

The inverse Laplace transform

* Definition of inverse Laplace transformation:

If the Laplace transform of a function $F(t)$ is $F(s)$, i.e. $L\{F(t)\} = F(s)$, then $F(t)$ is called an inverse Laplace transform of $F(s)$ and we write symbolically $F(t) = L^{-1}\{F(s)\}$ where L^{-1} is called the inverse Laplace transformation operator.

Example: Since $L\{e^{-bt}\} = \frac{1}{s+b}$ we can write

$$L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt}$$

AR-50 * prove that, i) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

$$(ii) L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, n > 1$$

proofs i) $L\left\{\frac{\sin at}{a}\right\} = \frac{1}{a} L\{\sin at\} = \frac{1}{a} \cdot \frac{a}{s^2+a^2} = \frac{1}{s^2+a^2}$

$$\therefore L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$$

ii) $L\left\{\frac{t^n}{n!}\right\} = \frac{1}{n!} L\{t^n\} = \frac{1}{n!} \cdot \frac{n!}{s^{n+1}} = \frac{1}{s^{n+1}}$

$$\therefore L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$$

AR-50 * prove that i) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at} \sin bt$

$$(ii) L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at} \cos bt.$$

proofs i) $L\left\{\frac{1}{b} e^{at} \sin bt\right\} = \frac{1}{b} L\{e^{at} \sin bt\} = \frac{1}{b} \cdot \frac{b}{(s-a)^2+b^2} = \frac{1}{(s-a)^2+b^2}$

$$\therefore L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at} \sin bt$$

ii) $L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

$$\therefore L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at} \cos bt.$$

6.6-10.29
*

Some important formulae:

1. $L^{-1}\left\{\frac{1}{s}\right\} = 1, s > 0$
2. $L^{-1}\left\{\frac{1}{s^n}\right\} = t, s > 0$
3. $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, n=1,2,\dots$, * $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$
4. $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
5. $L^{-1}\left\{\frac{1}{s^2+a^2}\right\}, s > 0 = \frac{\sin at}{a}$
6. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}, s > 0 = \cos at$
7. $L^{-1}\left\{\frac{1}{s^2-a^2}\right\}, s > |a| = \frac{\sinh at}{a}$
8. $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}, s > |a| = \cosh at$
9. $L^{-1}\left\{\frac{2s}{(s^2+a^2)^2}\right\} = \frac{t \sin at}{a}$
10. $L^{-1}\left\{\frac{s^2-a^2}{s^2+a^2}\right\} = t \cos at$
11. $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{e^{at} \sin bt}{b}$, * $L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{e^{at} \sinh bt}{b}$
12. $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at} \cos bt$, * $L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at} \cosh bt$
13. $L^{-1}\left\{\frac{1}{\sqrt{s^2+a^2}}\right\} = J_0(at)$
14. $L^{-1}\left\{\frac{s}{(s^2+a^2)^{3/2}}\right\} = t J_0(at)$
15. $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{t^{n-1} e^{at}}{(n-1)!}, n > 0$
16. $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3} (\sin at - at \cos at)$
17. $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{2a} (\sin at + at \cos at)$
18. $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t \sin at$
19. $L^{-1}\left\{\frac{s^2-a^2}{(s^2+a^2)^2}\right\} = t \cos at$

* Some properties of the inverse Laplace transform:

* Linearity property: If the Laplace transform of $F_1(t)$ and $F_2(t)$ be $F_1(s)$ and $F_2(s)$ respectively and c_1 and c_2 are constants then,

$$L^{-1}\{c_1 F_1(s)\} + L^{-1}\{c_2 F_2(s)\} = c_1 L^{-1}\{F_1(s)\} + c_2 L^{-1}\{F_2(s)\} \\ = c_1 F_1(t) + c_2 F_2(t).$$

✓ Problem: Find $L^{-1}\left\{\frac{2(s-a)}{(s-a)^2 + b^2} + \frac{s-6s}{16s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\}$.

