BECM 2203
Numerical Analysis and Computer Programming

Interpolation

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Introduction

- Interpolation, which is the process of **computing intermediate values of a function from a set of given or tabular values of that function.**

- Plays significant role in numerical research almost in all branches of science, humanities, commerce and in technical branches.

Forward differences

If y_0, y_1, y_2,…., y_n denote a set of values of any function y = f(x), then y_1 - y_0, y_2 - y_1,……., y_n - y_{n-1} are called the differences of the function y. Denoting these differences by Δy_0, Δy_1,…….. Δy_{n-1} respectively, we have

Δy_0 = y_1 - y_0, Δy_1 = y_2 - y_1,……, Δy_{n-1} = y_n - y_{n-1}

Where,

Δ is called the forward difference operator and

Δy_0, Δy_1,…….. Δy_{n-1} are called first forward differences.

The differences of the first forward differences are called second forward differences and denoted by $\Delta^2 y_0$, $\Delta^2 y_1$,…….Thus,

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,$

In like manner, the third and fourth forward differences are

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) = = y_3 - 3y_2 + 3y_1 - y_0$

$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$

Table 1 Forward Difference Table

x	**y**	**Δ**	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward Differences

The differences $y_1 - y_0, y_2 - y_1, \ldots\ldots, y_n - y_{n-1}$ are called first backward differences if they are denoted by $\nabla y_1, \nabla y_2, \ldots\ldots \nabla y_n$ respectively, so that

$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \ldots\ldots\ldots, \nabla y_n = y_n - y_{n-1}$

Where,

∇ is called the backward difference operator.

In a similar way, one can define backward differences of higher orders. Thus we obtain

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$

Table 2 Backward Difference Table

x	**y**	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$

Newton's Formula for Forward Interpolation

Let, y = f (x) denote a function which takes the values y_0, y_1, y_2,……. y_n for the equidistant values x_0, x_1, x_2,……x_n of the independent variable x, and let ϕ (x) denote a polynomial of the n^{th} degree. This polynomial may be written in the form

$\Phi\ (\mathrm{x}) = a_0 + a_1(x - x_0) + a_2(x - x_0)\ (x - x_1)\ + a_3(x - x_0)\ (x - x_1)\ (x - x_2) +$ ……. $+a_n(x - x_0)$ $(x - x_1)\ (x - x_2)$……. $(x - x_{n-1})$. …………………………(i)

To determine the coefficients $a_0, a_1, a_2, \ldots a_n$ so as to make

$\Phi\ (x_0) = y_0, \Phi\ (x_1) = y_1, \Phi\ (x_2) = y_2, \ldots . \Phi\ (x_n) = y_n.$

Substituting in equation (i) the successive values x_0, x_1, x_2,……x_n for x, at the same time putting

$\Phi\ (x_0) = y_0, \Phi\ (x_1) = y_1, \Phi\ (x_2) = y_2, \ldots . \Phi\ (x_n) = y_n$. We have

When, x = $\boldsymbol{x_0}$

$\Phi\ (x_0) = a_0$

$\boldsymbol{y_0 = a_0}$

When, x = $\boldsymbol{x_1}$

$\Phi\ (x_1) = a_0 + a_1(x_1 - x_0)$

$y_1 = y_0 + a_1 h$

$\boldsymbol{a_1} = \frac{y_1 - y_0}{h} = \frac{\boldsymbol{\Delta y_0}}{\boldsymbol{h}}$

As x is equidistant so

$x_i = x_0 + ih \quad [\ i = 1,2,3, \ldots n]$

$x_1 = x_0 + h$

$x_1 - x_0 = h\ ;\ x_2 - x_0 = 2h$

When, x = $\boldsymbol{x_2}$

$\Phi\ (x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)\ (x_2 - x_1)$

$y_2 = y_0 + \frac{\Delta y_0}{h}\,2h + a_2.2h.h = y_0 + 2\Delta y_0 + a_2.2h^2$

$y_2 - y_0 - 2\Delta y_0 = a_2.2h^2$

$\boldsymbol{a_2} = \frac{y_2 - y_0 - 2\Delta y_0}{2h^2} = \frac{y_2 - y_0 - 2(y_1 - y_0)}{2h^2} = \frac{y_2 - y_0 - 2y_1 + 2y_0)}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2} = \frac{\boldsymbol{\Delta^2 y_0}}{\boldsymbol{2!\, h^2}}$

