MATH

"Differentiation"

MNS

\# $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{x - \sin x}{x^3} = 1/6$

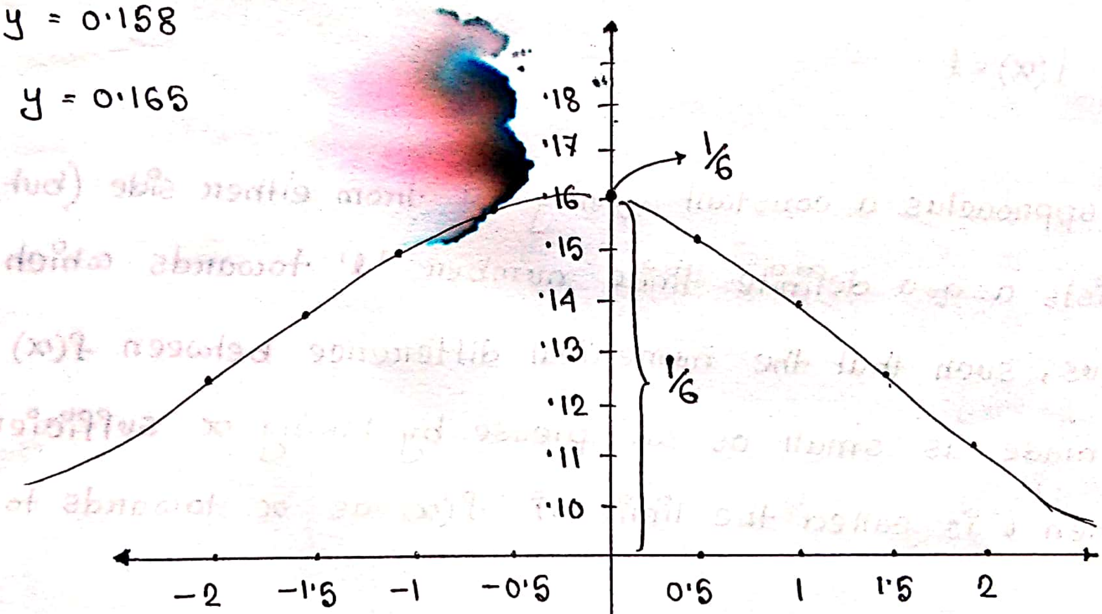
$\Rightarrow y = f(x) = \dfrac{x - \sin x}{x^3} \quad [-2 < x < 2]$

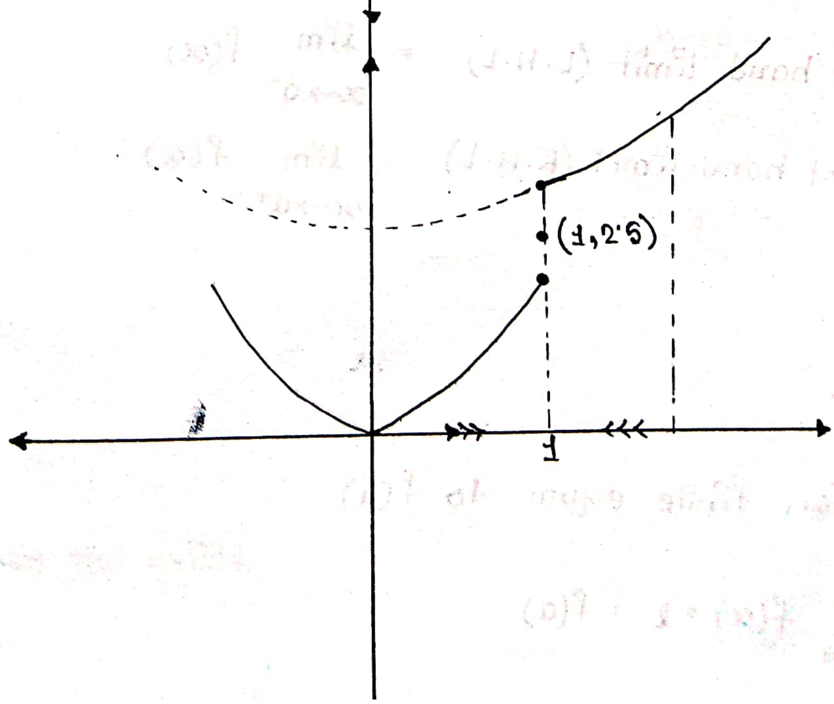
$x = 2 \Rightarrow y = 0.136$

$x = 1.5 \Rightarrow y = 0.148$

$x = 1 \Rightarrow y = 0.158$

$x = 0.5 \Rightarrow y = 0.165$





\# $\varphi(x) = x^2, \quad x < 1$

$= 2.5, \quad x = 1$

$= x^2 + 2, \quad x > 1$

$f(x) = |x|$

$$f(x) = x, \quad x > 0$$
$$= 0, \quad x = 0$$
$$= -x, \quad x < 0$$

Limit of a function:

$$\underset{x \to a}{Lt} f(x) = l$$

When x approaclus a constant quality 'a' from either side (but $\neq a$) if there exists a ~~qua~~ definite finite number 'l' towards which $f(x)$ approaclus, such that the numerical difference between $f(x)$ and l can be made as small as we please by taking x sufficiently close to a, then l is called the limit of $f(x)$ as x towards to a.

Left hand limit (L.H.L) $= \lim\limits_{x \to a^-} f(x)$

Right hand limit (R.H.L) $= \lim\limits_{x \to a^+} f(x)$

limit exist, finite, equal to $f(a)$

$$\lim_{x \to a} f(x) = l = f(a)$$

$f(x) = \frac{1}{2} - x$, when $0 < x < \frac{1}{2}$

$= \frac{1}{2}$, when $x = \frac{1}{2}$

$= \frac{3}{2} - x$, when $\frac{1}{2} < x < 1$

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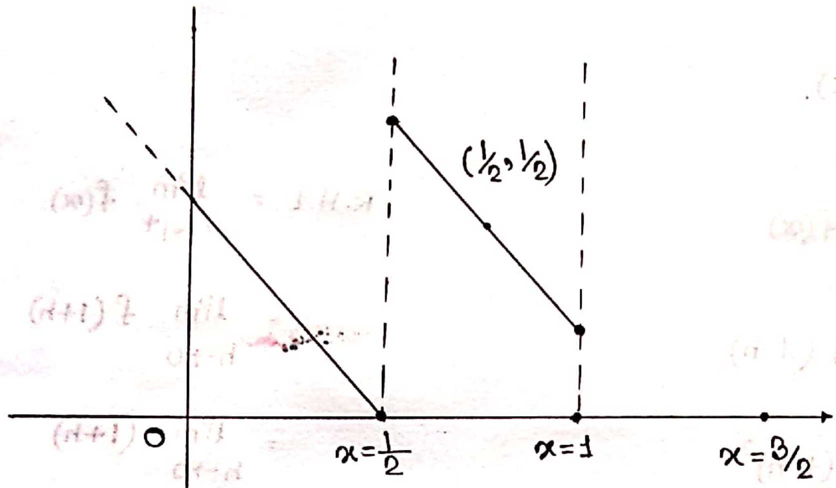
Show that $f(x)$ is discontinuous at $x = \frac{1}{2}$.



$$L.H.L = \lim_{x \to \frac{1}{2}^-} f(x)$$

let, $x = \frac{1}{2} - h$

$\Rightarrow h = \frac{1}{2} - x$

$$= \lim_{h \to 0} f(\tfrac{1}{2} - h)$$

$$= \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right]$$

$$= \lim_{h \to 0} (h)$$

$$= 0$$

$$R.H.L = \lim_{x \to \frac{1}{2}^+} f(x)$$

let, $x = \frac{1}{2} + h$

$$= \lim_{h \to 0} f(\tfrac{1}{2} + h)$$

$$= \lim_{h \to 0} \left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right]$$

$$= 1$$

$\therefore L.H.L \neq R.H.L$

$\therefore \lim_{x \to \frac{1}{2}} f(x)$ does not exists.

$\therefore$ function is discontinuous at $x = \frac{1}{2}$.

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$f(x) = x^2 \; ; \; 0 < x < 1$

$= x \; ; \; 1 \leq x < 2$

$= \frac{1}{4}x^3 \; ; \; 2 \leq x < 3$

Check the continity for $x=1$ and $x=2$.

