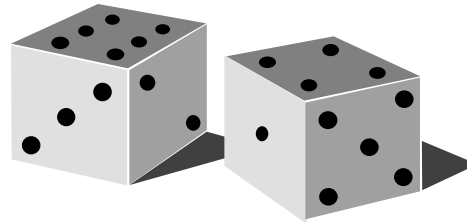


# Probability Concepts



# Learning Objectives

- Define Experiment, Outcome, Event, Sample Space, and Probability
- Explain How to Assign Probabilities
- Use a Venn Diagram, Contingency Table, or Tree Diagram to Find Probabilities
- Describe and Use Probability Rules

# Probability Basics

- Most applications of probability theory to statistical inference involve large populations.
- However, the fundamental concepts of probability are most easily illustrated and explained with relatively small populations and games of chance.
- Most examples in this chapter are designed expressly to demonstrate clearly the principles of probability.

# Probability Experiments

A **probability experiment** is an action or process whose outcome cannot be predicted with certainty.

**Example:** Rolling a die and observing the number that is rolled is a probability experiment.

The result of a single trial in a probability experiment is the **outcome**.

The set of all possible outcomes for an experiment is the **sample space**.

**Example:** The sample space when rolling a die has six outcomes.  $\{1, 2, 3, 4, 5, 6\}$

# Events

An **event** consists of one or more outcomes and is a subset of the sample space.

**Example:**

Events are represented by uppercase letters.



A die is rolled. Event **A** is rolling an even number.

A **simple event** is an event that consists of a single outcome.

**Example:** A die is rolled. Event **A** is rolling an even number.

This is not a simple event because the outcomes of event **A** are 2, 4, 6. That is,  $A = \{2, 4, 6\}$ .

# Probability of Events

Three methods for determining the probability of an event:

- 1) the classical method
- 2) the empirical method
- 3) the subjective method

# Classical Probability

**Classical** (or **theoretical**) **probability** is used when each outcome in a sample space is equally likely to occur.


The classical probability for the event  $E$  is given by

$$P(E) = \frac{\text{Number of outcomes in event}}{\text{Total number of outcomes in sample space}}$$

**Example:** A die is rolled. Find the probability of the event  $A$ : “rolling a 5”.

There is one outcome in the event  $A$ , namely,  $A = \{5\}$

$$P(A) = \frac{1}{6} \approx 0.167$$

“Probability of  
Event  $A$ .” 

# Classical Probability

**Example:** A die is rolled. Find the probability of Event  $E$ : rolling an even number.

In this case the event  $E$  is  $E = \{2,4,6\}$

$$P(E) = \frac{3}{6} = 0.5$$

# Empirical Probability

**Empirical** (or **statistical**) **probability** is based on observations obtained from probability experiments. The empirical frequency of an event  $E$  is the relative frequency of event  $E$ .

$$P(E) = \frac{\text{Frequency of Event E}}{\text{Total frequency}} = \frac{f}{N}$$

**Example:** A travel agent determines that in every 50 reservations she makes, 12 will be for a cruise. What is the probability that the next reservation she makes will be for a cruise?

$$P(\text{cruise}) = \frac{12}{50} = 0.24$$

# Probabilities with Frequency Distributions

**Example:** The following frequency distribution represents the ages of 30 students in a statistics class. If a student is selected at random, what is the probability that the student is between 26 and 33 years old?

Ages	Frequency $f$
18 – 25	13
26 – 33	8
34 – 41	4
42 – 49	3
50 – 57	2

$$\sum f = 30$$

$$P(\text{age 26 to 33}) = \frac{8}{30} \\ \approx 0.267$$

# Law of Large Numbers

As an experiment is repeated over and over, the empirical probability of an event approaches the theoretical (actual) probability of the event.

## **Example:**

Sally flips a coin 20 times and gets 3 heads. The empirical probability is  $3/20$ . This is not representative of the theoretical probability which is  $1/2$ . As the number of times Sally tosses the coin increases, the law of large numbers indicates that the empirical probability will get closer and closer to the theoretical probability.

# Basic Properties of Probabilities

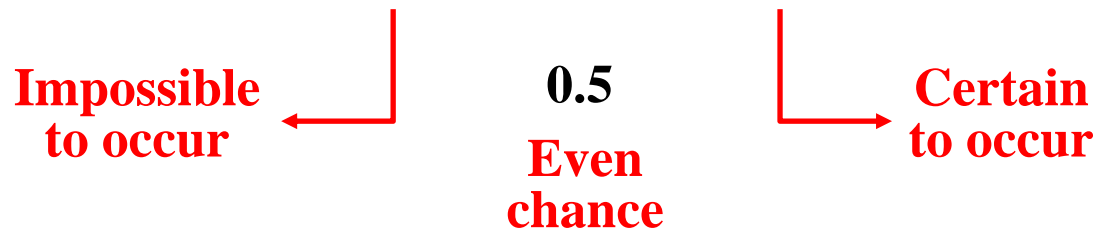
## Basic Properties of Probabilities

*Property 1:* The probability of an event is always between 0 and 1, inclusive.

*Property 2:* The probability of an event that cannot occur is 0. (An event that cannot occur is called an **impossible event**.)

*Property 3:* The probability of an event that must occur is 1. (An event that must occur is called a **certain event**.)

The probability of an event  $E$  is between 0 and 1, inclusive. That is  $0 \leq P(E) \leq 1$ .



# More Examples



# Example 1

Find the probability that

- a. The sum of the dice is 8.
- b. Doubles are rolled; that is, both dice come up the same number.

- In this experiment the sample space has  $N = 36$  possible outcomes.
- The event “sum of eight” can occur in 5 different ways. Therefore

$$P(\text{"sum of eight"}) = \frac{5}{36} \approx 0.139$$

- If we let  $D$  be the event doubles are rolled, then

$$P(D) = \frac{6}{36} = \frac{1}{6} \approx 0.167$$

## Example 2

- *Family Income.* The U.S. Bureau of the Census compiles data on family income and publishes its findings in *Current Population Reports*. The table in the next slide gives a frequency distribution of annual income for U.S. families.
- A U.S. family is selected **at random**, meaning that each family is equally likely to be the one obtained (simple random sample of size 1).

# Example 2

- Determine the probability that the family selected has an annual income of

Income	Frequency (1000s)
Under \$10,000	4,187
\$10,000–\$14,999	3,653
\$15,000–\$24,999	8,639
\$25,000–\$34,999	8,996
\$35,000–\$49,999	12,192
\$50,000–\$74,999	15,676
\$75,000 & over	18,192
	71,535

**a.** between \$50,000 and \$74,999,

**b.** between \$25,000 and \$74,999,

## Example 2

Let  $A$  be the event “the family selected has an annual income of between \$50,000 and \$74,999, inclusive”.

Income	Frequency (1000s)
Under \$10,000	4,187
\$10,000–\$14,999	3,653
\$15,000–\$24,999	8,639
\$25,000–\$34,999	8,996
\$35,000–\$49,999	12,192
\$50,000–\$74,999	15,676
\$75,000 & over	18,192
	71,535

$$P(A) = \frac{15676}{71535} = 0.219$$

## Example 2

Let  $B$  be the event “the family selected has an annual income of between \$25,000 and \$74,999, inclusive”.

