Chapter 5

VECTOR INTEGRATION

ORDINARY INTEGRALS OF VECTORS. Let $\mathbf{R}(u) = R_1(u)\mathbf{i} + R_2(u)\mathbf{j} + R_3(u)\mathbf{k}$ be a vector depending on a single scalar variable u, where $R_1(u)$, $R_2(u)$, $R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \mathbf{R}(u)\,du = \mathbf{i}\int R_1(u)\,du + \mathbf{j}\int R_2(u)\,du + \mathbf{k}\int R_3(u)\,du$$

is called an *indefinite integral* of $\mathbf{R}(u)$. If there exists a vector $\mathbf{S}(u)$ such that $\mathbf{R}(u) = \frac{d}{du}(\mathbf{S}(u))$, then

$$\int \mathbf{R}(u)\,du = \int \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}$$

where $\mathbf{c}$ is an *arbitrary constant vector* independent of u. The *definite integral* between limits $u=a$ and $u=b$ can in such case be written

$$\int_a^b \mathbf{R}(u)\,du = \int_a^b \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}\Big|_a^b = \mathbf{S}(b) - \mathbf{S}(a)$$

This integral can also be defined as a limit of a sum in a manner analogous to that of elementary integral calculus.

LINE INTEGRALS. Let $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where $\mathbf{r}(u)$ is the position vector of (x,y,z), define a curve C joining points P_1 and P_2, where $u=u_1$ and $u=u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $\mathbf{r}(u)$ has a continuous derivative. Let $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\mathbf{A}$ along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a *line integral.* If $\mathbf{A}$ is the force $\mathbf{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a *simple closed curve*, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint \mathbf{A}\cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

In aerodynamics and fluid mechanics this integral is called the *circulation* of $\mathbf{A}$ about C, where $\mathbf{A}$ represents the velocity of a fluid.

In general, any integral which is to be evaluated along a curve is called a line integral. Such integrals can be defined in terms of limits of sums as are the integrals of elementary calculus.

For methods of evaluation of line integrals, see the Solved Problems.

The following theorem is important.

THEOREM. If $\mathbf{A} = \nabla\phi$ everywhere in a region R of space, defined by $a_1 \leqq x \leqq a_2$, $b_1 \leqq y \leqq b_2$, $c_1 \leqq z \leqq c_2$, where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then

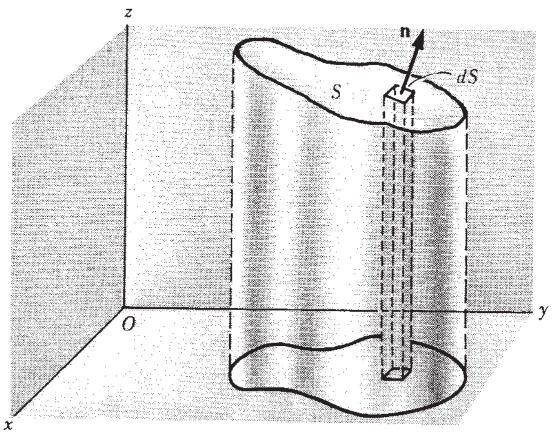
1. $\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r}$ is independent of the path C in R joining P_1 and P_2.

2. $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ around any closed curve C in R.

In such case $\mathbf{A}$ is called a *conservative vector field* and ϕ is its *scalar potential.*

A vector field $\mathbf{A}$ is conservative if and only if $\nabla\times\mathbf{A} = \mathbf{0}$, or equivalently $\mathbf{A} = \nabla\phi$. In such case $\mathbf{A}\cdot d\mathbf{r} = A_1\,dx + A_2\,dy + A_3\,dz = d\phi$, an exact differential. See Problems 10-14.

SURFACE INTEGRALS. Let S be a two-sided surface, such as shown in the figure below. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\mathbf{n}$ to any point of the positive side of S is called a *positive* or *outward drawn* unit normal.



Associate with the differential of surface area dS a vector $d\mathbf{S}$ whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\,dS$. The integral

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS$$

is an example of a surface integral called the *flux* of $\mathbf{A}$ over S. Other surface integrals are

$$\iint_S \phi\,dS, \quad \iint_S \phi\,\mathbf{n}\,dS, \quad \iint_S \mathbf{A}\times d\mathbf{S}$$

where ϕ is a scalar function. Such integrals can be defined in terms of limits of sums as in elementary calculus (see Problem 17).

The notation $\oiint_S$ is sometimes used to indicate integration over the closed surface S. Where no confusion can arise the notation $\oint_S$ may also be used.

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface in no more than one point. However, this does not pose any real problem since we can generally subdivide S into surfaces which do satisfy this restriction.

VOLUME INTEGRALS. Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V \mathbf{A}\,dV \quad \text{and} \quad \iiint_V \phi\,dV$$

are examples of *volume integrals* or *space integrals* as they are sometimes called. For evaluation of such integrals, see the Solved Problems.

SOLVED PROBLEMS

1. If $\mathbf{R}(u) = (u-u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}$, find (*a*) $\int \mathbf{R}(u)\,du$ and (*b*) $\int_1^2 \mathbf{R}(u)\,du$.

$$\begin{aligned}(a)\quad \int \mathbf{R}(u)\,du &= \int [(u-u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}]\,du \\ &= \mathbf{i}\int (u-u^2)\,du + \mathbf{j}\int 2u^3\,du + \mathbf{k}\int -3\,du \\ &= \mathbf{i}(\frac{u^2}{2} - \frac{u^3}{3} + c_1) + \mathbf{j}(\frac{u^4}{2} + c_2) + \mathbf{k}(-3u + c_3) \\ &= (\frac{u^2}{2} - \frac{u^3}{3})\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \\ &= (\frac{u^2}{2} - \frac{u^3}{3})\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}\end{aligned}$$

where $\mathbf{c}$ is the constant vector $c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

$$\begin{aligned}(b)\ \text{From }(a),\quad \int_1^2 \mathbf{R}(u)\,du &= (\frac{u^2}{2} - \frac{u^3}{3})\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}\,\Big|_1^2 \\ &= [(\frac{2^2}{2} - \frac{2^3}{3})\mathbf{i} + \frac{2^4}{2}\mathbf{j} - 3(2)\mathbf{k} + \mathbf{c}] - [(\frac{1^2}{2} - \frac{1^3}{3})\mathbf{i} + \frac{1^4}{2}\mathbf{j} - 3(1)\mathbf{k} + \mathbf{c}] \\ &= -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}\end{aligned}$$

Another Method.

$$\begin{aligned}\int_1^2 \mathbf{R}(u)\,du &= \mathbf{i}\int_1^2 (u-u^2)du + \mathbf{j}\int_1^2 2u^3\,du + \mathbf{k}\int_1^2 -3\,du \\ &= \mathbf{i}(\frac{u^2}{2} - \frac{u^3}{3})\Big|_1^2 + \mathbf{j}(\frac{u^4}{2})\Big|_1^2 + \mathbf{k}(-3u)\Big|_1^2 = -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}\end{aligned}$$

2. The acceleration of a particle at any time $t \geqq 0$ is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\cos 2t\,\mathbf{i} - 8\sin 2t\,\mathbf{j} + 16t\,\mathbf{k}$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

$$\begin{aligned}\text{Integrating,}\quad \mathbf{v} &= \mathbf{i}\int 12\cos 2t\,dt + \mathbf{j}\int -8\sin 2t\,dt + \mathbf{k}\int 16t\,dt \\ &= 6\sin 2t\,\mathbf{i} + 4\cos 2t\,\mathbf{j} + 8t^2\,\mathbf{k} + \mathbf{c}_1\end{aligned}$$

Putting $\mathbf{v}=\mathbf{0}$ when $t=0$, we find $\mathbf{0} = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} + \mathbf{c}_1$ and $\mathbf{c}_1 = -4\mathbf{j}$.

$$\begin{aligned}\text{Then}\quad \mathbf{v} &= 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\,\mathbf{k} \\ \text{so that}\quad \frac{d\mathbf{r}}{dt} &= 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\,\mathbf{k}.\end{aligned}$$

$$\begin{aligned}\text{Integrating,}\quad \mathbf{r} &= \mathbf{i}\int 6\sin 2t\,dt + \mathbf{j}\int (4\cos 2t - 4)\,dt + \mathbf{k}\int 8t^2\,dt \\ &= -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k} + \mathbf{c}_2\end{aligned}$$

Putting $\mathbf{r}=\mathbf{0}$ when $t=0$, $\mathbf{0} = -3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{c}_2$ and $\mathbf{c}_2 = 3\mathbf{i}$.

Then $\mathbf{r} = (3 - 3\cos 2t)\mathbf{i} + (2\sin 2t - 4t)\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k}$.

3. Evaluate $\displaystyle\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\,dt$.

$$\frac{d}{dt}(\mathbf{A}\times\frac{d\mathbf{A}}{dt}) = \mathbf{A}\times\frac{d^2\mathbf{A}}{dt^2} + \frac{d\mathbf{A}}{dt}\times\frac{d\mathbf{A}}{dt} = \mathbf{A}\times\frac{d^2\mathbf{A}}{dt^2}$$

Integrating, $\displaystyle\int \mathbf{A}\times\frac{d^2\mathbf{A}}{dt^2}\,dt = \int \frac{d}{dt}(\mathbf{A}\times\frac{d\mathbf{A}}{dt})\,dt = \mathbf{A}\times\frac{d\mathbf{A}}{dt} + \mathbf{c}$.

4. The equation of motion of a particle P of mass m is given by

$$m\,\frac{d^2\mathbf{r}}{dt^2} = f(r)\,\mathbf{r}_1$$

where $\mathbf{r}$ is the position vector of P measured from an origin O, $\mathbf{r}_1$ is a unit vector in the direction $\mathbf{r}$, and $f(r)$ is a function of the distance of P from O.

(*a*) Show that $\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \mathbf{c}$ where $\mathbf{c}$ is a constant vector.

(*b*) Interpret physically the cases $f(r) < 0$ and $f(r) > 0$.

(*c*) Interpret the result in (*a*) geometrically.

(*d*) Describe how the results obtained relate to the motion of the planets in our solar system.

(*a*) Multiply both sides of $m\,\frac{d^2\mathbf{r}}{dt^2} = f(r)\,\mathbf{r}_1$ by $\mathbf{r}\times$. Then

$$m\,\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = f(r)\,\mathbf{r}\times\mathbf{r}_1 = \mathbf{0}$$

since $\mathbf{r}$ and $\mathbf{r}_1$ are collinear and so $\mathbf{r}\times\mathbf{r}_1 = \mathbf{0}$. Thus

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = \mathbf{0} \quad\text{and}\quad \frac{d}{dt}(\mathbf{r}\times\frac{d\mathbf{r}}{dt}) = \mathbf{0}$$

Integrating, $\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \mathbf{c}$, where $\mathbf{c}$ is a constant vector. (Compare with Problem 3).

(*b*) If $f(r) < 0$ the acceleration $\frac{d^2\mathbf{r}}{dt^2}$ has direction opposite to $\mathbf{r}_1$; hence the force is directed toward O and the particle is always *attracted* toward O.

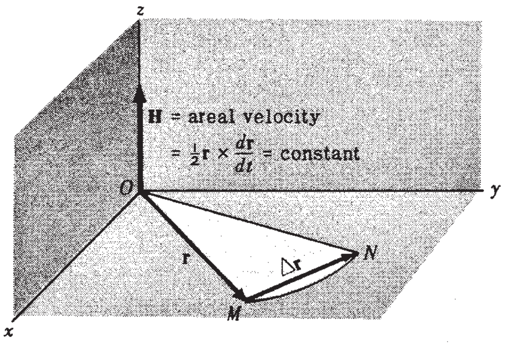
If $f(r) > 0$ the force is directed away from O and the particle is under the influence of a *repulsive* force at O.

A force directed toward or away from a fixed point O and having magnitude depending only on the distance r from O is called a *central force*.

(*c*) In time Δt the particle moves from M to N (see adjoining figure). The area swept out by the position vector in this time is approximately half the area of a parallelogram with sides $\mathbf{r}$ and $\Delta\mathbf{r}$, or $\frac{1}{2}\mathbf{r}\times\Delta\mathbf{r}$. Then the approximate area swept out by the radius vector per unit time is $\frac{1}{2}\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}$; hence the instantaneous time rate of change in area is

$$\lim_{\Delta t\to 0}\ \tfrac{1}{2}\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t} = \tfrac{1}{2}\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \tfrac{1}{2}\mathbf{r}\times\mathbf{v}$$

where $\mathbf{v}$ is the instantaneous velocity of the parti-



cle. The quantity $\mathbf{H} = \frac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v}$ is called the *areal velocity*. From part (*a*),

$$\text{Areal Velocity} = \mathbf{H} = \tfrac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \text{constant}$$

Since $\mathbf{r} \cdot \mathbf{H} = 0$, the motion takes place in a plane, which we take as the xy plane in the figure above.

(*d*) A planet (such as the earth) is attracted toward the sun according to Newton's universal law of gravitation, which states that any two objects of mass m and M respectively are attracted toward each other with a force of magnitude $F = \frac{GMm}{r^2}$, where r is the distance between objects and G is a universal constant. Let m and M be the masses of the planet and sun respectively and choose a set of coordinate axes with the origin O at the sun. Then the equation of motion of the planet is

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^2}\mathbf{r}_1 \qquad \text{or} \qquad \frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^2}\mathbf{r}_1$$

assuming the influence of the other planets to be negligible.

According to part (*c*), a planet moves around the sun so that its position vector sweeps out equal areas in equal times. This result and that of Problem 5 are two of Kepler's famous three laws which he deduced empirically from volumes of data compiled by the astronomer Tycho Brahe. These laws enabled Newton to formulate his universal law of gravitation. For Kepler's third law see Problem 36.

5. Show that the path of a planet around the sun is an ellipse with the sun at one focus.

From Problems 4(*c*) and 4(*d*),

$$(1) \qquad \frac{d\mathbf{v}}{dt} = -\frac{GM}{r^2}\mathbf{r}_1$$

$$(2) \qquad \mathbf{r} \times \mathbf{v} = 2\mathbf{H} = \mathbf{h}$$

Now $\mathbf{r} = r\,\mathbf{r}_1$, $\frac{d\mathbf{r}}{dt} = r\frac{d\mathbf{r}_1}{dt} + \frac{dr}{dt}\mathbf{r}_1$ so that

$$(3) \qquad \mathbf{h} = \mathbf{r} \times \mathbf{v} = r\,\mathbf{r}_1 \times \left(r\frac{d\mathbf{r}_1}{dt} + \frac{dr}{dt}\mathbf{r}_1\right) = r^2\,\mathbf{r}_1 \times \frac{d\mathbf{r}_1}{dt}$$

From (*1*),
$$\frac{d\mathbf{v}}{dt} \times \mathbf{h} = -\frac{GM}{r^2}\mathbf{r}_1 \times \mathbf{h} = -GM\,\mathbf{r}_1 \times \left(\mathbf{r}_1 \times \frac{d\mathbf{r}_1}{dt}\right)$$
$$= -GM\left[\left(\mathbf{r}_1 \cdot \frac{d\mathbf{r}_1}{dt}\right)\mathbf{r}_1 - (\mathbf{r}_1 \cdot \mathbf{r}_1)\frac{d\mathbf{r}_1}{dt}\right] = GM\frac{d\mathbf{r}_1}{dt}$$

using equation (*3*) and the fact that $\mathbf{r}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0$ (Problem 9, Chapter 3).

But since $\mathbf{h}$ is a constant vector, $\frac{d\mathbf{v}}{dt} \times \mathbf{h} = \frac{d}{dt}(\mathbf{v} \times \mathbf{h})$ so that

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{h}) = GM\frac{d\mathbf{r}_1}{dt}$$

Integrating,
$$\mathbf{v} \times \mathbf{h} = GM\,\mathbf{r}_1 + \mathbf{p}$$

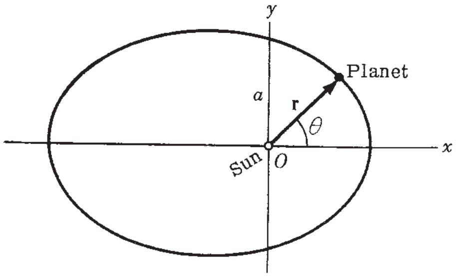
from which
$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = GM\,\mathbf{r} \cdot \mathbf{r}_1 + \mathbf{r} \cdot \mathbf{p}$$
$$= GMr + r\,\mathbf{r}_1 \cdot \mathbf{p} = GMr + rp\cos\theta$$

where $\mathbf{p}$ is an arbitrary constant vector with magnitude p, and θ is the angle between $\mathbf{p}$ and $\mathbf{r}_1$.

Since $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = h^2$, we have $h^2 = GMr + rp\cos\theta$ and

$$r = \frac{h^2}{GM + p\cos\theta} = \frac{h^2/GM}{1 + (p/GM)\cos\theta}$$

From analytic geometry, the polar equation of a conic section with focus at the origin and eccentricity ϵ is $r = \dfrac{a}{1+\epsilon\cos\theta}$ where a is a constant. Comparing this with the equation derived, it is seen that the required orbit is a conic section with eccentricity $\epsilon = p/GM$. The orbit is an ellipse, parabola or hyperbola according as ϵ is less than, equal to or greater than one. Since orbits of planets are closed curves it follows that they must be ellipses.



Ellipse $r = \dfrac{a}{1+\epsilon\cos\theta}$

LINE INTEGRALS

6. If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

(*a*) $x = t,\ y = t^2,\ z = t^3$.

(*b*) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

(*c*) the straight line joining (0,0,0) and (1,1,1).

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C \left[(3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}\right]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(*a*) If $x = t$, $y = t^2$, $z = t^3$, points (0,0,0) and (1,1,1) correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$

$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$

Another Method.

Along C, $\mathbf{A} = 9t^2\mathbf{i} - 14t^5\mathbf{j} + 20t^7\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$.

Then $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k})\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

(*b*) Along the straight line from (0,0,0) to (1,0,0) $y = 0$, $z = 0$, $dy = 0$, $dz = 0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \left(3x^2+6(0)\right)dx - 14(0)(0)(0) + 20x(0)^2(0) = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1$$

Along the straight line from (1,0,0) to (1,1,0) $x = 1$, $z = 0$, $dx = 0$, $dz = 0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2+6y)0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$

Along the straight line from (1,1,0) to (1,1,1) $x=1,\ y=1,\ dx=0,\ dy=0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} \left(3(1)^2+6(1)\right)0 \;-\; 14(1)\,z(0) \;+\; 20(1)\,z^2\,dz \;=\; \int_{z=0}^{1} 20z^2\,dz \;=\; \frac{20z^3}{3}\Big|_0^1 \;=\; \frac{20}{3}$$

Adding, $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; 1 + 0 + \frac{20}{3} \;=\; \frac{23}{3}$$

(*c*) The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x=t,\ y=t,\ z=t$. Then

$$\begin{aligned}\int_C \mathbf{A}\cdot d\mathbf{r} &= \int_{t=0}^{1} (3t^2+6t)\,dt \;-\; 14(t)(t)\,dt \;+\; 20(t)(t)^2\,dt \\ &= \int_{t=0}^{1} (3t^2+6t-14t^2+20t^3)\,dt \;=\; \int_{t=0}^{1} (6t-11t^2+20t^3)\,dt \;=\; \frac{13}{3}\end{aligned}$$

7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x=t^2+1,\ y=2t^2,\ z=t^3$ from $t=1$ to $t=2$.

$$\begin{aligned}\text{Total work} \;=\; \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) \\ &= \int_C 3xy\,dx \;-\; 5z\,dy \;+\; 10x\,dz \\ &= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,d(t^2+1) \;-\; 5(t^3)\,d(2t^2) \;+\; 10(t^2+1)\,d(t^3) \\ &= \int_1^2 (12t^5+10t^4+12t^3+30t^2)\,dt \;=\; 303\end{aligned}$$

8. If $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane, $y=2x^2$, from (0,0) to (1,2).

Since the integration is performed in the xy plane (z=0), we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j}) \\ &= \int_C 3xy\,dx \;-\; y^2\,dy\end{aligned}$$

First Method. Let $x=t$ in $y=2x^2$. Then the parametric equations of C are $x=t,\ y=2t^2$. Points (0,0) and (1,2) correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{t=0}^{1} 3(t)(2t^2)\,dt \;-\; (2t^2)^2\,d(2t^2) \;=\; \int_{t=0}^{1} (6t^3-16t^5)\,dt \;=\; -\frac{7}{6}$$

Second Method. Substitute $y=2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{x=0}^{1} 3x(2x^2)\,dx \;-\; (2x^2)^2\,d(2x^2) \;=\; \int_{x=0}^{1} (6x^3-16x^5)\,dx \;=\; -\frac{7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been 7/6 instead of $-7/6$.

9. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}$$

In the plane $z=0$, $\mathbf{F} = (2x-y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ so that the work done is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [(2x-y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j}]$$

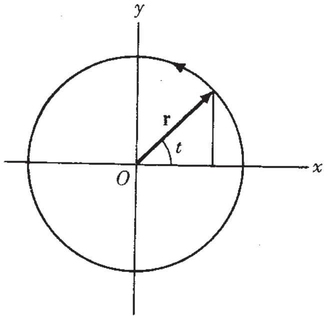
$$= \int_C (2x-y)\,dx + (x+y)\,dy$$

Choose the parametric equations of the circle as $x = 3\cos t$, $y = 3\sin t$ where t varies from 0 to 2π (see adjoining figure). Then the line integral equals

$$\int_{t=0}^{2\pi} [2(3\cos t) - 3\sin t][-3\sin t]\,dt + [3\cos t + 3\sin t][3\cos t]\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt = 9t - \frac{9}{2}\sin^2 t\Big|_0^{2\pi} = 18\pi$$

In traversing C we have chosen the counterclockwise direction indicated in the adjoining figure. We call this the *positive* direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be -18π.



$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$
$= 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$

10. (*a*) If $\mathbf{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

(*b*) Conversely, if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points, show that there exists a function ϕ such that $\mathbf{F} = \nabla\phi$.

(*a*) Work done $= \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\mathbf{r}$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them. This is true of course only if $\phi(x,y,z)$ is single-valued at all points P_1 and P_2.

(*b*) Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. By hypothesis, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points, which we take as (x_1, y_1, z_1) and (x,y,z) respectively. Then

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} F_1\,dx + F_2\,dy + F_3\,dz$$

is independent of the path joining (x_1, y_1, z_1) and (x,y,z). Thus

$$\phi(x+\Delta x, y, z) - \phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r} - \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{(x,y,z)}^{(x_1,y_1,z_1)} \mathbf{F}\cdot d\mathbf{r} + \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{(x,y,z)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1\,dx + F_2\,dy + F_3\,dz$$

Since the last integral must be independent of the path joining (x,y,z) and $(x+\Delta x,y,z)$, we may choose the path to be a straight line joining these points so that dy and dz are zero. Then

$$\frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1\,dx$$

Taking the limit of both sides as $\Delta x \to 0$, we have $\frac{\partial\phi}{\partial x} = F_1$.

Similarly, we can show that $\frac{\partial\phi}{\partial y} = F_2$ and $\frac{\partial\phi}{\partial z} = F_3$.

Then $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi$.

If $\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining P_1 and P_2, then $\mathbf{F}$ is called a *conservative field.* It follows that if $\mathbf{F} = \nabla\phi$ then $\mathbf{F}$ is conservative, and conversely.

Proof using vectors. If the line integral is independent of the path, then

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\,ds$$

By differentiation, $\frac{d\phi}{ds} = \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}$. But $\frac{d\phi}{ds} = \nabla\phi\cdot\frac{d\mathbf{r}}{ds}$ so that $(\nabla\phi - \mathbf{F})\cdot\frac{d\mathbf{r}}{ds} = 0$.

Since this must hold irrespective of $\frac{d\mathbf{r}}{ds}$, we have $\mathbf{F} = \nabla\phi$.

11. (*a*) If $\mathbf{F}$ is a conservative field, prove that $\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational).
(*b*) Conversely, if $\nabla\times\mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational), prove that $\mathbf{F}$ is conservative.

(*a*) If $\mathbf{F}$ is a conservative field, then by Problem 10, $\mathbf{F} = \nabla\phi$.
Thus $\text{curl}\,\mathbf{F} = \nabla\times\nabla\phi = \mathbf{0}$ (see Problem 27(*a*), Chapter 4).

(*b*) If $\nabla\times\mathbf{F} = \mathbf{0}$, then $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbf{0}$ and thus

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

We must prove that $\mathbf{F} = \nabla\phi$ follows as a consequence of this.

The work done in moving a particle from (x_1,y_1,z_1) to (x,y,z) in the force field $\mathbf{F}$ is

$$\int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

where C is a path joining (x_1, y_1, z_1) and (x,y,z). Let us choose as a particular path the straight line segments from (x_1, y_1, z_1) to (x, y_1, z_1) to (x,y,z_1) to (x,y,z) and call $\phi(x,y,z)$ the work done along this particular path. Then

$$\phi(x,y,z) = \int_{x_1}^{x} F_1(x, y_1, z_1)\,dx + \int_{y_1}^{y} F_2(x, y, z_1)\,dy + \int_{z_1}^{z} F_3(x,y,z)\,dz$$

It follows that

$$\frac{\partial\phi}{\partial z} = F_3(x,y,z)$$

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= F_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial F_3}{\partial y}(x,y,z)\,dz \\ &= F_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial F_2}{\partial z}(x,y,z)\,dz \\ &= F_2(x, y, z_1) + F_2(x,y,z)\Big|_{z_1}^{z} = F_2(x, y, z_1) + F_2(x,y,z) - F_2(x, y, z_1) = F_2(x,y,z)\end{aligned}$$

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= F_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial F_2}{\partial x}(x, y, z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_3}{\partial x}(x,y,z)\,dz \\ &= F_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial F_1}{\partial y}(x, y, z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_1}{\partial z}(x,y,z)\,dz \\ &= F_1(x, y_1, z_1) + F_1(x, y, z_1)\Big|_{y_1}^{y} + F_1(x,y,z)\Big|_{z_1}^{z} \\ &= F_1(x, y_1, z_1) + F_1(x, y, z_1) - F_1(x, y_1, z_1) + F_1(x,y,z) - F_1(x, y, z_1) = F_1(x,y,z)\end{aligned}$$

Then $\quad \mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi.$

Thus a necessary and sufficient condition that a field $\mathbf{F}$ be conservative is that $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$.

12. (*a*) Show that $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ is a conservative force field. (*b*) Find the scalar potential. (*c*) Find the work done in moving an object in this field from (1,–2,1) to (3,1,4).

(*a*) From Problem 11, a necessary and sufficient condition that a force will be conservative is that $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$.

Now $\quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = \mathbf{0}.$

Thus $\mathbf{F}$ *is a conservative force field.*

(*b*) *First Method.*

By Problem 10, $\mathbf{F} = \nabla\phi$ or $\dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$. Then

$$(1)\ \frac{\partial\phi}{\partial x} = 2xy + z^3 \qquad (2)\ \frac{\partial\phi}{\partial y} = x^2 \qquad (3)\ \frac{\partial\phi}{\partial z} = 3xz^2$$

Integrating, we find from (*1*), (*2*) and (*3*) respectively,

$$\begin{aligned} \phi &= x^2y + xz^3 + f(y,z) \\ \phi &= x^2y + g(x,z) \\ \phi &= xz^3 + h(x,y) \end{aligned}$$

These agree if we choose $f(y,z) = 0$, $g(x,z) = xz^3$, $h(x,y) = x^2y$ so that $\phi = x^2y + xz^3$ to which may be added any constant.

Second Method.

Since $\mathbf{F}$ is conservative, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining (x_1, y_1, z_1) and (x,y,z). Using the method of Problem 11(*b*),

$$\begin{aligned} \phi(x,y,z) &= \int_{x_1}^{x} (2xy_1 + z_1^3)\,dx + \int_{y_1}^{y} x^2\,dy + \int_{z_1}^{z} 3xz^2\,dz \\ &= (x^2y_1 + xz_1^3)\Big|_{x_1}^{x} + x^2y\Big|_{y_1}^{y} + xz^3\Big|_{z_1}^{z} \\ &= x^2y_1 + xz_1^3 - x_1^2y_1 - x_1z_1^3 + x^2y - x^2y_1 + xz^3 - xz_1^3 \\ &= x^2y + xz^3 - x_1^2y_1 - x_1z_1^3 = x^2y + xz^3 + \text{constant} \end{aligned}$$

Third Method. $\mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r} = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz = d\phi$

Then

$$\begin{aligned} d\phi = \mathbf{F}\cdot d\mathbf{r} &= (2xy + z^3)\,dx + x^2\,dy + 3xz^2\,dz \\ &= (2xy\,dx + x^2\,dy) + (z^3\,dx + 3xz^2\,dz) \\ &= d(x^2y) + d(xz^3) = d(x^2y + xz^3) \end{aligned}$$

and $\phi = x^2y + xz^3 + \text{constant}$.

(*c*) Work done $= \displaystyle\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$

$$\begin{aligned} &= \int_{P_1}^{P_2} (2xy + z^3)\,dx + x^2\,dy + 3xz^2\,dz \\ &= \int_{P_1}^{P_2} d(x^2y + xz^3) = x^2y + xz^3\Big|_{P_1}^{P_2} = x^2y + xz^3\Big|_{(1,-2,1)}^{(3,1,4)} = 202 \end{aligned}$$

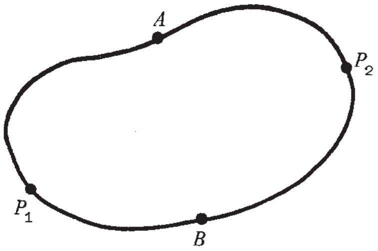
Another Method.

From part (*b*), $\phi(x,y,z) = x^2y + xz^3 + \text{constant}$.

Then work done $= \phi(3,1,4) - \phi(1,-2,1) = 202$.

13. Prove that if $\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed paths in the region and conversely.

Let $P_1AP_2BP_1$ (see adjacent figure) be a closed curve. Then



$$\oint \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} \;+\; \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r}$$
$$= \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} \;-\; \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r} \;=\; 0$$

since the integral from P_1 to P_2 along a path through A is the same as that along a path through B, by hypothesis.

Conversely if $\oint \mathbf{F}\cdot d\mathbf{r} = 0$, then

$$\int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} \;+\; \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} \;-\; \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r} \;=\; 0$$

so that, $\displaystyle\int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r}$.

14. (*a*) Show that a necessary and sufficient condition that $F_1\,dx + F_2\,dy + F_3\,dz$ be an exact differential is that $\nabla\times\mathbf{F} = \mathbf{0}$ where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

(*b*) Show that $(y^2z^3\cos x - 4x^3z)\,dx + 2z^3y\sin x\,dy + (3y^2z^2\sin x - x^4)\,dz$ is an exact differential of a function ϕ and find ϕ.

(*a*) Suppose $F_1\,dx + F_2\,dy + F_3\,dz = d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$, an exact differential. Then since x, y and z are independent variables,

$$F_1 = \frac{\partial\phi}{\partial x}, \qquad F_2 = \frac{\partial\phi}{\partial y}, \qquad F_3 = \frac{\partial\phi}{\partial z}$$

and so $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi$. Thus $\nabla\times\mathbf{F} = \nabla\times\nabla\phi = \mathbf{0}$.

Conversely if $\nabla\times\mathbf{F} = \mathbf{0}$ then by Problem 11, $\mathbf{F} = \nabla\phi$ and so $\mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r} = d\phi$, i.e. $F_1\,dx + F_2\,dy + F_3\,dz = d\phi$, an exact differential.

(*b*) $\mathbf{F} = (y^2z^3\cos x - 4x^3z)\mathbf{i} + 2z^3y\sin x\,\mathbf{j} + (3y^2z^2\sin x - x^4)\mathbf{k}$ and $\nabla\times\mathbf{F}$ is computed to be zero, so that by part (*a*)

$$(y^2z^3\cos x - 4x^3z)\,dx \;+\; 2z^3y\sin x\,dy \;+\; (3y^2z^2\sin x - x^4)\,dz \;=\; d\phi$$

By any of the methods of Problem 12 we find $\phi = y^2z^3\sin x - x^4z + \text{constant}$.

15. Let $\mathbf{F}$ be a conservative force field such that $\mathbf{F} = -\nabla\phi$. Suppose a particle of constant mass m to move in this field. If A and B are any two points in space, prove that

$$\phi(A) + \tfrac{1}{2}mv_A^2 \;=\; \phi(B) + \tfrac{1}{2}mv_B^2$$

where v_A and v_B are the magnitudes of the velocities of the particle at A and B respectively.

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{r}}{dt^2}. \qquad \text{Then} \quad \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = m\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2} = \frac{m}{2}\,\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2.$$

Integrating, $$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \frac{m}{2}v^2\Big|_A^B = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2.$$

If $\mathbf{F} = -\nabla\phi$, $$\int_A^B \mathbf{F}\cdot d\mathbf{r} = -\int_A^B \nabla\phi\cdot d\mathbf{r} = -\int_A^B d\phi = \phi(A) - \phi(B).$$

Then $\phi(A) - \phi(B) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$ and the result follows.

$\phi(A)$ is called the *potential energy* at A and $\frac{1}{2}mv_A^2$ is the *kinetic energy* at A. The result states that the total energy at A equals the total energy at B (conservation of energy). Note the use of the minus sign in $\mathbf{F} = -\nabla\phi$.

16. If $\phi = 2xyz^2$, $\mathbf{F} = xy\,\mathbf{i} - z\,\mathbf{j} + x^2\,\mathbf{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals (*a*) $\int_C \phi\, d\mathbf{r}$, (*b*) $\int_C \mathbf{F}\times d\mathbf{r}$.

(*a*) Along C,

$$\phi = 2xyz^2 = 2(t^2)(2t)(t^3)^2 = 4t^9,$$
$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = t^2\,\mathbf{i} + 2t\,\mathbf{j} + t^3\,\mathbf{k}, \quad \text{and}$$
$$d\mathbf{r} = (2t\,\mathbf{i} + 2\mathbf{j} + 3t^2\,\mathbf{k})\,dt. \quad \text{Then}$$

$$\int_C \phi\, d\mathbf{r} = \int_{t=0}^1 4t^9(2t\,\mathbf{i} + 2\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$
$$= \mathbf{i}\int_0^1 8t^{10}\,dt + \mathbf{j}\int_0^1 8t^9\,dt + \mathbf{k}\int_0^1 12t^{11}\,dt = \frac{8}{11}\mathbf{i} + \frac{4}{5}\mathbf{j} + \mathbf{k}$$

(*b*) Along C, $\mathbf{F} = xy\,\mathbf{i} - z\,\mathbf{j} + x^2\,\mathbf{k} = 2t^3\,\mathbf{i} - t^3\,\mathbf{j} + t^4\,\mathbf{k}$.

Then $$\mathbf{F}\times d\mathbf{r} = (2t^3\,\mathbf{i} - t^3\,\mathbf{j} + t^4\,\mathbf{k}) \times (2t\,\mathbf{i} + 2\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt = [(-3t^5 - 2t^4)\mathbf{i} + (2t^5 - 6t^5)\mathbf{j} + (4t^3 + 2t^4)\mathbf{k}]\,dt$$

and $$\int_C \mathbf{F}\times d\mathbf{r} = \mathbf{i}\int_0^1 (-3t^5 - 2t^4)\,dt + \mathbf{j}\int_0^1 (-4t^5)\,dt + \mathbf{k}\int_0^1 (4t^3 + 2t^4)\,dt$$
$$= -\frac{9}{10}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{7}{5}\mathbf{k}$$

SURFACE INTEGRALS.

17. Give a definition of $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$ over a surface S in terms of limit of a sum.

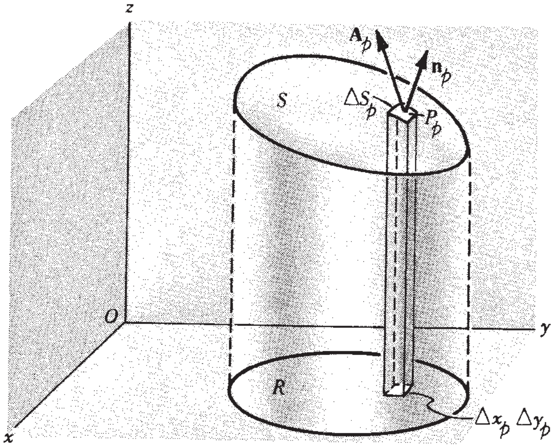
Subdivide the area S into M elements of area ΔS_p where $p = 1,2,3,\dots, M$. Choose any point P_p within ΔS_p whose coordinates are (x_p, y_p, z_p). Define $\mathbf{A}(x_p, y_p, z_p) = \mathbf{A}_p$. Let $\mathbf{n}_p$ be the positive unit normal to ΔS_p at P. Form the sum

$$\sum_{p=1}^{M} \mathbf{A}_p \cdot \mathbf{n}_p \, \Delta S_p$$

where $\mathbf{A}_p \cdot \mathbf{n}_p$ is the normal component of $\mathbf{A}_p$ at P_p.