So we given that,

$$\begin{aligned} & L^{-1}\left\{\frac{2(s-a)}{(s-a)^2 + b^2} + \frac{s-6s}{16s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\} \\ &= L^{-1}\left\{\frac{2(s-a)}{(s-a)^2 + b^2}\right\} + L^{-1}\left\{\frac{s-6s}{16s^2+9}\right\} + L^{-1}\left\{\frac{24-30\sqrt{s}}{s^4}\right\} \\ &= 2 L^{-1}\left\{\frac{s-a}{(s-a)^2 + b^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s^2 + 9/16}\right\} - \frac{3}{8} L^{-1}\left\{\frac{3}{s^2 + 9/16}\right\} \\ &\quad + 24 L^{-1}\left\{\frac{1}{s^4}\right\} - 30 L^{-1}\left\{\frac{1}{s^{7/2}}\right\} \\ &= 2 e^{at} \cos bt + \frac{1}{2} \sin \frac{bt}{4} - \frac{3}{8} \cos \frac{bt}{4} + 4t^3 - 30 \frac{t^{5/2}}{\frac{15}{2}\sqrt{\pi}} \\ &= 2 e^{at} \cos bt + \frac{1}{2} \sin \frac{bt}{4} - \frac{3}{8} \cos \frac{bt}{4} + 4t^3 - \frac{16}{\sqrt{\pi}} t^{5/2}. \end{aligned}$$

Pr. 51 * Find $L^{-1}\left\{\frac{2}{s-a} + \frac{3}{s^2} + \frac{4s}{s^2+a^2} + \frac{5s}{s^2-a^2}\right\}$

$$\begin{aligned} &= 2 L^{-1}\left\{\frac{1}{s-a}\right\} + 3 L^{-1}\left\{\frac{1}{s^2}\right\} + 4 L^{-1}\left\{\frac{s}{s^2+a^2}\right\} + 5 L^{-1}\left\{\frac{s}{s^2-a^2}\right\} \\ &= 2e^{at} + 3t + 4 \sin at + 5 \cosh at. \end{aligned}$$

Pr. 52 * Find the inverse Laplace transform of

$$\begin{aligned} & L^{-1}\left\{\frac{5}{(s-2)^2} + 2 \tan^{-1} \frac{1}{s} + \frac{s+2}{s^2+2s+13}\right\} \\ &= 5 L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 2 L^{-1}\left\{\tan^{-1} \frac{1}{s}\right\} + L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\} \\ &= 5 t e^{2t} + 2 \frac{\sin t}{t} + e^{-2t} \cos 3t. \end{aligned}$$

Q. 50 * Evaluate, $L^{-1} \left\{ \frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25} \right\}$

Sol: Given that,

$$\begin{aligned} & L^{-1} \left\{ \frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25} \right\} \\ &= L^{-1} \left\{ \frac{5s}{s^2+9} - \frac{6}{s^2+9} - \frac{s}{s^2-25} + \frac{15}{s^2-25} \right\} \\ &= 5 L^{-1} \left\{ \frac{s}{s^2+9} \right\} - 2 L^{-1} \left\{ \frac{3}{s^2+9} \right\} - L^{-1} \left\{ \frac{s}{s^2-25} \right\} + 3 L^{-1} \left\{ \frac{3}{s^2-25} \right\} \\ &= 5 \cos 3t - 2 \sin 3t - \cosh 5t + 3 \sinh 5t \end{aligned}$$

Q. 52 * First Translation (or shifting) property:

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{F(s-a)\} = e^{at} f(t)$

Q. 50 * Find (a) $L^{-1} \left\{ \frac{1}{\sqrt{2s+5}} \right\}$ (b) $L^{-1} \left\{ \frac{s+1}{s^2+6s+25} \right\}$

Sol: we have, $L^{-1} \left\{ \frac{1}{\sqrt{2s+5}} \right\} =$

$$\begin{aligned} \text{(b) we have, } & L^{-1} \left\{ \frac{s+1}{s^2+6s+25} \right\} \\ &= L^{-1} \left\{ \frac{s+3-2}{(s+3)^2+4} \right\} = L^{-1} \left\{ \frac{s+3}{(s+3)^2+4} \right\} - 2 L^{-1} \left\{ \frac{1}{(s+3)^2+4} \right\} \\ &= e^{-3t} \cos 4t - 2 e^{-3t} \frac{\sin 4t}{4} \\ &= e^{-3t} \left\{ \cos 4t - \frac{1}{2} \sin 4t \right\} \end{aligned}$$

