BECM 2203

Numerical Analysis and Computer Programming

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Numerical Integration

- Trapezoidal Rule
- Simpson's Rule

Numerical Integration

❖ The general formula

$$\int_{x_0}^{x_n} y\, dx = nh[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots\ldots] \quad \ldots\ldots\ldots\ldots\text{(i)}$$

❖ Trapezoidal Rule

Setting n = 1 in the general formula(Eq. (i)), all differences higher than the first will become zero and we obtain

$$\int_{x_0}^{x_1} y\, dx = h\,(y_0 + \frac{1}{2}\Delta y_0) = h\,[y_0 + \frac{1}{2}(y_1 - y_0) = \frac{h}{2}(y_0 + y_1) \quad \ldots\ldots\ldots\text{(a)}$$

For the next interval $[x_1, x_2]$, we deduce similarity

$$\int_{x_1}^{x_2} y\, dx = \frac{h}{2}(y_1 + y_2) \quad \ldots\ldots\ldots\ldots\text{(b)}$$

Numerical Integration

❖ Trapezoidal Rule

For the last interval $[x_{n-1}, x_n]$, we have

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}(y_{n-1} + y_n) \qquad \text{..............(c)}$$

Combining all these expressions, we obtain the rule

$$\boldsymbol{\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots\ldots + y_{n-1}) + y_n]}$$

This is known as the trapezoidal rule.

Numerical Integration

❖ Simpson's 1/3- Rule or Simpson's Rule

This rule is obtained by putting n =2 in Eq. (i) and we have

$$\int_{x_0}^{x_2} y\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= 2h\left(y_0 + y_1 - y_0 + \frac{1}{6}y_2 - \frac{2}{6}y_1 + \frac{1}{6}y_0\right) = 2h\left(\frac{1}{6}y_0 + \frac{2}{3}y_1 + \frac{1}{6}y_2\right)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

and finally

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n).$$

Summing up, we obtain

$$\boldsymbol{\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots + y_{n-2}) + y_n].}$$

Whis is known as Simpson's 1/3- Rule or simply Simpson's Rule.

Example -1

Use four segment for integrate y = f(x) = 0.2 + 25 x + $3x^2$ + $2x^4$ from a = 0 to b = 2 by using Trapezoidal rule and Simpson's 1/3-rule and find percentage of error from actual integrated value.

***Sol^n*:**

For four segment n = 4 and h = $\frac{2-0}{4}$ = 0.5 where x will be 0, 0.5, 1, 1.5 and 2.

x	0	0.5	1.0	1.5	2.0
y	0.2	13.58	30.2	54.58	94.2

(i) Trapezoidal rule:

$$\int_0^2 y\,dx = \frac{h}{2}[\,y_0 + 2(y_1 + y_2 + y_3) + y_4]$$

$$= \frac{0.5}{2}[\,0.2 + 2(13.58 + 30.2 + 54.58) + 94.2\,] = 72.78$$

(ii) Simpson's 1/3-rule

$$\int_0^2 y\,dx = \frac{h}{3}\left[\, y_0 + 4(y_1 + y_3\,) + 2y_2 + y_4\right]$$

$= \frac{0.5}{3}$ [0.2 + 4(13.58 + 54.58) + 2*30.2 +94.2] = 71.24

Actual integration value,

$$\int_0^2 y\,dx = \int_0^2 (0.2 + 25\,x + 3x^2 + 2x^4)\,dx = 71.2$$

Percentage of error for Trapezoidal rule = $\left|\frac{71.2-72.78}{71.2}\right|$ x 100% = 2.22%

Percentage of error for Simpson's rule = $\left|\frac{71.2-71.24}{71.2}\right|$ x 100% = 0.056%

Example -2

A reservoir discharging through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h (ft)	10	11	12	13	14
A (ft^2)	950	1070	1200	1350	1530

If t denotes the time in minutes, the rate of fall of the surface is given by

$$\frac{dh}{dt} = -\frac{48}{A}\sqrt{h}$$

Estimate the time taken for the water level to fall from 14 ft to 10 ft above the sluices.