When, x = x_3

$$\Phi\ (x_3) = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)\ (x_3 - x_1)\ + a_3(x_3 - x_0)\ (x_3 - x_1)\ (x_3 - x_2)$$

$$y_3 = y_0 + \frac{\Delta y_0}{h}\ 3h + \frac{\Delta^2 y_0}{2!\,h^2}\ 3h.2h + a_3.\ 3h.2h.h = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + a_3.6h^3$$

$$a_3 = \frac{y_3 - y_0 - 3\Delta y_0 - 3\Delta^2 y_0}{6h^3} = \frac{y_3 - y_0 - 3(y_1 - y_0) - 3(y_2 - 2y_1 + y_0)}{6h^3} = \frac{y_3 - y_0 - 3y_1 + 3y_0 - 3y_2 + 6y_1 - 3y_0}{6h^3}$$

$$a_3 = \frac{y_3 - 3y_2 + 3y_1 - y_0}{6h^3} = \frac{\Delta^3 y_0}{6h^3} = \frac{\boldsymbol{\Delta^3 y_0}}{\boldsymbol{3!\,h^3}}$$

Similarly we can write,

$$a_4 = \frac{\boldsymbol{\Delta^4 y_0}}{\boldsymbol{4!\,h^4}}\ ;\ a_5 = \frac{\boldsymbol{\Delta^5 y_0}}{\boldsymbol{5!\,h^5}}\ \ \ldots\ldots.\ a_n = \frac{\boldsymbol{\Delta^n y_0}}{\boldsymbol{n!\,h^n}}$$

Substituting these values of a_0, a_1, a_2,………. a_n in equation (i), we get

$$\Phi\ (x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x - x_0)\,(x - x_1) + \frac{\Delta^3 \boldsymbol{y_0}}{3!\,\boldsymbol{h^3}}(x - x_0)\,(x - x_1)\,(x - x_2) + \ldots\ldots. + \frac{\Delta^{\boldsymbol{n}} \boldsymbol{y_0}}{\boldsymbol{n}!\,\boldsymbol{h^n}}(x - x_0)$$

$$(x - x_1)\,(x - x_2)\ldots\ldots.\,(x - x_{n-1})\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Now put, p = $\frac{x-x_0}{h}$ or x = x_0 + ph

Since, x_1= x_0 + h ; x_2= x_0 + 2h , we have

$$\frac{x-x_1}{h} = \frac{x-x_0-h}{h} = \frac{x-x_0}{h} - 1 = p - 1$$

$$\frac{x-x_2}{h} = \frac{x-x_0-2h}{h} = \frac{x-x_0}{h} - 2 = p - 2$$

…………………………………………………………………

$$\frac{x-x_{n-1}}{h} = \frac{x-[x_0+(n-1)\,h]}{h} = \frac{x-x_0}{h} - (n-1) = p - n + 1$$

Substituting these values in equation (ii), we get

$$\boldsymbol{\Phi\ (x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\Delta^n y_0}$$

This is Newton's Forward Interpolation Formula

Example 1

Find the cubic polynomial which takes the following values:

y(1) = 24, y(3) = 120, y(5) = 336, and y(7) = 720. Hence, or otherwise, obtain the value of y(8).

Sol^n:

We form the forward difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Here,

$x_0 = 1$

$x_1 = 3$

$h = x_1 - x_0 = 2$

$P = \frac{x-x_0}{h} = \frac{x-1}{2}$

$y_0 = 24$

$\Delta y_0 = 96$

$\Delta^2 y_0 = 120$

$\Delta^3 y_0 = 48$

Example 1 continue.......

Now, we have the Newton's Forward Interpolation Formula,

$$\Phi(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots + \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0$$

$$\Phi(x) = 24 + \left(\frac{x-1}{2}\right).96 + \left(\frac{x-1}{2}\right)\left(\frac{x-1}{2} - 1\right).\frac{120}{2!} + \left(\frac{x-1}{2}\right)\left(\frac{x-1}{2} - 1\right)\left(\frac{x-1}{2} - 2\right).\frac{48}{3!}$$

$$= 24 + 48(x-1) + 15(x-1)(x-1-2) + (x-1)(x-1-2)(x-1-4)$$

$$= 24 + 48x - 48 + 15(x-1)(x-3) + (x-1)(x-3)(x-5)$$

$$= 24 + 48x - 48 + 15(x^2 - 4x + 3) + (x^2 - 4x + 3)(x-5)$$

$$= -24 + 48x + 15x^2 - 60x + 45 + x^3 - 5x^2 - 4x^2 + 20x + 3x - 15$$

$$= x^3 + 6x^2 + 11x + 6$$

$$y(x) = x^3 + 6x^2 + 11x + 6$$

$y(8) = 8^3 + 6.8^2 + 11.8 + 6 = 990$ (Ans:)

Newton's Formula for Backward Interpolation

Let,

$$\Phi\,(\mathrm{x}) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})\,(x - x_{n-2}) + \ldots\ldots + a_n(x - x_n)(x - x_{n-1})\ldots.\,(x - x_1)\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

