Example:

A standard deck of cards contains 52 cards. One card is randomly selected from the deck.

1. Compute the probability of randomly selecting a two or three from a deck of cards.

2. Compute the probability of randomly selecting a two or three or four from a deck of cards.

3. Compute the probability of randomly selecting a two or club from a deck of cards.

4. Compute the probability of randomly selecting a card other than a two from a deck of cards.

1. Compute the probability of randomly selecting a two or three from a deck of cards.

$$Let\ \mathrm{E} = \{2\}, \mathrm{F} = \{3\}$$

$$\mathrm{P(E\ or\ F)} = \mathrm{P(E \cup F)} = \mathrm{P(E)} + \mathrm{P(F)} - \mathrm{P(E \cap F)}$$

$$= \frac{4}{52} + \frac{4}{52} + 0 = \frac{8}{52} = \frac{2}{13}$$

2. Compute the probability of randomly selecting a two or three or four from a deck of cards.

Note: Mutually exclusive events as above.

$$Let\ \mathrm{G} = \{4\}$$

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$

$$= \frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{12}{52} = \frac{3}{13}$$

3. Compute the probability of randomly selecting a two or club from a deck of cards.

$$Let\ \mathrm{E} = \{2\},\ \ \mathrm{F} = \{\mathrm{club}\}$$

$$\mathrm{P}(\mathrm{E} \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

4. Compute the probability of randomly selecting a card other than a two from a deck of cards.

$$Let\ \mathrm{E} = \{2\},\ \ \mathrm{E}^{\mathrm{c}} = \{other\ \mathrm{than}\ 2\}$$

$$\mathrm{P}(\mathrm{E}^{\mathrm{c}}) = 1 - P(E)$$

$$= 1 - \frac{4}{52} = \frac{48}{52} = \frac{12}{13}$$

The Multiplication Rule

Conditional Probability

The notation $P(F \mid E)$ is read "the probability of event F given event E". It is the probability of an event F occurring given the occurrence of the event E.

The Multiplication Rule

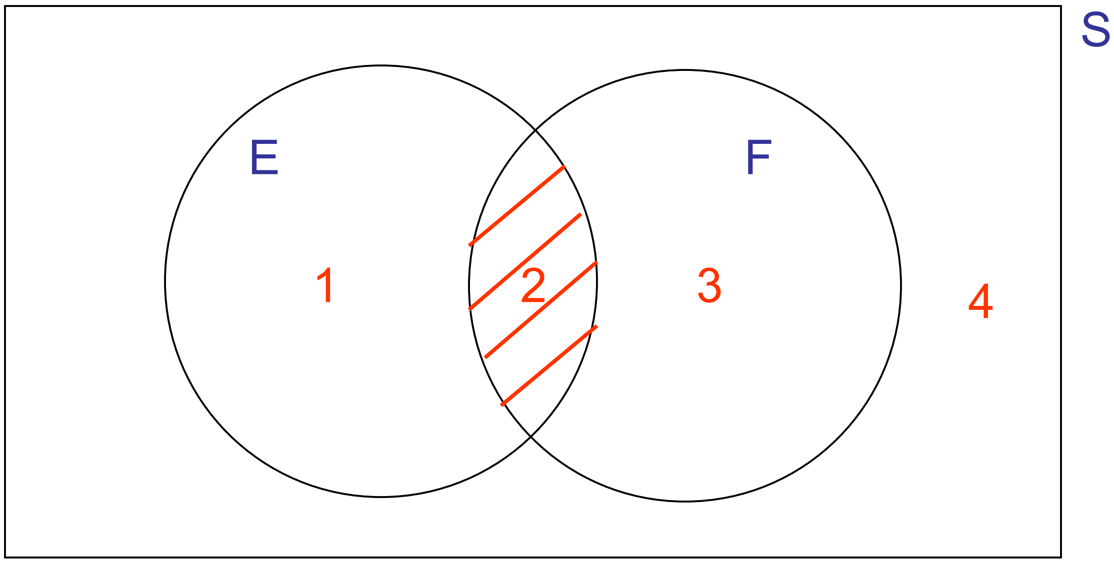
The probability that two events, E and F both occur is

$$P(E \text{ and } F) = P(E \cap F) = P(E) * P(F \mid E)$$

In words, the probability of E and F is the probability of event E occurring "times" the probability of event F occurring given the occurrence of event E.