$\Rightarrow$ a) $\lim\limits_{x \to 1} f(x)$.

$$L.H.L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{h \to 0} f(1-h)$$

$$= \lim_{h \to 0} (1-h)^2$$

$$= \lim_{h \to 0} (1-2h+h^2)$$

$$= 1$$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{h \to 0} f(1+h)$$

$$= \lim_{h \to 0} (1+h)$$

$$= 1$$

$L.H.L = R.H.L.$

$\therefore \lim\limits_{x \to 1} f(x) = 1 = f(1)$

$\therefore f(1) = 1$

$\therefore f(x)$ is continuous at $x = 1$.

continity → 4, 5, 9

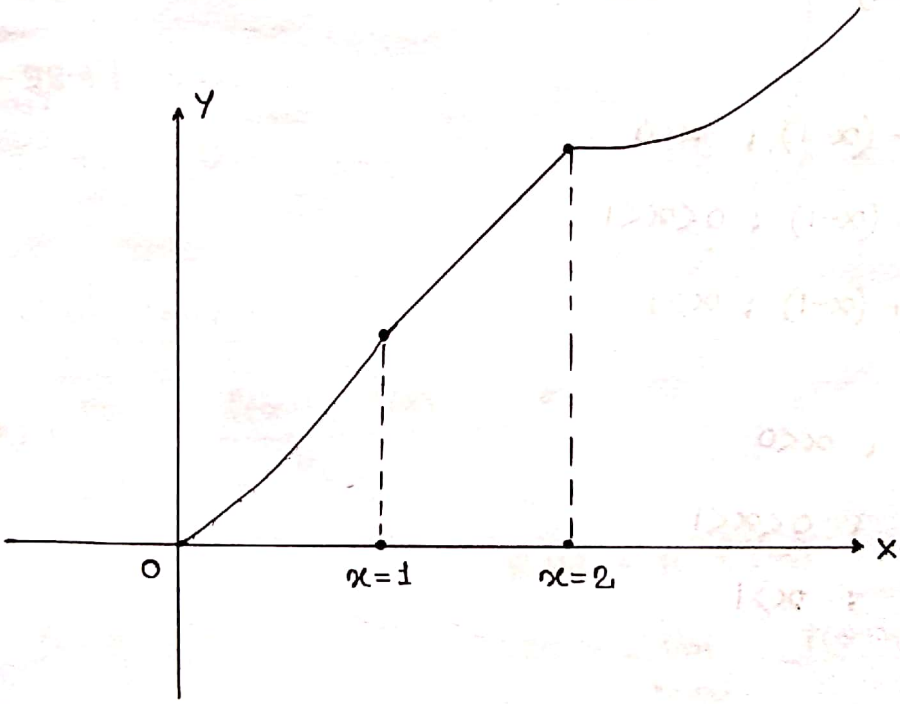
b) $\lim_{x \to 2} f(x)$

$$L.H.L = \lim_{x \to 2^-} f(x)$$

$$= \lim_{h \to 0} f(2-h)$$

$$= \lim_{h \to 0} (2-h)$$

$$= \lim_{h \to 0}$$



$|x|$

$|x-1|$

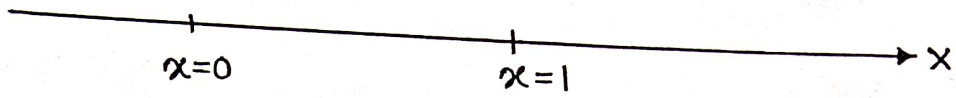
$x<1 \longrightarrow -(x-1)$

$x>1 \longrightarrow (x-1)$

$x=1 \longrightarrow 0$

$y = |x| + |x-1|$

$\Rightarrow$



$x<0$ $\downarrow$ $-x-(x-1)$

$0<x<1$ $\downarrow$ $+x-(x-1)$

$x>1$ $\downarrow$ $+x+(x-1)$

$$y = f(x) = -x-(x-1) \; ; \; x<0$$
$$= +x-(x-1) \; ; \; 0<x<1$$
$$= +x+(x-1) \; ; \; x>1$$

$$y = \begin{cases} -2x+1 & ; \; x<0 \\ +1 & ; \; 0<x<1 \\ 2x-1 & ; \; x>1 \end{cases}$$

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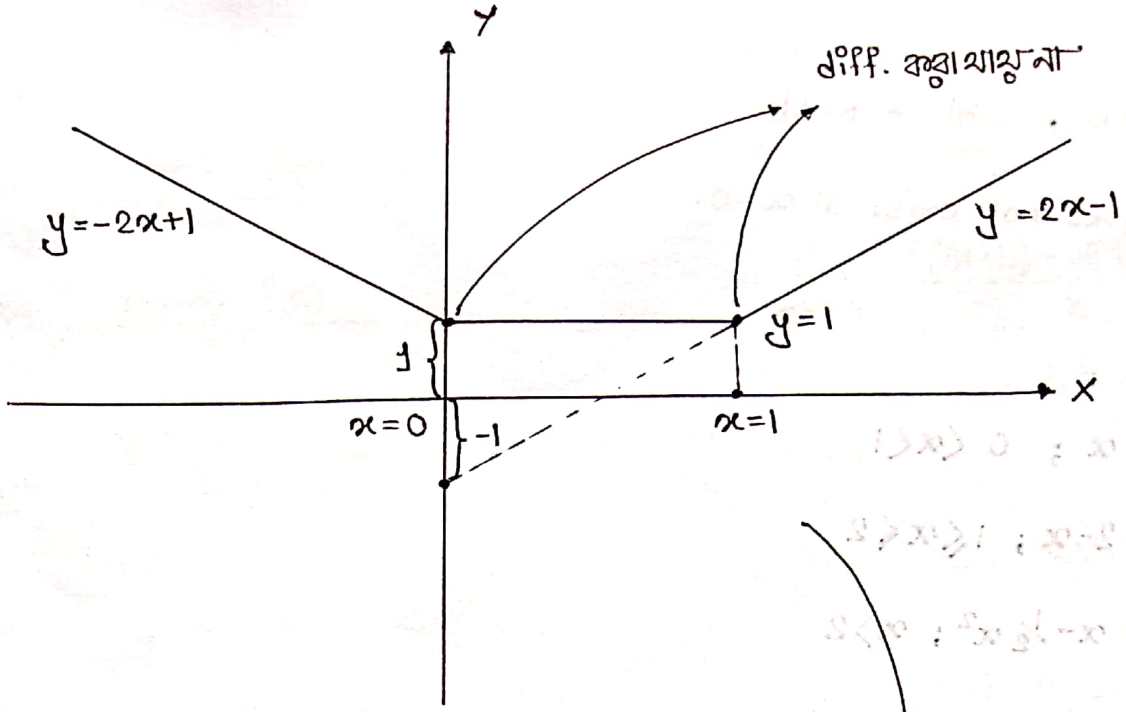


body temp.

$y = |x - 98.4|$

$= |104.4 - 98.4| = 6.0$

$= |94.4 - 98.4| = 4.0$

First Principal of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

L.H.L = $Lf'(x)$

$$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-2(0-h)+1-1}{-h}$$

$$= \lim_{h \to 0} \frac{2h}{-h}$$

$= -2$

R.H.L = $R.f'(x)$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h}$$

$= 0$

at $x=0$, L.HD $\neq$ R.H.D

$\therefore f'(x)$ does not exist at $x=0$.

\# $f(x) = x \; ; \; 0 < x < 1$

$= 2-x \; ; \; 1 \leq x \leq 2$

$= x - \frac{1}{2}x^2 \; ; \; x > 2$

Does $f'(1)$ and $f'(2)$ exists ?

$\Rightarrow$ L.H.D $= Lf'(x)$

$$= \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h}$$

$$= \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{(1-h)-1}{-h}$$

$$= \lim_{h\to 0} \frac{-h}{-h}$$

$= 1$

R.H.D $= R.f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{2-(1+h)-1}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h}$$

$= -1$.

$$L.H.D = Lf'(x)$$

$$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{2-2+h-0}{-h}$$

$$= -1$$

$$R.H.D = R.f'(x)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h) - \frac{1}{2}(x-h)^2 - 0}{h}$$

$$= \lim_{h \to 0} \frac{(2+h) - \frac{1}{2}(2+h)^2 - (2-2)}{h}$$

$$= \lim_{h \to 0} \frac{2+h-\frac{1}{2}(4+4h+h^2) - 0}{h}$$

$$= \lim_{h \to 0} \frac{2+h-2-2h+\frac{1}{2}h^2}{h}$$

$$= \lim_{h \to 0} \frac{-1-\frac{1}{2}h}{1}$$

$$= -1$$

$\frac{d}{dx}(\sin x) = \cos x$

$$\lim_{h \to 0} \frac{\log_e (x+h) - \log_e x}{h} = \frac{1}{x}$$

$$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \cos x$$

Find the derivative of $\sin^{-1}x$ using first principal of derivative.

$$\frac{d}{dx}(\sin^{-1}x) = \lim_{h \to 0} \frac{\sin^{-1}(x+h) - \sin^{-1}x}{h}$$

$\neq$

let,

$y = \sin^{-1}x$ —— (i)

$\Rightarrow x = \sin y$

$\Rightarrow x+h = \sin(y+K)$

$\therefore \sin^{-1}(x+h) = y+K$ —— (ii)

we get,

$x+h-x = \sin(y+K) - \sin y$

$\Rightarrow h = \sin(y+K) - \sin y$

If $h \to 0$, then $K \to 0$

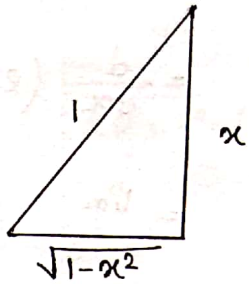
$$= \lim_{h \to 0} \frac{y+K-y}{h}$$

$$= \lim_{h \to 0} \frac{K}{h}$$

$$= \lim_{K \to 0} \frac{K}{\sin(y+K) - \sin y}$$

$$= \lim_{K \to 0} \frac{1}{\frac{\sin(y+K) - \sin y}{K}}$$

$$= \frac{1}{\lim_{K \to 0} \frac{\sin(y+K) - \sin y}{K}}$$



$$= \frac{1}{\cos y}$$

$$= \sec y$$

$$= \sec(\sin^{-1} x)$$

$$= \sec\left(\sec^{-1} \frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}}.$$

