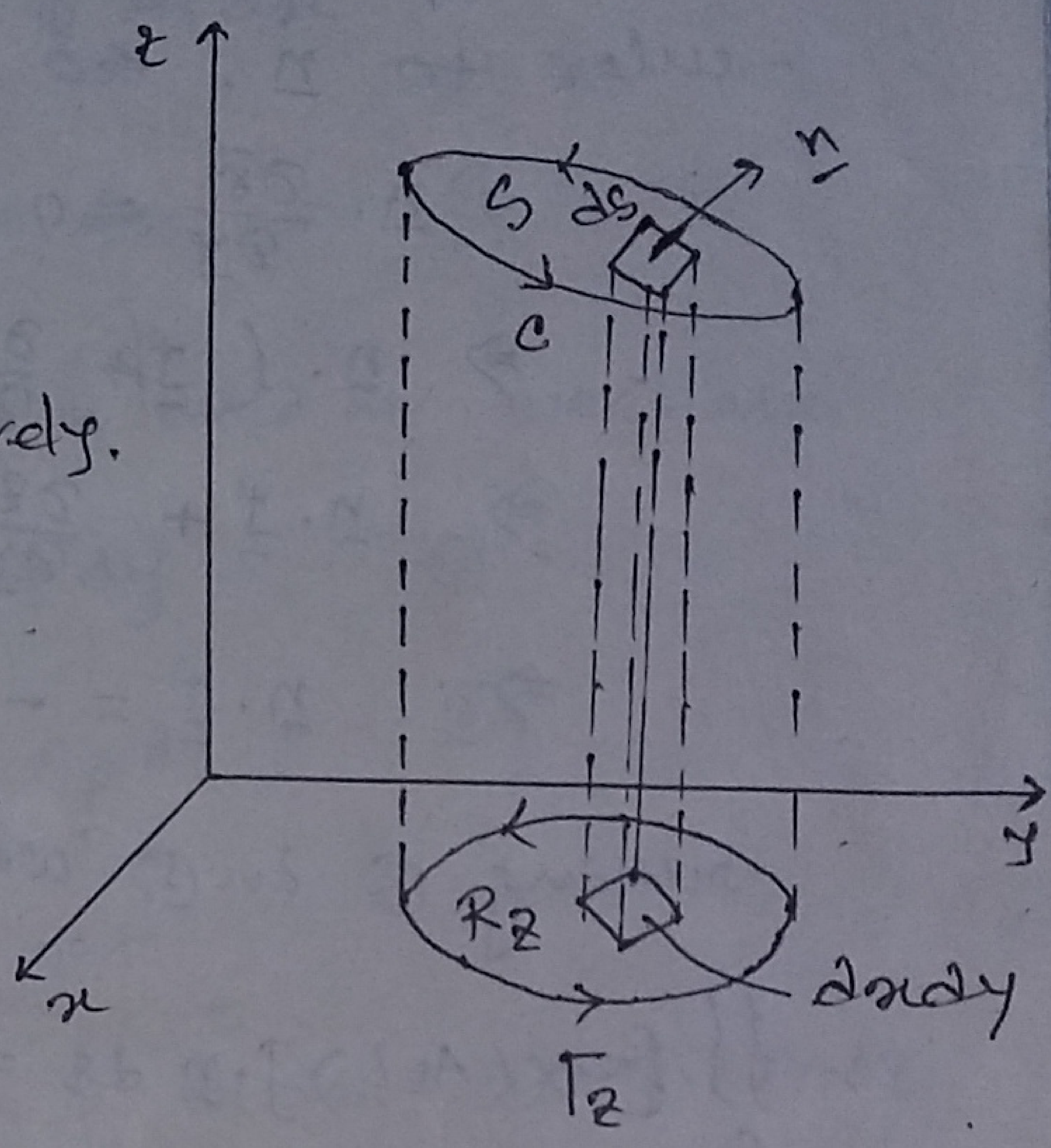
*State and prove Stoke's theorem.