Now take the limit of this sum as $M \to \infty$ in such a way that the largest dimension of each ΔS_p approaches zero. This limit, if it exists, is called the surface integral of the normal component of $\mathbf{A}$ over S and is denoted by

$$\iint_S \mathbf{A} \cdot \mathbf{n} \, dS$$



18. Suppose that the surface S has projection R on the xy plane (see figure of Prob.17). Show that

$$\iint_S \mathbf{A} \cdot \mathbf{n} \, dS \quad = \quad \iint_R \mathbf{A} \cdot \mathbf{n} \frac{dx\, dy}{|\mathbf{n} \cdot \mathbf{k}|}$$

By Problem 17, the surface integral is the limit of the sum

$$(1) \qquad \sum_{p=1}^{M} \mathbf{A}_p \cdot \mathbf{n}_p \, \Delta S_p$$

The projection of ΔS_p on the xy plane is $|(\mathbf{n}_p \, \Delta S_p) \cdot \mathbf{k}|$ or $|\mathbf{n}_p \cdot \mathbf{k}| \, \Delta S_p$ which is equal to $\Delta x_p \Delta y_p$ so that $\Delta S_p = \dfrac{\Delta x_p \Delta y_p}{|\mathbf{n}_p \cdot \mathbf{k}|}$. Thus the sum (1) becomes

$$(2) \qquad \sum_{p=1}^{M} \mathbf{A}_p \cdot \mathbf{n}_p \frac{\Delta x_p \Delta y_p}{|\mathbf{n}_p \cdot \mathbf{k}|}$$

By the fundamental theorem of integral calculus the limit of this sum as $M \to \infty$ in such a manner that the largest Δx_p and Δy_p approach zero is

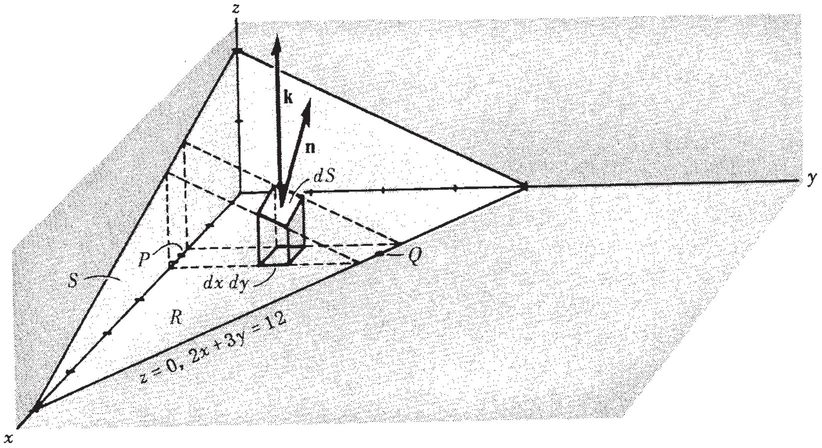
$$\iint_R \mathbf{A} \cdot \mathbf{n} \frac{dx\, dy}{|\mathbf{n} \cdot \mathbf{k}|}$$

and so the required result follows.

Strictly speaking, the result $\Delta S_p = \dfrac{\Delta x_p \Delta y_p}{|\mathbf{n}_p \cdot \mathbf{k}|}$ is only approximately true but it can be shown on closer examination that they differ from each other by infinitesimals of order higher than $\Delta x_p \Delta y_p$, and using this the *limits* of (1) and (2) can in fact be shown equal.

19. Evaluate $\displaystyle\iint_S \mathbf{A} \cdot \mathbf{n} \, dS$, where $\mathbf{A} = 18z\,\mathbf{i} - 12\mathbf{j} + 3y\,\mathbf{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

The surface S and its projection R on the xy plane are shown in the figure below.



From Problem 17,

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

To obtain **n** note that a vector perpendicular to the surface $2x+3y+6z = 12$ is given by $\nabla(2x+3y+6z) = 2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$ (see Problem 5 of Chapter 4). Then a unit normal to any point of S (see figure above) is

$$\mathbf{n} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

Thus $\mathbf{n}\cdot\mathbf{k} = (\frac{2}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k})\cdot\mathbf{k} = \frac{6}{7}$ and so $\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{7}{6}\,dx\,dy$.

Also $\mathbf{A}\cdot\mathbf{n} = (18z\,\mathbf{i} - 12\mathbf{j} + 3y\,\mathbf{k})\cdot(\frac{2}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}) = \dfrac{36z-36+18y}{7} = \dfrac{36-12x}{7}$,

using the fact that $z = \dfrac{12-2x-3y}{6}$ from the equation of S. Then

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_R \left(\frac{36-12x}{7}\right)\frac{7}{6}\,dx\,dy = \iint_R (6-2x)\,dx\,dy$$

To evaluate this double integral over R, keep x fixed and integrate with respect to y from $y=0$ (P in the figure above) to $y = \dfrac{12-2x}{3}$ (Q in the figure above); then integrate with respect to x from $x=0$ to $x=6$. In this manner R is completely covered. The integral becomes

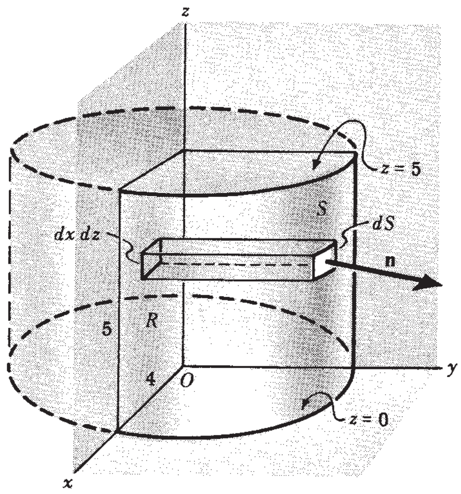
$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(6-2x)\,dy\,dx = \int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx = 24$$

If we had chosen the positive unit normal **n** opposite to that in the figure above, we would have obtained the result -24.

20. Evaluate $\displaystyle\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$, where $\mathbf{A} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2z\,\mathbf{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Project S on the xz plane as in the figure below and call the projection R. Note that the projection of S on the xy plane cannot be used here. Then

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS \;=\; \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$$



A normal to $x^2+y^2=16$ is $\nabla(x^2+y^2) = 2x\mathbf{i}+2y\mathbf{j}$. Thus the unit normal to S as shown in the adjoining figure, is

$$\mathbf{n} \;=\; \frac{2x\,\mathbf{i}+2y\,\mathbf{j}}{\sqrt{(2x)^2+(2y)^2}} \;=\; \frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}$$

since $x^2+y^2=16$ on S.

$$\mathbf{A}\cdot\mathbf{n} \;=\; (z\,\mathbf{i}+x\,\mathbf{j}-3y^2z\,\mathbf{k})\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}\right) \;=\; \frac{1}{4}(xz+xy)$$

$$\mathbf{n}\cdot\mathbf{j} \;=\; \frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}\cdot\mathbf{j} \;=\; \frac{y}{4}.$$

Then the surface integral equals

$$\iint_R \frac{xz+xy}{y}\,dx\,dz \;=\; \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dx\,dz \;=\; \int_{z=0}^{5}(4z+8)\,dz \;=\; 90$$

21. Evaluate $\displaystyle\iint_S \phi\mathbf{n}\,dS$ where $\phi = \frac{3}{8}xyz$ and S is the surface of Problem 20.

We have
$$\iint_S \phi\mathbf{n}\,dS \;=\; \iint_R \phi\mathbf{n}\,\frac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$$

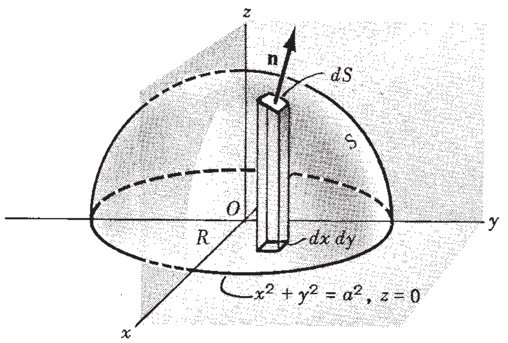
Using $\mathbf{n} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{4}$, $\mathbf{n}\cdot\mathbf{j} = \dfrac{y}{4}$ as in Problem 20, this last integral becomes

$$\iint_R \frac{3}{8}xz\,(x\,\mathbf{i}+y\,\mathbf{j})\,dx\,dz \;=\; \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4}(x^2z\,\mathbf{i}+xz\sqrt{16-x^2}\,\mathbf{j})\,dx\,dz$$

$$= \frac{3}{8}\int_{z=0}^{5}\left(\frac{64}{3}z\,\mathbf{i}+\frac{64}{3}z\,\mathbf{j}\right)dz \;=\; 100\,\mathbf{i} + 100\,\mathbf{j}$$

22. If $\mathbf{F} = y\,\mathbf{i}+(x-2xz)\mathbf{j}-xy\,\mathbf{k}$, evaluate $\displaystyle\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy plane.

$$\nabla\times\mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix} \;=\; x\,\mathbf{i}+y\,\mathbf{j}-2z\,\mathbf{k}$$



A normal to $x^2+y^2+z^2=a^2$ is

$$\nabla(x^2+y^2+z^2) \;=\; 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

Then the unit normal **n** of the figure above is given by

$$\mathbf{n} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{a}$$

since $x^2+y^2+z^2 = a^2$.

The projection of S on the xy plane is the region R bounded by the circle $x^2+y^2 = a^2,\ z=0$ (see figure above). Then

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$$= \iint_R (x\,\mathbf{i}+y\,\mathbf{j}-2z\,\mathbf{k})\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{a}\right)\frac{dx\,dy}{z/a}$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

using the fact that $z = \sqrt{a^2-x^2-y^2}$. To evaluate the double integral, transform to polar coordinates (ρ,ϕ) where $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $dy\,dx$ is replaced by $\rho\,d\rho\,d\phi$. The double integral becomes

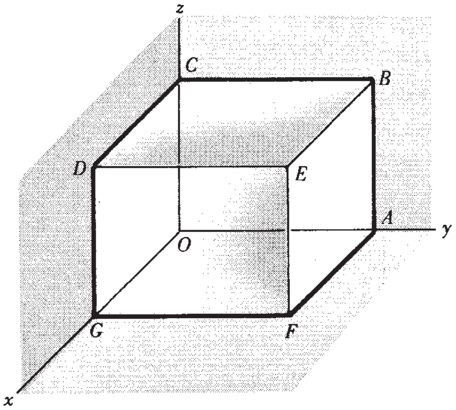
$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{3\rho^2-2a^2}{\sqrt{a^2-\rho^2}}\,\rho\,d\rho\,d\phi = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{3(\rho^2-a^2)+a^2}{\sqrt{a^2-\rho^2}}\,\rho\,d\rho\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left(-3\rho\sqrt{a^2-\rho^2} + \frac{a^2\rho}{\sqrt{a^2-\rho^2}}\right)d\rho\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\left[(a^2-\rho^2)^{3/2} - a^2\sqrt{a^2-\rho^2}\,\Big|_{\rho=0}^{a}\right]d\phi$$

$$= \int_{\phi=0}^{2\pi}(a^3-a^3)\,d\phi = 0$$

23. If $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$, evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.



Face DEFG: $\mathbf{n}=\mathbf{i}$, $x=1$. Then

$$\iint_{DEFG}\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4z\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k})\cdot\mathbf{i}\,dy\,dz$$

$$= \int_0^1\int_0^1 4z\,dy\,dz = 2$$

Face ABCO: $\mathbf{n}=-\mathbf{i}$, $x=0$. Then

$$\iint_{ABCO} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j}+yz\,\mathbf{k})\cdot(-\mathbf{i})\,dy\,dz = 0$$

Face ABEF: $\mathbf{n}=\mathbf{j}$, $y=1$. Then

$$\iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\,\mathbf{i}-\mathbf{j}+z\,\mathbf{k})\cdot\mathbf{j}\,dx\,dz = \int_0^1\int_0^1 -dx\,dz = -1$$

Face OGDC: $\mathbf{n}=-\mathbf{j}$, $y=0$. Then

$$\iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\,\mathbf{i})\cdot(-\mathbf{j})\,dx\,dz = 0$$

Face BCDE: $\mathbf{n}=\mathbf{k}$, $z=1$. Then

$$\iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4x\,\mathbf{i}-y^2\,\mathbf{j}+y\,\mathbf{k})\cdot\mathbf{k}\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = \tfrac{1}{2}$$

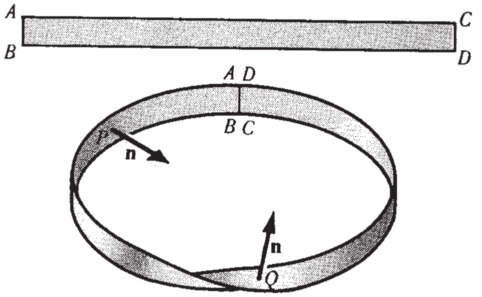
Face AFGO: $\mathbf{n}=-\mathbf{k}$, $z=0$. Then

$$\iint_{AFGO} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j})\cdot(-\mathbf{k})\,dx\,dy = 0$$

Adding, $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 2+0+(-1)+0+\tfrac{1}{2}+0 = \tfrac{3}{2}$.

24. In dealing with surface integrals we have restricted ourselves to surfaces which are two-sided. Give an example of a surface which is not two-sided.

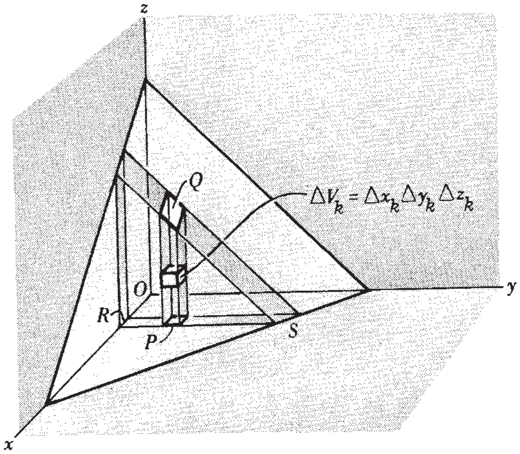
Take a strip of paper such as $ABCD$ as shown in the adjoining figure. Twist the strip so that points A and B fall on D and C respectively, as in the adjoining figure. If $\mathbf{n}$ is the positive normal at point P of the surface, we find that as $\mathbf{n}$ moves around the surface it reverses its original direction when it reaches P again. If we tried to color only one side of the surface we would find the whole thing colored. This surface, called a *Moebius strip*, is an example of a one-sided surface. This is sometimes called a *non-orientable* surface. A two-sided surface is *orientable*.



VOLUME INTEGRALS

25. Let $\phi = 45x^2y$ and let V denote the closed region bounded by the planes $4x+2y+z=8$, $x=0$, $y=0$, $z=0$. (a) Express $\displaystyle\iiint_V \phi\,dV$ as the limit of a sum. (b) Evaluate the integral in (a).

(*a*) Subdivide region V into M cubes having volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$ $k = 1, 2, \ldots, M$ as indicated in the adjoining figure and let (x_k, y_k, z_k) be a point within this cube. Define $\phi(x_k, y_k, z_k) = \phi_k$. Consider the sum



$$(1) \qquad \sum_{k=1}^{M} \phi_k \Delta V_k$$

taken over all possible cubes in the region. The limit of this sum, when $M \to \infty$ in such a manner that the largest of the quantities ΔV_k will approach zero, if it exists, is denoted by $\iiint_V \phi \, dV$. It can be shown that this limit is independent of the method of subdivision if ϕ is continuous throughout V.

In forming the sum (*1*) over all possible cubes in the region, it is advisable to proceed in an orderly fashion. One possibility is to add first all terms in (*1*) corresponding to volume elements contained in a column such as PQ in the above figure. This amounts to keeping x_k and y_k fixed and adding over all z_k's. Next, keep x_k fixed but sum over all y_k's. This amounts to adding all columns such as PQ contained in a slab RS, and consequently amounts to summing over all cubes contained in such a slab. Finally, vary x_k. This amounts to addition of all slabs such as RS.

In the process outlined the summation is taken first over z_k's then over y_k's and finally over x_k's. However, the summation can clearly be taken in any other order.

(*b*) The ideas involved in the method of summation outlined in (*a*) can be used in evaluating the integral. Keeping x and y constant, integrate from $z = 0$ (base of column PQ) to $z = 8 - 4x - 2y$ (top of column PQ). Next keep x constant and integrate with respect to y. This amounts to addition of columns having bases in the xy plane $(z = 0)$ located anywhere from R (where $y = 0$) to S (where $4x + 2y = 8$ or $y = 4 - 2x$), and the integration is from $y = 0$ to $y = 4 - 2x$. Finally, we add all slabs parallel to the yz plane, which amounts to integration from $x = 0$ to $x = 2$. The integration can be written

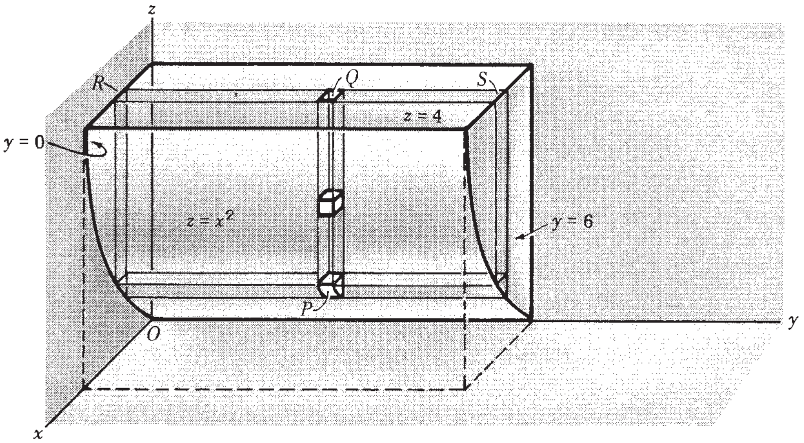
$$\int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2 y \, dz \, dy \, dx \;=\; 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2 y (8 - 4x - 2y) \, dy \, dx$$

$$= \; 45 \int_{x=0}^{2} \tfrac{1}{3} x^2 (4 - 2x)^3 \, dx \;=\; 128$$

Note: Physically the result can be interpreted as the mass of the region V in which the density ϕ varies according to the formula $\phi = 45x^2 y$.

26. Let $\mathbf{F} = 2xz\,\mathbf{i} - x\,\mathbf{j} + y^2\mathbf{k}$. Evaluate $\iiint_V \mathbf{F} \, dV$ where V is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, $z = 4$.

The region V is covered (*a*) by keeping x and y fixed and integrating from $z = x^2$ to $z = 4$ (base to top of column PQ), (*b*) then by keeping x fixed and integrating from $y = 0$ to $y = 6$ (R to S in the slab), (*c*) finally integrating from $x = 0$ to $x = 2$ (where $z = x^2$ meets $z = 4$). Then the required integral is

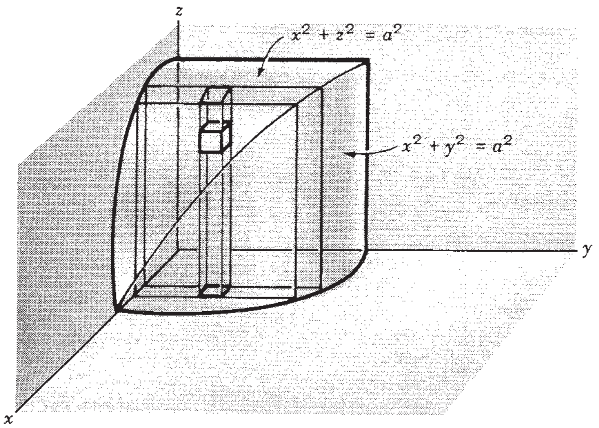


$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}(2xz\,\mathbf{i} - x\,\mathbf{j} + y^2\,\mathbf{k})\,dz\,dy\,dx$$

$$= \mathbf{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dz\,dy\,dx \;-\; \mathbf{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx \;+\; \mathbf{k}\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx$$

$$= 128\,\mathbf{i} - 24\,\mathbf{j} + 384\,\mathbf{k}$$

27. Find the volume of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.



Required volume = 8 times volume of region shown in above figure

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx \;=\; 8\int_{x=0}^{a}(a^2-x^2)\,dx \;=\; \frac{16a^3}{3}$$

SUPPLEMENTARY PROBLEMS

28. If $\mathbf{R}(t) = (3t^2-t)\mathbf{i} + (2-6t)\mathbf{j} - 4t\,\mathbf{k}$, find (*a*) $\int \mathbf{R}(t)\,dt$ and (*b*) $\int_2^4 \mathbf{R}(t)\,dt$.