$\frac{d}{dx}[\log(\sec x)] = \lim_{h \to 0} \frac{\log \sec(x+h) - \log \sec(x)}{h}$

let,

$y = \sec x$

$y+K = \sec(x+h)$

$\therefore K = \sec(x+h) - \sec x$

as $h \to 0$ then $K \to 0$

$$= \lim_{h \to 0} \frac{\log(y+K) - \log y}{h}$$

$$= \lim_{K \to 0} \frac{\log(y+K) - \log y}{K} \cancel{\lim_{h \to 0}}\left(\frac{K}{h}\right)$$

$$= \frac{1}{y} \cdot \lim_{h \to 0} \left(\frac{K}{h}\right)$$

$$= \frac{1}{y} \cdot \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \frac{1}{y} \cdot \sec x . \tan x$$

$$= \frac{1}{\sec x} \cdot \sec x . \tan x$$

$$= \tan x.$$

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$$\lim_{h \to 0} \frac{e^{n-1}}{n} = 1$$

$\frac{d}{dx}(x^x) = \frac{d}{dx}\left(e^{\ln x^x}\right)$

$$= \frac{d}{dx}\left(e^{x \log x}\right)$$

$$= \lim_{h \to 0} \frac{e^{(x+h)\log(x+h)} - e^{x \log x}}{h}$$

$$= e^{x \log x} \left[\lim_{h \to 0} \frac{e(x+h)\log(x+h) - x\log x - 1}{h}\right]$$

$$= e^{x\log x} \cdot \lim_{k \to 0} \frac{e^k - 1}{k} \cdot \lim_{h \to 0} \frac{k}{h}$$

let,

$$k = (x+h)\log(x+h) - x\log x$$

as $h \to 0$, then $k \to 0$

$$= e^{x\log x} \cdot 1 \cdot \lim_{h \to 0} \frac{(x+h)\log(x+h) - x\log x}{h}$$

$$= e^{x\log x} \cdot \left[\lim_{h \to 0} \frac{x\log(x+h) - x\log x}{h} + \lim_{h \to 0} \frac{h\log(x+h)}{h}\right]$$

$$= e^{x\log x}\left(x \cdot \frac{1}{x} + \log x\right)$$

$$= x^x(1 + \log x)$$

$\frac{d}{dx}\left[\log \sin\left(\frac{x}{a}\right)\right]$ $\left(\frac{x}{a}+h\right)$

$$= \lim_{h\to 0} \frac{\log \sin\left(\frac{x+\cancel{h}}{\cancel{a}}\right) - \log \sin\left(\frac{x}{a}\right)}{h}$$

$$= \lim_{K\to 0} \frac{\log(y+K) - \log(y)}{K} \cdot \lim_{h\to 0} \frac{\sin\left(\frac{x}{a}+h\right) - \sin\left(\frac{x}{a}\right)}{h}$$

$$= \frac{1}{y} \cdot \qquad \cos\left(\frac{x}{a}\right)\cdot \frac{1}{a}$$

$$= \frac{1}{y} \cdot \cos\left(\frac{x}{a}\right) \cdot \frac{1}{a}$$

let,

$y = \sin\left(\frac{x}{a}\right)$ $\left(\frac{x}{a}+h\right)$

$y+K = \sin\left(\frac{x\cancel{+h}}{\cancel{a}}\right)$ $\left(\frac{x}{a}+h\right)$

$\therefore K = \sin\left(\frac{x\cancel{+h}}{\cancel{a}}\right) - \sin\left(\frac{x}{a}\right)$

as $h \to 0$ then $K \to 0$

let,

$z+m = \sin\left(\frac{x}{a}+h\right)$

$*$ sin, cos, tan, uv

$\frac{d}{dx}\left[a \sin\left(\frac{x}{a}\right)\right]$

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$y = \tan^{-1}\left(\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right)$, then find $\frac{dy}{dx}$.

$$= \tan^{-1}\left(\frac{\left(\sqrt{1+\sin x} - \sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2 - \left(\sqrt{1-\sin x}\right)^2}\right)$$

$$= \tan^{-1} \frac{1+\cancel{\sin x} - 2.\sqrt{(1+\sin x)(1-\sin x)} + 1-\cancel{\sin x}}{2\sin x}$$

$$= \tan^{-1}\left(\frac{1-\sqrt{1-\sin^2 x}}{\sin x}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$$

$$= \tan^{-1} \frac{2\sin^2 x/2}{2\sin x/2 . \cos x/2}$$

$$= \tan^{-1}\left(\tan x/2\right)$$

$$y = x/2$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

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$y = \cot^{-1}\left(\frac{1+x}{1-x}\right)$, $\frac{dy}{dx} = ?$

let,

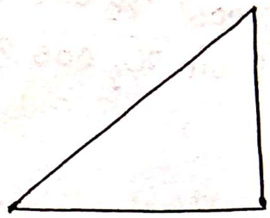
$x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$$= \cot^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \cot^{-1}\left(\frac{\tan \pi/4 + \tan\theta}{1-\tan \pi/4 \tan\theta}\right)$$

$$= \cot^{-1}\left(\tan\left(\pi/4+\theta\right)\right)$$



$$= \cot^{-1}\left[\cot\left\{\frac{\pi}{2}-\left(\pi/4+\theta\right)\right\}\right]$$

$$= \pi/2 - \left(\pi/4+\theta\right)$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} + c$$

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

let,

$x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$$= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}\left(\cos 2\theta\right)$$

$$= 2\theta$$

$$= 2.\tan^{-1}x$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

$y = \cot^{-1}(\operatorname{cosec} x + \cot x)$

$$= \cot^{-1}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2 x/2 \cos x/2}{2\sin x/2 \cos x/2}\right)$$

$$= x/2$$

$y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$; $\frac{dy}{dx} = ?$

$u = (\tan x)^{\cot x}$

$v = (\cot x)^{\tan x}$

~~$\ln y = \cot x . \ln(\tan x) + \tan x . \ln(\cot x)$~~

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$\Rightarrow \log y = \log (u+v)$

এখানে, $u = (\tan x)^{\cot x}$

$\Rightarrow \log u = \cot x \log(\tan x)$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{\cot x}{\tan x}(\sec^2 x) - \log(\tan x). \cancel{\cot x}. \operatorname{cosec}^2 x$

$\Rightarrow$

$\frac{dy}{dx} = (\tan x)^{\cot x}$ [ ]

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Differenciate $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$.

$u = x^{\sin^{-1}x}$

$v = \sin^{-1}x$.

$$\frac{du}{dx} =$$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$\log u = \sin^{-1}x \log x$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = \frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}$$

$$\Rightarrow \frac{du}{dx} = x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right]$$

$$= \frac{x^{\sin^{-1}x}\left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right]}{\frac{1}{\sqrt{1-x^2}}}$$

$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$.

$u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

$v = \tan^{-1}x$

$$\frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\frac{du}{dx} = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{x}$$

$$= \tan^{-1}\frac{\sec\theta-1}{\cancel{x}\tan\theta}$$

$$= \tan^{-1}\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}$$

$$= \tan^{-1}\frac{2\sin^2\theta/2}{2\sin\theta/2\cdot\cos\theta/2}$$

$$= \frac{\theta}{2}$$

$$= \frac{\tan^{-1}x}{2}$$

$$= \frac{1}{2}\cdot\frac{1}{1+x^2}$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{1/2}{1/(1+x^2)}$$

$$= \frac{1+x^2}{2}$$

ans: $1/2$

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$y = \tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}} \cdot \tan \frac{x}{2}\right\}$ $\frac{dy}{dx} = ?$

$$\Rightarrow \tan y = \sqrt{\frac{a-b}{a+b}} \cdot \tan x/2$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = \sqrt{\frac{a-b}{a+b}} \cdot \sec^2 x/2 \cdot (1/2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{a-b}{a+b}} \cdot \sec^2 x/2 \cdot \frac{1}{\sec^2 y}$$

$y = \frac{1}{a+x}$; $y_n = ?$ (Successive differentiation.)

$$\frac{dy}{dx} = y_1 = (-1)(a+x)^{-1-1} = (-1)(a+x)^{-2}$$

$$y_2 = (-1)(-2) \cdot (a+x)^{-3}$$

$$y_3 = (-1)(-2)(-3)(a+x)^{-4} = (-1)^3\, 3!\, (a+x)^{-4}$$

$\vdots$

$$y_n = (-1)^n\, n!\, (a+x)^{-(n+1)}$$

$y = \log(x+a)$, $y_n = ?$

$\frac{dy}{dx} = \frac{1}{x+a} = y_1 = y$ (আগের math এর y)

