

* Definition of the integral transformation- If there is a known function $K(s, t)$ of two variables s and t such that the integral

$$\int_0^{\infty} K(s, t) F(t) dt \quad \text{--- (1)}$$

is convergent, then the integral (1) is called the integral transformation of the function $F(t)$ and is denoted by $\dot{F}(s)$ or $T\{F(t)\}$ i.e

$$\dot{F}(s) = T\{F(t)\} = \int_0^{\infty} K(s, t) F(t) dt \quad \text{--- (2)}$$

the function $K(s, t)$ introduced here is sometimes known as the kernel of the transformation and s is a parameter (may be real or complex) independent of t .

* Deduction of the definition of the Laplace transformation from that of the integral transformation-

we have defined the Integral transform of $F(t)$ as

$$T\{F(t)\} = \int_0^{\infty} K(s, t) F(t) dt$$

where $K(s, t)$ is the kernel of the transformation.

If we take the kernel $K(s, t) = 0$ for $t < 0$
 $= e^{-st}$ for $t \geq 0$

then the transform

$$T\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt \quad \text{for } t \geq 0$$

is known as the Laplace transformation.

* Definition of the Laplace transformation-

If $F(t)$ be a function of t defined for all positive values of t i.e $t \geq 0$, then the Laplace transform of $F(t)$ denoted by $L\{F(t)\}$ or $\dot{F}(s)$ or $F(s)$ is defined by the expression,

$$L\{F(t)\} = F(s) = \int_0^{\infty} e^{-st} F(t) dt$$

where, s is a parameter real or complex.

If the integral $\int_0^{\infty} e^{st} F(t) dt$ converges for some value of s , then the Laplace transform of $F(t)$ is said to exist, otherwise it does not exist.

* Find the Laplace transformation of the function $F(t) = 1$.

Solⁿ: By the definition of Laplace transformation, we get

$$L\{F(t)\} = \int_0^{\infty} e^{st} F(t) dt \quad \dots \textcircled{1}$$

When $F(t) = 1$, then $\textcircled{1}$ becomes

$$\begin{aligned} L\{1\} &= \int_0^{\infty} e^{st} \cdot 1 \cdot dt \\ &= \left[\frac{e^{st}}{s} \right]_0^{\infty} = 0 + \frac{1}{s} = \frac{1}{s} \quad [\because e^{\infty} = 0] \end{aligned}$$

$$\therefore L\{1\} = \frac{1}{s} \text{ if } s > 0$$

* prove that, $L\{t\} = \frac{1}{s^2}, s > 0$

Solⁿ: By the definition of Laplace transform, of a function $F(t)$, we have,

$$L\{F(t)\} = \int_0^{\infty} e^{st} F(t) dt \quad \dots \textcircled{1}$$

When $F(t) = t$, then from above

$$\begin{aligned} L\{t\} &= \int_0^{\infty} e^{st} \cdot t \cdot dt \\ &= \left[t \cdot \frac{e^{st}}{s} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{st}}{s} dt \\ &= 0 + \frac{1}{s} \left[\frac{e^{st}}{s} \right]_0^{\infty} \\ &= \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}, \quad s > 0 \end{aligned}$$

$$\therefore L\{t\} = \frac{1}{s^2}, \quad s > 0.$$

* $F(t) = t^n$ or prove that $L\{t^n\} = \frac{n!}{s^{n+1}}, s > 0$.

Solⁿ: By the definition of Laplace transform, of a function $F(t)$, we get

$$L\{F(t)\} = \int_0^{\infty} e^{st} F(t) dt \quad \dots \textcircled{1}$$

When $F(t) = t^n$, then from $\textcircled{1}$,

$$L\{t^n\} = \int_0^{\infty} e^{st} \cdot t^n \cdot dt$$

$$= \left[t^2 \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 2t \cdot \frac{e^{-st}}{-s} dt$$

$$= 0 + \frac{2}{s} \int_0^\infty t \cdot e^{-st} dt \quad [\because e^{-\infty} = 0]$$

$$= \frac{2}{s} \cdot \frac{1}{s^2} \quad [\because \int_0^\infty t \cdot e^{-st} dt = \frac{1}{s^2}]$$

$$= \frac{2}{s^3}, \quad s > 0$$

$$\therefore L\{t^2\} = \frac{2}{s^3}, \quad s > 0$$

* Find the Laplace transform of the function $F(t) = t^n$, where, $n = 0, 1, 2, 3, \dots$

Solⁿ: By the definition of the Laplace transform of a function $F(t)$, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt \quad \text{--- (1)}$$

When $F(t) = t^n$, then from above

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n dt$$

$$= \left[\frac{e^{-st}}{-s} \cdot t^n \right]_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} \cdot t^{n-1} dt,$$

(Integrating by parts)

$$= 0 + \frac{n}{s} \left[\frac{e^{-st}}{-s} \cdot t^{n-1} \right]_0^\infty + \frac{n(n-1)}{s^2} \int_0^\infty e^{-st} \cdot t^{n-2} dt$$

(Integrating by parts)

$$= \frac{n(n-1)(n-2)}{s^3} \int_0^\infty e^{-st} \cdot t^{n-3} dt$$

Repeating the process of integration by parts, whence first integral vanishes for both the limits, we get

$$L\{t^n\} = \frac{n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1}{s^n} \int_0^\infty e^{-st} \cdot t^0 dt, \quad \text{where } t^0 = 1$$

$$= \frac{n!}{s^n} \int_0^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s}$$

$$= \frac{n!}{s^{n+1}}, \quad s > 0$$

$$= \frac{\overline{n+1}}{s^n}, \quad s > 0$$

Since $n! = \overline{n+1}$

* Find the Laplace transform of e^{at} , Here $F(t) = e^{at}$.

Solⁿ: By the definition of Laplace transform, we have

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} \cdot e^{at} dt$$

$$= \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$= \frac{1}{s-a}$, $s > a$ for which the integral converges otherwise it diverges for $s \leq a$.

* Find the Laplace transform of $\sin at$. Here $F(t) = \sin at$.

Solⁿ: By the definition of Laplace transformation

$$L\{F(t)\} = L\{\sin at\} = \int_0^{\infty} e^{-st} \cdot \sin at dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^{\infty}$$

$$\left[\because \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \right]$$

$$= \frac{a}{s^2 + a^2}, \quad s > 0$$

$$\therefore L\{\sin at\} = \frac{a}{s^2 + a^2}, \quad s > 0.$$

* Find the Laplace transform of $\cos at$. Here $F(t) = \cos at$.

