Continuous Random Variables and Standard Normal Distribution

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Determine probabilities from probability density functions.
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability density functions, and the reverse.
3. Calculate means and variances for continuous random variables.
4. Understand the assumptions for each of the continuous probability distributions presented.
5. Select an appropriate continuous probability distribution to calculate probabilities in specific applications.
6. Calculate probabilities, determine means and variances for each of the continuous probability distributions presented.
7. Standardize normal random variables.
8. Use the table for the cumulative distribution function of a standard normal distribution to calculate probabilities.
9. Approximate probabilities for some binomial and Poisson distributions.

Continuous Random Variables

Previously, we discussed the measurement of the current in a thin copper wire. We noted that the results might differ slightly in day-to-day replications because of small variations in variables that are not controlled in our experiment—changes in ambient temperatures, small impurities in the chemical composition of the wire, current source drifts, and so forth.

Another example is the selection of one part from a day's production and very accurately measuring a dimensional length. In practice, there can be small variations in the actual measured lengths due to many causes, such as vibrations, temperature fluctuations, operator differences, calibrations, cutting tool wear, bearing wear, and raw material changes. Even the measurement procedure can produce variations in the final results.

Probability Distributions and Probability Density Functions

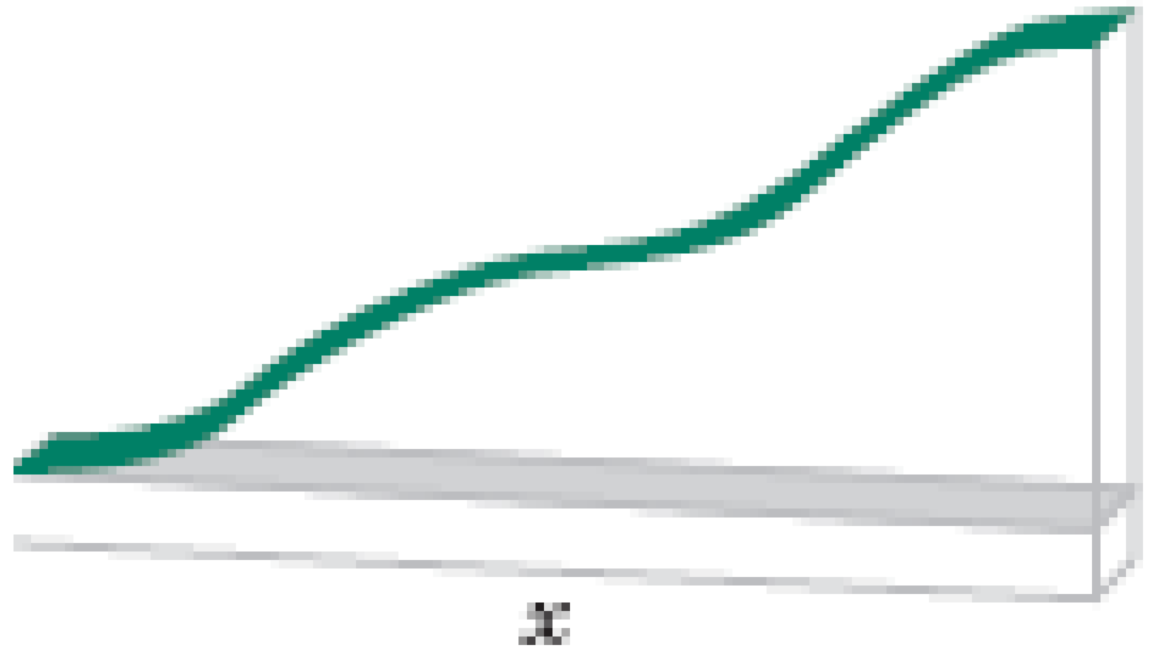


Figure Density function of a loading on a long, thin beam.

Probability Distributions and Probability Density Functions

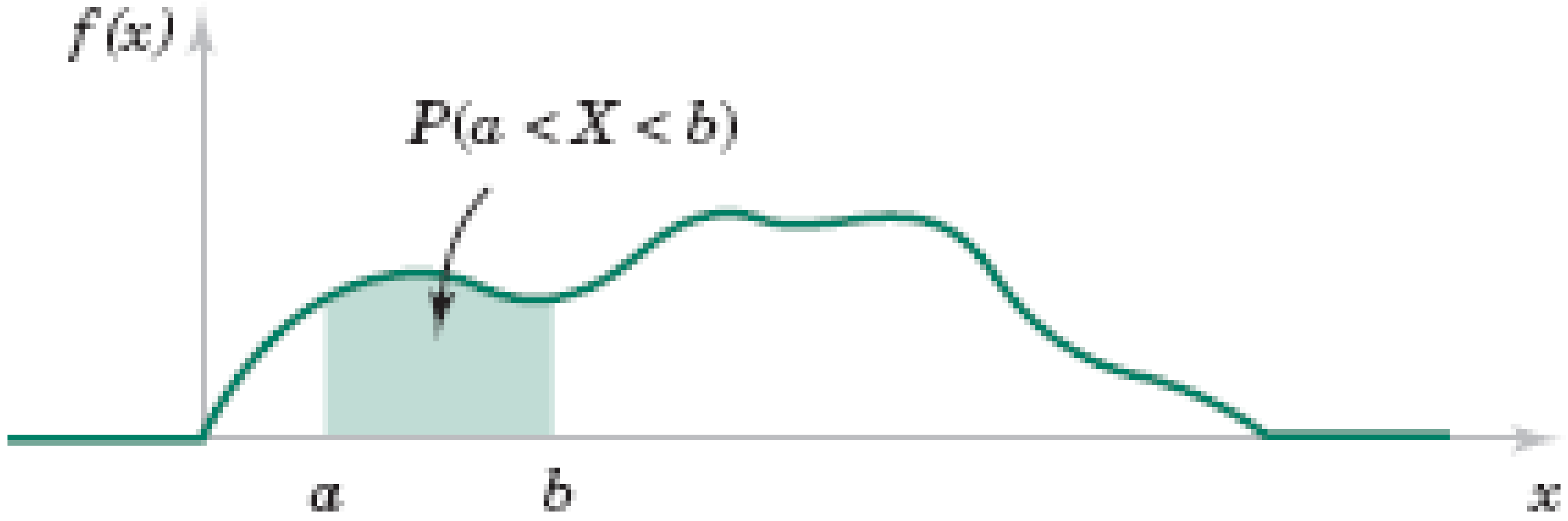


Figure Probability determined from the area under $f(x)$.

Probability Distributions and Probability Density Functions

Definition

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_a^b f(x)\,dx =$ area under $f(x)$ from a to b

for any a and b (4-1)

Probability Distributions and Probability Density Functions

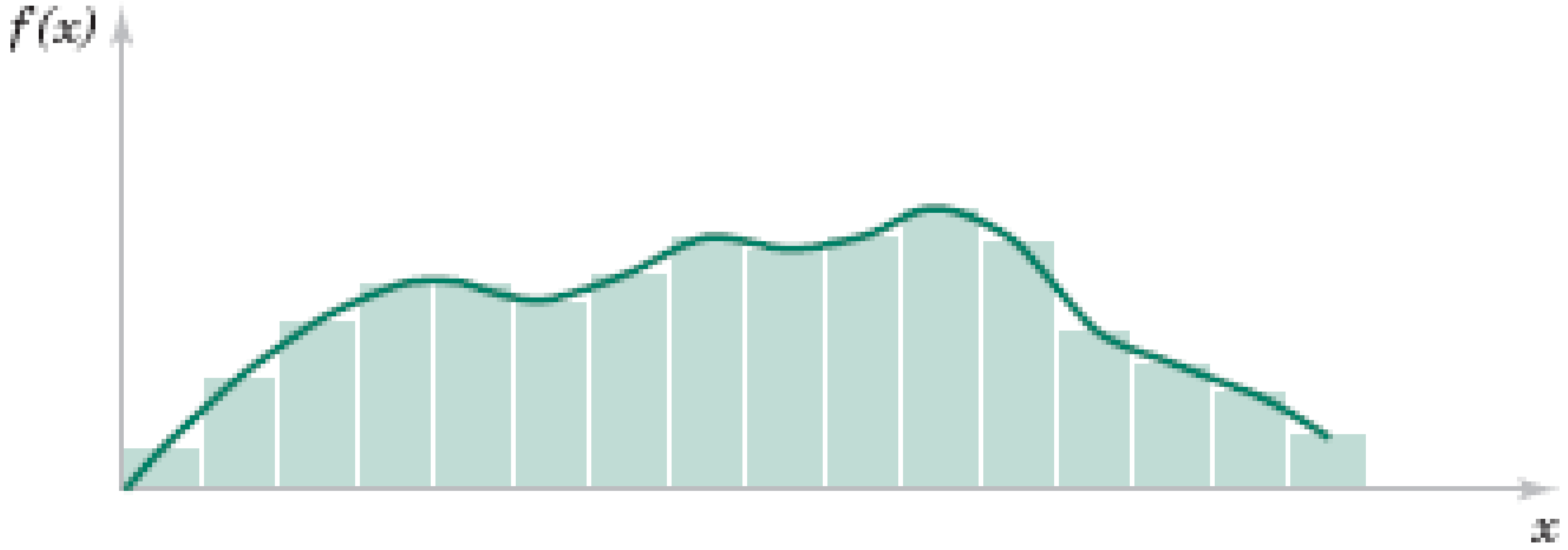


Figure Histogram approximates a probability density function.

Probability Distributions and Probability Density Functions

If X is a **continuous random variable,** for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2) \quad (4\text{-}2)$$

Probability Distributions and Probability Density Functions

Example

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped? The density function and the requested probability are shown in Fig. 4-5. A part is scrapped if $X > 12.60$. Now,

$$P(X > 12.60) = \int_{12.6}^{\infty} f(x)\,dx = \int_{12.6}^{\infty} 20e^{-20(x-12.5)}\,dx = -e^{-20(x-12.5)}\Big|_{12.6}^{\infty} = 0.135$$

Probability Distributions and Probability Density Functions

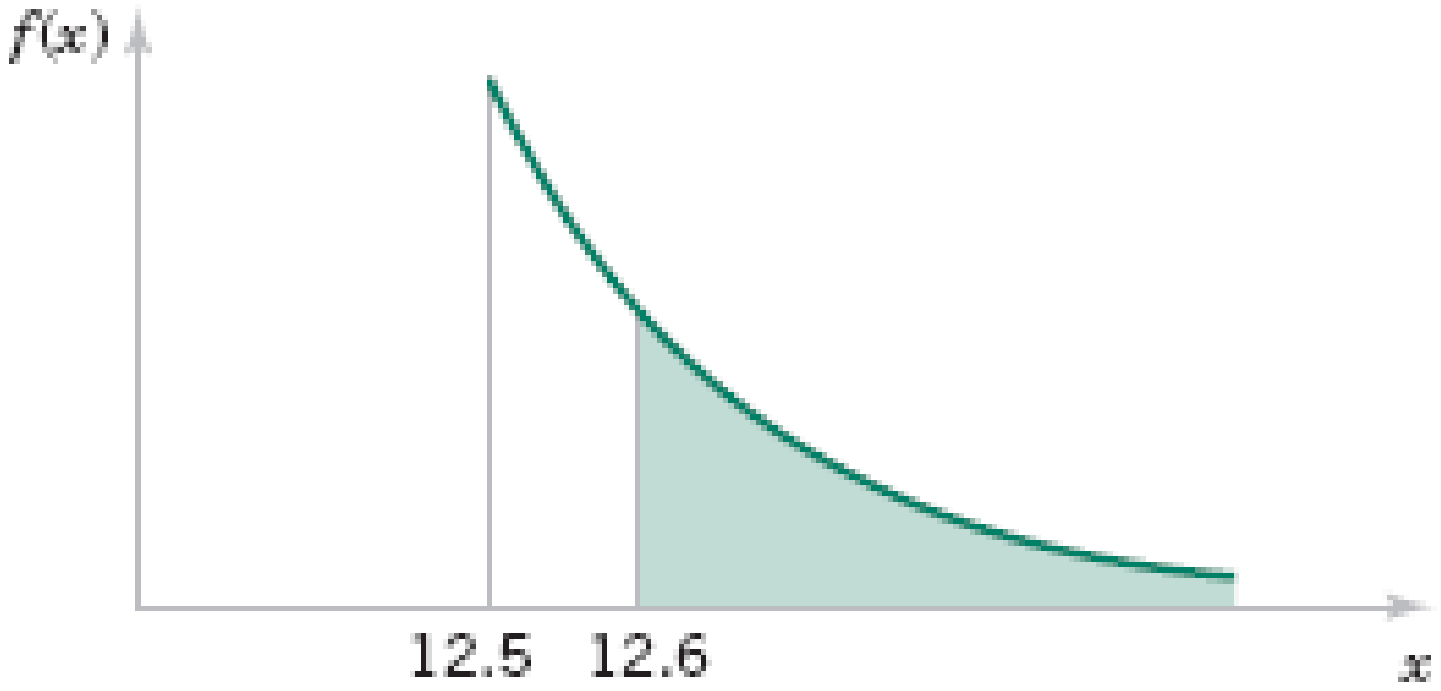


Figure Probability density function for above Example.

Probability Distributions and Probability Density Functions

Example (continued)

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)\,dx = -e^{-20(x-12.5)}\Big|_{12.5}^{12.6} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

Cumulative Distribution Functions

Definition

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\, du \qquad (4\text{-}3)$$

for $-\infty < x < \infty$.

Cumulative Distribution Functions

Example

For the drilling operation in Example 4-2, $F(x)$ consists of two expressions.

$$F(x) = 0 \quad \text{for} \quad x < 12.5$$

and for $12.5 \le x$

$$F(x) = \int_{12.5}^{x} 20e^{-20(u-12.5)}\, du$$
$$= 1 - e^{-20(x-12.5)}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \le x \end{cases}$$

Figure . . displays a graph of $F(x)$.