$y_2 = '$

$\vdots$

$y_n = (-1)^{n-1} (n-1)! \, (a+x)^{-n}$

$y = (ax+b)^m$, $y_n = ?$

$y_1 = m(ax+b)^{m-1} \cdot a$

$y_2 = m(m-1)(ax+b)^{m-2} \cdot a^2$

$y_3 = m(m-1)(m-2)(ax+b)^{m-3} \cdot a^3$

If $n > m \rightarrow$

$y_n = 0$

If $n = m \Rightarrow$

$y_n = n \cdot (n-1)(n-2)(n-3) \cdots [n-(n+1)] \, (ax+b)^{n-n} \, a^n$

$= n(n-1) \cdots 1 \cdot a^n$

$= n! \, a^n$

If $n < m \Rightarrow$

$y_n = m(m-1)(m-2) \cdots (m-(n-1)) \, (ax+b)^{m-n} \, a^n$

$= \frac{m(m-1) \cdots \cdots (m-n+1)(m-n)(m-n-1) \cdots 1}{(m-n)(m-n-1) \cdots 1} \, (ax+b)^{m-n} \cdot a^n$

$= \frac{m!}{(m-n)!} (ax+b)^{m-n} \cdot a^n$

$y = \sin(ax+b)$

$y_1 = a\cos(ax+b) = a\sin\left\{\pi/2 + (ax+b)\right\}$

$y_2 = -a^2\sin(ax+b) = a^2\cos(\pi/2 + (ax+b))$

$= a^2\sin(\pi/2 + \pi/2 + (ax+b))$

$= a^2\sin(2\cdot\pi/2 + (ax+b))$

$y_3 = a^3\cos(2\cdot\pi/2 + (ax+b))$

$= a^3\sin(3\cdot\pi/2 + (ax+b))$

$\therefore y_n = a^n\sin(n\cdot\pi/2 + (ax+b))$

$y = e^{3x}\cos 4x$

$y_1 = e^{3x}(-\sin 4x)\cdot 4 + \cos 4x\cdot 3\cdot e^{3x}$

$= e^{3x}(-\sin 4x) + \cos 4x\cdot 3\cdot e^{3x}$

$= 3e^{3x}\cos 4x - 4e^{3x}\sin 4x$

$= e^{3x}(3\cos 4x - 4\sin 4x)$

$= r e^{3x}(\cos\varphi\cdot\cos 4x - \sin 4x\cdot\sin\varphi)$

$= r e^{3x}\cos(4x+\varphi)$

let,

$3 = r\cos\varphi$

$4 = r\sin\varphi$

$r = \sqrt{3^2+4^2} = 5$

$\varphi = \tan^{-1} 4/3$

$$y_2 = r.\left[3.e^{3x}\cos(4x+\varphi) - 4e^{3x}\sin(4x+\varphi)\right]$$

$$= re^{3x}\left[3\cos(4x+\varphi) - 4\sin(4x+\varphi)\right]$$

$$= r^2 e^{3x}\left\{\cos\varphi.\cos(4x+\varphi) - \sin(4x+\varphi)\sin\varphi\right\}$$

$$= r^2 e^{3x}.\cos(4x+2\varphi)$$

$$\therefore y_n = r^n.e^{3x}.\cos(4x+n\varphi)$$

$$= (3^2+4^2)^{n/2}.e^{3x}.\cos(4x+n\tan^{-1}4/3)$$

$y = \frac{1}{x^2+16}$

$$= \frac{1}{x^2-(4i)^2}$$

$$= \frac{1}{(x+4i)(x-4i)}$$

$$= \frac{1}{8i}\left[\frac{1}{x+4i} - \frac{1}{x-4i}\right]$$

De-Movier's theorum

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

We know if, $y = \frac{1}{a+x}$

$$\therefore y_n = (-1)^n n!(a+x)^{-(n+1)}$$

$$\therefore y_n = \frac{1}{8i}\left[(-1)^n n!(x+4i)^{-(n+1)} - (-1)^n n!(x-4i)^{-(n+1)}\right]$$

let,

$$x = r\cos\varphi$$
$$4 = r\sin\varphi$$

$\therefore (x+4i)^{-(n+1)}$

$\Rightarrow (r\cos\varphi + i r\sin\varphi)^{-(n+1)}$

$\Rightarrow r^{-(n+1)} (\cos\varphi + i\sin\varphi)^{-(n+1)}$

$\Rightarrow r^{-(n+1)} \{\cos[-(n+1)\varphi] + i\sin[-(n+1)\varphi]$

$\Rightarrow r^{-(n+1)} \{\cos(n+1)\varphi - i\sin(n+1)\varphi\}$

for, $(x-4i)^{-(n+1)}$

$\Rightarrow r^{-(n+1)} [\cos(n+1)\varphi + i\sin(n+1)\varphi]$

for, $(x+4i)^{-(n+1)} - (x-4i)^{-(n+1)}$

$= -2i r^{-(n+1)} . \sin(n+1)\varphi$

$\therefore y_n = -\frac{1}{8i}(-1)^n n! (-2i) r^{-(n+1)} \sin(n+1)\varphi$

$= \frac{1}{4}(-1)^n . n! \; r^{-(n+1)} . \sin(n+1)\varphi$

$= \frac{1}{4}\cdot(-1)^n . n! \; (x^2+4^2)^{-\left(\frac{n+1}{2}\right)} . \sin(n+1)\tan^{-1} 4/x \;]$

$\dfrac{x^2+1}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

$$\Rightarrow x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$x=2$,

$5 = -B$

$\Rightarrow B = -5$

$x=1$,

$2 = 2A$

$\Rightarrow A = 1$

$x=0$,

$1 = 6A + 3B + 2C$

$= 6 - 15 + 2C$

$= -9 + 2C$

$\Rightarrow \dfrac{10}{2} = C$

$\Rightarrow C = 5$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}$$

$$\therefore y_n = (-1)^n n! (x-1)^{-(n+1)} - 5(-1)^n n! (x-2)^{-(n+1)} + 5(-1)^n n! (x-3)^{-(n+1)}$$

Leibnites Theorum:

If u and v are functions of x, then if

$$y = uv$$

the nth derivative will be given by,

$$y_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} \cdot v_2 + {}^nc_3 u_{n-3} v_3 + \cdots$$

$$+ \cdots {}^nc_r u_{n-r} v_r + \cdots + uv_n$$

where suffix no. denotes the time differentiation with respect to x.

Ex: $y = uv$

$$y_1 = u_1 v + u v_1$$

$$y_2 = u_2 v + v_1 u_1 + u v_1 + u v_2$$

$$= u_2 v + 2u_1 v_1 + u v_2$$

$$= u_2 v + 2c_1 u_1 v_1 + u v_2$$

$$y_3 = u_3 v + u_2 v_1 + 2u_2 v_1 + 2u_1 v_2 + u_1 v_2 + u v_3$$

$$= u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3$$

$$= u_3 v + 3c_1 u_2 v_1 + 3c_2 u_1 v_2 + u v_3$$

$n = n+1$ হলে, Mathmetical Induction theory

$$\therefore \; y_{n+1} = u_{n+1}v + u_n v_1 + {}^nC_1 (u_n v_1 + u_{n-1} v_2) + {}^nC_2 (u_{n-1} v_2 + u_{n-2} v_3) + \cdots$$

$$\cdots + {}^nC_{r-1}\, u_{n-r+1} v_{r-1} + {}^nC_r\, u_{n-r} v_r + \cdots + \cancel{u v_n}$$

$$+ {}^nC_{r+1}\, u_{n-r-1} v_{r+1}$$

$${}^nC_{r-1} (u_{n-r+2} v_{r-1} + v_{n-r+1} v_r) + {}^nC_r (u_{n-r+1} v_r + u_{n-r} v_{r+1})$$

$$+ u_1 v_n + u v_{n+1}$$

$$\boxed{{}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r}$$

$$y_{n+1} = u_{n+1}v + (1 + {}^nC_1) u_n v_1 + ({}^nC_1 + {}^nC_2) u_{n-1} v_2 + \cdots + ({}^nC_{r-1} + {}^nC_r)(u_{n-r+1} v_r)$$

$$+ \cdots + u v_{n+1}$$

$$= u_{n+1} v + {}^{n+1}C_1\, u_n v_1 + {}^{n+1}C_2\, u_{n-1} v_2 + {}^{n+1}C_r\, u_{n-r+1} v_r + \cdots + u v_{n+1}$$

According to Mathematical Induction theorum,

Leibnites theorum is true for every value of n.

