

Heaven's Light is Our Guide
Rajshahi University of Engineering & Technology
Department of Mathematics



Course No.: Math.-2201

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Lecture Topic

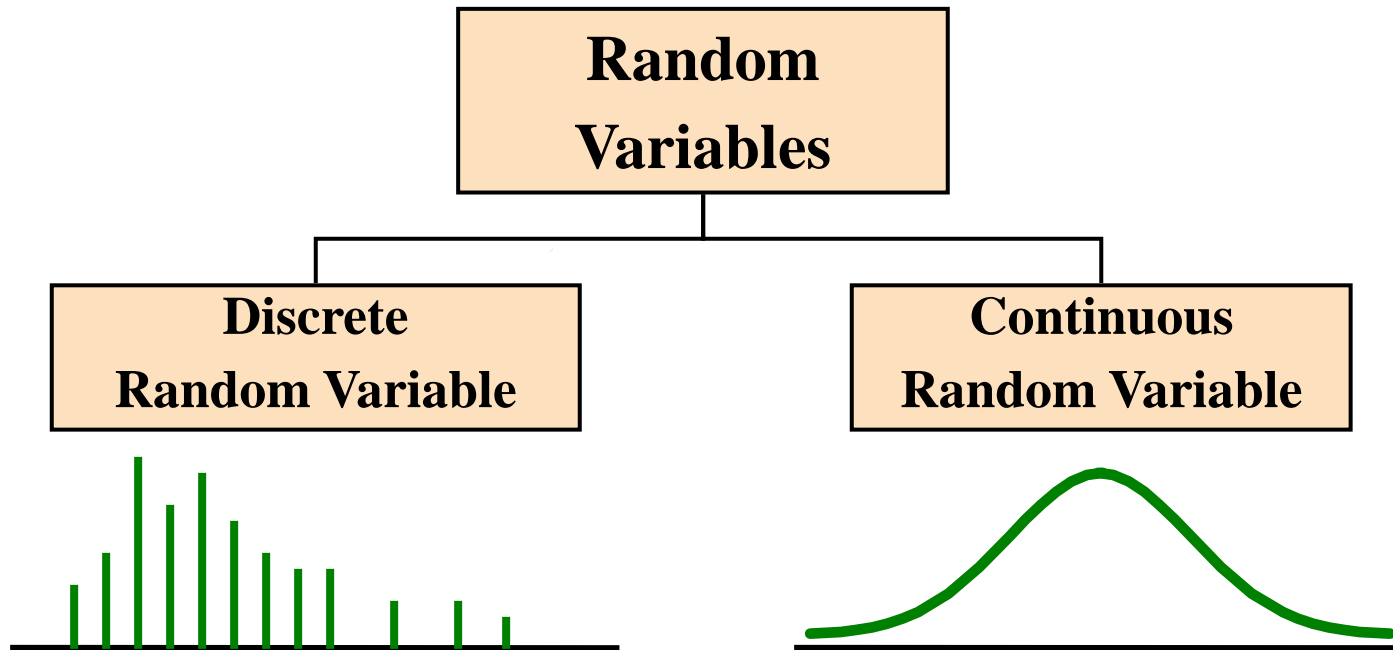
Random Variables
and
Distribution Functions

Random Variables

- Parameter that cannot be predicted with certainty
- Outcome of a random or uncertain process - flipping a coin or picking out a card from deck
- Can be discrete or continuous
- Data are usually discrete or quantized
- Usually easier to apply continuous distribution to discrete data that has been organized into bins

- **Random Variable**

- Represents a possible numerical value from an uncertain event



Discrete data

- Discrete data can only take exact values
- Examples:
- The number of cars passing a checkpoint in 30 minutes
- The shoe sizes of students in a class
- The number of tomatoes on each plant in a greenhouse
- Variables with **many repeated values** are treated as discrete

Continuous data

- Continuous data can be given values within a specified range or measured to a specified degree of accuracy.
- Examples:
 - The speed of a vehicle as it passes a checkpoint
 - The mass of a cooking apple
 - The time taken by a volunteer to perform a task
- Variables with **few repeated values** are treated as continuous

Discrete and Continuous Random Variables

- A random variable is discrete if it can assume a countable number of values.
- A random variable is continuous if it can assume an uncountable number of values.

Discrete random variable

After the first value is defined the second value, and any value thereafter are known.



Therefore, the number of values is countable

Continuous random variable

After the first value is defined, any number can be the next one



Therefore, the number of values is uncountable

Identify each random variable as discrete or continuous.

x = The number of people in a car

Discrete – you count the number of people in a car 0, 1, 2, 3...

Possible values can be listed.

x = The gallons of gas bought in a week

Continuous – you measure the gallons of gas. You cannot list the possible values.

x = The time it takes to drive from home to school

Continuous – you measure the amount of time. The possible values cannot be listed.

x = The number of trips to school you make per week

Discrete – you count the number of trips you make. The possible numbers can be listed.

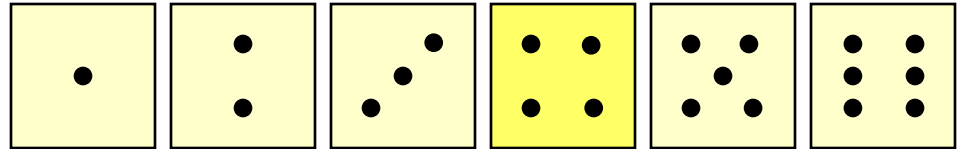
Probability distribution

A **probability distribution** of X provides the probability of each possible value of the random variable. If $P_X(x)$ is the probability that x is the value of the random variable, we know that $0 \leq P_X(x) \leq 1$ and that $\sum P_X(x) = 1$ where the summation is over all values that X takes on. This is because these values of X are mutually exclusive and one of them must occur.

Discrete Random Variables

- Can only assume a countable number of values

Examples:



– **Roll a die twice**

Let X be the number of times 4 comes up
(then X could be 0, 1, or 2 times)

– **Toss a coin 5 times.**

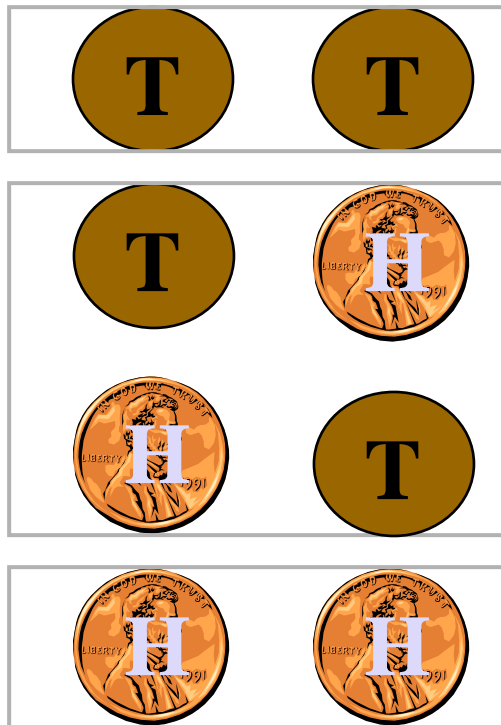
Let X be the number of heads
(then $X = 0, 1, 2, 3, 4, \text{ or } 5$)



Discrete Probability Distribution

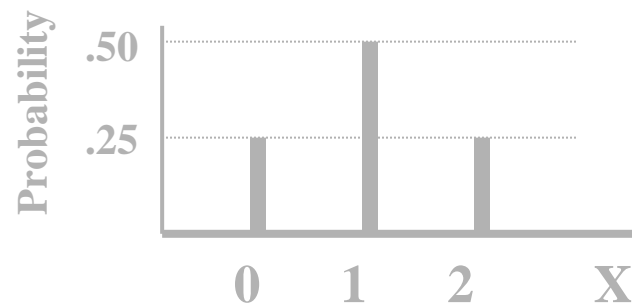
Experiment: Toss 2 Coins. Let $X = \text{no. of heads}$.

4 possible outcomes



Probability Distribution

<u>X Value</u>	<u>Probability</u>
0	$1/4 = .25$
1	$2/4 = .50$
2	$1/4 = .25$



Discrete Random Variable

Mathematical Expectation: If X be a random variable which can assume any one of the values x_1, x_2, \dots, x_n with respective probabilities p_1, p_2, \dots, p_n ; then the mathematical expectation of X usually called as expected value of X , denoted by $E(X)$ is defined as:

$$E(X) = p_1 x_1 + p_2 x_2 + \dots + p_n x_n = \sum p_i x_i ;$$

where $\sum p_i = 1$

Discrete Random Variable

- **Physical interpretation of $E(X)$**
- If \bar{x} denotes mean of set of observations x_1, x_2, \dots, x_n with respective frequencies f_1, f_2, \dots, f_n ; then $\bar{x} = \frac{\sum f_i x_i}{N}$, $N = \sum f_i$

$$\Rightarrow \bar{x} = \frac{f_1}{N} x_1 + \frac{f_2}{N} x_2 + \dots + \frac{f_n}{N} x_n \dots \dots \dots (1)$$

Now out of total N cases, f_i are favorable to x_i

$$\therefore P(X = x_i) = \frac{f_i}{N} = p_i, i = 1, 2, \dots, n$$

$$\Rightarrow \frac{f_1}{N} = p_1, \frac{f_2}{N} = p_2, \dots, \frac{f_n}{N} = p_n$$

Discrete Random Variable

- $\therefore \bar{x} = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = \sum p_i x_i$

Hence $\bar{x} = E(X)$

Hence mathematical expectation of a random variable is nothing but its arithmetic mean.