Example: Let S = {1,2,3,4} E = {1,2} F = {2,3} then (E∩F) = {2}

What is P(E∩F) ?



$$P(E \cap F)$$
$$= P(E) * P(F \mid E)$$
$$= \frac{2}{4} * \frac{1}{2} = \frac{1}{4}$$

$$P\{2\} = \frac{1}{4}$$

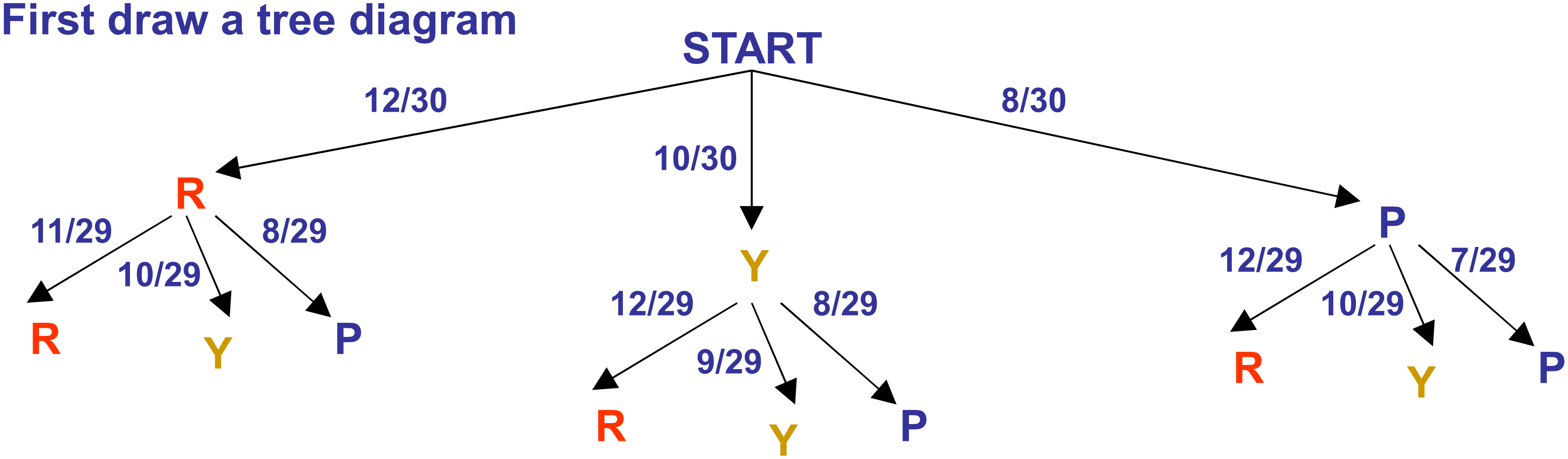
Note: P(E∩F) is referred to as the **joint probability** of E and F.
P(E) is referred to as the **marginal probability** of E.

Also note: Conditional = $\frac{\text{joint}}{\text{marginal}} \Rightarrow \mathrm{P(F \mid E)} = \frac{P(E \cap F)}{P(E)}$

Example:

A bag of 30 tulip bulbs was purchased from a nursery. The bag contains 12 red tulip bulbs, 10 yellow tulip bulbs and 8 purple tulip bulbs.