Cumulative Distribution Functions

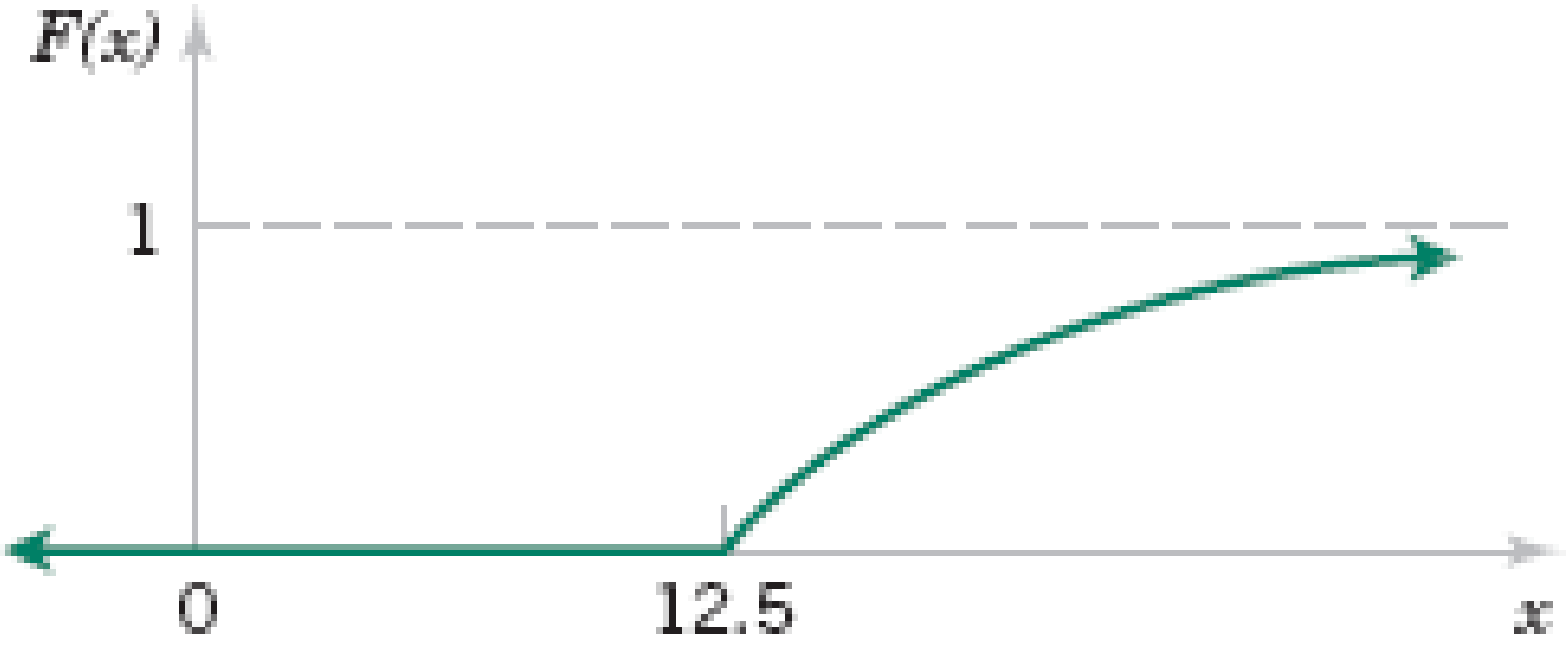


Figure Cumulative distribution function for above Example.

Example

The probability density function f (x) of a continuous random variable x is defined by

$$f(x) = \begin{cases} \dfrac{A}{x^3}, & 5 \le x \le 10 \\ 0, & otherwise \end{cases}$$

Find the value of A.

Solution. Here, $f(x) = \dfrac{A}{x^3}, \quad 5 \le x \le 10$

Since $f(x)$ is probability density function, so

$$\int_5^{10} \frac{A}{x^3}\,dx = 1 \quad \Rightarrow \quad \left[-\frac{A}{2x^2}\right]_5^{10} = 1$$

$$\frac{A}{2}\left[-\frac{1}{100}+\frac{1}{25}\right] = 1$$

$$\frac{A}{2}\left(\frac{3}{100}\right) = 1 \quad \Rightarrow \quad A = \frac{200}{3}$$

Ans.

Example

The diameter of an electric cable is assumed to be continuous random variate with probability density function:

$f(x) = 6x(1-x),\ 0 \le x \le 1$

(*i*) *verify that above is a p.d.f.* (*ii*) *find the mean and variance.*

Solution. (*i*) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 6x(1-x)dx = \int_0^1 (6x-6x^2)dx$

$= (3x^2-2x^3)_0^1 = 3-2=1$

Secondly $f(x) > 0$ for $0 \le x \le 1$.

Hence the given function is a probability density function.

(*ii*) Mean $= \int_{-\infty}^{\infty} x.f(x)\,dx = \int_0^1 x.6x(1-x)dx$

$= \int_0^1 (6x^2-6x^3)dx = \left(2x^3 - \frac{3}{2}x^4\right)_0^1 = 2-\frac{3}{2} = \frac{1}{2}$ **Ans.**

Variance $= \int_{-\infty}^{\infty} (x-\bar{x})^2.f(x)\,dx = \int_0^1 \left(x-\frac{1}{2}\right)^2 .6x(1-x)\,dx$

$= \int_0^1 \left(x^2-x+\frac{1}{4}\right)(6x-6x^2)dx = \int_0^1 \left(12x^3-6x^4-\frac{15}{2}x^2+\frac{3}{2}x\right)dx$

$= \left(3x^4-\frac{6}{5}x^5-\frac{5}{2}x^3+\frac{3x^2}{4}\right)_0^1 = \left(3-\frac{6}{5}-\frac{5}{2}+\frac{3}{4}\right) = \frac{1}{20}$ **Ans.**

Mean and Variance of a Continuous Random Variable

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{4-4}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Mean and Variance of a Continuous Random Variable

Example

For the copper current measurement in Example 4-1, the mean of X is

$$E(X) = \int_0^{20} xf(x)\,dx = 0.05x^2/2 \Big|_0^{20} = 10$$

The variance of X is

$$V(X) = \int_0^{20} (x - 10)^2 f(x)\,dx = 0.05(x - 10)^3/3 \Big|_0^{20} = 33.33$$

Mean and Variance of a Continuous Random Variable

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx \qquad (4\text{-}5)$$

Mean and Variance of a Continuous Random Variable

Example

For the drilling operation in Example 4-2, the mean of X is

$$E(X) = \int_{12.5}^{\infty} x f(x)\, dx = \int_{12.5}^{\infty} x\, 20 e^{-20(x-12.5)}\, dx$$

Integration by parts can be used to show that

$$E(X) = -x e^{-20(x-12.5)} - \frac{e^{-20(x-12.5)}}{20} \Bigg|_{12.5}^{\infty} = 12.5 + 0.05 = 12.55$$

The variance of X is

$$V(X) = \int_{12.5}^{\infty} (x - 12.55)^2 f(x)\, dx$$

Although more difficult, integration by parts can be used two times to show that $V(X) = 0.0025$.

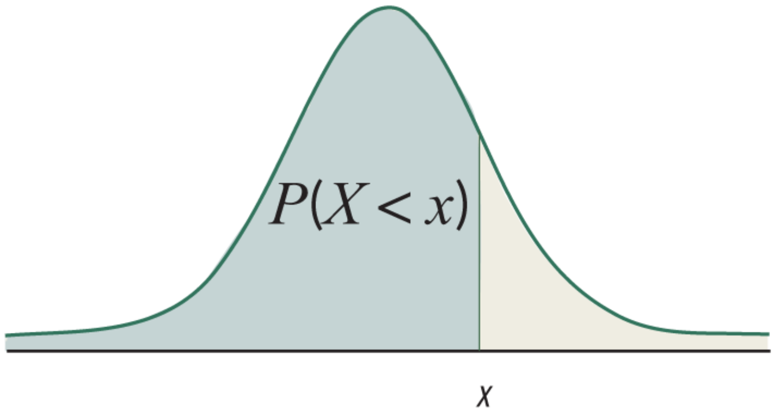
Normal Distribution:

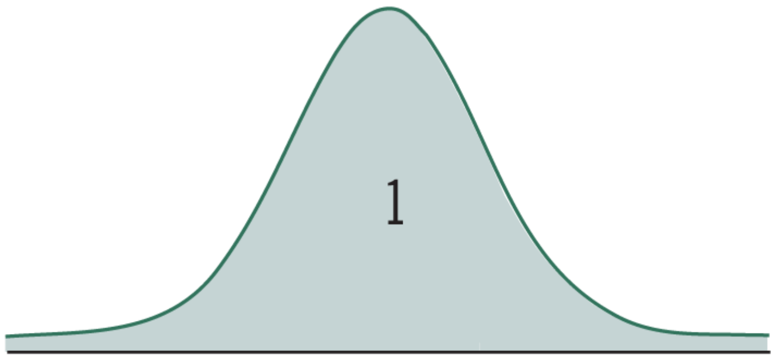
A continuous probability distribution for a given random variable, X, that is completely defined by its mean and variance.

Properties of a Normal Distribution:

1. A normal curve is symmetric and bell-shaped.
2. A normal curve is completely defined by its mean, μ, and variance, σ^2.
3. The total area under a normal curve equals 1.
4. The x-axis is a horizontal asymptote for a normal curve.

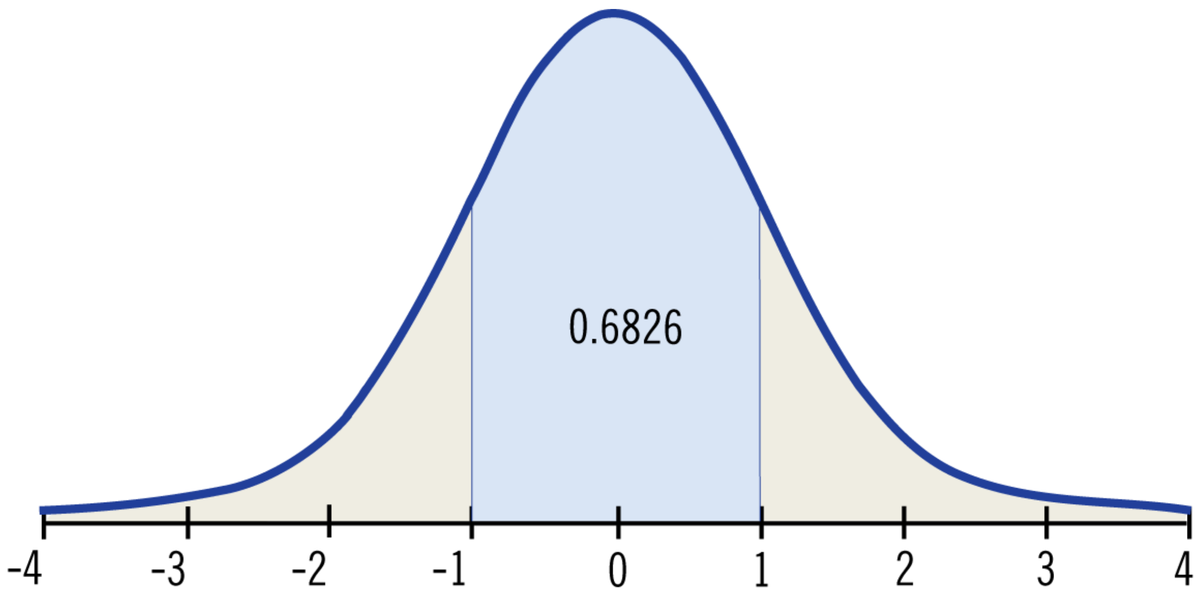
Total Area Under the Curve = 1:





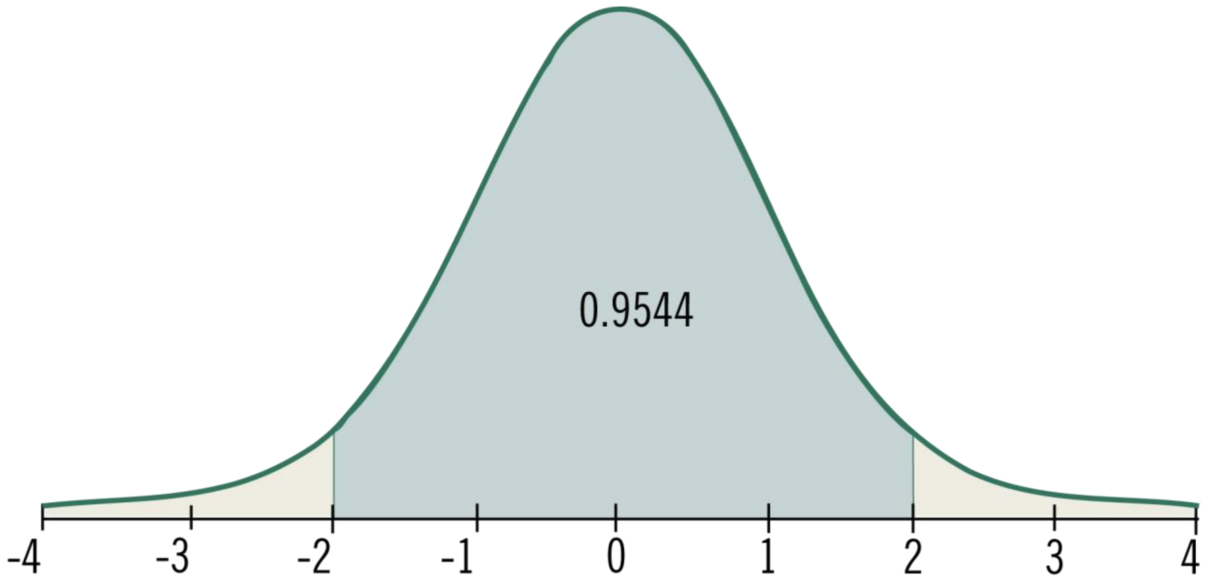
Area within One Standard Deviation:

The area under the curve and the probability of being within one standard deviation $(\pm 1\sigma)$ of the mean, μ, equals 0.6826.



