Lectures 7-8

On

Numerical Methods & Computer Programming

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Interpolation

Given that (x_0,y_0), (x_1, y_1), (x_2, y_2).... (x_n, y_n) satisfying the relation y = f(x), where the explicit nature of f(x) is not known.

It is required find a simpler function such as which will satisfy the tabulated points. This process of finding a simpler function is called interpolation.

If the f(x) is polynomial, then the process is called polynomial interpolation.

Interpolation

Finite Differences

Assume that we have a table of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$, the values of x being equally spaced, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$. Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x, or to obtain the derivative of $f(x)$ for some x in the range $x_0 \le x \le x_n$. The methods for the solution to these problems are based on the concept of the 'differences' of a function which we now proceed to define.

Interpolation

Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *differences* of y. Denoting these differences by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively, we have

First Forward Differences

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \ldots, \quad \Delta y_{n-1} = y_n - y_{n-1},$$

where Δ is called the *forward difference operator* and $\Delta y_0, \Delta y_1, \ldots,$ are called *first forward differences*. The differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$. Similarly, one can define *third forward differences*, *fourth forward differences*, etc. Thus,

Interpolation

Forward Differences

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$

Second Forward Differences

$$= y_2 - 2y_1 + y_0,$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$

Third Forward Differences

$$= y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$

Forth Forward Differences

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

Interpolation

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Newton's Formula for Interpolation

Given the set of $(n+1)$ values, viz., $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. let

$$x_i = x_0 + ih, \quad i = 0, 1, 2, \ldots, n.$$

Since $y_n(x)$ is a polynomial of the nth degree, it may be written as

$$\left.\begin{aligned} y_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots \\ &+ a_n(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1}). \end{aligned}\right\} \quad (3.9)$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain

Newton's Formula for Interpolation

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$$a_0 = y_0;\; a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h};\; a_2 = \frac{\Delta^2 y_0}{h^2 2!};\; a_3 = \frac{\Delta^3 y_0}{h^3 3!};\ldots;\; a_n = \frac{\Delta^n y_0}{h^n n!};$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \ldots, a_n$, Eq. (3.9) gives

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$
$$+ \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0, \qquad (3.10)$$

which is *Newton's forward difference interpolation formula* and is useful for interpolation *near the beginning* of a set of tabular values.

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0,$$

Example 3.4 Find the cubic polynomial which takes the following values: $y(1) = 24$, $y(3) = 120$, $y(5) = 336$, and $y(7) = 720$. Hence, or otherwise, obtain the value of $y(8)$.

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Example 3.4 Find the cubic polynomial which takes the following values: $y(1) = 24$, $y(3) = 120$, $y(5) = 336$, and $y(7) = 720$. Hence, or otherwise, obtain the value of $y(8)$.

We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

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x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Here $h = 2$. With $x_0 = 1$, we have $x = 1 + 2p$ or $p = (x-1)/2$. Substituting this value of p in Eq. (3.10), we obtain

$$y(x) = 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$$= x^3 + 6x^2 + 11x + 6.$$

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$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0,$$

$$= x^3 + 6x^2 + 11x + 6.$$

To determine $y(8)$, we observe that $p = 7/2$. Hence, formula (3.10) gives:

$$y(8) = 24 + \frac{7}{2}(96) + \frac{(7/2)(7/2-1)}{2}(120) + \frac{(7/2)(7/2-1)(7/2-2)}{6}(48) = 990.$$

Direct substitution in $y(x)$ also yields the same value.

Note: This process of finding the value of y for some value of x *outside* the given range is called extrapolation and this example demonstrates the fact that if a tabulated function is a polynomial, then both interpolation and extrapolation would give exact values.

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Newton's Formula for Interpolation

Example 3.8 The table below gives the values of $\tan x$ for $0.10 \le x \le 0.30$:

x	$y = \tan x$
0.10	0.1003
0.15	0.1511
0.20	0.2027
0.25	0.2553
0.30	0.3093

Find : (a) tan 0.12 ~~(b) tan 0.26, (c) tan 0.40 and (d) tan 0.50.~~

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Newton's Formula for Interpolation

The table of difference is

x	y	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

(a) To find tan (0.12), we have $0.12 = 0.10 + p(0.05)$, which gives $p = 0.4$. Hence formula (3.10) gives

$$\tan(0.12) = 0.1003 + 0.4(0.0508) + \frac{0.4(0.4-1)}{2}(0.0008) + \frac{0.4(0.4-1)(0.4-2)}{6}(0.0002) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(0.0002)$$

$$= 0.1205.$$

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