First draw a tree diagram



1. What is the probability that two randomly selected tulip bulbs will both be red?

2. What is the probability that the first bulb selected is red and the second is yellow?

3. What is the probability that the first bulb selected is yellow and the second is red?

4. What is the probability that one bulb is red and the other yellow? Tree on board

1. What is the probability that two randomly selected tulip bulbs will both be red?

$$P(R_1 \text{ than } \mathrm{R}_2) = \mathrm{P}(\mathrm{R}_1 \cap \mathrm{R}_2) = \mathrm{P}(\mathrm{R}_1) * P(R_2 \mid R_1)$$

$$= \frac{12}{30} * \frac{11}{29} = \frac{132}{870} = 0.152$$

2. What is the probability that the first bulb selected is red and the second is yellow?

$$P(R_1 \text{ than } \mathrm{Y}_2) = \mathrm{P}(\mathrm{R}_1 \cap \mathrm{Y}_2) = \mathrm{P}(\mathrm{R}_1) * P(Y_2 \mid R_1)$$

$$= \frac{12}{30} * \frac{10}{29} = \frac{120}{870} = 0.138$$

3. What is the probability that the first bulb selected is yellow and the second is red?

$$P(Y_1 \text{ than } \mathrm{R}_2) = \mathrm{P}(\mathrm{Y}_1 \cap \mathrm{R}_2) = \mathrm{P}(\mathrm{Y}_1) * P(R_2 \mid Y_1)$$

$$= \frac{10}{30} * \frac{12}{29} = \frac{120}{870} = 0.138$$

4. What is the probability that one bulb is red and the other yellow?

$$P(R \text{ and } \mathrm{Y}) = \mathrm{P}(\mathrm{R}_1 \text{ than } \mathrm{Y}_2) + \mathrm{P}(\mathrm{Y}_1 \text{ than } R_2)$$

$$= 0.138 + 0.138 = 0.276$$

Two events E and F are **independent** if the occurrence of event E in a probability experiment does not affect the probability of event F.

Two events are **dependent** if the occurrence of event E in a probability experiment affects the probability of event F.

From previous example: $P(R)=12/30 \neq P(R|Y) = 12/29$

Independent Events

Two events E and F are independent if and only if $P(F \mid E) = P(F)$ or $P(E \mid F) = P(E)$

Multiplication Rule for Independent Events

If E and F are independent events, the probability that E and F both occur is

$$P(E \text{ and } F) = P(E \cap F) = P(E)*P(F)$$

In words, the probability of E and F is the probability of event E occurring times the probability of event F occurring.

Example: 2 fair coins P(E = “head”) = 0.5

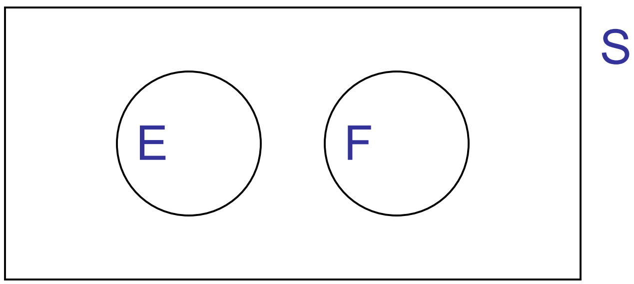
P(F = “tail”) = 0.5

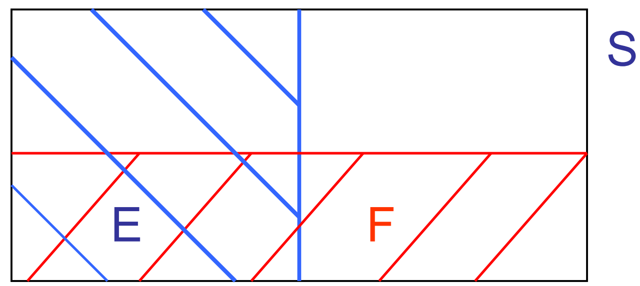
Each toss is an independent event. So P(E ∩ F) = P(E)*P(F) = 0.25

Mutually Exclusive **vs.** **Independent**

$$P(E|F) = P(E) = \frac{P(E \cap F)}{P(F)}$$

P(E U F) = P(E) + P(F)





Conditional Probability

Conditional Probability Rule
If E and F are any two events, then

$$P(F \mid E) = \frac{P(E \ and \ F)}{P(E)} = \frac{N(E \ and \ F)}{N(E)} \quad i.e. \ \text{Conditional} = \frac{\text{Joint}}{\text{Marginal}}$$

The probability of event F occurring given the occurrence of event E is found by dividing the probability of E and F by the probability of E. Or, the probability of event F occurring given the occurrence of event E is found by dividing the number of simple events in E and F by the number of simple events in E.

