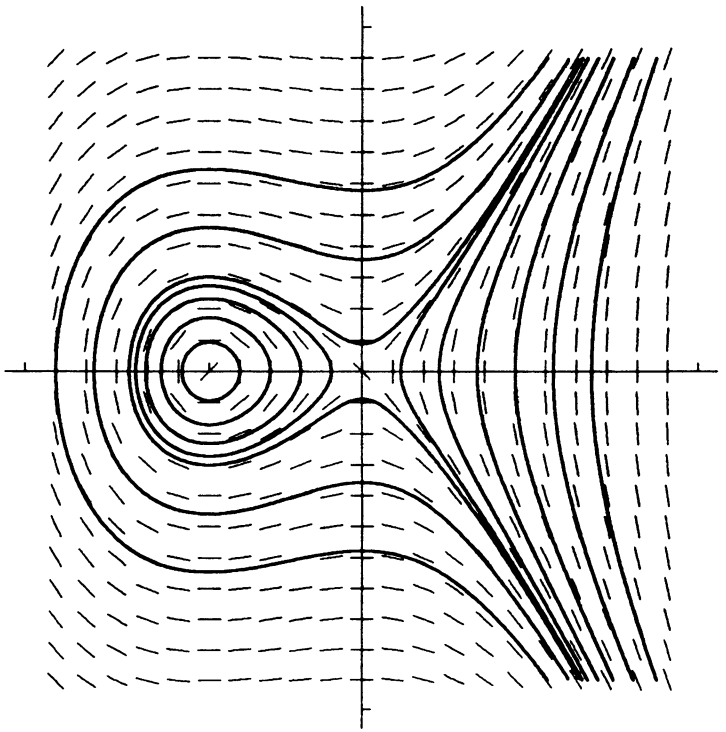
24. $\begin{cases} x_1 = c_1e^{-t} + 3c_2e^{t} + c_3\left(3t - \frac{1}{5}\right)e^{t}, \\ x_2 = 2c_1e^{-t} + 5c_2e^{t} + 5c_3te^{t}, \\ x_3 = -2c_1e^{-t} - 4c_2e^{t} + c_3\left(-4t + \frac{3}{5}\right)e^{t}. \end{cases}$

26.
$$\begin{cases} x_1 = c_1e^{2t} + c_2e^{2t} + c_3(2t + 1)e^{2t}, \\ x_2 = 2c_1e^{2t} + 2c_3te^{2t}, \\ x_3 = 2c_2e^{2t} + 2c_3te^{2t}. \end{cases}$$

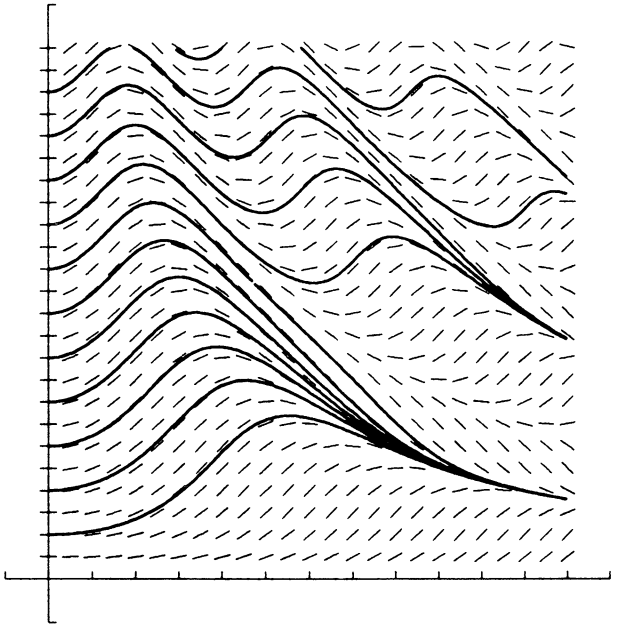
28.
$$\begin{cases} x_1 = 2c_1e^{2t} + c_2(2t + 1)e^{2t} + c_3(t^2 + t + 1)e^{2t}, \\ x_2 = -c_1e^{2t} - c_2te^{2t} - \frac{1}{2}c_3t^2e^{2t}, \\ x_3 = -2c_1e^{2t} - 2c_2te^{2t} + c_3(-t^2 + 1)e^{2t}. \end{cases}$$

**Section 8.1A, Page 421.**

2.



4.



