(*iv*) Generally, the volume of the parallelopiped is $V = |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$ for both right and left handed system i.e. for cyclic and anticyclic order of the vectors **a**, **b**, **c**.

(*v*) For the fundamental system of unit vectors **i**, **j**, **k**

$$V = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}] = [\mathbf{j}\ \mathbf{k}\ \mathbf{i}] = [\mathbf{k}\ \mathbf{i}\ \mathbf{j}] = 1$$

and $$V = [\mathbf{i}\ \mathbf{k}\ \mathbf{j}] = [\mathbf{j}\ \mathbf{i}\ \mathbf{k}] = [\mathbf{k}\ \mathbf{j}\ \mathbf{i}] = -1,$$

(*vi*) Also $[\lambda\mathbf{a}\ \mu\mathbf{b}\ \nu\mathbf{c}] = \lambda\mu\nu\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ where λ, μ, ν are scalars.

(*vii*) If the three vectors **a**, **b** and **c** lie in a plane, then no parallelopiped can be formed or height of the parallelopiped is zero. So, volume $V = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$ which is condition of coplanarity of the vectors **a**, **b** and **c** and the vectors will be linearly dependent if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$. This is also derivable from eqn. (1) when rows are identical.

(*viii*) If none of **a**, **b**, **c** is a null vector, the eqn. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$ in (*vii*) implies **a** is perpendicular to $\mathbf{b} \times \mathbf{c}$. Since $\mathbf{b} \times \mathbf{c}$ is $\perp$ to the plane of **b** and **c**, **da** $\perp$ to $\mathbf{b} \times \mathbf{c}$ means **a** lies in the plane of **b** & **c** or **a**, **b**, **c** are co-planar. Then **a** is of the form $\mathbf{a} = \lambda\mathbf{b} + \mu\mathbf{c}$, ($\lambda$, μ are scalars) Clearly $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = (\lambda\mathbf{b} + \mu\mathbf{c}).(\mathbf{b} \times \mathbf{c}) = 0$ varying (*vii*).

(*ix*) If the value of the determinant is zero, the three vectors **a**, **b**, **c** are *co-planar*. This happens when (a) any one of **a**, **b**, **c** is a null vector

(b) two or all are equal

(c) Two or all are parallel.

2.12 Volume of a tetrahedron

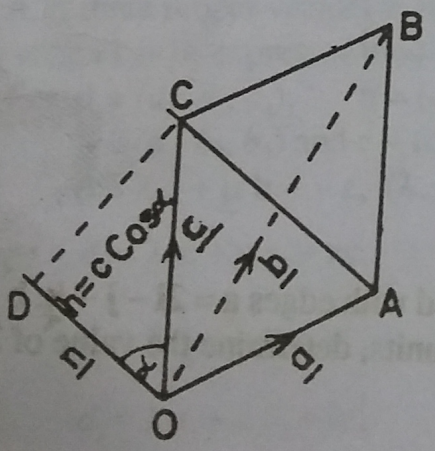

Fig. 41

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$ be the edges of the tetrahedron *OABC*. Let **n** the unit vector perpendicular to the plane of **a** and **b**.

From Fig. 41, $\mathbf{a} \times \mathbf{b} = ab \sin\theta$ **n** acts in the direction of **n**. If *V* be the volume of the tetrahedron, then

$$V = \frac{1}{3} \text{ (area of the base). (height)}$$

$$= \frac{1}{3}(\Delta OAB)(OD)$$

$$= \frac{1}{3}\left|\frac{1}{2}\mathbf{a}\times\mathbf{b}\right|(c\cos\alpha)$$

$$= \frac{1}{6}abc\sin\theta\cos\alpha$$

or $V = \frac{1}{6}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ or $\frac{1}{6}[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

If the position vectors of O be $\mathbf{r}$ then

$$\overrightarrow{OA} = \mathbf{a}-\mathbf{r}, \overrightarrow{OB} = \mathbf{b}-\mathbf{r} \text{ and } \overrightarrow{OC} = \mathbf{c}-\mathbf{r}.$$

$$\therefore \quad V = \frac{1}{6}[\mathbf{a}-\mathbf{r}\ \ \mathbf{b}-\mathbf{r}\ \ \mathbf{c}-\mathbf{r}].$$

EXAMPLES

1. If three vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \lambda\mathbf{i} -$ co-planar, find the value of λ.

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are co-planar then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$, that is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ \lambda & -1 & \lambda \end{vmatrix} = 0$$

or $3\lambda - 3 = 0$ i.e. $\lambda = 1$.

2. If the volume of a parallelopi