Likewise,

$$P(E \mid F) = \frac{P(E \ and \ F)}{P(F)} = \frac{N(E \ and \ F)}{N(F)}$$

Hence, we can use **Bayes Rule** to conclude,

$$P(E|F)*P(F) = P(F|E)*P(E) = P(E \cap F) \quad \text{i.e. Multiplication Rule}$$

Example:

A box contains 100 microchips, some of which were produced by factory 1 and the rest by factory two. Some of the chips are defective and some are good. An experiment consists of choosing one microchip at random from the box and testing whether it is good or defective. The data are presented in the following table.

	factory 1	factory 2	Total
defective	15	5	20
good	45	35	80
Total	60	40	100

Like a contingency table, but is this probability or statistics?

Population is known, therefore probability

Make a joint probability table:

	factory 1	factory 2	Total
defective	0.15	0.05	0.20
good	0.45	0.35	0.80
Total	0.60	0.40	1

Joint Probabilities

Marginal Probabilities

1. Find the probability of being defective.
2. Find the probability of being made in factory one.
3. Find the probability of being good.
4. Find the probability of being made in factory two.
5. Find the probability of being defective and made in factory one.
6. Find the probability of being defective given made in factory one.
7. Find the probability of made in factory one given defective.
8. Are the events of selecting a defective chip and one made at factory one independent events?
9. Are the events of selecting a defective chip and one made at factory one mutually exclusive events?

	factory 1	factory 2	Total
defective	0.15	0.05	0.20
good	0.45	0.35	0.80
Total	0.60	0.40	1

1. Find the probability of being defective.

$$P(D) = \frac{20}{100} = 0.2$$

2. Find the probability of being made in factory one.

$$P(Fac1) = \frac{60}{100} = 0.6$$

3. Find the probability of being good.

$$P(G) = \frac{80}{100} = 0.8 \qquad or \qquad P(D^c) = 1 - P(D) = 1 - 0.2 = 0.8$$

4. Find the probability of being made in factory two.

$$P(Fac2) = \frac{40}{100} = 0.4 \qquad or \qquad P(Fac1^c) = 1 - P(Fac1) = 1 - 0.6 = 0.4$$

	factory 1	factory 2	Total
defective	0.15	0.05	0.20
good	0.45	0.35	0.80
Total	0.60	0.40	1

5. Find the probability of being defective and made in factory one.

$$P(D \cap Fac1) = P(D) * P(Fac1 \mid D)$$

$$= 0.2 * \frac{15}{20} = 0.15$$

$$or \qquad = P(Fac1) * P(D \mid Fac1) = 0.6 * \frac{15}{60} = 0.15$$

$$\textit{Directly from } \text{Joint Table} \quad \mathrm{P(D \cap Fac1)} = 0.15 = \frac{15}{100}$$

6. Find the probability of being defective given made in factory one.

$$P(D \mid Fac1) = \frac{P(D \cap Fac1)}{P(Fac1)} = \frac{0.15}{0.60} = 0.25$$

7. Find the probability of made in factory one given defective.

$$P(Fac1 \mid D) = \frac{P(Fac1 \cap D)}{P(D)} = \frac{0.15}{0.20} = 0.75$$

	factory 1	factory 2	Total
defective	0.15	0.05	0.20
good	0.45	0.35	0.80
Total	0.60	0.40	1

8. Are the events of selecting a defective chip and one made at factory one independent events?

No.