\therefore We conclude Mean (\bar{x}) = $E(X) = \sum p_i x_i$

Similarly Variance (σ^2) = $\frac{\sum f_i x_i^2}{N} - \left(\frac{\sum f_i x_i}{N}\right)^2$
= $\sum p_i x_i^2 - (\sum p_i x_i)^2$
= $E(X^2) - (E(X))^2$

Discrete Random Variable Summary Measures

- **Expected Value (or mean)** of a discrete distribution (Weighted Average)

$$\mu = E(X) = \sum_{i=1}^N X_i P(X_i)$$

– **Example:** Toss 2 coins,
 $X =$ no. of heads,
compute expected value of X :

X	$P(X)$
0	.25
1	.50
2	.25

$$E(X) = (0 \times .25) + (1 \times .50) + (2 \times .25) = 1.0$$

$$= 1.0$$

Discrete Random Variable Summary Measures

(continued)

- **Variance** of a discrete random variable

$$\sigma^2 = \sum_{i=1}^N [X_i - E(X)]^2 P(X_i)$$

- **Standard Deviation** of a discrete random variable

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_{i=1}^N [X_i - E(X)]^2 P(X_i)}$$

where:

$E(X)$ = Expected value of the discrete random variable X

X_i = the i^{th} outcome of X

$P(X_i)$ = Probability of the i^{th} occurrence of X

Discrete Random Variable Summary Measures

(continued)

- **Example:** Toss 2 coins, X = no. of heads, compute standard deviation (recall $E(X) = 1$)

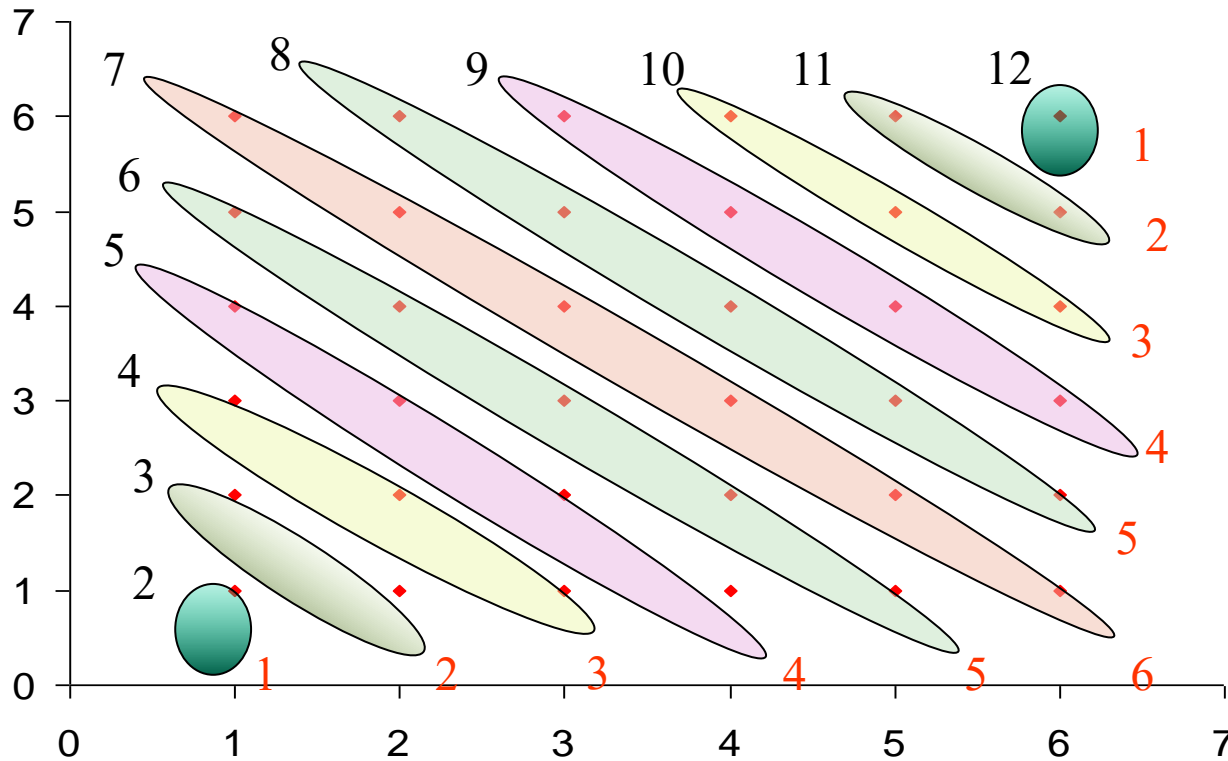
$$\sigma = \sqrt{\sum [X_i - E(X)]^2 P(X_i)}$$

$$\sigma = \sqrt{(0-1)^2(.25) + (1-1)^2(.50) + (2-1)^2(.25)} = \sqrt{.50} = .707$$

Possible number of heads = 0,
1, or 2

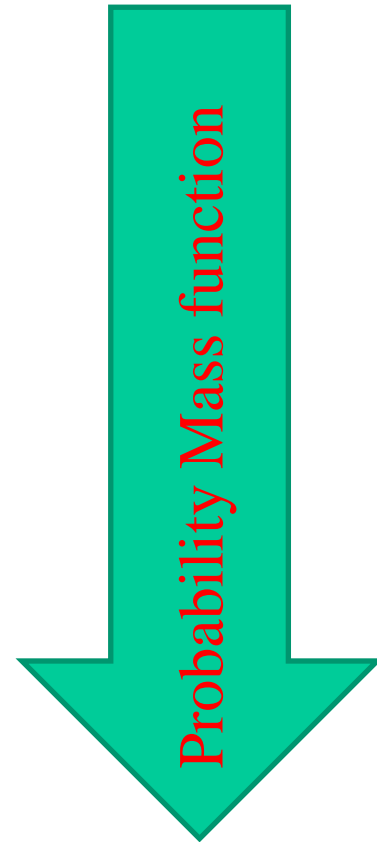
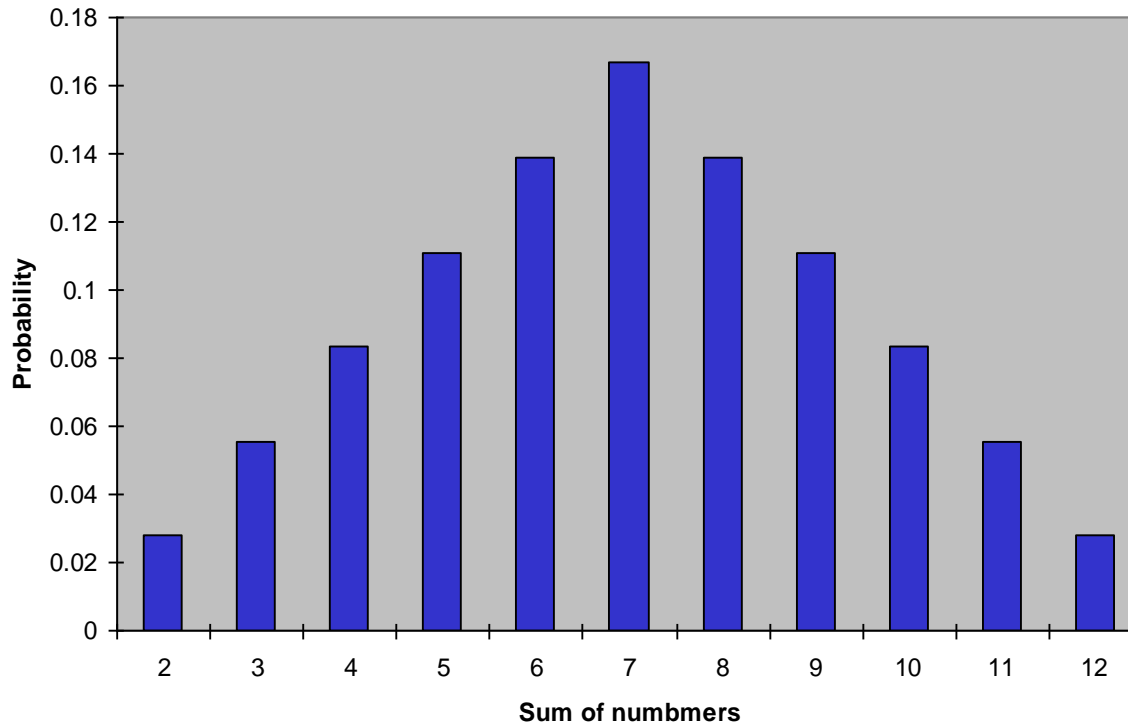


The sum of numbers showing on two true dices.



X	2	3	4	5	6	7	8	9	10	11	12	Total
$P_X(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

The probability distribution can be also presented by a graph.



And by a mathematical function: $P_X(x) = \frac{6 - |7 - x|}{36}$

Discrete Random Variable x

Discrete Random Variable

- **Example** For a random variable X , the probability mass function is

$$f(x) = kx, \text{ for } x = 1, 2, \dots, n \\ = 0, \text{ otherwise}$$

Find expectation of X .

Solution: Here $f(x)$ denotes probability mass function

$$\therefore \sum_{x=1}^n f(x) = \sum_{x=1}^n kx = k \sum_{x=1}^n x = 1$$

$$\Rightarrow k \frac{n(n+1)}{2} = 1 \quad \left[\because 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} \right]$$

$$\Rightarrow k = \frac{2}{n(n+1)}$$

Discrete Random Variable

$$\begin{aligned}\therefore E(X) &= \sum_{x=1}^n x f(x) = \sum_{x=1}^n x k x \\ &= \sum_{x=1}^n x^2 \frac{2}{n(n+1)} \\ &= \frac{2}{n(n+1)} \sum_{x=1}^n x^2 \\ &= \frac{2}{n(n+1)} (1^2 + 2^2 + \dots + n^2) \\ &= \frac{2}{n(n+1)} \frac{n(n+1)(2n+1)}{6} = \frac{(2n+1)}{3}\end{aligned}$$

Discrete Random Variable

5. A random variable X has the following probability distribution :

Values of X, x	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the value of a .

(ii) Find $P(X < 3)$, $P(X \geq 3)$, $P(0 < X < 5)$.

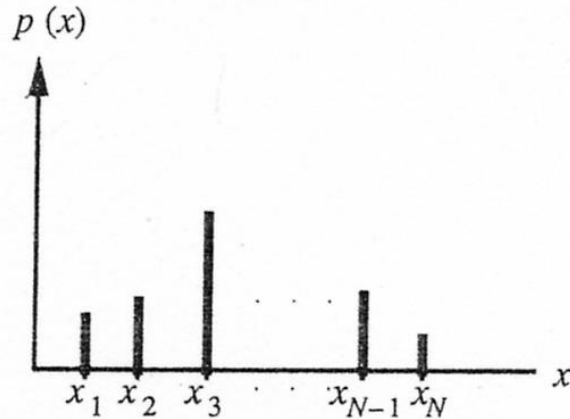
(iii) What is the smallest value of x for which $P(X \leq x) > 0.5$? and

(iv) Find out the distribution function of X ?

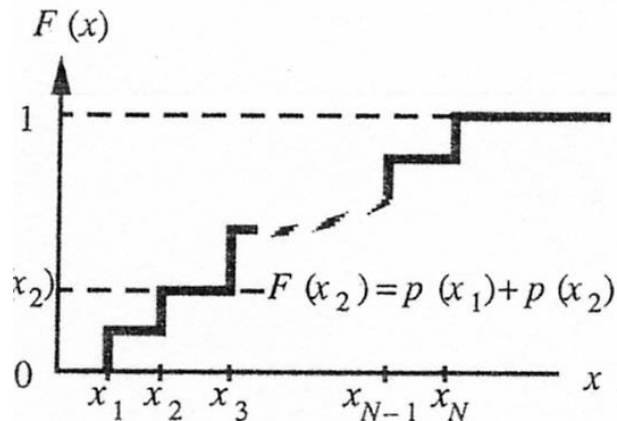
Ans. (i) $a = 1/81$, (ii) $9/81, 72/81, 24/81$, (iii) 6

Graphical Representation of pdf and cdf

Discrete Random Variables

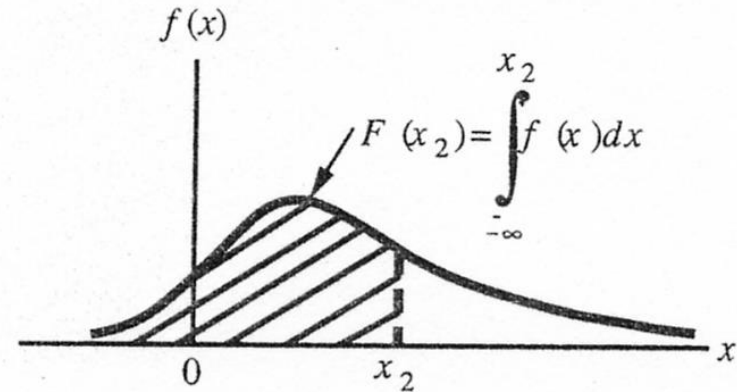


(a) Probability mass function (PMF) of a discrete random variable.