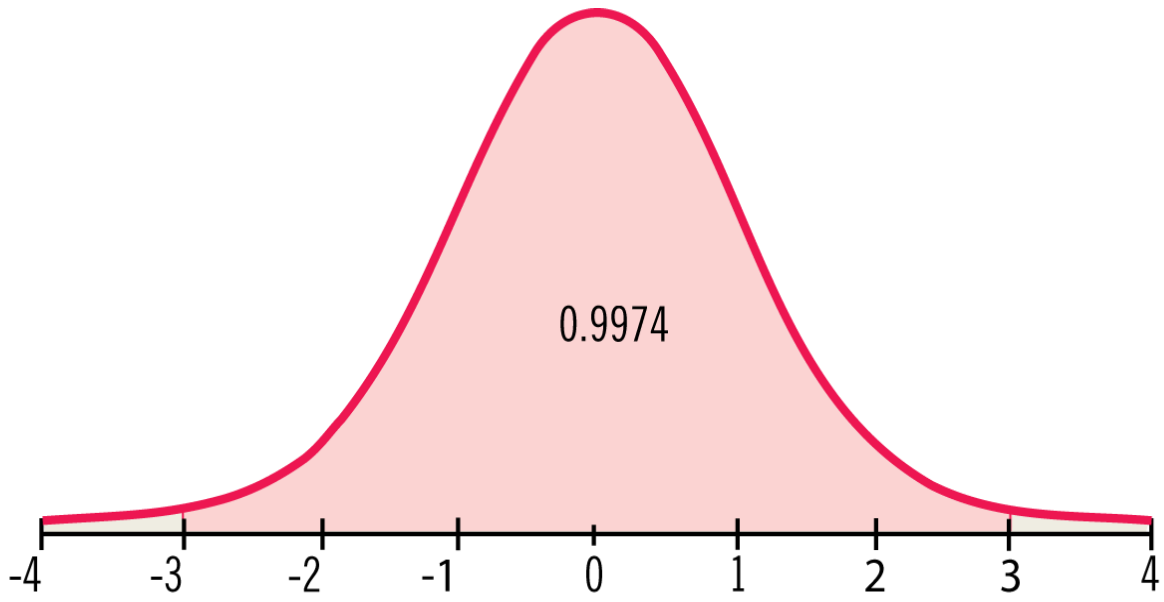
Area within Two Standard Deviations:

The area under the curve and the probability of being within two standard deviations $(\pm 2\sigma)$ of the mean, μ, equals 0.9544.



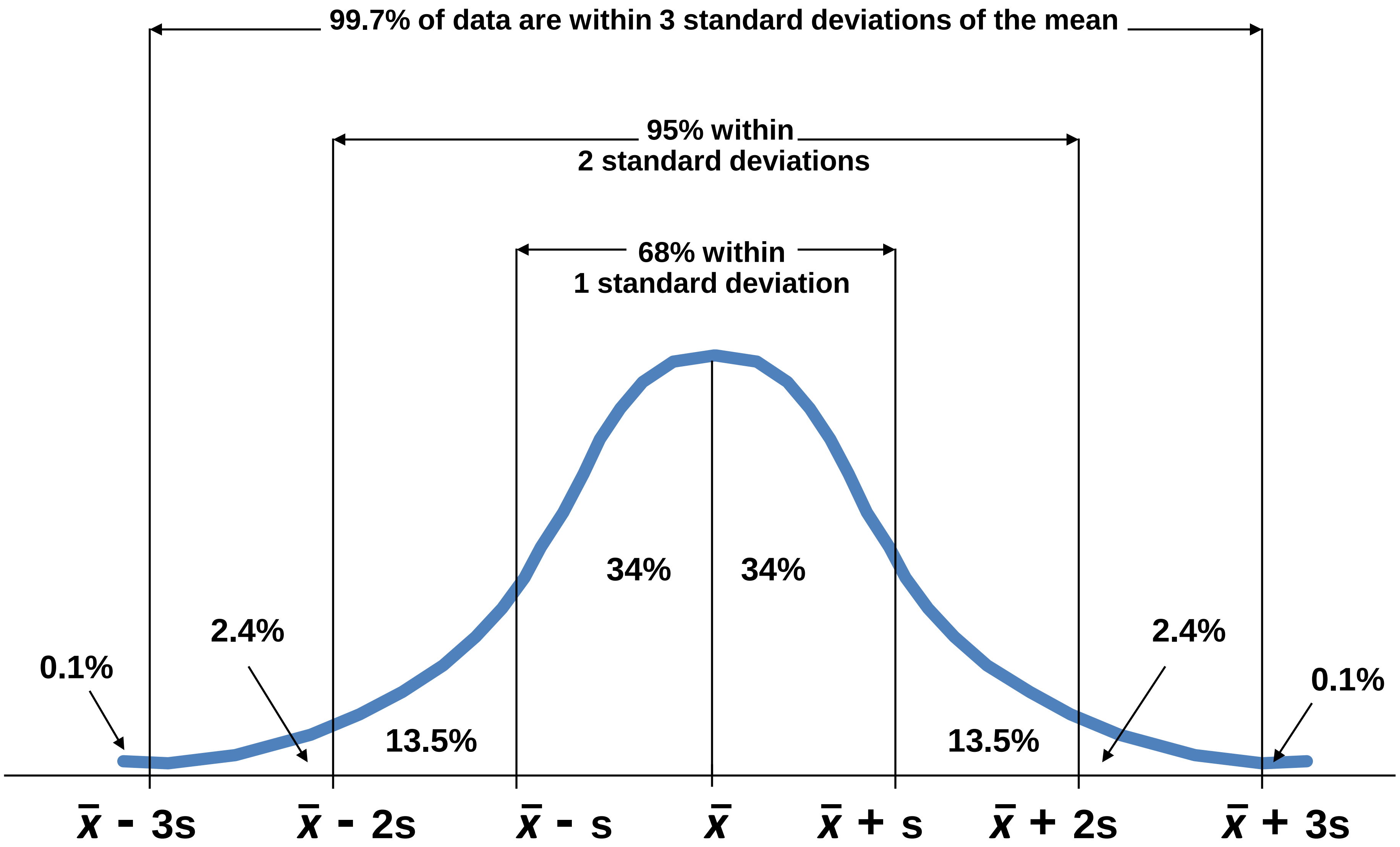
Area within Three Standard Deviations:

The area under the curve and the probability of being within three standard deviations $(\pm 3\sigma)$ of the mean, μ, equals 0.9974.



The Empirical Rule

Standard Normal Distribution: $\mu = 0$ and $\sigma = 1$



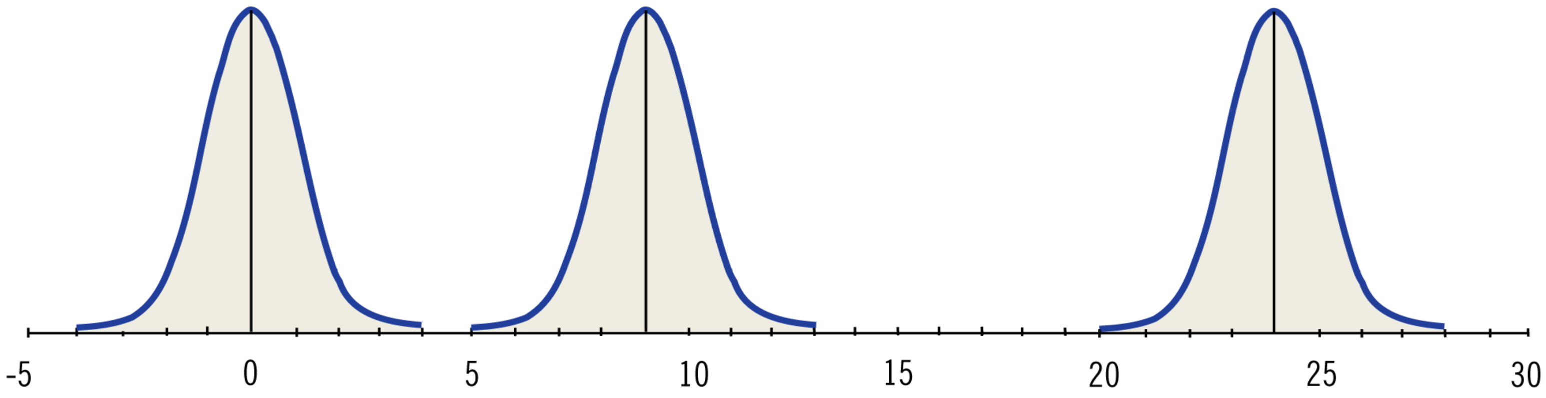
Definition:

- **Normal distribution** – a continuous probability density function completely defined by its mean and variance.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

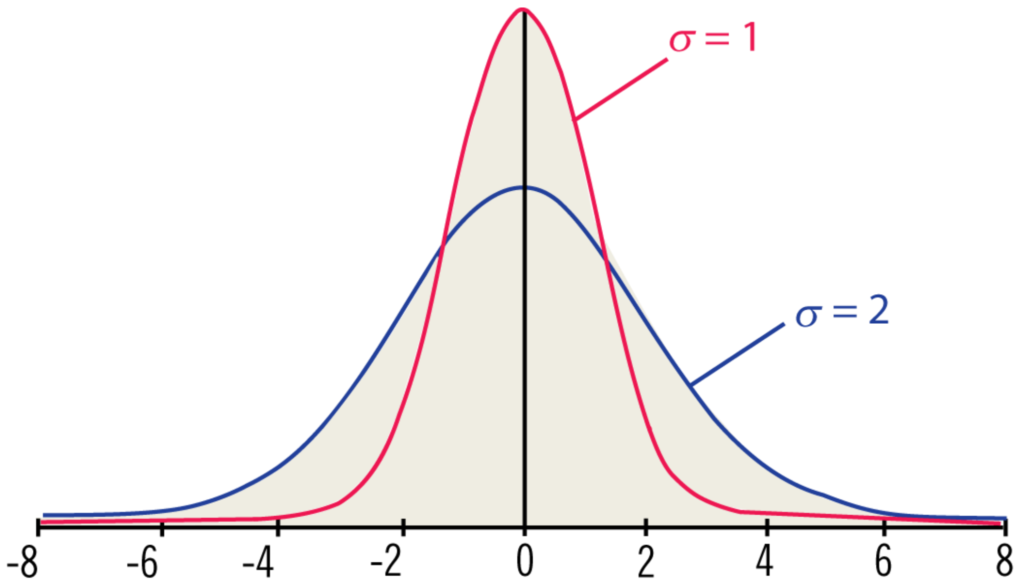
Normal Curves:

- The mean defines the location and the variance determines the dispersion.
- Below are three different normal curves with different means and identical variances.



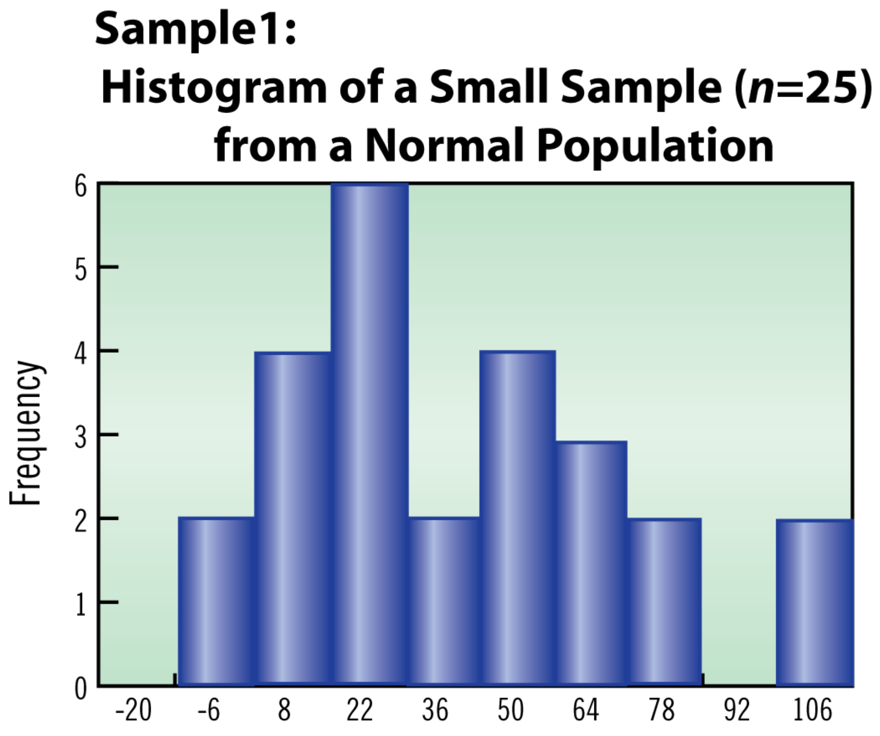
Normal Curves:

- Below are two different normal curves with identical means and different variances.
- Changing the variance parameter can have rather significant effects on the shape of the distribution.