Statement: If S is an open two-sided surface bounded by a simple closed curve C and if $\underline{A}$ has continuous derivatives, then

$$\oint_C \underline{A}\cdot d\underline{r} = \iint_S (\nabla\times\underline{A})\cdot\underline{n}\, ds$$

where, C is traversed in the positive direction and $\underline{n}$ is the positive outward drawn unit normal to S.

proof: Let S be a two sided surface bounded by a simple closed curve C. If projections on the xy, yz and zx planes are the regions R_z, R_x, and R_y bounded by the simple closed curves T_z, T_x and T_y respectively. Assume the surface S has the following representation $z=f(x,y)$ or $x=g(y,z)$ or $y=h(x,z)$, where f, g and h are single-valued, continuous and differentiable functions. Then we have to show that



$$\iint_S (\nabla\times\underline{A})\cdot\underline{n}\, ds = \iint_S [\nabla\times(A_1\underline{i}+A_2\underline{j}+A_3\underline{k})]\cdot\underline{n}\, ds$$
$$= \oint_C \underline{A}\cdot d\underline{r}.$$

Consider first, $\iint_S [\nabla\times(A_1\underline{i})]\cdot\underline{n}\, ds$

Since $\nabla\times(A_1\underline{i}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix}$

$$= \frac{\partial A_1}{\partial z}\underline{j} - \frac{\partial A_1}{\partial y}\underline{k}$$

$$\therefore \iint_S [\nabla\times(A_1\underline{i})]\cdot\underline{n}\, ds = \iint_S \left(\frac{\partial A_1}{\partial z}\underline{j} - \frac{\partial A_1}{\partial y}\underline{k}\right)\cdot\underline{n}\, ds$$

$$\therefore \iint_S [\nabla\times(A_1 \underline{i})]\cdot \underline{n}\, ds = \iint_S \left(\frac{\partial A_1}{\partial z}\, \underline{n}\cdot\underline{j} - \frac{\partial A_1}{\partial y}\, \underline{n}\cdot\underline{k} \right) ds \quad \text{----} (1)$$

Let $z = f(x,y)$ be the equation of S, then the position vector to any point of S is

$$\bar{r} = x\underline{i} + y\underline{j} + z\underline{k}$$
$$= x\underline{i} + y\underline{j} + f(x,y)\,\underline{k}$$

So that, $\dfrac{\partial \bar{r}}{\partial y} = \underline{j} + \dfrac{\partial z}{\partial y}\underline{k} = \underline{j} + \dfrac{\partial f}{\partial y}\underline{k}$.

But $\dfrac{\partial \bar{r}}{\partial y}$ is a vector tangent to S and thus perpendi-cular to $\underline{n}$, so that

$$\underline{n}\cdot\frac{\partial \bar{r}}{\partial y} = 0$$

$$\Rightarrow \underline{n}\cdot\left(\underline{j} + \frac{\partial f}{\partial y}\underline{k}\right) = 0$$

$$\Rightarrow \underline{n}\cdot\underline{j} + \frac{\partial z}{\partial y}\, \underline{n}\cdot\underline{k} = 0$$

$$\Rightarrow \underline{n}\cdot\underline{j} = -\frac{\partial z}{\partial y}\, \underline{n}\cdot\underline{k} \quad \text{----} (2)$$

putting (2) in (1) we get

$$\iint_S [\nabla\times(A_1 \underline{i})]\cdot \underline{n}\, ds = \iint_S \left\{ \frac{\partial A_1}{\partial z}\cdot\left(-\frac{\partial z}{\partial y}\, \underline{n}\cdot\underline{k}\right) - \frac{\partial A_1}{\partial y}\, \underline{n}\cdot\underline{k} \right\} ds$$

$$= -\iint_S \left(\frac{\partial A_1}{\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} \right) \underline{n}\cdot\underline{k}\, ds \quad \text{----} (3)$$