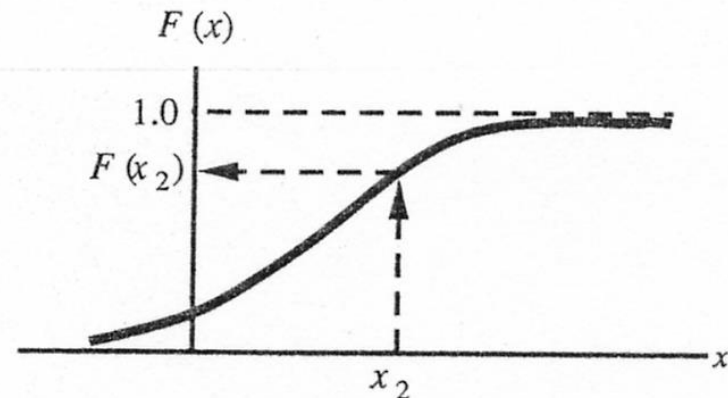


(b) Cumulative distribution function (CDF) of a discrete random variable.

Continuous Random Variables

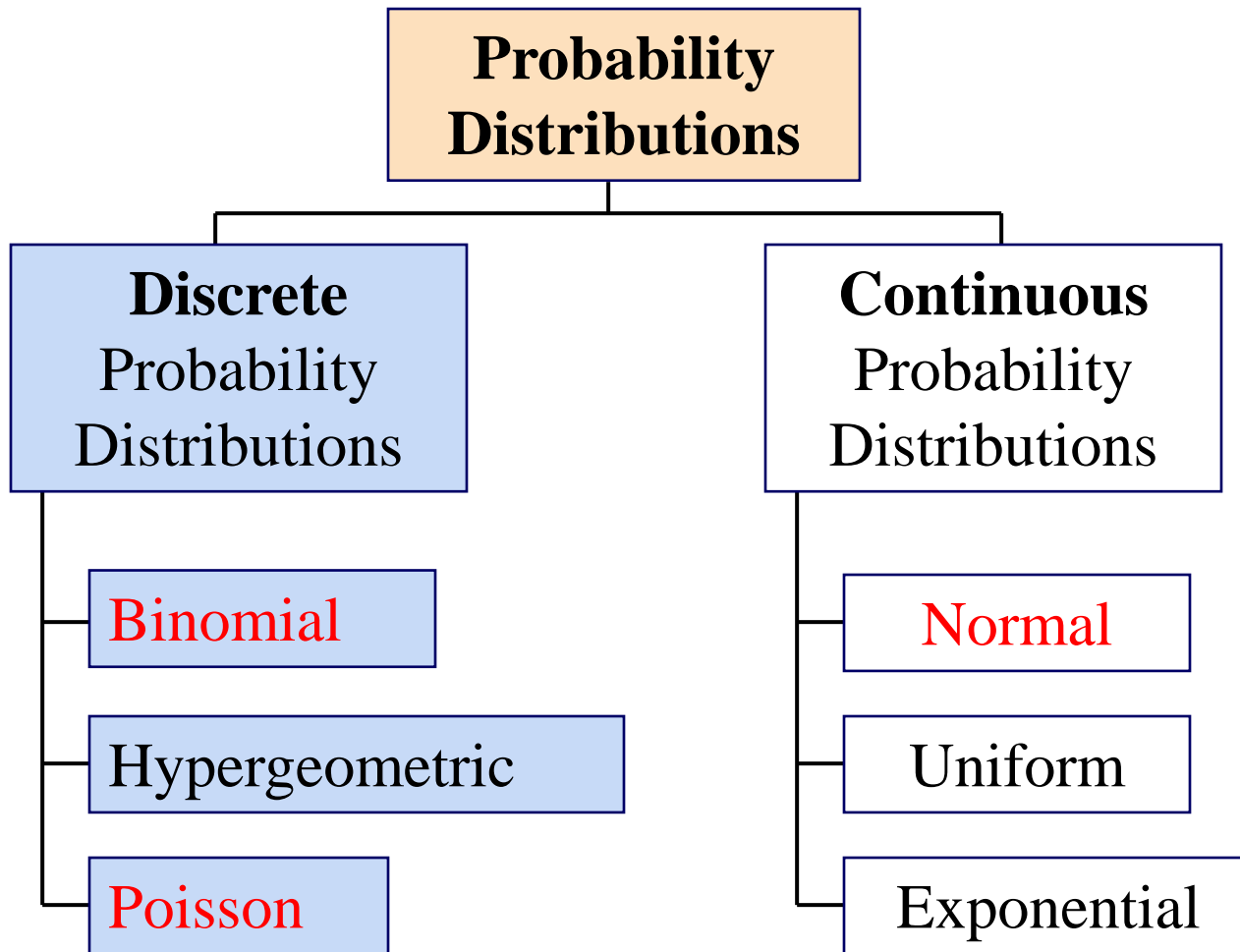


(c) Probability density function (PDF).



(d) Cumulative distribution function (CDF).

Probability Distributions



The Binomial Distribution

- The binomial experiment can result in only one of two possible outcomes.
- Typical cases where the binomial experiment applies:
 - A coin flipped results in heads or tails
 - An election candidate wins or loses
 - An employee is male or female

Binomial Distribution: $B(n, p)$

Consider a random sequence of n Bernoulli trials, each with:

Probability of success = p

Probability of failure = $q = 1 - p$

we evaluate the number of r success in the n trials by the number of combinations.

An outcome of the experiment like,

$\overbrace{SSS\dots SS}^{r \text{ times}} \overbrace{fff\dots ff}^{n-r \text{ times}}$

$$p^r q^{n-r}$$

There are $\binom{n}{r}$ of such combinations

each having a probability $p^r q^{n-r}$

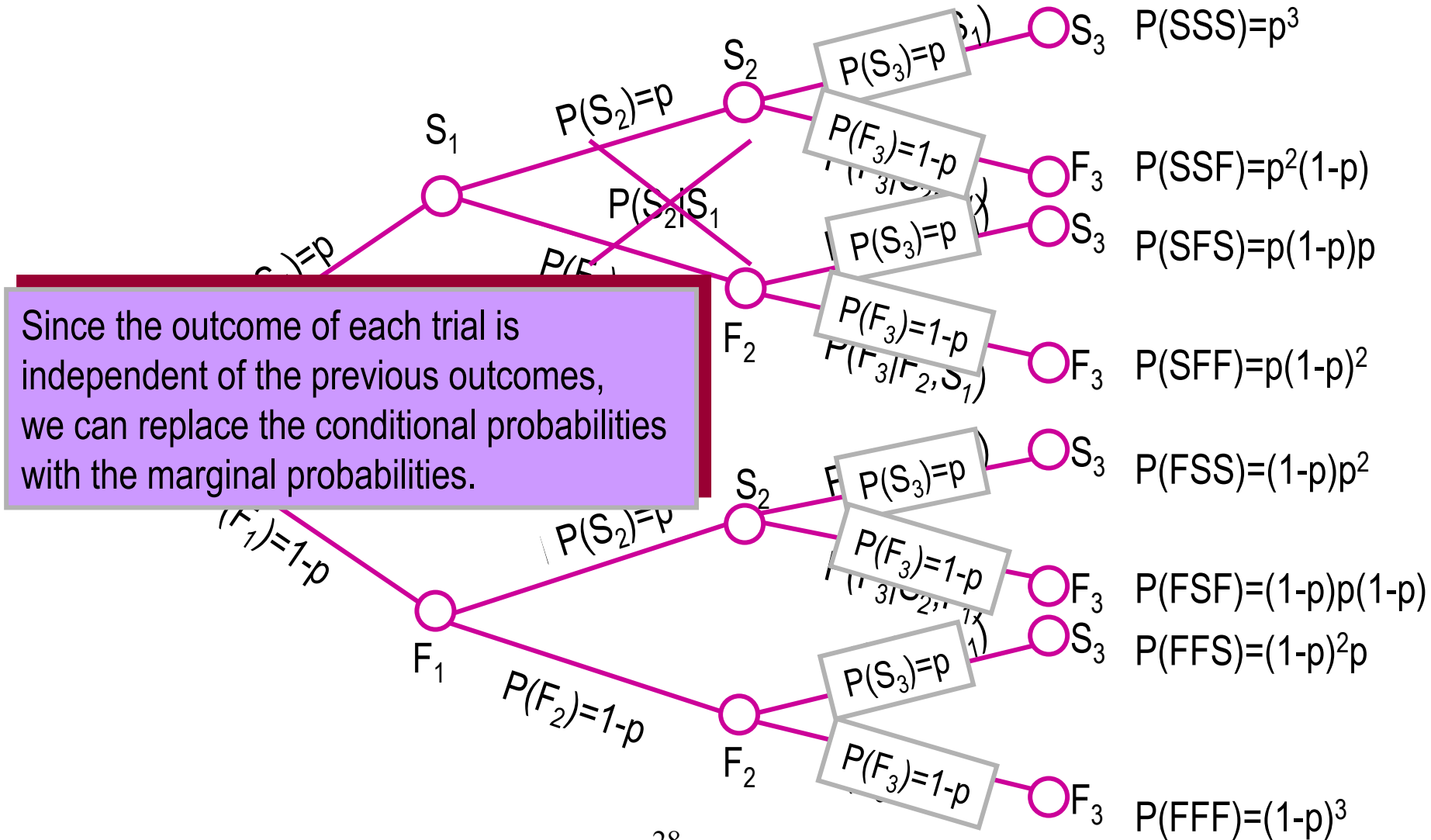
$$\Pr(X = r) = \binom{n}{r} p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r q^{n-r}, r = 0, 1, \dots, n$$

Example: throwing the coin several times (n times).