Solⁿ: By the definition of Laplace transformation,

$$L\{F(t)\} = L\{\cos at\} = \int_0^{\infty} e^{-st} \cdot \cos at dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^{\infty}$$

$$\left[\because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right]$$

$$= \frac{s}{s^2 + a^2}, \quad s > 0.$$

$$\therefore L\{\cos at\} = \frac{s}{s^2 + a^2}, \quad s > 0.$$

* Find the Laplace transform of $\sinh at$. Here $F(t) = \sinh at$
 Solⁿ: By the definition of the Laplace transform

$$\begin{aligned}
 L\{\sinh at\} &= \int_0^{\infty} e^{-st} \cdot \sinh at \, dt \\
 &= \int_0^{\infty} e^{-st} \cdot \frac{1}{2} (e^{at} - e^{-at}) \, dt \\
 &= \frac{1}{2} \int_0^{\infty} e^{-st} \cdot e^{at} \, dt - \frac{1}{2} \int_0^{\infty} e^{-st} \cdot e^{-at} \, dt \\
 &= \frac{1}{2} \int_0^{\infty} e^{-(s-a)t} \, dt - \frac{1}{2} \int_0^{\infty} e^{-(s+a)t} \, dt \\
 &= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} - \frac{1}{2} \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \\
 &= \frac{1}{2} \cdot \frac{1}{s-a} - \frac{1}{2} \cdot \frac{1}{s+a} \\
 &= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] \\
 &= \frac{1}{2} \left[\frac{s+a-s+a}{s^2-a^2} \right] = \frac{1}{2} \cdot \frac{2a}{s^2-a^2} \\
 &= \frac{a}{s^2-a^2}, \text{ if } s > |a|
 \end{aligned}$$

* Find the Laplace transform of $\cosh at$. Here $F(t) = \cosh at$.
 Solⁿ: By the definition of the Laplace transform

$$\begin{aligned}
 L\{\cosh at\} &= \int_0^{\infty} e^{-st} \cdot \cosh at \, dt \\
 &= \int_0^{\infty} e^{-st} \cdot \frac{1}{2} (e^{at} + e^{-at}) \, dt \\
 &= \frac{1}{2} \int_0^{\infty} e^{-st} \cdot e^{at} \, dt + \frac{1}{2} \int_0^{\infty} e^{-st} \cdot e^{-at} \, dt \\
 &= \frac{1}{2} \int_0^{\infty} e^{-(s-a)t} \, dt + \frac{1}{2} \int_0^{\infty} e^{-(s+a)t} \, dt \\
 &= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} + \frac{1}{2} \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \\
 &= \frac{1}{2} \cdot \frac{1}{s-a} + \frac{1}{2} \cdot \frac{1}{s+a} \\
 &= \frac{1}{2} \left[\frac{s+a+s-a}{s^2-a^2} \right] \\
 &= \frac{s}{s^2-a^2}, \text{ if } s > |a|
 \end{aligned}$$

* Find the Laplace transformation of the function $F(t) = t \sin at$.

Soln: By the definition of the Laplace transform of a function $F(t) = t \sin at$, we get

$$L\{F(t)\} = \int_0^{\infty} e^{-st} \cdot F(t) dt \quad \text{--- (1)}$$

When $F(t) = t \sin at$, equation (1) becomes

$$\begin{aligned} L\{t \sin at\} &= \int_0^{\infty} e^{-st} \cdot t \sin at dt \\ &= \int_0^{\infty} t \cdot e^{-st} \sin at dt \end{aligned}$$

Integrating by parts, taking t as the first function and $e^{-st} \sin at$ as the second function and apply the result,

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

we get

$$\begin{aligned} L\{t \sin at\} &= \left[t \cdot \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^{\infty} \\ &\quad - \int_0^{\infty} \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) dt \end{aligned}$$

while the first integral vanishes for both the limits, then

$$\begin{aligned} L\{t \sin at\} &= 0 + \int_0^{\infty} \frac{e^{-st}}{s^2 + a^2} (s \sin at + a \cos at) dt \\ &= \frac{s}{s^2 + a^2} \int_0^{\infty} e^{-st} \sin at dt + \frac{a}{s^2 + a^2} \int_0^{\infty} e^{-st} \cos at dt \\ &= \frac{s}{s^2 + a^2} \cdot \frac{a}{s^2 + a^2} + \frac{a}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2}, \text{ if } s > 0. \end{aligned}$$

[since, $L\{\sin at\} = \frac{a}{s^2 + a^2}$ and $L\{\cos at\} = \frac{s}{s^2 + a^2}$, $s > 0$]

$$\therefore L\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2}, \quad s > 0$$

* Find the Laplace transform of $t \cos at$. Here $F(t) = t \cos at$.
 Soln: By the definition of Laplace transform,

$$L\{t \cos at\} = \int_0^{\infty} e^{-st} \cdot t \cos at \, dt$$

On the right hand side, integrating by parts treating t as first function and $e^{-st} \cos at$ as second function and using the result

$$\int e^{ax} \cdot \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

while the first integral vanishes for both the limits, we are left with

$$\begin{aligned} L\{t \cos at\} &= \left[t \cdot \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^{\infty} \\ &\quad - \int_0^{\infty} \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \, dt \\ &= 0 + \frac{s}{s^2 + a^2} \int_0^{\infty} e^{-st} \cos at \, dt - \frac{a}{s^2 + a^2} \int_0^{\infty} e^{-st} \sin at \, dt \end{aligned}$$

$$= \frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2} - \frac{a}{s^2 + a^2} \cdot \frac{a}{s^2 + a^2}, \text{ for } s > a$$

$$\left[\text{Since } L\{\cos at\} = \frac{s}{s^2 + a^2}, L\{\sin at\} = \frac{a}{s^2 + a^2}, s > 0 \right]$$

$$\therefore L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2} \text{ for } s > 0.$$

* Find the Laplace transform of $e^{at} \sin bt$. Here $F(t) = e^{at} \sin bt$.

Soln: By the definition of Laplace transform

$$L\{e^{at} \sin bt\} = \int_0^{\infty} e^{-st} \cdot e^{at} \sin bt \, dt$$

$$\text{Let } L\{e^{at} \sin bt\} = \int_0^{\infty} e^{-st} \cdot e^{at} \sin bt \, dt = I \quad \dots (1)$$

$$\therefore L\{e^{at} \sin bt\} = \int_0^{\infty} e^{-(s-a)t} \sin bt \, dt$$

$$= \left[-e^{-(s-a)t} \cdot \frac{\cos bt}{b} \right]_0^{\infty} - \int_0^{\infty} -(s-a) e^{-(s-a)t} \cdot \left(-\frac{\cos bt}{b} \right) dt$$

$$= \frac{1}{b} - \frac{s-a}{b} \int_0^{\infty} e^{-(s-a)t} \cos bt \, dt$$

$$= \frac{1}{b} - \frac{s-a}{b} \left[\frac{e^{-(s-a)t}}{b} \cdot \frac{\sin bt}{b} \right]_0^\infty + \frac{s-a}{b} \int_0^\infty - (s-a) e^{-(s-a)t} \cdot \frac{\sin bt}{b} dt$$

$$= \frac{1}{b} - \frac{(s-a)^2}{b^2} \int_0^\infty e^{-(s-a)t} \cdot \sin bt dt$$

$$= \frac{1}{b} - \frac{(s-a)^2}{b^2} \cdot I \quad [\text{by } \textcircled{I}]$$

$$\therefore I = \frac{1}{b} - \frac{(s-a)^2}{b^2} \cdot I$$

$$\Rightarrow I \left\{ 1 + \frac{(s-a)^2}{b^2} \right\} = \frac{1}{b}$$

$$\Rightarrow I \left\{ \frac{b^2 + (s-a)^2}{b^2} \right\} = \frac{1}{b}$$

$$\therefore L\{e^{at} \cdot \sin bt\} = \frac{1}{b} \cdot \frac{b^2}{(s-a)^2 + b^2}$$

$$\therefore L\{e^{at} \cdot \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

Another method:

$$L\{e^{at} \cdot \sin bt\} = \int_0^\infty e^{-st} \cdot e^{at} \sin bt dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot \sin bt dt$$

$$= \left[\frac{e^{-(s-a)t}}{(s-a)^2 + b^2} \left\{ -(s-a) \sin bt - b \cos bt \right\} \right]_0^\infty$$

$$= \frac{b}{(s-a)^2 + b^2}$$

* Find the Laplace transform of $e^{at} \cdot \cos bt$. Here $F(t) = e^{at} \cdot \cos bt$.