$y = x^3 \sin x \; ; \; y_n = ?$

let,

$v = x^3$

$u = \sin x$

$$y_n = u_n x^3 + {}^nc_1 u_{n-1} \cdot 3x^2 + {}^nc_2 u_{n-2} \cdot 6x + {}^nc_3 u_{n-3} \cdot 6 + 0$$

$u = \sin x$

$u_n =$

$$= \frac{d^n}{dx^n}(\sin x) x^3 + {}^nc_1 \left[\frac{d^{n-1}}{dx^{n-1}}(\sin x) \cdot \frac{d}{dx}(x^3)\right]$$

$$+ {}^nc_2 \left[\frac{d^{n-2}}{dx^{n-2}}(\sin x) \frac{d^2}{dx^2}(x^3)\right] + {}^nc_3 \left[\frac{d^{n-3}}{dx^{n-3}}(\sin x) \frac{d^3}{dx^3}(x^3)\right]$$

we know,

$y = \sin(ax+b)$

$a=1, \; b=0$ হলে,

$y_n = a^n \sin(n \cdot \pi/2 + ax+b)$

$\Rightarrow y_n = \sin(n \pi/2 + x)$

$$y_n = x^3 \sin(n\pi/2 + x) + {}^nc_1 \, 3x^2 \sin[(n-1)\pi/2 + x] + {}^nc_2 \cdot 6x \sin[(n-2)\pi/2 + x]$$

$$+ {}^nc_3 \cdot 6 \sin[(n-3) \cdot \pi/2 + x] + 0$$

$\log = \ln \quad \log_{10} = \log_{10}$

$y = x^3 \log x$; $y_n = ?$

let,

$v = x^3$

$u = \log x$

$$\left\{ \begin{array}{l} y = \log(a+x) \\ \therefore y_n = (-1)^{n-1} (n-1)! \, (a+x)^{-n} \\ a = 0 \rightarrow \\ \quad y_n = (-1)^{n-1} (n-1)! \, x^{-n} \end{array} \right.$$

$$y_n = u_n v + {}^nc_1 \, u_{n-1} \cdot v_1 + {}^nc_2 \cdot u_{n-2} \cdot v_2 + {}^nc_3 \, u_{n-3} \cdot v_3 + \cdots$$

$$= (\log x)_n \cdot x^3 + {}^nc_1 \, (\log x)_{n-1} \cdot (x^3)_1 + {}^nc_2 \, (\log x)_{n-2} \, (x^3)_2 + \cdots$$

$$= (-1)^{n-1} (n-1)! \, x^{-n} \cdot x^3 + {}^nc_1 \cdot 3x^2 \cdot (-1)^{n-2} (n-2)! \cdot x^{-n+1} + {}^nc_2 \, 6x \, (-1)^{x-3} (n-3)! \, x^{-x+2}$$

$$+ {}^nc_3 \cdot 6 \, (-1)^{x-4} (n-4)! \, x^{-n+3}$$

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If $y = e^{a\sin^{-1}x}$ then prove that $(1-x^2)\,y_{n+2} - (2n+1)\,x.y_{n+1} - (n^2+a^2)y_n = 0$.

$$\Rightarrow y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(\sqrt{1-x^2}\right) y_1 = e^{a\sin^{-1}x} . a$$

$$\Rightarrow (1-x^2).\, y_1^2 = a^2 . e^{2a\sin^{-1}x}$$

$$\Rightarrow -2xy_1^2 + (1-x^2).\, 2y_1y_2 = a^2 . e^{2a\sin^{-1}x} \cdot \frac{2a}{\sqrt{1-x^2}}$$

$$\Rightarrow -2xy_1^2 + (1-x^2) \cdot 2y_1y_2 = a^2 . 2y^2 . y_1$$

$$\Rightarrow 2y_1\left[(1-x^2)\,y_2 - xy_1 - a^2 y\right] = 0$$

$$\Rightarrow \left[(1-x^2)\,y_2\right] - \left[xy_1\right] - \left[a^2 y\right] = 0$$

$y_2 = u$
$1 - x^2 = v$

From leibnit's theorum:

$$\left[(1-x^2)\,y_{n+2} + {}^nc_1\, y_{n+1}\,(-2x) + {}^nc_2\, y_n\,(-2)\right]$$

$$- \left[xy_{n+1} + {}^nc_1\, y_n \cdot 1\right] - \left[a^2 y_n\right] = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} + y_{n+1}\,(-2xn - x) + y_n\left\{(-2)\left(\frac{n(n-1)}{2}\right) - n - a^2\right\} = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} + y_n\left(-n^2 + n - n - a^2\right) = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1).\, x . y_{n+1} - (n^2+a^2)\,y_n = 0$$

$$\boxed{{}^nc_2 = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)(n-2)!}{2\,(n-2)!} = \frac{n(n-1)}{2}}$$

[Proved]

If $\log y = \tan^{-1} x$, then prove that; $(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$

$\Rightarrow \log y = \tan^{-1} x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{1+x^2}$

$\Rightarrow y_1 = y \cdot \frac{1}{1+x^2}$

$\Rightarrow (1+x^2) \cdot y_1 = y$

$\Rightarrow y_2 (1+x^2) + y_1 (2x) = y_1 \quad \Rightarrow y_2 (1+x^2) + y_1 (2x-1) = 0$

From leibnitz's theorum

$u = y_2$
$v = (1+x^2)$

$$\left[(1+x^2) \cdot y_{n+2} + {}^nc_1 \, y_{n+1} (2x) + {}^nc_2 \, y_n \cdot 2 + 0\right]$$

$$+ \left[(2x-1) \cdot y_{n+1} + {}^nc_1 \, y_n \cdot 2 + 0\right] = 0$$

$\Rightarrow (1+x^2) \cdot y_{n+2} + (y_n+1)(2nx+2x-1) + y_n \left(2 \cdot {}^nc_2 + 2 \cdot {}^nc_1\right) = 0$

$\Rightarrow (1+x^2) \cdot y_{n+2} + (2nx+2x-1) \cdot y_{n+1} + n(n+1) y_n = 0$

[Proved]

$y = f(x) = \dfrac{x}{\log x}$

Find the maximum and minimum value of $f(x)$.

$\Rightarrow$

$$y_1 = \frac{\log x . 1 - x . \frac{1}{x}}{(\log x)^2}$$

$$= \frac{\log x - 1}{(\log x)^2}$$

এখন,

$$\frac{\log x - 1}{(\log x)^2} = 0$$

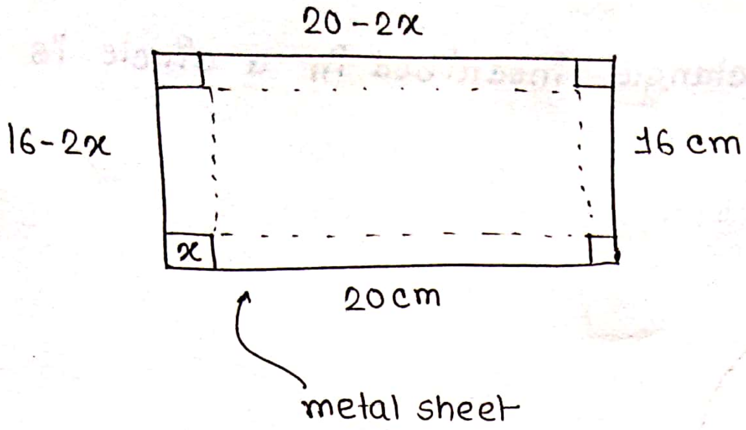
$$\Rightarrow \log x = 1$$

$$\Rightarrow x = \log^{-1}(1)$$

$$\Rightarrow x = e$$

$$y_2 = \frac{(\log x)^2 \frac{1}{x} - (\log x - 1) . 2 . \log x . \frac{1}{x}}{(\log x)^4}$$

#



total volume $V_1 = (20-2x)(16-2x)x$

$$V \cancel{\frac{dv}{dx}} = (20-2x).(16-2x).x$$

$$= (20x - 2x^2)(16-2x)$$

$$= 320x - 32x^2 - 40x^2 + 4x^3$$

$$= 4x^3 - 72x^2 + 320x$$

$$\frac{dv}{dx} = 12x^2 - 144x + 320$$

we know, $\frac{dv}{dx} = 0$

$$\Rightarrow 12x^2 - 144x + 320 = 0$$

$$\Rightarrow x = 9.056, 2.94$$

↳ গ্রহণযোগ্য নয়

$$\therefore x = 2.94$$

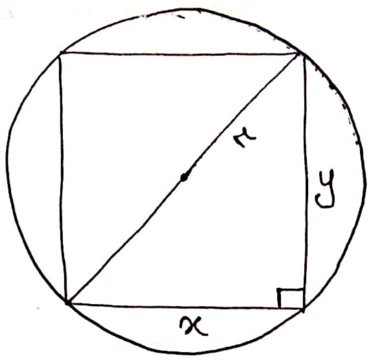
$$\frac{d^2v}{dx^2} = 24x - 144$$

If $x = 2.94$, $\frac{d^2v}{dx^2} = 24 \times 2.94 - 144 < 0$, maximum

$\therefore$ Maximum Volume $= 4(2.94)^3 - 72(2.94)^2 + 320 \times 2.94.$

Show that the maximum rectangle inscribed in a circle is a squre.

