Lecture 26

On

Numerical Methods & Computer Programming

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Runge Kutta Method

The Taylor series method has desirable features, particularly in its ability to keep the errors small, but that it also has the strong disadvantage of requiring the evaluation of higher derivatives of the function f(x,y). In the Taylor series method, each of these higher order derivatives is evaluated at the point x_i at the beginning of the step, in order to evaluate $y(x_i)$ at the end of the step. We observed that the Euler method could be improved by computing the function f(x,y) at a predicted point at the far end of the step in x. The Runge-Kutta approach is to aim for the desirable features of the Taylor series method, but with the replacement of the requirement for the evaluation of higher order derivatives with the requirement to evaluate f(x,y) at some points within the step x_i to x_{i+1}. Since it is not initially known at which points in the interval these evaluations should be done, it is possible to choose these points in such a way that the result is consistent with the Taylor series solution to some particular, which we shall call the order of the Runge-Kutta method. The Runge-Kutta method of order N = 4 is most popular. It is a good choice for common purposes because it is quite accurate, stable, and easy to program. Most authorities proclaim that it is not necessary to go to a higher-order method because the increased accuracy is offset by additional computational effort. If more accuracy is required, then either a smaller step size or an adaptive method should be used.

Runge Kutta Method

Second Order Runge-Kutta Method

Computationally, most efficient methods in terms of accuracy were developed by two German mathematicians, Carl Runge and Wilhelm Kutta. These methods are well known as Runge-Kutta methods (R-K methods). In this and the coming section we consider second and fourth order R-K methods.

There are several second order Runge-Kutta formulas and we consider one among them.

Working Method (Second Order Runge-Kutta Method)

Given the initial value problem (1). Suppose $x_0, x_1, x_2, \ldots$ be equally spaced x values with interval h. i.e.,

$$x_1 = x_0 + h, \quad x_2 = x_1 + h, \ldots$$

Runge Kutta Method

Also denote $y_0 = y(x_0),\ y_1 = y(x_1),\ y_2 = y(x_2), \ldots$

For $n = 0, 1, \cdots$ until termination do:

$$x_{n+1} = x_n + h$$

$$k_n = hf(x_n, y_n) \qquad \ldots(8)$$

$$l_n = hf(x_{n+1}, y_n + k_n) \qquad \ldots(9)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_n + l_n) \qquad \ldots(10)$$

Remark Modified Euler method is a special case of second order Runge-Kutta method given by (10).

Runge Kutta Method

Example Use second order Runge-Kutta method with $h=0.1$ to find $y(0.2)$, given $\frac{dy}{dx}=x^2+y^2$ with $y(0)=0$.

Here $f(x,y)=x^2+y^2$, $x_0=0$, $y_0=0$, $h=0.1$. Hence

$$x_1=x_0+h=0.1,\quad x_2=x_1+h=0.2.$$

To determine y_1, y_2 we use second order Runge-Kutta method and using (8) – (10),

$$k_n=hf(x_n,y_n)=0.1(x_n^2+y_n^2)$$

$$l_n=hf(x_{n+1},y_n+k_n)=0.1[x_{n+1}^2+(y_n+k_n)^2]$$

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and $$y_{n+1}=y_n+\frac{1}{2}(k_n+l_n)$$

$$k_0=0.2(x_0^2+y_0^2)=0.1(0^2+0^2)=0.$$

$$l_0=0.2(x_1^2+(y_0+k_0)^2)=0.1\left[(0.1)^2+(0+0)^2)\right]=0.001$$

Runge Kutta Method

and $$y_{n+1}=y_n+\frac{1}{2}(k_n+l_n)$$

$$k_0=0.2(x_0^2+y_0^2)=0.1(0^2+0^2)=0.$$

$$l_0=0.2(x_1^2+(y_0+k_0)^2)=0.1\left[(0.1)^2+(0+0)^2)\right]=0.001$$

and $$y_1=y_0+\frac{1}{2}(k_0+k_0)=0+\frac{1}{2}(0+0.001)=0.0005.$$

$$k_1=0.2(x_1^2+y_1^2)=0.1\left[(0.1)^2+(0.0005)^2\right]=0.001,$$ correct to three places of decimals.

$$l_1=0.2(x_2^2+(y_1+k_1)^2)=0.1\left[(0.2)^2+(0.0015)^2)\right]=0.004$$

and $$y_2=y_1+\frac{1}{2}(k_1+l_1)=0.0005+\frac{1}{2}(0.001+0.004)=0.003.$$

Hence $y(0.1)=0.0005$, $y(0.2)=0.003$.

Runge Kutta Method

Example Given the initial value problem $y'=x+y,\ y(0)=0$. Find the value of y approximately for $x=1$ by second order Runge-Kutta method in five steps. Compare the result with the exact value.

Here $f(x,y)=x+y$, $x_0=0$, $y_0=0$. As we have to calculate the value of y when $x=1$ in five steps, we have to take $h=\frac{x_n-x_0}{n}=\frac{1-0}{5}=0.2$. Hence

$$x_1=x_0+h=0.2,\quad x_2=x_1+h=0.4,\quad x_3=x_2+h=0.6,\quad x_4=x_3+h=0.8,\quad x_5=x_4+h=1.0.$$

We determine y_1, y_2, y_3, y_4, y_5 we use second order Runge-Kutta formula:

$$k_n=hf(x_n,y_n)=0.2(x_n+y_n)$$

$$l_n=hf(x_{n+1},y_n+k_1)=0.2(x_{n+1}+(y_n+k_n))$$

$$=0.2\left[x_n+0.2+y_n+0.2(x_n+y_n)\right],$$ as $x_{n+1}=x_n+h=x_n+a_2$ and $$y_{n+1}=y_n+\frac{1}{2}(k_n+l_n)$$

$$=y_n+\frac{1}{2}\left\{0.2(x_n+y_n)+0.2\left[x_n+0.2+y_n+0.2(x_n+y_n)\right]\right\}$$

$$=y_n+0.22(x_n+y_n)+0.02$$

Runge Kutta Method

The successive steps and calculations are plotted in the following table.

n	x_n	approximate value of y_n	$x_n + y_n$	$0.22(x_n + y_n) + 0.02$	y_{n+1}
0	0.0	0.0000	0.0000	0.0200	0.0200
1	0.2	0.0200	0.2200	0.0684	0.0884
2	0.4	0.0884	0.4884	0.1274	0.2158
3	0.6	0.2158	0.8158	0.1995	0.4153
4	0.8	0.4153	1.2153	0.2874	0.7027
5	1.0	0.7027			

Hence $y(1) = 0.7027$. In an earlier example we have noted that the exact value is 0.718.

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Thank You

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