Laplace Transformation

01. Elementary functions:

* $L(1) = \frac{1}{s}$
* $L(t) = \frac{1}{s^2}$
* 2014 $L(t^n) = \frac{n!}{s^{n+1}}$
* $L(e^{at}) = \frac{1}{s-a}$
* $L\{\sin at\} = \frac{a}{s^2+a^2}$
* $L\{\cos at\} = \frac{s}{s^2+a^2}$
* $L\{\sinh at\} = \frac{a}{s^2-a^2}$
* $L\{\cosh at\} = \frac{s}{s^2-a^2}$
* $L\{t \sin at\} = \frac{2as}{(s^2+a^2)^2}$; 2018, Q(4)
* $L\{t \cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$
* $L\{\cancel{t}\, e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}$; 2018, Q(3)
* $L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

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* $e^{iax} = \cos ax + i \sin ax$
* $\int e^{ax} \sin bx \, dx$
  $= \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$
* $\int e^{ax} \cos bx \, dx$
  $= \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$
* $\sinh at$
  $= \frac{1}{2}(e^{at} - e^{-at})$
* $\cosh at$
  $= \frac{1}{2}(e^{at} + e^{-at})$

$L\{t^n\} = \frac{n!}{s^{n+1}}$; $s>0$

Ans:
By the definition of the laplace transform of a function F(t). we have,

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt \quad \text{--(1)}$$

$F(t) = t^n$,

$$\therefore L\{t^n\} = \int_0^\infty e^{-st}.t^n dt$$

$$= t^n \int_0^\infty e^{-st} - \int\left[\frac{d}{dt}t^n \int e^{-st}\right].dt$$

$$= t^n \left[\frac{e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int t^{n-1}.e^{-st}$$

$$= 0 + \frac{n}{s}\left[t^{n-1}\frac{e^{-st}}{-s}\right]_0^\infty + \frac{n(n-1)}{s^2}\int t^{n-2}.e^{-st}dt$$

$$= \frac{n(n-1)(n-2)}{s^3}\int t^{-3}.e^{-st} dt$$

repeating the process,

$$L\{t^n\} = \frac{n(n-1)(n-2)(n-3)\ldots 3.2.1}{s^n}\int_0^\infty t^0.e^{-st}dt$$

$$= \frac{n!}{s^n}\int_0^\infty e^{-st}.1\,dt$$

$$= \frac{n!}{s^n}\frac{1}{s}$$

$$= \frac{n!}{s^{n+1}}$$

Ans

* $L\{\sin at\}$

By the defenition of Laplace transformation

$L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)\, dt$

$F(t) = \sin at$

$\therefore L\{\sin at\} = \int_0^\infty e^{-st} \cdot \sin at\, dt$

$$= \left[\frac{e^{-st}(-s\sin ax - a\cos ax)}{s^2+a^2}\right]_0^\infty$$

$$= \frac{0 - (-a\cos a\times 0)}{s^2+a^2}$$

$\therefore L\{\sin at\} = \frac{a}{s^2+a^2}$

Ans:

* $L\{t\sin at\}$

Soln:

$L\{t\sin at\} = \int_0^\infty e^{-st}\, t\sin at\, dt$

taking t and $(e^{-st}\cdot \sin at)$ differently,

$$= \left\{ t \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty - \int \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]\right.$$

$$= 0 + \int \frac{e^{-st}}{s^2+a^2}(s\sin at + a\cos at)\, dt$$

$$= \frac{s}{a^2+s^2}\int e^{-st}\sin at\, dt + \frac{a}{s^2+a^2}\int a \cdot e^{-st}\cos at\, dt$$

$$= \frac{s}{a^2+s^2}\cdot\frac{a}{a^2+s^2} + \frac{a}{s^2+a^2}\cdot\frac{s}{(s^2+a^2)} = \frac{2as}{(a^2+s^2)^2}$$

Ans

$L\{e^{at} \sin bt\}$

Soln:

$L\{e^{at} . \sin bt\}$

$$= \int_0^\infty e^{-st} . e^{at} \sin bt \, dt$$

$$= \int_0^\infty e^{(a-s)t} . \sin bt \, dt$$

$$= \int_0^\infty e^{-(s-a)t} . \sin bt . dt$$

$$= \left[\frac{e^{-(s-a)t}}{(s-a)^2+b^2} \left(-(s-a) \sin bt - b \cos bt\right) \right]_0^\infty$$

$$= \frac{b}{(s-a)^2+b^2}$$ Ans:

02. Properties of laplace transformation:

(i) Linearity property

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$

$$= c_1 F_1(s) + c_2 F_2(s)$$

where $F_1(s)$ and $F_2(s)$ are linear transforms of $F_1(t)$ and $F_2(t)$ respectively

$e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t$

By applying the linearity property,

$L\{e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t\}$

$= L\{e^{4t}\} + 4L\{t^3\} - 2L\{\sin 3t\} + 3L\{\cos 5t\}$

$= \frac{1}{s-4} + 4\cdot\frac{6}{s^4} - 2\cdot\frac{3}{s^2+9} + 3\cdot\frac{s}{s^2+25}$

$= \frac{1}{s-4} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+25}$ Ans.

(ii) First Translation (or Shifting Property):

if $L\{F(t)\} = F(s)$ then $L\{e^{at} F(t)\} = F(s-a)$

2013, 2014 Q7

* Find the laplace tranformation of $t^3 e^{5t}$

Soln:

$$L\{t^3\} = \frac{6}{s^4} = F(s)$$

by First shifting method,

$$L\{e^{5t} t^3\} = F(s-5) = \frac{6}{(s-5)^4} \quad \text{Ans}$$

2017

* Prove, (i) $L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$ (ii) $L\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$

Soln:

$L\{t\} = 1/s^2 = F(s)$

$$\therefore L\{t e^{ait}\} = F(s-ia) = \frac{1}{(s-ai)^2}$$

$$\Rightarrow L\{t\cos at + it\sin at\} = \frac{(s+ai)^2}{(s-ai)^2(s+ai)^2} = \frac{(s+ai)^2}{s^2+a^2} = \frac{s^2+2asi-a^2}{(s^2+a^2)}$$

$$\Rightarrow L\{t\cos at\} + iL\{t\sin at\} \longrightarrow = \frac{(s^2-a^2)+i2as}{(s^2+a^2)}$$

$$L\{t\cos at\} = \frac{s^2-a^2}{s^2+a^2}; \quad L\{t\sin at\} = \frac{2as}{s^2+a^2}$$

(iii) Second Translation or (Shifting):

2018, 2015 * If $L\{F(t)\} = F(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$ then show that $L\{G(t)\} = e^{-as} F(s)$

Proof:

We have,

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t)\, dt$$

$$= \int_0^a e^{-st} G(t) + \int_a^\infty e^{-st} G(t)\, dt$$

$$= \int_0^a e^{-st} . 0 . dt + \int_a^\infty e^{-st} F(t-a)\, dt$$

$$= \int_a^\infty e^{-st} F(t-a)\, dt \qquad \left| \begin{array}{l} t-a=u \\ \Rightarrow dt = du \end{array} \right.$$

$$= \int_0^\infty e^{-s(u+a)} F(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} . F(u)\, du$$

$$= e^{-as} . F(s)$$

$= R.H.S$

(Proved)

\# Find the Laplace transform of $F(t)$, where,

$$F(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$$

Soln:

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^{2\pi/3} e^{-st} F(t)\, dt + \int_{2\pi/3}^\infty e^{-st} F(t)\, dt$$

$$= 0 + \int_{2\pi/3}^\infty e^{-st} . \cos(t - 2\pi/3)\, dt$$

Now, $t - 2\pi/3 = u$

$dt = du$

$= \int_0^{\infty} e^{-s(u+2\pi/3)} \cos u \, du$

$= e^{-2\pi s/3} \int_0^{\infty} e^{-us} . \cos u \, du = e^{-2\pi/3} L\{\cos u\}$

$= e^{-2\pi s/3} \dfrac{s^2}{s^2+1}$ Ans:

(iv) * The change of scalar property:

* $L\{F(t)\} = F(s)$, then $L\{F(at)\} = 1/a \; F(s/a)$

* Find the Laplace transform of $\cos 5t$

$\therefore L\{\cos t\} = \dfrac{s}{s^2+1}$

$\therefore L\{\cos 5t\} = 1/5 \; \dfrac{s/5}{s^2/25+1} = \dfrac{s}{s^2+25}$ Ans!