Income	Frequency (1000s)
Under \$10,000	4,187
\$10,000–\$14,999	3,653
\$15,000–\$24,999	8,639
\$25,000–\$34,999	8,996
\$35,000–\$49,999	12,192
\$50,000–\$74,999	15,676
\$75,000 & over	18,192
	71,535

$$P(B) = \frac{8996 + 12192 + 15676}{71535}$$

$$P(B) = \frac{36864}{71535} = 0.515$$

# Quick Examples

## Experiment

- Toss a Coin, Note Face
- Toss 2 Coins, Note Faces
- Select 1 Card, Note Kind
- Select 1 Card, Note Color
- Play a Football Game
- Inspect a Part, Note Quality
- Observe Gender

## Sample Space

- {Head, Tail}
- {HH, HT, TH, TT}
- {2♥, 2♦, ..., A♠} (52)
- {Red, Black}
- {Win, Lose, Tie}
- {Defective, OK}
- {Male, Female}

# Quick Event Examples

**Experiment:** Toss 2 Coins. Note Faces.

## Event

## Outcomes in Event

- |                     |                  |
|---------------------|------------------|
| ■ 1 Head and 1 Tail | {HT, TH}         |
| ■ Heads on 1st Coin | {HH, HT}         |
| ■ At Least 1 Head   | {HH, HT, TH}     |
| ■ Heads on Both     | {HH}             |
| ■ Sample Space      | {HH, HT, TH, TT} |

# Visualizing Sample Space and Events

## 1. Listing

- $S = \{\text{Head, Tail}\}$

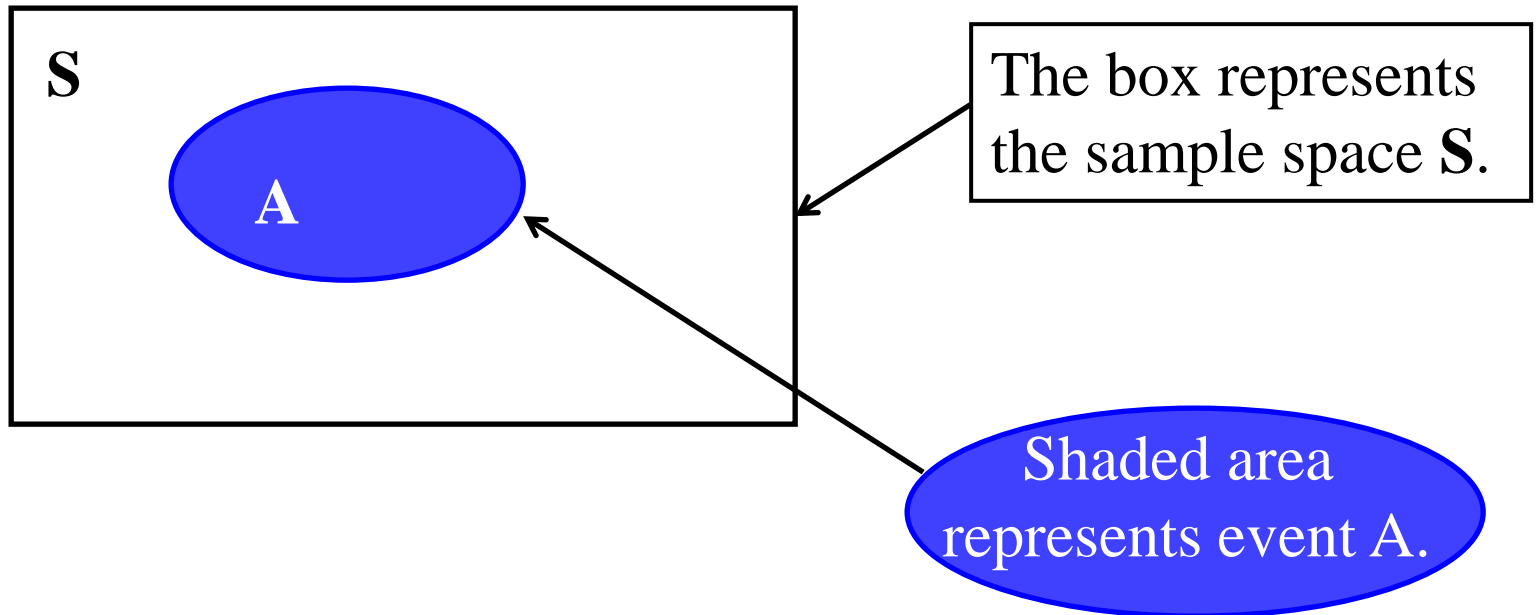
## 2. Venn Diagrams

## 3. Contingency Table

## 4. Decision Tree Diagram

# Venn Diagrams

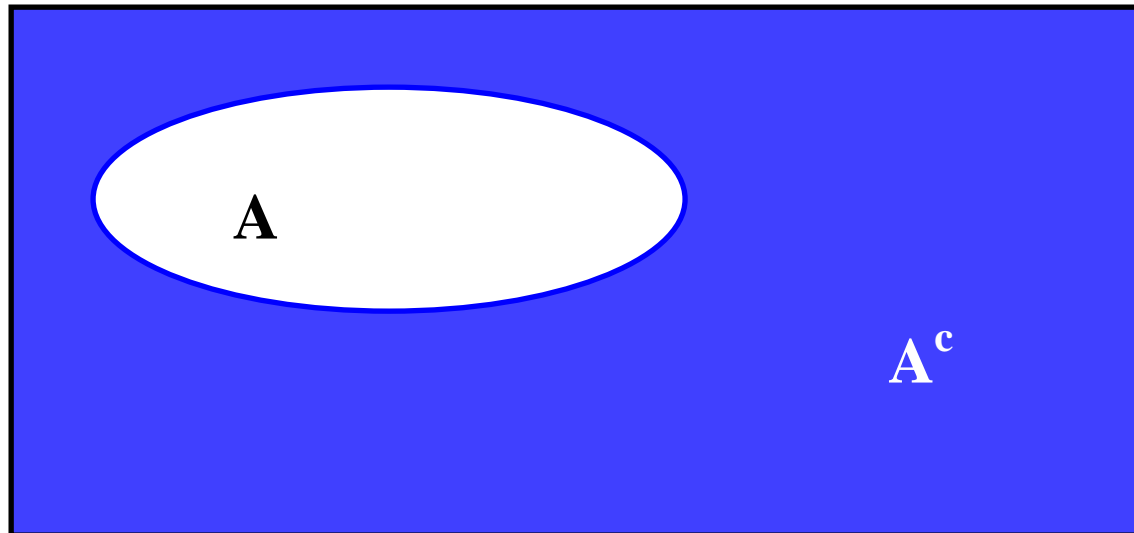
A Venn diagram is a graphical representation of the sample space and events



# The Complement of an Event

The complement of an event  $A$ , denoted by  $\bar{A}$ , is the event consisting of all outcomes that are **not** in  $A$