To determine the coefficients $a_0, a_1, a_2, \ldots a_n$ so as to make

$\Phi\,(x_n) = y_n$, $\Phi\,(x_{n-1}) = y_{n-1}$,…etc . Substituting in equation (i) the successive values x_n, x_{n-1}, x_{n-2},..etc for x, at the same time putting $\Phi\,(x_n) = y_n$, $\Phi\,(x_{n-1}) = y_{n-1}$,…etc., We have

When, x = $\boldsymbol{x_n}$

$\Phi\,(x_n) = a_0$

$\boldsymbol{y_n = a_0}$

When, x = $\boldsymbol{x_{n-1}}$

$$\Phi\ (x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$\boldsymbol{y_{n-1}} = y_n - a_1(x_n - x_{n-1}) = y_n - a_1 h$$

$$\boldsymbol{a_1} = \frac{y_n - y_{n-1}}{h} = \frac{\boldsymbol{\nabla y_n}}{\mathbf{h}}$$

For equidistant

$x_i = x_0 + ih$ [i = 1,2,3, . . .n]

$x_n - x_{n-1} = h$; $x_n - x_{n-2} = 2h$

When, x = $\boldsymbol{x_{n-2}}$

$$\Phi\ (x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$y_{n-2} = y_n + a_1(-2h) + a_2(-2h)(-h) = y_n + 2h.\frac{\nabla y_n}{h} + a_2.2h^2$$

$$\boldsymbol{a_2} = \frac{-\mathbf{y_n}+2\nabla \mathrm{y_n} + \mathbf{y_{n-2}}}{2h^2} = \frac{-\mathbf{y_n}+2\mathbf{y_n} - 2\mathbf{y_{n-1}}+\mathbf{y_{n-2}}}{2h^2} = \frac{\mathbf{y_n} - 2\mathbf{y_{n-1}}+\mathbf{y_{n-2}}}{2h^2} = \frac{\boldsymbol{\nabla^2 y_n}}{\mathbf{2!h^2}}$$

Similarly we can write that

$$a_3 = \frac{\nabla^3 y_n}{3!h^2}$$

$$a_4 = \frac{\nabla^4 y_n}{4!h^2}$$

$$a_n = \frac{\nabla^n y_n}{n!h^n}$$

Substituting these values of a_0, a_1, a_2,………. a_n in equation (i), we get

$$\Phi\ (\mathrm{x}) = \mathrm{y_n} + \frac{\nabla \mathrm{y_n}}{\mathrm{h}}(\mathrm{x}-\mathrm{x_n}) + \frac{\nabla^2 \mathrm{y_n}}{2!\mathrm{h}^2}(\mathrm{x}-\mathrm{x_n})(\mathrm{x}-\mathrm{x_{n-1}}) + \frac{\nabla^3 \mathrm{y_n}}{3!\mathrm{h}^3}(\mathrm{x}-\mathrm{x_n})(\mathrm{x}-\mathrm{x_{n-1}})\ (\mathrm{x}-\mathrm{x_{n-2}}) + \ldots\ldots. + \frac{\nabla^n \mathrm{y_n}}{\mathrm{n}!\mathrm{h}^\mathrm{n}}$$

$$(\mathrm{x}-\mathrm{x_n})\ (\mathrm{x}-\mathrm{x_{n-1}})\ldots.\ (\mathrm{x}-\mathrm{x_1})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(\mathrm{ii})$$

Now put, p = $\frac{x-x_n}{h}$ or x = x_n + ph

Since, x_n= x_{n-1} + h ; x_n= x_{n-2} + 2h , we have

$$\frac{x-x_{n-1}}{h} = \frac{x-x_n+h}{h} = \frac{x-x_n}{h} + 1 = p + 1$$

$$\frac{x-x_{n-2}}{h} = \frac{x-x_n+2h}{h} = \frac{x-x_n}{h} + 2 = p + 2$$

$$\frac{x-x_1}{h} = \frac{x-[x_n-(n-1)\ h]}{h} = \frac{x-x_n}{h} + (n-1) = p + n - 1$$

Substituting these values in equation (ii), we get

$$\boldsymbol{\Phi\ (\mathrm{x}) = \mathrm{y_n} + p\nabla \mathrm{y_n} + p(p+1)\,\frac{\nabla^2 \mathrm{y_n}}{2!} + p(p+1)(p+2)\,\frac{\nabla^3 \mathrm{y_n}}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots p(p+1)\ldots\ldots.(P+n-1)\ \frac{\nabla^n \mathrm{y_n}}{\mathrm{n}!}}$$

This is Newton's backward Interpolation Formula

Example 2

Values of x (in degrees) and $\sin x$ are given in the following table:

x (in degrees)	$\mathbf{\sin x}$
15	0.2588190
20	0.3420201
25	0.4226183
30	0.5
35	0.5735764
40	0.6427876

Determine the value of $\sin 38^\circ$.