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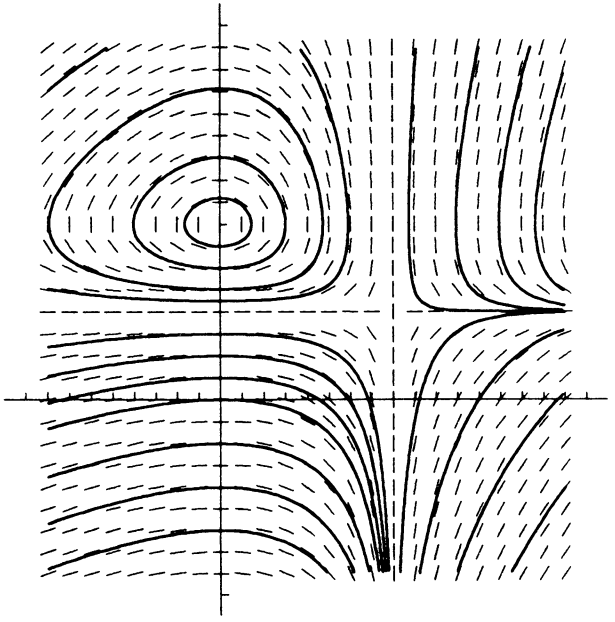

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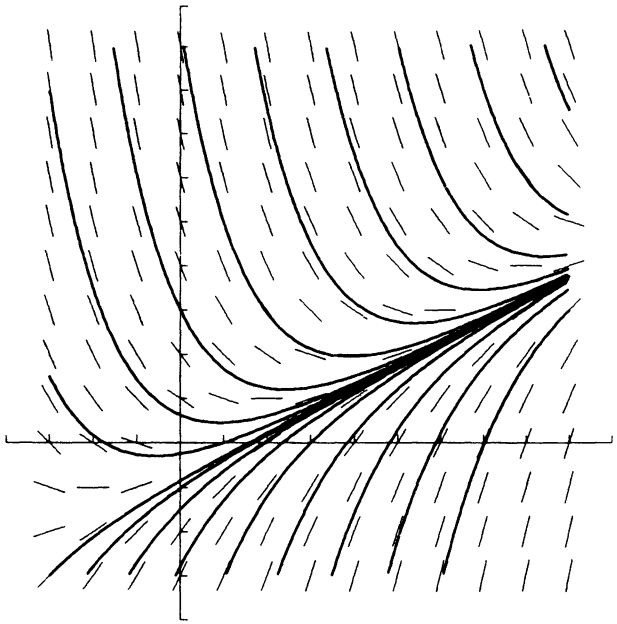

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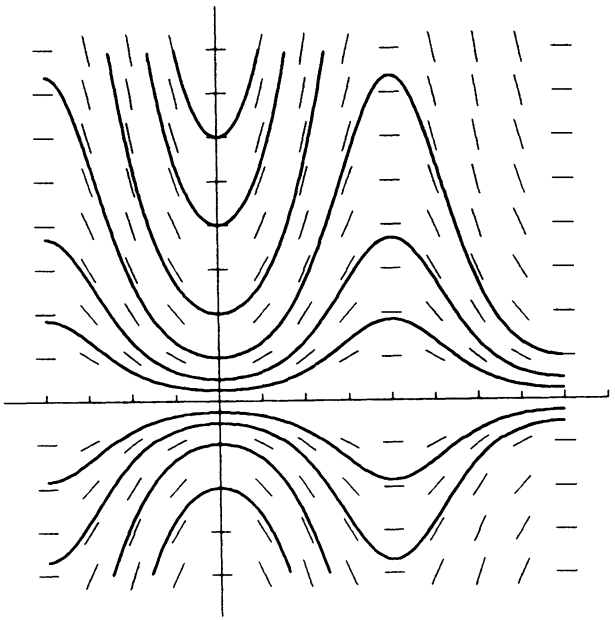
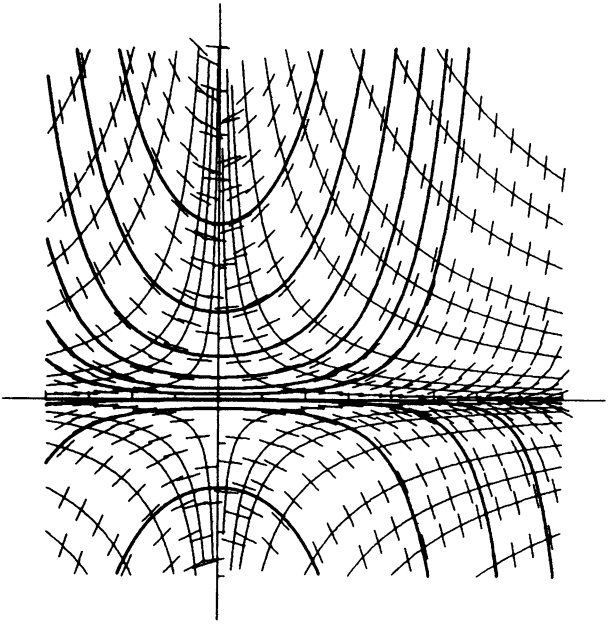