Sol^n:

Area, A = $\frac{h}{3}$ [y_0 + 4(y_1 + y_3) + 2y_2 + y_4]

= $\frac{1}{3}$ [950 + 4(1070 + 1350) + (2*1200) + 1530] = 4853.33 ft^2

Given,

$$\frac{dh}{dt} = - \frac{48}{A}\sqrt{h}$$

- ➢ $\frac{dh}{h^{1/2}} = - \frac{48}{A}$ dt
- ➢ $\int_{10}^{14} h^{-1/2} dh = - \frac{48}{A} \int_{0}^{t} dt$
- ➢ $[\frac{h^{1/2}}{1/2}]_{10}^{14} = - \frac{48}{4853.33}$ t
- ➢ $2\,[\sqrt{14} - \sqrt{10}\,] = - \frac{48}{4853.33}$ t
- ➢ $t = \frac{2\,[\sqrt{14} - \sqrt{10}\,] * 4853.33}{48} = - 117.16$

∴ t = 117. 16 minutes.

Exercise- 1: Use the two-segment trapezoidal rule to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a = 0 to b = 0.8. and estimate the percentage of error from actual integrated value.

Exercise- 2: Use Simpson's rule with n = 4 to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a = 0 to b = 0.8. and estimate the percentage of error from actual integrated value.

Numerical Solution of Ordinary Differential Equations

- **Picard's method**
- **Euler's method and**
- **Runge-Kutta method**

Picard's method of successive approximations

We consider the general first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{..(i)}$$

With the initial condition,

$$y(x_0) = y_0$$

Integrating the differential equation in Eq.(i), we obtain

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \quad \text{....................................(ii)}$$

First approximation: [Putting y_0 for y on right side of Eq.(ii)]

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

Second approximation: [Putting y_1 for y on right side of Eq.(ii)]

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

n^{th} approximation:

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

This method is failed when f (x, y) is not integrable under the given condition.

Example – 1: Using Picard's process of successive approximations, obtain a solution upto the fourth approximation of the equation $\frac{dy}{dx}$ = x + y, such that y = 1 when x = 0.

Sol^n:

Given,

$\frac{dy}{dx}$ = x + y = f(x, y)   with y_0 = 1 and x_0 = 0

1^{st} approximation:

y_1 = y_0 + $\int_{x_0}^{x} f(x, y_0)dx$

= y_0 + $\int_0^x (x + y_0)\, dx$= 1 + $\int_0^x (x + 1)\, dx$= 1 + x + $\frac{x^2}{2}$

2^{nd} approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$= y_0 + \int_0^x (x + \ y_1)\ dx = 1 + \int_0^x \left(x + (1 + \mathrm{x} + \frac{x^2}{2})\right)\ dx = 1 + \mathrm{x} + x^2 + \frac{x^3}{6}$$

3^{rd} approximation:

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

$$= y_0 + \int_0^x (x + \ y_2)\ dx = \ 1 + \int_0^x \left(x + (1 + \mathrm{x} + x^2 + \frac{x^3}{6}\ )\right)\ dx$$

$$= 1 + \mathrm{x} + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

4^{th} approximation:

$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3) dx$

$= y_0 + \int_0^x (x + y_3)\, dx = 1 + \int_0^x \left(x + (1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24})\right) dx$

$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$ (Answer)

Exercise- 1: Use Picard's method to solve the equation y' = x + y^2, subject to the condition y = 1 when x = 0.

Example – 2: **Use Picard's method to approximate the value of y when x = 0.1, given that y = 1 when x = 0 and $\frac{dy}{dx}$ = 3x + y^2.**

***Sol^n*:**

Given,

$\frac{dy}{dx}$ = 3x + y^2 = f(x, y) with initial value y_0 = 1 at x_0 = 0 and x = 0.1.

1^{st} approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= y_0 + \int_0^x (3x + y_0{}^2)\, dx = 1 + \int_0^x (3x + 1)\, dx = 1 + x + \frac{3x^2}{2}$$

2^{nd} approximation:

$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$

- $y_2 = y_0 + \int_0^x (3x + {y_1}^2)\, dx = 1 + \int_0^x \left(3x + (1 + x + \frac{3x^2}{2})^2\right) dx = 1 + \int_0^x \left(3x + (1 +\right.$

Since $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Euler's method

Suppose the general first order differential equation is

$$\frac{dy}{dx} = f(x, y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

With the initial condition,

$$y(x_0) = y_0$$

Taking $x_n = x_0 + nh$ (n = 1, 2,……) and Integrating the differential equation in Eq.(i), we obtain

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Assuming that f(x, y) = f(x_0, y_0) in $x_0 \leq x \leq x_1$, this gives Euler's formula

$$\mathbf{y_1 \approx y_0 + hf(x_0, y_0)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(a)}$$