Binomial Experiment

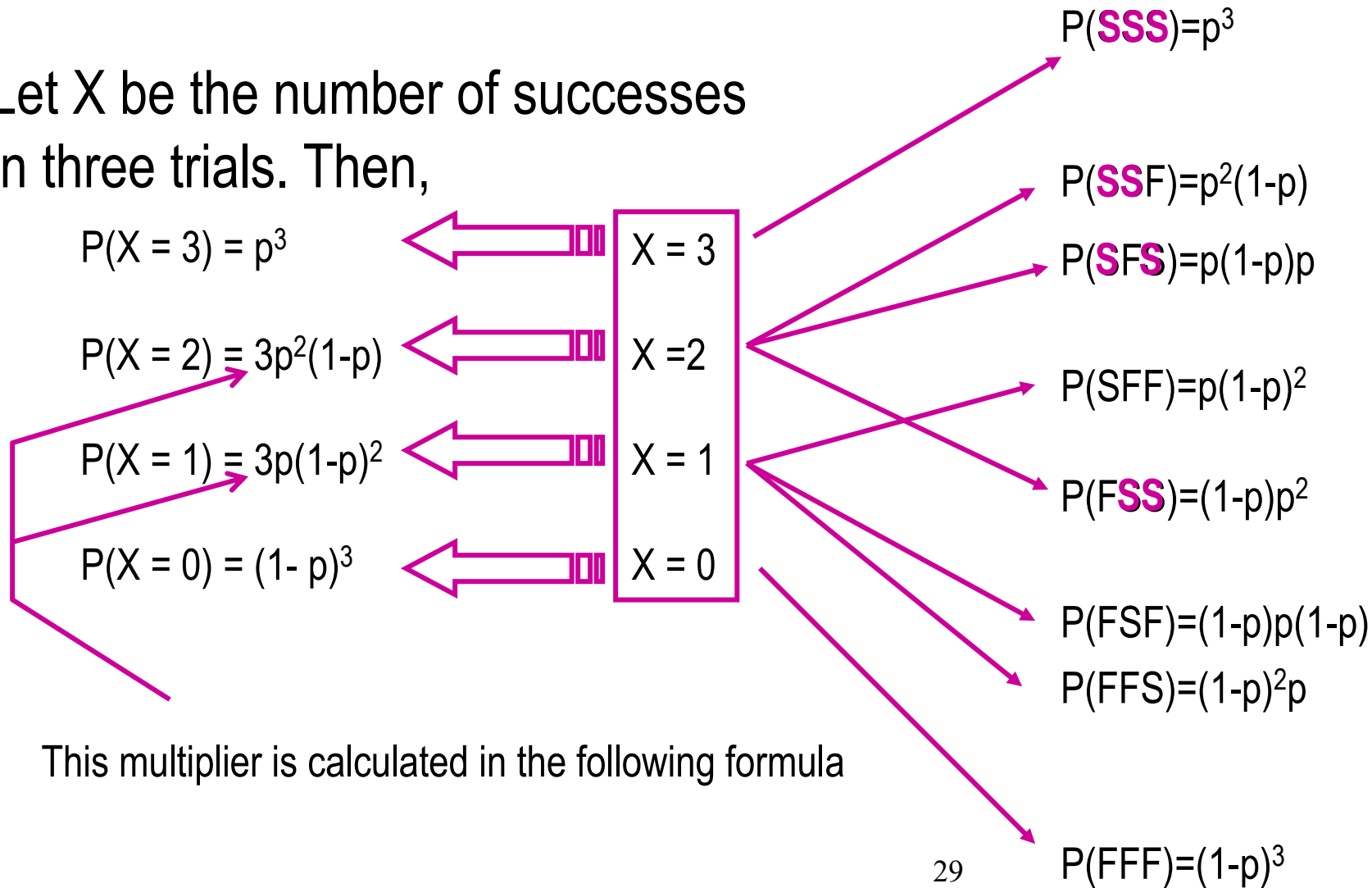
- There are n trials (n is finite and fixed).
- Each trial can result in a success or a failure.
- The probability p of success is the same for all the trials.
- All the trials of the experiment are independent.
- **Binomial Random Variable**
 - The binomial random variable *counts* the number of successes in n trials of the binomial experiment.
 - By definition, this is a discrete random variable.

Developing the Binomial Probability Distribution ($n = 3$)



Developing the Binomial Probability Distribution (n = 3)

Let X be the number of successes in three trials. Then,



Calculating the Binomial Probability

In general, The binomial probability is calculated by:

$$P(X = x) = p(x) = C_x^n p^x (1-p)^{n-x}$$

$$\text{where } C_x^n = \frac{n!}{x!(n-x)!}$$

Calculating the Binomial Probability

- **Example**

- A student is registered in a statistics course and intends to rely on luck to pass the next quiz.
- The quiz consists on 10 multiple choice questions with 5 possible choices for each question, only one of which is the correct answer.
- The student will guess the answer to each question
- Find the following probabilities
 - The student gets no answer correct
 - The student gets two answer correct?
 - The student fails the quiz

Calculating the Binomial Probability

- Solution
 - Checking the conditions
 - An answer can be either correct or incorrect.
 - There is a fixed finite number of trials ($n=10$)
 - Each answer is independent of the others.
 - The probability p of a correct answer (.20) does not change from question to question.

Calculating the Binomial Probability

- Solution - Continued
 - Determining the binomial probabilities:
Let X = the number of correct answers

$$P(X = 0) = \frac{10!}{0!(10-0)!} (.20)^0 (.80)^{10-0} = .1074$$

$$P(X = 2) = \frac{10!}{2!(10-2)!} (.20)^2 (.80)^{10-2} = .3020$$

Calculating the Binomial Probability

- Solution - Continued
 - Determining the binomial probabilities:
the student fails the test if the number of correct answers is less than 5, which means less than or equal to 4.

$$\begin{aligned}P(X \leq 4) &= p(0) + p(1) + p(2) + p(3) + p(4) \\ &= .1074 + .2684 + .3020 + .2013 + .0881 \\ &= .9672\end{aligned}$$

This is called cumulative probability

Calculating the Binomial Probability

What is the probability of one success in five observations if the probability of success is .1?

$$X = 1, n = 5, \text{ and } p = .1$$

$$\begin{aligned} P(X = 1) &= \frac{n!}{X!(n - X)!} p^X (1 - p)^{n - X} \\ &= \frac{5!}{1!(5 - 1)!} (.1)^1 (1 - .1)^{5 - 1} \\ &= (5)(.1)(.9)^4 \\ &= .32805 \end{aligned}$$

Calculating the Binomial Probability

Example

If the probability of success of an event is $\frac{1}{20}$; how many trials are required in order that the probability of getting at least one success, is just greater than $\frac{1}{2}$?

Solution: Here $p = \frac{1}{20}$, $q = \frac{19}{20}$

Let n be the required number of trials such that the probability of getting at least one success, is just greater than $\frac{1}{2}$

$$\text{i.e. } P(X \geq 1) > \frac{1}{2}$$

$$\Rightarrow 1 - P(X = 0) > \frac{1}{2}$$

Calculating the Binomial Probability

$$\Rightarrow P(X = 0) < 1 - \frac{1}{2}$$

$$\Rightarrow {}^n C_0 \left(\frac{19}{20}\right)^n \left(\frac{1}{20}\right)^0 < \frac{1}{2}$$

$$\Rightarrow \left(\frac{19}{20}\right)^n < \frac{1}{2}$$

$$\Rightarrow n \log_{10} \frac{19}{20} < \log_{10} \frac{1}{2}$$

$$\Rightarrow n > \frac{\log_{10} \frac{1}{2}}{\log_{10} \frac{19}{20}} \quad \because \log_{10} \frac{19}{20} < 0$$

$$\Rightarrow n > \frac{-0.3010}{-0.02228} = 13.5099 \quad \therefore n = 14$$

Calculating the Binomial Probability

Example

6 dice are thrown 729 times. How many times would you expect at least three dice to show 1 or 2?

Solution: Here the Binomial Distribution (B.D.) is given by $N(q + p)^n$

$$\text{Where } p = \frac{2}{6} = \frac{1}{3}, q = \frac{2}{3}, n = 6, N = 729$$

\therefore B.D. is given by $729 \left(\frac{2}{3} + \frac{1}{3} \right)^6$ and if X is random variable showing number of successes, then

Calculating the Binomial Probability

$$P(X \geq 3) = 729 \left[{}^6C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3 + {}^6C_4 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^4 + {}^6C_5 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^5 + {}^6C_6 \left(\frac{1}{3}\right)^6 \right]$$
$$= \frac{729}{3^6} [160 + 60 + 12 + 1] = 233$$

Calculating the Binomial Probability

Example

The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men, now 60, at least 7 will live to be 70 ?

Solution. The probability that a man aged 60 will live to be 70 = $p = 0.65$

$$q = 1 - p = 1 - 0.65 = 0.35$$

Number of men = $n = 10$

Probability that at least 7 men (7 or 8 or 9 or 10) will live to 70

$$= P(7) + P(8) + P(9) + P(10) = {}^{10}C_7 q^3 p^7 + {}^{10}C_8 q^2 p^8 + {}^{10}C_9 q p^9 + p^{10}$$

Calculating the Binomial Probability

$$\begin{aligned} &= \frac{10 \times 9 \times 8}{1 \times 2 \times 3} (0.35)^3 (0.65)^7 + \frac{10 \times 9}{1 \times 2} (0.35)^2 (0.65)^8 + 10 (0.35) (0.65)^9 + (0.65)^{10} \\ &= (0.65)^7 [120 (0.35)^3 + 45 (0.35)^2 (0.65) + 10 (0.35) (0.65)^2 + (0.65)^3] \\ &= (0.65)^7 \times 125 [120 \times (0.07)^3 + 45 \times (0.07)^2 (0.13) + 10 (0.07) (0.13)^2 + (0.13)^3] \\ &= 0.04902 \times 125 [0.04 + 0.028665 + 0.011830 + 0.002197] \\ &= 6.1275 \times 0.082692 = 0.5067 \end{aligned}$$

Ans.