**Alternative Notation:**  $A^c$  (*A-complement*) or  $\sim A$  (*not-A*)



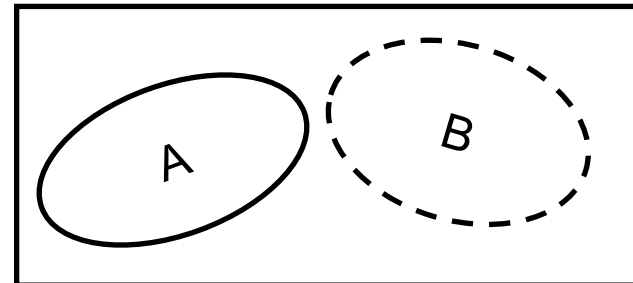
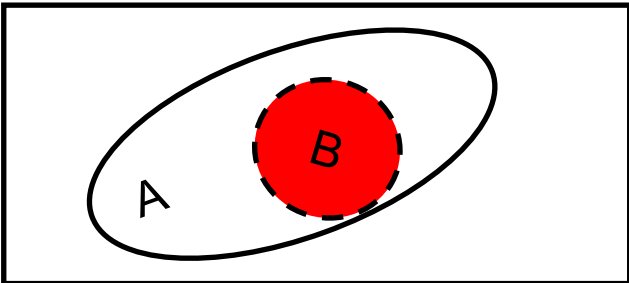
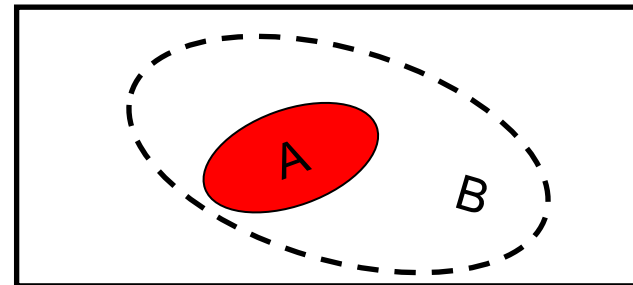
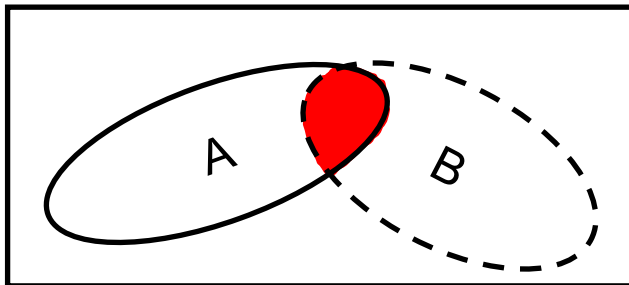
# Compound Events

- Intersection of events  $A$ ,  $B$ 
  - Set of outcomes **common to both** events  $A$  *and*  $B$
  - ‘**AND**’ Statement
  - $\cap$  Symbol (i.e.,  $A \cap B$ )
- Union of events  $A$ ,  $B$ 
  - Set of outcomes **in either events**  $A$  *or*  $B$  or both
  - ‘**OR**’ Statement
  - $\cup$  Symbol (i.e.,  $A \cup B$ )

# Intersection of Two events

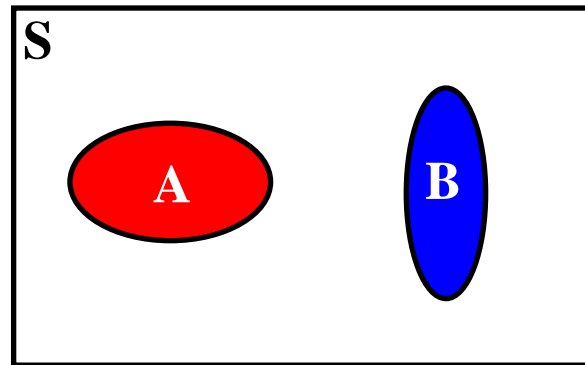
The intersection of the events  $A$  and  $B$  is the event consisting of all outcomes that are in both  $A$  and  $B$ .

**Notation:**  $A \cap B$ ,  $A$  and  $B$



# Mutually Exclusive Events

Events  $A$  and  $B$  are **disjoint or mutually exclusive** if they cannot occur simultaneously. Thus,  $A$  and  $B$  are mutually exclusive if and only if  $A \cap B = \emptyset$ .

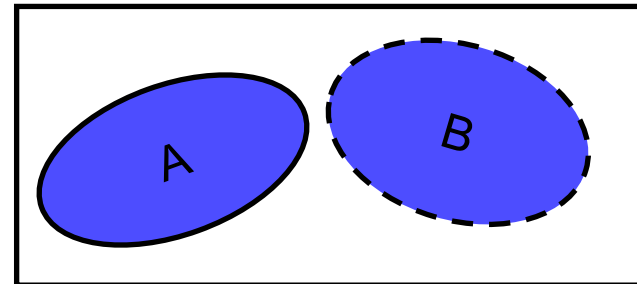
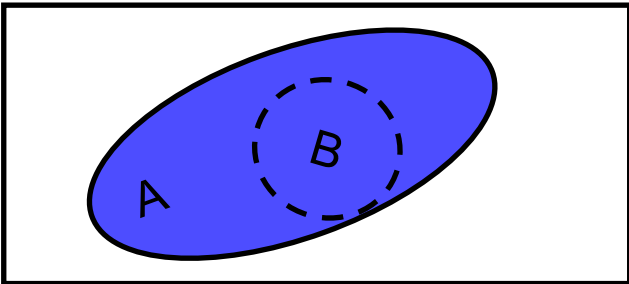
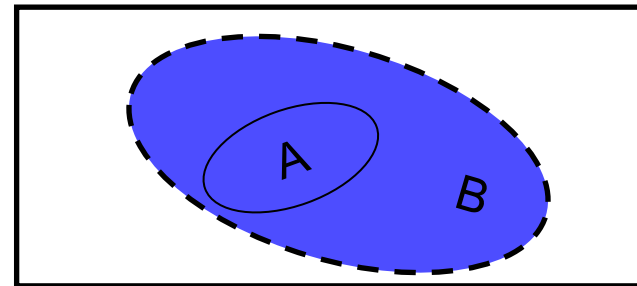
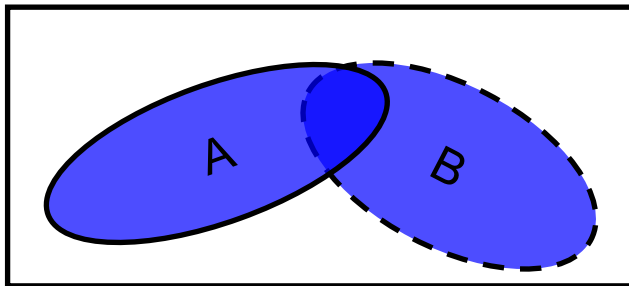


A **collection** of (two or more) events is **mutually exclusive** if all pairs of events from the collection are mutually exclusive.

# Union of Two events

The union of the events  $A$  and  $B$  is the event consisting of all outcomes that are in both  $A$  **or**  $B$ .

**Notation:**  $A \cup B$ ,  $A$  or  $B$



# Some Rules of Probability

## The Special Addition Rule

If event  $A$  and event  $B$  are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

More generally, if events  $A, B, C, \dots$  are mutually exclusive, then

$$P(A \text{ or } B \text{ or } C \text{ or } \dots) = P(A) + P(B) + P(C) + \dots.$$

In words, for mutually exclusive events, the probability that one or another of the events occurs equals the sum of the individual probabilities.

# Some Rules of Probability

## The Complementation Rule

For any event  $E$ ,

$$P(E) = 1 - P(\text{not } E).$$

In words, the probability that an event occurs equals 1 minus the probability that the event does not occur.