Soln: By the definition of Laplace transform

$$L\{e^{at} \cdot \cos bt\} = \int_0^\infty e^{-st} \cdot e^{at} \cos bt dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot \cos bt dt$$

$$= \left[e^{-(s-a)t} \cdot \frac{\sin bt}{b} \right]_0^\infty - \int_0^\infty - (s-a) e^{-(s-a)t} \cdot \frac{\sin bt}{b} dt$$

$$= 0 + \frac{s-a}{b} \int_0^\infty e^{-(s-a)t} \cdot \sin bt dt$$

$$= \frac{s-a}{b} \cdot \frac{b}{(s-a)^2 + b^2} \quad \left[\because L\{e^{at} \cdot \sin bt\} = \frac{b}{(s-a)^2 + b^2} \right]$$

$$\therefore L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \quad (\text{proved})$$

* Some properties of Laplace transformation.

* Linearity Property: A Laplace transform $L\{F(t)\}$ is said to be linear if for every pair of function $F_1(t)$ and $F_2(t)$ and for every pair of constant c_1 and c_2 , we have

$$\begin{aligned} L\{c_1 F_1(t) + c_2 F_2(t)\} &= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\} \\ &= c_1 F_1(s) + c_2 F_2(s) \end{aligned}$$

where $F_1(s)$ and $F_2(s)$ are linear transforms of $F_1(t)$ and $F_2(t)$ respectively.

proof: By the definition of Laplace transform, we have

$$L\{F_1(t)\} = F_1(s) = \int_0^{\infty} e^{-st} F_1(t) dt$$

$$\text{and } L\{F_2(t)\} = F_2(s) = \int_0^{\infty} e^{-st} F_2(t) dt$$

Now if c_1 and c_2 are any two constants then

$$L\{c_1 F_1(t)\} = c_1 F_1(s) = \int_0^{\infty} e^{-st} \cdot c_1 F_1(t) dt = c_1 L\{F_1(t)\}$$

$$\text{and } L\{c_2 F_2(t)\} = c_2 F_2(s) = \int_0^{\infty} e^{-st} \cdot c_2 F_2(t) dt = c_2 L\{F_2(t)\}$$

$$\therefore L\{c_1 F_1(t) + c_2 F_2(t)\} = \int_0^{\infty} e^{-st} \{c_1 F_1(t) + c_2 F_2(t)\} dt$$

by definition

$$= \int_0^{\infty} e^{-st} \cdot c_1 F_1(t) dt + \int_0^{\infty} e^{-st} \cdot c_2 F_2(t) dt$$

$$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$

$$= c_1 F_1(s) + c_2 F_2(s)$$

The result may be generalized for any number of functions and for the same number of arbitrary constants i.e

$$L\left\{\sum_{r=1}^n c_r F_r(t)\right\} = \sum_{r=1}^n c_r L\{F_r(t)\}$$

* Find the Laplace transform of

$$4e^{5t} + 6t^3 - 4 \cos 3t + 3 \sin 4t$$

Solⁿ: Applying linearity property, we have

$$\begin{aligned} & L\{4e^{5t} + 6t^3 - 4 \cos 3t + 3 \sin 4t\} \\ &= 4L\{e^{5t}\} + 6L\{t^3\} - 4L\{\cos 3t\} + 3L\{\sin 4t\} \\ &= 4 \cdot \frac{1}{s-5} + 6 \cdot \frac{3!}{s^4} - 4 \cdot \frac{s}{s^2+9} + 3 \cdot \frac{4}{s^2+16} \\ &= \frac{4}{s-5} + \frac{36}{s^4} - \frac{4s}{s^2+9} + \frac{12}{s^2+16} \quad \text{Ans.} \end{aligned}$$

* Find the Laplace transform of

$$e^{4t} + 4t^3 - 2 \sin 3t + 3 \cos 5t$$

Solⁿ: By applying linearity property, we have

$$\begin{aligned} & L\{e^{4t} + 4t^3 - 2 \sin 3t + 3 \cos 5t\} \\ &= L\{e^{4t}\} + 4L\{t^3\} - 2L\{\sin 3t\} + 3L\{\cos 5t\} \\ &= \frac{1}{s-4} + 4 \cdot \frac{3!}{s^4} - 2 \cdot \frac{3}{s^2+3^2} + 3 \cdot \frac{s}{s^2+5^2} \\ &= \frac{1}{s-4} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+25} \end{aligned}$$

* Find the Laplace transform of

$$3t^4 - 2t^3 + 4e^{-3t} - 2 \sin 5t + 3 \cos 2t$$

Solⁿ: By applying linearity property, we have

$$\begin{aligned} & L\{3t^4 - 2t^3 + 4e^{-3t} - 2 \sin 5t + 3 \cos 2t\} \\ &= 3L\{t^4\} - 2L\{t^3\} + 4L\{e^{-3t}\} - 2L\{\sin 5t\} \\ &\quad + 3L\{\cos 2t\} \\ &= 3 \cdot \frac{4!}{s^5} - 2 \cdot \frac{3!}{s^4} + 4 \cdot \frac{1}{s+3} - 2 \cdot \frac{5}{s^2+5^2} + 3 \cdot \frac{s}{s^2+2^2} \\ &= \frac{24}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4} \quad \text{Ans.} \end{aligned}$$

* Find the Laplace transform of $4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t$

Solⁿ: By applying linearity property, we have

$$\begin{aligned} L\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\} \\ &= 4L\{e^{5t}\} + 6L\{t^3\} - 3L\{\sin 4t\} + 2L\{\cos 2t\} \\ &= 4 \cdot \frac{1}{s-5} + 6 \cdot \frac{3!}{s^4} - 3 \cdot \frac{4}{s^2+4^2} + 2 \cdot \frac{s}{s^2+2^2} \\ &= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}, \text{ where } s > 5. \end{aligned}$$

Ans.

* First translation (or shifting) property:

If $F(s)$ be the Laplace transform of $F(t)$, then the Laplace transform of $e^{at} F(t)$ is $F(s-a)$, where a is any real or complex number i.e. if

$$L\{F(t)\} = F(s), \text{ then } L\{e^{at} F(t)\} = F(s-a).$$

Solⁿ: Given that,

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} F(t) dt = F(s) \\ \therefore L\{e^{at} F(t)\} &= \int_0^{\infty} e^{-st} \cdot e^{at} F(t) dt \\ &= \int_0^{\infty} e^{-(s-a)t} F(t) dt \\ &= \int_0^{\infty} e^{-ut} F(t) dt, \text{ by putting } u = s-a \\ &= F(u) = F(s-a). \end{aligned}$$

* Find the Laplace transform of $e^{-2t} \sin 3t$.