The Inverse Laplace transform

if $L\{F(t)\} = F(s)$ then $F(t) = L^{-1}\{F(s)\}$

(i) Linearity Property:

$\star$ find, $L^{-1}\left\{\frac{2(s-a)}{(s-a)^2+b^2} + \frac{8-6s}{16s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\}$

$$= L^{-1}\left\{\frac{2(s-a)}{(s-a)^2+b^2}\right\} + L^{-1}\left\{\frac{8-6s}{16s^2+9}\right\} + L^{-1}\left\{\frac{24-30\sqrt{s}}{s^4}\right\}$$

$$= 2L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} + L^{-1}\left\{\frac{8}{16s^2+9}\right\} - L^{-1}\left\{\frac{6s}{16s^2+9}\right\} + L^{-1}\left\{\frac{24}{s^4}\right\} - L^{-1}\left\{\frac{30\sqrt{s}}{s^4}\right\}$$

$$= 2e^{at}\cos bt + \frac{1}{2}L^{-1}\left\{\frac{1}{s^2+9/16}\right\} - \frac{3}{8}L^{-1}\left\{\frac{s}{s^2+9/16}\right\} + 24\{\frac{1}{s^4}\} - 30\{\frac{1}{s^{7/2}}\}$$

$$= 2e^{at}\cos bt + \frac{1}{2}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4} + \frac{24}{6}\cdot t^3 - \frac{30\,t^{5/2}}{5/2!}$$

Ans:

(ii) First Translation (shifting Property):

If $L^{-1}\{F(s)\} = F(t)$, then $L^{-1}\{F(s-a)\} = e^{at} F(t)$

2015, 2017

* $L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\}$

$= L^{-1}\left\{\frac{s+3-2}{(s+3)^2+16}\right\}$

$= L^{-1}\left\{\frac{s+3}{(s+3)^2+16}\right\} - 2L^{-1}\left\{\frac{1}{(s+3)^2+16}\right\}$

$= e^{-3t}\cos 4t - 2e^{-3t}\frac{\sin 4t}{4}$

$= e^{-3t}\left[\cos 4t - \frac{2\sin 4t}{4}\right]$ Ans:

*** * $L^{-1}\left\{\frac{1}{s^2+8s+16}\right\}$

$= L^{-1}\left\{\frac{1}{(s+4)^2}\right\}$ $[s+4 = s-(-4) = (s-a)]$

$= e^{-4t} L^{-1}\{1/s^2\}$

$= e^{-4t}\, t$ Ans:

* $L^{-1}\left\{\frac{s+3}{4s^2+4s+1}\right\}$

$= L^{-1}\left\{\frac{s+3}{4(s^2+s+\frac{1}{4})}\right\}$

$= \frac{1}{4}L^{-1}\left\{\frac{s+3}{(2s+1)^2}\right\}$

$= \frac{1}{4}L^{-1}\left\{\frac{s+3}{(s+1/2)^2}\right\}$

$= \frac{1}{4}L^{-1}\left\{\frac{s}{(s+1/2)^2}\right\} + \frac{1}{4}L^{-1}\left\{\frac{3}{(s+1/2)^2}\right\}$

$= \frac{1}{4}L^{-1}$

$\frac{s+3}{4\; s^2+s+\frac{1}{4}}$

$= \frac{s+3}{}$

2014

$*\ L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$= L^{-1}\left\{\frac{6s-12+8}{s^2-4s+4+16}\right\}$

$= L^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$

$= 6L^{-1}\left\{\frac{(s-2)}{(s-2)^2+16}\right\} + 2L^{-1}\left\{\frac{4}{(s-2)^2+16}\right\}$

$= 6\, e^{2t} \cos 4t + 2\, e^{2t} \sin 4t$ Ans.