Example 2 continue.......

The forward difference table is

x	$\sin x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
15	0.2588190					
		0.0832011				
20	0.3420201		-0.0026029			
		0.0805982		-0.0006136		
25	0.4226183		-0.0032165		0.0000248	
		0.0773817		-0.0005888		0.0000041
30	0.5		-0.0038053		0.0000249	
		0.0735764		-0.0005599		
35	0.5735764		-0.0043652			
		0.0692112				
40	0.6427876					

Example 2 continue.......

To find $\sin 38^{\circ}$,we use Newton's backward difference formula with x_n= 40, x = 38, x_{n-1}= 35 and y_n= 0.6427.

Therefore,

h = x_n- x_{n-1} = 40 – 35

P = $\frac{x - x_n}{h}$ = $\frac{38 - 40}{5}$ = -0.4

$$y\,(x) = y_n + p\nabla y_n + p(p+1)\frac{\nabla^2 y_n}{2!} + p(p+1)(p+2)\frac{\nabla^3 y_n}{3!} + p(p+1)(p+2)(p+3)\frac{\nabla^4 y_n}{4!} + p(p+1)(p+2)(p+3)(p+4)\frac{\nabla^5 y_n}{5!}$$

y(38) = 0.6156614

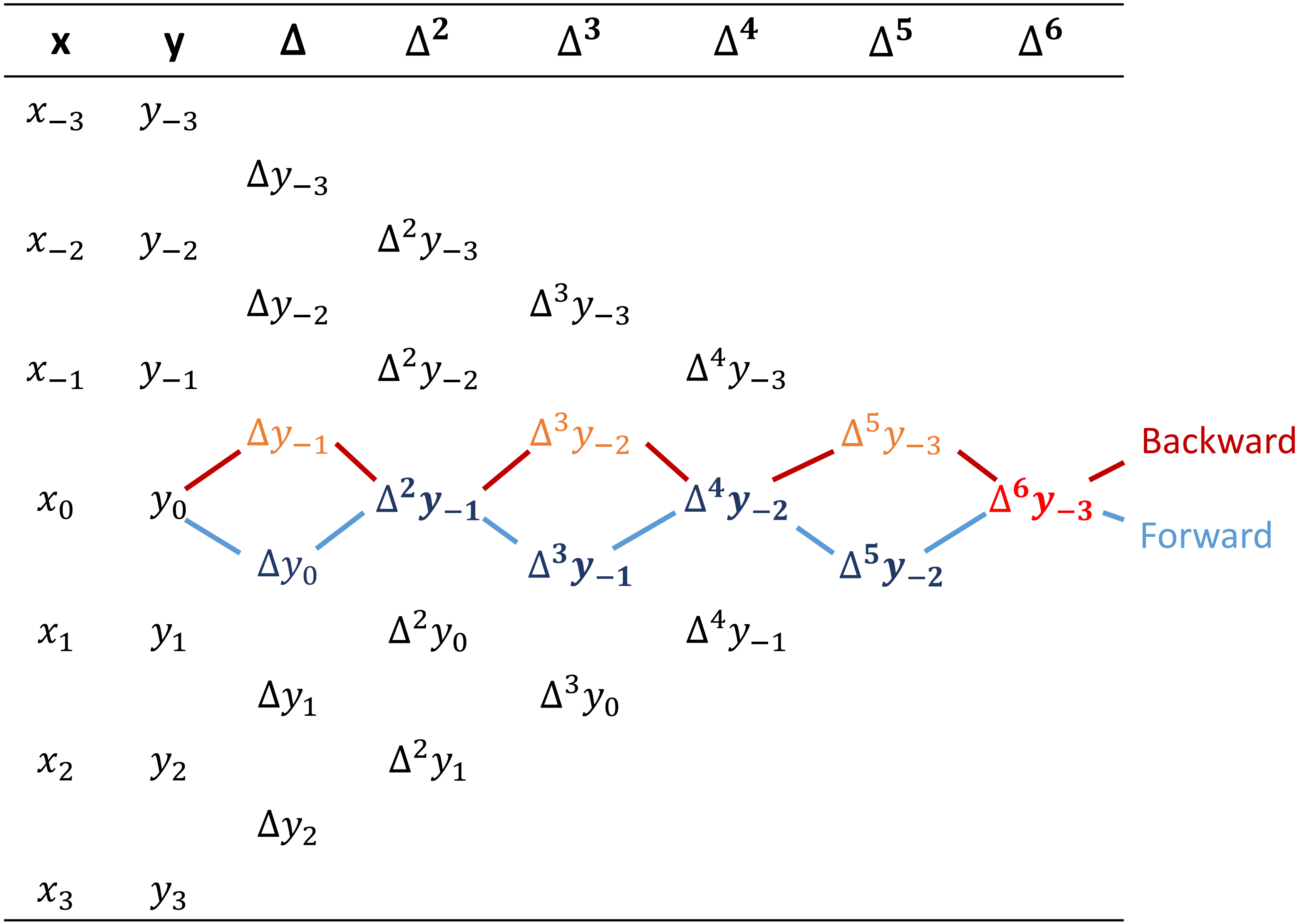
Exercise

The table below gives the values of $\tan x$ for $0.10 \leq \mathrm{x} \leq 0.30$:

x	$\mathrm{y} = \tan x$
0.10	0.1003
0.15	0.1511
0.20	0.2027
0.25	0.2553
0.30	0.3093

Find: (a) $\tan 0.12$ (b) $\tan 0.26$ (c) $\tan 0.40$ (d) $\tan 0.50$

Central Difference Interpolation Formulae



Difference table for Gauss’ Central Difference Formulae

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\boldsymbol{\Delta^2 y_{-1}}$		$\boldsymbol{\Delta^4 y_{-2}}$		$\boldsymbol{\Delta^6 y_{-3}}$
		Δy_0		$\boldsymbol{\Delta^3 y_{-1}}$		$\boldsymbol{\Delta^5 y_{-2}}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

Backward

Forward

Gauss' forward formula

Let, Gauss' forward formula can be expressed as

$y_p = y_0 + G_1\Delta y_0 + G_2\Delta^{\mathbf{2}}\boldsymbol{y_{-1}} + G_3\Delta^{\mathbf{3}}\boldsymbol{y_{-1}} + G_4\Delta^{\mathbf{4}}\boldsymbol{y_{-2}} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$

From L.H.S of Equation (i)

$y_p = E^p y_0$ [Where, E = Shift operator]

$= (1+\Delta)^p y_0$ [since, E = $1+\Delta$]

$$=(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 \ldots\ldots\ldots)\, y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$(\mathbf{1}+\boldsymbol{x})^{\boldsymbol{n}} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 \ldots\ldots\ldots)$$

