Chapter 4

GRADIENT, DIVERGENCE and CURL

THE VECTOR DIFFERENTIAL OPERATOR DEL, written ∇, is defined by

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

This vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the *gradient*, the *divergence* and the *curl*. The operator ∇ is also known as *nabla*.

THE GRADIENT. Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. ϕ defines a differentiable scalar field). Then the *gradient* of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ defines a vector field.

The component of $\nabla\phi$ in the direction of a unit vector $\mathbf{a}$ is given by $\boxed{\nabla\phi\cdot\mathbf{a}}$ and is called the *directional derivative* of ϕ in the direction $\mathbf{a}$. Physically, this is the rate of change of ϕ at (x,y,z) in the direction $\mathbf{a}$.

THE DIVERGENCE. Let $\mathbf{V}(x,y,z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. $\mathbf{V}$ defines a differentiable vector field). Then the *divergence* of $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$, is defined by

$$\begin{aligned}\nabla\cdot\mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\ &= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\end{aligned}$$

Note the analogy with $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$. Also note that $\nabla\cdot\mathbf{V} \neq \mathbf{V}\cdot\nabla$.

THE CURL. If $\mathbf{V}(x,y,z)$ is a differentiable vector field then the *curl* or *rotation* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, curl $\mathbf{V}$ or rot $\mathbf{V}$, is defined by

$$\begin{aligned}\nabla\times\mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3\end{vmatrix}\end{aligned}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} \mathbf{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

Note that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* V_1, V_2, V_3.

FORMULAS INVOLVING ∇. If **A** and **B** are differentiable vector functions, and ϕ and ψ are differentiable scalar functions of position (x, y, z), then

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$ or $\text{grad}(\phi + \psi) = \text{grad}\,\phi + \text{grad}\,\psi$
2. $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$ or $\text{div}(\mathbf{A}+\mathbf{B}) = \text{div}\,\mathbf{A} + \text{div}\,\mathbf{B}$
3. $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$ or $\text{curl}(\mathbf{A}+\mathbf{B}) = \text{curl}\,\mathbf{A} + \text{curl}\,\mathbf{B}$
4. $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$
5. $\nabla\times(\phi\mathbf{A}) = (\nabla\phi)\times\mathbf{A} + \phi(\nabla\times\mathbf{A})$
6. $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$
7. $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$
8. $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$
9. $\nabla\cdot(\nabla\phi) \equiv \nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

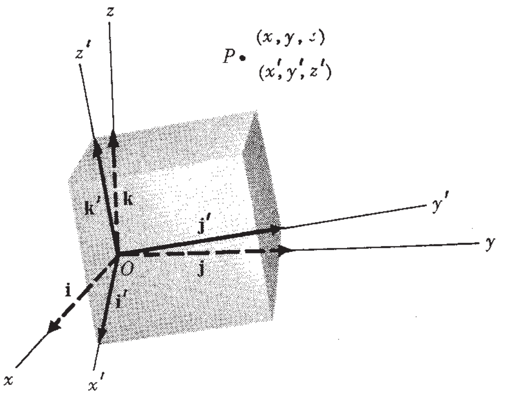
   where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator.*
10. $\nabla\times(\nabla\phi) = \mathbf{0}$. The curl of the gradient of ϕ is zero.
11. $\nabla\cdot(\nabla\times\mathbf{A}) = 0$. The divergence of the curl of **A** is zero.
12. $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

In Formulas 9-12, it is supposed that ϕ and **A** have continuous second partial derivatives.

INVARIANCE. Consider two rectangular coordinate systems or frames of reference xyz and $x'y'z'$ (see figure below) having the same origin O but with axes rotated with respect to each other.

A point P in space has coordinates (x, y, z) or (x', y', z') relative to these coordinate systems. The equations of transformation between coordinates or the *coordinate transformations* are given by

$$\begin{aligned} x' &= l_{11}x + l_{12}y + l_{13}z \\ y' &= l_{21}x + l_{22}y + l_{23}z \qquad (1) \\ z' &= l_{31}x + l_{32}y + l_{33}z \end{aligned}$$



where l_{jk}, $j, k = 1,2,3$, represent direction cosines of the x', y' and z' axes with respect to the x, y, and

z axes (see Problem 38). In case the origins of the two coordinate systems are not coincident the equations of transformation become

$$(2) \qquad \begin{cases} x' = l_{11}x + l_{12}y + l_{13}z + a_1' \\ y' = l_{21}x + l_{22}y + l_{23}z + a_2' \\ z' = l_{31}x + l_{32}y + l_{33}z + a_3' \end{cases}$$

where origin O of the xyz coordinate system is located at (a_1', a_2', a_3') relative to the $x'y'z'$ coordinate system.

The transformation equations (*1*) define a *pure rotation*, while equations (*2*) define a *rotation plus translation.* Any rigid body motion has the effect of a translation followed by a rotation. The transformation (*1*) is also called an *orthogonal transformation.* A general linear transformation is called an *affine transformation.*

Physically a scalar point function or scalar field $\phi(x,y,z)$ evaluated at a particular point should be independent of the coordinates of the point. Thus the temperature at a point is not dependent on whether coordinates (x,y,z) or (x',y',z') are used. Then if $\phi(x,y,z)$ is the temperature at point P with coordinates (x,y,z) while $\phi'(x',y',z')$ is the temperature at the same point P with coordinates (x',y',z'), we must have $\phi(x,y,z) = \phi'(x',y',z')$. If $\phi(x,y,z) = \phi'(x',y',z')$, where x,y,z and x',y',z' are related by the transformation equations (*1*) or (*2*), we call $\phi(x,y,z)$ an *invariant* with respect to the transformation. For example, $x^2+y^2+z^2$ is invariant under the transformation of rotation (*1*), since $x^2+y^2+z^2 = x'^2+y'^2+z'^2$.

Similarly, a vector point function or vector field $\mathbf{A}(x,y,z)$ is called an *invariant* if $\mathbf{A}(x,y,z) = \mathbf{A}'(x',y',z')$. This will be true if

$$A_1(x,y,z)\mathbf{i} + A_2(x,y,z)\mathbf{j} + A_3(x,y,z)\mathbf{k} = A_1'(x',y',z')\mathbf{i}' + A_2'(x',y',z')\mathbf{j}' + A_3'(x',y',z')\mathbf{k}'$$

In Chap. 7 and 8, more general transformations are considered and the above concepts are extended.

It can be shown (see Problem 41) that the gradient of an invariant scalar field is an invariant vector field with respect to the transformations (*1*) or (*2*). Similarly, the divergence and curl of an invariant vector field are invariant under this transformation.

SOLVED PROBLEMS

THE GRADIENT

1. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$.

$$\begin{aligned} \nabla\phi &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(3x^2y - y^3z^2) \\ &= \mathbf{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2) \\ &= 6xy\,\mathbf{i} + (3x^2 - 3y^2z^2)\mathbf{j} - 2y^3z\,\mathbf{k} \\ &= 6(1)(-2)\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\mathbf{j} - 2(-2)^3(-1)\mathbf{k} \\ &= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k} \end{aligned}$$

2. Prove (*a*) $\nabla(F+G) = \nabla F + \nabla G$, (*b*) $\nabla(FG) = F\,\nabla G + G\,\nabla F$ where F and G are differentiable scalar functions of x, y and z.

(*a*)
$$\begin{aligned}
\nabla(F+G) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(F+G) \\
&= \mathbf{i}\frac{\partial}{\partial x}(F+G) + \mathbf{j}\frac{\partial}{\partial y}(F+G) + \mathbf{k}\frac{\partial}{\partial z}(F+G) \\
&= \mathbf{i}\frac{\partial F}{\partial x} + \mathbf{i}\frac{\partial G}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{j}\frac{\partial G}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z} + \mathbf{k}\frac{\partial G}{\partial z} \\
&= \mathbf{i}\frac{\partial F}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z} + \mathbf{i}\frac{\partial G}{\partial x} + \mathbf{j}\frac{\partial G}{\partial y} + \mathbf{k}\frac{\partial G}{\partial z} \\
&= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)F + \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)G = \nabla F + \nabla G
\end{aligned}$$

(*b*)
$$\begin{aligned}
\nabla(FG) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(FG) \\
&= \frac{\partial}{\partial x}(FG)\mathbf{i} + \frac{\partial}{\partial y}(FG)\mathbf{j} + \frac{\partial}{\partial z}(FG)\mathbf{k} \\
&= \left(F\frac{\partial G}{\partial x} + G\frac{\partial F}{\partial x}\right)\mathbf{i} + \left(F\frac{\partial G}{\partial y} + G\frac{\partial F}{\partial y}\right)\mathbf{j} + \left(F\frac{\partial G}{\partial z} + G\frac{\partial F}{\partial z}\right)\mathbf{k} \\
&= F\left(\frac{\partial G}{\partial x}\mathbf{i} + \frac{\partial G}{\partial y}\mathbf{j} + \frac{\partial G}{\partial z}\mathbf{k}\right) + G\left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right) = F\nabla G + G\nabla F
\end{aligned}$$

3. Find $\nabla\phi$ if (*a*) $\phi = \ln|\mathbf{r}|$, (*b*) $\phi = \frac{1}{r}$.

(*a*) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\phi = \ln|\mathbf{r}| = \frac{1}{2}\ln(x^2+y^2+z^2)$.

$$\begin{aligned}
\nabla\phi &= \tfrac{1}{2}\nabla\ln(x^2+y^2+z^2) \\
&= \tfrac{1}{2}\left\{\mathbf{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \mathbf{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \mathbf{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\} \\
&= \tfrac{1}{2}\left\{\mathbf{i}\frac{2x}{x^2+y^2+z^2} + \mathbf{j}\frac{2y}{x^2+y^2+z^2} + \mathbf{k}\frac{2z}{x^2+y^2+z^2}\right\} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}
\end{aligned}$$

(*b*)
$$\begin{aligned}
\nabla\phi &= \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\{(x^2+y^2+z^2)^{-1/2}\} \\
&= \mathbf{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \mathbf{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \mathbf{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} \\
&= \mathbf{i}\left\{-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\right\} + \mathbf{j}\left\{-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\right\} + \mathbf{k}\left\{-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\right\} \\
&= \frac{-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}
\end{aligned}$$

4. Show that $\nabla r^n = nr^{n-2}\mathbf{r}$.

$$\begin{aligned}
\nabla r^n &= \nabla\left(\sqrt{x^2+y^2+z^2}\right)^n = \nabla(x^2+y^2+z^2)^{n/2} \\
&= \mathbf{i}\frac{\partial}{\partial x}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{j}\frac{\partial}{\partial y}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{k}\frac{\partial}{\partial z}\{(x^2+y^2+z^2)^{n/2}\}
\end{aligned}$$

$$= \mathbf{i}\left\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2x\right\} + \mathbf{j}\left\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2y\right\} + \mathbf{k}\left\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2z\right\}$$

$$= n\,(x^2+y^2+z^2)^{n/2-1}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

$$= n\,(r^2)^{n/2-1}\,\mathbf{r} = nr^{n-2}\,\mathbf{r}$$

Note that if $\mathbf{r} = r\mathbf{r}_1$ where $\mathbf{r}_1$ is a unit vector in the direction $\mathbf{r}$, then $\nabla r^n = nr^{n-1}\,\mathbf{r}_1$.

5. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where c is a constant.

Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector to any point $P(x,y,z)$ on the surface. Then $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ lies in the tangent plane to the surface at P.

But $d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz = 0$ or $\left(\dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = 0$

i.e. $\nabla\phi\cdot d\mathbf{r} = 0$ so that $\nabla\phi$ is perpendicular to $d\mathbf{r}$ and therefore to the surface.

6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

$\nabla(x^2y + 2xz) = (2xy + 2z)\mathbf{i} + x^2\,\mathbf{j} + 2x\,\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ at the point $(2,-2,3)$.

Then a unit normal to the surface $= \dfrac{-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}}{\sqrt{(-2)^2+(4)^2+(4)^2}} = -\dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$.

Another unit normal is $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$ having direction opposite to that above.

7. Find an equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$.

$$\nabla(2xz^2 - 3xy - 4x) = (2z^2 - 3y - 4)\mathbf{i} - 3x\,\mathbf{j} + 4xz\,\mathbf{k}$$

Then a normal to the surface at the point $(1,-1,2)$ is $7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$.

The equation of a plane passing through a point whose position vector is $\mathbf{r}_0$ and which is perpendicular to the normal $\mathbf{N}$ is $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{N} = 0$. (See Chap. 2, Prob. 18.) Then the required equation is

$$[(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})]\cdot(7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$$

or

$$7(x-1) - 3(y+1) + 8(z-2) = 0.$$

8. Let $\phi(x,y,z)$ and $\phi(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$ be the temperatures at two neighboring points $P(x,y,z)$ and $Q(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$ of a certain region.

(a) Interpret physically the quantity $\dfrac{\Delta\phi}{\Delta s} = \dfrac{\phi(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - \phi(x,y,z)}{\Delta s}$ where Δs is the distance between points P and Q.

(b) Evaluate $\displaystyle\lim_{\Delta s\to 0}\frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}$ and interpret physically.

(c) Show that $\dfrac{d\phi}{ds} = \nabla\phi\cdot\dfrac{d\mathbf{r}}{ds}$.

(a) Since $\Delta\phi$ is the change in temperature between points P and Q and Δs is the distance between these points, $\dfrac{\Delta\phi}{\Delta s}$ represents the average rate of change in temperature per unit distance in the direction from P to Q.

(*b*) From the calculus,

$$\Delta\phi = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z + \text{infinitesimals of order higher than } \Delta x, \Delta y \text{ and } \Delta z$$

Then
$$\lim_{\Delta s\to 0}\frac{\Delta\phi}{\Delta s} = \lim_{\Delta s\to 0}\frac{\partial\phi}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial\phi}{\partial y}\frac{\Delta y}{\Delta s} + \frac{\partial\phi}{\partial z}\frac{\Delta z}{\Delta s}$$

or
$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$$

$\frac{d\phi}{ds}$ represents the rate of change of temperature with respect to distance at point P in a direction toward Q. This is also called the *directional derivative* of ϕ.

(*c*)
$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right)$$

$$= \nabla\phi\cdot\frac{d\mathbf{r}}{ds}.$$

Note that since $\frac{d\mathbf{r}}{ds}$ is a unit vector, $\nabla\phi\cdot\frac{d\mathbf{r}}{ds}$ is the component of $\nabla\phi$ in the direction of this unit vector.

9. Show that the greatest rate of change of ϕ, i.e. the maximum directional derivative, takes place in the direction of, and has the magnitude of, the vector $\nabla\phi$.

By Problem 8(*c*), $\frac{d\phi}{ds} = \nabla\phi\cdot\frac{d\mathbf{r}}{ds}$ is the projection of $\nabla\phi$ in the direction $\frac{d\mathbf{r}}{ds}$. This projection will be a maximum when $\nabla\phi$ and $\frac{d\mathbf{r}}{ds}$ have the same direction. Then the maximum value of $\frac{d\phi}{ds}$ takes place in the direction of $\nabla\phi$ and its magnitude is $|\nabla\phi|$.

10. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$\nabla\phi = \nabla(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 8xz)\mathbf{k}$$
$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \quad \text{at} \quad (1,-2,-1).$$

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Then the required directional derivative is

$$\nabla\phi\cdot\mathbf{a} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k})\cdot(\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Since this is positive, ϕ is increasing in this direction.

11. (*a*) In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum?
(*b*) What is the magnitude of this maximum?

$$\nabla\phi = \nabla(x^2yz^3) = 2xyz^3\,\mathbf{i} + x^2z^3\,\mathbf{j} + 3x^2yz^2\,\mathbf{k}$$
$$= -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k} \quad \text{at} \quad (2,1,-1).$$

Then by Problem 9,

(*a*) the directional derivative is a maximum in the direction $\nabla\phi = -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$,

(*b*) the magnitude of this maximum is $|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + (12)^2} = \sqrt{176} = 4\sqrt{11}$.

12. Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

A normal to $x^2+y^2+z^2 = 9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

A normal to $z = x^2+y^2-3$ or $x^2+y^2-z = 3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2+y^2-z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2| \cos\theta$, where θ is the required angle. Then

$$(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\cdot(4\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = |4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}|\,|4\mathbf{i} - 2\mathbf{j} - \mathbf{k}| \cos\theta$$

$$16 + 4 - 4 = \sqrt{(4)^2+(-2)^2+(4)^2}\,\sqrt{(4)^2+(-2)^2+(-1)^2}\,\cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \text{arc cos } 0.5819 = 54^\circ 25'$.

13. Let R be the distance from a fixed point $A(a,b,c)$ to any point $P(x,y,z)$. Show that ∇R is a unit vector in the direction $\mathbf{AP} = \mathbf{R}$.

If $\mathbf{r}_A$ and $\mathbf{r}_P$ are the position vectors $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ of A and P respectively, then $\mathbf{R} = \mathbf{r}_P - \mathbf{r}_A = (x-a)\mathbf{i} + (y-b)\mathbf{j} + (z-c)\mathbf{k}$, so that $R = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$. Then

$$\nabla R = \nabla\left(\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}\right) = \frac{(x-a)\mathbf{i} + (y-b)\mathbf{j} + (z-c)\mathbf{k}}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} = \frac{\mathbf{R}}{R}$$

is a unit vector in the direction $\mathbf{R}$.

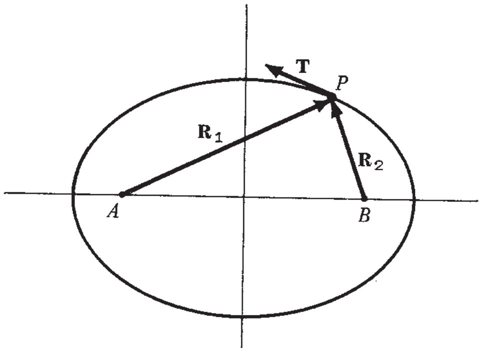
14. Let P be any point on an ellipse whose foci are at points A and B, as shown in the figure below. Prove that lines AP and BP make equal angles with the tangent to the ellipse at P.

Let $\mathbf{R}_1 = \mathbf{AP}$ and $\mathbf{R}_2 = \mathbf{BP}$ denote vectors drawn respectively from foci A and B to point P on the ellipse, and let $\mathbf{T}$ be a unit tangent to the ellipse at P.

Since an ellipse is the locus of all points P the sum of whose distances from two fixed points A and B is a constant p, it is seen that the equation of the ellipse is $R_1 + R_2 = p$.

By Problem 5, $\nabla(R_1+R_2)$ is a normal to the ellipse; hence $[\nabla(R_1+R_2)]\cdot\mathbf{T} = 0$ or $(\nabla R_2)\cdot\mathbf{T} = -(\nabla R_1)\cdot\mathbf{T}$.

Since ∇R_1 and ∇R_2 are unit vectors in direction $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively (Problem 13), the cosine of the angle between ∇R_2 and $\mathbf{T}$ is equal to the cosine of the angle between ∇R_1 and $-\mathbf{T}$; hence the angles themselves are equal.



The problem has a physical interpretation. Light rays (or sound waves) originating at focus A, for example, will be reflected from the ellipse to focus B.

THE DIVERGENCE

15. If $\mathbf{A} = x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}$, find $\nabla\cdot\mathbf{A}$ (or div $\mathbf{A}$) at the point $(1,-1,1)$.

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k})\\ &= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)\\ &= 2xz - 6y^2z^2 + xy^2 = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = -3 \quad \text{at } (1,-1,1).\end{aligned}$$

16. Given $\phi = 2x^3y^2z^4$. (*a*) Find $\nabla\cdot\nabla\phi$ (or div grad ϕ).
(*b*) Show that $\nabla\cdot\nabla\phi = \nabla^2\phi$, where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

(*a*)
$$\begin{aligned}\nabla\phi &= \mathbf{i}\frac{\partial}{\partial x}(2x^3y^2z^4) + \mathbf{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \mathbf{k}\frac{\partial}{\partial z}(2x^3y^2z^4)\\ &= 6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k}\end{aligned}$$

Then
$$\begin{aligned}\nabla\cdot\nabla\phi &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k})\\ &= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)\\ &= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2\end{aligned}$$

(*b*)
$$\begin{aligned}\nabla\cdot\nabla\phi &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi\end{aligned}$$

17. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) &= \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]\\ &= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}\end{aligned}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}} \quad\text{and}\quad \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0.$

The equation $\nabla^2\phi = 0$ is called *Laplace's equation.* It follows that $\phi = 1/r$ is a solution of this equation.