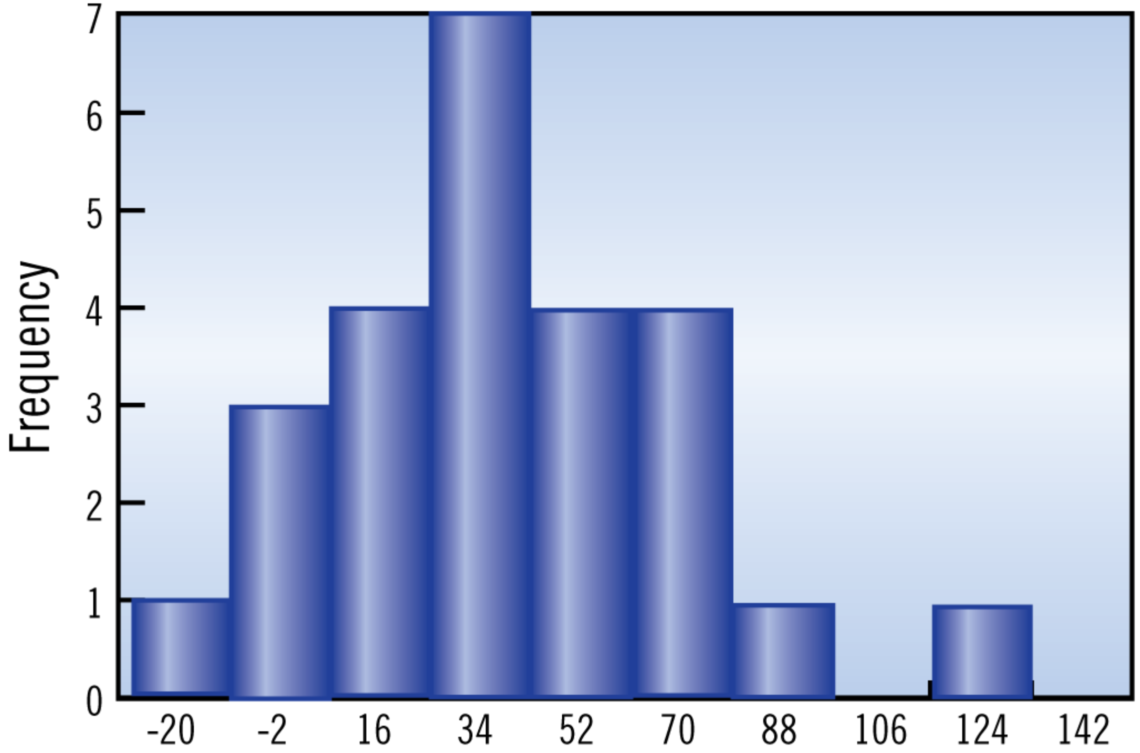


Data from Normal Distributions:

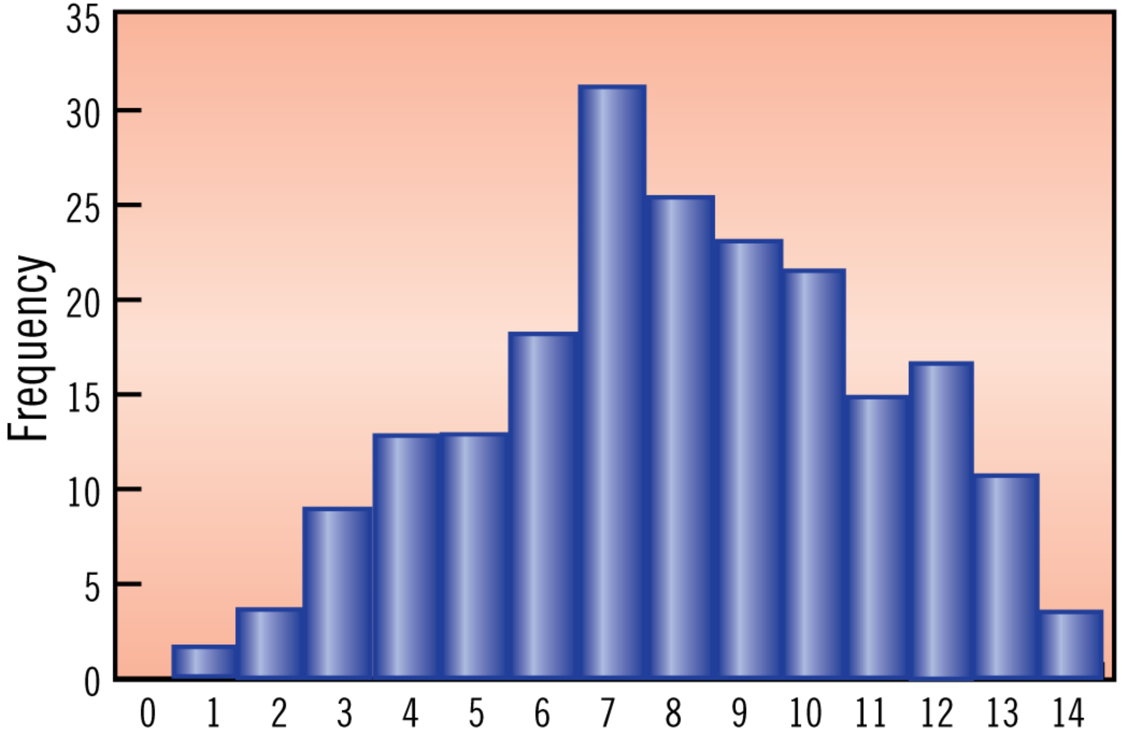
As the following three histograms demonstrate, data from a population that is assumed to come from a normal population will more closely represent a bell curve as the sample size *n* grows larger.



Sample 2:
Histogram of a Small Sample (*n*=25) from a Normal Population



Histogram of a Large Sample (n=200) from a Normal Population



Objectives:

- Understand the concept and characteristics of the standard normal distribution.
- To calculate the area underneath a standard normal distribution.

Standard Normal Distribution:

- A standard normal distribution has the same properties as the normal distribution; in addition, it has a mean of 0 and a variance of 1.

Properties of a Standard Normal Distribution:

1. The standard normal curve is symmetric and bell-shaped.
2. It is completely defined by its mean and standard deviation, $\mu = 0$ and $\sigma^2 = 1$.
3. The total area under a standard normal curve equals 1.
4. The *x*-axis is a horizontal asymptote for a standard normal curve.

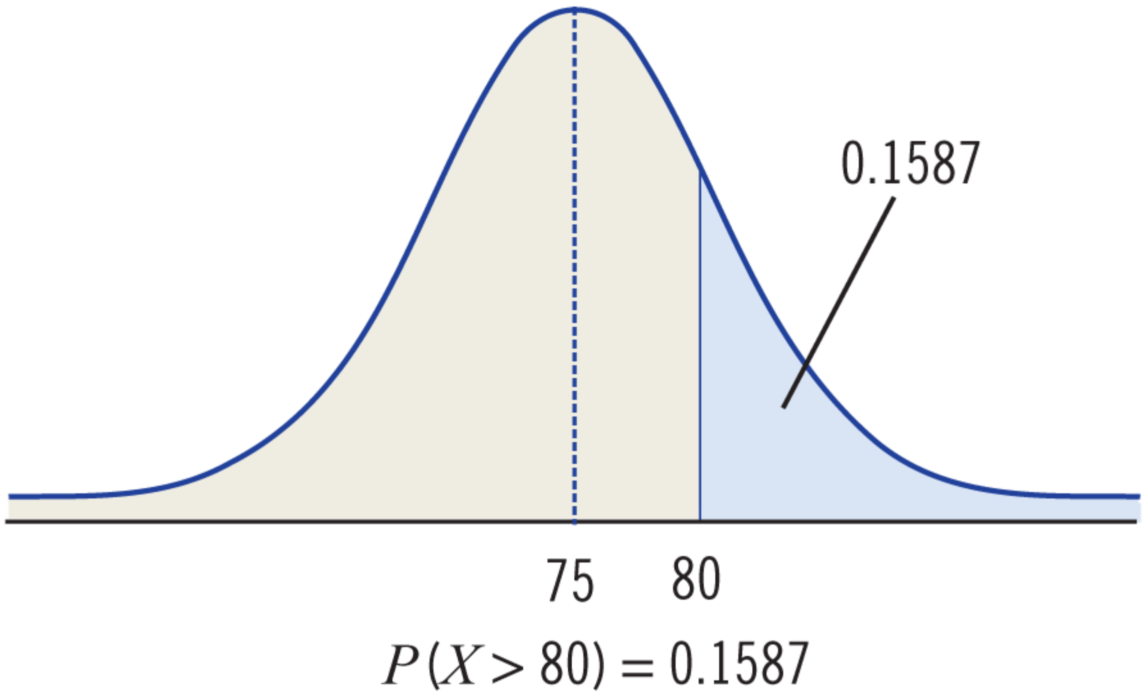
Tables for the standard normal curve:

There are two types of tables for calculating areas under the standard normal curve.

- The first contains probability calculations for various areas under the standard normal curve for a random variable between 0 and a specified value.
- The second contains probability calculations for various areas under the standard normal curve for a random variable between negative infinity $(-\infty)$ and a specified value.

Probability of a Normal Curve:

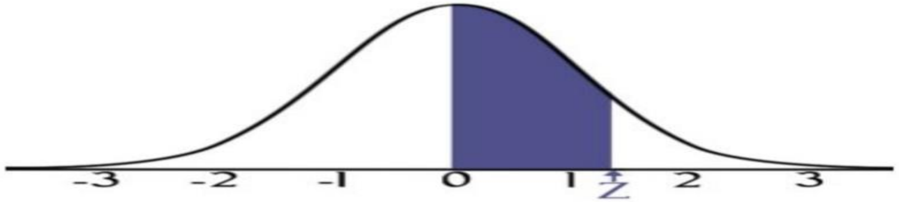
The probability of a random variable (X) having a value (e.g. 80) in a given range (e.g. 50 to 100) is equal to the area under the curve in that region.



Standard Normal Distribution Table:

Standard Normal Distribution Table from –∞ to positive z					
z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160
0.1	0.5398	0.5438	0.5478	0.5517	0.5557
0.2	0.5793	0.5832	0.5871	0.5910	0.5948
0.3	0.679	0.6217	0.6255	0.6293	0.6331
0.4	0.6554	0.6591	0.6628	0.6664	0.6700
0.5	0.6915	0.6950	0.6985	0.7019	0.7054
0.6	0.7257	0.7291	0.7324	0.7357	0.7389
0.7	0.7580	0.7611	0.7642	0.7673	0.7704
0.8	0.7881	0.7910	0.7939	0.7967	0.7995

Figure 1.3: Standard Normal Table

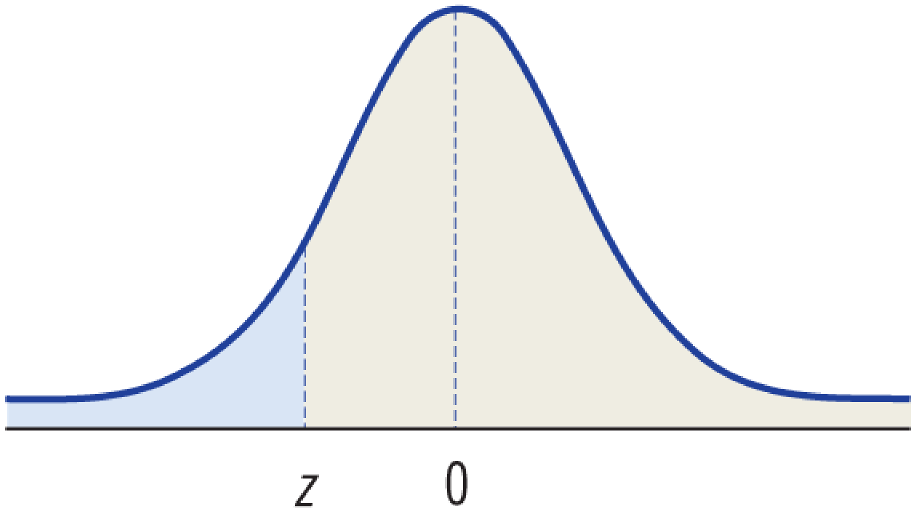


STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and *z* standard deviations above the mean. For example, for *z* = 1.25 the area under the curve between the mean (0) and *z* is 0.3944.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

Area to the Left of z:



Find the area to the left of z:

a. $z = 1.69$

0.9545

b. $z = -2.03$

0.0212

c. $z = 0$

0.5000

d. $z = 4.2$

Approximately 1

e. $z = -4.2$

Approximately 0

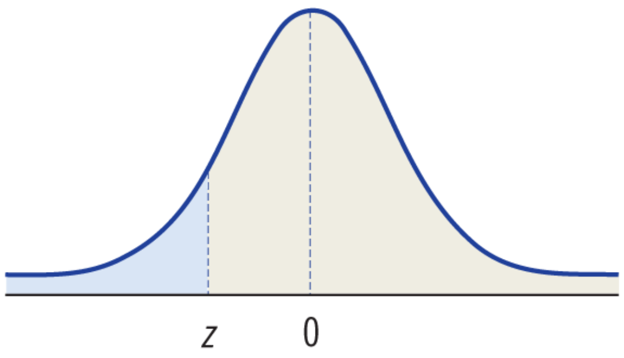
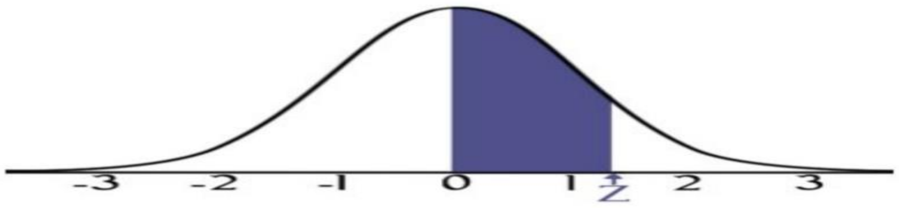