Q. 52 * Find $L^{-1} \left\{ \frac{s-2}{s^2-4s+20} \right\}$

Sol: we have, $L^{-1} \left\{ \frac{s-2}{(s-2)^2+4} \right\} = e^{2t} \cos 4t$

Q. 52 * Find $L^{-1} \left\{ \frac{1}{s^2+8s+16} \right\}$

$$\begin{aligned} &= L^{-1} \left\{ \frac{1}{(s+4)^2} \right\} \\ &= e^{-4t} L^{-1} \left\{ \frac{1}{s^2} \right\} = e^{-4t} \cdot t \\ &= t e^{-4t} \end{aligned}$$

AR-60
* prove that, $L^{-1} \left\{ \frac{6s-4}{s^2-4s+20} \right\} = 2e^{2t} (3 \cos 4t + \sin 4t)$

proof: Given that,

$$\begin{aligned} L^{-1} \left\{ \frac{6s-4}{s^2-4s+20} \right\} &= L^{-1} \left\{ \frac{6s-12+8}{s^2-4s+20} \right\} \\ &= L^{-1} \left\{ \frac{6(s-2)}{(s-2)^2+16} \right\} + L^{-1} \left\{ \frac{8}{(s-2)^2+16} \right\} \\ &= 6L^{-1} \left\{ \frac{s-2}{(s-2)^2+4^2} \right\} + 2L^{-1} \left\{ \frac{4}{(s-2)^2+4^2} \right\} \\ &= 6e^{2t} \cos 4t + 2e^{2t} \sin 4t \\ &= 2e^{2t} (3 \cos 4t + \sin 4t). \quad (\text{proved}) \end{aligned}$$

AR-65
* Evaluate $L^{-1} \left\{ \frac{8s+20}{s^2-12s+32} \right\}$

Solⁿ: Given, that,

$$\begin{aligned} L^{-1} \left\{ \frac{8s+20}{s^2-12s+32} \right\} &= L^{-1} \left\{ \frac{8(s-6)+68}{(s-6)^2-4} \right\} \\ &= 8L^{-1} \left\{ \frac{s-6}{(s-6)^2-4} \right\} + 34L^{-1} \left\{ \frac{2}{(s-6)^2-2^2} \right\} \\ &= 8e^{6t} L^{-1} \left\{ \frac{s}{s^2-2^2} \right\} + 34L^{-1} \left\{ \frac{2}{s^2-2^2} \right\} e^{6t} \\ &= 8e^{6t} \cosh 2t + 34e^{6t} \sinh 2t \\ &= 8e^{6t} \left\{ \frac{1}{2}(e^{2t} + e^{-2t}) \right\} + 34e^{6t} \left\{ \frac{1}{2}(e^{2t} - e^{-2t}) \right\} \\ &= 4e^{8t} + 4e^{-4t} + 17e^{8t} - 17e^{-4t} \\ &= 21e^{8t} - 13e^{-4t} \quad \text{---Ans.} \end{aligned}$$

AR-66
* Find $L^{-1} \left\{ \frac{3s+2}{(s-1)^5} \right\}$ as a function of t .

Solⁿ: Given that,

$$\begin{aligned} L^{-1} \left\{ \frac{3s+2}{(s-1)^5} \right\} &= L^{-1} \left\{ \frac{3s-3+5}{(s-1)^5} \right\} \\ &= L^{-1} \left\{ \frac{3(s-1)}{(s-1)^5} \right\} + L^{-1} \left\{ \frac{5}{(s-1)^5} \right\} \\ &= 3L^{-1} \left\{ \frac{1}{(s-1)^4} \right\} + 5L^{-1} \left\{ \frac{1}{(s-1)^5} \right\} \\ &= \frac{3}{2!} L^{-1} \left\{ \frac{2!}{(s-1)^4} \right\} + \frac{5}{4!} L^{-1} \left\{ \frac{4!}{(s-1)^5} \right\} \\ &= \frac{1}{2} L^{-1} \left\{ \frac{3!}{(s-1)^4} \right\} + \frac{5}{24} L^{-1} \left\{ \frac{4!}{(s-1)^5} \right\} \\ &= \frac{1}{2} t^3 e^t + \frac{5}{24} t^4 e^t, \quad (\text{Ans.}) \end{aligned}$$