Euler's method

Similarly for the range $x_1 \leq x \leq x_2$, we have

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x,y)dx$$

Substituting f(x_1, y_1) for f(x, y) in $x_1 \leq x \leq x_2$ we obtain

$$\mathbf{y_2 = y_1 + hf(x_1, y_1)} \qquad \text{.................................(b)}$$

Proceeding in this way, we obtain the general formula

$$\mathbf{y_{n+1} = y_n + hf(x_n, y_n),} \qquad n = 0, 1, 2, \ldots.. \qquad \text{.................................(b)}$$

It is very slow method for better accuracy taking a smaller value for h. Because of this restriction on h, this method is unsuitable for practical use and a modification of it, known as the modified Euler method, which gives more accurate results.

Example – 1: Using Euler's method compute y(0.04) for the differential equation $\frac{dy}{dx}$ = -y, with the condition y(0) = 1, take h = 0.01.

$\boldsymbol{Sol^n}$**:**

Given,

$\frac{dy}{dx}$ = -y = f(x, y) with initial condition, y_0= 1 at x_0= 0 and x = 0.04 , h = 0.01 .

Taking n = $\frac{x - x_0}{h}$ = $\frac{0.04 - 0}{0.01}$ = 4

Therefore, x_1= x_0 + h = 0 + 0.01 = 0.01, x_2= x_0 + 2h = 0.02, x_3= x_0 + 3h = 0.03 and

x_4= x_0 + 4h = 0.04

1^{st} approximation:

y_1= y_0 + hf(x_0, y_0)

y_1= y_0 + h(-y_0) = 1 + 0.01(-1) = 0.99 at x_1= 0.01

2^{nd} approximation:

y_2= y_1 + hf(x_1, y_1)

y_2= y_1 + h(-y_1) = 0.99 + 0.01(- 0.99) = 0.9801 at x_1= 0.02

3^{rd} approximation:

y_3= y_2 + hf(x_2, y_2)

y_3= y_2 + h(-y_2) = 0.9801 + 0.01(- 0.9801) = 0.9703 at x_1= 0.03

4^{th} approximation:

y_4= y_3 + hf(x_3, y_3)

y_4= y_3 + h(-y_3) = 0.9703 + 0.01(- 0.9703) = 0.9606 at x_1= 0.04

Therefore the value of **y at x = 0.04 is 0.9606.**

Exercise- 1: **Use Euler's method to find y(0.2) from the differential equation $\frac{dy}{dx} = xy$, y(0) = 1, take h = 0.1 .**

Modified Euler's method

1^{st} approximation:

By Euler's method we have,

$$y_1 = y_0 + hf(x_0, y_0)$$

Applying Euler modified formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1)]$$

[$y_1^{(1)}$ = first modified value of 1^{st} approximation]

$$y_1^{(2)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

[$y_1^{(2)}$ = 2^{nd} modified value of 1^{st} approximation]

$$y_1^{(3)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

……………………………………………………………

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(n)})]$$

Continuing this process when until two consecutive values are nearly equal, such that $y_1^{(3)} \approx y_1^{(2)} = y_1$ (Modified or improved y_1)

Modified Euler's method

2^{nd} approximation:

Using improved value of x_1 and y_1.

By Euler's method we have,

$$y_2 = y_1 + hf(x_1, y_1)$$

Applying Euler modified formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$ [$y_2^{(1)}$ = first modified value of 2^{nd} approximation]

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$ [$y_2^{(2)}$ = 2^{nd} modified value of 2^{nd} approximation]

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

…………………………………………………………….

$$y_2^{(n+1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(n)})]$$

Continuing this process when until two consecutive values are nearly equal, such that $y_2^{(3)} \approx y_2^{(2)} = y_2$ (Modified or improved y_1) at x_2.

For better accuracy taking at least n = 2.

Example – 1: **Given $\frac{dy}{dx} = 1 + \frac{y}{x}$**, y = 2 at x =1. Find approximate value of y at x = 1.4 by taking step size h = 0.2; apply modified Euler's method.

***Sol^n*:**

Given,

$\frac{dy}{dx}$ = $\mathbf{1} +$ $\frac{y}{x}$ = f(x, y)   with initial condition, y_0= 2 at x_0= 1 and x = 1.4 , h = 0.2 .