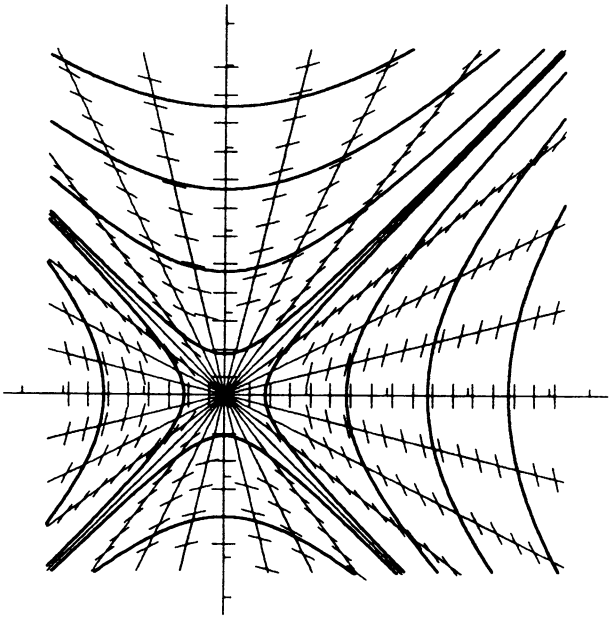
12. (a) 4; (b) 1; (c) 6; (d) 2; (e) 5; (f) 3.

## Section 8,.1B, Page 427

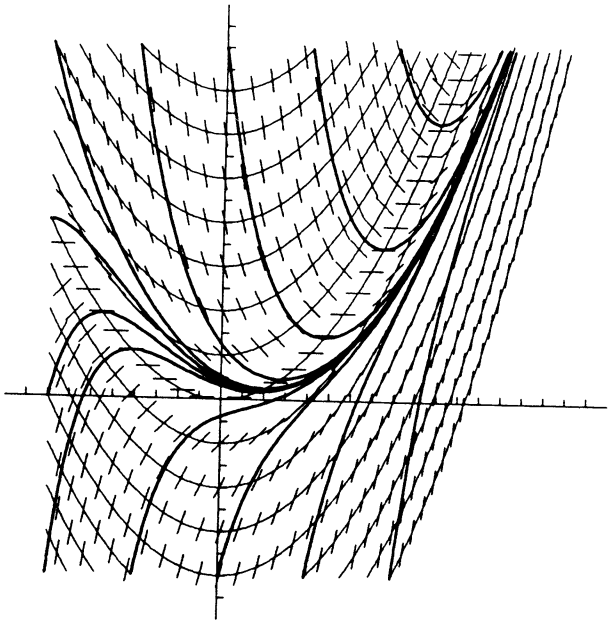
2.



4.



6.



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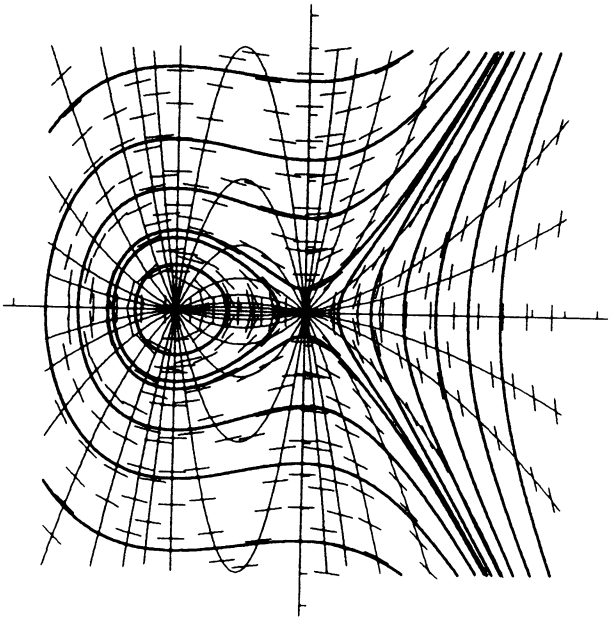