18. Prove: (*a*) $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$
(*b*) $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$.

(*a*) Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$.

$$\begin{aligned}\text{Then }\nabla\cdot(\mathbf{A}+\mathbf{B}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left[(A_1+B_1)\mathbf{i}+(A_2+B_2)\mathbf{j}+(A_3+B_3)\mathbf{k}\right]\\
&= \frac{\partial}{\partial x}(A_1+B_1)+\frac{\partial}{\partial y}(A_2+B_2)+\frac{\partial}{\partial z}(A_3+B_3)\\
&= \frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}\\
&= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})\\
&\qquad + \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k})\\
&= \nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}\end{aligned}$$

$$\begin{aligned}(b)\ \nabla\cdot(\phi\mathbf{A}) &= \nabla\cdot(\phi A_1\mathbf{i}+\phi A_2\mathbf{j}+\phi A_3\mathbf{k})\\
&= \frac{\partial}{\partial x}(\phi A_1)+\frac{\partial}{\partial y}(\phi A_2)+\frac{\partial}{\partial z}(\phi A_3)\\
&= \frac{\partial\phi}{\partial x}A_1+\phi\frac{\partial A_1}{\partial x}+\frac{\partial\phi}{\partial y}A_2+\phi\frac{\partial A_2}{\partial y}+\frac{\partial\phi}{\partial z}A_3+\phi\frac{\partial A_3}{\partial z}\\
&= \frac{\partial\phi}{\partial x}A_1+\frac{\partial\phi}{\partial y}A_2+\frac{\partial\phi}{\partial z}A_3+\phi\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)\\
&= \left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}) + \phi\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})\\
&= (\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A})\end{aligned}$$

19. Prove $\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) = 0$.

Let $\phi = r^{-3}$ and $\mathbf{A} = \mathbf{r}$ in the result of Problem 18(*b*).

$$\begin{aligned}\text{Then }\nabla\cdot(r^{-3}\mathbf{r}) &= (\nabla r^{-3})\cdot\mathbf{r} + (r^{-3})\nabla\cdot\mathbf{r}\\
&= -3r^{-5}\mathbf{r}\cdot\mathbf{r} + 3r^{-3} = 0, \quad\text{using Problem 4.}\end{aligned}$$

20. Prove $\nabla\cdot(U\nabla V - V\nabla U) = U\nabla^2 V - V\nabla^2 U.$

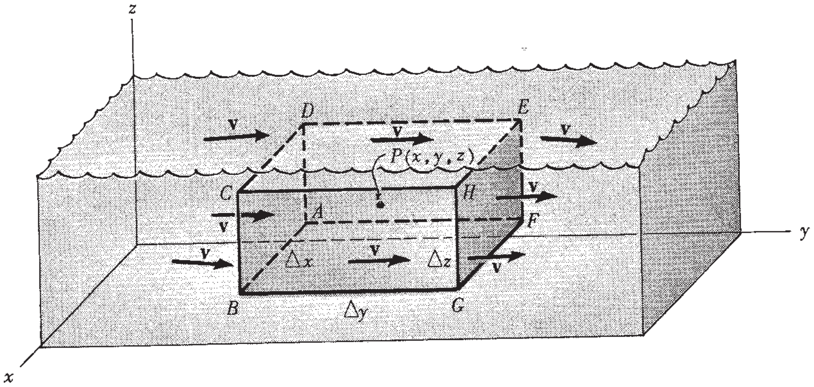
From Problem 18(*b*), with $\phi = U$ and $\mathbf{A} = \nabla V$,

$$\nabla\cdot(U\nabla V) = (\nabla U)\cdot(\nabla V) + U(\nabla\cdot\nabla V) = (\nabla U)\cdot(\nabla V) + U\nabla^2 V$$

Interchanging U and V yields $\nabla\cdot(V\nabla U) = (\nabla V)\cdot(\nabla U) + V\nabla^2 U.$

Then subtracting,
$$\begin{aligned}\nabla\cdot(U\nabla V) - \nabla\cdot(V\nabla U) &= \nabla\cdot(U\nabla V - V\nabla U)\\ &= (\nabla U)\cdot(\nabla V) + U\nabla^2 V - [(\nabla V)\cdot(\nabla U) + V\nabla^2 U]\\ &= U\nabla^2 V - V\nabla^2 U\end{aligned}$$

21. A fluid moves so that its velocity at any point is $\mathbf{v}(x,y,z)$. Show that the loss of fluid per unit volume per unit time in a small parallelepiped having center at $P(x,y,z)$ and edges parallel to the coordinate axes and having magnitude $\Delta x, \Delta y, \Delta z$ respectively, is given approximately by div $\mathbf{v} = \nabla\cdot\mathbf{v}$.



Referring to the figure above,

x component of velocity $\mathbf{v}$ at P $= v_1$

x component of $\mathbf{v}$ at center of face $AFED$ $= v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x$ approx.

x component of $\mathbf{v}$ at center of face $GHCB$ $= v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x$ approx.

Then (*1*) volume of fluid crossing $AFED$ per unit time $= (v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x)\,\Delta y\,\Delta z$

(*2*) volume of fluid crossing $GHCB$ per unit time $= (v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x)\,\Delta y\,\Delta z.$

Loss in volume per unit time in x direction $= (2) - (1) = \frac{\partial v_1}{\partial x}\Delta x\,\Delta y\,\Delta z.$

Similarly, loss in volume per unit time in y direction $= \frac{\partial v_2}{\partial y}\Delta x\,\Delta y\,\Delta z$

loss in volume per unit time in z direction $= \frac{\partial v_3}{\partial z}\Delta x\,\Delta y\,\Delta z.$

Then, total loss in volume per unit volume per unit time

$$= \frac{\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\Delta x\,\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z} = \text{div}\,\mathbf{v} = \nabla\cdot\mathbf{v}$$

This is true exactly only in the limit as the parallelepiped shrinks to P, i.e. as $\Delta x, \Delta y$ and Δz approach zero. If there is no loss of fluid anywhere, then $\nabla \cdot \mathbf{v} = 0$. This is called the *continuity equation* for an incompressible fluid. Since fluid is neither created nor destroyed at any point, it is said to have no sources or sinks. A vector such as $\mathbf{v}$ whose divergence is zero is sometimes called *solenoidal.*

22. Determine the constant a so that the vector $\mathbf{V} = (x+3y)\mathbf{i} + (y-2z)\mathbf{j} + (x+az)\mathbf{k}$ is solenoidal.

A vector $\mathbf{V}$ is solenoidal if its divergence is zero (Problem 21).

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 1 + 1 + a$$

Then $\nabla \cdot \mathbf{V} = a+2 = 0$ when $a = -2$.

THE CURL

23. If $\mathbf{A} = xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k}$, find $\nabla \times \mathbf{A}$ (or curl $\mathbf{A}$) at the point $(1,-1,1)$.

$$\nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]\mathbf{k}$$

$$= (2z^4 + 2x^2y)\mathbf{i} + 3xz^2\,\mathbf{j} - 4xyz\,\mathbf{k} = 3\mathbf{j} + 4\mathbf{k} \quad \text{at } (1,-1,1).$$

24. If $\mathbf{A} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find curl curl $\mathbf{A}$.

$$\text{curl curl } \mathbf{A} = \nabla \times (\nabla \times \mathbf{A})$$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = \nabla \times \left[(2x+2z)\mathbf{i} - (x^2+2z)\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+2z & 0 & -x^2-2z \end{vmatrix} = (2x+2)\mathbf{j}$$

25. Prove: (*a*) $\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

(*b*) $\nabla \times (\phi\mathbf{A}) = (\nabla\phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$.

(*a*) Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Then:

$$\nabla\times(\mathbf{A}+\mathbf{B}) = (\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}) \times [(A_1+B_1)\mathbf{i} + (A_2+B_2)\mathbf{j} + (A_3+B_3)\mathbf{k}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1+B_1 & A_2+B_2 & A_3+B_3 \end{vmatrix}$$

$$= [\frac{\partial}{\partial y}(A_3+B_3) - \frac{\partial}{\partial z}(A_2+B_2)]\mathbf{i} + [\frac{\partial}{\partial z}(A_1+B_1) - \frac{\partial}{\partial x}(A_3+B_3)]\mathbf{j} + [\frac{\partial}{\partial x}(A_2+B_2) - \frac{\partial}{\partial y}(A_1+B_1)]\mathbf{k}$$

$$= [\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}]\mathbf{i} + [\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}]\mathbf{j} + [\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}]\mathbf{k} + [\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}]\mathbf{i} + [\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}]\mathbf{j} + [\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}]\mathbf{k}$$

$$= \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$$

(*b*) $$\nabla\times(\phi\mathbf{A}) = \nabla\times(\phi A_1\mathbf{i} + \phi A_2\mathbf{j} + \phi A_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= [\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)]\mathbf{i} + [\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)]\mathbf{j} + [\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)]\mathbf{k}$$

$$= [\phi\frac{\partial A_3}{\partial y} + \frac{\partial\phi}{\partial y}A_3 - \phi\frac{\partial A_2}{\partial z} - \frac{\partial\phi}{\partial z}A_2]\mathbf{i} + [\phi\frac{\partial A_1}{\partial z} + \frac{\partial\phi}{\partial z}A_1 - \phi\frac{\partial A_3}{\partial x} - \frac{\partial\phi}{\partial x}A_3]\mathbf{j} + [\phi\frac{\partial A_2}{\partial x} + \frac{\partial\phi}{\partial x}A_2 - \phi\frac{\partial A_1}{\partial y} - \frac{\partial\phi}{\partial y}A_1]\mathbf{k}$$

$$= \phi[(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\mathbf{i} + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\mathbf{j} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}] + [(\frac{\partial\phi}{\partial y}A_3 - \frac{\partial\phi}{\partial z}A_2)\mathbf{i} + (\frac{\partial\phi}{\partial z}A_1 - \frac{\partial\phi}{\partial x}A_3)\mathbf{j} + (\frac{\partial\phi}{\partial x}A_2 - \frac{\partial\phi}{\partial y}A_1)\mathbf{k}]$$

$$= \phi(\nabla\times\mathbf{A}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \phi(\nabla\times\mathbf{A}) + (\nabla\phi)\times\mathbf{A}.$$

26. Evaluate $\nabla\cdot(\mathbf{A}\times\mathbf{r})$ if $\nabla\times\mathbf{A}=\mathbf{0}$.

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Then
$$\mathbf{A}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$
$$= (zA_2 - yA_3)\mathbf{i} + (xA_3 - zA_1)\mathbf{j} + (yA_1 - xA_2)\mathbf{k}$$

and
$$\nabla\cdot(\mathbf{A}\times\mathbf{r}) = \frac{\partial}{\partial x}(zA_2 - yA_3) + \frac{\partial}{\partial y}(xA_3 - zA_1) + \frac{\partial}{\partial z}(yA_1 - xA_2)$$
$$= z\frac{\partial A_2}{\partial x} - y\frac{\partial A_3}{\partial x} + x\frac{\partial A_3}{\partial y} - z\frac{\partial A_1}{\partial y} + y\frac{\partial A_1}{\partial z} - x\frac{\partial A_2}{\partial z}$$
$$= x(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) + y(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) + z(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})$$
$$= [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]\cdot[(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\mathbf{i} + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\mathbf{j} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}]$$

$= \mathbf{r}\cdot(\nabla\times\mathbf{A}) = \mathbf{r}\cdot\text{curl}\,\mathbf{A}$. If $\nabla\times\mathbf{A}=\mathbf{0}$ this reduces to zero.