Figure 1.3: Standard Normal Table

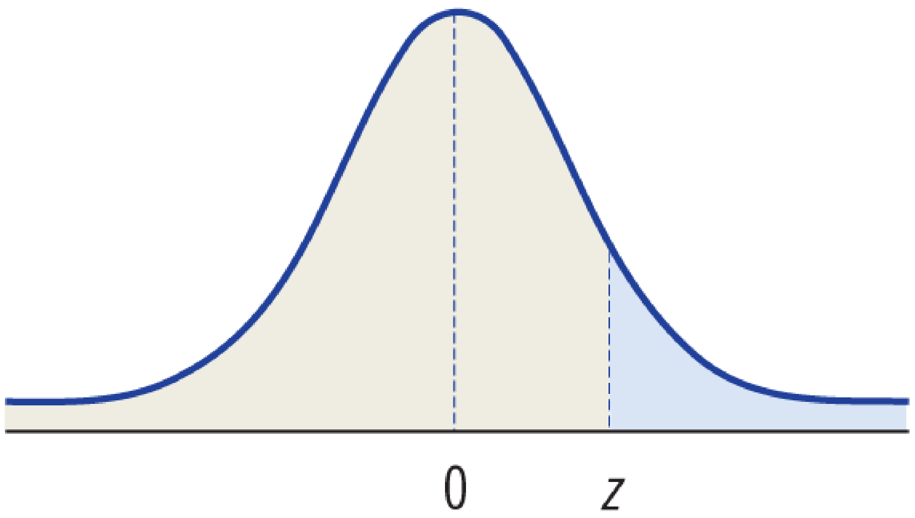


STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and *z* standard deviations above the mean. For example, for *z* = 1.25 the area under the curve between the mean (0) and *z* is 0.3944.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

Area to the Right of z:



Find the area to the right of z:

a. $z = 3.02$

0.0013

b. $z = -1.70$

0.9554 $z = 0$

0.5000

c. $z = 5.1$

Approximately 0

d. $z = -5.1$

Approximately 1

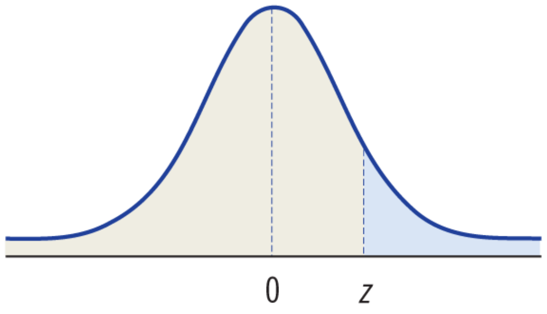
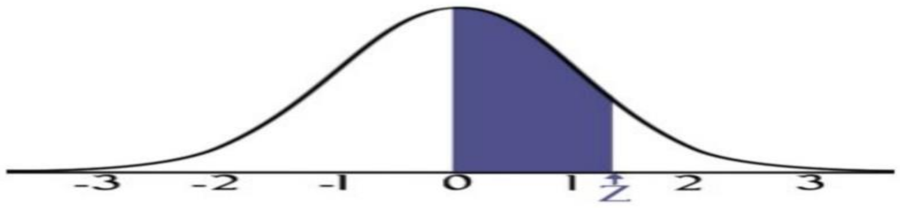


Figure 1.3: Standard Normal Table



STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and z standard deviations above the mean. For example, for $z = 1.25$ the area under the curve between the mean (0) and z is 0.3944.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
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0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
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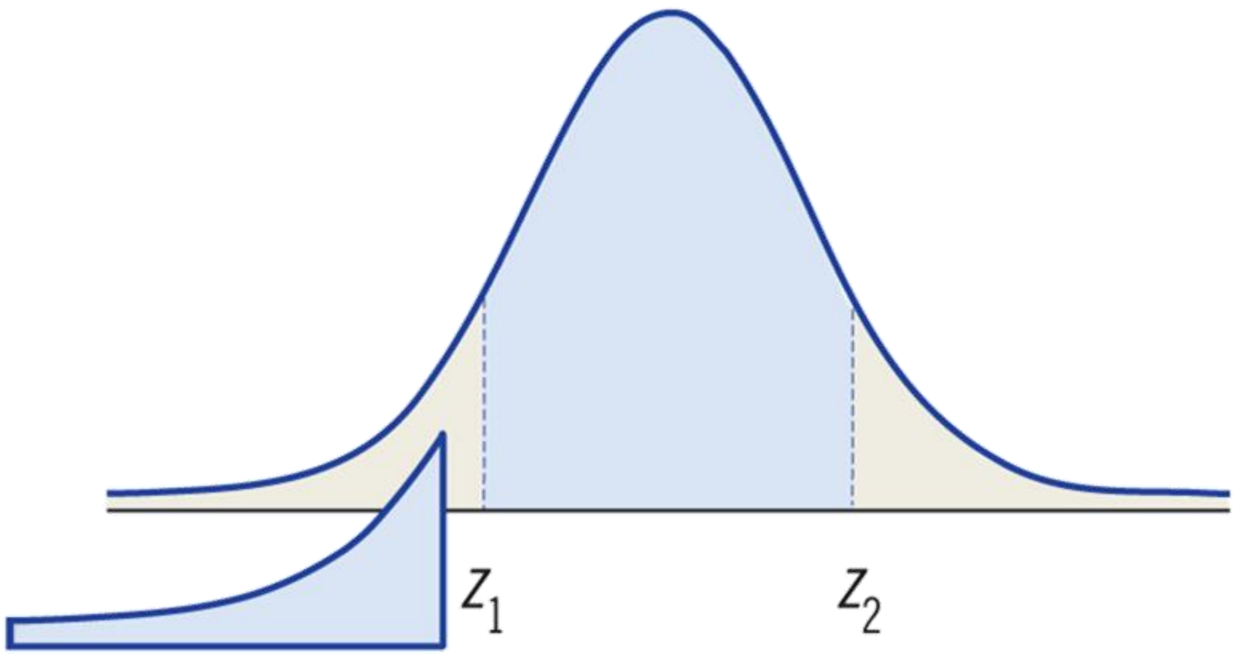
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

Area Between z_1 and z_2:



Find the area between z_1 and z_2:

a. $z_1 = 1.16$, $z_2 = 2.31$

0.1126

b. $z_1 = -2.76$, $z_2 = 0.31$

0.6188

c. $z_1 = -3.01$, $z_2 = -1.33$

0.0905

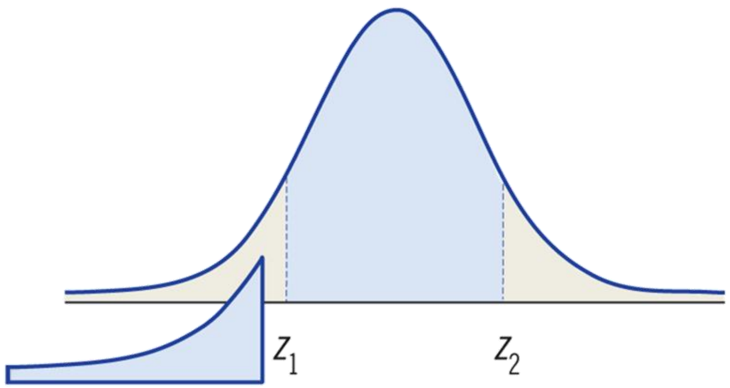
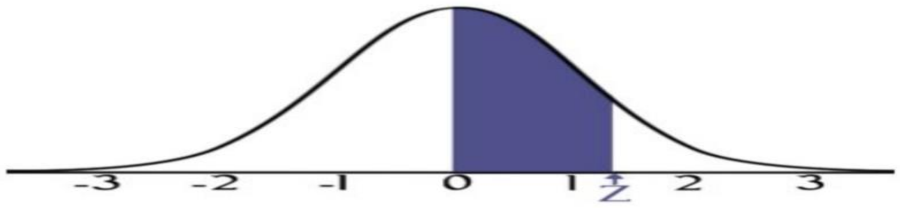


Figure 1.3: Standard Normal Table



STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and z standard deviations above the mean. For example, for $z = 1.25$ the area under the curve between the mean (0) and z is 0.3944.

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0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

Area in the Tails:

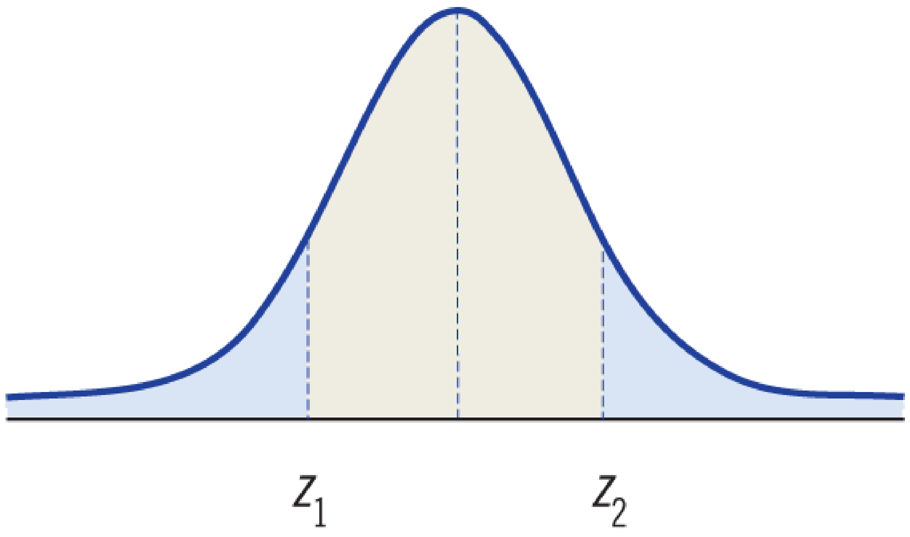
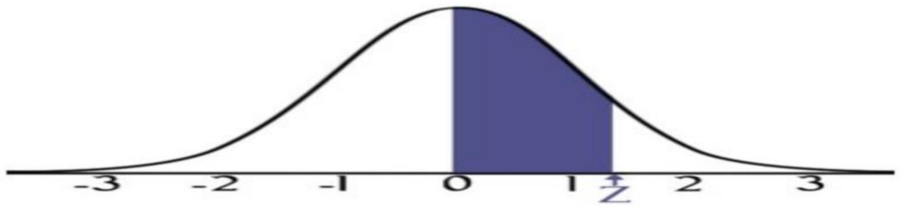


Figure 1.3: Standard Normal Table



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0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

Find the area in the tails:

a. $z_1 = 1.25$, $z_2 = 2.31$

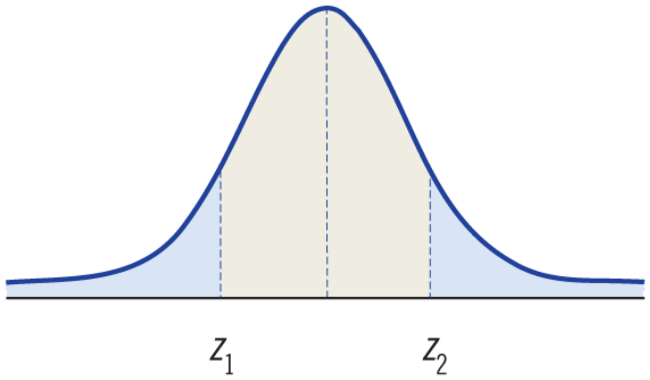
0.9048

b. $z_1 = -2.40$, $z_2 = -1.45$

0.9347

c. $z_1 = -1.05$, $z_2 = 1.05$

0.2937

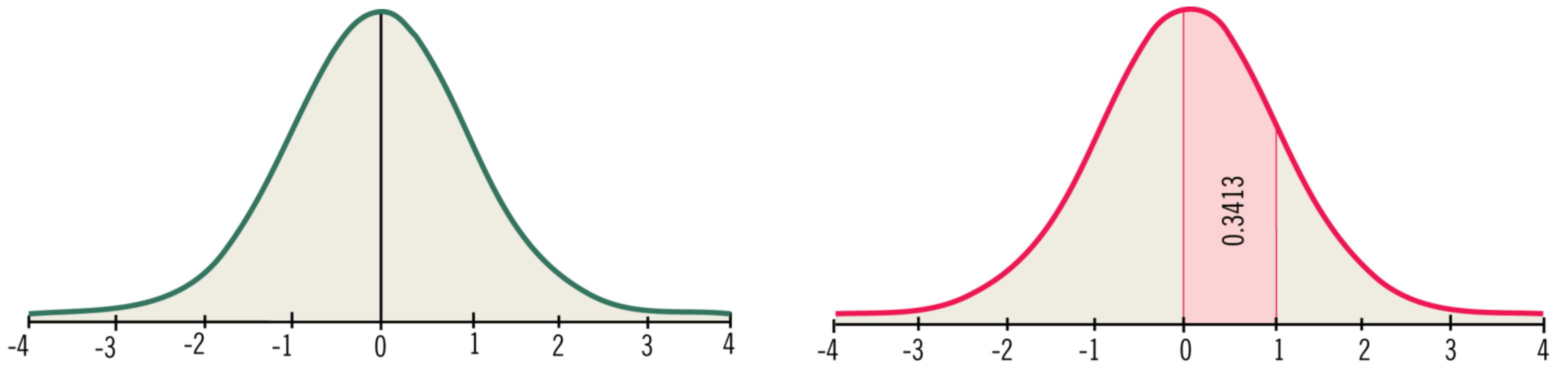


Example:

Calculate the probability that a standard normal random variable is between 0 and 1.

Solution:

- Look up the value of 1.00 in the table.
- The table value of .3413 is the area under the curve between 0 and 1.