$*\ L^{-1}\left\{\frac{s+1}{6s^2+7s+2}\right\}$

$= L^{-1}\left\{\frac{s+1}{6s^2+3s+4s+2}\right\}$

$= L^{-1}\left\{\frac{s+1}{(3s+2)(2s+1)}\right\}$

$= L^{-1}\left\{\frac{1}{2s+1} - \frac{1}{3s+2}\right\}$

$= L^{-1}\left\{\frac{1}{2s+1}\right\} - L^{-1}\left\{\frac{1}{3s+2}\right\}$

$= \frac{1}{2} L^{-1}\left\{\frac{1}{s+1/2}\right\} - \frac{1}{3} L^{-1}\left\{\frac{1}{s+2/3}\right\}$

$= \frac{1}{2} e^{-1/2\, t} - 1/3\, e^{-2/3\, t}$ Ans.

(iii) Second Translation (or shifting) property:

If $L^{-1}\{F(s)\} = f(t)$ then, $L^{-1}\{e^{-as} F(s)\} = \begin{cases} f(t-a) ; & t>a \\ 0 ; & t<a \end{cases}$

2017 # Evaluate $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$

soln:

Since, $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du = \left[-\cos u\right]_0^t = 1 - \cos t$

Again, $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)\,du = t - \sin t$

Again, $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (u - \sin u)\,du = \frac{t^2}{2} + \cos t - 1$ Ans.

Partial fraction:

* Find: $L^{-1}\left\{\frac{s^2+2}{s(s^2+4)}\right\}$

$=L^{-1}\left\{\frac{s^2+4-2}{s(s^2+4)}\right\}$

$=L^{-1}\left\{\frac{1}{s}-\frac{2}{s(s^2+4)}\right\}$

$=L^{-1}\left\{\frac{1}{s}-2\cdot\frac{1}{s(s^2+4)}\right\}$

$\cancel{=L^{-1}\left\{\frac{1}{s}\right\}-2L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}}$

$=L^{-1}\left\{\frac{1}{s}-\frac{2}{4}\cdot\frac{s^2+4-s^2}{s(s^2+4)}\right\}$

$=L^{-1}\left\{\frac{1}{s}-\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right\}$

$=L^{-1}\frac{1}{s}-\frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{2}L^{-1}\left\{\frac{s}{s^2+4}\right\}$

$=\frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{2}L^{-1}\left\{\frac{s}{s^2+4}\right\}$

$=\frac{1}{2}\,1.+\frac{1}{2}\cos 2t$

$=\frac{1}{2}(1+\cos 2t)$ Ans.

2018

* $L^{-1}\left\{\frac{2s+5}{(s^2+1)(s-4)}\right\}$

$=$

ODE:

Application of Laplace Transform

2014] solve: $y'' - 3y' + 2y = 2e^{-t}$; $y(0) = 2$, $y'(0) = -1$

$L\{F''(t)\} = s^2 F(s) - sF(0) - F'(0)$

$L\{F'(t)\} = s\,F(s) - F(0)$

$L\{t\,F(t)\} = -F'(s)$

Soln:

$L\{y''\} - 3L\{y'\} + 2L\{y\} = 2L\{e^{-t}\}$

$\Rightarrow s^2 Y(s) - s\,Y(0) - Y'(0) - 3\left(s\,Y(s) - Y(0)\right) + 2\,Y(s) = \frac{2}{s+1}$

$\Rightarrow s^2 Y(s) - 2s + 1 - 3\,(s\,Y(s) - 2) + 2Y(s) = \frac{2}{(s+1)}$

$\Rightarrow s^2 Y(s) - 2s + 1 - 3s\,Y(s) + 6 + 2Y(s) = \frac{2}{(s+1)}$

$\Rightarrow Y(s)\,[s^2 - 3s + 2] = 2s - 7 + \frac{2}{(s+1)}$

$\Rightarrow Y(s)\,(s-1)(s-2) = 2s - 7 + \frac{2}{s+1}$

$\Rightarrow Y(s) = \frac{2s}{(s-1)(s-2)} - \frac{7}{(s-1)(s-2)} + \frac{2}{(s-1)(s-2)(s+1)}$