Now, on S,

$$A_1(x,y,z) = A_1(x,y,f(x,y)) = F(x,y) \text{ (say)}$$

$$\therefore \frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$$

Hence (3) become

$$\iint_S [\nabla\times(A_1 \underline{i})]\cdot \underline{n}\, ds = -\iint_R \frac{\partial F}{\partial y}\, \underline{n}\cdot\underline{k}\, ds,$$ where R is the projection of S on the xy plane.

$$= -\iint_{R_2} \frac{\partial F}{\partial y}\, dx\,dy.$$

$$= \oint_{T_z} F\,dx \quad \Big[\text{By Green's theorem}$$

$$\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint M\,dx = - \iint_R \frac{\partial M}{\partial y}\,dx\,dy \Big]$$

Since at each point (x, y) of T_{xy} the value of F is the same as the value of A_1 at each point (x, y, z) of c. and since dx is the same for both curves T_z and c. then we must have

$$\iint_S [\nabla \times (A_1 \underline{i})] \cdot \underline{n}\, ds = \oint_{T_z} F\,dx = \oint_c A_1\,dx$$

$$\Rightarrow \iint_S [\nabla \times (A_1 \underline{i})] \cdot \underline{n}\, ds = \oint_c A_1\,dx \quad \text{---} (4)$$

Similarly, for the projections R_x and R_y, we have

$$\iint_S [\nabla \times (A_2 \underline{j})] \cdot \underline{n}\, ds = \oint_c A_2\,dy \quad \text{--} (5)$$

and

$$\iint_S [\nabla \times (A_3 \underline{k})] \cdot \underline{n}\, ds = \oint_c A_3\,dz \quad \text{---} (6)$$

Now, adding (4) (5) and (6) we get

$$\iint_S [\nabla \times (A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{k})] \cdot \underline{n}\, ds = \oint_c A_1\,dx + A_2\,dy + A_3\,dz$$

$$\Rightarrow \iint_S (\nabla \times \underline{A}) \cdot \underline{n}\, ds = \oint_c \underline{A} \cdot d\underline{r}$$

Hence the theorem is proved.

*SP-32: Verify Stokes's theorem for $\underline{A} = (2x-y)\underline{i} - yz^2\underline{j} - y^2z\underline{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and c its boundary.

Sol^n: Here given, $\underline{A} = (2x-y)\underline{i} - yz^2\underline{j} - y^2z\underline{k}$. Now we are to prove that

$$\oint_c \underline{A}\, d\underline{r} = \iint_S (\nabla\times\underline{A})\cdot\underline{n}\, ds$$

Here the boundary c of S is a circle in the xy-plane of radius one and the centre of this circle at the origin.

Let $x=\cos\theta$, $y=\sin\theta$, $z=0$ be the parametric equation of c, where $0\le\theta\le 2\pi$. Then

$$\oint_c \underline{A}\cdot d\underline{r} = \oint_c (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\, d\theta)$$

$$= \int_0^{2\pi} (\sin^2\theta - \sin 2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left\{ \tfrac{1}{2}(1-\cos 2\theta) - \sin 2\theta \right\} d\theta$$

$$= \tfrac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} + \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \tfrac{1}{2}\cdot 2\pi = \pi$$

Again, we have,

$$\nabla\times\underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz+2yz)\underline{i} + (0-0)\underline{j} + (0+1)\underline{k}$$

$$= \underline{k}$$

Now, $\iint_S (\nabla\times\underline{A})\cdot\underline{n}\, ds = \iint_S \underline{k}\cdot\underline{n}\, ds = \iint_R dx\,dy$

since $\underline{n}\cdot\underline{k}\, ds = dx\,dy$ and R is the projection of S on the xy-plane.

$$\therefore \iint_R dx\,dy = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx$$

$$= \int_{x=-1}^{1} 2\sqrt{1-x^2}\,dx$$

$$= 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_{-1}^{1}$$

$$= 2\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2}\right] = 2\cdot\frac{\pi}{2} = \pi;$$

Hence, the theorem is verified.