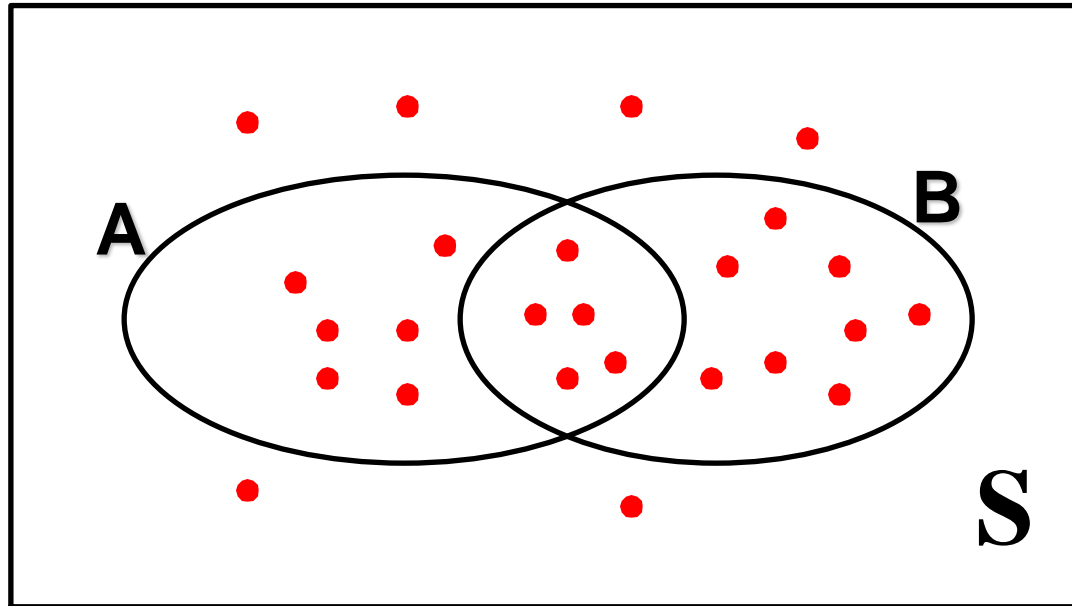
## The General Addition Rule

If  $A$  and  $B$  are any two events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \& B).$$

In words, for any two events, the probability that one or the other (or both) occurs equals the sum of the individual probabilities less the probability that both occur.

# Some Rules of Probability



**A Classical Example**  
**The Standard deck of 52 playing  
cards**

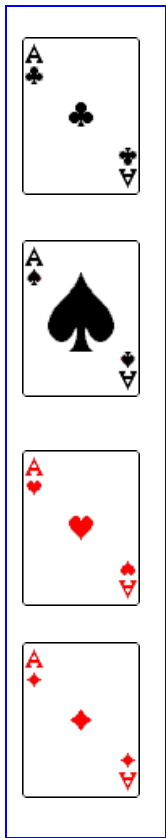
- The cards are ranked from high to low in the following order:  
Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.
- Aces are ALWAYS high. Aces are worth more than Kings which are worth more than Queens which are worth more than Jack, and so on.
- The cards are also separated into four suits. The suits are:

Clubs: 

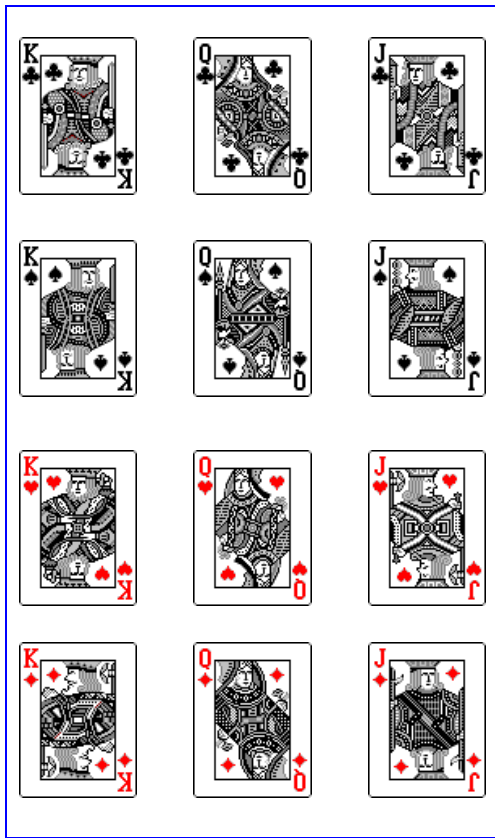
Spades: 

Hearts: 

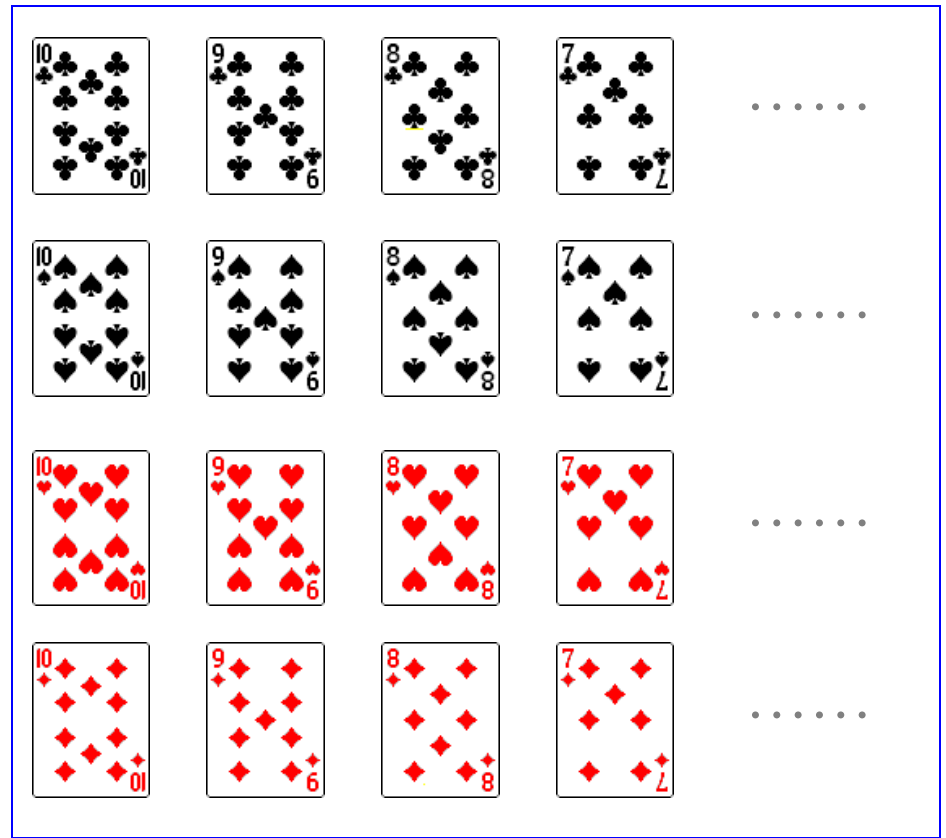
Diamonds: 