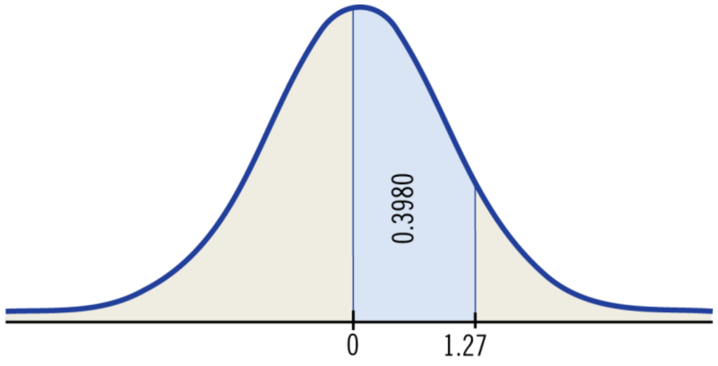


Example:

Calculate the probability that a standard normal random variable is between 0 and 1.27.

Solution:

- Look up the value of 1.27 in the table.
- The table value of .3980 is the area under the curve between 0 and 1.27.



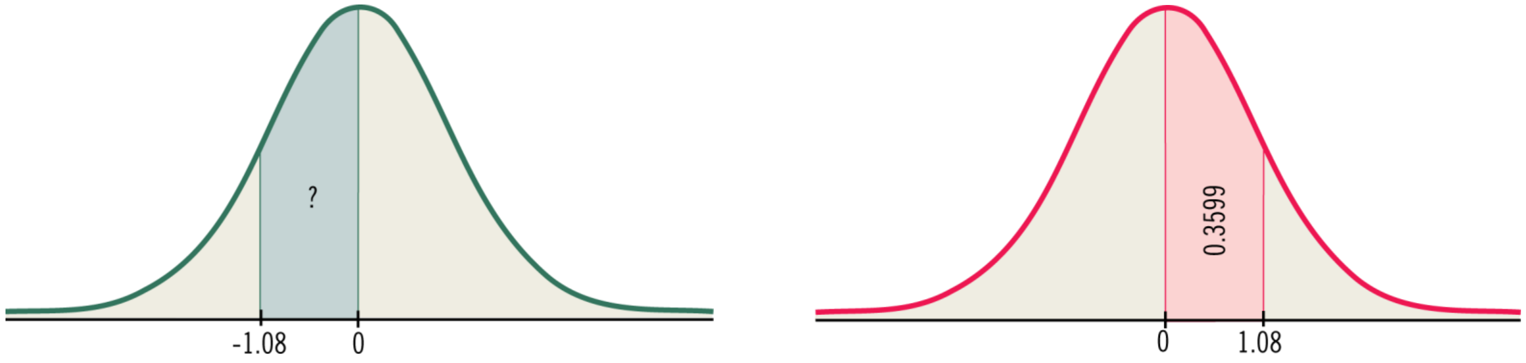
$$P(0 < z < 1.27) = .3980$$

Example:

Calculate the probability that a standard normal random variable is between −1.08 and 0.

Solution:

- The value −1.08 is not given in the table.
- Since the distribution is symmetric, the probability that the random variable is between −1.08 and 0 is equal to the probability the random variable is between 0 and 1.08.
- The table value of 0.3599 is the area under the curve between 0 and 1.08.



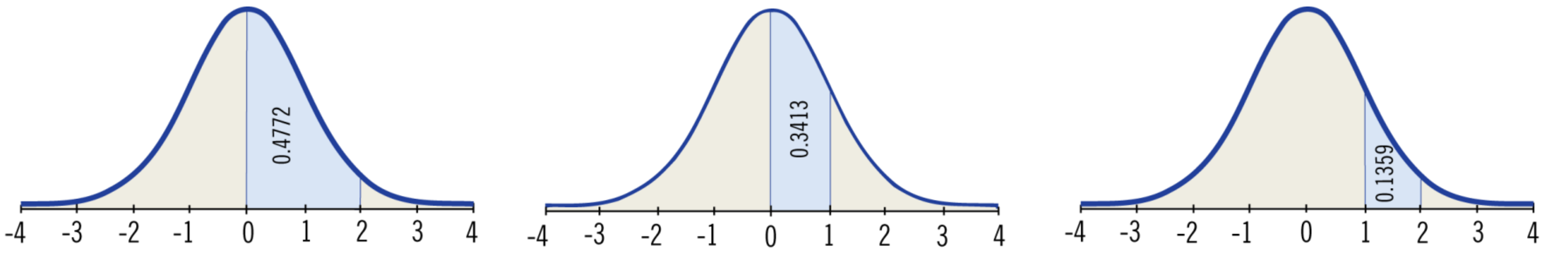
$$P(-1.08 < z < 0) \quad = \quad P(0 < z < 1.08) \; = .3599$$

Example:

Calculate the probability that a standard normal random variable is between 1.0 and 2.0.

Solution:

- First determine the probability that *z* is between 0 and 2.0, which the table gives as .4772.
- Then determine the probability that *z* is between 0 and 1.0, which the table gives as .3413.
- The final step is to subtract the probability *z* is between 0 and 1.0 from the probability that *z* is between 0 and 2.0.



$P(0 < z < 2) \quad - \quad P(0 < z < 1) \quad = \quad P(1 < z < 2)$

$.4772 \quad - \quad .3413 \quad = \quad .1359$

Objectives:

- Understand how to perform a z-Transformation.
- To calculate the probability of a normal random variable.

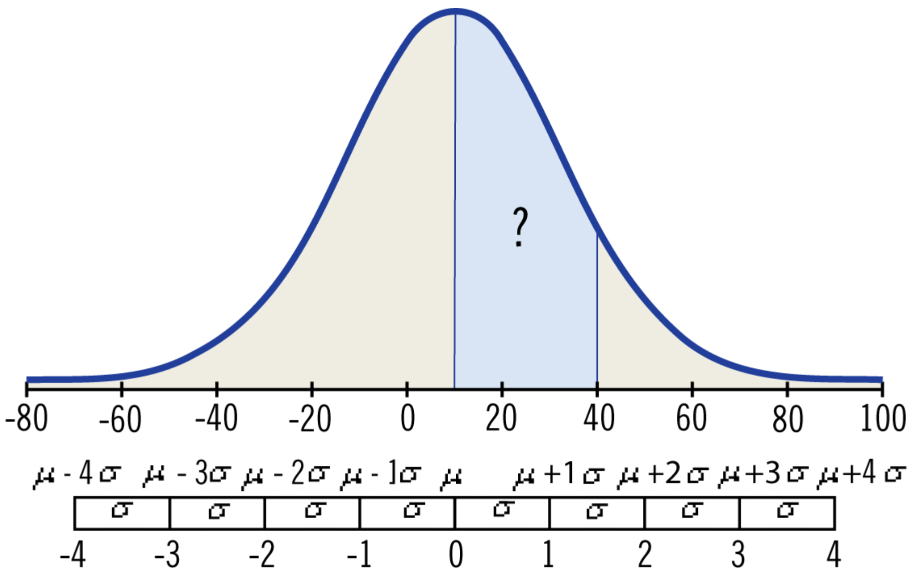
Definition:

- ***z*-Transformation** – a transformation of any normal variable into a standard normal variable. The *z*-transformation is denoted by *z* and is given by the formula

$$z = \frac{x - \mu}{\sigma}.$$

Example:

Calculate the probability that a normal random variable with a mean of 10 and a standard deviation of 20 will lie between 10 and 40.



Example:

Calculate the probability that a normal random variable with a mean of 10 and a standard deviation of 20 will lie between 10 and 40.

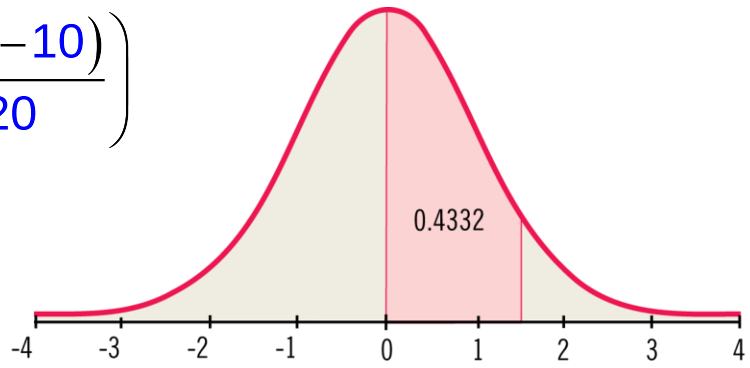
Solution:

Applying the z-transformation yields

$$P(10 \le X \le 40) = P\left(\frac{(10-10)}{20} \le \frac{x-\mu}{\sigma} \le \frac{(40-10)}{20}\right)$$

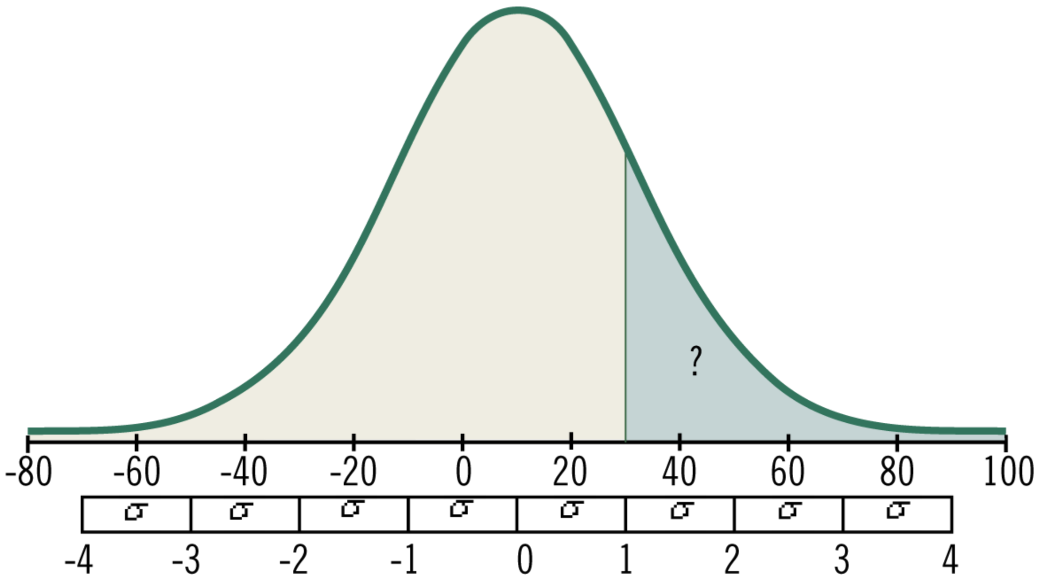
$$= P(0 \le z \le 1.5)$$

$$= 0.4332$$



Example:

Calculate the probability that a normal random variable with a mean of 10 and a standard deviation of 20 will be greater than 30.



Example:

Calculate the probability that a normal random variable with a mean of 10 and a standard deviation of 20 will be greater than 30.

Solution:

Applying the *z*-transformation yields

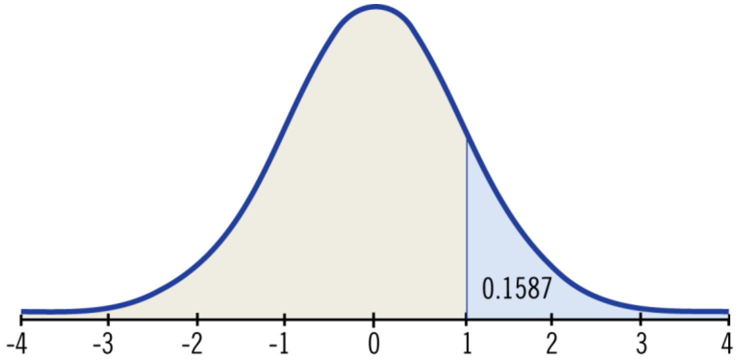
$$P(X > 30) = P\left(z > \frac{30-10}{20}\right)$$

$$= P(z > 1)$$

$$P(z > 1) = P(0 < z < \infty) - P(0 < z < 1)$$

$$= .5 - .3413$$

$$= .1587$$

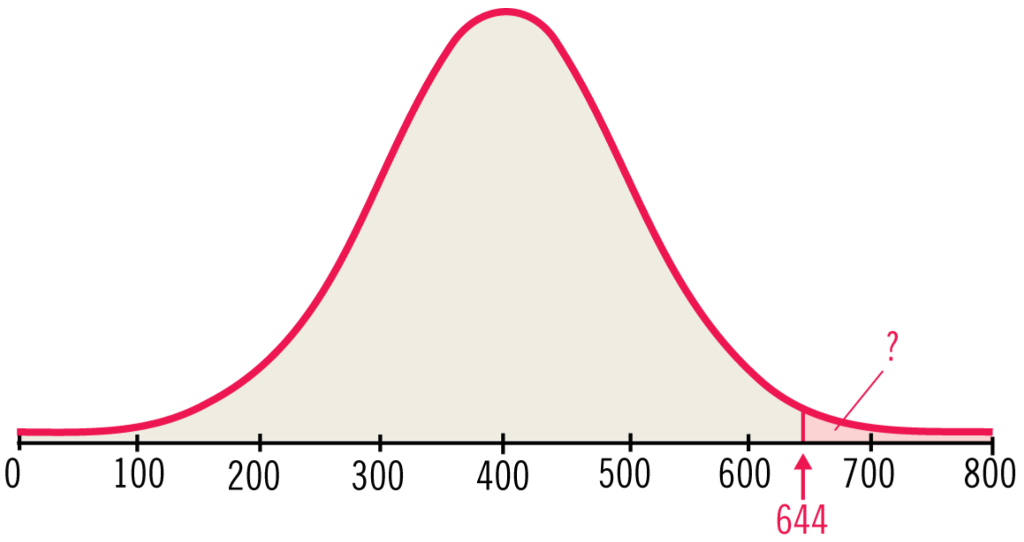


Example:

Suppose that a national testing service gives a test in which the results are normally distributed with a mean of 400 and a standard deviation of 100. If you score a 644 on the test, what fraction of the students taking the test exceeded your score?

Solution:

Let X = a student's score on the test.