Mean of Binomial Distribution

Mean of Binomial Distribution

$$\text{Mean} = \sum_{r=0}^n r P(r)$$

$$= \sum_{r=0}^n r {}^n C_r q^{n-r} p^r$$

$$= {}^n C_1 q^{n-1} p^1 + 2 {}^n C_2 q^{n-2} p^2 + 3 {}^n C_3 q^{n-3} p^3 + \dots + n {}^n C_n q^0 p^n$$

$$= n q^{n-1} p + \frac{2n(n-1)}{2!} q^{n-2} p^2 + \frac{3n(n-1)(n-2)}{3!} q^{n-3} p^3 + \dots + n p^n$$

$$= np \left[q^{n-1} + (n-1) q^{n-2} p + \frac{(n-1)(n-2)}{2!} q^{n-3} p^2 + \dots + p^{n-1} \right]$$

Mean of Binomial Distribution

$$= np \left[{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2}p + {}^{n-1}C_2 q^{n-3}p^2 + \dots + {}^{n-1}C_{n-1} p^{n-1} \right]$$

$$= np (q + p)^{n-1}$$

$$= np \quad \because q + p = 1$$

Variance of Binomial Distribution

Variance of Binomial Distribution

$$\text{Variance} = \sum_{r=0}^n r^2 P(r) - (\text{mean})^2$$

$$\text{Now } \sum_{r=0}^n r^2 P(r) = \sum_{r=0}^n r^2 {}^n C_r q^{n-r} p^r$$

$$= {}^n C_1 q^{n-1} p + 2^2 {}^n C_2 q^{n-2} p^2 + 3^2 {}^n C_3 q^{n-3} p^3 + \dots + n^2 {}^n C_n p^n$$

$$= n q^{n-1} p + \frac{4n(n-1)}{2!} q^{n-2} p^2 + \frac{9n(n-1)(n-2)}{3!} q^{n-3} p^3 + \dots + n^2 p^n$$

$$= np \left[q^{n-1} + 2(n-1) q^{n-2} p + \frac{3(n-1)(n-2)}{2!} q^{n-3} p^2 + \dots + np^{n-1} \right]$$

Variance of Binomial Distribution

$$\begin{aligned} &= np \left[q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2!} q^{n-3}p^2 + \dots + p^{n-1} \right] \\ &+ np \left[(n-1)q^{n-2}p + (n-1)(n-2)q^{n-3}p^2 + \dots + (n-1)p^{n-1} \right] \\ &= np \left[{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2}p + {}^{n-1}C_2 q^{n-3}p^2 + \dots + {}^{n-1}C_{n-1} p^{n-1} \right] \\ &+ np(n-1)p \left[{}^{n-2}C_0 q^{n-2} + {}^{n-2}C_1 q^{n-3}p + \dots + {}^{n-2}C_{n-2} p^{n-2} \right] \end{aligned}$$

Variance of Binomial Distribution

$$= np[(q + p)^{n-1} + (n - 1)p(q + p)^{n-2}]$$

$$= np[1 + (n - 1)p] \quad \because q + p = 1$$

$$\therefore \text{Variance} = np[1 + np - p] - n^2p^2 = np[q + np] - n^2p^2 \quad \because 1 - p = q$$

$$= npq$$

Mean and Variance of Binomial Variable

$$\begin{aligned} E(X) &= \mu = np \\ V(X) &= \sigma^2 = np(1-p) \end{aligned}$$

- **Example**

- If all the students in class intend to guess the answers to the quiz, what is the mean and the standard deviation of the quiz mark?

- **Solution**

- $m = np = 10(.2) = 2.$
- $s = [np(1-p)]^{1/2} = [10(.2)(.8)]^{1/2} = 1.26.$

Calculating the Binomial Probability

Example

If mean and variance of a binomial distribution are 4 and 2 respectively, find the probability of (i) exactly 2 successes (ii) less than 2 successes (iii) at least 2 successes.

Solution.

$$\text{Mean} = 4 \quad \Rightarrow \quad np = 4$$

$$\text{Variance} = 2 \quad npq = 2$$

Dividing (2) by (1), we get

$$\frac{npq}{np} = \frac{2}{4} \quad \Rightarrow \quad q = \frac{1}{2}$$

$$p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$$

Putting the value of p in (1), we get

$$n \left(\frac{1}{2} \right) = 4 \quad \Rightarrow \quad n = 8$$

Calculating the Binomial Probability

(i) Probability of r successes $= {}^n C_r p^r q^{n-r}$

$$P(2) = {}^8 C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{8-2} = {}^8 C_2 \left(\frac{1}{2}\right)^8 = \frac{8 \times 7}{2} \frac{1}{256} = \frac{7}{64}$$

(ii) P (less than 2 successes) $= P(0) + P(1) = {}^8 C_0 p^0 q^8 + {}^8 C_1 p^1 q^7$

$$= \frac{1}{256} + 8 \frac{1}{2} \left(\frac{1}{2}\right)^7 = \frac{9}{256}$$

(iii) P (at least 2 successes) $= P(2) + P(3) + \dots + P(8)$

$$= P(0) + P(1) + P(2) + P(3) + \dots + P(8) - P(0) - P(1)$$

$$= 1 - P(0) - P(1) = 1 - [P(0) + P(1)] = 1 - \frac{9}{256} = \frac{247}{256}$$

Ans.

Binomial Distribution $B(n,p)$

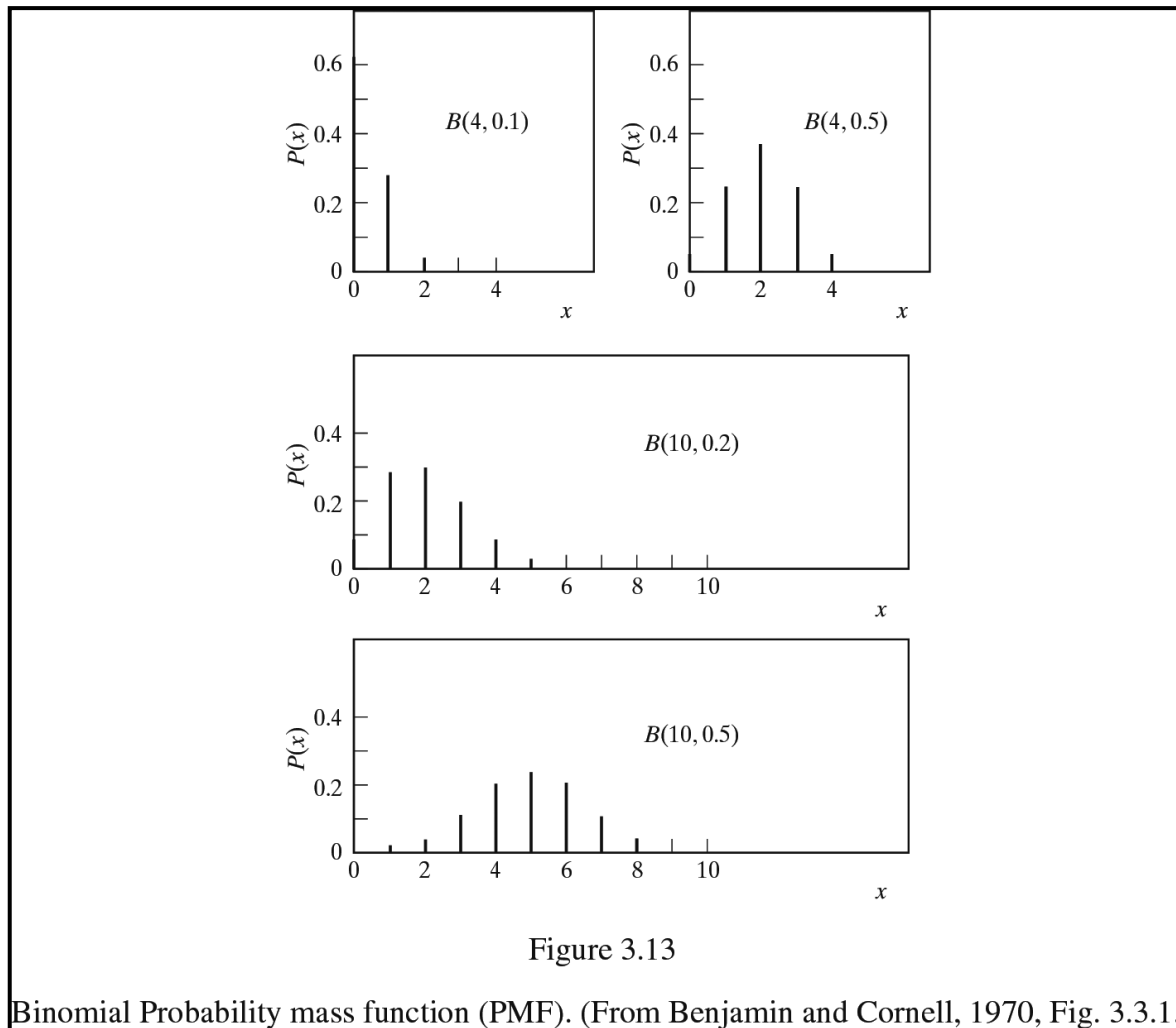


Chart 1

$$P(X = x) = p(x) = C_x^n p^x (1 - p)^{n-x}$$

Binomial distribution for n=4 and p=0.9

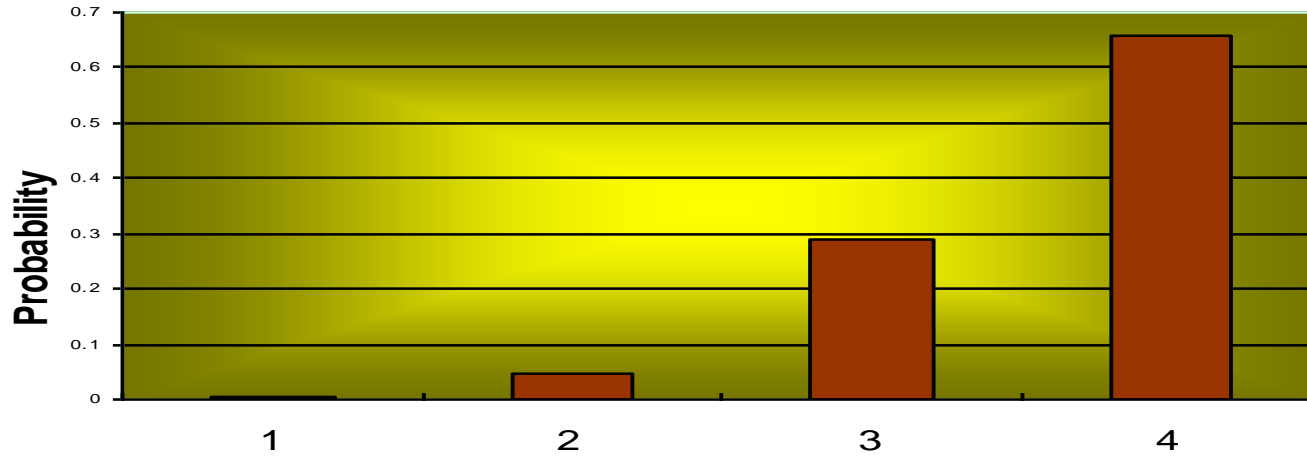
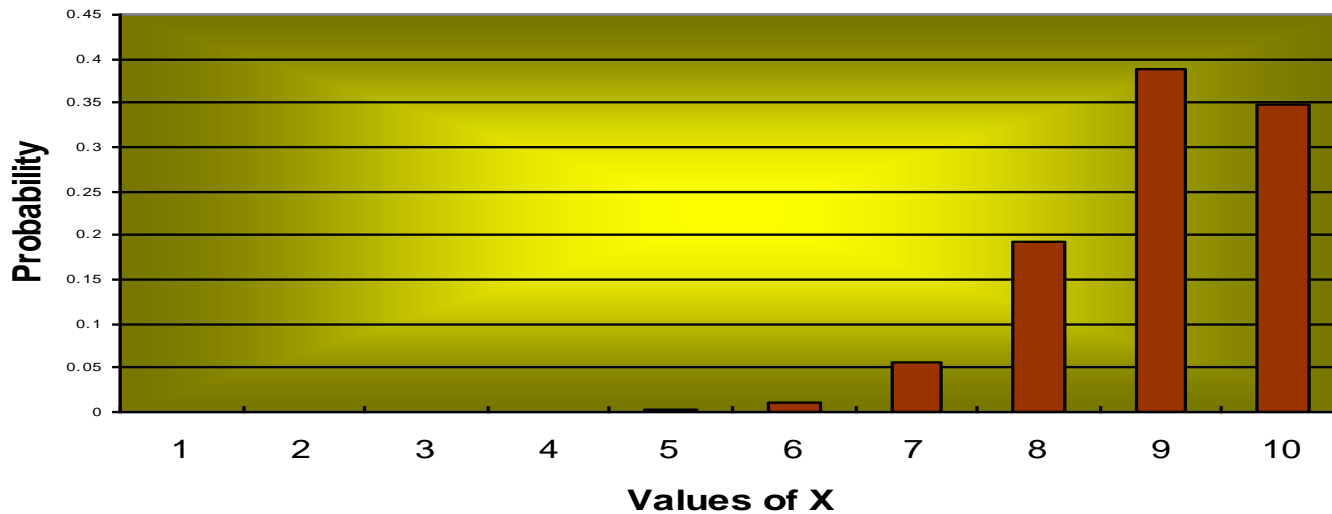