**4 Aces**



**12 Face Cards**

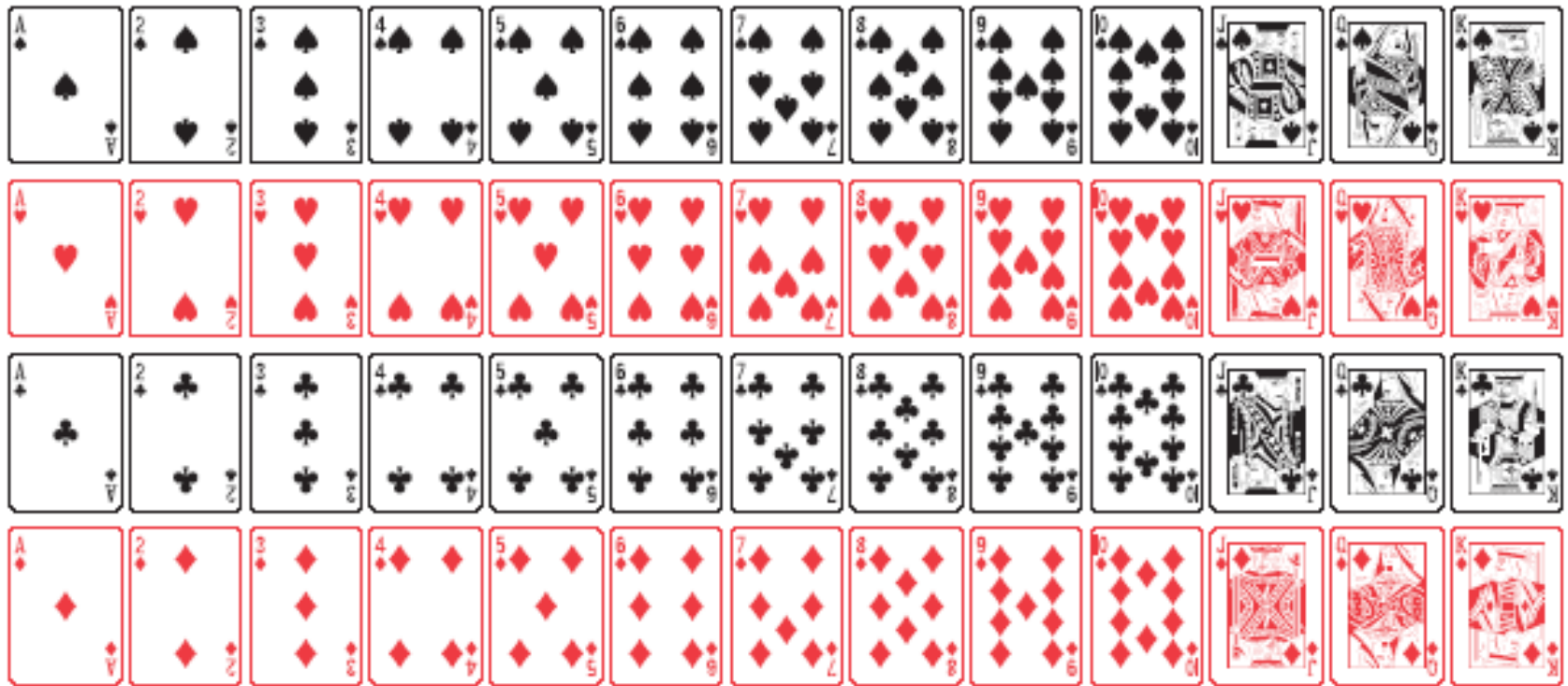


**Numbered Cards From 2 to 10**

**16 Lettered Cards**

**36 Numbered Cards**

- When we perform the experiment of randomly selecting one card from the deck, **one** of the 52 cards of the deck will be obtained. Thus, the sample space is



# Events - Unions - Intersections

- Randomly select one card from the deck. What is the probability that the selected card
- Is a face card?
- Is a club?
- Is a heart and less than 8?
- Is a heart or less than 8?
- Is a black ace?
- Is red and higher than ten?

# Independent Events

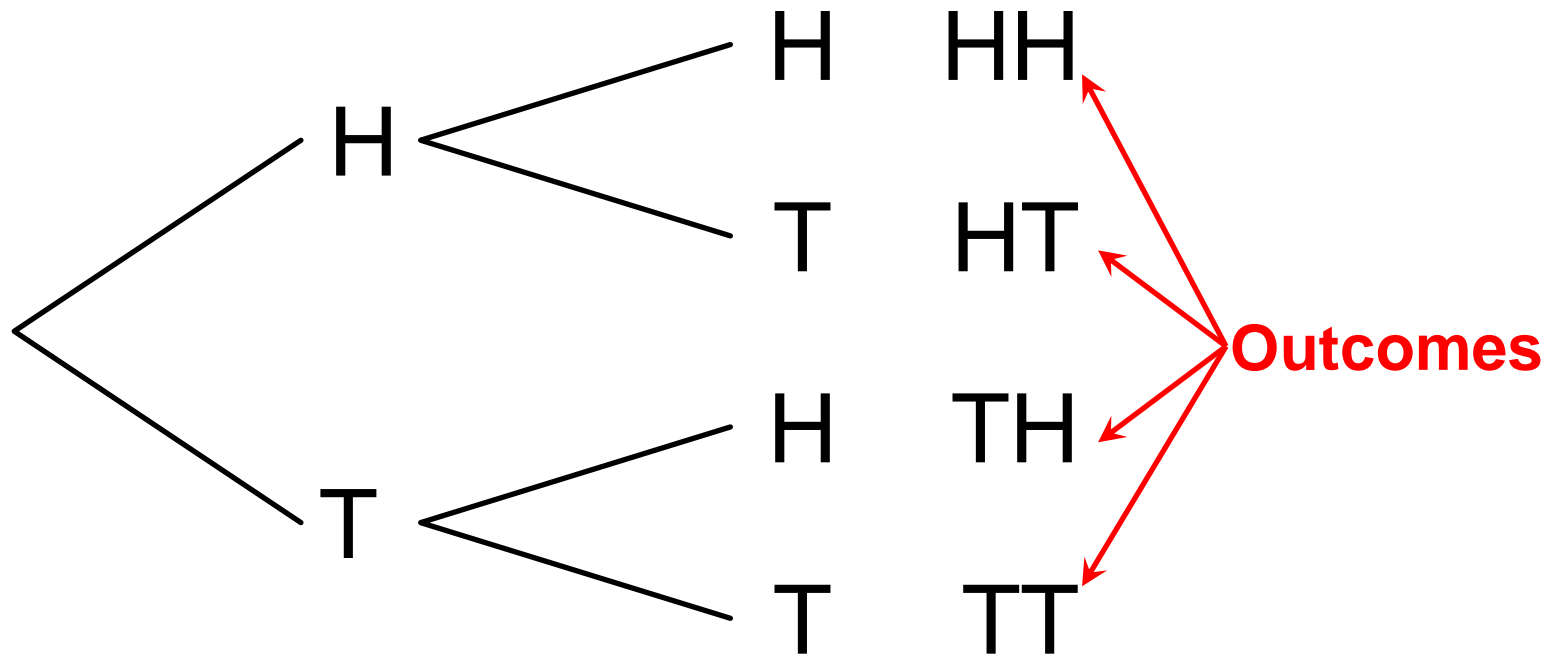
- Two events  $A$  and  $B$  are **independent** if the occurrence of one does not affect the probability of the occurrence of the other.
- Similarly, several events are independent if the occurrence of any one of them does not affect the probabilities of occurrence of the others.
- If  $A$  and  $B$  are not independent, they are said to be **dependent**.

# Examples

- You select two cards, one at a time, from the standard deck of cards. What is the probability that both are red if
  - Before selecting the second card you put back in the deck the first one you selected? (selection with replacement)
  - You do not replace the first in the deck?
- Are these events independent, dependent?