AR-69 * prove that, $L^{-1} \left\{ \frac{s}{(s^2-2s+5)(s^2+2s+5)} \right\} = \frac{1}{4} \sin 2t \sinh t$

proof: we have,

$$\begin{aligned} & L^{-1} \left\{ \frac{s}{(s^2-2s+5)(s^2+2s+5)} \right\} \\ &= L^{-1} \left\{ \frac{1}{4} \cdot \frac{(s^2+2s+5) - (s^2-2s+5)}{(s^2-2s+5)(s^2+2s+5)} \right\} \\ &= \frac{1}{4} L^{-1} \left\{ \frac{1}{s^2-2s+5} - \frac{1}{s^2+2s+5} \right\} \\ &= \frac{1}{4} L^{-1} \left\{ \frac{1}{s^2-2s+5} \right\} - \frac{1}{4} L^{-1} \left\{ \frac{1}{s^2+2s+5} \right\} \\ &= \frac{1}{4} L^{-1} \left\{ \frac{1}{(s-1)^2+2^2} \right\} - \frac{1}{4} L^{-1} \left\{ \frac{1}{(s+1)^2+2^2} \right\} \\ &= \frac{1}{4} \cdot \frac{1}{2} e^t \sin 2t - \frac{1}{4} \cdot \frac{1}{2} e^{-t} \sin 2t \\ &= \frac{1}{4} \sin 2t \cdot \frac{1}{2} (e^t - e^{-t}) \\ &= \frac{1}{4} \sin 2t \cdot \sinh t. \text{ (proved.)} \end{aligned}$$

Q-10.31 * second Translation (or shifting) property:

$$\text{If } L^{-1}\{F(s)\} = f(t), \text{ then } L^{-1}\{e^{-as} F(s)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

Q-10.32 * Find $L^{-1} \left\{ \frac{s e^{-2\pi s/3}}{s^2+9} \right\}$

solving since, $L^{-1} \left\{ \frac{s}{s^2+9} \right\} = \cos 3t$

$$\therefore L^{-1} \left\{ \frac{s e^{-2\pi s/3}}{s^2+9} \right\} = \begin{cases} \cos 3(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$$

AR-53 * Find $L^{-1} \left\{ \frac{e^{-5s}}{(s-2)^4} \right\}$

solving since, $L^{-1} \left\{ \frac{1}{(s-2)^4} \right\} = e^{2t} L^{-1} \left\{ \frac{1}{s^4} \right\} = e^{2t} \cdot \frac{t^3}{3!}$
 $= \frac{1}{6} t^3 e^{2t}$

$$\therefore L^{-1} \left\{ \frac{e^{-5s}}{(s-2)^4} \right\} = \begin{cases} \frac{1}{6} (t-5)^3 e^{2(t-5)}, & t > 5 \\ 0, & t < 5 \end{cases}$$

or 30.34
 $\times$ Find $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

Soln: Since, $L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u \, du = 1 - \cos t$$

and hence, $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1 - \cos u) \, du$
 $= t - \sin t.$

AP-60 $\times$ Evaluate $L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$

Soln: By the theorem, we have

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t F(u) \, du$$

$$\therefore L^{-1}\left\{\frac{1}{s^2+4}\right\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{\sin 2u}{2} \, du$$

$$= \frac{1}{2} \left\{ -\frac{1}{2} \cos 2u \right\}_0^t$$

$$= -\frac{1}{4} (\cos 2t - 1)$$

$$= \frac{1}{4} (1 - \cos 2t)$$

and $L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \frac{1}{4} (1 - \cos 2u) \, du$

$$= \frac{1}{4} \left[u - \frac{1}{2} \sin 2u \right]_0^t$$

$$= \frac{1}{4} (t - 0) - \frac{1}{8} (\sin 2t - 0)$$

$$= \frac{1}{4} t - \frac{1}{8} \sin 2t$$

also, $L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \left(\frac{1}{4} u - \frac{1}{8} \sin 2u \right) \, du$

$$= \left[\frac{1}{8} u^2 + \frac{1}{16} \cos 2u \right]_0^t$$

$$= \frac{1}{8} (t^2 - 0) + \frac{1}{16} (\cos 2t - 1)$$

$$= \frac{1}{8} t^2 + \frac{1}{16} \cos 2t - \frac{1}{16}$$

AP-65 * Inverse Laplace transform of integrals:

If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\int_s^\infty F(u) du\right\} = \frac{f(t)}{t}$$

AP-66 * Find the inverse Laplace transform of

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du\right\}$$

Solⁿ: Since, $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\}$
 $= t - \sin t$

$$\begin{aligned} \therefore L^{-1}\left\{\int_s^\infty \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du\right\} &= L^{-1}\left\{-\frac{1}{u} + \tan^{-1} u\right\}_s^\infty \\ &= L^{-1}\left\{\frac{1}{s} - \frac{\pi}{2} + \tan^{-1} s\right\} \\ &= L^{-1}\left\{\frac{1}{s} - \cot^{-1} s\right\} \quad \text{Since, } \cot^{-1} s + \tan^{-1} s = \frac{\pi}{2} \\ &= L^{-1}\left\{\frac{1}{s} - \tan^{-1} \frac{1}{s}\right\} \\ &= 1 - \frac{\sin t}{t} = \frac{t - \sin t}{t} \end{aligned}$$

Hence, $L^{-1}\left\{\int_s^\infty \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du\right\} = \frac{t - \sin t}{t}$

AP-67 * Prove that, $L^{-1}\left\{\log\left(1 + \frac{1}{s^2}\right)\right\} = \frac{2}{t}(1 - \cos t)$.

Proof: We know that,

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!} \quad \text{for } n = 0, 1, 2, 3, \dots$$

and $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$$\therefore \log\left(1 + \frac{1}{s^2}\right) = \frac{1}{s^2} - \frac{1}{2} \cdot \frac{1}{s^4} + \frac{1}{3} \cdot \frac{1}{s^6} - \frac{1}{4} \cdot \frac{1}{s^8} + \dots \quad \text{--- (1)}$$

Taking the inverse Laplace transform of both sides of (1), we get

$$\begin{aligned} L^{-1}\left\{\log\left(1 + \frac{1}{s^2}\right)\right\} &= L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{s^4}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{s^6}\right\} \\ &\quad - \frac{1}{4} L^{-1}\left\{\frac{1}{s^8}\right\} + \dots \\ &= t - \frac{1}{2} \cdot \frac{t^3}{3!} + \frac{1}{3} \cdot \frac{t^5}{5!} - \frac{1}{4} \cdot \frac{t^7}{7!} + \dots \\ &= \frac{2t^2}{2t} - \frac{2}{t} \cdot \frac{t^4}{4!} + \frac{2}{t} \cdot \frac{t^6}{6!} - \frac{2}{t} \cdot \frac{t^8}{8!} + \dots \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{t} \left[\frac{t^2}{2} - \frac{t^4}{4} + \frac{t^6}{6} - \frac{t^8}{8} + \dots \right] \\
&= \frac{2}{t} \left[1 - 1 + \frac{t^2}{2} - \frac{t^4}{4} + \frac{t^6}{6} - \frac{t^8}{8} + \dots \right] \\
&= \frac{2}{t} \left[1 - \left(1 - \frac{t^2}{2} + \frac{t^4}{4} - \frac{t^6}{6} + \frac{t^8}{8} - \dots \right) \right] \\
&= \frac{2}{t} (1 - \cos t) \quad \text{Since, } \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots
\end{aligned}$$

Hence, $L^{-1} \left\{ \log \left(1 + \frac{1}{s^2} \right) \right\} = \frac{2}{t} (1 - \cos t).$

~~60~~ * prove that, $L^{-1} \left\{ \frac{1}{s} \log \left(1 + \frac{1}{s^2} \right) \right\} = 2 \int_0^t \frac{1 - \cos u}{u} du.$

proof: By the theorem, we have

$$L^{-1} \left\{ \frac{F(s)}{s} \right\} = \int_0^t F(u) du \quad \text{--- (1)}$$

$$\text{Let } F(s) = \log \left(1 + \frac{1}{s^2} \right) = L \{ F(t) \} \quad \text{--- (2)}$$

$$\text{Now, } L^{-1} \left\{ \log \left(1 + \frac{1}{s^2} \right) \right\} = \frac{2}{t} (1 - \cos t) \quad \text{--- (3)}$$