Ans. (*a*) $(t^3-t^2/2)\mathbf{i} + (2t-3t^2)\mathbf{j} - 2t^2\,\mathbf{k} + \mathbf{c}$ (*b*) $50\mathbf{i} - 32\mathbf{j} - 24\mathbf{k}$

29. Evaluate $\int_0^{\pi/2} (3\sin u\,\mathbf{i} + 2\cos u\,\mathbf{j})\,du$ *Ans.* $3\mathbf{i} + 2\mathbf{j}$

30. If $\mathbf{A}(t) = t\,\mathbf{i} - t^2\,\mathbf{j} + (t-1)\mathbf{k}$ and $\mathbf{B}(t) = 2t^2\,\mathbf{i} + 6t\,\mathbf{k}$, evaluate (*a*) $\int_0^2 \mathbf{A}\cdot\mathbf{B}\,dt$, (*b*) $\int_0^2 \mathbf{A}\times\mathbf{B}\,dt$.

Ans. (*a*) 12 (*b*) $-24\,\mathbf{i} - \frac{40}{3}\mathbf{j} + \frac{64}{5}\mathbf{k}$

31. Let $\mathbf{A} = t\,\mathbf{i} - 3\mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{C} = 3\mathbf{i} + t\,\mathbf{j} - \mathbf{k}$. Evaluate (*a*) $\int_1^2 \mathbf{A}\cdot\mathbf{B}\times\mathbf{C}\,dt$, (*b*) $\int_1^2 \mathbf{A}\times(\mathbf{B}\times\mathbf{C})\,dt$.

Ans. (*a*) 0 (*b*) $-\frac{87}{2}\mathbf{i} - \frac{44}{3}\mathbf{j} + \frac{15}{2}\mathbf{k}$

32. The acceleration $\mathbf{a}$ of a particle at any time $t \geqq 0$ is given by $\mathbf{a} = e^{-t}\mathbf{i} - 6(t+1)\mathbf{j} + 3\sin t\,\mathbf{k}$. If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

Ans. $\mathbf{v} = (1-e^{-t})\mathbf{i} - (3t^2+6t)\mathbf{j} + (3-3\cos t)\mathbf{k}$, $\mathbf{r} = (t-1+e^{-t})\mathbf{i} - (t^3+3t^2)\mathbf{j} + (3t-3\sin t)\mathbf{k}$

33. The acceleration $\mathbf{a}$ of an object at any time t is given by $\mathbf{a} = -g\,\mathbf{j}$, where g is a constant. At $t=0$ the velocity is given by $\mathbf{v} = v_0\cos\theta_0\,\mathbf{i} + v_0\sin\theta_0\,\mathbf{j}$ and the displacement $\mathbf{r} = \mathbf{0}$. Find $\mathbf{v}$ and $\mathbf{r}$ at any time $t>0$. This describes the motion of a projectile fired from a cannon inclined at angle θ_0 with the positive x-axis with initial velocity of magnitude v_0.

Ans. $\mathbf{v} = v_0\cos\theta_0\,\mathbf{i} + (v_0\sin\theta_0 - gt)\mathbf{j}$, $\mathbf{r} = (v_0\cos\theta_0)t\,\mathbf{i} + [(v_0\sin\theta_0)t - \frac{1}{2}gt^2]\,\mathbf{j}$

34. Evaluate $\int_2^3 \mathbf{A}\cdot\frac{d\mathbf{A}}{dt}\,dt$ if $\mathbf{A}(2) = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{A}(3) = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. *Ans.* 10

35. Find the areal velocity of a particle which moves along the path $\mathbf{r} = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$ where a,b,ω are constants and t is time. *Ans.* $\frac{1}{2}ab\omega\,\mathbf{k}$

36. Prove that the squares of the periods of planets in their motion around the sun are proportional to the cubes of the major axes of their elliptical paths (Kepler's third law).

37. If $\mathbf{A} = (2y+3)\mathbf{i} + xz\,\mathbf{j} + (yz-x)\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ along the following paths C:

(*a*) $x=2t^2$, $y=t$, $z=t^3$ from $t=0$ to $t=1$,
(*b*) the straight lines from (0,0,0) to (0,0,1), then to (0,1,1), and then to (2,1,1),
(*c*) the straight line joining (0,0,0) and (2,1,1).

Ans. (*a*) 288/35 (*b*) 10 (*c*) 8

38. If $\mathbf{F} = (5xy-6x^2)\mathbf{i} + (2y-4x)\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C in the xy plane, $y=x^3$ from the point (1,1) to (2,8). *Ans.* 35

39. If $\mathbf{F} = (2x+y)\mathbf{i} + (3y-x)\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane consisting of the straight lines from (0,0) to (2,0) and then to (3,2). *Ans.* 11

40. Find the work done in moving a particle in the force field $\mathbf{F} = 3x^2\,\mathbf{i} + (2xz-y)\mathbf{j} + z\,\mathbf{k}$ along

(*a*) the straight line from (0,0,0) to (2,1,3).
(*b*) the space curve $x=2t^2$, $y=t$, $z=4t^2-t$ from $t=0$ to $t=1$.
(*c*) the curve defined by $x^2=4y$, $3x^3=8z$ from $x=0$ to $x=2$.

Ans. (*a*) 16 (*b*) 14.2 (*c*) 16

41. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (x-3y)\mathbf{i} + (y-2x)\mathbf{j}$ and C is the closed curve in the xy plane, $x = 2\cos t$, $y = 3\sin t$ from $t = 0$ to $t = 2\pi$. *Ans.* 6π, if C is traversed in the positive (counterclockwise) direction.

42. If $\mathbf{T}$ is a unit tangent vector to the curve C, $\mathbf{r} = \mathbf{r}(u)$, show that the work done in moving a particle in a force field $\mathbf{F}$ along C is given by $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where s is the arc length.

43. If $\mathbf{F} = (2x+y^2)\mathbf{i} + (3y-4x)\mathbf{j}$, evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around the triangle C of Figure 1, (*a*) in the indicated direction, (*b*) opposite to the indicated direction. *Ans.* (*a*) $-14/3$ (*b*) $14/3$

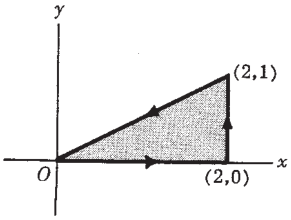


Fig. 1

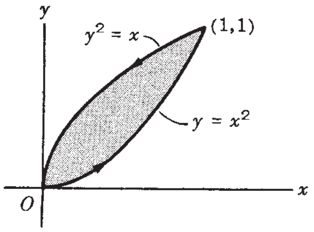


Fig. 2

44. Evaluate $\oint_C \mathbf{A}\cdot d\mathbf{r}$ around the closed curve C of Fig. 2 above if $\mathbf{A} = (x-y)\mathbf{i} + (x+y)\mathbf{j}$. *Ans.* $2/3$

45. If $\mathbf{A} = (y-2x)\mathbf{i} + (3x+2y)\mathbf{j}$, compute the circulation of $\mathbf{A}$ about a circle C in the xy plane with center at the origin and radius 2, if C is traversed in the positive direction. *Ans.* 8π

46. (*a*) If $\mathbf{A} = (4xy - 3x^2z^2)\mathbf{i} + 2x^2\mathbf{j} - 2x^3z\,\mathbf{k}$, prove that $\int_C \mathbf{A}\cdot d\mathbf{r}$ is independent of the curve C joining two given points. (*b*) Show that there is a differentiable function ϕ such that $\mathbf{A} = \nabla\phi$ and find it.
Ans. (*b*) $\phi = 2x^2y - x^3z^2 + \text{constant}$

47. (*a*) Prove that $\mathbf{F} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}$ is a conservative force field.
(*b*) Find the scalar potential for $\mathbf{F}$.
(*c*) Find the work done in moving an object in this field from $(0,1,-1)$ to $(\pi/2,-1,2)$.
Ans. (*b*) $\phi = y^2\sin x + xz^3 - 4y + 2z + \text{constant}$ (*c*) $15 + 4\pi$

48. Prove that $\mathbf{F} = r^2\mathbf{r}$ is conservative and find the scalar potential. *Ans.* $\phi = \frac{r^4}{4} + \text{constant}$

49. Determine whether the force field $\mathbf{F} = 2xz\,\mathbf{i} + (x^2-y)\mathbf{j} + (2z-x^2)\mathbf{k}$ is conservative or non-conservative.
Ans. non-conservative

50. Show that the work done on a particle in moving it from A to B equals its change in kinetic energies at these points whether the force field is conservative or not.

51. Evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ along the curve $x^2+y^2 = 1$, $z = 1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$ if $\mathbf{A} = (yz+2x)\mathbf{i} + xz\,\mathbf{j} + (xy+2z)\mathbf{k}$. *Ans.* 1

52. (*a*) If $\mathbf{E} = r\mathbf{r}$, is there a function ϕ such that $\mathbf{E} = -\nabla\phi$? If so, find it. (*b*) Evaluate $\oint_C \mathbf{E}\cdot d\mathbf{r}$ if C is any simple closed curve. *Ans.* (*a*) $\phi = -\frac{r^3}{3} + \text{constant}$ (*b*) 0

53. Show that $(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2\sin y)\,dy + x\sin y\,dz$ is an exact differential. Hence

solve the differential equation $(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2\sin y)\,dy + x\sin y\,dz = 0$.
Ans. $x^2\cos y + xz\sin y = \text{constant}$

54. Solve (*a*) $(e^{-y} + 3x^2y^2)\,dx + (2x^3y - xe^{-y})\,dy = 0$,
(*b*) $(z - e^{-x}\sin y)\,dx + (1 + e^{-x}\cos y)\,dy + (x - 8z)\,dz = 0$.
Ans. (*a*) $xe^{-y} + x^3y^2 = \text{constant}$ (*b*) $xz + e^{-x}\sin y + y - 4z^2 = \text{constant}$

55. If $\phi = 2xy^2z + x^2y$, evaluate $\int_C \phi\, d\mathbf{r}$ where C
(*a*) is the curve $x = t,\ y = t^2,\ z = t^3$ from $t = 0$ to $t = 1$
(*b*) consists of the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).
Ans. (*a*) $\frac{19}{45}\mathbf{i} + \frac{11}{15}\mathbf{j} + \frac{75}{77}\mathbf{k}$ (*b*) $\frac{1}{2}\mathbf{j} + 2\mathbf{k}$

56. If $\mathbf{F} = 2y\,\mathbf{i} - z\,\mathbf{j} + x\,\mathbf{k}$, evaluate $\int_C \mathbf{F}\times d\mathbf{r}$ along the curve $x = \cos t,\ y = \sin t,\ z = 2\cos t$ from $t = 0$ to $t = \pi/2$. *Ans.* $(2 - \frac{\pi}{4})\mathbf{i} + (\pi - \frac{1}{2})\mathbf{j}$

57. If $\mathbf{A} = (3x + y)\mathbf{i} - x\,\mathbf{j} + (y - 2)\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, evaluate $\oint_C (\mathbf{A}\times\mathbf{B})\times d\mathbf{r}$ around the circle in the xy plane having center at the origin and radius 2 traversed in the positive direction. *Ans.* $4\pi(7\mathbf{i} + 3\mathbf{j})$

58. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$ for each of the following cases.
(*a*) $\mathbf{A} = y\,\mathbf{i} + 2x\,\mathbf{j} - z\,\mathbf{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.
(*b*) $\mathbf{A} = (x + y^2)\mathbf{i} - 2x\,\mathbf{j} + 2yz\,\mathbf{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.
Ans. (*a*) 108 (*b*) 81

59. If $\mathbf{F} = 2y\,\mathbf{i} - z\,\mathbf{j} + x^2\,\mathbf{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$. *Ans.* 132

60. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$, if $\mathbf{A} = 6z\,\mathbf{i} + (2x + y)\mathbf{j} - x\,\mathbf{k}$. *Ans.* 18π

61. Evaluate $\iint_S \mathbf{r}\cdot\mathbf{n}\,dS$ over: (*a*) the surface S of the unit cube bounded by the coordinate planes and the planes $x = 1,\ y = 1,\ z = 1$; (*b*) the surface of a sphere of radius a with center at (0,0,0).
Ans. (*a*) 3 (*b*) $4\pi a^3$

62. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$ over the entire surface of the region above the xy plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\mathbf{A} = 4xz\,\mathbf{i} + xyz^2\,\mathbf{j} + 3z\,\mathbf{k}$. *Ans.* 320π

63. (*a*) Let R be the projection of a surface S on the xy plane. Prove that the surface area of S is given by $\iint_R \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\,dx\,dy$ if the equation for S is $z = f(x,y)$.

(*b*) What is the surface area if S has the equation $F(x,y,z)=0$? *Ans.* $\displaystyle\iint_R \frac{\sqrt{(\frac{\partial \mathbf{F}}{\partial x})^2+(\frac{\partial \mathbf{F}}{\partial y})^2+(\frac{\partial \mathbf{F}}{\partial z})^2}}{|\frac{\partial \mathbf{F}}{\partial z}|}\,dx\,dy$

64. Find the surface area of the plane $x+2y+2z=12$ cut off by: (*a*) $x=0, y=0, x=1, y=1$; (*b*) $x=0, y=0$, and $x^2+y^2=16$. *Ans.* (*a*) 3/2 (*b*) 6π

65. Find the surface area of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. *Ans.* $16a^2$

66. Evaluate (*a*) $\displaystyle\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ and (*b*) $\displaystyle\iint_S \phi\,\mathbf{n}\,dS$ if $\mathbf{F}=(x+2y)\mathbf{i}-3z\,\mathbf{j}+x\,\mathbf{k}$, $\phi=4x+3y-2z$, and S is the surface of $2x+y+2z=6$ bounded by $x=0, x=1, y=0$ and $y=2$.
Ans. (*a*) 1 (*b*) $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

67. Solve the preceding problem if S is the surface of $2x+y+2z=6$ bounded by $x=0, y=0$, and $z=0$.
Ans. (*a*) 9/2 (*b*) $72\mathbf{i}+36\mathbf{j}+72\mathbf{k}$

68. Evaluate $\displaystyle\iint_R \sqrt{x^2+y^2}\,dx\,dy$ over the region R in the xy plane bounded by $x^2+y^2=36$. *Ans.* 144π

69. Evaluate $\displaystyle\iiint_V (2x+y)\,dV$, where V is the closed region bounded by the cylinder $z=4-x^2$ and the planes $x=0, y=0, y=2$ and $z=0$. *Ans.* 80/3

70. If $\mathbf{F}=(2x^2-3z)\mathbf{i}-2xy\,\mathbf{j}-4x\,\mathbf{k}$, evaluate (*a*) $\displaystyle\iiint_V \nabla\cdot\mathbf{F}\,dV$ and (*b*) $\displaystyle\iiint_V \nabla\times\mathbf{F}\,dV$, where V is the closed region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$. *Ans.* (*a*) $\frac{8}{3}$ (*b*) $\frac{8}{3}(\mathbf{j}-\mathbf{k})$

Chapter 6

The DIVERGENCE THEOREM, STOKES' THEOREM, and RELATED INTEGRAL THEOREMS

THE DIVERGENCE THEOREM OF GAUSS states that if V is the volume bounded by a closed surface S and $\mathbf{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla\cdot\mathbf{A}\,dV \quad = \quad \iint_S \mathbf{A}\cdot\mathbf{n}\,dS \quad = \quad \oiint_S \mathbf{A}\cdot d\mathbf{S}$$

where $\mathbf{n}$ is the positive (outward drawn) normal to S.

STOKES' THEOREM states that if S is an open, two-sided surface bounded by a closed, non-intersecting curve C (simple closed curve) then if $\mathbf{A}$ has continuous derivatives

$$\oint_C \mathbf{A}\cdot d\mathbf{r} \quad = \quad \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS \quad = \quad \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where C is traversed in the positive direction. The direction of C is called *positive* if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of the positive normal to S, has the surface on his left.

GREEN'S THEOREM IN THE PLANE. If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy \quad = \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive (counterclockwise) direction. Unless otherwise stated we shall always assume $\oint$ to mean that the integral is described in the positive sense.

Green's theorem in the plane is a special case of Stokes' theorem (see Problem 4). Also, it is of interest to notice that Gauss' divergence theorem is a generalization of Green's theorem in the plane where the (plane) region R and its closed boundary (curve) C are replaced by a (space) region V and its closed boundary (surface) S. For this reason the divergence theorem is often called *Green's theorem in space* (see Problem 4).

Green's theorem in the plane also holds for regions bounded by a finite number of simple closed curves which do not intersect (see Problems 10 and 11).

RELATED INTEGRAL THEOREMS.

1. $$\iiint_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dV = \iint_S (\phi\nabla\psi)\cdot d\mathbf{S}$$

This is called *Green's first identity or theorem.*

2. $$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

This is called *Green's second identity or symmetrical theorem.* See Problem 21.

3. $$\iiint_V \nabla\times\mathbf{A}\,dV = \iint_S (\mathbf{n}\times\mathbf{A})\,dS = \iint_S d\mathbf{S}\times\mathbf{A}$$

Note that here the dot product of Gauss' divergence theorem is replaced by the cross product. See Problem 23.

4. $$\oint_C \phi\,d\mathbf{r} = \iint_S (\mathbf{n}\times\nabla\phi)\,dS = \iint_S d\mathbf{S}\times\nabla\phi$$

5. Let ψ represent either a vector or scalar function according as the symbol $\circ$ denotes a dot or cross, or an ordinary multiplication. Then

$$\iiint_V \nabla\circ\psi\,dV = \iint_S \mathbf{n}\circ\psi\,dS = \iint_S d\mathbf{S}\circ\psi$$

$$\oint_C d\mathbf{r}\circ\psi = \iint_S (\mathbf{n}\times\nabla)\circ\psi\,dS = \iint_S (d\mathbf{S}\times\nabla)\circ\psi$$

Gauss' divergence theorem, Stokes' theorem and the results *3* and *4* are special cases of these. See Problems 22, 23, and 34.

INTEGRAL OPERATOR FORM FOR ∇. It is of interest that, using the terminology of Problem 19, the operator ∇ can be expressed symbolically in the form

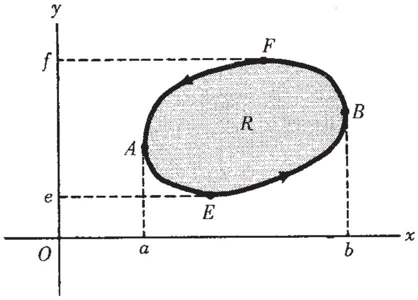
$$\nabla\circ \equiv \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint\!\!\!\oint_{\Delta S} d\mathbf{S}\,\circ$$

where $\circ$ denotes a dot, cross or an ordinary multiplication (see Problem 25). The result proves useful in extending the concepts of gradient, divergence and curl to coordinate systems other than rectangular (see Problems 19, 24 and also Chapter 7).