Solⁿ: we have

$$\begin{aligned} L\{\sin 3t\} &= \frac{3}{s^2+9} = F(s) \\ \therefore L\{e^{-2t} \sin 3t\} &= F(s+2) = \frac{3}{(s+2)^2+9} \quad [\text{Using first shifting theorem}] \\ &= \frac{3}{s^2+4s+4+9} \\ &= \frac{3}{s^2+4s+13}. \quad \text{Ans.} \end{aligned}$$

* Find the Laplace transform of $t^3 e^{5t}$.

Soln: we have,

$$L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} = F(s) \text{ (say)}$$

then by using first shifting theorem, we get

$$L\{e^{5t} t^3\} = F(s-5) = \frac{6}{(s-5)^4} \text{ where } s > 5.$$

* Find the Laplace transformation of

$$e^{3t} (2 \cos 5t - 3 \sin 5t)$$

Soln: we have,

$$\begin{aligned} L\{2 \cos 5t - 3 \sin 5t\} &= 2 \cdot \frac{s}{s^2 + 5^2} - 3 \cdot \frac{5}{s^2 + 5^2} \\ &= \frac{2s}{s^2 + 5^2} - \frac{15}{s^2 + 5^2} \\ &= \frac{2s - 15}{s^2 + 25} = F(s) \text{ say:} \end{aligned}$$

then by applying the first shifting theorem, we get

$$\begin{aligned} L\{e^{3t} (2 \cos 5t - 3 \sin 5t)\} &= F(s-3) \\ &= \frac{2(s-3) - 15}{(s-3)^2 + 25} \\ &= \frac{2s - 6 - 15}{s^2 - 6s + 9 + 25} \\ &= \frac{2s - 21}{s^2 - 6s + 34}. \text{ Ans.} \end{aligned}$$

* Find the Laplace transform, of $e^{4t} \cosh 5t$.

Soln: we have, $L\{\cosh 5t\} = \frac{s}{s^2 - 25} = F(s) \text{ (say)}$

now using the first shifting theorem, we get

$$\begin{aligned} L\{e^{4t} \cosh 5t\} &= F(s-4) = \frac{s-4}{(s-4)^2 - 25} \\ &= \frac{s-4}{s^2 - 8s + 16 - 25} \\ &= \frac{s-4}{s^2 - 8s - 9}. \text{ Ans.} \end{aligned}$$

* prove that, i) $L\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2}$

ii) $L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$

by using first shifting theorem.

proof: we have,

$$L\{t\} = 1/s^2 = F(s) \text{ (say)}$$

then by applying the first shifting theorem, we get

$$L\{t e^{iat}\} = F(s - ai) = \frac{i}{(s - ia)^2}$$

or, $L\{t \cos at + it \sin at\} = \frac{(s + ia)^2}{(s - ia)^2 (s + ia)^2}$

$$= \frac{(s + ia)^2}{\{(s - ia)(s + ia)\}^2}$$

$$= \frac{(s + ia)^2}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2 + i2as}{(s^2 + a^2)^2}$$

or, $L\{t \cos at\} + i L\{t \sin at\} = \frac{(s^2 - a^2) + i(2as)}{(s^2 + a^2)^2}$

Equating real and imaginary parts from both sides, we get,

$$L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2} \quad \text{and}$$

$$L\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2} \quad (\text{proved})$$

similar applications of the first shifting property leads us to the following results:

1. $L\{e^{at} t^n\} = \frac{n!}{(s - a)^{n+1}}$

2. $L\{e^{at} \sin bt\} = \frac{b}{(s - a)^2 + b^2}$

3. $L\{e^{at} \cos bt\} = \frac{s - a}{(s - a)^2 + b^2}$

4. $L\{e^{at} \sinh bt\} = \frac{b}{(s - a)^2 - b^2}$

5. $L\{e^{at} \cosh bt\} = \frac{s - a}{(s - a)^2 - b^2}$

* Second Translation (or shifting) property:

$$\text{If } L\{F(t)\} = F(s) \text{ and } g(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\text{then } L\{g(t)\} = e^{-as} F(s)$$

proof: we have,

$$\begin{aligned} L\{g(t)\} &= \int_0^{\infty} e^{-st} g(t) dt \\ &= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt \\ &= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} F(t-a) dt \\ &= \int_a^{\infty} e^{-st} F(t-a) dt \\ &= \int_0^{\infty} e^{-s(u+a)} F(u) du, \text{ by taking } u = t-a \text{ i.e.} \\ &= e^{-as} \int_0^{\infty} e^{-su} F(u) du \quad du = dt \\ &= e^{-as} F(s) \end{aligned}$$

* Find the Laplace transform of $F(t)$, where

$$F(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$$

soln: we have,

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} F(t) dt \\ &= \int_0^{2\pi/3} e^{-st} F(t) dt + \int_{2\pi/3}^{\infty} e^{-st} F(t) dt \\ &= \int_0^{2\pi/3} e^{-st} \cdot 0 dt + \int_{2\pi/3}^{\infty} e^{-st} F(t) dt \\ &= \int_{2\pi/3}^{\infty} e^{-st} \cos(t - 2\pi/3) dt \end{aligned}$$

putting, $t - 2\pi/3 = u \therefore t = u + 2\pi/3$
and $dt = du$

$$\begin{aligned}
 \therefore L\{F(t)\} &= \int_0^{\infty} e^{-s(u+\frac{2\pi}{3})} \cdot \cos u \, du \\
 &= e^{-\frac{2\pi s}{3}} \int_0^{\infty} e^{-su} \cdot \cos u \, du \\
 &= e^{-\frac{2\pi s}{3}} \cdot L\{\cos u\} \\
 &= e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}, \quad \text{Ans.}
 \end{aligned}$$

* Find the Laplace transform of $F(t)$, where

$$F(t) = \begin{cases} \sin(t-\frac{\pi}{3}), & t > \frac{\pi}{3} \\ 0 & , t < \frac{\pi}{3} \end{cases}$$

Soln: we have, $L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) \, dt$

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{3}} e^{-st} F(t) \, dt + \int_{\frac{\pi}{3}}^{\infty} e^{-st} F(t) \, dt \\
 &= \int_{\frac{\pi}{3}}^{\infty} e^{-st} \cdot \sin(t-\frac{\pi}{3}) \, dt
 \end{aligned}$$

putting, $t-\frac{\pi}{3} = u \quad \therefore t = u + \frac{\pi}{3}$

$$\therefore dt = du$$

$$\begin{aligned}
 \therefore L\{F(t)\} &= \int_0^{\infty} e^{-s(u+\frac{\pi}{3})} \cdot \sin u \, du \\
 &= e^{-\frac{\pi s}{3}} \int_0^{\infty} e^{-su} \cdot \sin u \, du \\
 &= e^{-\frac{\pi s}{3}} \cdot L\{\sin u\} = e^{-\frac{\pi s}{3}} \cdot \frac{1}{s^2+1}
 \end{aligned}$$