$$(\mathbf{1}+\boldsymbol{x})^{-\mathbf{1}} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots)$$

$$(\mathbf{1}+\boldsymbol{x})^{-\mathbf{2}} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots)$$

From R.H.S of Equation (i)

We have

$\Delta^2 \boldsymbol{y_{-1}} = \Delta^2 E^{-1} y_0 = \Delta^2 (1+\Delta)^{-1} y_0 = \Delta^2(1-\Delta+\Delta^2-\Delta^3+\ldots\ldots\ldots) y_0$

$= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots\ldots\ldots$

$\Delta^3 \boldsymbol{y_{-1}} = \Delta^3 E^{-1} y_0 = \Delta^3 (1+\Delta)^{-1} y_0 = \Delta^3(1-\Delta+\Delta^2-\Delta^3+\ldots\ldots\ldots) y_0$

$= \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \ldots\ldots\ldots\ldots$

$\Delta^4 \boldsymbol{y_{-2}} = \Delta^4 E^{-2} y_0 = \Delta^4 (1+\Delta)^{-2} y_0 = \Delta^4(1-2\Delta+3\Delta^2-4\Delta^3+\ldots\ldots\ldots) y_0$

$= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots$

Hence, Equation (i) gives the identity

$y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots = y_0 + G_1 \Delta y_0 + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots.) + G_3(\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \ldots) + G_4(\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \ldots) + \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$

Equating the coefficients of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ and $\Delta^4 y_0$ on both sides of equation (ii), we obtain

$\boldsymbol{G_1} = \mathbf{p}$

$$\boldsymbol{G_2} = \frac{\boldsymbol{p(p-1)}}{\boldsymbol{2}!}$$

$$-G_2 + G_3 = \frac{p(p-1)(p-2)}{3!}\ ;\ G_3 = \frac{p(p-1)(p-2)}{3!} + G_2 = \frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)}{2!} = \frac{2!p(p-1)(p-2)+3!p(p-1)}{3!2!}$$

$$\boldsymbol{G_3} = \frac{2p(p-1)\,(p-2+3)}{3!2!} = \frac{\boldsymbol{(p+1)p(p-1)}}{\boldsymbol{3}!}$$

$$G_2 - G_3 + G_4 = \frac{p(p-1)(p-2)(p-3)}{4!}\ ;\ G_4 = \frac{p(p-1)(p-2)(p-3)}{4!} - G_2 + G_3 = \frac{p(p-1)(p-2)(p-3)}{4!} - \frac{p(p-1)}{2!} + \frac{(p+1)p(p-1)}{3!}$$