SP-34* prove $\oint d\underline{r}\times\underline{B} = \iint_S (\underline{n}\times\nabla)\times\underline{B}\,ds$

proof: By the stokes's theorem, we have

$$\oint_c \underline{A}\cdot d\underline{r} = \iint_S (\nabla\times\underline{A})\cdot\underline{n}\,ds$$

$$\Rightarrow \oint_c d\underline{r}\cdot\underline{A} = \iint_S (\nabla\times\underline{A})\cdot\underline{n}\,ds$$

Now, putting $\underline{A} = \underline{B}\times\underline{c}$, where c is a constant vector, then,

$$\oint_c d\underline{r}\cdot(\underline{B}\times\underline{c}) = \iint_S [\nabla\times(\underline{B}\times\underline{c})]\cdot\underline{n}\,ds$$

$$\Rightarrow \oint_c \underline{c}\cdot(d\underline{r}\times\underline{B}) = \iint_S [(\underline{c}\cdot\nabla)\underline{B} - \underline{c}(\nabla\cdot\underline{B})]\cdot\underline{n}\,ds$$

$$\Rightarrow c\oint_c (d\underline{r}\times\underline{B}) = \iint_S [(\underline{c}\cdot\nabla)\underline{B}]\cdot\underline{n}\,ds - \iint_S [\underline{c}(\nabla\cdot\underline{B})]\cdot\underline{n}\,ds$$

$$= \iint_S c\,[\nabla(\underline{B}\cdot\underline{n})]\,ds - \iint_S c\cdot[\underline{n}(\nabla\cdot\underline{B})]\,ds$$

$$= c\iint_S [\nabla(\underline{B}\cdot\underline{n}) - \underline{n}(\nabla\cdot\underline{B})]\,ds$$

$$= c\iint_S (\underline{n}\times\nabla)\times\underline{B}\,ds$$

Since c is an arbitrary constant vector, then we can write

$$\oint_c d\underline{r}\times\underline{B} = \iint_S (\underline{n}\times\nabla)\times\underline{B}\,ds.$$