* The change of scalar property:

If $L\{F(t)\} = F(s)$, then $L\{F(at)\} = \frac{1}{a} F(s/a)$.

proof: we have, $L\{F(t)\} = \int_0^{\infty} e^{-st} \cdot F(t) \, dt$

on replacing t by at , we get

$$\begin{aligned}
 L\{F(at)\} &= \int_0^{\infty} e^{-st} \cdot F(at) \, dt \\
 &= \int_0^{\infty} e^{-su/a} \cdot F(u) \, du/a \quad \text{by taking } at = u.
 \end{aligned}$$

$$\begin{aligned}
 \therefore L\{F(at)\} &= \frac{1}{a} \int_0^{\infty} e^{-pu} \cdot F(u) du, \text{ where } p = s/a \\
 &= \frac{1}{a} \int_0^{\infty} e^{-pt} F(t) dt, \text{ replacing } u \text{ by } t \\
 &= \frac{1}{a} \cdot F(p) \\
 &= \frac{1}{a} \cdot F(s/a) \quad \because p = s/a.
 \end{aligned}$$

* Find the Laplace transform of $\cos 5t$.

Solⁿ: By the definition Laplace transform, we get

$$L\{\cos t\} = \frac{s}{s^2 + 1}, \quad s > 0$$

$$\therefore L\{\cos 5t\} = \frac{1}{5} \cdot \frac{s/5}{(s/5)^2 + 1} = \frac{s}{s^2 + 25}$$

* Given that, $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/s)$, find $L\left\{\frac{\sin at}{at}\right\}$.

Solⁿ: we have,

$$\begin{aligned}
 L\left\{\frac{\sin at}{at}\right\} &= \frac{1}{a} L\left\{\frac{\sin at}{t}\right\} \\
 &= \frac{1}{a} \tan^{-1}\left\{\frac{1}{(s/a)}\right\} \\
 &= \frac{1}{a} \tan^{-1}(a/s)
 \end{aligned}$$

then, $L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}(a/s).$

* Laplace transform of derivatives:

If $F(t)$ is continuous for $t \geq 0$ and of exponential order as $t \rightarrow \infty$ while $F'(t)$ is sectionally continuous i.e. $F'(t)$ is of class A for $t \geq 0$, and if $L\{F(t)\} = F(s)$ then $L\{F'(t)\} = sF(s) - F(0)$.

In general if $L\{F(t)\} = F(s)$ and $F(t), F'(t), F''(t), \dots, F^{(n-1)}(t)$ are continuous for $t \geq 0$ and of exponential order as $t \rightarrow \infty$ while $F^n(t)$ is sectionally continuous for $t \geq 0$, then

$$\begin{aligned}
 L\{F^n(t)\} &= s^n F(s) - \sum_{r=0}^{n-1} s^{n-r-1} F^{(r)}(0) \\
 &= s^n F(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \dots - F^{(n-1)}(0).
 \end{aligned}$$

Solⁿ: By the definition of Laplace transformation, we have, $L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt = F(s)$

$$\begin{aligned}
 \therefore L\{f'(t)\} &= \int_0^{\infty} e^{-st} f'(t) dt \\
 &= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} -s e^{-st} f(t) dt \\
 &\quad \text{(integrating by parts)} \\
 &= 0 - f(0) + s \int_0^{\infty} e^{-st} f(t) dt \\
 &= 0 - f(0) + sF(s) \\
 &= sF(s) - f(0) \quad \text{--- ①}
 \end{aligned}$$

$$\begin{aligned}
 \text{Again, } L\{f''(t)\} &= \int_0^{\infty} e^{-st} f''(t) dt \\
 &= \left[e^{-st} f'(t) \right]_0^{\infty} - \int_0^{\infty} -s e^{-st} f'(t) dt \\
 &= 0 - f'(0) + s \int_0^{\infty} e^{-st} f'(t) dt \\
 &= -f'(0) + s \{ sF(s) - f(0) \} \quad [\text{by ①}] \\
 &= s^2 F(s) - s f(0) - f'(0)
 \end{aligned}$$

$$\text{Similarly, } L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

Generalizing it, we find

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

* Find the Laplace transform of $f'(t)$ when $f(t) = e^{3t}$,

Soln: Given that, $f(t) = e^{3t} \quad \therefore f(0) = 1$

$$\text{and } f'(t) = 3e^{3t}.$$

We know that,

$$\begin{aligned}
 L\{f'(t)\} &= sF(s) - f(0) \\
 &= s \cdot \frac{1}{s-3} - 1 = \frac{s}{s-3} - 1 \\
 &= \frac{s-s+3}{s-3} = \frac{3}{s-3} \quad \text{--- Ans.}
 \end{aligned}$$

* Find the Laplace transform of $f''(t)$ when $f(t) = \cos at$.

Soln: Given that, $f(t) = \cos at \quad \therefore f(0) = 1$

$$\text{then, } f'(t) = -a \sin at \quad \therefore f'(0) = 0$$

$$\int_0^{\infty} e^{-st} \{t^K F(t)\} dt = (-1)^K F^K(s) \quad \text{--- (2)}$$

Differentiating both sides w.r. to s , we get

$$\frac{d}{ds} \int_0^{\infty} e^{-st} \{t^K F(t)\} dt = (-1)^K F^{K+1}(s)$$

Now applying Leibnitz's rule we get

$$- \int_0^{\infty} e^{-st} \{t^{K+1} F(t)\} dt = (-1)^K F^{K+1}(s)$$

$$\therefore L\{t^{K+1} F(t)\} = (-1)^{K+1} F^{K+1}(s)$$

Thus it follows that if the theorem holds for $n=K$ then it also holds for $n=K+1$. Therefore the theorem holds for all positive integral values of n .

$$\therefore L\{t^n F(t)\} = (-1)^n F^n(s).$$

* Find the Laplace transform of $t^3 e^t$.

Soln Since, $L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$

$$\therefore L\{e^t\} = \frac{1}{s-1}$$

$$\begin{aligned} \therefore L\{t^3 e^t\} &= (-1)^3 \frac{d^3}{ds^3} \left(\frac{1}{s-1} \right) \\ &= (-1)^3 \cdot \frac{(-1)(-2)(-3)}{(s-1)^4} = \frac{6}{(s-1)^4} \end{aligned}$$

* Find the Laplace transform of $t^3 \cos at$.