SOLVED PROBLEMS

GREEN'S THEOREM IN THE PLANE

1. Prove Green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points.



Let the equations of the curves AEB and AFB (see adjoining figure) be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. If R is the region bounded by C, we have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dy\right]dx = \int_{x=a}^{b} M(x,y)\Big|_{y=Y_1(x)}^{Y_2(x)}\,dx = \int_a^b \Big[M(x,Y_2) - M(x,Y_1)\Big]\,dx$$

$$= -\int_a^b M(x,Y_1)\,dx - \int_b^a M(x,Y_2)\,dx = -\oint_C M\,dx$$

Then $$(1) \qquad \oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

Similarly let the equations of curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

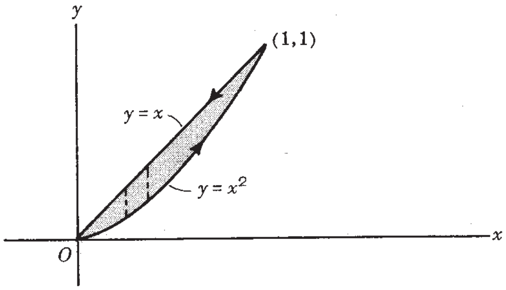
$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x}\,dx\right]dy = \int_e^f \Big[N(X_2,y) - N(X_1,y)\Big]\,dy$$

$$= \int_f^e N(X_1,y)\,dy + \int_e^f N(X_2,y)\,dy = \oint_C N\,dy$$

Then $$(2) \qquad \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy$$

Adding (1) and (2), $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy.$$

2. Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.



$y = x$ and $y = x^2$ intersect at (0,0) and (1,1). The positive direction in traversing C is as shown in the adjacent diagram.

Along $y = x^2$, the line integral equals

$$\int_0^1 \left((x)(x^2) + x^4\right) dx \;+\; (x^2)(2x)\,dx \;=\; \int_0^1 (3x^3 + x^4)\,dx \;=\; \frac{19}{20}$$

Along $y = x$ from (1,1) to (0,0) the line integral equals

$$\int_1^0 \left((x)(x) + x^2\right) dx \;+\; x^2\,dx \;=\; \int_1^0 3x^2\,dx \;=\; -1$$

Then the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \;=\; \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

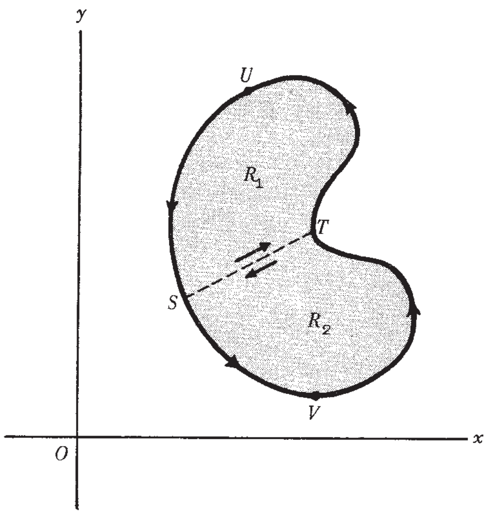
$$= \iint_R (x - 2y)\,dx\,dy \;=\; \int_{x=0}^1 \int_{y=x^2}^x (x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[\int_{x^2}^x (x - 2y)\,dy\right] dx \;=\; \int_0^1 (xy - y^2)\Big|_{x^2}^x\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx \;=\; -\frac{1}{20}$$

so that the theorem is verified.

3. Extend the proof of Green's theorem in the plane given in Problem 1 to the curves C for which lines parallel to the coordinate axes may cut C in more than two points.



Consider a closed curve C such as shown in the adjoining figure, in which lines parallel to the axes may meet C in more than two points. By constructing line ST the region is divided into two regions R_1 and R_2 which are of the type considered in Problem 1 and for which Green's theorem applies, i.e.,

$$(1)\qquad \int_{STUS} M\,dx + N\,dy \;=\; \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$(2)\qquad \int_{SVTS} M\,dx + N\,dy \;=\; \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Adding the left hand sides of (1) and (2), we have, omitting the integrand $M\,dx + N\,dy$ in each case,

$$\int_{STUS} + \int_{SVTS} \;=\; \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} \;=\; \int_{TUS} + \int_{SVT} \;=\; \int_{TUSVT}$$

using the fact that $\displaystyle\int_{ST} = -\int_{TS}$

Adding the right hand sides of (1) and (2), omitting the integrand,

$$\iint_{R_1} + \iint_{R_2} = \iint_{R}$$

where R consists of regions R_1 and R_2.

Then $\displaystyle\int_{TUSVT} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ and the theorem is proved.

A region R such as considered here and in Problem 1, for which any closed curve lying in R can be continuously shrunk to a point without leaving R, is called a *simply-connected region.* A region which is not simply-connected is called *multiply-connected.* We have shown here that Green's theorem in the plane applies to simply-connected regions bounded by closed curves. In Problem 10 the theorem is extended to multiply-connected regions.

For more complicated simply-connected regions it may be necessary to construct more lines, such as ST, to establish the theorem.

4. Express Green's theorem in the plane in vector notation.

We have $M\,dx + N\,dy = (M\mathbf{i}+N\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j}) = \mathbf{A}\cdot d\mathbf{r}$, where $\mathbf{A} = M\mathbf{i} + N\mathbf{j}$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

Also, if $\mathbf{A} = M\mathbf{i} + N\mathbf{j}$ then

$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\mathbf{i} + \frac{\partial M}{\partial z}\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

so that $(\nabla\times\mathbf{A})\cdot\mathbf{k} = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$.

Then Green's theorem in the plane can be written

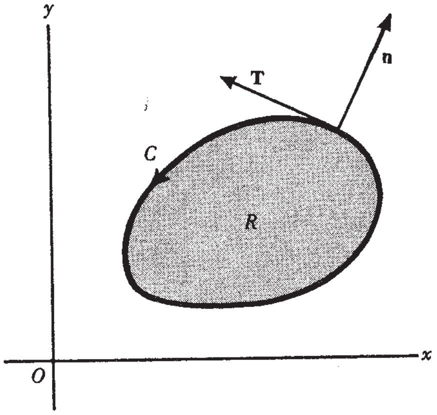
$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_R (\nabla\times\mathbf{A})\cdot\mathbf{k}\,dR$$

where $dR = dx\,dy$.

A generalization of this to surfaces S in space having a curve C as boundary leads quite naturally to *Stokes' theorem* which is proved in Problem 31.

Another Method.

As above, $M\,dx + N\,dy = \mathbf{A}\cdot d\mathbf{r} = \mathbf{A}\cdot\dfrac{d\mathbf{r}}{ds}\,ds = \mathbf{A}\cdot\mathbf{T}\,ds$, where $\dfrac{d\mathbf{r}}{ds} = \mathbf{T}$ = unit tangent vector to C (see adjacent figure). If $\mathbf{n}$ is the outward drawn unit normal to C, then $\mathbf{T} = \mathbf{k}\times\mathbf{n}$ so that



$$M\,dx + N\,dy = \mathbf{A}\cdot\mathbf{T}\,ds = \mathbf{A}\cdot(\mathbf{k}\times\mathbf{n})\,ds = (\mathbf{A}\times\mathbf{k})\cdot\mathbf{n}\,ds$$

Since $\mathbf{A} = M\mathbf{i} + N\mathbf{j}$, $\mathbf{B} = \mathbf{A}\times\mathbf{k} = (M\mathbf{i}+N\mathbf{j})\times\mathbf{k} = N\mathbf{i}-M\mathbf{j}$ and $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = \nabla\cdot\mathbf{B}$. Then Green's theorem in the plane becomes

$$\oint_C \mathbf{B}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{B}\,dR$$

where $dR = dx\,dy$.

Generalization of this to the case where the differential arc length ds of a closed curve C is replaced by the differential of surface area dS of a closed surface S, and the corresponding plane region R enclosed by C is replaced by the volume V enclosed by S, leads to *Gauss' divergence theorem* or *Green's theorem in space.*

$$\iint_S \mathbf{B}\cdot\mathbf{n}\,dS \quad = \quad \iiint_V \nabla\cdot\mathbf{B}\,dV$$

5. Interpret physically the first result of Problem 4.

If $\mathbf{A}$ denotes the force field acting on a particle, then $\oint_C \mathbf{A}\cdot d\mathbf{r}$ is the work done in moving the particle around a closed path C and is determined by the value of $\nabla\times\mathbf{A}$. It follows in particular that if $\nabla\times\mathbf{A} = \mathbf{0}$ or equivalently if $\mathbf{A} = \nabla\phi$, then the integral around a closed path is zero. This amounts to saying that the work done in moving the particle from one point in the plane to another is independent of the path in the plane joining the points or that the force field is conservative. These results have already been demonstrated for force fields and curves in space (see Chapter 5).

Conversely, if the integral is independent of the path joining any two points of a region, i.e. if the integral around any closed path is zero, then $\nabla\times\mathbf{A} = \mathbf{0}$. In the plane, the condition $\nabla\times\mathbf{A} = \mathbf{0}$ is equivalent to the condition $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ where $\mathbf{A} = M\mathbf{i} + N\mathbf{j}$.

6. Evaluate $\displaystyle\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\,dx \; - \; 3x^2y^2\,dy$ along the path $x^4 - 6xy^3 = 4y^2$.

A direct evaluation is difficult. However, noting that $M = 10x^4 - 2xy^3$, $N = -3x^2y^2$ and $\dfrac{\partial M}{\partial y} = -6xy^2 = \dfrac{\partial N}{\partial x}$, it follows that the integral is independent of the path. Then we can use any path, for example the path consisting of straight line segments from (0,0) to (2,0) and then from (2,0) to (2,1).

Along the straight line path from (0,0) to (2,0), $y = 0$, $dy = 0$ and the integral equals $\displaystyle\int_{x=0}^{2} 10x^4\,dx = 64$.

Along the straight line path from (2,0) to (2,1), $x = 2$, $dx = 0$ and the integral equals $\displaystyle\int_{y=0}^{1} -12y^2\,dy = -4$.

Then the required value of the line integral $= 64 - 4 = 60$.

Another Method.

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $(10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$ is an exact differential (of $2x^5 - x^2y^3$). Then

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy \quad = \quad \int_{(0,0)}^{(2,1)} d(2x^5 - x^2y^3) \quad = \quad 2x^5 - x^2y^3\Big|_{(0,0)}^{(2,1)} \quad = \quad 60$$

7. Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

In Green's theorem, put $M = -y$, $N = x$. Then

$$\oint_C x\,dy - y\,dx \quad = \quad \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dx\,dy \quad = \quad 2\iint_R dx\,dy \quad = \quad 2A$$

where A is the required area. Thus $A = \frac{1}{2}\oint_C x\,dy - y\,dx$.

8. Find the area of the ellipse $x = a\cos\theta,\ y = b\sin\theta$.

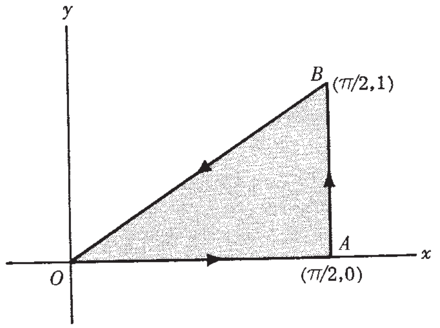
$$\text{Area} = \tfrac{1}{2}\oint_C x\,dy - y\,dx = \tfrac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)\,d\theta - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)\,d\theta = \tfrac{1}{2}\int_0^{2\pi} ab\,d\theta = \pi ab$$

9. Evaluate $\oint_C (y - \sin x)\,dx + \cos x\,dy$, where C is the triangle of the adjoining figure:

(*a*) directly,

(*b*) by using Green's theorem in the plane.



(*a*) Along OA, $y = 0$, $dy = 0$ and the integral equals

$$\int_0^{\pi/2} (0 - \sin x)\,dx + (\cos x)(0) = \int_0^{\pi/2} -\sin x\,dx$$

$$= \cos x\Big|_0^{\pi/2} = -1$$

Along AB, $x = \frac{\pi}{2}$, $dx = 0$ and the integral equals

$$\int_0^1 (y-1)0 + 0\,dy = 0$$

Along BO, $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi}dx$ and the integral equals

$$\int_{\pi/2}^0 \left(\frac{2x}{\pi} - \sin x\right)dx + \frac{2}{\pi}\cos x\,dx = \left(\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right)\Big|_{\pi/2}^0 = 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Then the integral along $C = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$.

(*b*) $M = y - \sin x$, $N = \cos x$, $\frac{\partial N}{\partial x} = -\sin x$, $\frac{\partial M}{\partial y} = 1$ and

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \iint_R (-\sin x - 1)\,dy\,dx$$

$$= \int_{x=0}^{\pi/2}\left[\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\right]dx = \int_{x=0}^{\pi/2} (-y\sin x - y)\Big|_0^{2x/\pi}\,dx$$

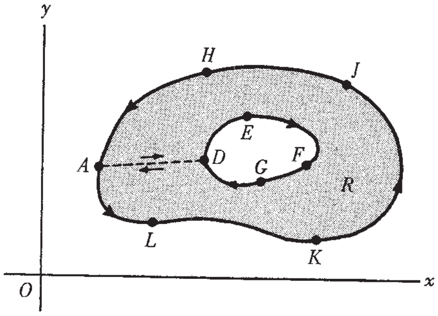
$$= \int_0^{\pi/2} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right)dx = -\frac{2}{\pi}(-x\cos x + \sin x) - \frac{x^2}{\pi}\Big|_0^{\pi/2} = -\frac{2}{\pi} - \frac{\pi}{4}$$

in agreement with part (*a*).

Note that although there exist lines parallel to the coordinate axes (coincident with the coordinate axes in this case) which meet C in an *infinite* number of points, Green's theorem in the plane still holds. In general the theorem is valid when C is composed of a finite number of straight line segments.

10. Show that Green's theorem in the plane is also valid for a multiply-connected region R such as shown in the figure below.

The shaded region R, shown in the figure below, is multiply-connected since not every closed curve lying in R can be shrunk to a point without leaving R, as is observed by considering a curve surrounding $DEFGD$ for example. The boundary of R, which consists of the exterior boundary $AHJKLA$ and the interior boundary $DEFGD$, is to be traversed in the positive direction, so that a person traveling in this direction always has the region on his left. It is seen that the positive directions are those indicated in the adjoining figure.



In order to establish the theorem, construct a line, such as AD, called a *cross-cut*, connecting the exterior and interior boundaries. The region bounded by $ADEFGDALKJHA$ is simply-connected, and so Green's theorem is valid. Then

$$\oint_{ADEFGDALKJHA} M\,dx + N\,dy \quad = \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

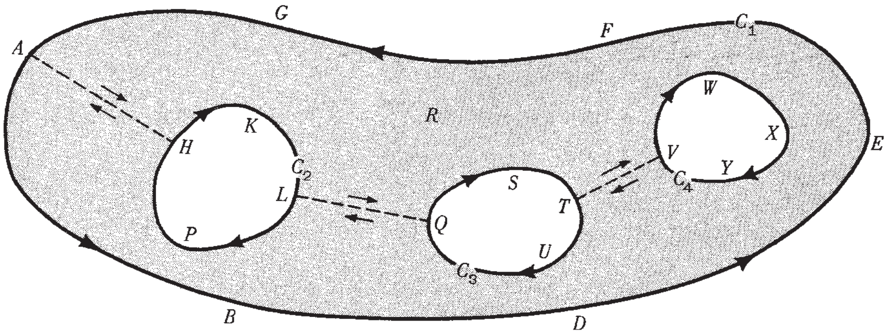
But the integral on the left, leaving out the integrand, is equal to

$$\int_{AD} + \int_{DEFGD} + \int_{DA} + \int_{ALKJHA} \quad = \quad \int_{DEFGD} + \int_{ALKJHA}$$

since $\int_{AD} = -\int_{DA}$. Thus if C_1 is the curve $ALKJHA$, C_2 is the curve $DEFGD$ and C is the boundary of R consisting of C_1 and C_2 (traversed in the positive directions), then $\int_{C_1} + \int_{C_2} = \int_C$ and so

$$\oint_C M\,dx + N\,dy \quad = \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

11. Show that Green's theorem in the plane holds for the region R, of the figure below, bounded by the simple closed curves $C_1(ABDEFGA)$, $C_2(HKLPH)$, $C_3(QSTUQ)$ and $C_4(VWXYV)$.



Construct the cross-cuts AH, LQ and TV. Then the region bounded by $AHKLQSTVWXYVTUQLPHABDEFGA$ is simply-connected and Green's theorem applies. The integral over this boundary is equal to

$$\int_{AH} + \int_{HKL} + \int_{LQ} + \int_{QST} + \int_{TV} + \int_{VWXYV} + \int_{VT} + \int_{TUQ} + \int_{QL} + \int_{LPH} + \int_{HA} + \int_{ABDEFGA}$$

Since the integrals along AH and HA, LQ and QL, TV and VT cancel out in pairs, this becomes

$$\int_{HKL} + \int_{QST} + \int_{VWXYV} + \int_{TUQ} + \int_{LPH} + \int_{ABDEFGA}$$

$$= \left(\int_{HKL} + \int_{LPH}\right) + \left(\int_{QST} + \int_{TUQ}\right) + \int_{VWXYV} + \int_{ABDEFGA}$$

$$= \int_{HKLPH} + \int_{QSTUQ} + \int_{VWXYV} + \int_{ABDEFGA}$$

$$= \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_1} = \int_{C}$$

where C is the boundary consisting of C_1, C_2, C_3 and C_4. Then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

as required.

12. Prove that $\oint_C M\,dx + N\,dy = 0$ around every closed curve C in a simply-connected region if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ everywhere in the region.

Assume that M and N are continuous and have continuous partial derivatives everywhere in the region R bounded by C, so that Green's theorem is applicable. Then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R, then clearly $\oint_C M\,dx + N\,dy = 0$.

Conversely, suppose $\oint_C M\,dx + N\,dy = 0$ for all curves C. If $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} > 0$ at a point P, then from the continuity of the derivatives it follows that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} > 0$ in some region A surrounding P. If Γ is the boundary of A then

$$\oint_\Gamma M\,dx + N\,dy = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy > 0$$

which contradicts the assumption that the line integral is zero around every closed curve. Similarly the assumption $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} < 0$ leads to a contradiction. Thus $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ at all points.

Note that the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is equivalent to the condition $\nabla \times \mathbf{A} = \mathbf{0}$ where $\mathbf{A} = M\mathbf{i} + N\mathbf{j}$ (see Problems 10 and 11, Chapter 5). For a generalization to space curves, see Problem 31.

13. Let $\mathbf{F} = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$. (*a*) Calculate $\nabla\times\mathbf{F}$. (*b*) Evaluate $\oint \mathbf{F}\cdot d\mathbf{r}$ around any closed path and explain the results.

(*a*) $$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = \mathbf{0} \quad \text{in any region excluding } (0,0).$$

(*b*) $\oint \mathbf{F}\cdot d\mathbf{r} = \oint \dfrac{-y\,dx + x\,dy}{x^2+y^2}$. Let $x = \rho\cos\phi$, $y = \rho\sin\phi$, where (ρ,ϕ) are polar coordinates. Then

$$dx = -\rho\sin\phi\,d\phi + d\rho\cos\phi, \qquad dy = \rho\cos\phi\,d\phi + d\rho\sin\phi$$

and so $$\frac{-y\,dx + x\,dy}{x^2+y^2} = d\phi = d\left(\arctan\frac{y}{x}\right)$$

For a closed curve *ABCDA* (see Figure (*a*) below) surrounding the origin, $\phi = 0$ at *A* and $\phi = 2\pi$ after a complete circuit back to *A*. In this case the line integral equals $\int_0^{2\pi} d\phi = 2\pi$.