27. Prove: (*a*) $\nabla\times(\nabla\phi) = \mathbf{0}$ (curl grad $\phi = \mathbf{0}$), (*b*) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ (div curl $\mathbf{A} = 0$).

(*a*)
$$\nabla\times(\nabla\phi) = \nabla\times(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$
$$= [\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial z}) - \frac{\partial}{\partial z}(\frac{\partial\phi}{\partial y})]\mathbf{i} + [\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial x}) - \frac{\partial}{\partial x}(\frac{\partial\phi}{\partial z})]\mathbf{j} + [\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial y}) - \frac{\partial}{\partial y}(\frac{\partial\phi}{\partial x})]\mathbf{k}$$
$$= (\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y})\mathbf{i} + (\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z})\mathbf{j} + (\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x})\mathbf{k} = \mathbf{0}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

(*b*)
$$\nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \nabla\cdot[(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\mathbf{i} + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\mathbf{j} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}]$$
$$= \frac{\partial}{\partial x}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})$$

$$= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y} = 0$$

assuming that **A** has continuous second partial derivatives.

Note the similarity between the above results and the results $(\mathbf{C}\times\mathbf{C}m) = (\mathbf{C}\times\mathbf{C})m = \mathbf{0}$, where m is a scalar and $\mathbf{C}\cdot(\mathbf{C}\times\mathbf{A}) = (\mathbf{C}\times\mathbf{C})\cdot\mathbf{A} = 0$.

28. Find $\operatorname{curl}(\mathbf{r}f(r))$ where $f(r)$ is differentiable.

$$\operatorname{curl}(\mathbf{r}\,f(r)) = \nabla\times(\mathbf{r}\,f(r))$$

$$= \nabla\times(x\,f(r)\mathbf{i} + y\,f(r)\mathbf{j} + z\,f(r)\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\,f(r) & y\,f(r) & z\,f(r) \end{vmatrix}$$

$$= (z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z})\mathbf{i} + (x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x})\mathbf{j} + (y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y})\mathbf{k}$$

But $\frac{\partial f}{\partial x} = (\frac{\partial f}{\partial r})(\frac{\partial r}{\partial x}) = \frac{\partial f}{\partial r}\frac{\partial}{\partial x}(\sqrt{x^2+y^2+z^2}) = \frac{f'(r)\,x}{\sqrt{x^2+y^2+z^2}} = \frac{f'x}{r}$. Similarly, $\frac{\partial f}{\partial y} = \frac{f'y}{r}$ and $\frac{\partial f}{\partial z} = \frac{f'z}{r}$.

Then the result $= (z\frac{f'y}{r} - y\frac{f'z}{r})\mathbf{i} + (x\frac{f'z}{r} - z\frac{f'x}{r})\mathbf{j} + (y\frac{f'x}{r} - x\frac{f'y}{r})\mathbf{k} = \mathbf{0}$.

29. Prove $\nabla\times(\nabla\times\mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})$.

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla\times\left[(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\mathbf{i} + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\mathbf{j} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}) - \frac{\partial}{\partial z}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\right]\mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) - \frac{\partial}{\partial x}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) - \frac{\partial}{\partial y}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\right]\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_3}{\partial z\,\partial y} + \frac{\partial^2 A_1}{\partial x\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z}\right)\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_1}{\partial x\,\partial y} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_3}{\partial z\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$+ \mathbf{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2\mathbf{A} + \nabla\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})$$

If desired, the labor of writing can be shortened in this as well as other derivations by writing only the **i** components since the others can be obtained by symmetry.

The result can also be established formally as follows. From Problem 47(*a*), Chapter 2,

$$(1) \qquad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

Placing $\mathbf{A} = \mathbf{B} = \nabla$ and $\mathbf{C} = \mathbf{F}$,

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - (\nabla\cdot\nabla)\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$$

Note that the formula (*1*) must be written so that the operators **A** and **B** precede the operand **C**, otherwise the formalism fails to apply.

30. If $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, prove $\boldsymbol{\omega} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$ where $\boldsymbol{\omega}$ is a constant vector.

$$\text{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = \nabla\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix}$$

$$= \nabla\times\left[(\omega_2 z - \omega_3 y)\mathbf{i} + (\omega_3 x - \omega_1 z)\mathbf{j} + (\omega_1 y - \omega_2 x)\mathbf{k}\right]$$

$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x\end{vmatrix} = 2(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) = 2\boldsymbol{\omega}\,.$$

Then $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$.

This problem indicates that the curl of a vector field has something to do with rotational properties of the field. This is confirmed in Chapter 6. If the field $\mathbf{F}$ is that due to a moving fluid, for example, then a paddle wheel placed at various points in the field would tend to rotate in regions where $\text{curl}\,\mathbf{F} \neq \mathbf{0}$, while if $\text{curl}\,\mathbf{F} = \mathbf{0}$ in the region there would be no rotation and the field $\mathbf{F}$ is then called *irrotational.* A field which is not irrotational is sometimes called a *vortex field.*

31. If $\nabla\cdot\mathbf{E} = 0$, $\nabla\cdot\mathbf{H} = 0$, $\nabla\times\mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$, $\nabla\times\mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$, show that $\mathbf{E}$ and $\mathbf{H}$ satisfy $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$.

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla\times\left(-\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

By Problem 29, $\nabla\times(\nabla\times\mathbf{E}) = -\nabla^2\mathbf{E} + \nabla(\nabla\cdot\mathbf{E}) = -\nabla^2\mathbf{E}$. Then $\nabla^2\mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$.

Similarly, $\nabla\times(\nabla\times\mathbf{H}) = \nabla\times\left(\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial^2 \mathbf{H}}{\partial t^2}$.

But $\nabla\times(\nabla\times\mathbf{H}) = -\nabla^2\mathbf{H} + \nabla(\nabla\cdot\mathbf{H}) = -\nabla^2\mathbf{H}$. Then $\nabla^2\mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2}$.

The given equations are related to *Maxwell's equations of electromagnetic theory.* The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$ is called the *wave equation.*

MISCELLANEOUS PROBLEMS.

32. (*a*) A vector $\mathbf{V}$ is called irrotational if $\text{curl}\,\mathbf{V} = \mathbf{0}$ (see Problem 30). Find constants a,b,c so that

$$\mathbf{V} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$$

is irrotational.

(*b*) Show that $\mathbf{V}$ can be expressed as the gradient of a scalar function.

(*a*) $$\text{curl}\,\mathbf{V} = \nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = (c+1)\mathbf{i} + (a-4)\mathbf{j} + (b-2)\mathbf{k}$$

This equals zero when $a = 4$, $b = 2$, $c = -1$ and

$$\mathbf{V} = (x + 2y + 4z)\mathbf{i} + (2x - 3y - z)\mathbf{j} + (4x - y + 2z)\mathbf{k}$$

(*b*) Assume $\mathbf{V} = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

Then (*1*) $\frac{\partial\phi}{\partial x} = x + 2y + 4z$, (*2*) $\frac{\partial\phi}{\partial y} = 2x - 3y - z$, (*3*) $\frac{\partial\phi}{\partial z} = 4x - y + 2z$.

Integrating (*1*) partially with respect to x, keeping y and z constant,

(*4*) $$\phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$$

where $f(y,z)$ is an arbitrary function of y and z. Similarly from (2) and (3),

$$\phi = 2xy - \frac{3y^2}{2} - yz + g(x,z) \tag{5}$$

$$\phi = 4xz - yz + z^2 + h(x,y). \tag{6}$$

Comparison of (4), (5) and (6) shows that there will be a common value of ϕ if we choose

$$f(y,z) = -\frac{3y^2}{2} + z^2, \qquad g(x,z) = \frac{x^2}{2} + z^2, \qquad h(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

so that

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$$

Note that we can also add any constant to ϕ. In general if $\nabla \times \mathbf{V} = \mathbf{0}$, then we can find ϕ so that $\mathbf{V} = \nabla\phi$. A vector field $\mathbf{V}$ which can be derived from a scalar field ϕ so that $\mathbf{V} = \nabla\phi$ is called a *conservative vector field* and ϕ is called the *scalar potential*. Note that conversely if $\mathbf{V} = \nabla\phi$, then $\nabla \times \mathbf{V} = \mathbf{0}$ (see Prob.27*a*).

33. Show that if $\phi(x,y,z)$ is any solution of Laplace's equation, then $\nabla\phi$ is a vector which is both solenoidal and irrotational.

By hypothesis, ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, i.e. $\nabla\cdot(\nabla\phi) = 0$. Then $\nabla\phi$ is solenoidal (see Problems 21 and 22).

From Problem 27*a*, $\nabla \times (\nabla\phi) = \mathbf{0}$ so that $\nabla\phi$ is also irrotational.