$$\boldsymbol{G_4} = \frac{[12p(p-1)(p-2)(p-3)\} - 144[p(p-1)] + [48(p+1)p(p-1)]}{4!2!3!} = \frac{12p(p-1)\,\{(p-2)(p-3)-12+4(p+1)\}}{288}$$

$$= \frac{12p(p-1)\,\{p^2-5p+6-12+4p+4\}}{288} = \frac{p(p-1)\,\{p^2-p-2\}}{24} = \frac{p(p-1)\,(p^2-2p+p-2)}{24} = \frac{p(p-1)\,\{p(p-2)+1\,(p-2)\}}{24}$$

$$= \frac{p(p-1)\,(p+1)\,(p-2)}{24} = \frac{\boldsymbol{(p+1)p(p-1)(p-2)}}{\boldsymbol{4}!}$$

Substituting these values of G_1, G_2, G_3, G_4………. in equation (i), we get

$$y_p = y_0 + P\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots$$

This is called Gauss' forward formula

Gauss' Backward Formula

Gauss' backward formula can be assumed to be of the form

$y_p = y_0 + G_1'\Delta y_{-1} + G_2'\Delta^2 y_{-1} + G_3'\Delta^3 y_{-2} + G_4'\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots$(i)

Where G_1', G_2', G_3'……. have to be determined. Following the same procedure as in Gauss' forward formula, we obtain,

$$G_1' = p$$

$$G_2' = \frac{p(p+1)}{2!}$$

$$G_3' = \frac{(p+1)p(p-1)}{3!}$$

$$G_4' = \frac{(p+2)(p+1)p(p-1)}{4!}$$

Substituting these values of G_1', G_2', G_3', ………. in equation (i), we get

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots$$

This is called Gauss' backward formula

Stirling's Formula

This is the mean of Gauss' forward and backward formulae, that is

$$y_p = y_0 + p\frac{\Delta y_{-1}+\Delta y_0}{2} + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots\ldots$$

Example 3

From the following table, find the value of $e^{1.17}$ using (a) Gauss' forward formula (b) Gauss' backward formula (c) Stirling's Formula

x	$\boldsymbol{e^x}$
1.00	2.7183
1.05	2.8577
1.10	3.0042
1.15	3.1582
1.20	3.3201
1.25	3.4903
1.30	3.6693

Example 3 continue.......

Sol^n:

We form the difference table:

Here,

x_0= 1.15

x_1= 1.20

x = 1.17

y_0= 3.1582

h = x_1- x_0= 0.05

P = $\frac{x - x_0}{h} = \frac{1.17 - 1.15}{0.05} = \frac{2}{5}$

x	y = e^x	Δ	Δ^2	Δ^3	Δ^4
1.00	2.7183				
		0.1394			
1.05	2.8577		0.0071		
		0.1465		0.0004	
1.10	3.0042		0.0075		0
		0.1540		0.0004	
1.15	3.1582		0.0079		0
		0.1619		0.0004	
1.20	3.3201		0.0083		0.0001
		0.1702		0.0005	
1.25	3.4903		0.0088		
		0.1790			
1.30	3.6693				

Example 3 continue…….

(a)Gauss' forward formula

$$y_p = y_0 + \mathrm{P}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$e^{1.17} = 3.1582 + \frac{2}{5}(0.1619) + \frac{\frac{2}{5}(\frac{2}{5}-1)}{2!}(0.0079) + \frac{(\frac{2}{5}+1)\frac{2}{5}\left(\frac{2}{5}-1\right)}{3!}(0.0004) = 3.222$$

(b) Gauss' backward formula

$$y_p = y_0 + \mathrm{p}\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$e^{1.17} = 3.1582 + \frac{2}{5}(0.1540) + \frac{\frac{2}{5}(\frac{2}{5}+1)}{2!}(0.0079) + \frac{(\frac{2}{5}+1)\frac{2}{5}\left(\frac{2}{5}-1\right)}{3!}(0.0004) = 3.222$$

Example 3 continue.......

(c) Stirling's Formula

$$y_p = y_0 + p\frac{\Delta y_{-1}+\Delta y_0}{2} + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\,\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$e^{1.17} = 3.1582 + \frac{2}{5}\left(\frac{0.1540 + 0.1619}{2}\right) + \frac{\frac{2^2}{5}}{2}(0.0079) + \frac{\frac{2}{5}\left(\left(\frac{2}{5}\right)^2-1\right)}{3!}\left(\frac{0.0004 + 0.0004}{2}\right) = 3.222$$

Lagrange’s Interpolation Formula

Let, y = f (x) denote a function which takes the values y_0, y_1, y_2,……. y_n corresponding to values x_0, x_1, x_2,…….x_n. Since there are (n + 1) values of y corresponding to (n + 1) values of x, we can represent the function f(x) by a polynomial (φ (x)) of n^{th} degree.