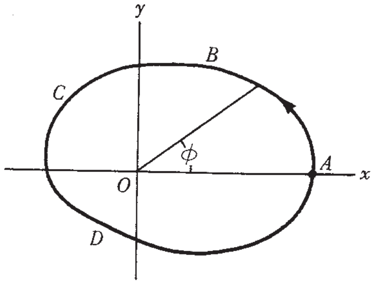


Fig. (*a*)

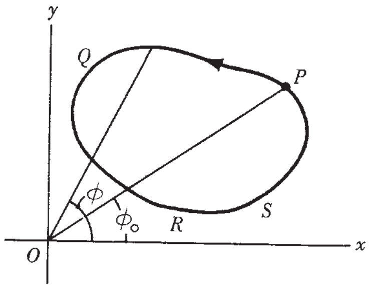


Fig. (*b*)

For a closed curve *PQRSP* (see Figure (*b*) above) not surrounding the origin, $\phi = \phi_0$ at *P* and $\phi = \phi_0$ after a complete circuit back to *P*. In this case the line integral equals $\int_{\phi_0}^{\phi_0} d\phi = 0$.

Since $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, $\nabla\times\mathbf{F} = \mathbf{0}$ is equivalent to $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and the results would seem to contradict those of Problem 12. However, no contradiction exists since $M = \dfrac{-y}{x^2+y^2}$ and $N = \dfrac{x}{x^2+y^2}$ do not have continuous derivatives throughout any region including (0,0), and this was assumed in Prob. 12.

THE DIVERGENCE THEOREM

14. (*a*) Express the divergence theorem in words and (*b*) write it in rectangular form.

(*a*) The surface integral of the normal component of a vector **A** taken over a closed surface is equal to the integral of the divergence of **A** taken over the volume enclosed by the surface.

(*b*) Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. Then $\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \dfrac{\partial A_1}{\partial x} + \dfrac{\partial A_2}{\partial y} + \dfrac{\partial A_3}{\partial z}$.

The unit normal to S is $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$. Then $n_1 = \mathbf{n}\cdot\mathbf{i} = \cos\alpha$, $n_2 = \mathbf{n}\cdot\mathbf{j} = \cos\beta$ and $n_3 = \mathbf{n}\cdot\mathbf{k} = \cos\gamma$, where α, β, γ are the angles which $\mathbf{n}$ makes with the positive x, y, z axes or $\mathbf{i}, \mathbf{j}, \mathbf{k}$ directions respectively. The quantities $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\mathbf{n}$. Then

$$\begin{aligned}\mathbf{A}\cdot\mathbf{n} &= (A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k})\cdot(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}) \\ &= A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma\end{aligned}$$

and the divergence theorem can be written

$$\iiint\limits_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dx\,dy\,dz \quad = \quad \iint\limits_S (A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma)\,dS$$

15. Demonstrate the divergence theorem physically.

Let $\mathbf{A}$ = velocity $\mathbf{v}$ at any point of a moving fluid. From Figure (*a*) below:

$$\begin{aligned}&\text{Volume of fluid crossing } dS \text{ in } \Delta t \text{ seconds} \\ &\quad = \text{volume contained in cylinder of base } dS \text{ and slant height } \mathbf{v}\Delta t \\ &\quad = (\mathbf{v}\Delta t)\cdot\mathbf{n}\,dS \;=\; \mathbf{v}\cdot\mathbf{n}\,dS\,\Delta t\end{aligned}$$

Then, volume per second of fluid crossing dS $=$ $\mathbf{v}\cdot\mathbf{n}\,dS$

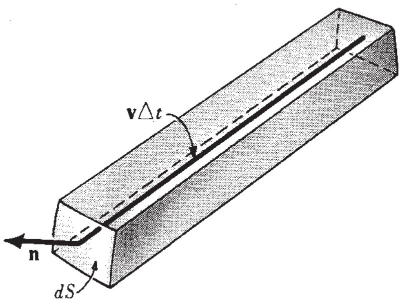


Fig. (*a*)

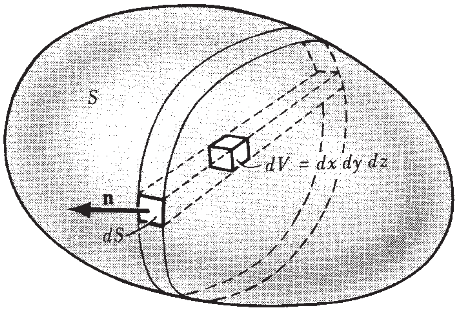


Fig. (*b*)

From Figure (*b*) above:

Total volume per second of fluid emerging from closed surface S

$$= \iint\limits_S \mathbf{v}\cdot\mathbf{n}\,dS$$

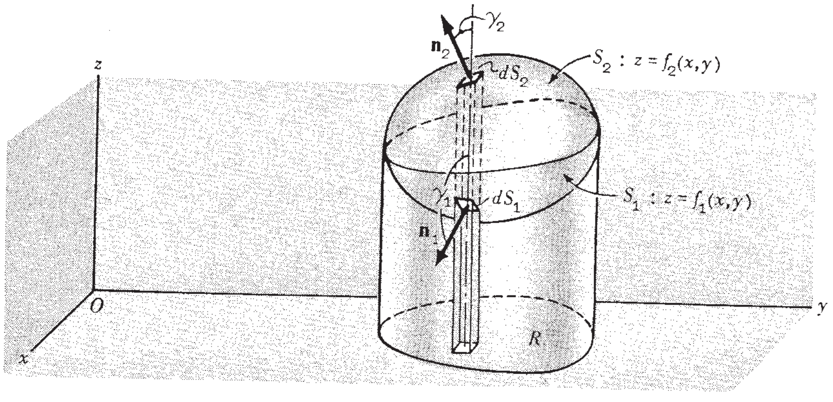
From Problem 21 of Chapter 4, $\nabla\cdot\mathbf{v}\,dV$ is the volume per second of fluid emerging from a volume element dV. Then

Total volume per second of fluid emerging from all volume elements in S

$$= \iiint\limits_V \nabla\cdot\mathbf{v}\,dV$$

Thus
$$\iint\limits_S \mathbf{v}\cdot\mathbf{n}\,dS \;=\; \iiint\limits_V \nabla\cdot\mathbf{v}\,dV$$

16. Prove the divergence theorem.



Let S be a closed surface which is such that any line parallel to the coordinate axes cuts S in at most two points. Assume the equations of the lower and upper portions, S_1 and S_2, to be $z = f_1(x,y)$ and $z = f_2(x,y)$ respectively. Denote the projection of the surface on the xy plane by R. Consider

$$\iiint_V \frac{\partial A_3}{\partial z}\, dV \;=\; \iiint_V \frac{\partial A_3}{\partial z}\, dz\, dy\, dx \;=\; \iint_R \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z}\, dz \right] dy\, dx$$

$$= \iint_R A_3(x,y,z)\Big|_{z=f_1}^{f_2} dy\, dx \;=\; \iint_R \left[A_3(x,y,f_2) - A_3(x,y,f_1)\right] dy\, dx$$

For the upper portion S_2, $dy\, dx = \cos\gamma_2\, dS_2 = \mathbf{k}\cdot\mathbf{n}_2\, dS_2$ since the normal $\mathbf{n}_2$ to S_2 makes an acute angle γ_2 with $\mathbf{k}$.

For the lower portion S_1, $dy\, dx = -\cos\gamma_1\, dS_1 = -\mathbf{k}\cdot\mathbf{n}_1\, dS_1$ since the normal $\mathbf{n}_1$ to S_1 makes an obtuse angle γ_1 with $\mathbf{k}$.

Then

$$\iint_R A_3(x,y,f_2)\, dy\, dx \;=\; \iint_{S_2} A_3\, \mathbf{k}\cdot\mathbf{n}_2\, dS_2$$

$$\iint_R A_3(x,y,f_1)\, dy\, dx \;=\; -\iint_{S_1} A_3\, \mathbf{k}\cdot\mathbf{n}_1\, dS_1$$

and

$$\iint_R A_3(x,y,f_2)\, dy\, dx - \iint_R A_3(x,y,f_1)\, dy\, dx \;=\; \iint_{S_2} A_3\, \mathbf{k}\cdot\mathbf{n}_2\, dS_2 + \iint_{S_1} A_3\, \mathbf{k}\cdot\mathbf{n}_1\, dS_1$$

$$= \iint_S A_3\, \mathbf{k}\cdot\mathbf{n}\, dS$$

so that

$$(1) \qquad \iiint_V \frac{\partial A_3}{\partial z}\, dV \;=\; \iint_S A_3\, \mathbf{k}\cdot\mathbf{n}\, dS$$

Similarly, by projecting S on the other coordinate planes,

$$(2) \qquad \iiint_V \frac{\partial A_1}{\partial x}\, dV = \iint_S A_1\, \mathbf{i}\cdot\mathbf{n}\, dS$$

$$(3) \qquad \iiint_V \frac{\partial A_2}{\partial y}\, dV = \iint_S A_2\, \mathbf{j}\cdot\mathbf{n}\, dS$$

Adding *(1)*, *(2)* and *(3)*,

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_S (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\mathbf{n}\, dS$$

or

$$\iiint_V \nabla\cdot\mathbf{A}\, dV = \iint_S \mathbf{A}\cdot\mathbf{n}\, dS$$

The theorem can be extended to surfaces which are such that lines parallel to the coordinate axes meet them in more than two points. To establish this extension, subdivide the region bounded by S into subregions whose surfaces do satisfy this condition. The procedure is analogous to that used in Green's theorem for the plane.

17. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

By the divergence theorem, the required integral is equal to

$$\iiint_V \nabla\cdot\mathbf{F}\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\, dz\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz\,\Big|_{z=0}^{1}\, dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\, dy\, dx = \frac{3}{2}$$

The surface integral may also be evaluated directly as in Problem 23, Chapter 5.

18. Verify the divergence theorem for $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z=0$ and $z=3$.

$$\text{Volume integral} = \iiint_V \nabla\cdot\mathbf{A}\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4 - 4y + 2z)\, dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\, dz\, dy\, dx = 84\pi$$

The surface S of the cylinder consists of a base S_1 $(z=0)$, the top S_2 $(z=3)$ and the convex portion S_3 $(x^2+y^2=4)$. Then

$$\text{Surface integral} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 + \iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 + \iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3$$

On S_1 $(z=0)$, $\mathbf{n} = -\mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\mathbf{j}$ and $\mathbf{A}\cdot\mathbf{n} = 0$, so that $\displaystyle\iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 = 0$.

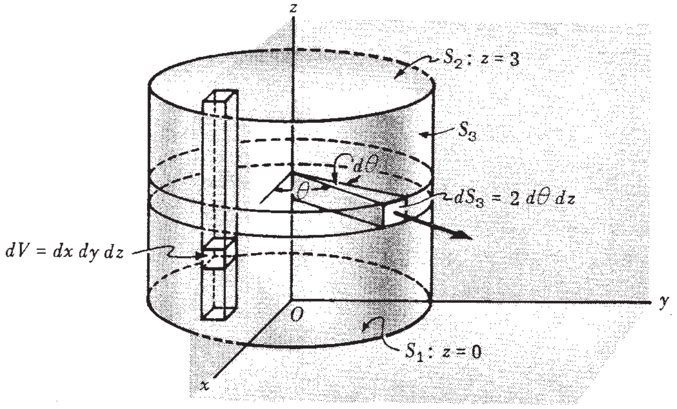
On S_2 $(z=3)$, $\mathbf{n} = \mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\mathbf{j} + 9\mathbf{k}$ and $\mathbf{A}\cdot\mathbf{n} = 9$, so that

$$\iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 = 9\iint_{S_2} dS_2 = 36\pi, \qquad \text{since area of } S_2 = 4\pi$$

On S_3 $(x^2+y^2=4)$. A perpendicular to $x^2+y^2=4$ has the direction $\nabla(x^2+y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

Then a unit normal is $\mathbf{n} = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{4x^2+4y^2}} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{2}$ since $x^2+y^2=4$.

$$\mathbf{A}\cdot\mathbf{n} = (4x\,\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k})\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}}{2}\right) = 2x^2 - y^3$$



From the figure above, $x = 2\cos\theta$, $y = 2\sin\theta$, $dS_3 = 2\,d\theta\,dz$ and so

$$\iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \left[2(2\cos\theta)^2 - (2\sin\theta)^3\right]\,2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\,d\theta = \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi$$

Then the surface integral $= 0 + 36\pi + 48\pi = 84\pi$, agreeing with the volume integral and verifying the divergence theorem.

Note that evaluation of the surface integral over S_3 could also have been done by projection of S_3 on the xz or yz coordinate planes.

19. If $\operatorname{div}\mathbf{A}$ denotes the divergence of a vector field $\mathbf{A}$ at a point P, show that

$$\operatorname{div}\mathbf{A} = \lim_{\Delta V\to 0} \frac{\displaystyle\iint_{\Delta S} \mathbf{A}\cdot\mathbf{n}\,dS}{\Delta V}$$

where ΔV is the volume enclosed by the surface ΔS and the limit is obtained by shrinking ΔV to the point P.

By the divergence theorem, $\iiint\limits_{\Delta V} \operatorname{div} \mathbf{A}\, dV = \iint\limits_{\Delta S} \mathbf{A}\cdot\mathbf{n}\, dS$

By the mean-value theorem for integrals, the left side can be written

$$\overline{\operatorname{div} \mathbf{A}} \iiint\limits_{\Delta V} dV = \overline{\operatorname{div} \mathbf{A}}\, \Delta V$$

where $\overline{\operatorname{div} \mathbf{A}}$ is some value intermediate between the maximum and minimum of $\operatorname{div} \mathbf{A}$ throughout ΔV. Then

$$\overline{\operatorname{div} \mathbf{A}} = \frac{\iint\limits_{\Delta S} \mathbf{A}\cdot\mathbf{n}\, dS}{\Delta V}$$

Taking the limit as $\Delta V \to 0$ such that P is always interior to ΔV, $\overline{\operatorname{div} \mathbf{A}}$ approaches the value $\operatorname{div} \mathbf{A}$ at point P; hence

$$\operatorname{div} \mathbf{A} = \lim_{\Delta V \to 0} \frac{\iint\limits_{\Delta S} \mathbf{A}\cdot\mathbf{n}\, dS}{\Delta V}$$

This result can be taken as a starting point for defining the divergence of $\mathbf{A}$, and from it all the properties may be derived including proof of the divergence theorem. In Chapter 7 we use this definition to extend the concept of divergence of a vector to coordinate systems other than rectangular. Physically,

$$\frac{\iint\limits_{\Delta S} \mathbf{A}\cdot\mathbf{n}\, dS}{\Delta V}$$

represents the flux or net outflow per unit volume of the vector $\mathbf{A}$ from the surface ΔS. If $\operatorname{div} \mathbf{A}$ is positive in the neighborhood of a point P it means that the outflow from P is positive and we call P a *source*. Similarly, if $\operatorname{div} \mathbf{A}$ is negative in the neighborhood of P the outflow is really an inflow and P is called a *sink*. If in a region there are no sources or sinks, then $\operatorname{div} \mathbf{A} = 0$ and we call $\mathbf{A}$ a *solenoidal* vector field.

20. Evaluate $\iint\limits_{S} \mathbf{r}\cdot\mathbf{n}\, dS$, where S is a closed surface.

By the divergence theorem,

$$\iint\limits_{S} \mathbf{r}\cdot\mathbf{n}\, dS = \iiint\limits_{V} \nabla\cdot\mathbf{r}\, dV$$

$$= \iiint\limits_{V} \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\, dV$$

$$= \iiint\limits_{V} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dV = 3\iiint\limits_{V} dV = 3V$$

where V is the volume enclosed by S.

21. Prove $\iiint\limits_{V} (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint\limits_{S} (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$.

Let $\mathbf{A} = \phi\nabla\psi$ in the divergence theorem. Then

$$\iiint_V \nabla\cdot(\phi\nabla\psi)\,dV = \iint_S (\phi\nabla\psi)\cdot\mathbf{n}\,dS = \iint_S (\phi\nabla\psi)\cdot d\mathbf{S}$$

But $\qquad \nabla\cdot(\phi\nabla\psi) = \phi(\nabla\cdot\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi) = \phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)$

Thus $\qquad \displaystyle\iiint_V \nabla\cdot(\phi\nabla\psi)\,dV = \iiint_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dV$

or

$$(1) \qquad \iiint_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dV = \iint_S (\phi\nabla\psi)\cdot d\mathbf{S}$$

which proves *Green's first identity*. Interchanging ϕ and ψ in *(1)*,

$$(2) \qquad \iiint_V [\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)]\,dV = \iint_S (\psi\nabla\phi)\cdot d\mathbf{S}$$

Subtracting *(2)* from *(1)*, we have

$$(3) \qquad \iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

which is *Green's second identity* or *symmetrical theorem.* In the proof we have assumed that ϕ and ψ are scalar functions of position with continuous derivatives of the second order at least.

22. Prove $\displaystyle\iiint_V \nabla\phi\,dV = \iint_S \phi\mathbf{n}\,dS.$

In the divergence theorem, let $\mathbf{A} = \phi\mathbf{C}$ where $\mathbf{C}$ is a constant vector. Then

$$\iiint_V \nabla\cdot(\phi\mathbf{C})\,dV = \iint_S \phi\mathbf{C}\cdot\mathbf{n}\,dS$$

Since $\nabla\cdot(\phi\mathbf{C}) = (\nabla\phi)\cdot\mathbf{C} = \mathbf{C}\cdot\nabla\phi$ and $\phi\mathbf{C}\cdot\mathbf{n} = \mathbf{C}\cdot(\phi\mathbf{n})$,

$$\iiint_V \mathbf{C}\cdot\nabla\phi\,dV = \iint_S \mathbf{C}\cdot(\phi\mathbf{n})\,dS$$

Taking $\mathbf{C}$ outside the integrals,

$$\mathbf{C}\cdot\iiint_V \nabla\phi\,dV = \mathbf{C}\cdot\iint_S \phi\mathbf{n}\,dS$$

and since $\mathbf{C}$ is an arbitrary constant vector,

$$\iiint_V \nabla\phi\,dV = \iint_S \phi\mathbf{n}\,dS$$

23. Prove $\displaystyle\iiint_V \nabla\times\mathbf{B}\,dV = \iint_S \mathbf{n}\times\mathbf{B}\,dS.$

In the divergence theorem, let $\mathbf{A} = \mathbf{B}\times\mathbf{C}$ where $\mathbf{C}$ is a constant vector. Then

$$\iiint_V \nabla\cdot(\mathbf{B}\times\mathbf{C})\,dV = \iint_S (\mathbf{B}\times\mathbf{C})\cdot\mathbf{n}\,dS$$

Since $\nabla\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\nabla\times\mathbf{B})$ and $(\mathbf{B}\times\mathbf{C})\cdot\mathbf{n} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{n}) = (\mathbf{C}\times\mathbf{n})\cdot\mathbf{B} = \mathbf{C}\cdot(\mathbf{n}\times\mathbf{B})$,

$$\iiint_V \mathbf{C}\cdot(\nabla\times\mathbf{B})\,dV = \iint_S \mathbf{C}\cdot(\mathbf{n}\times\mathbf{B})\,dS$$

Taking $\mathbf{C}$ outside the integrals,

$$\mathbf{C}\cdot\iiint_V \nabla\times\mathbf{B}\,dV = \mathbf{C}\cdot\iint_S \mathbf{n}\times\mathbf{B}\,dS$$

and since $\mathbf{C}$ is an arbitrary constant vector,

$$\iiint_V \nabla\times\mathbf{B}\,dV = \iint_S \mathbf{n}\times\mathbf{B}\,dS$$

24. Show that at any point P

$$(a)\quad \nabla\phi = \lim_{\Delta V\to 0}\frac{\displaystyle\iint_{\Delta S}\phi\mathbf{n}\,dS}{\Delta V} \qquad\text{and}\qquad (b)\quad \nabla\times\mathbf{A} = \lim_{\Delta V\to 0}\frac{\displaystyle\iint_{\Delta S}\mathbf{n}\times\mathbf{A}\,dS}{\Delta V}$$

where ΔV is the volume enclosed by the surface ΔS, and the limit is obtained by shrinking ΔV to the point P.