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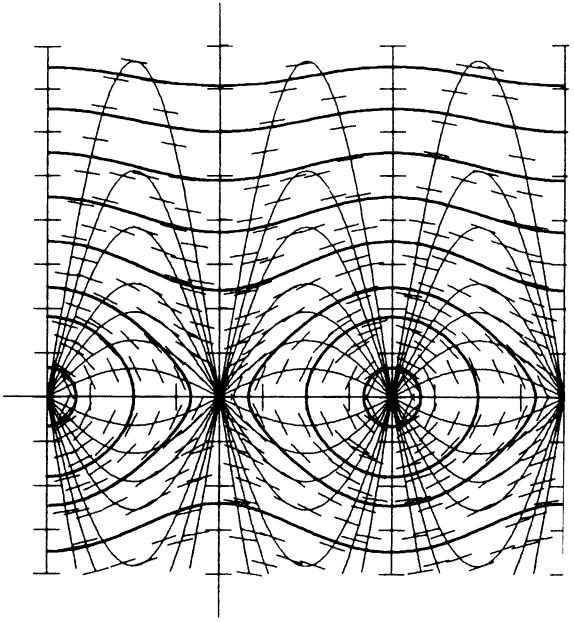
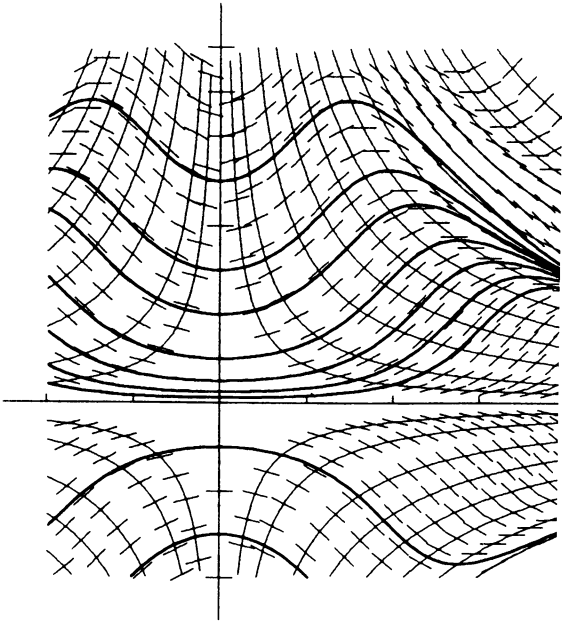


12.



**Section 8.2, Page 434.**

2. $y = 4 + 32x + 256x^2 + \frac{6145}{3}x^3 + \cdots$.

4. $y = 3 + 27x + \frac{729}{2}x^2 + \frac{10,935}{2}x^3 + \cdots$.

6. $y = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots$.

8. $y = 1 + x + 2x^2 + \frac{14}{3}x^3 + \frac{35}{3}x^4 + \cdots$.

10. $y = 1 + 2(x - 1) + \frac{7(x - 1)^2}{2} + \frac{14(x - 1)^3}{3}$
$+ \frac{73(x - 1)^4}{12} + \cdots$.

12. $y = \pi + \frac{(x - 1)^2}{2} + \frac{(x - 1)^5}{40} + \cdots$.

14. $y = 2 + (e + 2)(x - 1) + \left(\frac{3}{2}e + 1\right)(x - 1)^2$
$+ \left(e + \frac{1}{3}\right)(x - 1)^3 + \left(\frac{5}{12}e + \frac{1}{12}\right)(x - 1)^4 + \cdots$.

**Section 8.3, Page 438.**

2. $\phi_1(x) = 1 + x + \frac{x^2}{2}$, $\phi_2(x) = 1 + x + x^2 + \frac{x^3}{6}$,
$\phi_3(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$.

4. $\phi_1(x) = x$, $\phi_2(x) = x + \frac{x^4}{4}$, $\phi_3(x) = x + \frac{x^4}{4} + \frac{x^7}{14} + \frac{x^{10}}{160}$.

6. $\phi_1(x) = 1 - \cos x,$

$$\phi_2(x) = 1 + \frac{3}{2}x - \cos x - 2\sin x + \frac{1}{4}\sin 2x.$$

8. $\phi_1(x) = x,\ \phi_2(x) = x + x^6,$

$$\phi_3(x) = x + x^6 + \frac{24}{11}x^{11} + \frac{9}{4}x^{16} + \frac{8}{7}x^{21} + \frac{6}{25}x^{25}.$$

**Section 8.4**

| | $x_n$ | Exact Sol'n | Euler Approx. | Error | % Rel Errors |
|---|---|---|---|---|---|
| 2. | 0.25 | 0.026633 | 0.000000 | 0.026633 | 100.0 |
| | 1.50 | 0.512447 | 0.503906 | 0.008541 | 1.666615 |
| 4. | -0.80 | 1.268869 | 1.200000 | 0.068869 | 5.427554 |
| | 0.00 | 5.291792 | 3.783680 | 1.508112 | 28.499080 |
| 6. | 1.50 | 2.598076 | 2.625000 | 0.026924 | 1.036297 |
| | 3.00 | 4.242641 | 4.304352 | 0.061711 | 1.454539 |
| 8. | 0.20 | 0.510067 | 0.500000 | 0.010067 | 1.973606 |
| | 1.00 | 0.791798 | 0.713110 | 0.078687 | 9.937829 |
| 10. | 1.20 | 0.692821 | 1.200000 | 0.507179 | 73.204864 |
| | 2.00 | 1.743560 | 2.000000 | 0.256440 | 14.707844 |
| 12. | 0.20 | 1.019739 | 1.000000 | 0.019739 | 1.935654 |
| | 2.00 | 1.957624 | 1.935192 | 0.022432 | 1.145899 |