Φ (x) = a_0 $(x - x_1)$ $(x - x_2)$ $(x - x_3)$…. $(x - x_n)$ + $a_1(x - x_0)$ $(x - x_2)$ $(x - x_3)$….. $(x - x_n)$ + $a_2(x - x_0)$ $(x - x_1)$ $(x - x_3)$…… $(x - x_n)$ + ……. +$a_n(x - x_0)$ $(x - x_1)$ $(x - x_2)$……. $(x - x_{n-1})$. ……………………….(i)

To determine the coefficients a_0, a_1, a_2, . . . a_n so as to make

Φ (x_0) = y_0, Φ (x_1) = y_1, Φ (x_2) = y_2, ... Φ (x_n) = y_n.

Substituting in equation (i) the successive values x_0, x_1, x_2,……x_n for x, at the same time putting

Φ (x_0) = y_0, Φ (x_1) = y_1, Φ (x_2) = y_2, ... Φ (x_n) = y_n. We get

When, $\mathrm{x} = \mathrm{x}_0$

$y_0 = a_0\,(x_0 - x_1)\,(x_0 - x_2)\,(x_0 - x_3)\ldots.\,(x_0 - x_n)$

$a_0 = \dfrac{y_0}{(x_0-x_1)\,(x_0-x_2)\,(x_0-x_3)\ldots.\,(x_0-x_n)}$

When, $\mathrm{x} = \mathrm{x}_1$

$y_1 = a_1(\mathrm{x}_1 - x_0)\,(\mathrm{x}_1 - x_2)\,(\mathrm{x}_1 - x_3)\ldots..\,(\mathrm{x}_1 - x_n)$

$a_1 = \dfrac{y_1}{(\mathrm{x}_1-x_0)\,(\mathrm{x}_1-x_2)\,(\mathrm{x}_1-x_3)\ldots..\,(\mathrm{x}_1-x_n)}$

Similarly,

$a_2 = \dfrac{y_2}{(x_2-x_0)\,(x_2-x_1)\,(x_2-x_3)\ldots...\,(x_2-x_n)}$

……………………………………………………………………………

$a_n = \dfrac{y_n}{(x_n-x_0)\,(x_n-x_1)\,(x_n-x_2)\ldots....\,(x_n-x_{n-1})}$

Substituting these values of a_0, a_1, a_2,………. a_n in equation (i), we get

$$\Phi(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}\,y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}\,y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)}\,y_2 + \ldots\ldots + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})}\,y_n$$

This formula is known as Lagrange's Interpolation Formula.

Divided Differences

Let (x_0, y_0), (x_1, y_1), (x_2, y_2),……….. (x_n, y_n) be the given (n +1) points. Then the divided differences of order 1, 2, 3,…..,n are defined by the relations:

$$\delta\,(x_0, x_1) = \frac{y_1 - y_0}{x_1 - x_0}$$

$$\delta\,(x_0, x_1, x_2) = \frac{\delta(x_1, x_2) - \delta\,(x_0, x_1)}{x_2 - x_0}$$

$$\delta\,(x_0, x_1, x_2, x_3) = \frac{\delta(x_1, x_2, x_3) - \delta\,(x_0, x_1, x_2)}{x_3 - x_0}$$

…………………………………………………………………………………………

$$\delta\,(x_0, x_1, \ldots, x_n) = \frac{\delta(x_1, x_2, \ldots\ldots, x_n) - \delta\,(x_0, x_1, \ldots\ldots, x_{n-1})}{x_n - x_0}$$

Divided Difference Table

x	y	First order differences	Second order differences	Third order differences
x_0	y_0			
		$\delta\ (x_0, x_1)$		
x_1	y_1		$\delta\ (x_0, x_1, x_2)$	
		$\delta\ (x_1, x_2)$		$\delta\ (x_0, x_1, x_2, x_3)$
x_2	y_2		$\delta\ (x_1, x_2, x_3)$	
		$\delta\ (x_2, x_3)$		$\delta\ (x_1, x_2, x_3, x_4)$
x_3	y_3		$\delta\ (x_2, x_3, x_4)$	
		$\delta\ (x_3, x_4)$		$\delta\ (x_2, x_3, x_4, x_5)$
x_4	y_4		$\delta\ (x_3, x_4, x_5)$	
		$\delta\ (x_4, x_5)$		
x_5	y_5			