proved

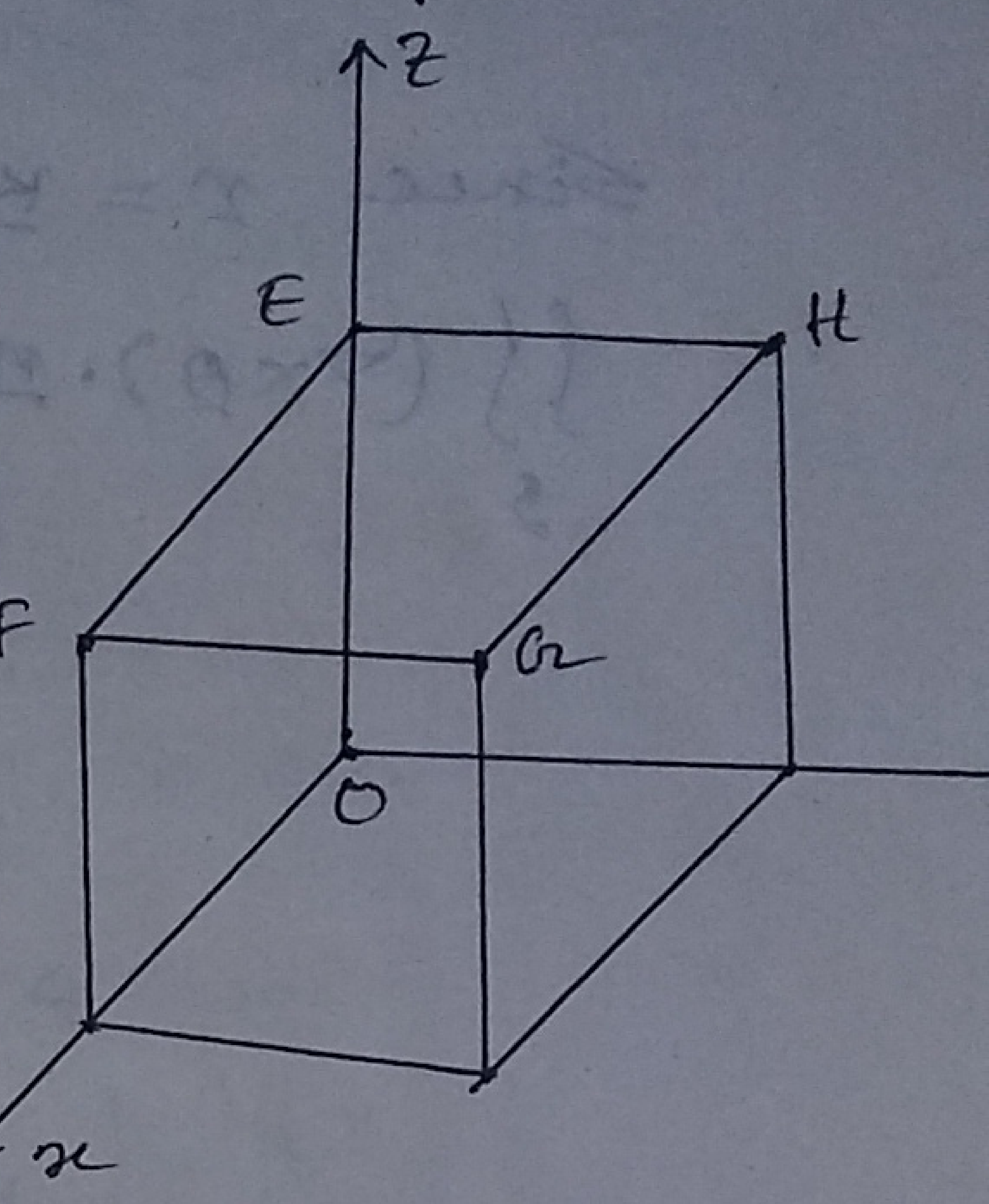
*SP-63: Verify Stokes's theorem for $\underline{A} = (y-z+2)\underline{i} + (yz+4)\underline{j} - xz\underline{k}$ where S is the surface of the cube $x=0,\ y=0,\ z=0$, $x=2,\ y=2,\ z=2$ above the xy-plane.

Soln: Here given that

$$\underline{A} = (y-z+2)\underline{i} + (yz+4)\underline{j} - xz\underline{k}$$

We are to show that

$$\oint_c \underline{A}\cdot d\underline{r} = \iint_S (\nabla\times\underline{A})\cdot\underline{n}\, ds.$$



Let EFGH be the required surface S above the xy plane.

Here, $E \equiv (0,0,2)$, $F(2,0,2)$

$G \equiv (2,2,2)$ and $H \equiv (0,2,2)$

$$\therefore \oint_c \underline{A}\cdot d\underline{r} = \int_{EF} \underline{A}\cdot d\underline{r} + \int_{FG} \underline{A}\cdot d\underline{r} + \int_{GH} \underline{A}\cdot d\underline{r} + \int_{HE} \underline{A}\cdot d\underline{r}$$

Here, $\underline{A}\cdot d\underline{r} = (y-z+2)\,dx + (yz+4)\,dy - xz\,dz$.