## Section 8.5

| | $x_n$ | Exact Sol'n | Improved Euler | Error | % Rel Errors |
|---|---|---|---|---|---|
| 2. | 0.25 | 0.026633 | 0.031250 | 0.004617 | 17.337112 |
| | 1.50 | 0.512447 | 0.514901 | 0.002454 | 0.478956 |
| 4. | -0.80 | 1.268869 | 1.260000 | 0.008869 | 0.698932 |
| | 0.00 | 5.291792 | 5.075616 | 0.216176 | 4.085122 |
| 6. | 1.50 | 2.598076 | 2.602679 | 0.004602 | 0.177145 |
| | 3.00 | 4.242641 | 4.251370 | 0.008730 | 0.205757 |
| 8. | 0.20 | 0.510067 | 0.509933 | 0.000133 | 0.026123 |
| | 1.00 | 0.791798 | 0.789224 | 0.002574 | 0.325092 |
| 10. | 1.20 | 0.692821 | 0.800000 | 0.107179 | 15.469910 |
| | 2.00 | 1.743560 | 1.789042 | 0.045482 | 2.608548 |
| 12. | 0.20 | 1.019739 | 1.019867 | 0.000128 | 0.012583 |
| | 2.00 | 1.957624 | 1.955473 | 0.002152 | 0.109918 |

## Section 8.6

| | $x_n$ | Exact Sol'n | Runge-Kutta | Error | % Rel Errors |
|---|---|---|---|---|---|
| 2. | 0.25 | 0.026633 | 0.026693 | 0.000060 | 0.225450 |
| | 1.50 | 0.512447 | 0.512476 | 0.000030 | 0.005776 |
| 4. | -0.80 | 1.268869 | 1.268800 | 0.000069 | 0.005400 |
| | 0.00 | 5.291792 | 5.290095 | 0.001697 | 0.032064 |

| | | | | | |
|---|---|---|---|---|---|
| 6. | 1.50 | 2.598076 | 2.598127 | 0.000051 | 0.001961 |
| | 3.00 | 4.242641 | 4.242717 | 0.000077 | 0.001805 |
| 8. | 0.20 | 0.510067 | 0.510067 | 0.000000 | 0.000000 |
| | 1.00 | 0.791798 | 0.791796 | 0.000001 | 0.000139 |
| 10. | 1.20 | 0.692821 | 0.725782 | 0.032960 | 4.757407 |
| | 2.00 | 1.743560 | 1.756940 | 0.013380 | 0.767424 |
| 12. | 0.20 | 1.019739 | 1.019739 | 0.000001 | 0.000064 |
| | 2.00 | 1.957624 | 1.957627 | 0.000003 | 0.000137 |

## Section 8.7

| | $x_n$ | Exact Sol'n | ABAM approx. | Error | % Rel Errors |
|---|---|---|---|---|---|
| 2. | 1.00 | 0.283834 | 0.283619 | 0.000215 | 0.075706 |
| | 1.50 | 0.512447 | 0.512201 | 0.000245 | 0.047869 |
| 4. | -0.20 | 3.564774 | 3.563793 | 0.000981 | 0.027531 |
| | 0.00 | 5.291792 | 5.289874 | 0.001918 | 0.036247 |
| 6. | 3.00 | 4.242641 | 4.242756 | 0.000116 | 0.002724 |
| 8. | 0.80 | 0.677156 | 0.677153 | 0.000003 | 0.000440 |
| | 1.00 | 0.791798 | 0.791787 | 0.000011 | 0.001363 |
| 10. | 1.80 | 1.509967 | 1.539989 | 0.030021 | 1.988212 |
| | 2.00 | 1.743560 | 1.769706 | 0.026146 | 1.499599 |
| 12. | 0.80 | 1.267512 | 1.267597 | 0.000085 | 0.006677 |
| | 2.00 | 1.957624 | 1.957674 | 0.000050 | 0.002540 |