# Tree Diagram

**Experiment:** Toss 2 Coins. Note Faces.

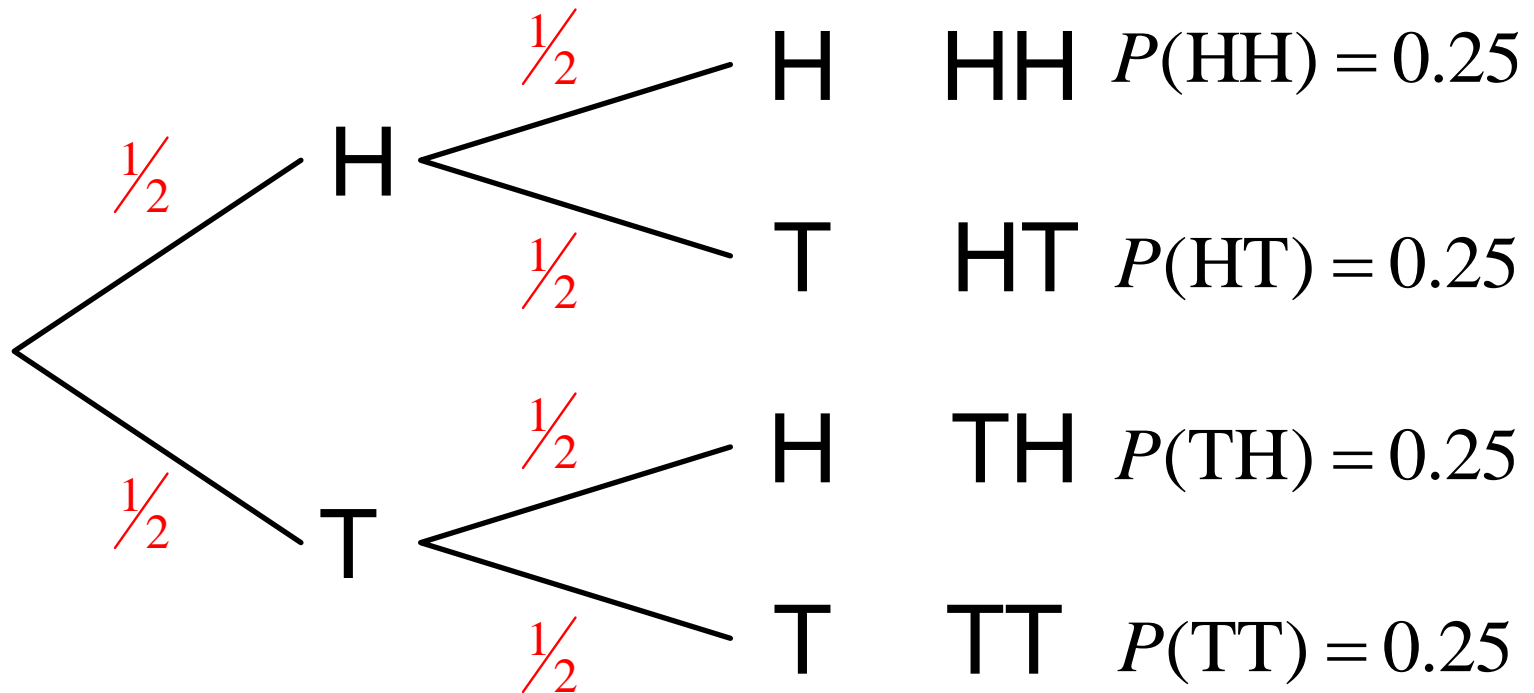


$S = \{HH, HT, TH, TT\}$

**Sample Space**

# Probability Tree

**Experiment:** Toss 2 Coins. Note Faces.

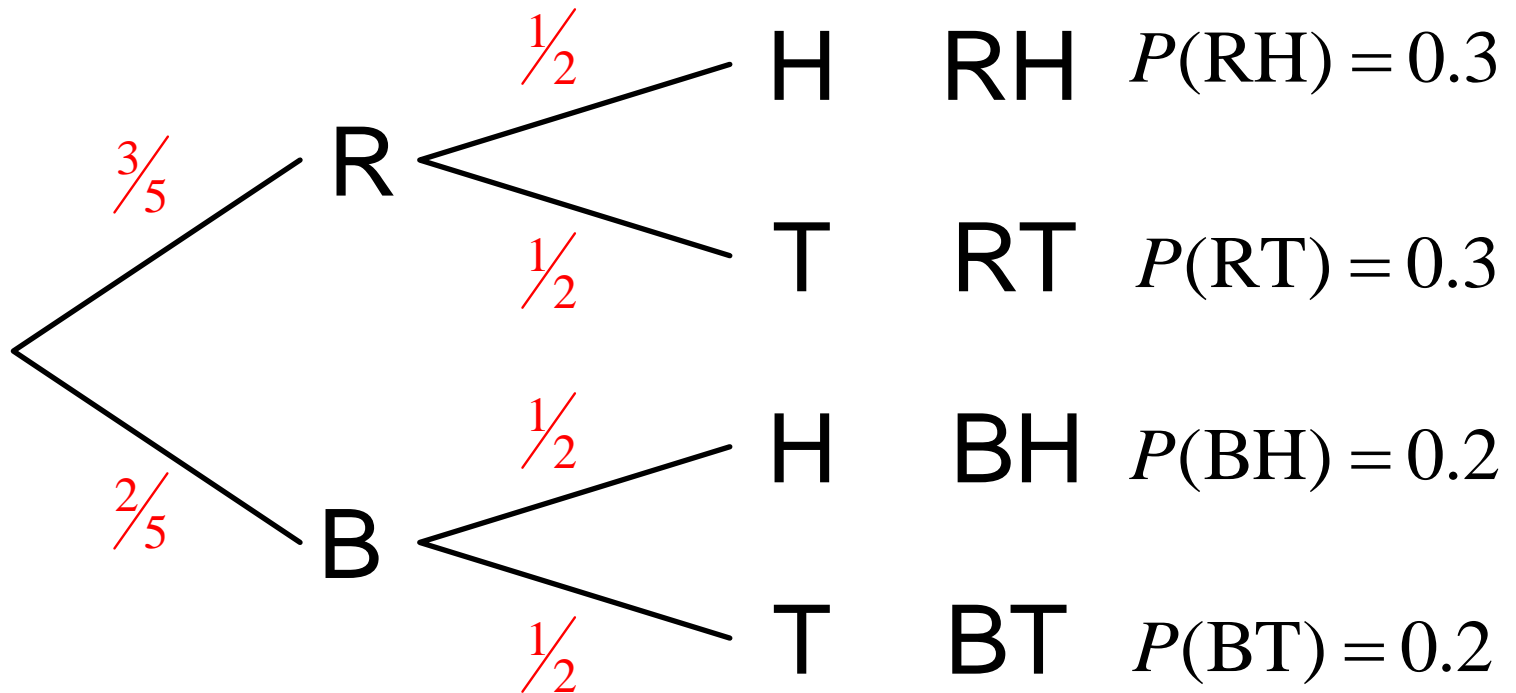


$S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$

# Another Example

- In a bag there are three red balls and two blue balls.
- The experiment consists in selecting at random one ball from the bag and tossing a coin.
- What is the probability of getting a red ball and a head?
- What is the probability of getting a blue ball?

First we draw the probability tree for the experiment



$S = \{\text{RH}, \text{RT}, \text{BH}, \text{BT}\}$

Sample Space

# Event Probability Using Contingency Table

- Suppose every the element in the sample space  $S$  can be classified as being either in  $A_1$  or in  $A_2$  but not both. That is  $S = A_1 \cup A_2$  and  $A_1 \cap A_2 = \emptyset$
- Also, suppose every the element in the sample space  $S$  can be classified as being either in  $B_1$  or in  $B_2$  but not both. That is  $S = B_1 \cup B_2$  and  $B_1 \cap B_2 = \emptyset$

# Event Probability Using Contingency Table

	$B_1$	$B_2$	<i>Total</i>
$A_1$	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	$P(A_1)$
$A_2$	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	$P(A_2)$
<i>Total</i>	$P(B_1)$	$P(B_2)$	1

**Joint Probability**

**Marginal Probability**

# Example: ASU Faculty

Data from one year on the variables age and rank of ASU faculty members yielded the contingency table shown in following table.