Example:

Suppose that a national testing service gives a test in which the results are normally distributed with a mean of 400 and a standard deviation of 100. If you score a 644 on the test, what fraction of the students taking the test exceeded your score?

Solution:

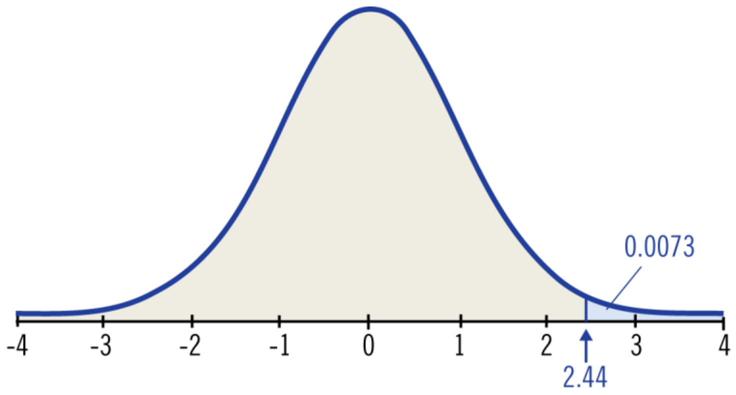
The first step is to apply the *z*-transformation.

$$P(X > 644) = P\left(z > \frac{(644 - 400)}{100}\right)$$

$$= P(z > 2.44)$$

$$= .5 - .4927$$

$$= .0073$$



Thus, only 0.73% if the students scored higher than your score of 644.

Four basic types of probability problems:

1. Probability less than some value
2. Probability greater than some value
3. Probability between two values
4. Probability less than one value and greater than another value

Determine the probability:

Deviation IQ scores, sometimes called Wechsler IQ scores, are scores with a mean of 100 and a standard deviation of 15. What percentage of the general population have IQ's lower than 92?

Solution:

$$\mu = 100,\ \sigma = 15,\ x = 92$$

$$z = \frac{x - \mu}{\sigma} = \frac{92 - 100}{15} = -0.53$$

$$P(z < -0.53) = 0.2981 = 29.81\%$$

Determine the probability:

Deviation IQ scores, sometimes called Wechsler IQ scores, are scores with a mean of 100 and a standard deviation of 15. What percentage of the general population have IQ's larger than 130?

Solution:

$$\mu = 100,\ \sigma = 15,\ x = 130$$

$$z = \frac{x - \mu}{\sigma} = \frac{130 - 100}{15} = 2.00$$

$$P(z > 2.00) = 0.0228 = 2.28\%$$

Determine the probability:

Deviation IQ scores, sometimes called Wechsler IQ scores, are scores with a mean of 100 and a standard deviation of 15. What percentage of the general population have IQ's between 90 and 110?

Solution:

$$\mu = 100,\ \sigma = 15,\ x_1 = 90 \text{ and } x_2 = 110$$

$$z = \frac{x - \mu}{\sigma} = \frac{90 - 100}{15} = -0.67$$

$$z = \frac{110 - 100}{15} = 0.67$$

$$P(-0.67 < z < 0.67) = 0.4972 = 49.72\%$$

Determine the probability:

Deviation IQ scores, sometimes called Wechsler IQ scores, are scores with a mean of 100 and a standard deviation of 15. What percentage of the general population have IQ's less than 80 and greater than 120?

Solution:

$$\mu = 100,\ \sigma = 15,\ x_1 = 80 \text{ and } x_2 = 120$$

$$z = \frac{x - \mu}{\sigma} = \frac{80 - 100}{15} = -1.33$$

$$z = \frac{120 - 100}{15} = 1.33$$

$$P(z < -1.33 \text{ or } z > 1.33) = 0.1835 = 18.35\%$$

Determine the probability:

In a recent year, the ACT scores for high school students with a 3.50 to 4.00 GPA were normally distributed, with a mean of 24.2 and a standard deviation of 4.2. A student who took the ACT during this time is selected. Find the probability that the student's ACT score is less than 20.

Solution:

$$\mu = 24.2,\ \sigma = 4.2,\ x = 20$$

$$z = \frac{x - \mu}{\sigma} = \frac{20 - 24.2}{4.2} = -1.00$$

$$P(z < -1.00) = 0.1587 = 15.87\%$$

Determine the probability:

In a recent year, the ACT scores for high school students with a 3.50 to 4.00 GPA were normally distributed, with a mean of 24.2 and a standard deviation of 4.2. A student who took the ACT during this time is selected. Find the probability that the student's ACT score is greater than 31.

Solution:

$$\mu = 24.2,\ \sigma = 4.2,\ x = 31$$

$$z = \frac{x - \mu}{\sigma} = \frac{31 - 24.2}{4.2} = 1.62$$

$$P(z > 1.62) = 0.0526 = 5.26\%$$

Determine the probability:

In a recent year, the ACT scores for high school students with a 3.50 to 4.00 GPA were normally distributed, with a mean of 24.2 and a standard deviation of 4.2. A student who took the ACT during this time is selected. Find the probability that the student's ACT score is between 25 and 32.

Solution:

$$\mu = 24.2,\ \sigma = 4.2,\ x_1 = 25 \text{ and } x_2 = 32$$

$$z = \frac{x - \mu}{\sigma} = \frac{25 - 24.2}{4.2} = 0.19$$

$$z = \frac{32 - 24.2}{4.2} = 1.86$$

$$P(0.19 < z < 1.86) = 0.3933 = 39.33\%$$

Example

Find the area under the normal curve in each of the cases

(a) $z = 0$ and $z = 1.2$; *(b) $z = -0.68$ and $z = 0$;*

(c) $z = -0.46$ and $z = 2.21$; *(d) $z = 0.81$ and $z = 1.94$;*

(e) To the left of $z = 0.6$; *(f) Right of $z = -1.28$.*

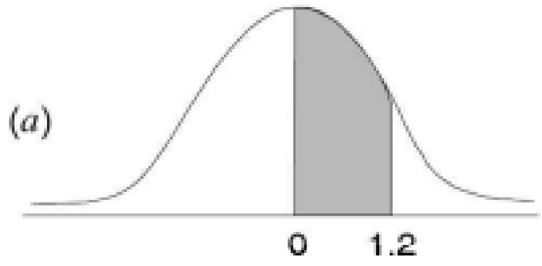
Solution.

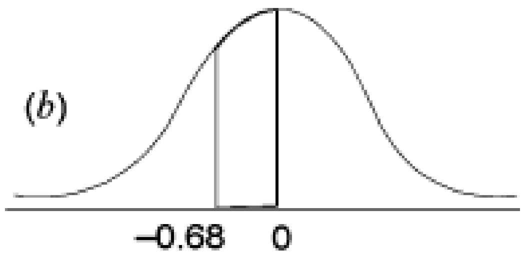
(*a*) Area between $z = 0$ and $z = 1.2$

$= .3849$

(*b*) Area between $z = 0$ and $z = -0.68$

$= 0.2518$





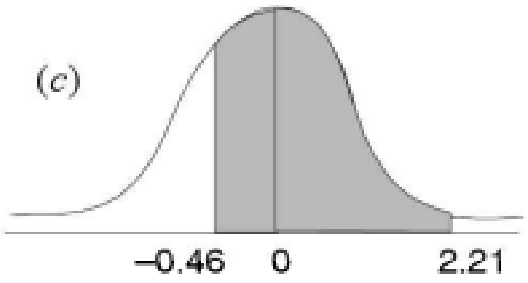
(*c*) Required area = (Area between $z = 0$ and $z = 2.21$)

+ (Area between $z = 0$ and $z = -0.46$)

= (Area between $z = 0$ and $z = 2.21$)

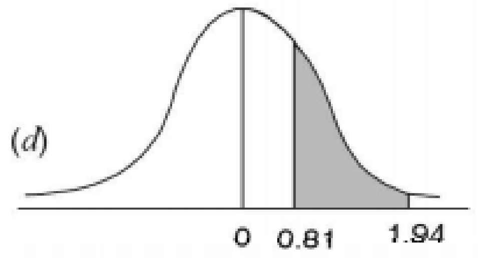
+ (Area between $z = 0$ and $z = 0.46$)

$= 0.4865 + 0.1772 = 0.6637.$



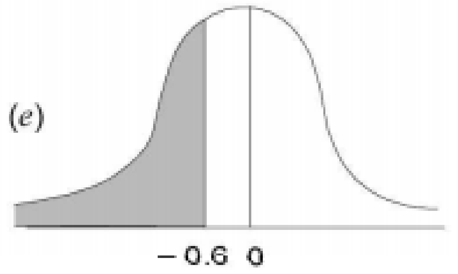
(d) Required area = (Area between $z = 0$ and $z = 1.94$) – (Area between $z = 0$ and $z = 0.81$)

$= 0.4738 - 0.2910 = 0.1828$



(e) Required area $= 0.5 -$ (Area between $z = 0$ and $z = 0.6$)

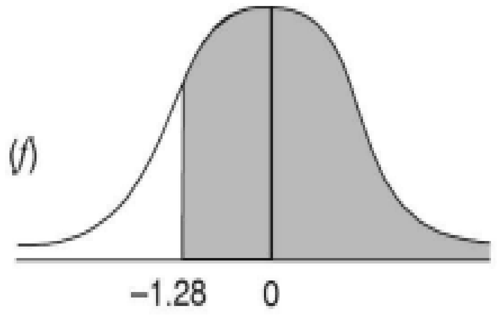
$= 0.5 - 0.2257 = 0.2743$



(f) Required area = (Area between $z = 0$ and $z = -1.28$) + 0.5

$= 0.3997 + 0.5$

$= 0.8997.$



Example

In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5. Assuming the distribution to be normal, find

(*i*) *how many students score between 12 and 15 ?*

(*ii*) *how many score above 18 ?* (*iii*) *how many score below 8 ?*

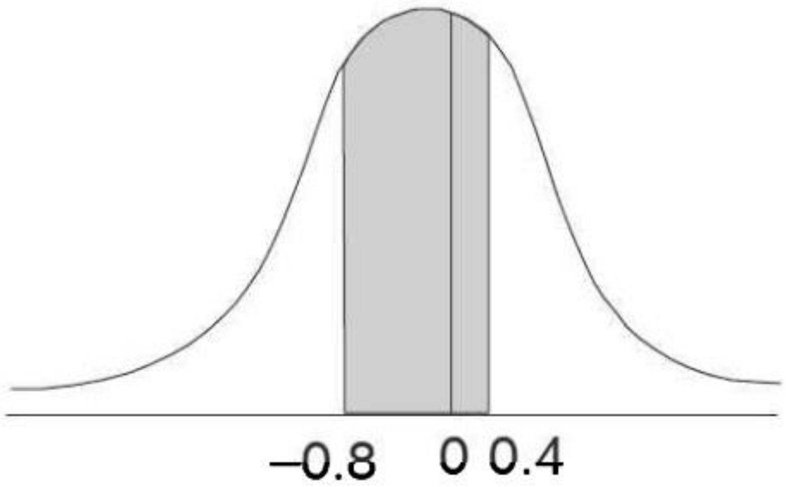
(*iv*) *how many score 16 ?*

Solution. $n = 1000, \ \bar{x} = 14, \sigma = 2.5$

(*i*)

$$z_1 = \frac{x - \bar{x}}{\sigma} = \frac{12 - 14}{2.5} = -0.8$$

$$_2 = \frac{15 - 14}{2.5} = \frac{1}{2.5} = 0.4$$



The area lying between -0.8 to 0.4 = Area from 0 to -0.8 + area from 0 to 0.4

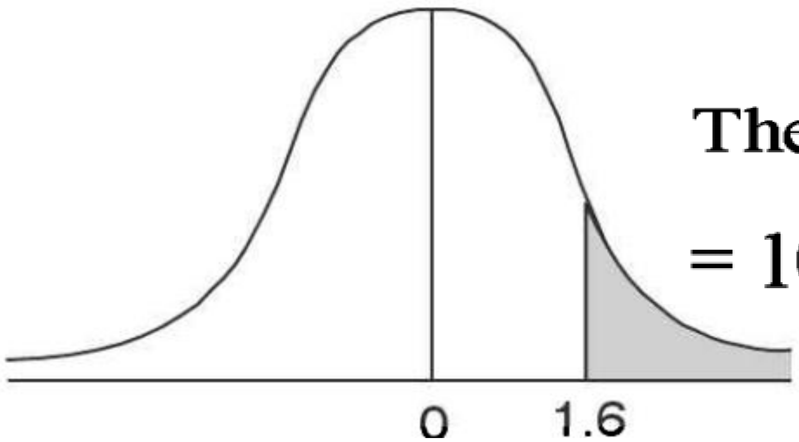
$$= 0.2881 + 0.1554 = 0.4435$$

The required number of students = $1000 \times 0.4435 = 443.5 = 444$ (say)

(*ii*) $$z_1 = \frac{18-14}{2.5} = \frac{4}{2.5} = 1.6$$

Area right to 1.6 $= 0.5 -$ Area between 0 and 1.6

$$= 0.5 - 0.4452 = 0.0548$$



The required number of students

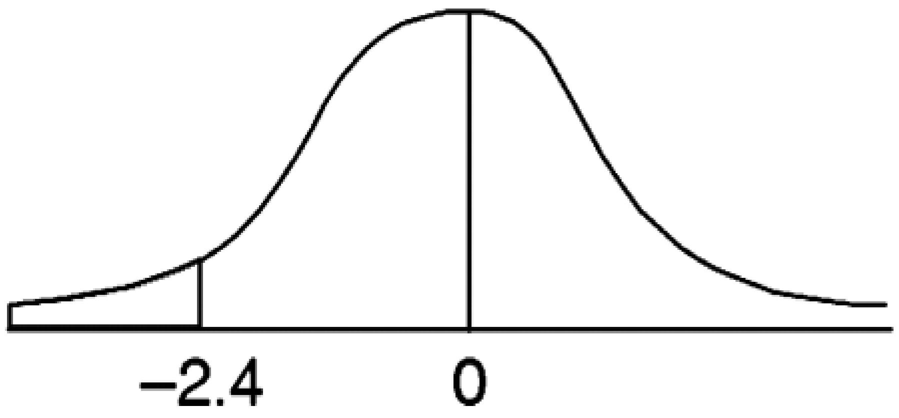
$$= 1000 \times 0.0548 = 54.8 = 55 \text{ (say)}$$