On, EF, x varies from 0 to 2 and $y=0,\ z=2$

" FG, y " " 0 to 2 " $x=2,\ z=2$

" GH, x " " 2 to 0 " $y=2,\ z=2$

" HE, y " " 2 to 0, $x=0,\ z=2$

$$\therefore \oint_E \underline{A}\cdot d\underline{r} = \int_E^F \underline{A}\cdot d\underline{r} = \int_0^2 (0-2+2)\,dx + 0 - 0 = 0$$

$$\int_{FG} \underline{A}\cdot d\underline{r} = \int_F^G \underline{A}\cdot d\underline{r} = \int_0^2 (2y+4)\,dy = [y^2+4y]_0^2 = 12$$

$$\int_{GH} \underline{A}\cdot d\underline{r} = \int_G^H \underline{A}\cdot d\underline{r} = \int_2^0 (2-2+2)\,dx = 2x|_2^0 = -4$$

$$\int_{HE} \underline{A}\cdot d\underline{r} = \int_H^E \underline{A}\cdot d\underline{r} = \int_2^0 (2y+4)\,dy = y^2+4y|_2^0 = -12$$

$$\therefore \oint_c A\cdot dr = 0+12-4-12 = -4.$$

Again $\nabla\times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$

$$= -y\,i + (-1+z)\,j - k$$

Since $n = k$, then

$$\iint_S (\nabla\times A)\cdot n\, ds = \iint_S \{-y\,i + (-1+z)\,j - k)\}\, k\, ds$$