⋮

partial fraction এর মতো

2018] $ty'' + y' + 4ty = 0$ wher $y'(0) = 0$, $y(0) = 3$

Doint the laplace transform on both sides,

$L\{ty''\} + L\{y'\} + 4L\{ty\} = 0$

$$\Rightarrow -\frac{d}{ds}\{s^2Y(s) - sY(0) - Y'(0)\} + \{sY(s) - Y(0)\} - 4\frac{dY}{ds} = 0$$

$$\Rightarrow -\frac{d}{ds}\{s^2Y(s) - 3s - 0\} + \{sY(s) - 3\} - 4\frac{dY}{ds} = 0$$

$$\Rightarrow -2sY(s) - s^2\frac{dY}{ds} + \cancel{3} + sY - \cancel{3} - 4\frac{dY}{ds} = 0$$

$$\Rightarrow -s^2\frac{dY}{ds} - 4\frac{dY}{ds} - sY = 0$$

$$\Rightarrow (s^2+4)\frac{dY}{ds} + sY = 0$$

$$\Rightarrow (s^2+4)\frac{dY}{ds} = -sY$$

$$\Rightarrow (s^2+4)\,dY = -ds.\,sY$$

$$\Rightarrow \frac{dY}{Y} + \frac{1}{2}\frac{2s\,ds}{s^2+4} = 0$$

Integrating,

$$\log Y + \frac{1}{2}\log(s^2+4) = \log C$$

$$\Rightarrow \log Y = \log\frac{C}{\sqrt{s^2+4}}$$

$$\therefore Y = \frac{C}{\sqrt{s^2+4}}$$

Inverting $= CJ_0(at)$ $\quad\because L^{-1}\left\{\frac{1}{\sqrt{s^2+a^2}}\right\} = J_0(at)$

PDE: For the function $U(x,t)$

* $L\left\{\frac{\partial u}{\partial t}\right\} = su(x,s) - U(x,0) = su - U(x,0)$.

where, $u = u(x,s) = L\{U(x,t)\}$

* $L\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial u}{\partial x}$

* $L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 u(x,s) - sU(x,0) - U_t(x,0)$

* $L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2 u}{dx^2}$

~~2018~~, 2013, ~~2018~~
2017

Solve the PDE $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ $u(0,t)=0$, $U(3,t)=0$,

$u(x,0) = 10\sin 2\pi x - 6\sin 4\pi x$

Ans: Here, $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$

$\Rightarrow L\left\{\frac{\partial u}{\partial t}\right\} = 4L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$

$\Rightarrow su(x,s) - u(x,0) = 4\frac{\partial^2 u}{\partial x^2}$

$\Rightarrow 4\frac{\partial^2 u}{\partial x^2} - su = -u(x,0)$

$\Rightarrow \frac{d^2 u}{dx^2} - \frac{s}{4}u = -\frac{1}{4}u(x,0) = -\frac{1}{4}(10\sin 2\pi x - 6\sin 4\pi x)$

Aux. eq: $m^2 - \frac{s}{4} = 0$

$\therefore m = \pm\sqrt{s/4}$

$\therefore C.F = c_1 e^{\sqrt{s/4}\,x} + c_2 e^{-\sqrt{s/4}\,x}$

Particular Integral is,

$$U_p = \frac{1}{D^2 - 5/4}\{-1/4(10\sin 2\pi x - 6\sin 4\pi x\}$$

$$= -\frac{10}{4}\cdot\frac{1}{(D^2-5/4)}\sin 2\pi x + \frac{6}{4(D^2-5/4)}\sin 4\pi x$$

$$= -\frac{5}{2(-4\pi^2-5/4)}\sin 2\pi x + \frac{6}{4(-16\pi^2-5/4)}\sin 4\pi x$$

$$= \frac{10\sin 2\pi x}{16\pi^2+5} - \frac{6\sin 4\pi x}{64\pi^2+5}$$

Pr.