$\Rightarrow$



Area of the rectangle, $A = xy$ —— (i)

$r =$ constant

From figure,

$$(2r)^2 = x^2 + y^2$$

$$\Rightarrow y = \sqrt{4r^2 - x^2} \text{ —— (ii)}$$

$$\therefore A = x . \sqrt{4r^2 - x^2}$$

$$\Rightarrow \cancel{\frac{dA}{dx}} \quad A^2 = x^2 (4r^2 - x^2) = 4r^2x^2 - x^4$$

$$\Rightarrow 2 . A . \frac{dA}{dx} = 8r^2 x - 4x^3$$

For. max or min $\rightarrow$

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 8r^2x - 4x^3 = 0$$

$$\Rightarrow x(4r^2 - x^2) = 0$$

$\Rightarrow x = 0 \longrightarrow$ গ্রহণযোগ্য নয়

$\Rightarrow x = \pm\sqrt{2r^2}$

$= \pm\sqrt{2}r \longrightarrow$ গ্রহণযোগ্য নয়

$= \sqrt{2}r$

$$\frac{d^2A}{dx^2} = \frac{1}{2A}(8r^2 - 12x^2)$$

(ii) →

$\therefore y = \sqrt{4r^2 - 2r^2}$

$= \sqrt{2}\, r$

$\therefore x = y$

$x = \sqrt{2}\, r \rightarrow$

$= \frac{1}{2A}(8r^2 - 12 \cdot 2r^2) < 0$

↳ maximum পাওয়া যাবে।

Find the point on the parabola $2y = x^2$ which is nearest to the point $(0,3)$.

⇒ let,
(x_1, y_1) is the point on the parabola $2y = x^2$ which is nearest to the poin $(0,3)$.

$\therefore 2y_1 = x_1^2 \longrightarrow (1)$

$D = \sqrt{x_1^2 + (y_1 - 3)^2}$

$\Rightarrow D^2 = x_1^2 + (y_1 - 3)^2$

$\Rightarrow D^2 = 2y_1 + (y_1 - 3)^2$

$\therefore x_1^2 = 2 \cdot 2$

$\Rightarrow x_1^2 = 4$

$\Rightarrow x_1 = \pm 2$

we get;

point → $(2,2)$ $(-2,2)$

$$2D \cdot \frac{dD}{dy_1} = 2 + 2(y_1 - 3)$$

For maximum or minimum,

$\frac{dD}{dy_1} = 0$

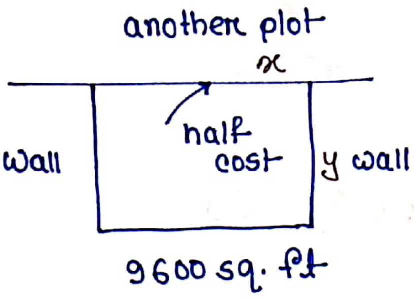
$\Rightarrow 2 + 2(y_1 - 3) = 0$

$\Rightarrow y_1 = 2$

$\frac{d^2D}{dy^2} > 0$

for $y_1 = 2$

(37) #



cost minimum?

$\Rightarrow \quad xy = 9600$

Cost per ft is $= p$.

$$\text{cost } c = 2yp + xp + \frac{xp}{2}$$

$$\therefore c = 2 \cdot \frac{9600}{x} p + \frac{3xp}{2} \qquad [p \neq 0]$$

$$\therefore \frac{dc}{dx} = 2p \cdot 9600 \left(-\frac{1}{x^2}\right) + \frac{3}{2}p$$

$$\frac{d^2c}{dx^2} = \frac{19200p}{x^3}$$

আবার,

$$\frac{dc}{dx} = 0 \quad \Rightarrow -\frac{2 \times 9600}{x^2} = -\frac{3}{2}$$

[যেহেতু, p হচ্ছে cost যা 0 এর চেয়ে বড় হবে, তাই $\frac{d^2c}{dx^2} > 0$

∴ minimum পাওয়া যাবে।]

$$\Rightarrow x = 80\sqrt{2}$$

$$\therefore y = \frac{9600}{80\sqrt{2}}$$

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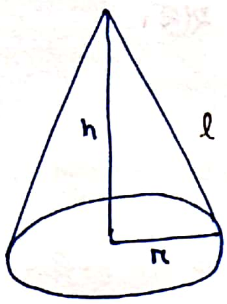
$S, \pi \longrightarrow$ constant

#



Surface area is constant. maximum volume = ?

$\Rightarrow \quad V = \frac{1}{3}\pi r^2 h$ — (i) [function]

$S = \pi r\sqrt{h^2+r^2}$ [condition] — (ii)

$= \pi r l$

$\Rightarrow S^2 = \pi^2 r^2 (h^2+r^2)$

$\Rightarrow h^2 = \frac{S^2}{\pi^2 r^2} - r^2$

$\therefore V^2 = \frac{1}{9}\pi^2 r^4 h^2$

$\Rightarrow \frac{1}{9}\pi^2 r^4 \cdot \left(\frac{S^2}{\pi^2 r^2} - r^2\right) = V^2$

$\Rightarrow \frac{1}{9}\pi^2 r^4 \left(\frac{S^2 - r^4\pi^2}{\pi^2 r^2}\right) = V^2$

$\Rightarrow \frac{1}{9} r^2 (S^2 - r^4\pi^2) = V^2$

$\Rightarrow \frac{1}{9} r^2 S^2 - \frac{1}{9} r^6 \pi^2 = V^2$ — (iii)

$\frac{d}{dr}$ (iii) $\Rightarrow$

$2V \cdot \frac{dV}{dr} = \frac{1}{9} S^2 \cdot 2r - \frac{1}{9}\pi^2 6r^5$

$\Rightarrow \frac{dV}{dr} = \frac{r}{9 \cdot V}(S^2 - 3\pi^2 r^4)$

$\frac{dV}{dr} = 0$

$\Rightarrow (S^2 - 3\pi^2 r^4) r = 0$

$\Rightarrow S^2 = 3\pi^2 r^4$

$\Rightarrow r = \sqrt[4]{\frac{S^2}{3\pi^2}}$

$\Rightarrow r = \left(\frac{S^2}{3\pi^2}\right)^{\frac{1}{4}}$

$\Rightarrow r^2 = \frac{1}{\sqrt{3}} \cdot \frac{S}{\pi}$

r এর মান বসিয়ে,

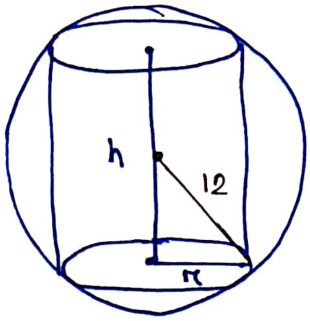
$h^2 = \frac{S^2}{\pi^2 r^2} - \frac{S}{\sqrt{3}\pi}$

$\Rightarrow h = \frac{S^2}{\pi^2} \times \sqrt{3}\,\frac{\pi}{S} - \frac{S}{\sqrt{3}\pi} = \frac{\sqrt{3}S}{\pi} - \frac{S}{\sqrt{3}\pi}$

$\Rightarrow h^2 = \frac{S}{\pi}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$

$\therefore h : r = \sqrt{2} : 1$

#



Volume maximum হবে।

radius of the sphere is 12 cm. $h = ?$, $r = ?$

$\Rightarrow$ maximam,

$$V = \pi r^2 h$$

$$\left(\frac{h}{2}\right)^2 + r^2 = 12^2$$

$$\Rightarrow r^2 = 144 - \frac{h^2}{4}$$

r^2 এর মান বসিয়ে,

$$V = \pi\left(144 - \frac{h^2}{4}\right)h$$

$$= \pi\left(144h - \frac{h^3}{4}\right)$$

$$= 144\pi h - \frac{\pi}{4}h^3$$

$$\therefore \frac{dV}{dh} = 144.\pi - \frac{\pi}{4}.3h^2$$

$$\frac{dV}{dh} = 0$$

$$144\pi = \frac{\pi}{4}3h^2$$

বা, $\sqrt{\dfrac{144 \times 4}{3}} = h$

বা, $h = 8\sqrt{3}$

$$\therefore r = \sqrt{\frac{V}{\pi h}}$$

$$r^2 = 144 - \frac{192}{4} = 96$$

$$\therefore r = 4\sqrt{6}$$

Rolle's Theorum:

if →

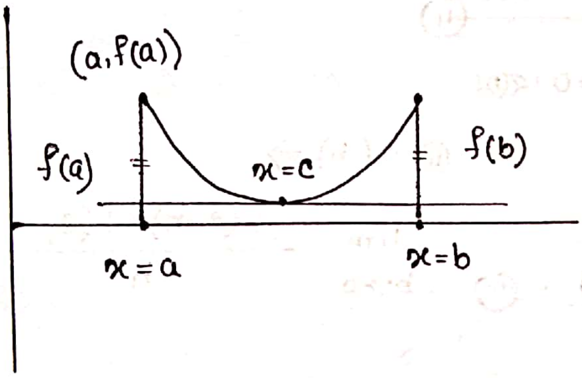
(i) $f(x)$ is continuous in the closed interval $a \leq x \leq b$.

(ii) $f'(x)$ exists in the open interval $a < x < b$.