34. Give a possible definition of grad $\mathbf{B}$.

Assume $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Formally, we can define grad $\mathbf{B}$ as

$$\begin{aligned} \nabla\mathbf{B} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\ &= \frac{\partial B_1}{\partial x}\mathbf{ii} + \frac{\partial B_2}{\partial x}\mathbf{ij} + \frac{\partial B_3}{\partial x}\mathbf{ik} \\ &\quad + \frac{\partial B_1}{\partial y}\mathbf{ji} + \frac{\partial B_2}{\partial y}\mathbf{jj} + \frac{\partial B_3}{\partial y}\mathbf{jk} \\ &\quad + \frac{\partial B_1}{\partial z}\mathbf{ki} + \frac{\partial B_2}{\partial z}\mathbf{kj} + \frac{\partial B_3}{\partial z}\mathbf{kk} \end{aligned}$$

The quantities $\mathbf{ii}$, $\mathbf{ij}$, etc., are called *unit dyads*. (Note that $\mathbf{ij}$, for example, is not the same as $\mathbf{ji}$.) A quantity of the form

$$a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} + a_{23}\mathbf{jk} + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk}$$

is called a *dyadic* and the coefficients $a_{11}, a_{12}, \ldots$ are its *components*. An array of these nine components in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is called a 3 by 3 *matrix*. A dyadic is a generalization of a vector. Still further generalization leads to *triadics* which are quantities consisting of 27 terms of the form $a_{111}\,\mathbf{iii} + a_{211}\,\mathbf{jii} + \ldots$. A study of how the components of a dyadic or triadic transform from one system of coordinates to another leads to the subject of *tensor analysis* which is taken up in Chapter 8.

35. Let a vector $\mathbf{A}$ be defined by $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and a dyadic $\mathbf{\Phi}$ by

$$\mathbf{\Phi} = a_{11}\mathbf{i}\mathbf{i} + a_{12}\mathbf{i}\mathbf{j} + a_{13}\mathbf{i}\mathbf{k} + a_{21}\mathbf{j}\mathbf{i} + a_{22}\mathbf{j}\mathbf{j} + a_{23}\mathbf{j}\mathbf{k} + a_{31}\mathbf{k}\mathbf{i} + a_{32}\mathbf{k}\mathbf{j} + a_{33}\mathbf{k}\mathbf{k}$$

Give a possible definition of $\mathbf{A}\cdot\mathbf{\Phi}$.

Formally, assuming the distributive law to hold,

$$\mathbf{A}\cdot\mathbf{\Phi} = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\mathbf{\Phi} = A_1\mathbf{i}\cdot\mathbf{\Phi} + A_2\mathbf{j}\cdot\mathbf{\Phi} + A_3\mathbf{k}\cdot\mathbf{\Phi}$$

As an example, consider $\mathbf{i}\cdot\mathbf{\Phi}$. This product is formed by taking the dot product of $\mathbf{i}$ with each term of $\mathbf{\Phi}$ and adding results. Typical examples are $\mathbf{i}\cdot a_{11}\mathbf{i}\mathbf{i}$, $\mathbf{i}\cdot a_{12}\mathbf{i}\mathbf{j}$, $\mathbf{i}\cdot a_{21}\mathbf{j}\mathbf{i}$, $\mathbf{i}\cdot a_{32}\mathbf{k}\mathbf{j}$, etc. If we give meaning to these as follows

$$\begin{array}{lllll}
\mathbf{i}\cdot a_{11}\mathbf{i}\mathbf{i} & = a_{11}(\mathbf{i}\cdot\mathbf{i})\mathbf{i} & = a_{11}\mathbf{i} & \text{since } \mathbf{i}\cdot\mathbf{i} & = 1 \\
\mathbf{i}\cdot a_{12}\mathbf{i}\mathbf{j} & = a_{12}(\mathbf{i}\cdot\mathbf{i})\mathbf{j} & = a_{12}\mathbf{j} & \text{since } \mathbf{i}\cdot\mathbf{i} & = 1 \\
\mathbf{i}\cdot a_{21}\mathbf{j}\mathbf{i} & = a_{21}(\mathbf{i}\cdot\mathbf{j})\mathbf{i} & = 0 & \text{since } \mathbf{i}\cdot\mathbf{j} & = 0 \\
\mathbf{i}\cdot a_{32}\mathbf{k}\mathbf{j} & = a_{32}(\mathbf{i}\cdot\mathbf{k})\mathbf{j} & = 0 & \text{since } \mathbf{i}\cdot\mathbf{k} & = 0
\end{array}$$

and give analogous interpretation to the terms of $\mathbf{j}\cdot\mathbf{\Phi}$ and $\mathbf{k}\cdot\mathbf{\Phi}$, then

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{\Phi} &= A_1(a_{11}\mathbf{i} + a_{12}\mathbf{j} + a_{13}\mathbf{k}) + A_2(a_{21}\mathbf{i} + a_{22}\mathbf{j} + a_{23}\mathbf{k}) + A_3(a_{31}\mathbf{i} + a_{32}\mathbf{j} + a_{33}\mathbf{k}) \\
&= (A_1a_{11} + A_2a_{21} + A_3a_{31})\,\mathbf{i} + (A_1a_{12} + A_2a_{22} + A_3a_{32})\,\mathbf{j} + (A_1a_{13} + A_2a_{23} + A_3a_{33})\,\mathbf{k}
\end{aligned}$$

which is a vector.

36. (*a*) Interpret the symbol $\mathbf{A}\cdot\nabla$. (*b*) Give a possible meaning to $(\mathbf{A}\cdot\nabla)\mathbf{B}$. (*c*) Is it possible to write this as $\mathbf{A}\cdot\nabla\mathbf{B}$ without ambiguity?

(*a*) Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. Then, formally,

$$\begin{aligned}
\mathbf{A}\cdot\nabla &= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\
&= A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}
\end{aligned}$$

is an operator. For example,

$$(\mathbf{A}\cdot\nabla)\phi = \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)\phi = A_1\frac{\partial\phi}{\partial x} + A_2\frac{\partial\phi}{\partial y} + A_3\frac{\partial\phi}{\partial z}$$

Note that this is the same as $\mathbf{A}\cdot\nabla\phi$.

(*b*) Formally, using (*a*) with ϕ replaced by $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$,

$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\mathbf{B} &= \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)\mathbf{B} = A_1\frac{\partial\mathbf{B}}{\partial x} + A_2\frac{\partial\mathbf{B}}{\partial y} + A_3\frac{\partial\mathbf{B}}{\partial z} \\
&= \left(A_1\frac{\partial B_1}{\partial x} + A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z}\right)\mathbf{i} + \left(A_1\frac{\partial B_2}{\partial x} + A_2\frac{\partial B_2}{\partial y} + A_3\frac{\partial B_2}{\partial z}\right)\mathbf{j} + \left(A_1\frac{\partial B_3}{\partial x} + A_2\frac{\partial B_3}{\partial y} + A_3\frac{\partial B_3}{\partial z}\right)\mathbf{k}
\end{aligned}$$

(*c*) Use the interpretation of $\nabla\mathbf{B}$ as given in Problem 34. Then, according to the symbolism established in Problem 35,

$$\begin{aligned}
\mathbf{A}\cdot\nabla\mathbf{B} &= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\nabla\mathbf{B} = A_1\mathbf{i}\cdot\nabla\mathbf{B} + A_2\mathbf{j}\cdot\nabla\mathbf{B} + A_3\mathbf{k}\cdot\nabla\mathbf{B} \\
&= A_1\left(\frac{\partial B_1}{\partial x}\mathbf{i} + \frac{\partial B_2}{\partial x}\mathbf{j} + \frac{\partial B_3}{\partial x}\mathbf{k}\right) + A_2\left(\frac{\partial B_1}{\partial y}\mathbf{i} + \frac{\partial B_2}{\partial y}\mathbf{j} + \frac{\partial B_3}{\partial y}\mathbf{k}\right) + A_3\left(\frac{\partial B_1}{\partial z}\mathbf{i} + \frac{\partial B_2}{\partial z}\mathbf{j} + \frac{\partial B_3}{\partial z}\mathbf{k}\right)
\end{aligned}$$

which gives the same result as that given in part (*b*). It follows that $(\mathbf{A}\cdot\nabla)\mathbf{B} = \mathbf{A}\cdot\nabla\mathbf{B}$ without ambiguity provided the concept of dyadics is introduced with properties as indicated.

37. If $\mathbf{A} = 2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\,\mathbf{k}$, $\mathbf{B} = x^2\mathbf{i} + yz\,\mathbf{j} - xy\,\mathbf{k}$ and $\phi = 2x^2yz^3$, find (*a*) $(\mathbf{A}\cdot\nabla)\phi$, (*b*) $\mathbf{A}\cdot\nabla\phi$, (*c*) $(\mathbf{B}\cdot\nabla)\mathbf{A}$, (*d*) $(\mathbf{A}\times\nabla)\phi$, (*e*) $\mathbf{A}\times\nabla\phi$.

(*a*)
$$
\begin{aligned}
(\mathbf{A}\cdot\nabla)\phi &= \left[(2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k})\cdot\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\right]\phi \\
&= \left(2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)(2x^2yz^3) \\
&= 2yz\frac{\partial}{\partial x}(2x^2yz^3) - x^2y\frac{\partial}{\partial y}(2x^2yz^3) + xz^2\frac{\partial}{\partial z}(2x^2yz^3) \\
&= (2yz)(4xyz^3) - (x^2y)(2x^2z^3) + (xz^2)(6x^2yz^2) \\
&= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4
\end{aligned}
$$

(*b*)
$$
\begin{aligned}
\mathbf{A}\cdot\nabla\phi &= (2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k})\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \\
&= (2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k})\cdot(4xyz^3\,\mathbf{i} + 2x^2z^3\,\mathbf{j} + 6x^2yz^2\mathbf{k}) \\
&= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4
\end{aligned}
$$

Comparison with (*a*) illustrates the result $(\mathbf{A}\cdot\nabla)\phi = \mathbf{A}\cdot\nabla\phi$.