[Use the result of the above problem]

$$\therefore F(t) = \frac{2}{t} (1 - \cos t)$$

$$\therefore F(u) = \frac{2}{u} (1 - \cos u)$$

then, from (1) we get

$$L^{-1} \left\{ \frac{1}{s} \log \left(1 + \frac{1}{s^2} \right) \right\} = 2 \int_0^t \frac{1 - \cos u}{u} du.$$

* Partial fraction:

* problem: Find $\mathcal{L}^{-1} \left\{ \frac{s^2+3}{s(s^2+9)} \right\}$

Soln: Given that,

$$\begin{aligned}\mathcal{L}^{-1} \left\{ \frac{s^2+3}{s(s^2+9)} \right\} &= \mathcal{L}^{-1} \left\{ \frac{s^2+9-6}{s(s^2+9)} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{6}{s(s^2+9)} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{6}{9} \left(\frac{s^2+9-s^2}{s(s^2+9)} \right) \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{2}{3} \cdot \frac{1}{s} + \frac{2}{3} \left(\frac{s}{s^2+9} \right) \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{3} \cdot \frac{1}{s} + \frac{2}{3} \cdot \frac{s}{s^2+9} \right\}\end{aligned}$$

$$\begin{aligned}\therefore \mathcal{L}^{-1} \left\{ \frac{s^2+3}{s(s^2+9)} \right\} &= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} \\&= \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \cos 3t \\&= \frac{1}{3} (1 + 2 \cos 3t) \quad \text{--- Answer.}\end{aligned}$$

* problem: Find $\mathcal{L}^{-1} \left\{ \frac{s^2+2}{s(s^2+4)} \right\}$

Soln: Given that

$$\begin{aligned}\mathcal{L}^{-1} \left\{ \frac{s^2+2}{s(s^2+4)} \right\} &= \mathcal{L}^{-1} \left\{ \frac{s^2+4-2}{s(s^2+4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} - 2 \cdot \frac{1}{s(s^2+4)} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{2}{4} \cdot \frac{s^2+4-s^2}{s(s^2+4)} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2+4} \right) \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{s}{s^2+4} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{s}{s^2+4} \right\} \\&= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+2^2} \right\} \\&= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \cos 2t \\&= \frac{1}{2} (1 + \cos 2t) \\&= \cos^2 t \quad [\because \cos 2t = 2 \cos^2 t - 1]\end{aligned}$$

* problem: Find $L^{-1} \left\{ \frac{1}{s^3(s^2+1)} \right\}$

Soln: Given that,

$$L^{-1} \left\{ \frac{1}{s^3(s^2+1)} \right\} = L^{-1} \left\{ \frac{(s^2+1) - s^2}{s^3(s^2+1)} \right\}$$

$$= L^{-1} \left\{ \frac{1}{s^3} - \frac{1}{s(s^2+1)} \right\}$$

$$= L^{-1} \left\{ \frac{1}{2} \cdot \frac{2}{s^3} - \frac{(s^2+1) - s^2}{s(s^2+1)} \right\}$$

$$= L^{-1} \left\{ \frac{1}{2} \cdot \frac{2}{s^3} - \frac{1}{s} + \frac{s}{s^2+1} \right\}$$

$$= \frac{1}{2} L^{-1} \left\{ \frac{2}{s^3} \right\} - L^{-1} \left\{ \frac{1}{s} \right\} + L^{-1} \left\{ \frac{s}{s^2+1} \right\}$$

$$= \frac{1}{2} \cdot t^2 - 1 + \cos t$$

$$= \frac{t^2}{2} + \cos t - 1, \text{ Ans.}$$

* problem: Find $L^{-1} \left\{ \frac{s^2}{(s^2+a^2)(s^2+b^2)} \right\}$

$$= L^{-1} \left\{ \frac{a^2}{a^2-b^2} \cdot \frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2} \cdot \frac{1}{s^2+b^2} \right\}$$

$$= \frac{1}{a^2-b^2} L^{-1} \left\{ \frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2} \right\}$$

$$= \frac{1}{a^2-b^2} (a^2 \cdot \frac{1}{a} \sin at - b^2 \cdot \frac{1}{b} \sin bt)$$