Chart 2

Binomial distribution for n=10 and p=0.9



$$P(X = x) = p(x) = C_x^n p^x (1 - p)^{n-x}$$

Chart 3

Binomial distribution for n=20 and p=0.9

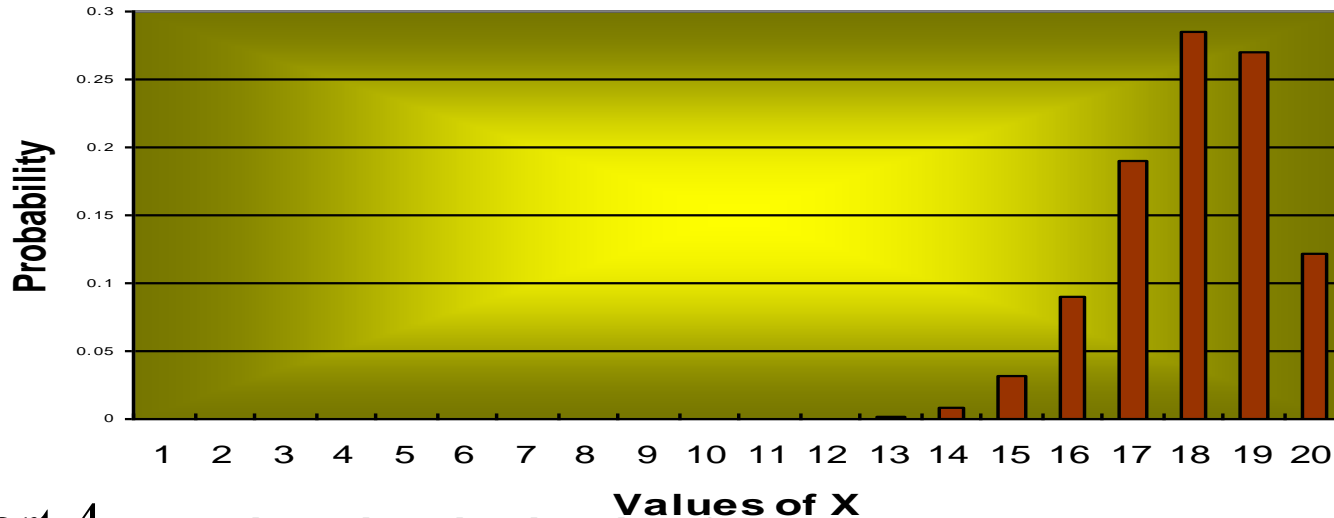
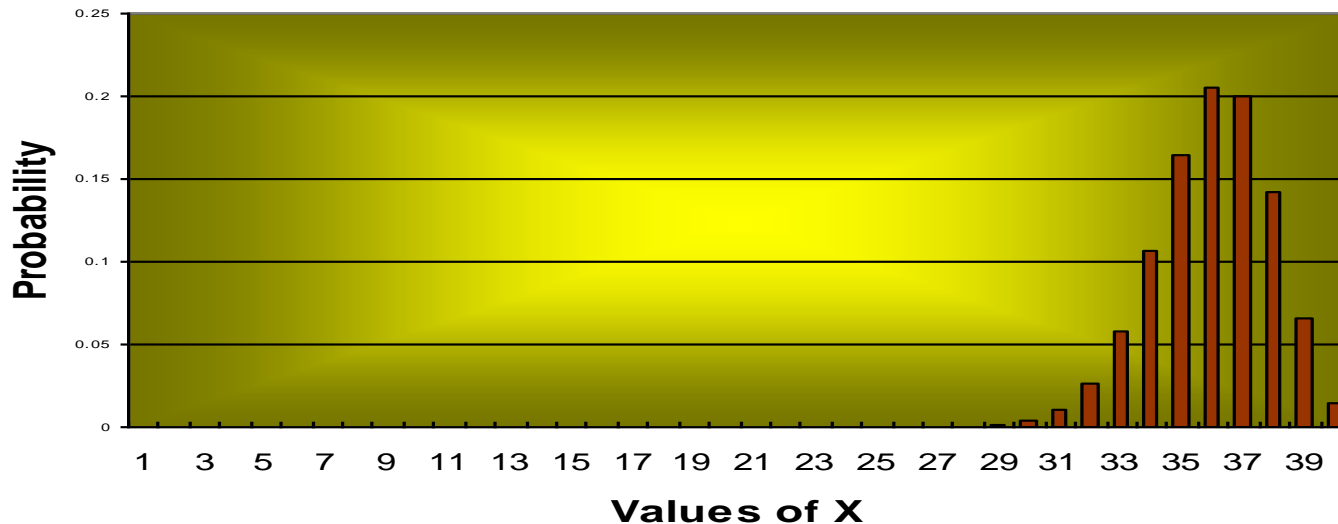


Chart 4

Binomial distribution for n=40 and p=0.9



$$P(X = x) = p(x) = C_x^n p^x (1 - p)^{n-x}$$

Chart 5

Binomial distribution for n=80 and p=0.9

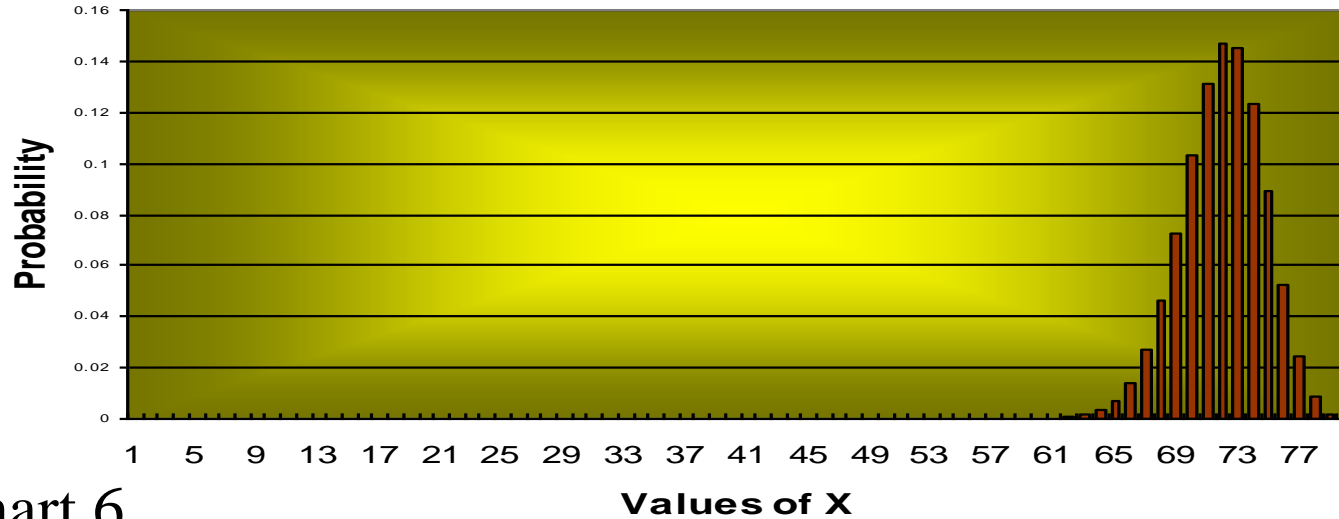
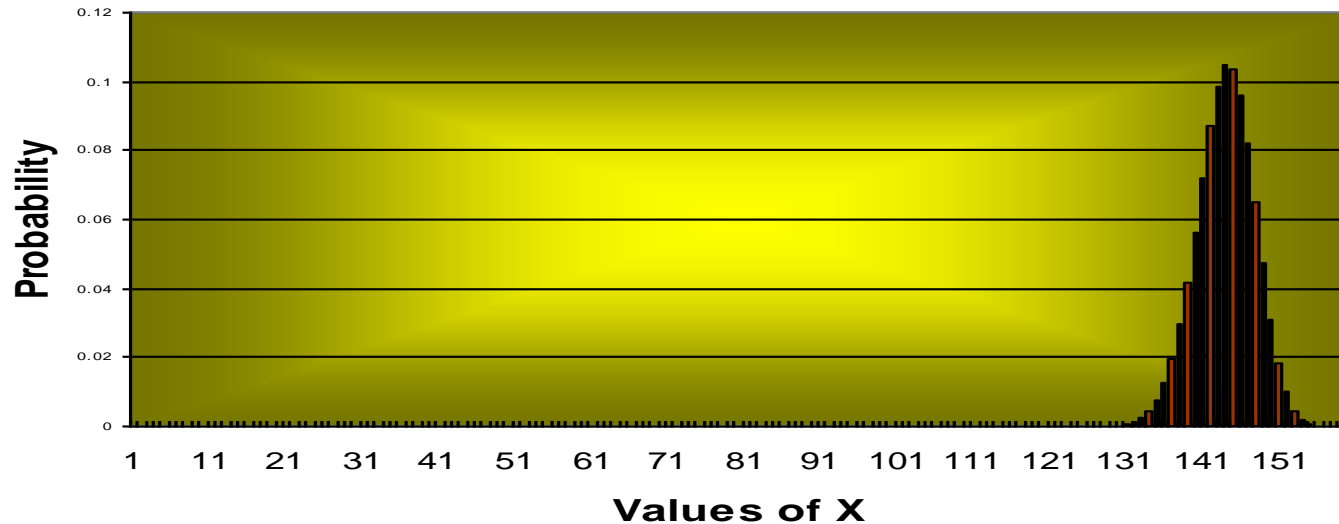