(*c*)
$$
\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \left[(x^2\mathbf{i} + yz\,\mathbf{j} - xy\,\mathbf{k})\cdot\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\right]\mathbf{A} \\
&= \left(x^2\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}\right)\mathbf{A} \;=\; x^2\frac{\partial\mathbf{A}}{\partial x} + yz\frac{\partial\mathbf{A}}{\partial y} - xy\frac{\partial\mathbf{A}}{\partial z} \\
&= x^2(-2xy\,\mathbf{j} + z^2\mathbf{k}) + yz(2z\,\mathbf{i} - x^2\mathbf{j}) - xy(2y\,\mathbf{i} + 2xz\,\mathbf{k}) \\
&= (2yz^2 - 2xy^2)\mathbf{i} - (2x^3y + x^2yz)\mathbf{j} + (x^2z^2 - 2x^2yz)\mathbf{k}
\end{aligned}
$$

For comparison of this with $\mathbf{B}\cdot\nabla\mathbf{A}$, see Problem 36(*c*).

(*d*)
$$
\begin{aligned}
(\mathbf{A}\times\nabla)\phi &= \left[(2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k})\times\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\right]\phi \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2yz & -x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}\phi \\
&= \left[\mathbf{i}\left(-x^2y\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial y}\right) + \mathbf{j}\left(xz^2\frac{\partial}{\partial x} - 2yz\frac{\partial}{\partial z}\right) + \mathbf{k}\left(2yz\frac{\partial}{\partial y} + x^2y\frac{\partial}{\partial x}\right)\right]\phi \\
&= -\left(x^2y\frac{\partial\phi}{\partial z} + xz^2\frac{\partial\phi}{\partial y}\right)\mathbf{i} + \left(xz^2\frac{\partial\phi}{\partial x} - 2yz\frac{\partial\phi}{\partial z}\right)\mathbf{j} + \left(2yz\frac{\partial\phi}{\partial y} + x^2y\frac{\partial\phi}{\partial x}\right)\mathbf{k} \\
&= -(6x^4y^2z^2 + 2x^3z^5)\mathbf{i} + (4x^2yz^5 - 12x^2y^2z^3)\mathbf{j} + (4x^2yz^4 + 4x^3y^2z^3)\mathbf{k}
\end{aligned}
$$

(*e*) $$\mathbf{A}\times\nabla\phi = (2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k})\times(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2yz & -x^2y & xz^2 \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= (-x^2y\frac{\partial\phi}{\partial z} - xz^2\frac{\partial\phi}{\partial y})\mathbf{i} + (xz^2\frac{\partial\phi}{\partial x} - 2yz\frac{\partial\phi}{\partial z})\mathbf{j} + (2yz\frac{\partial\phi}{\partial y} + x^2y\frac{\partial\phi}{\partial x})\mathbf{k}$$

$$= -(6x^4y^2z^2 + 2x^3z^5)\mathbf{i} + (4x^2yz^5 - 12x^2y^2z^3)\mathbf{j} + (4x^2yz^4 + 4x^3y^2z^3)\mathbf{k}$$

Comparison with (*d*) illustrates the result $(\mathbf{A}\times\nabla)\phi = \mathbf{A}\times\nabla\phi$.

INVARIANCE

38. Two rectangular xyz and $x'y'z'$ coordinate systems having the same origin are rotated with respect to each other. Derive the transformation equations between the coordinates of a point in the two systems.

Let $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors of any point P in the two systems (see figure on page 58). Then since $\mathbf{r} = \mathbf{r}'$,

(*1*) $$x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

Now for any vector $\mathbf{A}$ we have (Problem 20, Chapter 2),

$$\mathbf{A} = (\mathbf{A}\cdot\mathbf{i}')\,\mathbf{i}' + (\mathbf{A}\cdot\mathbf{j}')\,\mathbf{j}' + (\mathbf{A}\cdot\mathbf{k}')\,\mathbf{k}'$$

Then letting $\mathbf{A} = \mathbf{i},\mathbf{j},\mathbf{k}$ in succession,

(*2*) $$\begin{cases} \mathbf{i} = (\mathbf{i}\cdot\mathbf{i}')\,\mathbf{i}' + (\mathbf{i}\cdot\mathbf{j}')\,\mathbf{j}' + (\mathbf{i}\cdot\mathbf{k}')\,\mathbf{k}' = l_{11}\,\mathbf{i}' + l_{21}\,\mathbf{j}' + l_{31}\,\mathbf{k}' \\ \mathbf{j} = (\mathbf{j}\cdot\mathbf{i}')\,\mathbf{i}' + (\mathbf{j}\cdot\mathbf{j}')\,\mathbf{j}' + (\mathbf{j}\cdot\mathbf{k}')\,\mathbf{k}' = l_{12}\,\mathbf{i}' + l_{22}\,\mathbf{j}' + l_{32}\,\mathbf{k}' \\ \mathbf{k} = (\mathbf{k}\cdot\mathbf{i}')\,\mathbf{i}' + (\mathbf{k}\cdot\mathbf{j}')\,\mathbf{j}' + (\mathbf{k}\cdot\mathbf{k}')\,\mathbf{k}' = l_{13}\,\mathbf{i}' + l_{23}\,\mathbf{j}' + l_{33}\,\mathbf{k}' \end{cases}$$

Substituting equations (*2*) in (*1*) and equating coefficients of $\mathbf{i}',\mathbf{j}',\mathbf{k}'$ we find

(*3*) $$x' = l_{11}x + l_{12}y + l_{13}z, \qquad y' = l_{21}x + l_{22}y + l_{23}z, \qquad z' = l_{31}x + l_{32}y + l_{33}z$$

the required transformation equations.

39. Prove $$\begin{aligned} \mathbf{i}' &= l_{11}\mathbf{i} + l_{12}\mathbf{j} + l_{13}\mathbf{k} \\ \mathbf{j}' &= l_{21}\mathbf{i} + l_{22}\mathbf{j} + l_{23}\mathbf{k} \\ \mathbf{k}' &= l_{31}\mathbf{i} + l_{32}\mathbf{j} + l_{33}\mathbf{k} \end{aligned}$$

For any vector $\mathbf{A}$ we have $\mathbf{A} = (\mathbf{A}\cdot\mathbf{i})\mathbf{i} + (\mathbf{A}\cdot\mathbf{j})\mathbf{j} + (\mathbf{A}\cdot\mathbf{k})\mathbf{k}$.

Then letting $\mathbf{A} = \mathbf{i}',\mathbf{j}',\mathbf{k}'$ in succession,

$$\begin{aligned} \mathbf{i}' &= (\mathbf{i}'\cdot\mathbf{i})\,\mathbf{i} + (\mathbf{i}'\cdot\mathbf{j})\,\mathbf{j} + (\mathbf{i}'\cdot\mathbf{k})\,\mathbf{k} = l_{11}\,\mathbf{i} + l_{12}\,\mathbf{j} + l_{13}\,\mathbf{k} \\ \mathbf{j}' &= (\mathbf{j}'\cdot\mathbf{i})\,\mathbf{i} + (\mathbf{j}'\cdot\mathbf{j})\,\mathbf{j} + (\mathbf{j}'\cdot\mathbf{k})\,\mathbf{k} = l_{21}\,\mathbf{i} + l_{22}\,\mathbf{j} + l_{23}\,\mathbf{k} \\ \mathbf{k}' &= (\mathbf{k}'\cdot\mathbf{i})\,\mathbf{i} + (\mathbf{k}'\cdot\mathbf{j})\,\mathbf{j} + (\mathbf{k}'\cdot\mathbf{k})\,\mathbf{k} = l_{31}\,\mathbf{i} + l_{32}\,\mathbf{j} + l_{33}\,\mathbf{k} \end{aligned}$$

40. Prove that $\sum_{p=1}^{3} l_{pm} l_{pn} = 1$ if $m = n$, and 0 if $m \neq n$, where m and n can assume any of the values 1, 2, 3.

From equations (2) of Problem 38,

$$\begin{aligned} \mathbf{i}\cdot\mathbf{i} = 1 &= (l_{11}\mathbf{i}' + l_{21}\mathbf{j}' + l_{31}\mathbf{k}')\cdot(l_{11}\mathbf{i}' + l_{21}\mathbf{j}' + l_{31}\mathbf{k}') \\ &= l_{11}^2 + l_{21}^2 + l_{31}^2 \end{aligned}$$

$$\begin{aligned} \mathbf{i}\cdot\mathbf{j} = 0 &= (l_{11}\mathbf{i}' + l_{21}\mathbf{j}' + l_{31}\mathbf{k}')\cdot(l_{12}\mathbf{i}' + l_{22}\mathbf{j}' + l_{32}\mathbf{k}') \\ &= l_{11}l_{12} + l_{21}l_{22} + l_{31}l_{32} \end{aligned}$$

$$\begin{aligned} \mathbf{i}\cdot\mathbf{k} = 0 &= (l_{11}\mathbf{i}' + l_{21}\mathbf{j}' + l_{31}\mathbf{k}')\cdot(l_{13}\mathbf{i}' + l_{23}\mathbf{j}' + l_{33}\mathbf{k}') \\ &= l_{11}l_{13} + l_{21}l_{23} + l_{31}l_{33} \end{aligned}$$

These establish the required result where $m=1$. By considering $\mathbf{j}\cdot\mathbf{i}, \mathbf{j}\cdot\mathbf{j}, \mathbf{j}\cdot\mathbf{k}, \mathbf{k}\cdot\mathbf{i}, \mathbf{k}\cdot\mathbf{j}$ and $\mathbf{k}\cdot\mathbf{k}$ the result can be proved for $m=2$ and $m=3$.

By writing $\delta_{mn} = \begin{cases} 1 \text{ if } m=n \\ 0 \text{ if } m\neq n \end{cases}$ the result can be written $\sum_{p=1}^{3} l_{pm} l_{pn} = \delta_{mn}$.

The symbol δ_{mn} is called *Kronecker's symbol.*

41. If $\phi(x,y,z)$ is a scalar invariant with respect to a rotation of axes, prove that gradϕ is a vector invariant under this transformation.