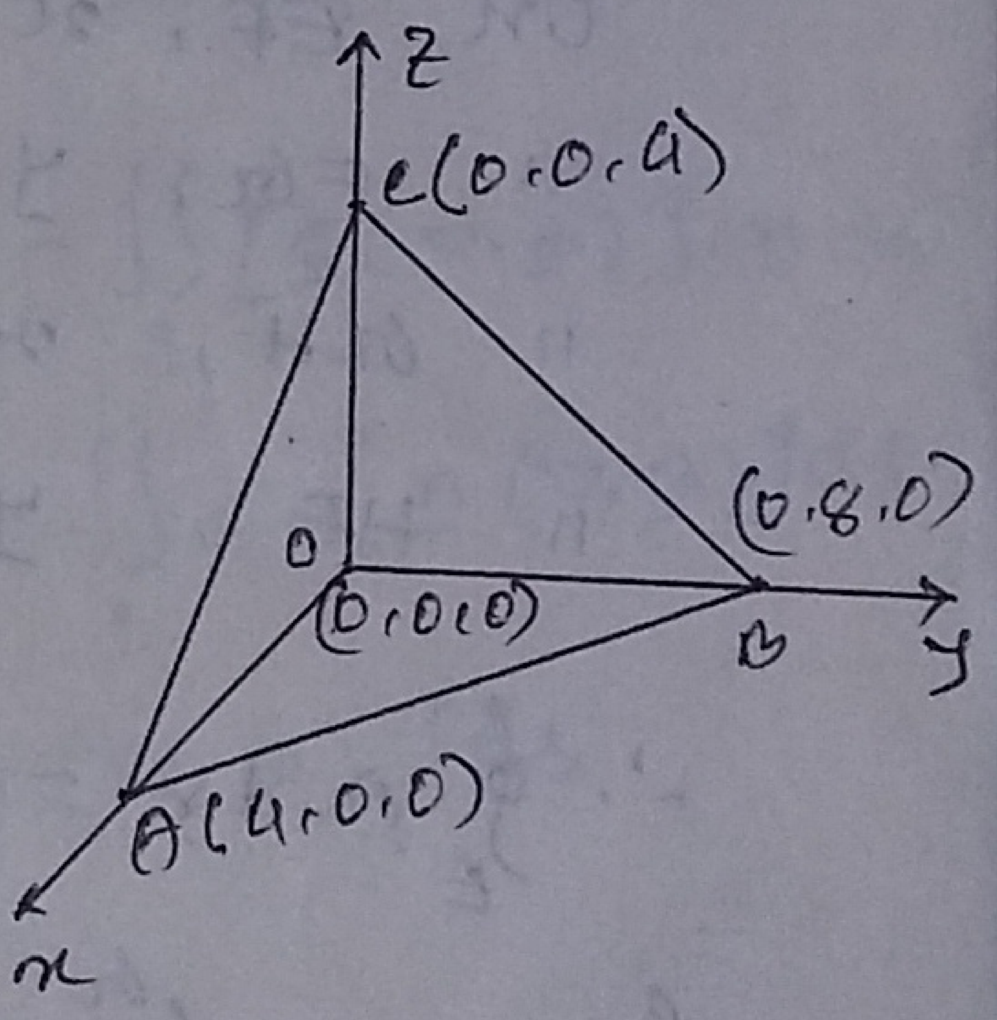
$$= \iint_S -1\, ds = -\iint_R dx\,dy$$

$$= -\int_{x=0}^{2}\int_{y=0}^{2} dy\,dx$$

$$= -4.$$

Hence, the stokes's theorem varified. (proved)

Ex-64 * Verify Stoke's theorem for $F = xz\,i - y\,j + x^2y\,k$, where S is the surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x+y+2z=8$, which is not included in the xz plane.



Sol^n: we have to prove that

$$\oint F\cdot dr = \iint_S (\nabla\times F)\cdot n\, ds.$$

The plane $2x+y+2z=8$ cuts the axes at $A(4,0,0)$, $B(0,8,0)$ and $C(0,0,4)$.

Hence,

$$\oint_{ABC} F\cdot dr = \int_{AB} F\cdot dr + \int_{BC} F\cdot dr + \int_{CA} F\cdot dr.$$

Here, $F\cdot dr = xz\,dx + (-y)\,dy + x^2y\,dz$

Now, the equation of AB is

$$\frac{x-4}{4} = \frac{y}{-8} = \frac{z}{0}$$

$$\Rightarrow \quad x-4 = \frac{y}{-2} = \frac{z}{0} = t \text{ (say)}$$

$\therefore \; x = t+4, \quad y = -2t, \quad z = 0$

where, t varies from 0 to -4.

$$\therefore \int_{AB} \underline{F}\cdot d\underline{r} = \int_0^{-4} -(-2t)(-2)\,dt = \int_0^{-4} -4t\,dt$$

$$= [-2t^2]_0^{-4} = -32.$$

Also, the equation of BC is

$$\frac{x}{0} = \frac{y-8}{8} = \frac{z-0}{-4}$$

$$\Rightarrow \frac{x}{0} = \frac{y-8}{2} = -z = t \text{ (say)}$$

$\therefore \; x = 0, \quad y = 2t+8, \quad z = -t$

where t varies from 0 to -4

$$\therefore \int_{BC} \underline{F}\cdot d\underline{r} = \int_0^{-4} -(2t+8)\,2dt = \int_0^{-4} (-4t-16)\,dt$$

$$= [-2t^2 - 16t]_0^{-4}$$

$$= -32 + 64 = 32$$

Again, the equation of CA is

$$\frac{x}{-4} = \frac{y}{0} = \frac{z-4}{4}$$

$$\Rightarrow -x = \frac{y}{0} = z-4 = t \text{ (say)}$$

$\therefore \; x = -t, \quad y = 0, \quad z = t+4$

where, t varies from 0 to -4

$$\therefore \int_{CA} \underline{F}\cdot d\underline{r} = \int_0^{-4} -t(t+4)(-dt) = \int_0^{-4} (t^2+4t)\,dt$$

$$= \left[\frac{t^3}{3} + 2t^2\right]_0^{-4} = \left[-\frac{64}{3} + 32\right]$$

$$= \frac{-64+96}{3} = \frac{32}{3}$$

Now, $\nabla\times\underline{F} = x^2\hat{i} + (x-2xy)\hat{j}$

$\nabla\times\underline{F} = \underline{A}$ say $\therefore \underline{A} = x^2\hat{i} + (x-2xy)\hat{j}$