Chart 6

Binomial distribution for n=160 and p=0.9



$$P(X = x) = p(x) = C_x^n p^x (1 - p)^{n-x}$$

Chart 7

Binomial distribution for n=160 and p=0.7

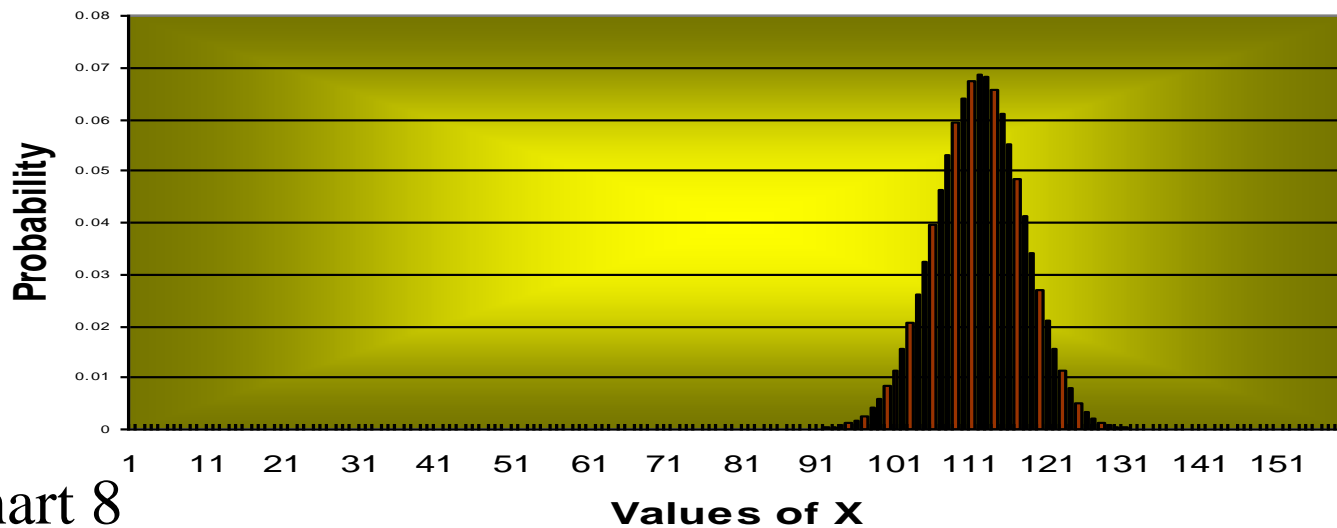
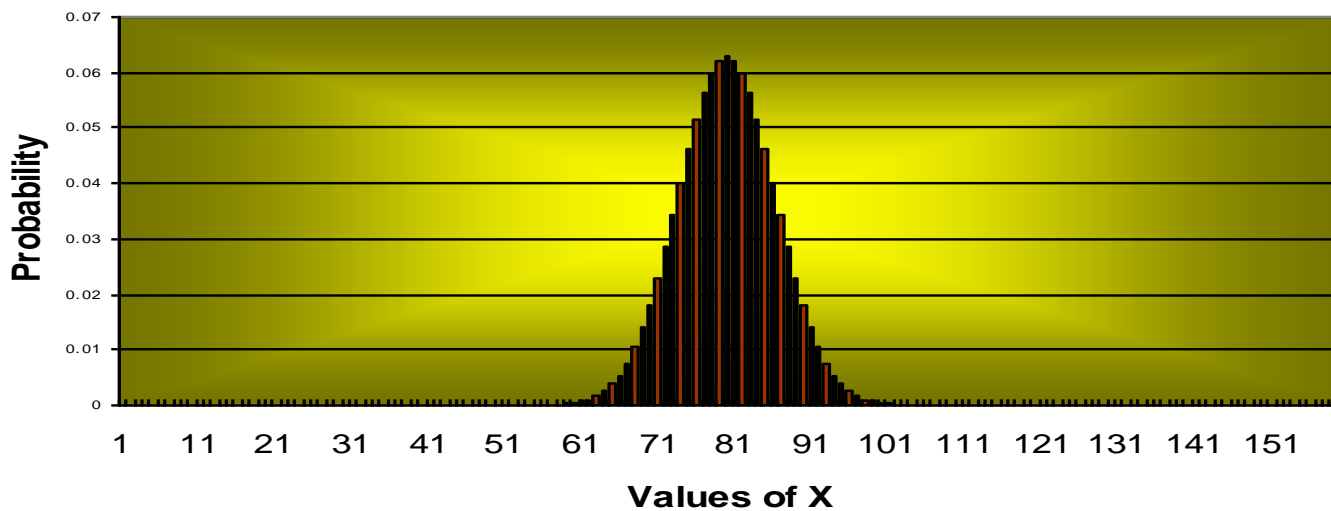


Chart 8

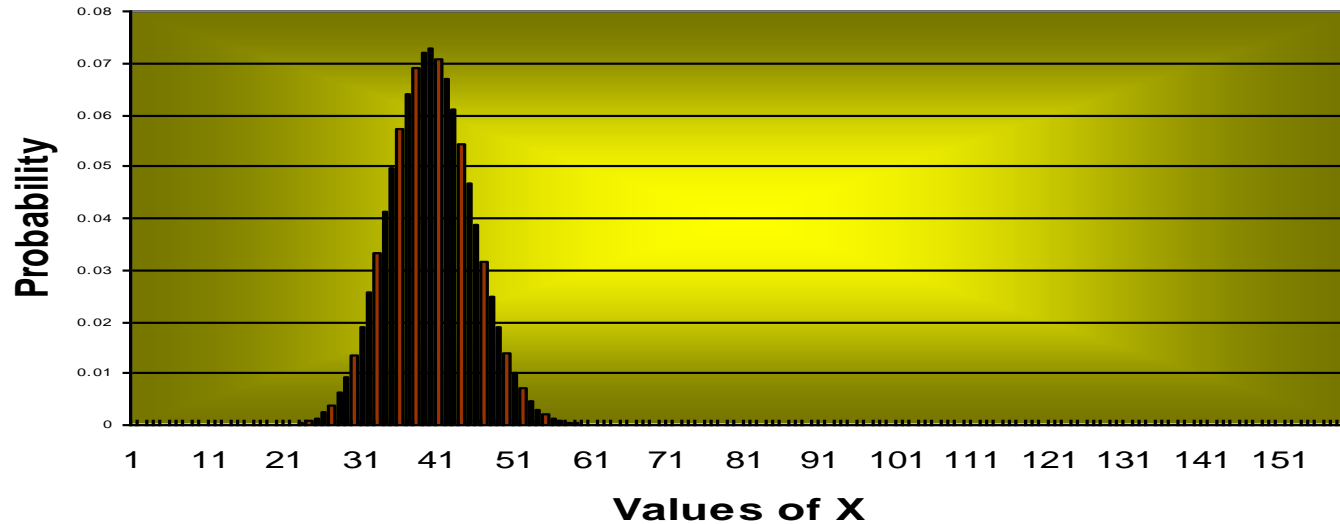
Binomial distribution for n=160 and p=0.5



$$P(X = x) = p(x) = C_x^n p^x (1-p)^{n-x}$$

Chart 9

Binomial distribution for n=160 and p=0.25



Poisson Distribution

- The Poisson experiment typically fits cases of rare events that occur over a fixed amount of time or within a specified region
- Typical cases
 - The number of errors a typist makes per page
 - The number of customers entering a service station per hour
 - The number of telephone calls received by a switchboard per hour.
 - The number of Rainy days in Rajshahi within a year.

Properties of the Poisson Experiment

- The number of successes (events) that occur in a certain time interval is independent of the number of successes that occur in another time interval.
- The probability of a success in a certain time interval is: the same for all time intervals of the same size, proportional to the length of the interval.
- The probability that two or more successes will occur in an interval approaches zero as the interval becomes smaller.

Poisson Distribution

Poisson distribution with $P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$ is a limiting case of Binomial distribution, under the conditions **i.** $n \rightarrow \infty$ **ii.** $p \rightarrow 0$ **iii.** $np = \lambda$ is finite

Proof: In a Binomial distribution

$$\begin{aligned} P(r) &= {}^n C_r q^{n-r} p^r \\ &= {}^n C_r (1-p)^{n-r} p^r \\ &= {}^n C_r \left(1 - \frac{\lambda}{n}\right)^{n-r} \left(\frac{\lambda}{n}\right)^r \quad \because np = \lambda \end{aligned}$$

Poisson Distribution

$$\begin{aligned} &= \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} \left(1 - \frac{\lambda}{n}\right)^{n-r} \left(\frac{\lambda}{n}\right)^r \\ &= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(r-1)}{n}\right)}{\left(1-\frac{\lambda}{n}\right)^r r!} \left(1 - \frac{\lambda}{n}\right)^n \lambda^r \end{aligned}$$

Taking limit as $n \rightarrow \infty$

$$P(r) = \frac{\lambda^r}{r!} \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n$$

Poisson Distribution

$$= \frac{\lambda^r}{r!} \lim_{n \rightarrow \infty} \left[\left(1 - \frac{\lambda}{n} \right)^{\frac{-n}{\lambda}} \right]^{-\lambda}$$

$$\therefore P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e$$

Poisson Distribution

Mean of Poisson Distribution

$$\text{Mean} = \sum_{r=0}^{\infty} rP(r)$$

$$= \sum_{r=0}^{\infty} \frac{re^{-\lambda}\lambda^r}{r!}$$

$$= \frac{e^{-\lambda}\lambda^1}{1!} + \frac{2e^{-\lambda}\lambda^2}{2!} + \frac{3e^{-\lambda}\lambda^3}{3!} + \frac{4e^{-\lambda}\lambda^4}{4!} + \dots$$

$$= e^{-\lambda} \lambda \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right)$$

$$= e^{-\lambda} \lambda e^{\lambda} = \lambda$$

Poisson Distribution

Variance of Poisson Distribution

$$\text{Variance} = \sum_{r=0}^n r^2 P(r) - (\text{mean})^2$$

$$\begin{aligned}\text{Now } \sum_{r=0}^n r^2 P(r) &= \sum_{r=0}^{\infty} \frac{r^2 e^{-\lambda} \lambda^r}{r!} \\ &= \frac{e^{-\lambda} \lambda^1}{1!} + \frac{2^2 e^{-\lambda} \lambda^2}{2!} + \frac{3^2 e^{-\lambda} \lambda^3}{3!} + \frac{4^2 e^{-\lambda} \lambda^4}{4!} + \dots \\ &= e^{-\lambda} \lambda \left(1 + 2\lambda + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \dots \right)\end{aligned}$$

Poisson Distribution

$$= e^{-\lambda} \lambda \left[\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right) + \left(\lambda + \lambda^2 + \frac{3\lambda^3}{3!} + \dots \right) \right]$$

$$= e^{-\lambda} \lambda \left[\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right) + \lambda \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right) \right]$$

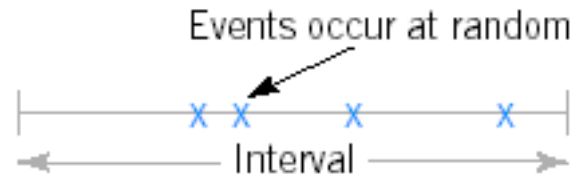
$$= e^{-\lambda} \lambda [e^{\lambda} + \lambda e^{\lambda}]$$

$$\Rightarrow \sum_{r=0}^n r^2 P(r) = e^{-\lambda} \lambda e^{\lambda} [1 + \lambda] = \lambda + \lambda^2$$

$$\therefore \text{Variance} = \lambda + \lambda^2 - \lambda^2 = \lambda$$

The Poisson Variable and Distribution

Figure 3-32 In a Poisson process, events occur at random in an interval.