Soln Since, $L\{\cos at\} = \frac{s}{s^2 + a^2}$ 370

$$\begin{aligned} \therefore L\{t^3 \cos at\} &= (-1)^3 \frac{d^3}{ds^3} \left(\frac{s}{s^2 + a^2} \right) \\ &= \frac{d}{ds} \left\{ \frac{-2s^3}{(s^2 + a^2)^2} + \frac{1}{s^2 + a^2} \right\} \\ &= \frac{6s^3}{(s^2 + a^2)^3} - \frac{6s}{(s^2 + a^2)^2} \\ &= \frac{6s^3 - 6s^3 - 6a^2 s}{(s^2 + a^2)^3} \\ &= \frac{-6a^2 s}{(s^2 + a^2)^3} \end{aligned}$$

* prove that, $\int_0^{\infty} t e^{-st} \cos at \, dt = \frac{s^2 - a^2}{(s^2 + a^2)^2}$

proof: By the definition of Laplace transform, we get

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} F(t) \, dt \\ \text{L.H.S} &= \int_0^{\infty} t e^{-st} \cos at \, dt = \int_0^{\infty} e^{-st} t \cos at \, dt \\ &= L\{t \cos at\} = -\frac{d}{ds} L\{\cos at\} \\ &= -\frac{d}{ds} \left\{ \frac{s}{s^2 + a^2} \right\} \\ &= - \left[\frac{(s^2 + a^2) \cdot 1 - s \cdot 2s}{(s^2 + a^2)^2} \right] \\ &= - \left\{ \frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right\} \\ &= \frac{s^2 - a^2}{(s^2 + a^2)^2} = \text{R.H.S} \end{aligned}$$

* prove that, $\int_0^{\infty} t e^{-st} \sin at \, dt = \frac{2as}{(s^2 + a^2)^2}$

soln: By the definition of Laplace transform, we get

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} F(t) \, dt \\ \therefore \text{L.H.S} &= \int_0^{\infty} e^{-st} t \sin at \, dt \\ &= L\{t \sin at\} \\ &= (-1) \frac{d}{ds} \left\{ \frac{a}{s^2 + a^2} \right\} \\ &= - \left\{ \frac{-2as}{(s^2 + a^2)^2} \right\} \\ &= \frac{2as}{(s^2 + a^2)^2} \quad (\text{proved}) \end{aligned}$$

* Find the Laplace transform of $F(t) = t^n (\sin at + e^{at})$

soln: Since $L\{\sin at + e^{at}\} = \frac{a}{s^2 + a^2} + \frac{1}{s-a}$

$$\begin{aligned} \therefore L\{t^n (\sin at + e^{at})\} &= (-1)^n \frac{d^n}{ds^n} \left\{ \frac{a}{s^2 + a^2} + \frac{1}{s-a} \right\} \\ &= \frac{d}{ds} \left\{ \frac{-2as}{(s^2 + a^2)^2} - \frac{1}{(s-a)^2} \right\} \\ &= \frac{(s^2 + a^2)^2 \cdot (-2a) + 2as \cdot 2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4} + \frac{2}{(s-a)^3} \end{aligned}$$

$$\begin{aligned}
&= \frac{-2a(s^2+a^2) + 8as^2}{(s^2+a^2)^3} + \frac{2}{(s-a)^3} \\
&= \frac{-2as^2 - 2a^3 + 8as^2}{(s^2+a^2)^3} + \frac{2}{(s-a)^3} \\
&= \frac{6as^2 - 2a^3}{(s^2+a^2)^3} + \frac{2}{(s-a)^3} \\
&= \frac{2a(3s^2 - a^2)}{(s^2+a^2)^3} + \frac{2}{(s-a)^3}
\end{aligned}$$

Hence, $L\{t^2(\sin at + e^{at})\} = \frac{2a(3s^2 - a^2)}{(s^2+a^2)^3} + \frac{2}{(s-a)^3}$

* Laplace transform of integrals:

i) If $L\{F(t)\} = F(s)$, then $L\left\{\int_0^t F(u) du\right\} = \frac{F(s)}{s}$

proof: Let $G(t) = \int_0^t F(u) du$

then, $G'(t) = \frac{d}{dt} \left\{ \int_0^t F(u) du \right\} = F(t)$

and $G(0) = \int_0^0 F(u) du = 0$

Now, applying Laplace transform of derivatives

$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$

or, $L\{F(t)\} = sL\left\{\int_0^t F(u) du\right\}$

or, $F(s) = sL\left\{\int_0^t F(u) du\right\}$

$$\therefore L\left\{\int_0^t F(u) du\right\} = \frac{F(s)}{s}$$

ii) If $L\{F(t)\} = F(s)$, then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(u) du$

proof: Let $G(t) = \frac{F(t)}{t}$, so that $F(t) = tG(t)$

Taking Laplace transform,

$$\begin{aligned}
L\{F(t)\} &= L\{tG(t)\} \\
&= (-1) \frac{d}{ds} L\{G(t)\}
\end{aligned}$$

or, $-F(s) = \frac{d}{ds} L\{G(t)\}$

Integrating both sides with respect to s to ∞ , we get

$$[L\{G(t)\}]_s^\infty = - \int_s^\infty F(u) du$$

$$\text{or, } -\lim_{s \rightarrow \infty} L\{g(t)\} + L\{g(t)\} = \int_s^{\infty} F(s) ds$$

$$\text{since, } \lim_{s \rightarrow s} L\{g(t)\} = L\{g(t)\}$$

$$\text{and } \lim_{s \rightarrow \infty} L\{g(t)\} = \lim_{s \rightarrow \infty} \int_0^{\infty} e^{-st} g(t) dt = 0$$

$$\therefore L\{g(t)\} = \int_s^{\infty} F(s) ds$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} F(s) ds = \int_s^{\infty} F(u) du$$

* Find the Laplace transform of a) $\int_0^{\infty} \frac{\sin t}{t} dt$
b) $\frac{\sin at}{t}$.

$$\text{Solving we have, } L\{\sin t\} = \frac{1}{s^2 + 1} = F(s) \text{ (say)}$$

$$\text{and } L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{du}{u^2 + 1}$$

$$= [\tan^{-1} u]_s^{\infty}$$

$$= \tan^{-1} \infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s \quad \text{since } \tan^{-1} s + \cot^{-1} s = \frac{\pi}{2}$$

$$= \tan^{-1} \frac{1}{s}$$

$$\therefore L\left\{\int_0^{\infty} \frac{\sin t}{t} dt\right\} = \frac{1}{s} \tan^{-1} \frac{1}{s}$$

b) we have

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} \frac{adu}{u^2 + a^2}$$

$$= a [\tan^{-1} \frac{u}{a}]_s^{\infty}$$

$$= a \left[\frac{\pi}{2} - \tan^{-1} \frac{s}{a} \right]$$

$$= a \cot^{-1} \frac{s}{a}$$

* Evaluate, $L\left\{\frac{\sinh t}{t}\right\}$.

Soln: By the theorem, we have, if $L\{F(t)\} = F(s)$,

$$\text{then } L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(u) du \quad \text{--- (1)}$$

Let $F(t) = \sinh t$, then $L\{\sinh t\} = L\left\{\frac{1}{2}(e^t - e^{-t})\right\}$
 $= \frac{1}{2} \left\{ \frac{1}{s-1} - \frac{1}{s+1} \right\} = \frac{1}{s^2-1}$

$\therefore F(s) = \frac{1}{s^2-1}$. Therefore, $F(u) = \frac{1}{u^2-1}$.