**Section 8.8** using the Euler method

| | $t_n$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | Euler $\begin{cases} x_n \\ y_n \end{cases}$ | Error | % Rel Errors |
|---|---|---|---|---|---|
| 2. | 0.100 | 2.69972 | 2.60000 | 0.09972 | 3.694 |
| | | 2.69972 | 2.60000 | 0.09972 | 3.694 |
| | 0.500 | 8.96338 | 7.42586 | 1.53752 | 17.15 |
| | | 8.96338 | 7.42586 | 1.53752 | 17.15 |
| 4. | 0.100 | 0.90484 | 0.90000 | 0.00484 | 0.535 |
| | | -0.90484 | -0.90000 | 0.00484 | 0.535 |
| | 0.500 | 0.60653 | 0.59049 | 0.01604 | 2.645 |
| | | -0.60653 | -0.59049 | 0.01604 | 2.645 |
| 6. | 0.100 | 3.11518 | 3.10000 | 0.01518 | 0.487 |
| | | 2.01001 | 2.00000 | 0.01001 | 0.498 |
| | 0.500 | 3.90397 | 3.81151 | 0.09246 | 2.368 |
| | | 2.25525 | 2.20100 | 0.05425 | 2.406 |
| 8. | 0.100 | 2.44281 | 2.40000 | 0.04281 | 1.752 |
| | | 3.66421 | 3.60000 | 0.06421 | 1.752 |
| | 0.500 | 5.43656 | 5.03860 | 0.39797 | 7.320 |
| | | 8.15485 | 7.43811 | 0.71673 | 8.789 |
| 10. | 0.100 | 1.34264 | 1.30000 | 0.04264 | 3.176 |
| | | 3.87874 | 3.80000 | 0.07874 | 2.030 |
| | 0.500 | 3.69453 | 3.45744 | 0.23709 | 6.417 |
| | | 7.38906 | 8.45152 | 1.06246 | 14.38 |
| 12. | 0.100 | 8.84140 | 8.80000 | 0.04140 | 0.468 |
| | | 8.84281 | 8.80000 | 0.04281 | 0.484 |
| | 0.500 | 13.21828 | 12.89295 | 0.32533 | 2.461 |
| | | 13.43656 | 13.03010 | 0.40646 | 3.025 |

**Section 8.8** using the Runge-Kutta method

| | $t_n$ | Exact $\begin{cases} x(t_n) \\ y(t_n) \end{cases}$ | R-K $\begin{cases} x_n \\ y_n \end{cases}$ | Error | % Rel Errors |
|---|---|---|---|---|---|
| 2. | 0.100 | 2.6997176 | 2.6996750 | 0.0000426 | 0.00158 |
| | | 2.6997176 | 2.6996750 | 0.0000426 | 0.00158 |
| | 0.500 | 8.9633781 | 8.9626707 | 0.0007074 | 0.00789 |
| | | 8.9633781 | 8.9626707 | 0.0007074 | 0.00789 |
| 4. | 0.100 | 0.9048374 | 0.9048375 | 0.0000001 | 0.00001 |
| | | -0.9048374 | -0.9048375 | 0.0000001 | 0.00001 |
| | 0.500 | 0.6065307 | 0.6065309 | 0.0000003 | 0.00005 |
| | | -0.6065307 | -0.6065309 | 0.0000003 | 0.00005 |
| 6. | 0.100 | 3.1151793 | 3.1151792 | 0.0000001 | 0.00000 |
| | | 2.0100083 | 2.0100083 | 0.0000000 | 0.00000 |
| | 0.500 | 3.9039732 | 3.9039722 | 0.0000010 | 0.00003 |
| | | 2.2552519 | 2.2552516 | 0.0000004 | 0.00002 |
| 8. | 0.100 | 2.4428055 | 2.4428285 | 0.0000230 | 0.00094 |
| | | 3.6642083 | 3.6642073 | 0.0000010 | 0.00003 |
| | 0.500 | 5.4365637 | 5.4371347 | 0.0005710 | 0.01050 |
| | | 8.1548455 | 8.1550023 | 0.0001568 | 0.00192 |
| 10. | 0.100 | 1.3426422 | 1.3426667 | 0.0000244 | 0.00182 |
| | | 3.8787442 | 3.8789333 | 0.0001891 | 0.00488 |
| | 0.500 | 3.6945280 | 3.6960378 | 0.0015098 | 0.04087 |
| | | 7.3890561 | 7.3973576 | 0.0083015 | 0.11235 |
| 12. | 0.100 | 8.8414028 | 8.8414025 | 0.0000003 | 0.00000 |
| | | 8.8428055 | 8.8428058 | 0.0000003 | 0.00000 |
| | 0.500 | 13.2182818 | 13.2182719 | 0.0000100 | 0.00008 |
| | | 13.4365637 | 13.4365489 | 0.0000148 | 0.00011 |

## Section 9.1A, Page 488

2. $\dfrac{1}{s^2 - 1}$.

4. $\dfrac{2}{s}(2 - e^{-3s})$.

6. $(e^{-s} - e^{-2s})\left(\dfrac{1}{s} + \dfrac{1}{s^2}\right)$.