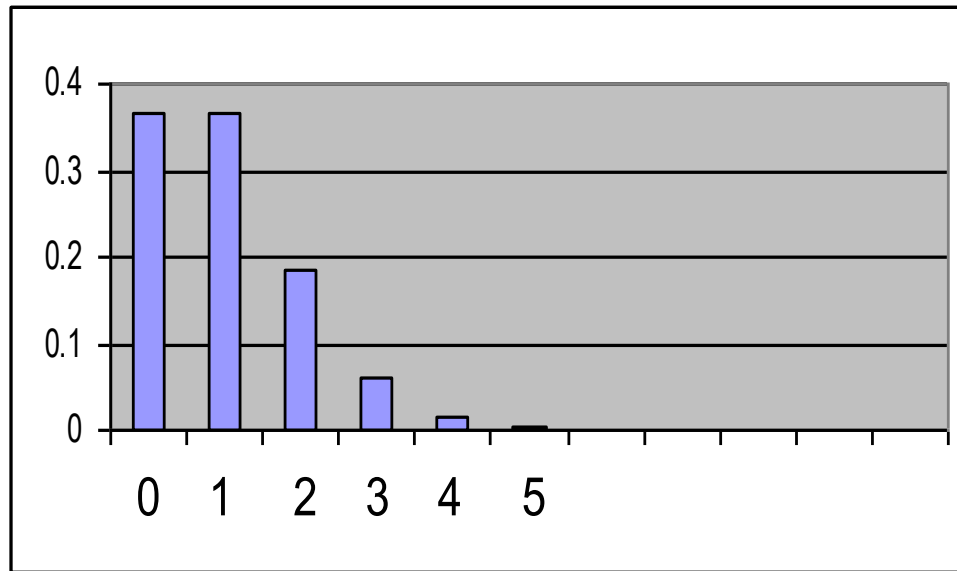
- **The Poisson Random Variable**
 - The Poisson variable indicates the number of successes that occur during a given time interval or in a specific region in a Poisson experiment

- **Probability Distribution of the Poisson Random Variable.**

$$P(X = x) = p(x) = \frac{e^{-\mu} \mu^x}{x!} \quad x = 0, 1, 2, \dots$$

$$E(X) = V(X) = \mu$$

Poisson Distributions (Graphs)



$$P(X = 0) = p(0) = \frac{e^{-1}1^0}{0!} = e^{-1} = .3678$$

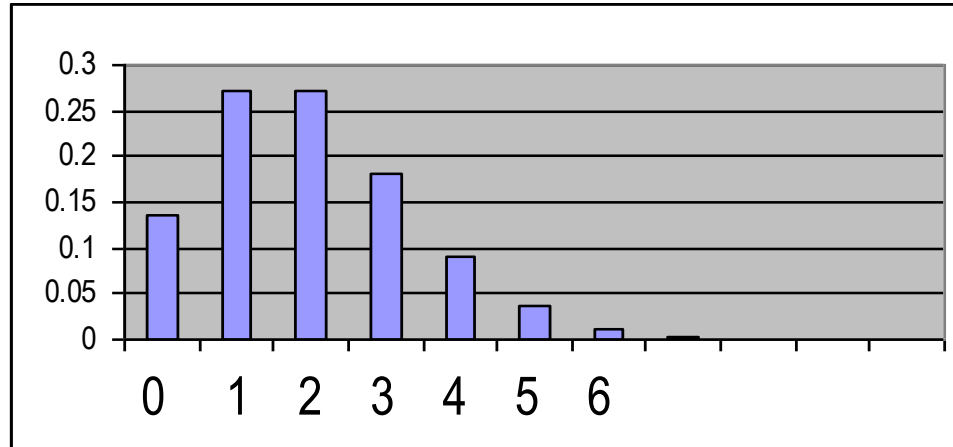
$$P(X = 1) = p(1) = \frac{e^{-1}1^1}{1!} = e^{-1} = .3678$$

$$P(X = 2) = p(2) = \frac{e^{-1}1^2}{2!} = \frac{e^{-1}}{2} = .1839$$

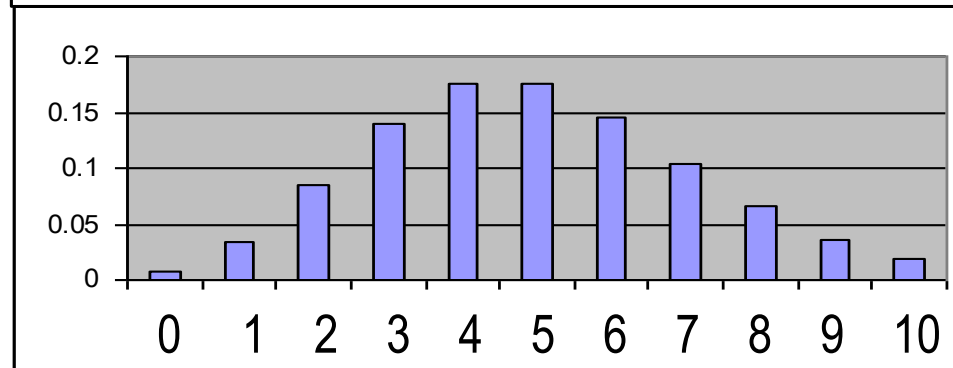
$$P(X = 3) = p(3) = \frac{e^{-1}1^3}{3!} = \frac{e^{-1}}{6} = .0613$$

Poisson Distributions (Graphs)

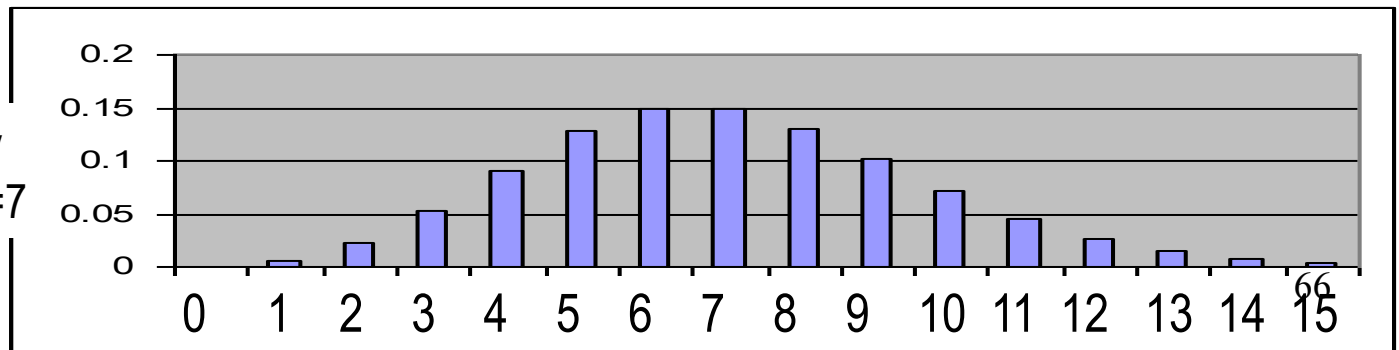
Poisson probability distribution with $\mu = 2$



Poisson probability distribution with $\mu = 5$



Poisson probability distribution with $\mu = 7$



Poisson Distribution (Example 1)

- **Example**

- The number of Typographical errors in new editions of textbooks is Poisson distributed with a mean of 1.5 per 100 pages.
- 100 pages of a new book are randomly selected.
- What is the probability that there are no typos?

- **Solution**

- $$P(X=0) = \frac{e^{-\mu} \mu^x}{x!} = \frac{e^{-1.5} 1.5^0}{0!} = .2231$$

Poisson Distribution (Example 2)

- **Example**

- For a 400 page book calculate the following probabilities

- There are no typos
- There are five or fewer typos

Important!

A mean of 1.5 typos per 100 pages, is equivalent to 6 typos per 400 pages.

- **Solution**

- $P(X=0) = \frac{e^{-\mu} \mu^x}{x!} = \frac{e^{-6} 6^0}{0!} = .002479$

- $P(X \leq 5) =$ <use the formula to find $p(0)$, $p(1), \dots, p(5)$, then calculate $p(0) + p(1) + \dots + p(5) = .4457$

Poisson Distribution (Example 3)

If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals

(a) exactly 3 (b) more than 2 individuals (c) None (d) More than one individual will suffer a bad reaction.

Solution. $p = 0.001$, $n = 2000$

$$m = np = 2000 \times 0.001 = 2$$

$$\therefore P(r) = \frac{e^{-m} m^r}{r!} = e^{-2} \frac{2^r}{r!} = \frac{1}{e^2} \times \frac{2^r}{r!}$$

$$(a) P(\text{more than 3}) P(3) = \frac{1}{e^2} \cdot \frac{2^3}{3!} = \frac{1}{(2.718)^2} \times \frac{8}{6} = (0.135) \times \frac{4}{3} = 0.18$$

Poisson Distribution (Example 2)

$$(b) P(\text{more than } 2) = P(3) + P(4) + P(5) + \dots + P(2000)$$

$$= 1 - [P(0) + P(1) + P(2)] = 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!} + \frac{e^{-2}(2)^2}{2!} \right]$$

$$= 1 - e^{-2}[1 + 2 + 2] = 1 - \frac{5}{e^2} = 1 - 5 \times 0.135 = 1 - 0.675 = 0.325 \quad \text{Ans.}$$

Poisson Distribution (Example 2)

$$(c) \quad P(\text{none}) = P(0) = \frac{e^{-2}(2)^0}{0!} = 0.135$$

$$(d) \quad P(\text{more than 1}) = P(2) + P(3) + P(4) + \dots + P(2000) = 1 - [P(0) + P(1)]$$

$$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!} \right] = 1 - 3e^{-2} = 1 - 3 \times 0.135 = 1 - 0.405 = 0.595 \text{ Ans.}$$

The difference between Binomial and Poisson distributions.

The parameter p and n in the Poisson distribution are dependent.

The Binomial distribution is bounded by $X=0$, $X=n$ with $p(X=0)=(1-p)^n$ and $p(X=n)=p^n$

The Poisson distribution is bounded by $X=0$ for $p(X=0)=e^{-\lambda}$

The Poisson distribution only the occurrence of an event can be counted; the number of its non-occurrences cannot.

Probability Calculation

- Probability: Numerical measure of the likelihood that an event will occur
- Some probabilities that can be calculated using Excel:
 - Binomial Probabilities
 - Poisson Probabilities
 - Hypergeometric Probabilities
 - Normal Probabilities
 - Exponential Probabilities

Binomial Probabilities

- Use BINOMDIST to compute binomial distribution probabilities and cumulative binomial probabilities
Cell =BINOMDIST(number_s, trials, probability_s, cumulative)
 - Calculates the probability that a sequence of independent trials with two possible outcomes will have a given number of successes
 - Cumulative is either TRUE or FALSE

Poisson Probabilities

- Use POISSON to compute Poisson Probabilities
Cell = POISSON (x, mean, cumulative)
 - Shows the probability of x occurrences of an event over a specified interval of time or space

Thank You All