$P(D|Fac1) = 0.25 \neq 0.20 = P(D)$

9. Are the events of selecting a defective chip and one made at factory one mutually exclusive events?

No.

$P(D \cap Fac1) = 0.15 \neq 0$

Note: Joint Tables may be constructed by two means.

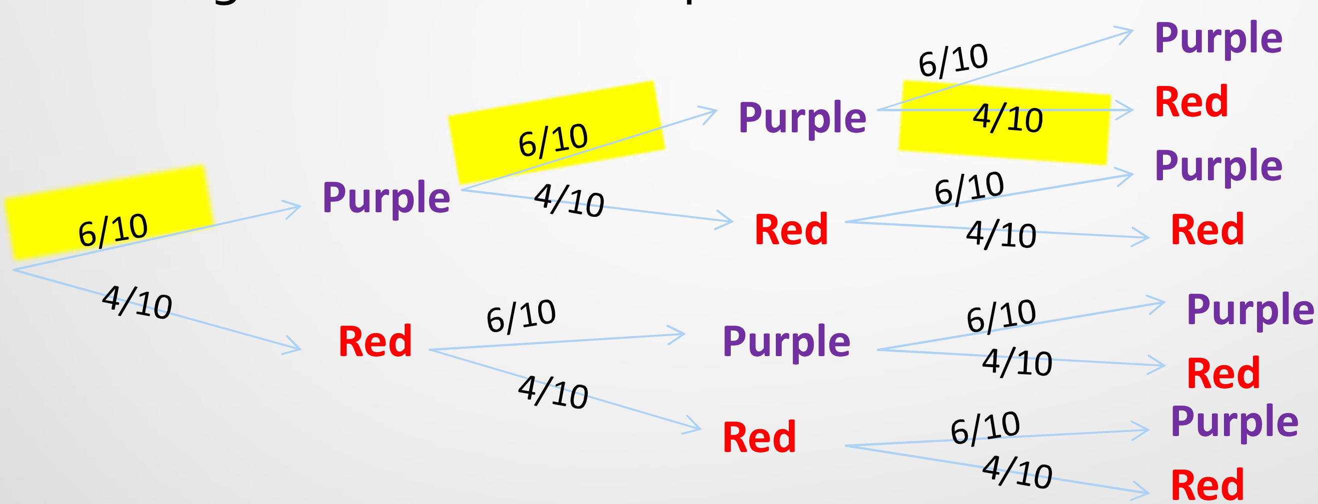
- Empirical
- Theoretical

Example 5: Let's look at the different of Independent vs. Dependent Example

INDEPENDENT

A box contains 4 red marbles and 6 purple marbles. You are going to choose 3 marbles ***with replacement .***

a.) Draw a tree diagram and write the probabilities on each branch.



b.) What is the probability of drawing 2 purple marbles and 1 red marble in succession (aka in order)?

P(1st draw purple) • P(2nd draw purple) • P(3rd draw red)