Taking n = $\frac{x - x_0}{h}$ = $\frac{1.4 - 1}{0.2}$ = 2

Therefore, x_1= x_0 + h = 1.2, x_2= x_0 + 2h = 1.4

1^{st} approximation:

By Euler's method ,

y_1= y_0 + hf(x_0, y_0)

- y_1= 2 + 0.2 f(1, 2) = 2 + 0.2(1 + $\frac{2}{1}$) = 2.6
- **y_1 = 2.6   at x_1 = 1.2**

By Euler modified formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1)]$$

- $y_1^{(1)} = 2 + \frac{0.2}{2}[\,f(1, 2) + f(1.2, 2.6)] = 2 + 0.1\,[(1 + \frac{2}{1}) + (1 + \frac{2.6}{1.2})] = 2.6167$
- $y_1^{(1)}$ **= 2.6167**

$$y_1^{(2)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

- $y_1^{(2)} = 2 + \frac{0.2}{2}[\,f(1, 2) + f(1.2, 2.6167)] = 2 + 0.1\,[(1 + \frac{2}{1}) + (1 + \frac{2.6167}{1.2})] = 2.61806$
- $y_1^{(2)}$ **= 2.61806**

$$y_1^{(3)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$y_1^{(3)} = 2 + \frac{0.2}{2}[\,f(1, 2) + f(1.2, 2.61806)] = 2 + 0.1\,[(1 + \frac{2}{1}) + (1 + \frac{2.61806}{1.2})] = 2.61817$

$y_1^{(3)}$ **= 2.61817**

Say $y_1^{(3)} = y_1^{(2)}$ **= 2.618 =** y_1 **at** x_1 **= 1.2**

2^{nd} approximation:

Using improved value of y_1 = 2.618 at x_1 = 1.2.

By Euler's method,

$$y_2 = y_1 + hf(x_1, y_1)$$

- y_2= 2.618 + 0.2 f(1.2, 2.618) = 2.618 + 0.2$(1 + \frac{2.618}{1.2})$ = 3.2543
- y_2 = 3.2543 at x_2 = 1.4

By Euler modified formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

- $y_2^{(1)} = 2.618 + \frac{0.2}{2}$ [f(1.2, 2.618) + f(1.4, 3.2543)] = 2.618 + 0.1 $[(1 + \frac{2.618}{1.2}) + (1 + \frac{3.2543}{1.4})]$ = 3.2686
- $y_2^{(1)}$ **= 3.2686**

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

- $y_2^{(2)} = 2.618 + \frac{0.2}{2}$ [f(1.2, 2.618) + f(1.4, 3.2686)] = 2.618 + 0.1 $[(1 + \frac{2.618}{1.2}) + (1 + \frac{3.2686}{1.4})]$ = 3.2696
- $y_2^{(2)}$ **= 3.2696**

$y_2^{(3)} = y_1 + \frac{h}{2}$ [f(x_1, y_1) + f($x_2, y_2^{(2)}$)]

- $y_2^{(3)}$ = 2.618 +$\frac{0.2}{2}$ [f(1.2, 2.618) + f(1.4, 3.2696)] = 2.618 + 0.1 [(1 + $\frac{2.618}{1.2}$) + (1 + $\frac{3.2696}{1.4}$)] = 3.2697
- $y_2^{(3)}$ = **3.2697**

Say $y_2^{(3)} = y_2^{(2)}$ = 3.269 at x_2 = **1.4 .**

Hence, y (1.4) = 3.269 (Ans.)

Example – 2: Use Euler's modified form to determine the value of y when x = 0. 1 given that y(0) = 1 and y'= x^2 + y, taken h = 0.05.

$\boldsymbol{Sol^n}$:

Given,

$\frac{dy}{dx}$ = x^2 + y = f(x, y) with initial condition, y_0= 1 and x_0= 0

and x = 0.1 , h = 0.05. [Given]

Therefore, x_1= 0.05 and x_2= 0.1

1^{st} approximation:

By Euler's method ,

y_1= y_0 + hf(x_0, y_0)

- y_1= 1 + 0.05 f(0, 1) = 1 + 0.05(0 + 1) = 1.05
- **y_1 = 1.05 at x_1 = 0.05**

By Euler modified formula,

$$\boldsymbol{y_1}^{(1)} = \boldsymbol{y_0} + \frac{\boldsymbol{h}}{\boldsymbol{2}} [\, \mathbf{f}(\boldsymbol{x_0}, \boldsymbol{y_0}) + \mathbf{f}(\boldsymbol{x_1}, \boldsymbol{y_1})]$$