8. $\dfrac{2}{s^2}(1 - e^{-s} - e^{-3s})$.

## Section 9.1B, Page 496.

2. $\dfrac{2abs^2}{[s^2 + (a - b)^2][s^2 + (a + b)^2]}$.

4. $\dfrac{s^3 + 7a^2s}{(s^2 + a^2)(s^2 + 9a^2)}$; $\dfrac{as^2 + 3a^3}{(s^2 + a^2)(s^2 + 9a^2)}$.

6. $\dfrac{24}{s^5}$.

8. $\dfrac{3s + 11}{s^2 + 4s - 8}$.

10. $\dfrac{-s^3 + s + 1}{s^4 - s^2}$.

12. $\dfrac{12s - 2}{3s^2 - 5s + 7} + \dfrac{2}{(3s^2 - 5s + 7)(s^2 + 4)}$.

14. $\dfrac{2b^2}{(s - a)[(s - a)^2 + 4b^2]}$.

16. $\dfrac{24bs(s^2 - b^2)}{(s^2 + b^2)^4}$.

18. $\dfrac{24}{(s - a)^5}$.

**Section 9.2A, Page 504.**

2. $4e^{-2t} + 7$.

4. $2\cos 3t$.

6. $5t + 3t^2$.

8. $e^{-4t}(3\sin 2t + \cos 2t)$.

10. $2\cosh 2t + \frac{3}{2}\sinh 2t$.

12. $-\frac{1}{2}e^{-t/2} + \frac{3}{4}e^{3t/4}$.

14. $\frac{1}{2} - \frac{1}{2}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$.

16. $t^2e^{-3t}\left(1 + \frac{t}{6}\right)$.

18. $te^{-t/2}\left(1 + \frac{1}{4}t\right)$.

20. $t(\cos 2t - \sin 2t)$.

22. $e^{-2t}\left(5\cos 3t + \frac{7}{3}\sin 3t\right)$.

24. $\frac{5}{2}e^{t/2} - 2e^{-t}$.

26. $-\frac{13}{4} + \frac{5}{2}t + 4e^{-t} - \frac{3}{4}e^{-2t}$.

28. $2t^2 + 3te^{2t}$.

30. $\sin 3t - 3t\sin 3t + 2t\cos 3t$.

**Section 9.2B, Page 509.**

2. $\frac{(e^t - e^{-4t})}{5}$.

4. $\frac{[3 - e^{-2t}(2\sin 3t + 3\cos 3t)]}{39}$.

6. $\frac{(2\sin t - \cos t + e^{-2t})}{5}$.

**Section 9.3, Page 519.**

2. $y = \sin t - \cos t$.

4. $y = 8t^2 - 8t + 4 + 3e^{-2t}$.

6. $y = \dfrac{(15e^{3t} + 13e^{-4t})}{7}$.

8. $y = e^{-t}(3 \sin 2t + 2 \cos 2t)$.

10. $y = 3 \sin 3t + 2e^{-3t}$.

12. $y = -\dfrac{1}{2} + 2e^{-t/2} + \dfrac{1}{2} e^{2t}$.

14. $y = (t + 1)e^{t} + (t - 1)e^{-t}$.

16. $y = -e^{-t} + 2te^{-t} + 2e^{-2t} + te^{-2t}$.

18. $y = 3e^{3t} - 2e^{5t} + 4e^{2t} + 3te^{2t}$.

20. $y = (t + 1)^2 e^{-t}$.

22. $y = -4e^{t} + 14e^{3t} + 6te^{4t} - 11e^{4t}$.

**Section 9.4A, Page 527.**

2. $\dfrac{-3e^{-10s}}{s}$.

4. $\dfrac{2}{s}(1 - e^{-5s})$.

6. $\dfrac{6}{s}(e^{-9s} - e^{-3s})$.

8. $\dfrac{3}{s}(3 - e^{-5s} - e^{-10s} - e^{-15s})$.

10. $\dfrac{1}{s}(4 - 4e^{-5s} + 3e^{-10s})$.

12. $3e^{-4s}\left(\dfrac{1}{s^2} + \dfrac{4}{s}\right)$.

14. $\frac{2}{s^2}(1 - e^{-5s})$.

16. $\frac{e^{-2(s+1)}}{s + 1}$.

18. $\frac{6}{s} - \frac{2e^{-s}}{s^2} + \frac{2e^{-3s}}{s^2}$.

20. $\frac{e^{-3\pi s/2} + e^{-9\pi s/2}}{s^2 + 1}$.

22. $\frac{1 - e^{-s} - se^{-2s}}{s^2(1 - e^{-2s})}$.

24. $\frac{1}{1 - e^{-\pi s}}\left[\frac{e^{-\pi s} + 1}{s^2 + 1}\right]$.