Hence, by (1) we get

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(u) du$$

$$\begin{aligned} \Rightarrow L\left\{\frac{\sinh t}{t}\right\} &= \int_s^\infty \frac{1}{u^2-1} du = \frac{1}{2} \int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du \\ &= \frac{1}{2} [\log(u-1) - \log(u+1)]_s^\infty \\ &= \frac{1}{2} [\log(\infty-1) - \log(\infty+1) - \log(s-1) + \log(s+1)] \\ &= \frac{1}{2} \log\left(\frac{s+1}{s-1}\right) \quad (\text{Ans.}) \end{aligned}$$

* Using Laplace transform prove that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a} \quad \text{as } s \rightarrow 0$$

proof: Let $F(t) = e^{-at} - e^{-bt}$
 $\therefore F(s) = L\{F(t)\} = L\{e^{-at}\} - L\{e^{-bt}\}$
 $= \frac{1}{s+a} - \frac{1}{s+b}$

then by theorem, we have

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(u) du$$

or, $\int_0^\infty e^{-st} \frac{F(t)}{t} dt = \int_s^\infty \left(\frac{1}{u+a} - \frac{1}{u+b} \right) du$

$$\begin{aligned} \text{or, } \int_0^\infty e^{-st} \frac{e^{-at} - e^{-bt}}{t} dt &= \int_s^\infty \left(\frac{1}{u+a} - \frac{1}{u+b} \right) du \\ &= [\log(u+a) - \log(u+b)]_s^\infty \\ &= \log(\infty+a) - \log(\infty+b) - \log(s+a) + \log(s+b) \\ &= \log(\infty) - \log(\infty) + \log\left(\frac{s+b}{s+a}\right) = \log\left(\frac{s+b}{s+a}\right) \end{aligned}$$

Now taking limit as $s \rightarrow 0$, we have

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a}.$$

* prove that, $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)$.

proof: Let $F(t) = \cos at - \cos bt$

then, $F(s) = L\{F(t)\} = L\{\cos at - \cos bt\}$
 $= \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$

then, by the theorem, we have

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(u) du$$

$$\begin{aligned} \therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} &= \int_s^\infty \left(\frac{u}{u^2 + a^2} - \frac{u}{u^2 + b^2}\right) du \\ &= \frac{1}{2} \int_s^\infty \left(\frac{2u}{u^2 + a^2} - \frac{2u}{u^2 + b^2}\right) du \\ &= \frac{1}{2} [\log(u^2 + a^2) - \log(u^2 + b^2)]_s^\infty \\ &= \frac{1}{2} [\log \infty - \log \infty - \log(s^2 + a^2) + \log(s^2 + b^2)] \\ &= \frac{1}{2} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right) \end{aligned}$$

Hence, $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)$
(proved)

$$\begin{aligned} & * \frac{1}{2} [\log(u^2 + a^2) - \log(u^2 + b^2)]_s^\infty \\ &= \frac{1}{2} \left[\log\left(\frac{u^2 + a^2}{u^2 + b^2}\right) \right]_s^\infty \\ &= \frac{1}{2} \left[\log \frac{u^2 (1 + \frac{a^2}{u^2})}{u^2 (1 + \frac{b^2}{u^2})} \right]_s^\infty \\ &= \frac{1}{2} [\log(1 + \frac{a^2}{u^2}) - \log(1 + \frac{b^2}{u^2})]_s^\infty \\ &= \frac{1}{2} [\log(1 + \frac{a^2}{\infty^2}) - \log(1 + \frac{b^2}{\infty^2}) - \log(1 + \frac{a^2}{s^2}) \\ &\quad - \log(1 + \frac{b^2}{s^2})] \\ &= \frac{1}{2} \left[-\log \frac{s^2 + a^2}{s^2 + b^2} \right] \\ &= \frac{1}{2} \log \frac{s^2 + b^2}{s^2 + a^2} \quad \text{(proved)} \end{aligned}$$

* Periodic function:

If $F(t)$ is a periodic function with period $T > 0$, so that $F(t+T) = F(t)$, then $L\{F(t)\} = \frac{\int_0^T e^{-st} F(t) dt}{1 - e^{-sT}}$

proof: If $F(t)$ is a periodic function with period T , then $F(t+T) = F(t)$.

Now by definition of Laplace transform, we have

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} F(t) dt \\ &= \int_0^T e^{-st} F(t) dt + \int_T^{2T} e^{-st} F(t) dt + \int_{2T}^{3T} e^{-st} F(t) dt + \dots \end{aligned}$$

In the second integral let $t = u + T$, in the third integral let $t = u + 2T$ etc. we have

$$\begin{aligned} L\{F(t)\} &= \int_0^T e^{-su} F(u) du + \int_0^T e^{-s(u+T)} F(u+T) du \\ &\quad + \int_0^T e^{-s(u+2T)} F(u+2T) du + \dots \\ &= \int_0^T e^{-su} F(u) du + \int_0^T e^{-su} e^{-sT} F(u) du \\ &\quad + \int_0^T e^{-su} e^{-2sT} F(u) du + \dots \\ &= \int_0^T e^{-su} F(u) du + e^{-sT} \int_0^T e^{-su} F(u) du \\ &\quad + e^{-2sT} \int_0^T e^{-su} F(u) du + \dots \\ &= (1 + e^{-sT} + e^{-2sT} + \dots) \int_0^T e^{-su} F(u) du \\ &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-su} F(u) du \quad \left[(1 - e^{-sT})^{-1} = 1 + e^{-sT} + e^{-2sT} + \dots \right. \\ &\quad \left. \text{when } |e^{-sT}| < 1 \right] \\ &= \frac{\int_0^T e^{-st} F(t) dt}{1 - e^{-sT}} \end{aligned}$$

* Find the Laplace transform of $F(t)$, when $F(t)$ is a periodic function with period 2π , such that,

$$F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

proof: If $F(t)$ has period $T > 0$ then

$$L\{F(t)\} = \frac{\int_0^T e^{st} F(t) dt}{1 - e^{sT}}$$

In the given problem $F(t)$ has period $T = 2\pi$

$$\begin{aligned} \therefore L\{F(t)\} &= \frac{1}{1 - e^{2\pi s}} \int_0^{2\pi} e^{st} F(t) dt \\ &= \frac{1}{1 - e^{2\pi s}} \left[\int_0^{\pi} e^{st} F(t) dt + \int_{\pi}^{2\pi} e^{st} F(t) dt \right] \\ &= \frac{1}{1 - e^{2\pi s}} \left[\int_0^{\pi} e^{st} \cdot \sin t dt + \int_{\pi}^{2\pi} e^{st} \cdot 0 dt \right] \\ &= \frac{1}{1 - e^{2\pi s}} \left[\frac{e^{st}(-s \sin t - \cos t)}{s^2 + 1} \right]_0^{\pi} + 0 \\ &= \frac{1}{1 - e^{2\pi s}} \left[\frac{e^{\pi s} + 1}{s^2 + 1} \right] \\ &= \frac{1}{(1 - e^{2\pi s})(s^2 + 1)} \end{aligned}$$

* If $F(t) = t^n$, $0 < t < 2$ and $F(t+2) = F(t)$. Then find $L\{F(t)\}$.