(iii) $f(a) = f(b)$

Then there exists at least one point $x = c$ (say)

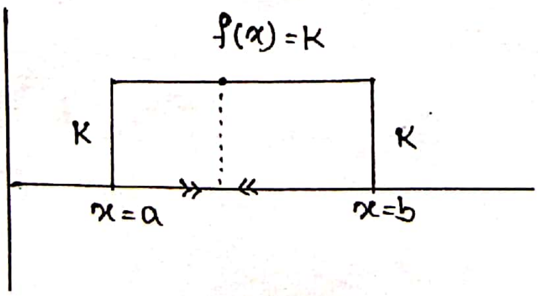
where, $a < c < b$, $f'(c) = 0$



* properties of limit
পড়তে হবে।

if $f(x) = K$ (K is a constant)

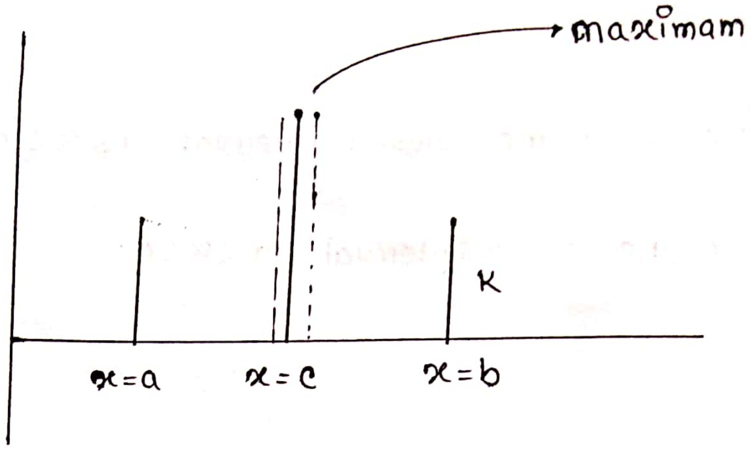
$\therefore y = K$



for constant function, we know, continuity exists for all x and $f'(x)$ also exists for all value of x.

We know,

$f'(x) = 0$; for all values of x, in the interval $a < x < b$.



f(c) is the maximum value.

If

$$\lim_{h\to 0} f(c+h) - f(c) \leq 0 \quad \text{——— (i)}$$

$$\lim_{h\to 0} f(c-h) - f(c) \leq 0 \quad \text{——— (ii)}$$

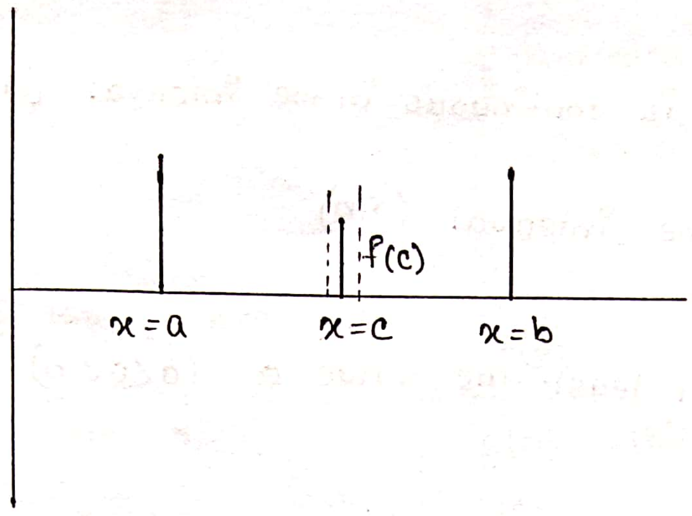
h→0 হলে

(i) ÷ h ⇒

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{— (iii)}$$

(ii) ÷ (−h) ⇒

$$\lim_{h\to 0} \frac{f(c-h) - f(c)}{-h} \geq 0 \quad \text{— (iv)}$$



f(c) is minimum

$f(c+h)$

$f(c-h)$

$\lim_{h\to 0} f(c+h) - f(c) \geq 0$ —— (v)

$\lim_{h\to 0} f(c-h) - f(c) \geq 0$ —— (vi)

$\lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h} \geq 0$ —— (vii)

$\lim_{h\to 0} \dfrac{f(c-h) - f(c)}{-h} \leq 0$ —— (viii)

$* \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} -$

Mean-value Theorum:

(I) If a function $f(x)$ is continuous in the interval $[a,b]$.

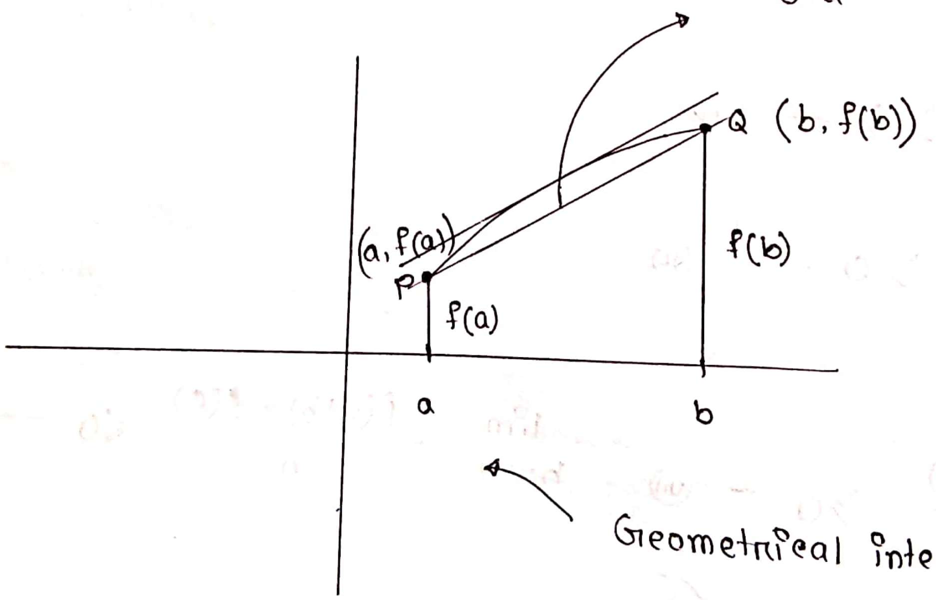
(II) $f'(x)$ exists in the interval (a,b)

then there exists at least one value c. $(a<c<b)$

such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{f(b)-f(a)}{b-a}$$



Geometrical interpretation of mean-value theorum.

Slope of the St. line.

$$PQ = \frac{f(b)-f(a)}{b-a}$$

$x^2 + 5x + 3$

Let,

$g(x) = f(x) + Ax$ ——— (1)

where, A is a constant

polynomial functions are continuous for all values of x and also differentiable for all values of x.

We know,

∵ $f(x)$ is continuous in the interval $[a,b]$ and x is a polynomial function.

So $g(x)$ is continuous in the interval $[a,b]$

We know,

$f'(x)$ exists in the interval (a,b) and x is polynomial function and is differentiable for all values of x.

∴ $g'(x)$ exists in the interval (a,b).

$g(x)$ will follow the Rolle's Theorem,

if $g(a) = g(b)$.

$\Rightarrow f(a) + Aa = f(b) + Ab$

$\Rightarrow A = \dfrac{f(b) - f(a)}{a-b}$

$g(a) = f(a) + Aa$

$g(b) = f(b) + Ab$

if,

$g(a) = g(b)$

$\Rightarrow f(a) + Aa = f(b) + Ab$

$\Rightarrow A = \dfrac{f(b) - f(a)}{a-b}$

$g'(c) = 0$

$\Rightarrow f'(c) + \dfrac{f(b) - f(a)}{a-b} = 0$

$\Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b-a}$

Putting the value of A in ①

$$g(x) = f(x) + \frac{f(b) - f(a)}{a-b} x$$

$\therefore$ $g(x)$ follow all the three conditions of Rolle's theorum.

So there exist at least one value $x = c$, $a < c < b$.

Such that,

$$g'(c) = 0.$$

$$\therefore \boxed{f'c = \frac{f(b) - f(a)}{b-a}}$$

[Proved]

$f(b) - f(a) = (b-a) f'(c)$ —— ①

If,

$$c = a + \theta (b-a) —— (ii) \quad [0 < \theta < 1]$$

and,

$$h = b - a$$

From, (ii) →

$$c = a + h\theta$$

From ① →

$$\boxed{f(a+h) - f(a) = hf'(a+h\theta)} \quad \text{where } 0 < \theta < 1$$

Another form of mean-value theorum.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a+h\theta)$$

If $a = x$,

$$\therefore f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^n(x+h\theta)$$

Taylor's Series.

or, generalised mean-value theorem.

Verify mean-value theorum for $f(x) = 2x - x^2$ in the interval $[\underset{a}{0}, \underset{b}{1}]$.