By hypothesis $\phi(x,y,z) = \phi'(x',y',z')$. To establish the desired result we must prove that

$$\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \frac{\partial\phi'}{\partial x'}\mathbf{i}' + \frac{\partial\phi'}{\partial y'}\mathbf{j}' + \frac{\partial\phi'}{\partial z'}\mathbf{k}'$$

Using the chain rule and the transformation equations (3) of Problem 38, we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\phi'}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial\phi'}{\partial z'}\frac{\partial z'}{\partial x} = \frac{\partial\phi'}{\partial x'}l_{11} + \frac{\partial\phi'}{\partial y'}l_{21} + \frac{\partial\phi'}{\partial z'}l_{31}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial\phi'}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial\phi'}{\partial z'}\frac{\partial z'}{\partial y} = \frac{\partial\phi'}{\partial x'}l_{12} + \frac{\partial\phi'}{\partial y'}l_{22} + \frac{\partial\phi'}{\partial z'}l_{32}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial\phi'}{\partial y'}\frac{\partial y'}{\partial z} + \frac{\partial\phi'}{\partial z'}\frac{\partial z'}{\partial z} = \frac{\partial\phi'}{\partial x'}l_{13} + \frac{\partial\phi'}{\partial y'}l_{23} + \frac{\partial\phi'}{\partial z'}l_{33}$$

Multiplying these equations by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively, adding and using Problem 39, the required result follows.

SUPPLEMENTARY PROBLEMS

42. If $\phi = 2xz^4 - x^2y$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2,-2,-1)$. *Ans.* $10\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}$, $2\sqrt{93}$

43. If $\mathbf{A} = 2x^2\,\mathbf{i} - 3yz\,\mathbf{j} + xz^2\,\mathbf{k}$ and $\phi = 2z - x^3y$, find $\mathbf{A}\cdot\nabla\phi$ and $\mathbf{A}\times\nabla\phi$ at the point $(1,-1,1)$.
Ans. 5, $7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$

44. If $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find (a) $\nabla(F+G)$ and (b) $\nabla(FG)$ at the point $(1,0,-2)$.
Ans. (a) $-4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$, (b) $-8\mathbf{j}$

45. Find $\nabla|\mathbf{r}|^3$. *Ans.* $3r\mathbf{r}$

46. Prove $\nabla f(r) = \dfrac{f'(r)\,\mathbf{r}}{r}$.

47. Evaluate $\nabla(3r^2 - 4\sqrt{r} + \dfrac{6}{\sqrt[3]{r}})$. *Ans.* $(6 - 2r^{-3/2} - 2r^{-7/3})\,\mathbf{r}$

48. If $\nabla U = 2r^4\,\mathbf{r}$, find U. *Ans.* $r^6/3$ + constant

49. Find $\phi(r)$ such that $\nabla\phi = \dfrac{\mathbf{r}}{r^5}$ and $\phi(1) = 0$. *Ans.* $\phi(r) = \dfrac{1}{3}(1 - \dfrac{1}{r^3})$

50. Find $\nabla\psi$ where $\psi = (x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}$. *Ans.* $(2 - r)\,e^{-r}\,\mathbf{r}$

51. If $\nabla\phi = 2xyz^3\,\mathbf{i} + x^2z^3\,\mathbf{j} + 3x^2yz^2\,\mathbf{k}$, find $\phi(x,y,z)$ if $\phi(1,-2,2) = 4$. *Ans.* $\phi = x^2yz^3 + 20$

52. If $\nabla\psi = (y^2 - 2xyz^3)\mathbf{i} + (3 + 2xy - x^2z^3)\mathbf{j} + (6z^3 - 3x^2yz^2)\mathbf{k}$, find ψ.
Ans. $\psi = xy^2 - x^2yz^3 + 3y + (3/2)\,z^4$ + constant

53. If U is a differentiable function of x,y,z, prove $\nabla U\cdot d\mathbf{r} = dU$.

54. If F is a differentiable function of x,y,z,t where x,y,z are differentiable functions of t, prove that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla F\cdot\frac{d\mathbf{r}}{dt}$$

55. If $\mathbf{A}$ is a constant vector, prove $\nabla(\mathbf{r}\cdot\mathbf{A}) = \mathbf{A}$.

56. If $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, show that $d\mathbf{A} = (\nabla A_1\cdot d\mathbf{r})\mathbf{i} + (\nabla A_2\cdot d\mathbf{r})\mathbf{j} + (\nabla A_3\cdot d\mathbf{r})\mathbf{k}$.

57. Prove $\nabla(\dfrac{F}{G}) = \dfrac{G\nabla F - F\nabla G}{G^2}$ if $G \neq 0$.

58. Find a unit vector which is perpendicular to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(1,2,5)$. *Ans.* $\dfrac{2\mathbf{i} + 4\mathbf{j} - \mathbf{k}}{\pm\sqrt{21}}$

59. Find the unit outward drawn normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3,1,-4)$.
Ans. $(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$

60. Find an equation for the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1,-3,2)$.
Ans. $2x - y - 3z + 1 = 0$

61. Find equations for the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(2,-1,5)$.
Ans. $4x - 2y - z = 5$, $\dfrac{x-2}{4} = \dfrac{y+1}{-2} = \dfrac{z-5}{-1}$ or $x = 4t + 2,\ y = -2t - 1,\ z = -t + 5$

62. Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
Ans. $376/7$

63. Find the directional derivative of $P = 4e^{2x - y + z}$ at the point $(1,1,-1)$ in a direction toward the point $(-3,5,6)$. *Ans.* $-20/9$

64. In what direction from the point (1,3,2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum? *Ans.* In the direction of the vector $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, $2\sqrt{14}$

65. Find the values of the constants a,b,c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the z axis. *Ans.* $a=6$, $b=24$, $c=-8$

66. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1,-2,1)$.

Ans. $\arccos \dfrac{3}{\sqrt{14}\sqrt{21}} = \arccos \dfrac{\sqrt{6}}{14} = 79^\circ 55'$

67. Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$. *Ans.* $a = 5/2$, $b=1$

68. (*a*) Let u and v be differentiable functions of x,y and z. Show that a necessary and sufficient condition that u and v are functionally related by the equation $F(u,v) = 0$ is that $\nabla u \times \nabla v = \mathbf{0}$.

(*b*) Determine whether $u = \arctan x + \arctan y$ and $v = \dfrac{x+y}{1-xy}$ are functionally related.

Ans. (*b*) Yes ($v = \tan u$)

69. (*a*) Show that a necessary and sufficient condition that $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ be functionally related through the equation $F(u,v,w) = 0$ is $\nabla u \cdot \nabla v \times \nabla w = 0$.

(*b*) Express $\nabla u \cdot \nabla v \times \nabla w$ in determinant form. This determinant is called the Jacobian of u,v,w with respect to x,y,z and is written $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ or $J(\dfrac{u,v,w}{x,y,z})$.

(*c*) Determine whether $u = x+y+z$, $v = x^2+y^2+z^2$ and $w = xy+yz+zx$ are functionally related.

Ans. (*b*) $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$ (*c*) Yes ($u^2 - v - 2w = 0$)

70. If $\mathbf{A} = 3xyz^2\,\mathbf{i} + 2xy^3\,\mathbf{j} - x^2yz\,\mathbf{k}$ and $\phi = 3x^2 - yz$, find (*a*) $\nabla\cdot\mathbf{A}$, (*b*) $\mathbf{A}\cdot\nabla\phi$, (*c*) $\nabla\cdot(\phi\mathbf{A})$, (*d*) $\nabla\cdot(\nabla\phi)$, at the point $(1,-1,1)$. *Ans.* (*a*) 4, (*b*) -15, (*c*) 1, (*d*) 6

71. Evaluate $\operatorname{div}(2x^2z\,\mathbf{i} - xy^2z\,\mathbf{j} + 3yz^2\,\mathbf{k})$. *Ans.* $4xz - 2xyz + 6yz$

72. If $\phi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$, find $\nabla^2\phi$. *Ans.* $6z + 24xy - 2z^3 - 6y^2z$

73. Evaluate $\nabla^2(\ln r)$. *Ans.* $1/r^2$

74. Prove $\nabla^2 r^n = n(n+1)r^{n-2}$ where n is a constant.

75. If $\mathbf{F} = (3x^2y - z)\mathbf{i} + (xz^3 + y^4)\mathbf{j} - 2x^3z^2\mathbf{k}$, find $\nabla(\nabla\cdot\mathbf{F})$ at the point $(2,-1,0)$. *Ans.* $-6\mathbf{i} + 24\mathbf{j} - 32\mathbf{k}$

76. If $\boldsymbol{\omega}$ is a constant vector and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, prove that $\operatorname{div}\mathbf{v} = 0$.

77. Prove $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$.

78. If $U = 3x^2y$, $V = xz^2 - 2y$ evaluate $\operatorname{grad}[(\operatorname{grad} U)\cdot(\operatorname{grad} V)]$. *Ans.* $(6yz^2 - 12x)\mathbf{i} + 6xz^2\mathbf{j} + 12xyz\,\mathbf{k}$

79. Evaluate $\nabla\cdot(r^3\mathbf{r})$. *Ans.* $6r^3$

80. Evaluate $\nabla\cdot[r\nabla(1/r^3)]$. *Ans.* $3r^{-4}$

81. Evaluate $\nabla^2[\nabla\cdot(\mathbf{r}/r^2)]$. *Ans.* $2r^{-4}$

82. If $\mathbf{A} = \mathbf{r}/r$, find grad div $\mathbf{A}$. *Ans.* $-2r^{-3}\,\mathbf{r}$

83. (*a*) Prove $\nabla^2 f(r) = \dfrac{d^2f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$. (*b*) Find $f(r)$ such that $\nabla^2 f(r) = 0$.

Ans. $f(r) = A + B/r$ where A and B are arbitrary constants.

84. Prove that the vector $\mathbf{A} = 3y^4z^2\,\mathbf{i} + 4x^3z^2\,\mathbf{j} - 3x^2y^2\,\mathbf{k}$ is solenoidal.

85. Show that $\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$ is not solenoidal but $\mathbf{B} = xyz^2\,\mathbf{A}$ is solenoidal.

86. Find the most general differentiable function $f(r)$ so that $f(r)\mathbf{r}$ is solenoidal.
Ans. $f(r) = C/r^3$ where C is an arbitrary constant.

87. Show that the vector field $\mathbf{V} = \dfrac{-x\,\mathbf{i} - y\,\mathbf{j}}{\sqrt{x^2 + y^2}}$ is a "sink field". Plot and give a physical interpretation.