$(D^2-1)y = \sin 2x$

Aux eqn : $m^2 - 1 = 0$

$\therefore m = \pm 1$

$\therefore y_c = c_1 e^x + c_2 e^{-x}$

$y_p = \frac{1}{D^2-1}\sin 2x$

$= \frac{1}{-2^2-1}\sin 2x$

$\left[y_p = \frac{1}{f(D^2)} = \frac{1}{f(-a^2)} \right.$ for $\sin ax$

$\therefore$ The general equation is

$$u = u_c + u_p$$

$$= c_1 e^{\sqrt{5/4}\,x} + c_2 e^{-\sqrt{5/4}\,x} + \frac{10\sin 2\pi x}{16\pi^2+5} - \frac{6\sin 4\pi x}{64\pi^2+5} \quad \text{——— (I)}$$

Taking the laplace transform of those boundary conditions, which involve t, we have,

$$L\{u(0,t)\} = u(0,s) = 0$$

$$L\{u(3,t)\} = u(3,s) = 0$$

Using the first condition is $u(0,s) = 0$ in (I),

$$0 = c_1 e^0 + c_2 e^0 + 0 - 0$$

$$\therefore c_1 + c_2 = 0 \quad \text{——— (II)}$$

using the second condition, $u(3,s)=0$

$$0 = c_1 e^{\sqrt{5/4}\,3} + c_2 e^{-3\sqrt{5/4}} \quad \text{(iii)}$$

solving (ii) & (iii)

$$c_1 = 0, \quad c_2 = 0$$

Thus (i) becomes,

$$u = \frac{10\sin 2\pi x}{s+16\pi^2} - \frac{6\sin 4\pi x}{64\pi^2+s}$$

Now taking inverse laplace tranformations,

$$L^{-1}\{u\} = 10\sin 2\pi x\, L^{-1}\left\{\frac{1}{s+16\pi^2}\right\} - 6\sin 4\pi x\, L^{-1}\left\{\frac{1}{64\pi^2+s}\right\}$$

$$= 10\sin 2\pi x\, e^{-16\pi^2 t} - 6\sin 4\pi x\, e^{-64\pi^2 t}$$

$$\therefore u(x,t) = 10\sin 2\pi x\, e^{-16\pi^2 t} - 6\sin 4\pi x\, e^{-64\pi^2 t}$$

2018, 2014, 2013

* Sin $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = 3\sin 2\pi x$, $u(0,t) = 0$, $u(1,t) = 0$ wh
$0 < x < 1$, $t > 0$

Solution:

Taking the laplace transforms,

$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$

$\Rightarrow su - u(x,0) = \frac{\partial^2 u}{\partial x^2}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} - su = -3\sin 2\pi x$

$\Rightarrow$ ∴ Auxillary equation,

$m^2 - s = 0$

$\therefore m = \pm\sqrt{s}$

$\therefore$ $u_c = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x}$

Particular integral,

$u_p = \frac{1}{(D^2 - s)}(-3\sin 2\pi x)$

$= \frac{-3}{-4\pi^2 - s} \sin 2\pi x$

$= \frac{3\sin 2\pi x}{s + 4\pi^2}$

∴ The general equation is,

$u = u_c + u_p = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{3\sin 2\pi x}{s + 4\pi^2}$ —(1)

using the laplace transform on boundary condition,

$L\{u(0,t)\} = u(0,s) = 0$

$L\{u(1,t)\} = u(1,s) = 0$

Using the conditions on (I),

$0 = c_1 e^0 + c_2 e^0 + 0$

$\therefore c_1 + c_2 = 0$ —— (II)

Again,

$0 = c_1 e^{\sqrt{5}} + c_2^{-\sqrt{5}} + 0$

$\therefore c_1 e^{\sqrt{5}} + c_2^{-\sqrt{5}} = 0$ —— (III)

Solving (II) & (III)

$c_1 = 0, \; c_2 = 0$

$\therefore$ (I) becomes,

$$u = \frac{3\sin 2\pi x}{s + 4\pi^2}$$

Apply inverse laplace transform,

$$L^{-1}\{u\} = 3\sin 2\pi x \, L^{-1}\left\{\frac{1}{s+4\pi^2}\right\}$$

$$= 3\sin 2\pi x \; e^{-4\pi^2}$$

$$\therefore V(x,t) = 3e^{-4\pi^2} \sin 2\pi x$$