Then,

$$\iint_S (\nabla\times \underline{A})\, \underline{n}.\, ds = \oint_C \underline{A}.\, d\underline{r}$$

$$= \int_0^{2\pi} (x^2+y-4)\, dx + 3xy\, dy + (2xz+z^2)\, dz$$

$$= \int_0^{2\pi} (4\cos^2\theta + 2\sin\theta - 4)(-2\sin\theta\, d\theta)$$
$$+ 3.2\cos\theta.2\sin\theta.2\cos\theta d\theta$$

$$= \int_0^{2\pi} -8\cos^2\theta.\sin\theta - 4\sin^2\theta + 8\sin\theta$$
$$+ 24\cos^2\theta.\sin\theta\, d\theta$$

$$= \int_0^{2\pi} (16\cos^2\theta\sin\theta - 4\sin^2\theta + 8\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} -16\cos^2\theta\{d(\cos\theta)\} - 2\int_0^{2\pi} (1-\cos 2\theta)\, d\theta$$
$$+ 8\int_0^{2\pi} \sin\theta\, d\theta$$

$$= \left[-\tfrac{16}{3}\cos^3\theta - 2\left(\theta - \frac{\sin 2\theta}{2}\right) - 8\cos\theta\right]_0^{2\pi}$$

$$= [(-\tfrac{16}{3} - 2.2\pi - 8) - (-\tfrac{16}{3} - 8)]$$

$$= -\tfrac{16}{3} - 4\pi - 8 + \tfrac{16}{3} + 8$$

$$= -4\pi. \quad \text{Ans.}$$

If $\oint_C \underline{E}.d\underline{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \underline{H}.d\underline{s}$, where S is any surface
bounded by the curve C, show that $\nabla\times\underline{E} = -\frac{1}{c}.\frac{\partial H}{\partial t}$.

A normal to $2x+y+2z=8$ is

$$\nabla(2x+y+2z) = 2\underline{i}+\underline{j}+2\underline{k}.$$

Thus, the unit normal $= \dfrac{2\underline{i}+\underline{j}+2\underline{k}}{\sqrt{4+1+4}} = \frac{2}{3}\underline{i}+\frac{1}{3}\underline{j}+\frac{2}{3}\underline{k}$

$$\therefore \underline{n}\cdot\underline{k} = \frac{2}{3}.$$

$\therefore \iint_S (\nabla\times\underline{F})\cdot\underline{n}\,ds$

$= \iint_S \underline{A}\cdot\underline{n}\,ds = \iint_R \underline{A}\cdot\underline{n}\,\dfrac{dx\,dy}{|\underline{n}\cdot\underline{k}|}$, R is the projection on the xy plane, $z=0$

$$= \int_{x=0}^{4}\int_{y=0}^{8-2x} \left\{\frac{2}{3}\cdot x^2 + (x-2xy)\cdot\frac{1}{3}\right\}\frac{dx\,dy}{2/3}$$

$$= \int_{x=0}^{4}\int_{y=0}^{8-2x} (2x^2+x-2xy)\cdot\frac{1}{3}\cdot\frac{dx\,dy}{2/3}$$

$$= \frac{1}{2}\int_{x=0}^{4} \left(2x^2y+xy-xy^2\right)\Big|_0^{8-2x} dx$$

$$= \frac{1}{2}\int_{x=0}^{4} \left\{2x^2(8-2x)+x(8-2x)-x(8-2x)^2\right\} dx$$

$$= \frac{1}{2}\int_{x=0}^{4} \left(16x^2-4x^3+8x-2x^2-64x+32x^2-4x^3\right) dx$$

$$= \frac{1}{2}\int_{x=0}^{4} \left(46x^2-8x^3-56x\right) dx$$

$$= \left[\frac{46}{3}x^3 - 8\frac{x^4}{4} - 28x^2\right]_0^4$$

$$= \frac{46}{3}\cdot 64 - 2\cdot 256 - 448$$

$$= \frac{2944}{3} - 512 - 448$$

$$= \frac{32}{3}.$$

Hence, the Stokes theorem is verified.

(proved)