Soln: By the theorem, we have if $F(t)$ is a periodic function with period $T > 0$, then

$$L\{F(t)\} = \frac{\int_0^T e^{st} F(t) dt}{1 - e^{sT}}$$

Here, $F(t)$ is a periodic function with period $T = 2$

$$\begin{aligned} \therefore L\{F(t)\} &= \frac{\int_0^2 e^{st} t^n dt}{1 - e^{2s}} \\ &= \frac{\left[t^n \frac{e^{st}}{s} \right]_0^2 + \frac{n}{s} \int_0^2 t e^{st} dt}{1 - e^{2s}} \end{aligned}$$

$$\begin{aligned}
&= \frac{-\frac{4}{s} e^{-2s} + \frac{2}{s} \left[-\frac{t}{s} e^{-st} \right]_0^2 + \frac{2}{s} \int_0^2 e^{-st} dt}{1 - e^{-2s}} \\
&= \frac{-\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} + \frac{2}{s} \left[-\frac{e^{-st}}{s} \right]_0^2}{1 - e^{-2s}} \\
&= \frac{-\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} e^{-2s} + \frac{2}{s^3}}{1 - e^{-2s}} \\
&= \frac{-4s e^{-2s} - 4s^2 e^{-2s} - 2 e^{-2s} + 2}{s^3 (1 - e^{-2s})}
\end{aligned}$$

* Laplace transform of the error function:

we know that,

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du \quad \dots \textcircled{1}$$

writing $\sqrt{t}$ for t in $\textcircled{1}$, we get

$$\begin{aligned}
\operatorname{erf}(\sqrt{t}) &= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - \frac{u^2}{1!} + \frac{u^4}{2!} - \frac{u^6}{3!} + \dots \right) du \\
&= \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3!} + \frac{u^5}{5 \cdot 2!} - \frac{u^7}{7 \cdot 3!} + \dots \right]_0^{\sqrt{t}} \\
&= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5 \cdot 2!} - \frac{t^{7/2}}{7 \cdot 3!} + \dots \right]
\end{aligned}$$

Taking Laplace transform of both sides we get

$$\begin{aligned}
L\{\operatorname{erf}(\sqrt{t})\} &= \frac{2}{\sqrt{\pi}} L\left\{ t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5 \cdot 2!} - \frac{t^{7/2}}{7 \cdot 3!} + \dots \right\} \\
&= \frac{2}{\sqrt{\pi}} \left\{ \frac{\Gamma_{3/2}}{s^{3/2}} - \frac{1}{3} \cdot \frac{\Gamma_{5/2}}{s^{5/2}} + \frac{1}{5 \cdot 2!} \cdot \frac{\Gamma_{7/2}}{s^{7/2}} - \frac{1}{7 \cdot 3!} \cdot \frac{\Gamma_{9/2}}{s^{9/2}} + \dots \right\} \\
&= \frac{2}{\sqrt{\pi}} \left\{ \frac{\frac{1}{2} \cdot \Gamma_{1/2}}{s^{3/2}} - \frac{1}{3} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \Gamma_{1/2}}{s^{5/2}} + \frac{1}{5 \cdot 2!} \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma_{1/2}}{s^{7/2}} - \frac{1}{7 \cdot 3!} \cdot \frac{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma_{1/2}}{s^{9/2}} + \dots \right\} \\
&= \frac{1}{s^{3/2}} - \frac{1}{2} \cdot \frac{1}{s^{5/2}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{s^{7/2}} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{s^{9/2}} + \dots \\
&= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2} \cdot \frac{1}{s} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{s^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{s^3} + \dots \right] \\
&= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s} \right)^{-1/2} \text{ (using binomial theorem)} \\
&= \frac{1}{s^{3/2}} \cdot \left(\frac{s+1}{s} \right)^{-1/2}
\end{aligned}$$

$$= \frac{1}{s^{3/2}} \cdot \frac{s^{1/2}}{(s+1)^{1/2}} = \frac{1}{s\sqrt{s+1}}$$

$$\text{Hence } L\{e^{-t} J_0(\sqrt{t})\} = \frac{1}{s\sqrt{s+1}}$$

* Laplace transform of Bessel functions:

$$i) \text{ prove that } L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

proof: By the definition of Bessel function of order n we have,

$$J_n(t) = \frac{t^n}{2^n n!} \left\{ 1 - \frac{t^2}{2 \cdot (2n+2)} + \frac{t^4}{2 \cdot 4 \cdot (2n+2)(2n+4)} - \dots \right\}$$

putting $n=0$, we get

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

Taking Laplace transform on both sides, we have

$$L\{J_0(t)\} = L\{1\} - L\left\{\frac{t^2}{2^2}\right\} + L\left\{\frac{t^4}{2^2 \cdot 4^2}\right\} - L\left\{\frac{t^6}{2^2 \cdot 4^2 \cdot 6^2}\right\} + \dots$$

$$= \frac{1}{s} - \frac{1}{2^2} \frac{1!}{s^3} + \frac{1}{2^2 \cdot 4^2} \cdot \frac{4!}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{6!}{s^7} + \dots$$

$$= \frac{1}{s} \left[1 - \frac{1}{2} \left(\frac{1}{s^2}\right) + \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{1}{s^4}\right) - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{1}{s^6}\right) + \dots \right]$$

$$= \frac{1}{s} \left(1 + \frac{1}{s^2} \right)^{-1/2}$$

$$= \frac{1}{s} \left(\frac{s^2+1}{s^2} \right)^{-1/2} = \frac{1}{s} \cdot \frac{s}{\sqrt{s^2+1}}$$

$$\text{Hence, } L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

$$ii) \text{ prove that } L\{J_0(at)\} = \frac{1}{\sqrt{s^2+a^2}}$$

proof: We know that,

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

By the change of scale property, we have if $L\{F(t)\} = F(s)$, then,

$$L\{F(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

$$\therefore L\{J_0(at)\} = \frac{1}{a} \cdot \frac{1}{\sqrt{(s/a)^2+1}} = \frac{1}{a} \cdot \frac{a}{\sqrt{s^2+a^2}}$$

$$\therefore L\{J_0(at)\} = \frac{1}{\sqrt{s^2+a^2}} \quad (\text{proved})$$

* prove that, $L\{e^{-at} J_0(bt)\} = \frac{1}{\sqrt{(s+a)^2 + b^2}}$.

proof: by formula, we have

$$\text{if } L\{F(t)\} = F(s) \text{ then } L\{F(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

$$L\{J_0(t)\} = \frac{1}{s^2 + 1}$$

$$\therefore L\{J_0(at)\} = \frac{1}{a} \cdot \frac{1}{\sqrt{\left(\frac{s}{a}\right)^2 + 1}} = \frac{1}{\sqrt{s^2 + a^2}}$$

$$\text{Similarly, } L\{J_0(bt)\} = \frac{1}{\sqrt{s^2 + b^2}}$$

Again, by formula, we have

$$L\{e^{-at} F(t)\} = F(s+a) \text{ where } F(s) = L\{F(t)\}$$

$$\therefore L\{e^{-at} J_0(bt)\} = \frac{1}{\sqrt{(s+a)^2 + b^2}}, \quad (\text{proved})$$