(iii) $$z = \frac{8-14}{2.5} = -\frac{6}{2.5} = -2.4$$

Area left to –2.4 = 0.5 – area between 0 and –2.4

= 0.5 – 0.4918 = 0.0082

The required number of students = 1000 × 0.0082 = 8.2 = 8 (say)

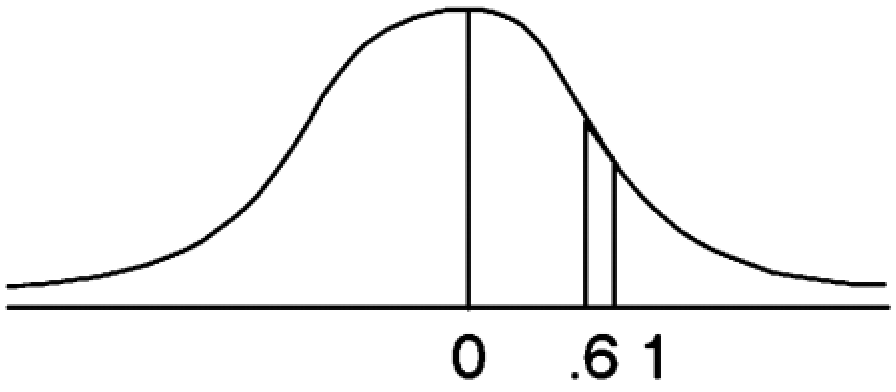


(*iv*) Area between 15.5 and 16.5

$$z_1 = \frac{15.5 - 14}{2.5} = 0.6$$

$$z_2 = \frac{16.5 - 14}{2.5} = 1$$

Area between 0.6 and 1 $= 0.3413 - 0.2257 = 0.1156$



The required number of students $= 0.1156 \times 1000 = 115.6 = 116$ say

Objectives:

- Understand the concept of using the normal distribution to approximate discrete distributions.
- Learn how to use the continuity correction factor.
- Use the normal approximation to calculate a binomial probability.
- Use the normal approximation to calculate a Poisson probability.

Review of Binomial Distribution:

- The experiment consists of n identical trials.
- Each trial is independent of the others.
- For each trial, there are only two possible outcomes. For counting purposes, one outcome is labeled a *success*, the other a *failure*.
- For every trial, the probability of getting a success is called p. The probability of getting a failure is then $1 – p$.
- The binomial random variable, X, is the number of successes in n trials.

Normal Distribution Approximation of a Binomial Distribution:

- If the conditions that $np \geq 5$ and $n(1 - p) \geq 5$ are met for a given binomial distribution, then a normal distribution can be used to approximate its probability distribution with the given mean and variance:

$$\mu = E(X) = np, \text{ and}$$

$$\sigma^2 = V(X) = np(1-p).$$

Example:

Approximate a binomial with n = 20 and p =.5.

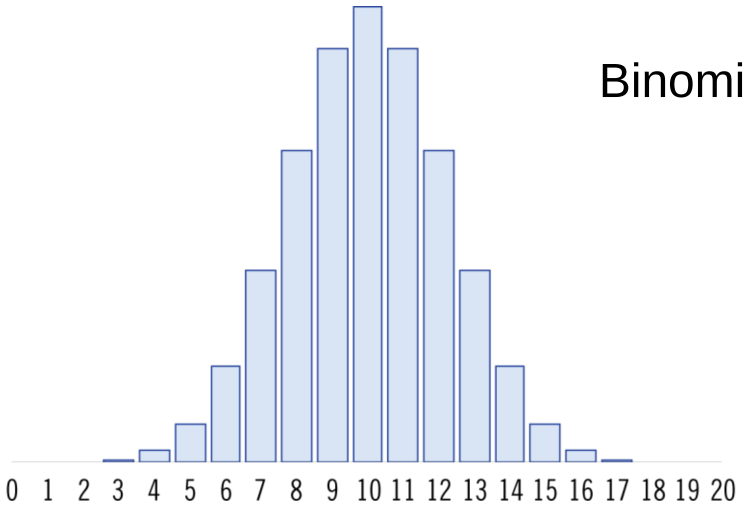
Solution:

To approximate a binomial with n = 20 and p = .5 would require a normal distribution with

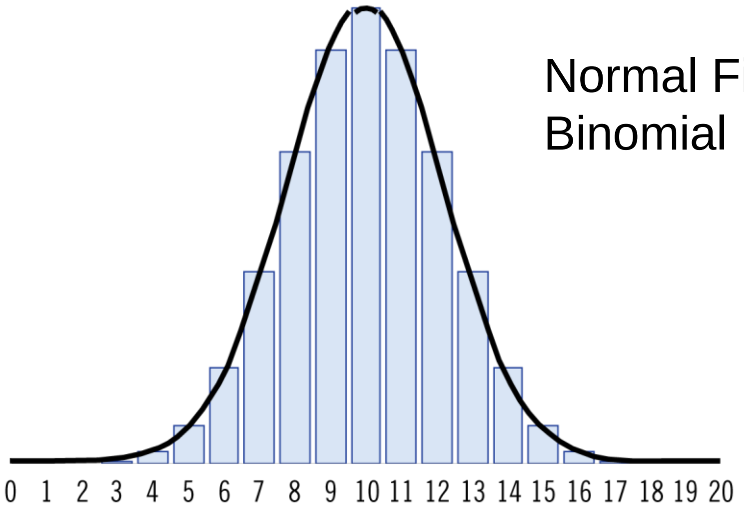
$$\mu = (20)(.5) = 10,$$

$$\sigma^2 = (20)(.5)(1-.5) = 5, \text{ and}$$

$$\sigma = \sqrt{5} = 2.236.$$



Binomial



Normal Fit of Binomial

Continuity Correction:

A **continuity correction** is a correction factor employed when using a continuous distribution to approximate a discrete distribution.

Examples of the Continuity Correction		
Statement	**Symbolically**	**Area**

Process for Using the Normal Curve to Approximate the Binomial Distribution:

1. Determine the values of n and p.
2. Verify that the conditions $np \geq 5$ and $n(1 - p) \geq 5$ are met.
3. Calculate the values of the mean and variance using the formulas

$$\mu = np$$

and

$$\sigma^2 = np(1 - p).$$

4. Use a continuity correction to determine the interval corresponding to the value of x.
5. Draw a normal curve labeled with the information in the problem.
6. Convert the value of the random variable(s) to a z-value(s).
7. Use the normal curve table to find the appropriate area under the curve.

Example:

Assuming n = 20, p = .5, use the normal distribution to approximate the probability that a binomial random variable was 5 or less.

Solution:

np = 10 and $n(1-p)$ = 10 which are both greater than or equal to 5.

$$\mu = E(X) = np = (20)(.5) = 10$$

$$\sigma^2 = V(X) = np(1-p) = 20(.5)(1-.5) = 5$$

$$\sigma = \sqrt{5} = 2.236$$

Using the continuity correction, add 0.5 to 5.

Example:

Assuming n = 20, p = .5, use the normal distribution to approximate the probability that a binomial random variable was 5 or less.

Solution:

Using the normal distribution, called Y, with mean 10 and variance 5, to approximate the binomial using continuity correction,

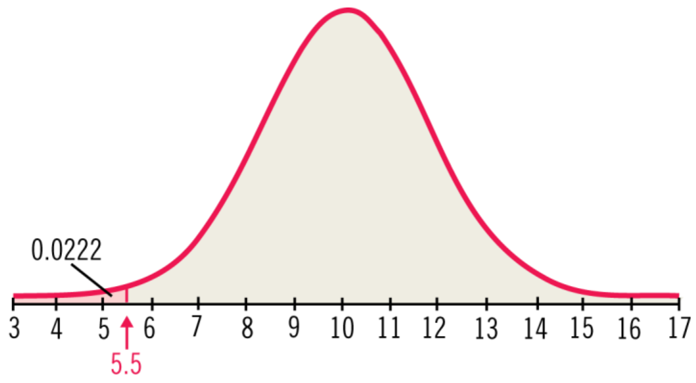
$$P(Y \leq 5.5) = P\left(z \leq \frac{(5.5 - 10)}{2.236}\right)$$

$$= P(z \leq -2.01)$$

$$= .5 - P(0 \leq z \leq 2.01)$$

$$= .5 - 0.4778$$

$$= 0.0222.$$



Example:

Assuming n = 20, p = .5, use the normal distribution to approximate the probability that a binomial random variable was greater than 4.

Solution:

np = 10 and $n(1-p)$ = 10 which are both greater than or equal to 5.

$$\mu = E(X) = np = (20)(.5) = 10$$

$$\sigma^2 = V(X) = np(1-p) = 20(.5)(1-.5) = 5$$

$$\sigma = \sqrt{5} = 2.236$$

Using the continuity correction add 0.5 to 4.

Example:

Assuming n = 20, p = .5, use the normal distribution to approximate the probability that a binomial random variable was greater than 4.

Solution:

Using the normal distribution, called Y, with mean 10 and variance 5, to approximate the binomial using continuity correction,

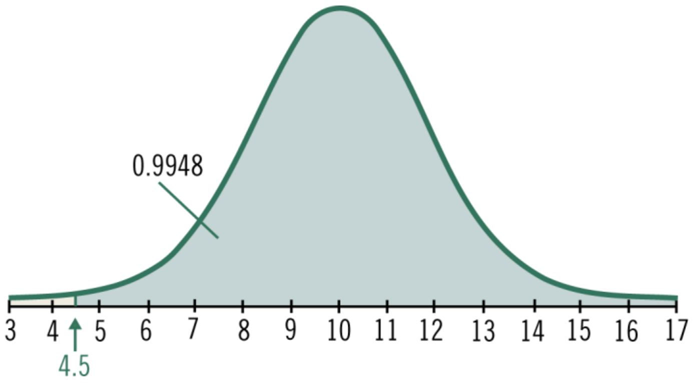
$$P(Y \geq 4.5) = P\left(z \geq \frac{(4.5 - 10)}{2.236}\right)$$

$$= P(z \geq -2.46)$$

$$= .5 + P(0 \leq z \leq 2.46)$$

$$= .5 + .4948$$

$$= .9948.$$



Example:

Assuming n = 20, p = .5, we found that using the normal distribution to approximate the probability with the continuity correction that a binomial random variable was 5 or less is 0.0222. Find the probability that the random variable is 5 or less *without* the continuity correction.

Solution:

$$\mu = E(X) = np = (20)(.5) = 10$$

$$\sigma^2 = V(X) = np(1-p) = 20(.5)(1-.5) = 5$$

$$\sigma = \sqrt{5} = 2.236$$

Example:

Assuming n = 20, p = .5, we found that using the normal distribution to approximate the probability with the continuity correction that a binomial random variable was 5 or less is 0.0222. Find the probability that the random variable is 5 or less *without* the continuity correction.

Solution:

$$
\begin{aligned}
P(Y \le 5) &= P\left(z \le \frac{(5-10)}{2.236}\right) \\
&= P(z \le -2.24) \\
&= .5 - P(0 \le z \le 2.24) \\
&= .5 - .4875 \\
&= .0125.
\end{aligned}
$$

Normal Distribution Approximation of a Poisson Distribution:

- Approximating the Poisson distribution is similar to approximating the binomial distribution.
- To use this distribution, the mean and variance of the normal should be set to the mean and variance of the Poisson.

$$\mu = \lambda,\ \sigma^2 = \lambda,\ \sigma = \sqrt{\lambda}.$$

Example:

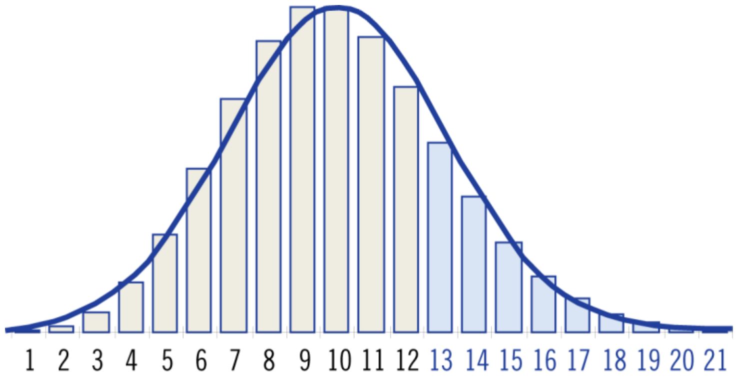
Suppose that calls arrive following a Poisson distribution with an average number of 10 calls per hour. What is the probability that in a given hour more than 12 calls will be received? Use a normal approximation to find the desired probability.

Solution:

Let *X* = the number of telephone calls in an hour. The random variable *X* has a Poisson distribution with

$$\mu = 10, \text{ and}$$

$$\sigma = \sqrt{10} = 3.16.$$



Example:

Suppose that calls arrive following a Poisson distribution with an average number of 10 calls per hour. What is the probability that in a given hour more than 12 calls will be received? Use a normal approximation to find the desired probability.

Solution:

If Y is a random normal variable with mean 10 and standard deviation 3.16, it should be a good approximation to the Poisson.

$$P(Y > 12) = P\left(z > \frac{(12-10)}{3.16}\right)$$

$$= P(z > 0.6329)$$

$$= .5 - P(0 \le z \le 0.6329)$$

$$= .5 - .2357$$

$$= .2643.$$