		Rank				Total
		Full professor $R_1$	Associate professor $R_2$	Assistant professor $R_3$	Instructor $R_4$	
Age	Under 30 $A_1$	2	3	57	6	68
	30–39 $A_2$	52	170	163	17	402
	40–49 $A_3$	156	125	61	6	348
	50–59 $A_4$	145	68	36	4	253
	60 & over $A_5$	75	15	3	0	93
	Total	430	381	320	33	1164

# Example: ASU Faculty

## Joint Frequency Distribution

		Rank				Total
		Full professor $R_1$	Associate professor $R_2$	Assistant professor $R_3$	Instructor $R_4$	
Age	Under 30 $A_1$	2	3	57	6	68
	30–39 $A_2$	52	170	163	17	402
	40–49 $A_3$	156	125	61	6	348
	50–59 $A_4$	145	68	36	4	253
	60 & over $A_5$	75	15	3	0	93
	Total	430	381	320	33	1164

# Example: ASU Faculty

## Joint Probability Distribution

		Rank				$P(A_i)$
		Full professor $R_1$	Associate professor $R_2$	Assistant professor $R_3$	Instructor $R_4$	
Age	Under 30 $A_1$	0.002	0.003	0.049	0.005	0.058
	30–39 $A_2$	0.045	0.146	0.140	0.015	0.345
	40–49 $A_3$	0.134	0.107	0.052	0.005	0.299
	50–59 $A_4$	0.125	0.058	0.031	0.003	0.217
	60 & over $A_5$	0.064	0.013	0.003	0.000	0.080
	$P(R_j)$	0.369	0.327	0.275	0.028	1.000

# Example: ASU Faculty

- Suppose that an ASU faculty member is selected at random.
- Determine the probability that the faculty member selected is in his or her 50s.
- Determine the probability that the faculty member selected is at least associate in his or her 40s.

# Example: ASU Faculty

- Determine the probability that the faculty member selected is in his or her 50s.

		Rank				Total
		Full professor $R_1$	Associate professor $R_2$	Assistant professor $R_3$	Instructor $R_4$	
Age	Under 30 $A_1$	2	3	57	6	68
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	50–59 $A_4$	145	68	36	4	253
	60 & over $A_5$	75	15	3	0	93
	Total	430	381	320	33	1164

$$P(A_4) = \frac{253}{1164}$$
$$P(A_4) = 0.217$$

# Example: ASU Faculty

- Determine the probability that the faculty member selected is at least associate in his or her 40s.

		Rank				Total
		Full professor $R_1$	Associate professor $R_2$	Assistant professor $R_3$	Instructor $R_4$	
Age	Under 30 $A_1$	2	3	57	6	68
	30-39 $A_2$	52	170	163	17	402
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	60 & over $A_5$	75	15	3	0	93
	Total	430	381	320	33	1164

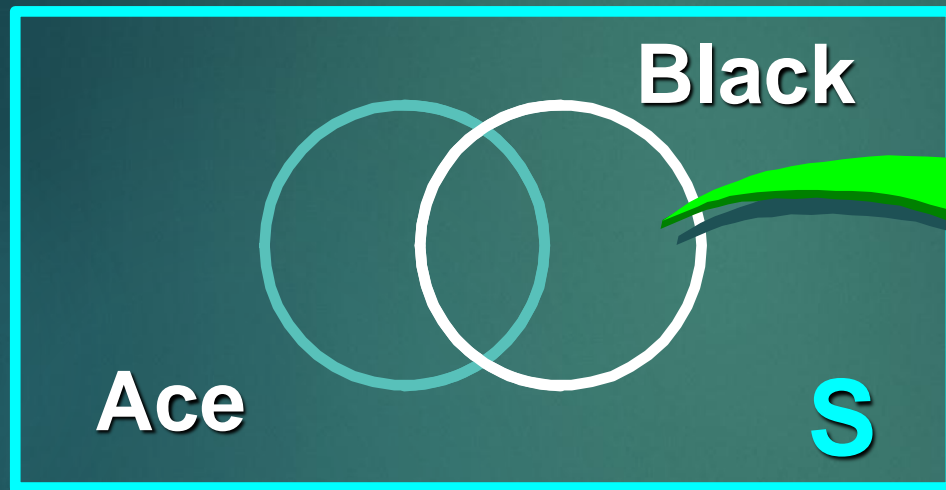
$$\begin{aligned} P(A_3 \text{ and } (R_1 \text{ or } R_2)) &= \\ &= \frac{156 + 125}{1164} \\ &= 0.241 \end{aligned}$$

# Conditional Probability

# Conditional Probability

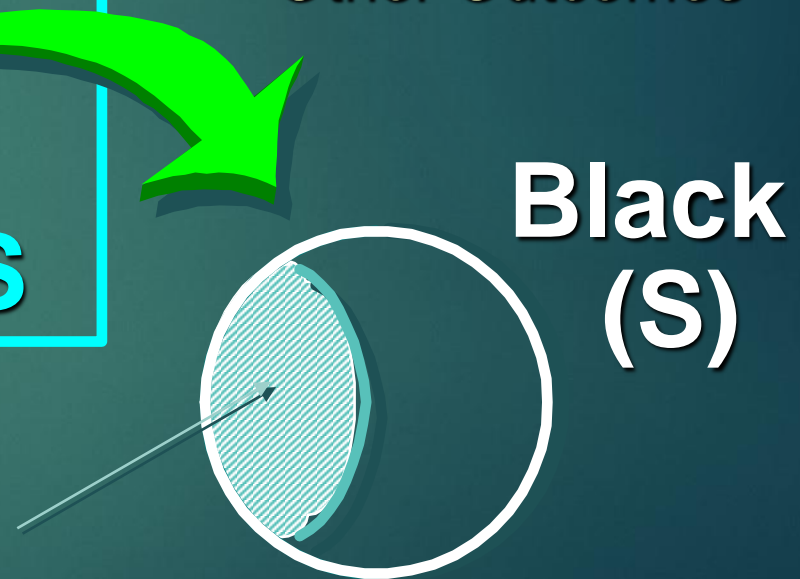
- ▶ 1. Event Probability **Given** that Another Event Occurred
- ▶ 2. Revise Original Sample Space to Account for **New** Information
  - ▶ Eliminates Certain Outcomes
- ▶ 3.  $P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$

# Conditional Probability Using Venn Diagram



Black 'Happens':  
Eliminates All  
Other Outcomes

Event (Ace AND Black)



# Conditional Probability Using Contingency Table

Experiment: Draw 1 Card. Note Kind, Color & Suit.