$$= \frac{1}{a^2-b^2} (a \sin at - b \sin bt),$$

* problem: Find $L^{-1} \left\{ \frac{s-2}{6s^2+20} \right\}$

Soln: Given that,

$$L^{-1} \left\{ \frac{s-2}{6s^2+20} \right\} = L^{-1} \left\{ \frac{1}{6} \left(\frac{s-2}{s^2+20/6} \right) \right\}$$

$$= \frac{1}{6} L^{-1} \left[\frac{s}{s^2+10/3} - \frac{2}{s^2+10/3} \right]$$

$$= \frac{1}{6} L^{-1} \left[\frac{s}{s^2+(\sqrt{10/3})^2} - 2 \cdot \sqrt{\frac{3}{10}} \cdot \frac{\sqrt{10/3}}{s^2+(\sqrt{10/3})^2} \right]$$

$$= \frac{1}{6} L^{-1} \left\{ \frac{s}{s^2+(\sqrt{10/3})^2} \right\} - \frac{2}{3} \cdot \sqrt{\frac{3}{10}} L^{-1} \left\{ \frac{\sqrt{10/3}}{s^2+(\sqrt{10/3})^2} \right\}$$

$$= \frac{1}{6} \cos \sqrt{\frac{10}{3}} t - \frac{1}{\sqrt{30}} \sin \sqrt{\frac{10}{3}} t \text{ Ans.}$$

AR-66 Evaluate $\mathcal{L}^{-1} \left\{ \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} \right\}$

Soln: Let $\frac{2s^2 - 4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3} \dots \text{--- (1)}$

$\therefore 2s^2 - 4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2) \dots \text{--- (2)}$

putting $s = -1$ in (2), we get

$-2 = 12A \quad \therefore A = -\frac{1}{6}$

putting $s = 2$ in (2), we get

$8 - 4 = -3B \quad \therefore B = -\frac{4}{3}$

also putting $s = 3$ in (2) we get

$14 = 4C \quad \therefore C = \frac{7}{2}$

Hence, from (1) we get,

$$\frac{2s^2 - 4}{(s+1)(s-2)(s-3)} = \frac{-1/6}{s+1} + \frac{-4/3}{s-2} + \frac{7/2}{s-3}$$

Taking inverse Laplace transformation on both sides we get,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} \right\} &= \mathcal{L}^{-1} \left\{ \frac{-1/6}{s+1} + \frac{-4/3}{s-2} + \frac{7/2}{s-3} \right\} \\ &= -\frac{1}{6} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{4}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} + \frac{7}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s-3} \right\} \\ &= -\frac{1}{6} e^{-t} - \frac{4}{3} e^{2t} + \frac{7}{2} e^{3t} \end{aligned}$$

AR-67 Find the inverse Laplace transformation of

$$\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$$

Soln: Given that,

$$\begin{aligned} \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} &= \frac{2s^2 - 6s + 5}{s^2 \{s-1\} - 5s(s-1) + 6(s-1)} \\ &= \frac{2s^2 - 6s + 5}{(s-1)(s^2 - 5s + 6)} = \frac{2s^2 - 6s + 5}{(s-1)\{s^2 - 5s + 6\}} \\ &= \frac{2s^2 - 6s + 5}{(s-1)\{s(s-3) - 2(s-3)\}} \\ &= \frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} \end{aligned}$$

Let,

$$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)} \dots \text{--- (1)}$$

$$\therefore 2s^2 - 6s + 5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \quad \text{--- (2)}$$

putting $s=1$ in (2), we get
 $1 = 2A \quad \therefore A = \frac{1}{2}$

also putting $s=2$ in (2), we get
 $5 - 12 + 5 = -B \quad \therefore B = -1$

again, putting $s=3$ in (2) we get
 $18 - 18 + 5 = 2C \quad \therefore C = \frac{5}{2}$

putting the values of A , B and C in (1) we get

$$\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2} \cdot \frac{1}{s-3}$$

Taking inverse Laplace transformation we get

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2} \cdot \frac{1}{s-3} \right\} \\ &= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} + \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s-3} \right\} \\ &= \frac{1}{2} e^t - e^{2t} + \frac{5}{2} e^{3t}, \quad \text{Ans.} \end{aligned}$$