- $y_1^{(1)}$ = 1 + $\frac{0.05}{2}$ [f(0, 1) + f(0.05, 1.05)] = 1 + 0.025 [(0 + 1) + (0.05^2 + 1.05)] = 1.0513
- $y_1^{(1)}$ = 1.0513

$$\boldsymbol{y_1}^{(2)} = \boldsymbol{y_0} + \frac{\boldsymbol{h}}{\boldsymbol{2}} [\, \mathbf{f}(\boldsymbol{x_0}, \boldsymbol{y_0}) + \mathbf{f}(\boldsymbol{x_1}, \boldsymbol{y_1}^{(1)})]$$

- $y_1^{(2)}$ = 1 + $\frac{0.05}{2}$ [f(0, 1) + f(0.05, 1.0513)] = 1 + 0.025 [(0 + 1) + (0.05^2 + 1.0513)] = 1.0513
- $y_1^{(2)}$ = 1.0513

Say $y_1^{(2)}$ = $y_1^{(1)}$ = 1.0513 = y_1 at x_1 = 0.05

2^{nd} approximation:

Using improved value of y_1 = 1.0513 at x_1 = 0.05.

By Euler's method,

$y_2 = y_1 + hf(x_1, y_1)$

- y_2= 1.0513 + 0.05 f(0.05, 1.0513) = 1.0513 + 0.05(0.05^2 + 1.0513) = 1.1040
- y_2 = 1.1040 **at x_2 = 0.1**

By Euler modified formula,

$y_2^{(1)} = y_1 + \frac{h}{2} [\, f(x_1, y_1) + f(x_2, y_2)]$

- $y_2^{(1)} = 1.0513 + \frac{0.05}{2} [\, f(0.05, 1.0513) + f(0.1, 1.1040)] = 1.0513 + 0.025\, [(0.05^2 + 1.0513) + (0.1^2 + 1.1040)]$
- $\boldsymbol{y_2^{(1)}}$ **= 1.1055**

$\boldsymbol{y_2^{(2)} = y_1 + \frac{h}{2} [\, f(x_1, y_1) + f(x_2, y_2^{(1)})]}$

- $y_2^{(2)} = 1.0513 + \frac{0.05}{2} [\, f(0.05, 1.0513) + f(0.1, 1.1055)] = 1.0513 + 0.025\, [(0.05^2 + 1.0513) + (0.1^2 + 1.1055)]$
- $\boldsymbol{y_2^{(2)}}$ **= 1.1055**

Say $y_2^{(2)} = y_2^{(1)} = 1.1055$ at $x_2 = 0.1$.

Hence, the value of y when x = 0. 1 is 1.1055 (Ans.)

Runge-Kutta method

Suppose the general first order differential equation is

$$\frac{dy}{dx} = f(x, y) \quad \text{...(i)}$$

With the initial condition,

$$y(x_0) = y_0$$

If the initial values of the variables are x_0 and y_0; the first increment is

$$x_1 = x_0 + h, \qquad y_1 = y_0 + \Delta y$$

…………………………………………………….

$$x_n = x_0 + nh, \qquad y_n = y_0 + n\Delta y \qquad (n = 1, 2, \ldots\ldots)$$

1^{st} approximation or 1^{st} interval:

k_1 = f (x_0, y_0)h

k_2 = f$(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$h

k_3 = f$(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$h

k_4 = f $(x_0$ + h, $y_0 + k_3)$h

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Then,

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

2^{nd} approximation or 2^{nd} interval:

k_1 = f (x_1, y_1)h

k_2 = f$(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$h

k_3 = f$(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$h

k_4 = f $(x_1$ + h, $y_1 + k_3)$h

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Then,

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Example – 1: Given $\frac{dy}{dx} = 1 + y^2$, where y = 0 when x = 0, find y(0.2), y(0.4) and y(0.6) by using Runge-Kutta method.

Sol^n:

Given,

$\frac{dy}{dx} = 1 + y^2 = f(x, y)$ with initial condition, $y_0 = 0$ and $x_0 = 0$

h = 0.4 – 0.2 = 0.2

$x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

1^{st} approximation:

$k_1 = f(x_0, y_0)h$

- $k_1 = f(0,0) * 0.2 = 0.2(1 + 0^2) = 0.2$
- $k_1 = 0.2$

$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})h$

- $k_2 = f(0 + \frac{.2}{2}, 0 + \frac{.2}{2}) * 0.2 = 0.2\, f(0.1, 0.1) = 0.2(1 + 0.1^2) = 0.202$
- $k_2 = 0.202$

k_3 = f($x_0 + \frac{h}{2}$, $y_0 + \frac{k_2}{2}$)h

- k_3 = f(0+ $\frac{.2}{2}$, 0+ $\frac{0.202}{2}$)* 0.2 = 0.2 f(0.1, 0.101) = 0.2(1 + 0.101^2) = 0.20204
- k_3 = 0.20204

k_4 = f (x_0 + h, y_0+ k_3)h

- k_4 = f (0+ 0.2, 0+.20204)* 0.2 = 0.2 (1 + 0.20204^2) = 0.20816
- k_4 = 0.20816

∴ $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ = 0 + $\frac{1}{6}$(0.2 + 2(0.202) + 2(0.20204) +0.20816)

y_1 = 0.2027 at x_1= 0.2

2^{nd} approximation (with y_1 = 0.2027 at x_1= 0.2):

k_1 = f (x_1, y_1)h

- k_1 = f (0.2, 0.2027)* 0.2 = 0.2 (1 + 0.2027^2) = 0.2082
- k_1 = 0.2082

k_2 = f($x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}$)h

- k_2 = f(0.2+ $\frac{.2}{2}$, 0.2027 + $\frac{0.2082}{2}$)*0.2 = 0.2 f(0.3, 0.3068) = 0.2 (1 + 0.3068^2) = 0.2188
- k_2 = 0.2188

k_3 = f($x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}$)h

- k_3 = f(0.2+ $\frac{.2}{2}$, 0.2027 + $\frac{0.2188}{2}$)*0.2 = 0.2 f(0.3, 0.3121) = 0.2 (1 + 0.3121^2) = 0.2195
- k_3 = 0.2195

k_4 = f $(x_1$ + h, y_1+ $k_3)$h

- k_4 = f (0.2+ 0.2, 0.2027+ 0.2195)* 0.2 = 0.2f(0.4, 0.4222) = 0.2 (1 + 0.4222^2) = 0.2356
- k_4 = 0.2356

$\therefore \; y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

y_2 = 0.2027 + $\frac{1}{6}$ [0.2082+ 2(0.2188)+ 2(0.2195)+ 0.2356] = 0.4228

y_2 = 0.4228 at x_2= 0.4

3^{rd} approximation (with y_2 = 0.4228 at x_2= 0.4):

k_1 = f (x_2, y_2)h

- k_1 = f (0.4, 0.4228)* 0.2 = 0.2 (1 + 0.4228^2) = 0.2357
- k_1 = 0.2357

k_2 = f($x_2 + \frac{h}{2}$, $y_2 + \frac{k_1}{2}$)h

- k_2 = f(0.4+ $\frac{.2}{2}$, 0.4228 + $\frac{0.2357}{2}$)*0.2 = 0.2 f(0.5, 0.5406) = 0.2 (1 + 0.5406^2) = 0.2585
- k_2 = 0.2584

k_3 = f($x_2 + \frac{h}{2}$, $y_2 + \frac{k_2}{2}$)h

- k_3 = f(0.4+ $\frac{.2}{2}$, 0.4228 + $\frac{0.2584}{2}$)*0.2 = 0.2 f(0.5, 0.552) = 0.2 (1 + 0.552^2) = 0.2609
- k_3 = 0.2609

k_4 = f (x_2 + h, y_2+ k_3)h

- k_4 = f (0.4+ 0.2, 0.4228 + 0.2609)* 0.2 = 0.2f(0.6, 0.6837) = 0.2 (1 + 0.6837^2) = 0.2935
- k_4 = 0.2935

$\therefore \quad y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

- $y_3 = 0.4228 + \frac{1}{6}[0.2357 + 2(0.2584) + 2(0.2609) + 0.2935] = 0.6841$
- $y_3 = 0.6841$ at $x_3 = 0.6$

Therefore, y(0.2) = 0.2027, y(0.4) = 0.4228 and y(0.6) = 0.6841 (Ans.)

Exercise- 1: Use Runge – Kutta method to solve the equation $\frac{dy}{dx}$ = 1 +y^2 for x = 0.2 to x = 0.4 with h = 0.2, given that y(0) = 0.5.

Answer: y(0.2) = 0.7819 and y(0.4) = 1.1699

Exercise- 2: Use Runge – Kutta method of fourth order to find approximate value y when x = 0.1, given that y = 1 at x = 0 and h = 0.1 for $\frac{dy}{dx}$ = 3x + y^2.

Answer: y(0.1) = 1.12725

Thank You All