**Section 9.4B, Page 530.**

2. $\begin{cases} 0, & 0 < t < 5, \\ e^{2(t-5)}(7t - 32), & t > 5. \end{cases}$

4. $\begin{cases} 0, & 0 < t < 4, \\ 4e^{t-4} - 4e^{-2(t-4)}, & t > 4. \end{cases}$

6. $\begin{cases} 0, & 0 < t < 2, \\ -e^{-4(t-2)} + 2e^{2(t-2)}, & t > 2. \end{cases}$

8. $\begin{cases} 0, & 0 < t < 3, \\ e^{-2(t-3)}\left[2\cos 3(t - 3) + \frac{5}{3}\sin 3(t - 3)\right], & t > 3. \end{cases}$

10. $\begin{cases} 0, & 0 < t < 3, \\ \frac{(t - 3)^2}{2}, & 3 < t < 8, \\ 5t - \frac{55}{2}, & t > 8. \end{cases}$

12. $$\begin{cases} \dfrac{2(\sin 3t)}{3}, & 0 < t < 3, \\ \dfrac{2(\sin 3t)}{3} - \sin 3(t-3), & t > 3. \end{cases}$$

14. $$\begin{cases} \cos 2t - 1, & 0 < t < 2, \\ \cos 2t - \cos 2(t-2), & t > 2. \end{cases}$$

**Section 9.4C, Page 533.**

2. $$y = \begin{cases} 4e^{5t/3}, & 0 < t < 6, \\ -2 + (4 + 2e^{-10})e^{5t/3}, & t > 6. \end{cases}$$

4. $$y = \begin{cases} 1 - 3e^{-2t} + 2e^{-3t}, & 0 < t < 2. \\ 3(e^4 - 1)e^{-2t} + 2(1 - e^6)e^{-3t}, & t > 2. \end{cases}$$

6. $$y = \begin{cases} \dfrac{3}{8} + \dfrac{3}{4}e^{-2t} - \dfrac{1}{8}e^{-4t}, & 0 < t < 2\pi, \\ \dfrac{3}{4}e^{-2t}(1 + e^{4\pi}) - \dfrac{1}{8}e^{-4t}(1 + 3e^{8\pi}), & t > 2\pi. \end{cases}$$

8. $$y = \begin{cases} 2\sin t + 2\cos t + t, & 0 < t < \pi. \\ \sin t + 2\cos t + \pi, & t > \pi. \end{cases}$$

10. $$y = \begin{cases} 3e^{-3t}\cos 4t, & 0 < t < 2, \\ 3e^{-3t}\cos 4t + 1 - e^{-3(t-2)}\cos 4(t-2) \\ \quad - \frac{3}{4}e^{-3(t-2)}\sin 4(t-2), & 2 < t < 4, \\ 3e^{-3t}\cos 4t - e^{-3(t-2)}\cos 4(t-2) \\ \quad - \frac{3}{4}e^{-3(t-2)}\sin 4(t-2) + e^{-3(t-4)}\cos 4(t-2) \\ \quad + \frac{3}{4}e^{-3(t-4)}\sin 4(t-2), & t > 4. \end{cases}$$

12. $$y = \begin{cases} (t-1)^2, & 0 < t < 2, \\ -1 + 2e^{t-2}, & t > 2. \end{cases}$$

**Section 9.4D, Page 539.**

2. $$y = \begin{cases} 0, & 0 < t < 2\pi, \\ e^{-2(t-2\pi)}\sin t, & t > 2\pi. \end{cases}$$

4. $$y = \begin{cases} -2e^{-t} + 4e^{-2t}, & 0 < t < 4, \\ (e^4 - 2)e^{-t} - (e^8 - 4)e^{-2t}, & t > 4. \end{cases}$$

6. $$y = \begin{cases} e^{-2t}\cos t, & 0 < t < \pi, \\ e^{-2t}(\cos t - e^{2\pi}\sin t), & t > \pi. \end{cases}$$

**Section 9.5, Page 542.**

2. $\begin{cases} x = 2 + 2e^{t} - e^{2t}, \\ y = e^{t} - e^{2t}. \end{cases}$

4. $\begin{cases} x = -3t - e^{-t}, \\ y = 2t - 1 + e^{-t}. \end{cases}$

6. $\begin{cases} x = e^{2t} + 2\cos 2t - \sin 2t, \\ y = 2e^{2t} + 5\sin 2t. \end{cases}$

8. $\begin{cases} x = 1 - t + 3e^{4t} - e^{-2t}, \\ y = t - 3e^{4t} - e^{-2t}. \end{cases}$

10. $\begin{cases} x = 2e^{t} - e^{2t} - 1, \\ y = e^{t} - 2. \end{cases}$