Newton’s General Interpolation formula

We have, from the definition of divided differences,

δ (x, x_0) = $\frac{y - y_0}{x - x_0}$ ……………………………………………. (a)

δ (x, x_0, x_1) = $\frac{\delta(x, x_0) - \delta(x_0,\ x_1)}{x - x_1}$ …………………………………………………(b)

δ (x, x_0, x_1, x_2) = $\frac{\delta(x, x_0,\ x_1\ ) - \delta\ (x_0, x_1, x_2)}{x - x_2}$ ………………………………(c)

δ (x, x_0, x_1, x_2, x_3) = $\frac{\delta(x, x_0,\ x_1, x_2) - \delta\ (x_0, x_1, x_2, x_3)}{x - x_3}$ ………………………………(d)

<u>From equation (a):</u>

y = y_0 + ($x - x_0$) δ (x, x_0) …………………………(e)

From equation (b):

$\delta(x, x_0) = \delta(x_0, x_1) + (x - x_1)\,\delta(x, x_0, x_1)$ ……………………..(f)

Putting this value in equation (e), we get,

$y = y_0 + (x - x_0)\{\delta(x_0, x_1) + (x - x_1)\,\delta(x, x_0, x_1)\}$ ………………..(g)

From equation (c):

$\delta(x, x_0, x_1) = \delta(x_0, x_1, x_2) + (x - x_2)\,\delta(x, x_0, x_1, x_2)$ ……………………..(h)

Putting this value in equation (g), we get,

$y = y_0 + (x - x_0)\delta(x_0, x_1) + (x - x_0)(x - x_1)\,\delta(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)\,\delta(x, x_0, x_1, x_2)$

……….(i)

From equation (d):

$\delta(x, x_0, x_1, x_2) = \delta(x_0, x_1, x_2, x_3) + (x - x_3)\,\delta(x, x_0, x_1, x_2, x_3)$ …………………………………….(j)

Putting this value in equation (i), we get,

$y = y_0 + (x - x_0)\delta(x_0, x_1) + (x - x_0)(x - x_1)\,\delta(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)\delta(x_0, x_1, x_2, x_3) +$
$(x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\delta(x, x_0, x_1, x_2, x_3)$

This is called Newton's General Interpolation formula

Example 4

The discharge of a hydraulic structure for different values of head (H) is shown in table

H (feet)	1.2	2.1	2.7	4.0
Q (cft/sec.)	25	60	90	155

Calculate the discharge, Q for H = 3 ft. using **Newton's divided formula** and **Lagrange's Interpolation Formula.**

Example 4 continue.......

Sol^n:

We form the Divided Difference Table

x	y	First order differences	Second order differences	Third order differences
1.2	25			
		δ (x_0, x_1) = 38.89		
2.1	60		δ (x_0, x_1, x_2) = 7.41	
		δ (x_1, x_2)= 50		δ (x_0, x_1, x_2, x_3) = -2.65
2.7	90		δ (x_1, x_2, x_3) = 0	
		δ (x_2, x_3) = 50		
4.0	155			

Example 4 continue.......

Newton's General Interpolation formula **H = x = 3**

$y = y_0 + (x - x_0)\delta(x_0, x_1) + (x - x_0)(x - x_1)\,\delta\,(x_0, x_1, x_2) + (x - x_0)(x - x_1)\,(x - x_2)\delta\,(x_0, x_1, x_2, x_3) +$
$(x - x_0)(x - x_1)\,(x - x_2)(x - x_3)\,\delta\,(x, x_0, x_1, x_2, x_3)$

= 25 + (3 – 1.2)*38.89 + (3 – 1.2)(3 – 2.1)*7.41 + (3 – 1.2)(3 – 2.1)(3 – 2.7)*(-2.65)

= 105. 72 cft/sec.

Lagrange's Interpolation Formula

$$y = \frac{(x-x_1)\,(x-x_2)\,(x-x_3)\ldots.\,(x-x_n)}{(x_0-x_1)\,(x_0-x_2)\,(x_0-x_3)\ldots.\,(x_0-x_n)}\,y_0 + \frac{(x-x_0)\,(x-x_2)\,(x-x_3)\ldots..\,(x-x_n)}{(x_1-x_0)\,(x_1-x_2)\,(x_1-x_3)\ldots..\,(x_1-x_n)}\,y_1 +$$

$$\frac{(x-x_0)\,(x-x_1)\,(x-x_3)\ldots\ldots\,(x-x_n)}{(x_2-x_0)\,(x_2-x_1)\,(x_2-x_3)\ldots\ldots\,(x_2-x_n)}\,y_2$$

$=\frac{25}{14} - \frac{600}{19} + \frac{1620}{13}$ + 10.89 =105. 72 cft/sec.

Thank You All