88. If U and V are differentiable scalar fields, prove that $\nabla U \times \nabla V$ is solenoidal.

89. If $\mathbf{A} = 2xz^2\,\mathbf{i} - yz\,\mathbf{j} + 3xz^3\,\mathbf{k}$ and $\phi = x^2yz$, find
(*a*) $\nabla \times \mathbf{A}$, (*b*) curl $(\phi\mathbf{A})$, (*c*) $\nabla \times (\nabla \times \mathbf{A})$, (*d*) $\nabla[\mathbf{A} \cdot \text{curl}\,\mathbf{A}]$, (*e*) curl grad $(\phi\mathbf{A})$ at the point (1,1,1).
Ans. (*a*) $\mathbf{i} + \mathbf{j}$, (*b*) $5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$, (*c*) $5\mathbf{i} + 3\mathbf{k}$, (*d*) $-2\mathbf{i} + \mathbf{j} + 8\mathbf{k}$, (*e*) $\mathbf{0}$

90. If $F = x^2yz$, $G = xy - 3z^2$, find (*a*) $\nabla[(\nabla F)\cdot(\nabla G)]$, (*b*) $\nabla \cdot [(\nabla F) \times (\nabla G)]$, (*c*) $\nabla \times [(\nabla F) \times (\nabla G)]$.
Ans. (*a*) $(2y^2z + 3x^2z - 12xyz)\mathbf{i} + (4xyz - 6x^2z)\mathbf{j} + (2xy^2 + x^3 - 6x^2y)\mathbf{k}$
(*b*) 0
(*c*) $(x^2z - 24xyz)\mathbf{i} - (12x^2z + 2xyz)\mathbf{j} + (2xy^2 + 12yz^2 + x^3)\mathbf{k}$

91. Evaluate $\nabla \times (\mathbf{r}/r^2)$. *Ans.* $\mathbf{0}$

92. For what value of the constant a will the vector $\mathbf{A} = (axy - z^3)\,\mathbf{i} + (a-2)x^2\,\mathbf{j} + (1-a)xz^2\,\mathbf{k}$ have its curl identically equal to zero? *Ans.* $a = 4$

93. Prove curl $(\phi$ grad $\phi) = \mathbf{0}$.

94. Graph the vector fields $\mathbf{A} = x\,\mathbf{i} + y\,\mathbf{j}$ and $\mathbf{B} = y\,\mathbf{i} - x\,\mathbf{j}$. Compute the divergence and curl of each vector field and explain the physical significance of the results obtained.

95. If $\mathbf{A} = x^2z\,\mathbf{i} + yz^3\,\mathbf{j} - 3xy\,\mathbf{k}$, $\mathbf{B} = y^2\,\mathbf{i} - yz\,\mathbf{j} + 2x\,\mathbf{k}$ and $\phi = 2x^2 + yz$, find
(*a*) $\mathbf{A} \cdot (\nabla\phi)$, (*b*) $(\mathbf{A} \cdot \nabla)\phi$, (*c*) $(\mathbf{A} \cdot \nabla)\mathbf{B}$, (*d*) $\mathbf{B}(\mathbf{A} \cdot \nabla)$, (*e*) $(\nabla \cdot \mathbf{A})\mathbf{B}$.
Ans. (*a*) $4x^3z + yz^4 - 3xy^2$, (*b*) $4x^3z + yz^4 - 3xy^2$ (same as (*a*)),
(*c*) $2y^2z^3\,\mathbf{i} + (3xy^2 - yz^4)\mathbf{j} + 2x^2z\,\mathbf{k}$,
(*d*) the operator $(x^2y^2z\,\mathbf{i} - x^2yz^2\,\mathbf{j} + 2x^3z\,\mathbf{k})\dfrac{\partial}{\partial x} + (y^3z^3\,\mathbf{i} - y^2z^4\,\mathbf{j} + 2xyz^3\,\mathbf{k})\dfrac{\partial}{\partial y}$

$$+ (-3xy^3\,\mathbf{i} + 3xy^2z\,\mathbf{j} - 6x^2y\,\mathbf{k})\frac{\partial}{\partial z}$$

(*e*) $(2xy^2z + y^2z^3)\mathbf{i} - (2xyz^2 + yz^4)\mathbf{j} + (4x^2z + 2xz^3)\mathbf{k}$

96. If $\mathbf{A} = yz^2\,\mathbf{i} - 3xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$, $\mathbf{B} = 3x\,\mathbf{i} + 4z\,\mathbf{j} - xy\,\mathbf{k}$ and $\phi = xyz$, find
(*a*) $\mathbf{A} \times (\nabla\phi)$, (*b*) $(\mathbf{A} \times \nabla)\phi$, (*c*) $(\nabla \times \mathbf{A}) \times \mathbf{B}$, (*d*) $\mathbf{B} \cdot \nabla \times \mathbf{A}$.
Ans. (*a*) $-5x^2yz^2\,\mathbf{i} + xy^2z^2\,\mathbf{j} + 4xyz^3\,\mathbf{k}$
(*b*) $-5x^2yz^2\,\mathbf{i} + xy^2z^2\,\mathbf{j} + 4xyz^3\,\mathbf{k}$ (same as (*a*))
(*c*) $16z^3\,\mathbf{i} + (8x^2yz - 12xz^2)\mathbf{j} + 32xz^2\,\mathbf{k}$ (*d*) $24x^2z + 4xyz^2$

97. Find $\mathbf{A} \times (\nabla \times \mathbf{B})$ and $(\mathbf{A} \times \nabla) \times \mathbf{B}$ at the point (1,–1,2), if $\mathbf{A} = xz^2\,\mathbf{i} + 2y\,\mathbf{j} - 3xz\,\mathbf{k}$ and $\mathbf{B} = 3xz\,\mathbf{i} + 2yz\,\mathbf{j} - z^2\,\mathbf{k}$.
Ans. $\mathbf{A} \times (\nabla \times \mathbf{B}) = 18\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}$, $(\mathbf{A} \times \nabla) \times \mathbf{B} = 4\mathbf{j} + 76\mathbf{k}$

98. Prove $(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v})$.

99. Prove $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$.

100. Prove $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$.

101. Prove $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$.

102. Show that $\mathbf{A} = (6xy + z^3)\mathbf{i} + (3x^2 - z)\mathbf{j} + (3xz^2 - y)\mathbf{k}$ is irrotational. Find ϕ such that $\mathbf{A} = \nabla\phi$.
Ans. $\phi = 3x^2y + xz^3 - yz +$ constant

103. Show that $\mathbf{E} = \mathbf{r}/r^2$ is irrotational. Find ϕ such that $\mathbf{E} = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$.
Ans. $\phi = \ln(a/r)$

104. If **A** and **B** are irrotational, prove that $\mathbf{A}\times\mathbf{B}$ is solenoidal.

105. If $f(r)$ is differentiable, prove that $f(r)\mathbf{r}$ is irrotational.

106. Is there a differentiable vector function **V** such that (*a*) curl $\mathbf{V} = \mathbf{r}$, (*b*) curl $\mathbf{V} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$? If so, find **V**.
Ans. (*a*) No, (*b*) $\mathbf{V} = 3x\,\mathbf{j} + (2y - x)\mathbf{k} + \nabla\phi$, where ϕ is an arbitrary twice differentiable function.

107. Show that solutions to Maxwell's equations

$$\nabla\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \quad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \quad \nabla\cdot\mathbf{H} = 0, \quad \nabla\cdot\mathbf{E} = 4\pi\rho$$

where ρ is a function of x,y,z and c is the velocity of light, assumed constant, are given by

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{H} = \nabla\times\mathbf{A}$$

where **A** and ϕ, called the *vector and scalar potentials* respectively, satisfy the equations

$$(1)\ \nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0, \quad (2)\ \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \quad (3)\ \nabla^2\mathbf{A} = \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}$$

108. (*a*) Given the dyadic $\mathbf{\Phi} = \mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j} + \mathbf{k}\mathbf{k}$, evaluate $\mathbf{r}\cdot(\mathbf{\Phi}\cdot\mathbf{r})$ and $(\mathbf{r}\cdot\mathbf{\Phi})\cdot\mathbf{r}$. (*b*) Is there any ambiguity in writing $\mathbf{r}\cdot\mathbf{\Phi}\cdot\mathbf{r}$? (*c*) What does $\mathbf{r}\cdot\mathbf{\Phi}\cdot\mathbf{r} = 1$ represent geometrically?
Ans. (*a*) $\mathbf{r}\cdot(\mathbf{\Phi}\cdot\mathbf{r}) = (\mathbf{r}\cdot\mathbf{\Phi})\cdot\mathbf{r} = x^2+y^2+z^2$, (*b*) No, (*c*) Sphere of radius one with center at the origin.

109. (*a*) If $\mathbf{A} = xz\,\mathbf{i} - y^2\,\mathbf{j} + yz^2\,\mathbf{k}$ and $\mathbf{B} = 2z^2\,\mathbf{i} - xy\,\mathbf{j} + y^3\,\mathbf{k}$, give a possible significance to $(\mathbf{A}\times\nabla)\mathbf{B}$ at the point $(1,-1,1)$.
(*b*) Is it possible to write the result as $\mathbf{A}\times(\nabla\mathbf{B})$ by use of dyadics?
Ans. (*a*) $-4\mathbf{i}\mathbf{i} - \mathbf{i}\mathbf{j} + 3\mathbf{i}\mathbf{k} - \mathbf{j}\mathbf{j} - 4\mathbf{j}\mathbf{i} + 3\mathbf{k}\mathbf{k}$
(*b*) Yes, if the operations are suitably performed.

110. Prove that $\phi(x,y,z) = x^2 + y^2 + z^2$ is a scalar invariant under a rotation of axes.

111. If $\mathbf{A}(x,y,z)$ is an invariant differentiable vector field with respect to a rotation of axes, prove that (*a*) div **A** and (*b*) curl **A** are invariant scalar and vector fields respectively under the transformation.

112. Solve equations (*3*) of Solved Problem 38 for x,y,z in terms of x',y',z'.
Ans. $x = l_{11}x' + l_{21}y' + l_{31}z', \quad y = l_{12}x' + l_{22}y' + l_{32}z', \quad z = l_{13}x' + l_{23}y' + l_{33}z'$

113. If **A** and **B** are invariant under rotation show that $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ are also invariant.

114. Show that under a rotation

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \mathbf{i}'\frac{\partial}{\partial x'} + \mathbf{j}'\frac{\partial}{\partial y'} + \mathbf{k}'\frac{\partial}{\partial z'} = \nabla'$$

115. Show that the Laplacian operator is invariant under a rotation.