(*a*) From Problem 22, $\displaystyle\iiint_{\Delta V}\nabla\phi\,dV = \iint_{\Delta S}\phi\mathbf{n}\,dS$. Then $\displaystyle\iiint_{\Delta V}\nabla\phi\cdot\mathbf{i}\,dV = \iint_{\Delta S}\phi\mathbf{n}\cdot\mathbf{i}\,dS$.

Using the same principle employed in Problem 19, we have

$$\overline{\nabla\phi\cdot\mathbf{i}} = \frac{\displaystyle\iint_{\Delta S}\phi\mathbf{n}\cdot\mathbf{i}\,dS}{\Delta V}$$

where $\overline{\nabla\phi\cdot\mathbf{i}}$ is some value intermediate between the maximum and minimum of $\nabla\phi\cdot\mathbf{i}$ throughout ΔV. Taking the limit as $\Delta V\to 0$ in such a way that P is always interior to ΔV, $\overline{\nabla\phi\cdot\mathbf{i}}$ approaches the value

$$(1)\qquad \nabla\phi\cdot\mathbf{i} = \lim_{\Delta V\to 0}\frac{\displaystyle\iint_S\phi\mathbf{n}\cdot\mathbf{i}\,dS}{\Delta V}$$

Similarly we find

$$(2)\qquad \nabla\phi\cdot\mathbf{j} = \lim_{\Delta V\to 0}\frac{\displaystyle\iint_S\phi\mathbf{n}\cdot\mathbf{j}\,dS}{\Delta V}$$

$$(3)\qquad \nabla\phi\cdot\mathbf{k} = \lim_{\Delta V\to 0}\frac{\displaystyle\iint_S\phi\mathbf{n}\cdot\mathbf{k}\,dS}{\Delta V}$$

Multiplying (*1*), (*2*), (*3*) by $\mathbf{i},\mathbf{j},\mathbf{k}$ respectively, and adding, using

$$\nabla\phi = (\nabla\phi\cdot\mathbf{i})\mathbf{i} + (\nabla\phi\cdot\mathbf{j})\mathbf{j} + (\nabla\phi\cdot\mathbf{k})\mathbf{k}, \qquad \mathbf{n} = (\mathbf{n}\cdot\mathbf{i})\mathbf{i} + (\mathbf{n}\cdot\mathbf{j})\mathbf{j} + (\mathbf{n}\cdot\mathbf{k})\mathbf{k}$$

(see Problem 20, Chapter 2) the result follows.

(*b*) From Problem 23, replacing $\mathbf{B}$ by $\mathbf{A}$, $\displaystyle\iiint_{\Delta V} \nabla\times\mathbf{A}\, dV = \iint_{\Delta S} \mathbf{n}\times\mathbf{A}\, dS$.

Then as in part (*a*), we can show that

$$(\nabla\times\mathbf{A})\cdot\mathbf{i} = \lim_{\Delta V\to 0} \frac{\displaystyle\iint_{\Delta S} (\mathbf{n}\times\mathbf{A})\cdot\mathbf{i}\, dS}{\Delta V}$$

and similar results with $\mathbf{j}$ and $\mathbf{k}$ replacing $\mathbf{i}$. Multiplying by $\mathbf{i},\mathbf{j},\mathbf{k}$ and adding, the result follows.

The results obtained can be taken as starting points for definition of gradient and curl. Using these definitions, extensions can be made to coordinate systems other than rectangular.

25. Establish the operator equivalence

$$\nabla\circ \equiv \lim_{\Delta V\to 0} \frac{1}{\Delta V} \oiint_{\Delta S} d\mathbf{S}\,\circ$$

where $\circ$ indicates a dot product, cross product or ordinary product.

To establish the equivalence, the results of the operation on a vector or scalar field must be consistent with already established results.

If $\circ$ is the dot product, then for a vector $\mathbf{A}$,

$$\nabla\circ\mathbf{A} = \lim_{\Delta V\to 0} \frac{1}{\Delta V} \iint_{\Delta S} d\mathbf{S}\circ\mathbf{A}$$

or

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \lim_{\Delta V\to 0} \frac{1}{\Delta V} \iint_{\Delta S} d\mathbf{S}\cdot\mathbf{A} \\ &= \lim_{\Delta V\to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathbf{A}\cdot\mathbf{n}\, dS\end{aligned}$$

established in Problem 19.

Similarly if $\circ$ is the cross product,

$$\begin{aligned}\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} &= \lim_{\Delta V\to 0} \frac{1}{\Delta V} \iint_{\Delta S} d\mathbf{S}\times\mathbf{A} \\ &= \lim_{\Delta V\to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathbf{n}\times\mathbf{A}\, dS\end{aligned}$$

established in Problem 24 (*b*).

Also if $\circ$ is ordinary multiplication, then for a scalar ϕ,

$$\nabla\circ\phi = \lim_{\Delta V\to 0} \frac{1}{\Delta V} \iint_{\Delta S} d\mathbf{S}\circ\phi \quad \text{or} \quad \nabla\phi = \lim_{\Delta V\to 0} \frac{1}{\Delta V} \iint_{\Delta S} \phi\, d\mathbf{S}$$

established in Problem 24(*a*).

26. Let S be a closed surface and let $\mathbf{r}$ denote the position vector of any point (x,y,z) measured from an origin O. Prove that

$$\iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS$$

is equal to (a) zero if O lies outside S; (b) 4π if O lies inside S. This result is known as *Gauss' theorem.*

(a) By the divergence theorem, $\displaystyle\iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = \iiint_V \nabla\cdot\frac{\mathbf{r}}{r^3}\,dV.$

But $\nabla\cdot\frac{\mathbf{r}}{r^3} = 0$ (Problem 19, Chapter 4) everywhere within V provided $r \neq 0$ in V, i.e. provided O is outside of V and thus outside of S. Then $\displaystyle\iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = 0.$

(b) If O is inside S, surround O by a small sphere s of radius a. Let τ denote the region bounded by S and s. Then by the divergence theorem

$$\iint_{S+s} \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = \iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS + \iint_s \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = \iiint_\tau \nabla\cdot\frac{\mathbf{r}}{r^3}\,dV = 0$$

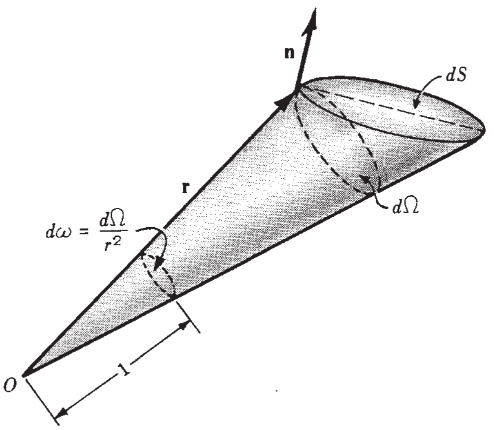
since $r \neq 0$ in τ. Thus

$$\iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = -\iint_s \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS$$

Now on s, $r = a$, $\mathbf{n} = -\frac{\mathbf{r}}{a}$ so that $\frac{\mathbf{n}\cdot\mathbf{r}}{r^3} = \frac{(-\mathbf{r}/a)\cdot\mathbf{r}}{a^3} = -\frac{\mathbf{r}\cdot\mathbf{r}}{a^4} = -\frac{a^2}{a^4} = -\frac{1}{a^2}$ and

$$\iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = -\iint_s \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = \iint_s \frac{1}{a^2}\,dS = \frac{1}{a^2}\iint_s dS = \frac{4\pi a^2}{a^2} = 4\pi$$

27. Interpret Gauss' theorem (Problem 26) geometrically.



Let dS denote an element of surface area and connect all points on the boundary of dS to O (see adjoining figure), thereby forming a cone. Let $d\Omega$ be the area of that portion of a sphere with O as center and radius r which is cut out by this cone; then the *solid angle* subtended by dS at O is defined as $d\omega = \frac{d\Omega}{r^2}$ and is numerically equal to the area of that portion of a sphere with center O and unit radius cut out by the cone. Let $\mathbf{n}$ be the positive unit normal to dS and call θ the angle between $\mathbf{n}$ and $\mathbf{r}$; then $\cos\theta = \frac{\mathbf{n}\cdot\mathbf{r}}{r}$. Also, $d\Omega = \pm\, dS\cos\theta = \pm\,\frac{\mathbf{n}\cdot\mathbf{r}}{r}\,dS$ so that $d\omega = \pm\,\frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS$, the $+$ or $-$ being chosen according as $\mathbf{n}$ and $\mathbf{r}$ form an acute or an obtuse angle θ with each other.

Let S be a surface, as in Figure (a) below, such that any line meets S in not more than two points. If O lies outside S, then at a position such as 1, $\frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = d\omega$; whereas at the corresponding position 2,

$\frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = -d\omega$. An integration over these two regions gives zero, since the contributions to the solid angle cancel out. When the integration is performed over S it thus follows that $\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = 0$, since for every positive contribution there is a negative one.

In case O is inside S, however, then at a position such as 3, $\frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = d\omega$ and at 4, $\frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = d\omega$ so that the contributions add instead of cancel. The total solid angle in this case is equal to the area of a unit sphere which is 4π, so that $\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = 4\pi$.

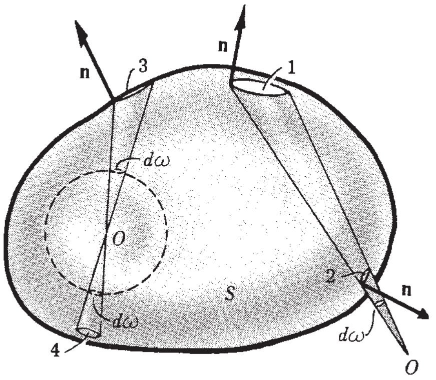


Fig. (a)

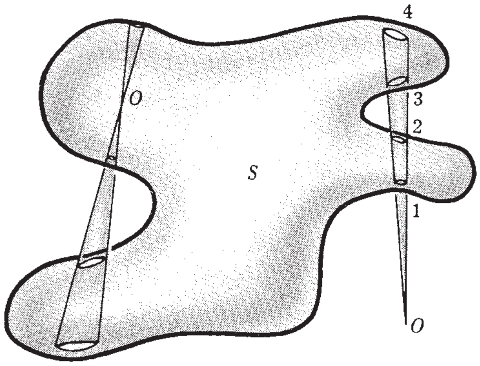


Fig. (b)

For surfaces S, such that a line may meet S in more than two points, an exactly similar situation holds as is seen by reference to Figure (b) above. If O is outside S, for example, then a cone with vertex at O intersects S at an even number of places and the contribution to the surface integral is zero since the solid angles subtended at O cancel out in pairs. If O is inside S, however, a cone having vertex at O intersects S at an odd number of places and since cancellation occurs only for an even number of these, there will always be a contribution of 4π for the entire surface S.

28. A fluid of density $\rho(x,y,z,t)$ moves with velocity $\mathbf{v}(x,y,z,t)$. If there are no sources or sinks, prove that

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{where } \mathbf{J} = \rho\mathbf{v}$$

Consider an arbitrary surface enclosing a volume V of the fluid. At any time the mass of fluid within V is

$$M = \iiint\limits_V \rho\,dV$$

The time rate of increase of this mass is

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\iiint\limits_V \rho\,dV = \iiint\limits_V \frac{\partial\rho}{\partial t}\,dV$$

The mass of fluid per unit time leaving V is

$$\iint\limits_S \rho\mathbf{v}\cdot\mathbf{n}\,dS$$

(see Problem 15) and the time rate of increase in mass is therefore

$$-\iint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS \quad = \quad -\iiint_V \nabla\cdot(\rho\mathbf{v})\,dV$$

by the divergence theorem. Then

$$\iiint_V \frac{\partial\rho}{\partial t}\,dV \quad = \quad -\iiint_V \nabla\cdot(\rho\mathbf{v})\,dV$$

or

$$\iiint_V (\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t})\,dV \quad = \quad 0$$

Since V is arbitrary, the integrand, assumed continuous, must be identically zero, by reasoning similar to that used in Problem 12. Then

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} \quad = \quad 0 \qquad \text{where } \mathbf{J} = \rho\mathbf{v}$$

The equation is called the *continuity equation.* If ρ is a constant, the fluid is incompressible and $\nabla\cdot\mathbf{v} = 0$, i.e. $\mathbf{v}$ is solenoidal.

The continuity equation also arises in electromagnetic theory, where ρ is the *charge density* and $\mathbf{J} = \rho\mathbf{v}$ is the *current density.*

29. If the temperature at any point (x,y,z) of a solid at time t is $U(x,y,z,t)$ and if κ, ρ and c are respectively the thermal conductivity, density and specific heat of the solid, assumed constant, show that

$$\frac{\partial U}{\partial t} \quad = \quad k\,\nabla^2 U \qquad \text{where } k = \kappa/\rho c$$

Let V be an arbitrary volume lying within the solid, and let S denote its surface. The total flux of heat across S, or the quantity of heat leaving S per unit time, is

$$\iint_S (-\kappa\,\nabla U)\cdot\mathbf{n}\,dS$$

Thus the quantity of heat entering S per unit time is

$$(1) \qquad \iint_S (\kappa\,\nabla U)\cdot\mathbf{n}\,dS \quad = \quad \iiint_V \nabla\cdot(\kappa\,\nabla U)\,dV$$

by the divergence theorem. The heat contained in a volume V is given by

$$\iiint_V c\rho\,U\,dV$$

Then the time rate of increase of heat is

$$(2) \qquad \frac{\partial}{\partial t}\iiint_V c\rho\,U\,dV \quad = \quad \iiint_V c\rho\,\frac{\partial U}{\partial t}\,dV$$

Equating the right hand sides of (*1*) and (*2*),

$$\iiint_V [c\rho\,\frac{\partial U}{\partial t} - \nabla\cdot(\kappa\,\nabla U)]\,dV \quad = \quad 0$$

and since V is arbitrary, the integrand, assumed continuous, must be identically zero so that

$$c\rho\,\frac{\partial U}{\partial t} \;=\; \nabla\cdot(\kappa\,\nabla U)$$

or if κ, c, ρ are constants,

$$\frac{\partial U}{\partial t} \;=\; \frac{\kappa}{c\rho}\,\nabla\cdot\nabla U \;=\; k\,\nabla^2 U$$

The quantity k is called the *diffusivity*. For steady-state heat flow (i.e. $\frac{\partial U}{\partial t} = 0$ or U is independent of time) the equation reduces to Laplace's equation $\nabla^2 U = 0$.

STOKES' THEOREM

30. (*a*) Express Stokes' theorem in words and (*b*) write it in rectangular form.

(*a*) The line integral of the tangential component of a vector $\mathbf{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\mathbf{A}$ taken over any surface S having C as its boundary.

(*b*) As in Problem 14(*b*),

$$\mathbf{A} \;=\; A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}, \qquad \mathbf{n} \;=\; \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

Then

$$\nabla\times\mathbf{A} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \;=\; \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

$$(\nabla\times\mathbf{A})\cdot\mathbf{n} \;=\; \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma$$

$$\mathbf{A}\cdot d\mathbf{r} \;=\; (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) \;=\; A_1\,dx + A_2\,dy + A_3\,dz$$

and Stokes' theorem becomes

$$\iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right] dS \;=\; \oint_C A_1\,dx + A_2\,dy + A_3\,dz$$

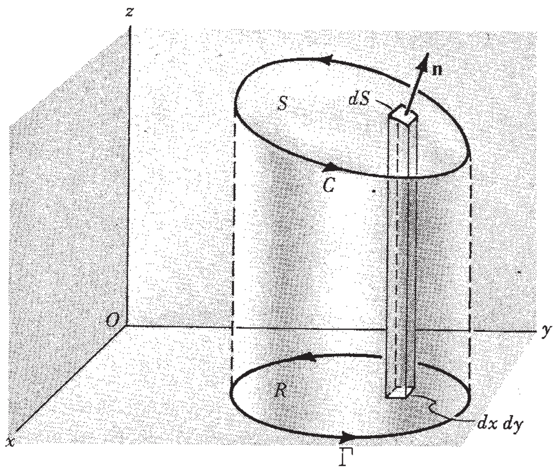
31. Prove Stokes' theorem.

Let S be a surface which is such that its projections on the xy, yz and xz planes are regions bounded by simple closed curves, as indicated in the adjoining figure. Assume S to have representation $z = f(x,y)$ or $x = g(y,z)$ or $y = h(x,z)$, where f, g, h are single-valued, continuous and differentiable functions. We must show that

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS \;=\; \iint_S [\nabla\times(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})]\cdot\mathbf{n}\,dS$$

$$=\; \oint_C \mathbf{A}\cdot d\mathbf{r}$$

where C is the boundary of S.



Consider first $\displaystyle\iint_S [\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\, dS$.

Since $$\nabla\times(A_1\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z}\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{k},$$

(1) $$[\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\, dS = (\frac{\partial A_1}{\partial z}\mathbf{n}\cdot\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}\cdot\mathbf{k})\, dS$$

If $z = f(x,y)$ is taken as the equation of S, then the position vector to any point of S is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$ so that $\frac{\partial\mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$. But $\frac{\partial\mathbf{r}}{\partial y}$ is a vector tangent to S (see Problem 25, Chapter 3) and thus perpendicular to $\mathbf{n}$, so that

$$\mathbf{n}\cdot\frac{\partial\mathbf{r}}{\partial y} = \mathbf{n}\cdot\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{n}\cdot\mathbf{k} = 0 \qquad \text{or} \qquad \mathbf{n}\cdot\mathbf{j} = -\frac{\partial z}{\partial y}\mathbf{n}\cdot\mathbf{k}$$

Substitute in (1) to obtain

$$(\frac{\partial A_1}{\partial z}\mathbf{n}\cdot\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}\cdot\mathbf{k})\, dS = (-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\mathbf{n}\cdot\mathbf{k} - \frac{\partial A_1}{\partial y}\mathbf{n}\cdot\mathbf{k})\, dS$$

or

(2) $$[\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\, dS = -(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y})\,\mathbf{n}\cdot\mathbf{k}\, dS$$

Now on S, $A_1(x,y,z) = A_1(x,y,f(x,y)) = F(x,y)$; hence $\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$ and (2) becomes

$$[\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\, dS = -\frac{\partial F}{\partial y}\,\mathbf{n}\cdot\mathbf{k}\, dS = -\frac{\partial F}{\partial y}\, dx\, dy$$

Then

$$\iint_S [\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\, dS = \iint_R -\frac{\partial F}{\partial y}\, dx\, dy$$

where R is the projection of S on the xy plane. By Green's theorem for the plane the last integral equals $\displaystyle\oint_\Gamma F\, dx$ where Γ is the boundary of R. Since at each point (x,y) of Γ the value of F is the same as the value of A_1 at each point (x,y,z) of C, and since dx is the same for both curves, we must have

$$\oint_\Gamma F\, dx = \oint_C A_1\, dx$$

or

$$\iint_S [\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\, dS = \oint_C A_1\, dx$$

Similarly, by projections on the other coordinate planes,

$$\iint_S [\nabla\times(A_2\mathbf{j})]\cdot\mathbf{n}\, dS = \oint_C A_2\, dy$$

$$\iint_S [\nabla\times(A_3\mathbf{k})]\cdot\mathbf{n}\, dS = \oint_C A_3\, dz$$

Thus by addition,

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \oint_C \mathbf{A}\cdot d\mathbf{r}$$

The theorem is also valid for surfaces S which may not satisfy the restrictions imposed above. For assume that S can be subdivided into surfaces $S_1, S_2, \ldots S_k$ with boundaries $C_1, C_2, \ldots C_k$ which do satisfy the restrictions. Then Stokes' theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_1, C_2, \ldots C_k$, the line integral over C is obtained.