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Revised  
Sample  
Space

$$P(\text{Ace} \mid \text{Black}) = \frac{P(\text{Ace AND Black})}{P(\text{Black})} = \frac{2 / 52}{26 / 52} = \frac{2}{26}$$

# Statistical Independence

▶ 1. Event Occurrence Does **Not** Affect Probability of Another Event

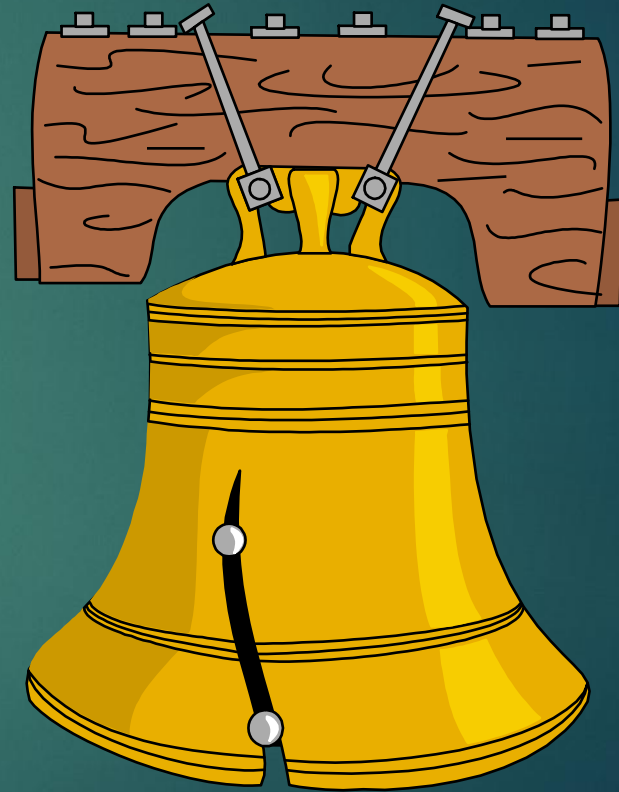
▶ Toss 1 Coin Twice

▶ 2. Causality Not Implied

▶ 3. Tests For

▶  $P(A | B) = P(A)$

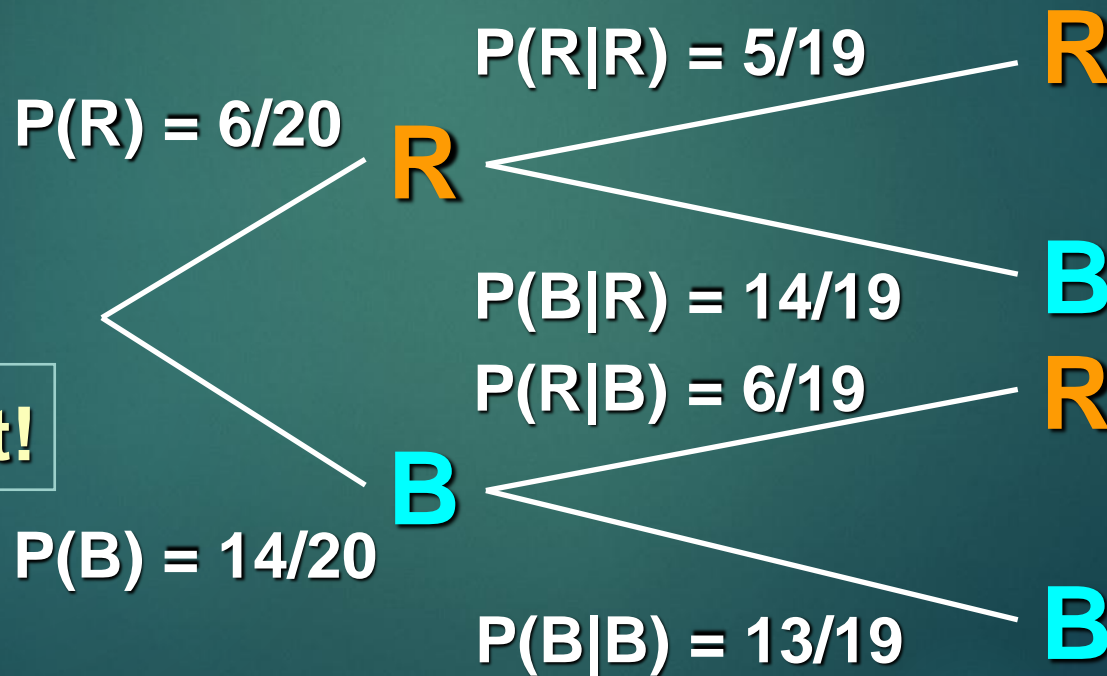
▶  $P(A \text{ and } B) = P(A) * P(B)$



# Tree Diagram



Experiment: Select 2 Pens from 20 Pens:  
14 Blue & 6 Red. Don't Replace.



**Dependent!**

# Thinking Challenge

▶ Using the Table Then the Formula, What's the Probability?

▶  $P(A | D) =$

▶  $P(C | B) =$

▶ Are C & B Independent?

Event	Event		Total
	C	D	
A	4	2	6
B	1	3	4
Total	5	5	10

# Solution\*

► Using the Formula, the Probabilities Are:

$$P(A | D) = \frac{P(A \cap D)}{P(D)} = \frac{2/10}{5/10} = \frac{2}{5}$$

$$P(C | B) = \frac{P(C \cap B)}{P(B)} = \frac{1/10}{4/10} = \frac{1}{4}$$

$$P(C) = \frac{5}{10} \neq \frac{1}{4} \quad \text{Dependent}$$

# Multiplicative Rule

# Multiplicative Rule

- ▶ 1. Used to Get Compound Probabilities for **Intersection** of Events
  - ▶ Called Joint Events
- ▶ 2. 
$$\begin{aligned} P(A \text{ and } B) &= P(A \cap B) \\ &= P(\mathbf{A}) * P(B | \mathbf{A}) \\ &= P(\mathbf{B}) * P(A | \mathbf{B}) \end{aligned}$$
- ▶ 3. For Independent Events:  
$$P(A \text{ and } B) = P(A \cap B) = P(A) * P(B)$$

# Multiplicative Rule

## Example

Experiment: Draw 1 Card. Note Kind, Color & Suit.

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

$$\begin{aligned}P(\text{Ace AND Black}) &= P(\text{Ace}) \cdot P(\text{Black} \mid \text{Ace}) \\ &= \left(\frac{4}{52}\right) \left(\frac{2}{4}\right) = \left(\frac{2}{52}\right)\end{aligned}$$

# Thinking Challenge

▶ Using the Multiplicative Rule, What's the Probability?

▶  $P(C \cap B) =$

▶  $P(B \cap D) =$

▶  $P(A \cap B) =$

	Event		
Event	C	D	Total
A	4	2	6
B	1	3	4
Total	5	5	10

# Solution\*

► Using the Multiplicative Rule, the Probabilities Are:

$$P(C \cap B) = P(C) \cdot P(B|C) = 5/10 * 1/5 = 1/10$$

$$P(B \cap D) = P(B) \cdot P(D|B) = 4/10 * 3/4 = 3/10$$

$$P(A \cap B) = P(A) \cdot P(B|A) = 0$$



**Example 3** An urn contains 5 white, 6 red and 4 blackballs. Two balls are drawn at random. Find the probability that both are red. Also find the probability of one white and one black ball.

**Example 5** Four cards are drawn without replacement from a well shuffled pack of 52 cards. Find the probability that

- (i) All cards are spades
- (ii) There are two spades and two hearts
- (iii) All cards are black

Also compute the probabilities if four cards are drawn with replacement.

**1.35.** A box contains 8 red, 3 white, and 9 blue balls. If 3 balls are drawn at random without replacement, determine the probability that (a) all 3 are red, (b) all 3 are white, (c) 2 are red and 1 is white, (d) at least 1 is white, (e) 1 of each color is drawn, (f) the balls are drawn in the order red, white, blue.