6/10 • 6/10 • 4/10

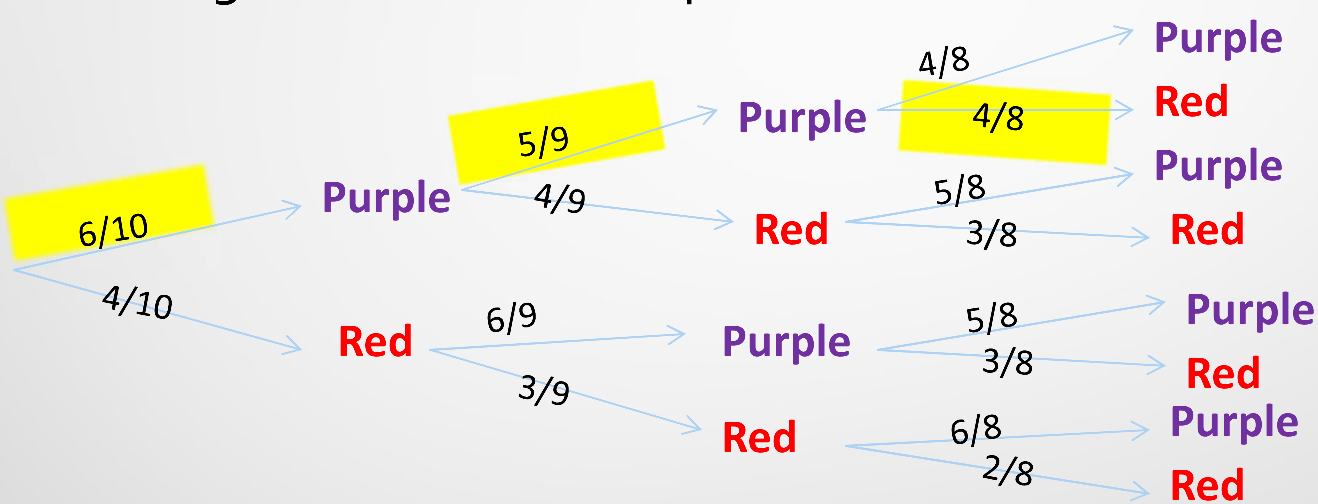
= 18/125 or .144 or 14.4%

Example 5: Let's look at the different of Independent vs. Dependent Example

<u>DEPENDENT</u>

A box contains 4 red marbles and 6 purple marbles. You are going to choose 3 marbles ***<u>without replacement .</u>***

a.) Draw a tree diagram and write the probabilities on each branch.



b.) What is the probability of drawing 2 purple marbles and 1 red marble in succession (aka in order)?

P(1st draw purple) • P(2nd draw purple) • P(3rd draw red)

6/10 • 5/9 • 4/8

= 1/6 or .167 or 16.7%

P(1st draw purple) • P(2nd draw purple) • P(3rd draw red)

$$= \frac{1}{\frac{3!}{2!}} \left(\frac{\binom{6}{2} \times \binom{4}{1}}{\binom{10}{3}} \right)$$

In the above example, what is the probability of first drawing all 4 red marbles in succession and then drawing all 6 purple marbles in succession *without replacement*?

- P(4 red then 6 purple) = (4/10)(3/9)(2/8)(1/7)(6/6)(5/5)(4/4)(3/3)(2/2)(1/1) = 1/210 or .0048
- The probability of drawing 4 red then 6 purple without replacement is 0.48%
- Explain why the last 6 probabilities above were all equivalent to 1.
- This is because there were only purple marbles left, so the probability for drawing a purple marble was 1.

$$P(\text{4 red then 6 purple}) = \frac{1}{\frac{10!}{4!\times 6!}}\left(\frac{\binom{6}{2}\times\binom{4}{1}}{\binom{10}{3}}\right)$$

A bag contains 6 white and 4 black balls. 2 balls are drawn at random. Find the probability that they are of same color.

A bag contains 12 white and 18 black balls. Two balls are drawn in succession without replacement.
What is the probability that first is white and second is black?

In a class, there are 15 boys and 10 girls. Three students are selected at random. The probability that 1 girl and 2 boys are selected, is

Example 3 An urn contains 5 white, 6 red and 4 blackballs. Two balls are drawn at random. Find the probability that both are red. Also find the probability of one white and one black ball.

Example5 Four cards are drawn without replacement from a well shuffled pack of 52 cards. Find the probability that

(i) All cards are spades
(ii) There are two spades and two hearts
(iii) All cards are black

Also compute the probabilities if four cards are drawn with replacement.

1.35. A box contains 8 red, 3 white, and 9 blue balls. If 3 balls are drawn at random without replacement, determine the probability that (a) all 3 are red, (b) all 3 are white, (c) 2 are red and 1 is white, (d) at least 1 is white, (e) 1 of each color is drawn, (f) the balls are drawn in the order red, white, blue.