32. Verify Stokes' theorem for $\mathbf{A} = (2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.

The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leqq t < 2\pi$ be parametric equations of C. Then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$

Also,

$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{k}$$

Then

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S \mathbf{k}\cdot\mathbf{n}\,dS = \iint_R dx\,dy$$

since $\mathbf{n}\cdot\mathbf{k}\,dS = dx\,dy$ and R is the projection of S on the xy plane. This last integral equals

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

and Stokes' theorem is verified.

33. Prove that a necessary and sufficient condition that $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ for every closed curve C is that $\nabla\times\mathbf{A} = \mathbf{0}$ identically.

Sufficiency. Suppose $\nabla\times\mathbf{A} = \mathbf{0}$. Then by Stokes' theorem

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = 0$$

Necessity. Suppose $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ around every closed path C, and assume $\nabla\times\mathbf{A} \neq \mathbf{0}$ at some point P. Then assuming $\nabla\times\mathbf{A}$ is continuous there will be a region with P as an interior point, where $\nabla\times\mathbf{A} \neq \mathbf{0}$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point has the same direction as $\nabla\times\mathbf{A}$, i.e. $\nabla\times\mathbf{A} = \alpha\mathbf{n}$ where α is a positive constant. Let C be the boundary of S. Then by Stokes' theorem

$$\oint_C \mathbf{A}\cdot d\mathbf{r} \;=\; \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS \;=\; \alpha\iint_S \mathbf{n}\cdot\mathbf{n}\,dS \;>\; 0$$

which contradicts the hypothesis that $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ and shows that $\nabla\times\mathbf{A} = \mathbf{0}$.

It follows that $\nabla\times\mathbf{A} = \mathbf{0}$ is also a necessary and sufficient condition for a line integral $\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r}$ to be independent of the path joining points P_1 and P_2. (See Problems 10 and 11, Chapter 5.)

34. Prove $\oint d\mathbf{r}\times\mathbf{B} = \iint_S (\mathbf{n}\times\nabla)\times\mathbf{B}\,dS$.

In Stokes' theorem, let $\mathbf{A} = \mathbf{B}\times\mathbf{C}$ where $\mathbf{C}$ is a constant vector, Then

$$\oint d\mathbf{r}\cdot(\mathbf{B}\times\mathbf{C}) \;=\; \iint_S [\nabla\times(\mathbf{B}\times\mathbf{C})]\cdot\mathbf{n}\,dS$$

$$\oint \mathbf{C}\cdot(d\mathbf{r}\times\mathbf{B}) \;=\; \iint_S [(\mathbf{C}\cdot\nabla)\mathbf{B} - \mathbf{C}(\nabla\cdot\mathbf{B})]\cdot\mathbf{n}\,dS$$

$$\mathbf{C}\cdot\oint d\mathbf{r}\times\mathbf{B} \;=\; \iint_S [(\mathbf{C}\cdot\nabla)\mathbf{B}]\cdot\mathbf{n}\,dS \;-\; \iint_S [\mathbf{C}(\nabla\cdot\mathbf{B})]\cdot\mathbf{n}\,dS$$

$$=\; \iint_S \mathbf{C}\cdot[\nabla(\mathbf{B}\cdot\mathbf{n})]\,dS \;-\; \iint_S \mathbf{C}\cdot[\mathbf{n}(\nabla\cdot\mathbf{B})]\,dS$$

$$=\; \mathbf{C}\cdot\iint_S [\nabla(\mathbf{B}\cdot\mathbf{n}) - \mathbf{n}(\nabla\cdot\mathbf{B})]\,dS \;=\; \mathbf{C}\cdot\iint_S (\mathbf{n}\times\nabla)\times\mathbf{B}\,dS$$

Since $\mathbf{C}$ is an arbitrary constant vector $\oint d\mathbf{r}\times\mathbf{B} = \iint_S (\mathbf{n}\times\nabla)\times\mathbf{B}\,dS$

35. If ΔS is a surface bounded by a simple closed curve C, P is any point of ΔS not on C and $\mathbf{n}$ is a unit normal to ΔS at P, show that at P

$$(\text{curl}\,\mathbf{A})\cdot\mathbf{n} \;=\; \lim_{\Delta S\to 0} \frac{\oint_C \mathbf{A}\cdot d\mathbf{r}}{\Delta S}$$

where the limit is taken in such a way that ΔS shrinks to P.

By Stokes' theorem, $\iint_{\Delta S} (\text{curl}\,\mathbf{A})\cdot\mathbf{n}\,dS = \oint_C \mathbf{A}\cdot d\mathbf{r}$.

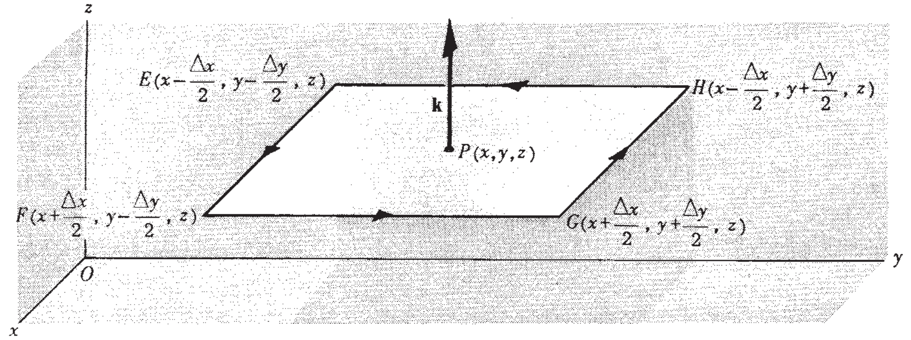
Using the mean value theorem for integrals as in Problems 19 and 24, this can be written

$$\overline{(\text{curl}\,\mathbf{A})\cdot\mathbf{n}} \;=\; \frac{\oint_C \mathbf{A}\cdot d\mathbf{r}}{\Delta S}$$

and the required result follows upon taking the limit as $\Delta S \to 0$.

This can be used as a starting point for defining curl **A** (see Problem 36) and is useful in obtaining curl **A** in coordinate systems other than rectangular. Since $\oint_C \mathbf{A} \cdot d\mathbf{r}$ is called the circulation of **A** about C, the normal component of the curl can be interpreted physically as the limit of the circulation per unit area, thus accounting for the synonym rotation of **A** (rot **A**) instead of curl of **A**.

36. If curl **A** is defined according to the limiting process of Problem 35, find the z component of curl **A**.



Let $EFGH$ be a rectangle parallel to the xy plane with interior point $P(x,y,z)$ taken as midpoint, as shown in the figure above. Let A_1 and A_2 be the components of **A** at P in the positive x and y directions respectively.

If C is the boundary of the rectangle, then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \int_{EF} \mathbf{A}\cdot d\mathbf{r} + \int_{FG} \mathbf{A}\cdot d\mathbf{r} + \int_{GH} \mathbf{A}\cdot d\mathbf{r} + \int_{HE} \mathbf{A}\cdot d\mathbf{r}$$

But

$$\int_{EF} \mathbf{A}\cdot d\mathbf{r} = (A_1 - \frac{1}{2}\frac{\partial A_1}{\partial y}\Delta y)\Delta x \qquad \int_{GH} \mathbf{A}\cdot d\mathbf{r} = -(A_1 + \frac{1}{2}\frac{\partial A_1}{\partial y}\Delta y)\Delta x$$

$$\int_{FG} \mathbf{A}\cdot d\mathbf{r} = (A_2 + \frac{1}{2}\frac{\partial A_2}{\partial x}\Delta x)\Delta y \qquad \int_{HE} \mathbf{A}\cdot d\mathbf{r} = -(A_2 - \frac{1}{2}\frac{\partial A_2}{\partial x}\Delta x)\Delta y$$

except for infinitesimals of higher order than $\Delta x\, \Delta y$.

Adding, we have approximately $\oint_C \mathbf{A}\cdot d\mathbf{r} = (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\Delta x\, \Delta y$.

Then, since $\Delta S = \Delta x \Delta y$,

$$z \text{ component of curl } \mathbf{A} = (\text{curl } \mathbf{A})\cdot\mathbf{k} = \lim_{\Delta S \to 0} \frac{\oint \mathbf{A}\cdot d\mathbf{r}}{\Delta S}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\Delta x\, \Delta y}{\Delta x\, \Delta y}$$

$$= \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

SUPPLEMENTARY PROBLEMS

37. Verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region defined by: (*a*) $y=\sqrt{x}$, $y=x^2$; (*b*) $x=0$, $y=0$, $x+y=1$.
Ans. (*a*) common value = 3/2 (*b*) common value = 5/3

38. Evaluate $\oint_C (3x+4y)\,dx + (2x-3y)\,dy$ where C, a circle of radius two with center at the origin of the xy plane, is traversed in the positive sense. *Ans.* -8π

39. Work the previous problem for the line integral $\oint_C (x^2+y^2)\,dx + 3xy^2\,dy$. *Ans.* 12π

40. Evaluate $\oint (x^2-2xy)\,dx + (x^2y+3)\,dy$ around the boundary of the region defined by $y^2=8x$ and $x=2$ (*a*) directly, (*b*) by using Green's theorem. *Ans.* 128/5

41. Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy-y^2)\,dx + (3x^2-2xy)\,dy$ along the cycloid $x=\theta-\sin\theta$, $y=1-\cos\theta$.
Ans. $6\pi^2-4\pi$

42. Evaluate $\oint (3x^2+2y)\,dx - (x+3\cos y)\,dy$ around the parallelogram having vertices at (0,0), (2,0), (3,1) and (1,1). *Ans.* -6

43. Find the area bounded by one arch of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, $a>0$, and the x axis.
Ans. $3\pi a^2$

44. Find the area bounded by the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$, $a>0$.
Hint: Parametric equations are $x=a\cos^3\theta$, $y=a\sin^3\theta$. *Ans.* $3\pi a^2/8$

45. Show that in polar coordinates (ρ,ϕ) the expression $x\,dy-y\,dx = \rho^2\,d\phi$. Interpret $\frac{1}{2}\int x\,dy - y\,dx$.

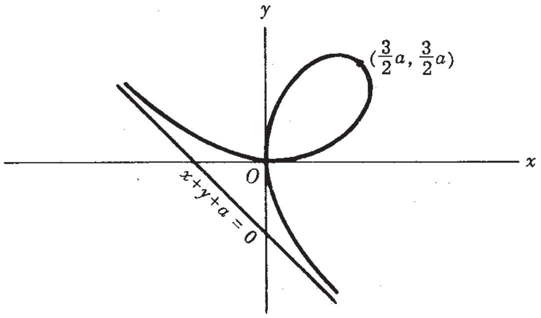
46. Find the area of a loop of the four-leafed rose $\rho = 3\sin 2\phi$. *Ans.* $9\pi/8$

47. Find the area of both loops of the lemniscate $\rho^2 = a^2\cos 2\phi$. *Ans.* a^2

48. Find the area of the loop of the folium of Descartes $x^3+y^3=3axy$, $a>0$ (see adjoining figure).
Hint: Let $y=tx$ and obtain the parametric equations of the curve. Then use the fact that

$$\text{Area} = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\oint x^2\,d\left(\frac{y}{x}\right) = \frac{1}{2}\oint x^2\,dt$$

Ans. $3a^2/2$



49. Verify Green's theorem in the plane for $\oint_C (2x-y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$. *Ans.* common value = 60π

50. Evaluate $\int_{(1,0)}^{(-1,0)} \frac{-y\,dx + x\,dy}{x^2+y^2}$ along the following paths:

(*a*) straight line segments from (1,0) to (1,1), then to (−1,1), then to (−1,0).
(*b*) straight line segments from (1,0) to (1,−1), then to (−1,−1), then to (−1,0).
Show that although $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the line integral *is dependent* on the path joining (1,0) to (−1,0) and explain.
Ans. (*a*) π (*b*) $-\pi$

51. By changing variables from (x,y) to (u,v) according to the transformation $x = x(u,v)$, $y = y(u,v)$, show that the area A of a region R bounded by a simple closed curve C is given by

$$A = \iint_R \left| J\left(\frac{x,y}{u,v}\right) \right| du\, dv \qquad \text{where} \qquad J\left(\frac{x,y}{u,v}\right) \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of x and y with respect to u and v. What restrictions should you make ? Illustrate the result where u and v are polar coordinates.
Hint: Use the result $A = \frac{1}{2}\int x\,dy - y\,dx$, transform to u,v coordinates and then use Green's theorem.

52. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = 2xy\,\mathbf{i} + yz^2\,\mathbf{j} + xz\,\mathbf{k}$ and S is:

(*a*) the surface of the parallelepiped bounded by $x=0$, $y=0$, $z=0$, $x=2$, $y=1$ and $z=3$,
(*b*) the surface of the region bounded by $x=0$, $y=0$, $y=3$, $z=0$ and $x+2z=6$.
Ans. (*a*) 30 (*b*) 351/2

53. Verify the divergence theorem for $\mathbf{A} = 2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k}$ taken over the region in the first octant bounded by $y^2+z^2 = 9$ and $x=2$. *Ans.* 180

54. Evaluate $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS$ where (*a*) S is the sphere of radius 2 with center at (0,0,0), (*b*) S is the surface of the cube bounded by $x=-1$, $y=-1$, $z=-1$, $x=1$, $y=1$, $z=1$, (*c*) S is the surface bounded by the paraboloid $z = 4-(x^2+y^2)$ and the xy plane. *Ans.* (*a*) 32π (*b*) 24 (*c*) 24π

55. If S is any closed surface enclosing a volume V and $\mathbf{A} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, prove that $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = (a+b+c)V$.

56. If $\mathbf{H} = \text{curl}\,\mathbf{A}$, prove that $\iint_S \mathbf{H}\cdot\mathbf{n}\, dS = 0$ for any closed surface S.

57. If $\mathbf{n}$ is the unit outward drawn normal to any closed surface of area S, show that $\iiint_V \text{div}\,\mathbf{n}\, dV = S$.

58. Prove $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS$.

59. Prove $\iint_S r^5\mathbf{n}\, dS = \iiint_V 5r^3\mathbf{r}\, dV$.

60. Prove $\iint_S \mathbf{n}\, dS = \mathbf{0}$ for any closed surface S.

61. Show that Green's second identity can be written $\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S (\phi\frac{d\psi}{dn} - \psi\frac{d\phi}{dn})\,dS$

62. Prove $\iint_S \mathbf{r}\times d\mathbf{S} = \mathbf{0}$ for any closed surface S.

63. Verify Stokes' theorem for $\mathbf{A} = (y-z+2)\mathbf{i} + (yz+4)\mathbf{j} - xz\,\mathbf{k}$, where S is the surface of the cube $x=0$, $y=0$, $z=0$, $x=2$, $y=2$, $z=2$ above the xy plane. *Ans.* common value = -4

64. Verify Stokes' theorem for $\mathbf{F} = xz\,\mathbf{i} - y\,\mathbf{j} + x^2y\,\mathbf{k}$, where S is the surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x+y+2z=8$ which is not included in the xz plane. *Ans.* common value = 32/3

65. Evaluate $\displaystyle\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$, where $\mathbf{A} = (x^2+y-4)\mathbf{i} + 3xy\,\mathbf{j} + (2xz+z^2)\mathbf{k}$ and S is the surface of (*a*) the hemisphere $x^2+y^2+z^2 = 16$ above the xy plane, (*b*) the paraboloid $z = 4-(x^2+y^2)$ above the xy plane. *Ans.* (*a*) -16π, (*b*) -4π

66. If $\mathbf{A} = 2yz\,\mathbf{i} - (x+3y-2)\mathbf{j} + (x^2+z)\mathbf{k}$, evaluate $\displaystyle\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$ over the surface of intersection of the cylinders $x^2+y^2 = a^2$, $x^2+z^2 = a^2$ which is included in the first octant. *Ans.* $-\dfrac{a^2}{12}(3\pi+8a)$

67. A vector $\mathbf{B}$ is always normal to a given closed surface S. Show that $\displaystyle\iiint_V \operatorname{curl}\mathbf{B}\,dV = \mathbf{0}$, where V is the region bounded by S.

68. If $\displaystyle\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \mathbf{H}\cdot d\mathbf{S}$, where S is any surface bounded by the curve C, show that $\nabla\times\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial\mathbf{H}}{\partial t}$.

69. Prove $\displaystyle\oint_C \phi\,d\mathbf{r} = \iint_S d\mathbf{S}\times\nabla\phi$.

70. Use the operator equivalence of Solved Problem 25 to arrive at (*a*) $\nabla\phi$, (*b*) $\nabla\cdot\mathbf{A}$, (*c*) $\nabla\times\mathbf{A}$ in rectangular coordinates.

71. Prove $\displaystyle\iiint_V \nabla\phi\cdot\mathbf{A}\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS - \iiint_V \phi\nabla\cdot\mathbf{A}\,dV$.

72. Let $\mathbf{r}$ be the position vector of any point relative to an origin O. Suppose ϕ has continuous derivatives of order two, at least, and let S be a closed surface bounding a volume V. Denote ϕ at O by ϕ_o. Show that

$$\iint_S \left[\tfrac{1}{r}\nabla\phi - \phi\nabla\left(\tfrac{1}{r}\right)\right]\cdot d\mathbf{S} = \iiint_V \frac{\nabla^2\phi}{r}\,dV + \alpha$$

where $\alpha = 0$ or $4\pi\phi_o$ according as O is outside or inside S.

73. The potential $\phi(P)$ at a point $P(x,y,z)$ due to a system of charges (or masses) $q_1, q_2, \ldots, q_n$ having position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ with respect to P is given by

$$\phi = \sum_{m=1}^{n}\frac{q_m}{r_m}$$

Prove *Gauss' law*

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi Q$$

where $\mathbf{E} = -\nabla\phi$ is the electric field intensity, S is a surface enclosing all the charges and $Q = \sum_{m=1}^{n} q_m$ is the total charge within S.

74. If a region V bounded by a surface S has a continuous charge (or mass) distribution of density ρ, the potential $\phi(P)$ at a point P is defined by $\phi = \displaystyle\iiint_V \frac{\rho\,dV}{r}$. Deduce the following under suitable assumptions:

(*a*) $\displaystyle\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi\iiint_V \rho\,dV$, where $\mathbf{E} = -\nabla\phi$.

(*b*) $\nabla^2\phi = -4\pi\rho$ (Poisson's equation) at all points P where charges exist, and $\nabla^2\phi = 0$ (Laplace's equation) where no charges exist.