$\Rightarrow$ $f(b) = f(1) = 2-1 = 1$

$f(a) = f(0) = 0$

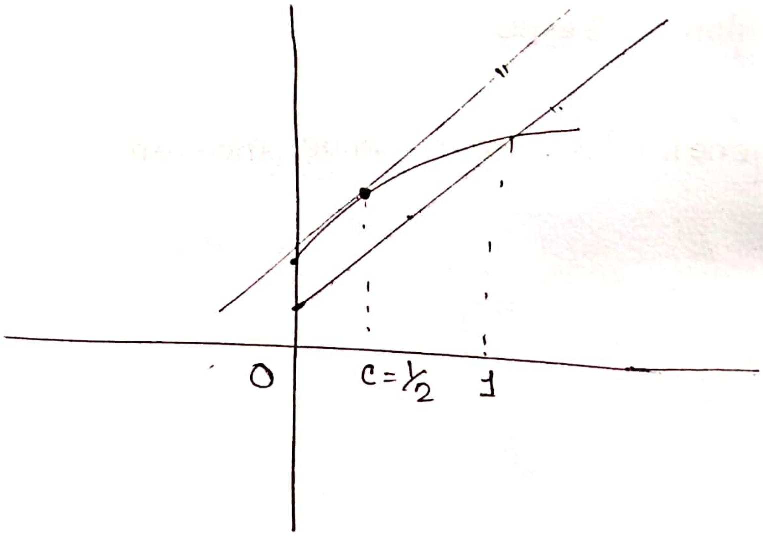
$f'(x) = 2 - 2x$

$f'(c) = 2 - 2c$

$$f'(c) = \frac{f(b) - f(a)}{(b-a)} = \frac{1-0}{1-0} = 1$$

$\Rightarrow 2 - 2c = 1$

$\Rightarrow c = \frac{1}{2}$



$$f(x) = \frac{2x - x^2}{x - \cancel{3/2}\ 1/3} \quad [0,1]$$

$\longrightarrow$ $x = 1/3$ বিন্দুতে continuity নাই।

$= \infty$

Verify Rolle's theorum for $f(x) = 2x - x^2$ $[0,1]$

$\Rightarrow$ $f(a) = f(b)$

$\Rightarrow 0 \neq 1$

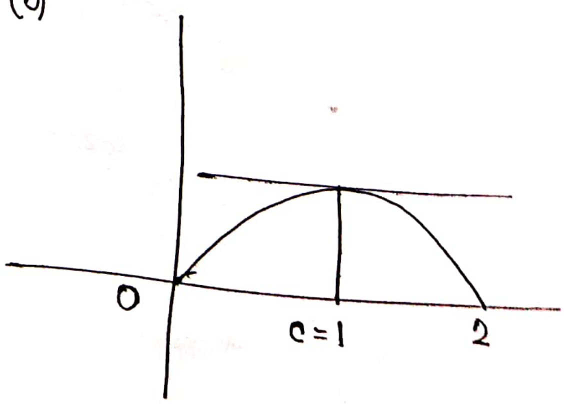
$(0,2] \Rightarrow f(a) = f(b)$

$\Rightarrow 0 = 0$

$f'(c) = 0$

$\Rightarrow 2 - 2c = 0$

$\Rightarrow c = 1$



$f(x) = (x-1)(x-2)(x-3)$, Verify mean value theorum in the interval $[0,4]$

$$= (x^2 - 2x - x + 2)(x-3)$$

$$= (x^2 - 3x + 2)(x-3)$$

$$= x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$$

$$= x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$\therefore f'(c) = 3c^2 - 12c + 11$$

$$f(a) = f(0) = \cancel{\#} -6$$

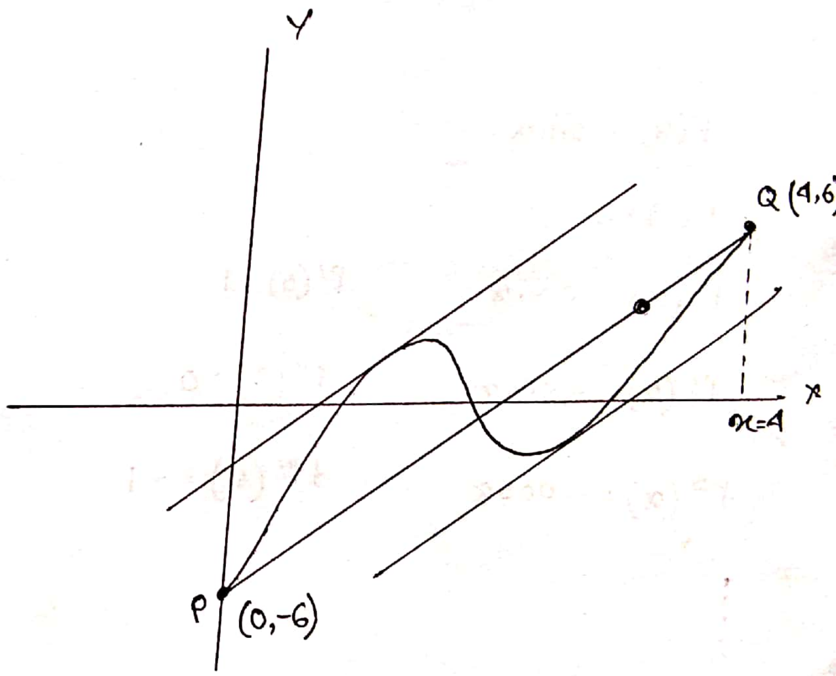
$$f(b) = f(4) = 6$$

$$\therefore f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow 3c^2 - 12c + 11 = \frac{\cancel{-6} + 12}{4} = 3$$

$$\Rightarrow \cancel{12c^2 - 48c + 49} = 0$$

$$\Rightarrow c = \frac{6+2\sqrt{3}}{3}, \frac{6-2\sqrt{3}}{3}$$



Expand $\sin x$ in a power series of x.

Maclaurin's series:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(\theta x)$$

where $0 < \theta < 1$

Putting, $x = 0$, $h = x$ in Taylor's series.

$f(x) = \sin x$

$f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$\vdots$

$f^n(x) = \sin(n\pi/2 + x) \quad \therefore \quad f^n(\theta x) = \sin(n\pi/2 + \theta x)$

$$\Rightarrow f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{x^n}{n!} \sin\left(n\pi/2 + \theta x\right)$$

e^x

$\log(1-x)$

$\tan^{-1} x$

Partial differentiation:

$$z = f(x, y)$$

$$y = f(x)$$

$x \longrightarrow$ independent variable

$y \longrightarrow$ dependent variable

$x^2+y^2+z^2=a^2 \longrightarrow$ sphere

$$\Rightarrow z^2 = (x^2+y^2)+a^2$$

$$\Rightarrow z = \sqrt{a^2-x^2-y^2}$$

$$\left(\frac{\partial z}{\partial y}\right)_x \quad \left(\frac{\partial z}{\partial x}\right)_y$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left(a^2-x^2-y^2\right)^{-1/2}(-2x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}}$$

$$\left(\frac{\partial z}{\partial y}\right)_x = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right)$$

$$= (-y)\left(-\tfrac{1}{2}\right)\left(a^2-x^2-y^2\right)^{-3/2}(-2x)$$

$$= -\frac{xy}{(a^2-x^2-y^2)^{3/2}}$$

Now

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{a^2-x^2-y^2}}\right)$$

$$= (-x)\left(-\frac{1}{2}\right)(a^2-x^2-y^2)^{-3/2}(-2y)$$

$$= \frac{-xy}{(a^2-x^2-y^2)^{3/2}}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

If $v = z\tan^{-1} y/x$, then prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

$\Rightarrow$ $$\frac{\partial v}{\partial x} = z \cdot \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}(y/x)$$

$$= \frac{z}{\frac{x^2+y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{-yz}{x^2+y^2}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{yz}{x^2+y^2}\right)$$

$$= -yz(-1)(x^2+y^2)^{-2} \cdot 2x = \frac{2xyz}{(x^2+y^2)^2}$$

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$$\frac{\partial v}{\partial y} = z \cdot \frac{1}{1+ y^2/x^2} \cdot \frac{1}{x}$$

$$= \frac{2x^2}{x(x^2+y^2)}$$

$$= \frac{2x}{(x^2+y^2)}$$

$$\frac{\partial^2 v}{\partial y^2} = 2x(-1)(x^2+y^2)^{-2}(2y)$$

$$= -\frac{2xyz}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial z} = \tan^{-1} y/x$$

$$\frac{\partial^2 v}{\partial z^2} = 0$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$u = \log(x^3+y^3+z^3-3xyz)$

Find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

$$\frac{\partial u}{\partial x} = \frac{3x^2+0+0-3yz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^3+y^3+z^3-3xyz)(6x)-(3x^2-3yz)(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(3y^2-3xz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^3+y^3+z^3-3xyz)(6y)-(3y^2-3xz)(3y^2-3xz)}{(x^3+y^3+z^3-3xyz)^2}$$

$$\frac{\partial u}{\partial z} = \frac{(3z^2-3xy)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{(x^3+y^3+z^3-3xyz)(6z)-(3z^2-3xy)(3z^2-3xy)}{(x^3+y^3+z^3-3xyz)^2}$$

Show that if $u(x,y,z)$ satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{——— (1)}$$

then,

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$, satisfy it and also

(ii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ satisfy it.

$$\Rightarrow \text{(i)} \quad \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 u}{\partial z^2 \partial x} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x} \cdot 0 = 0$$

(ii) $$\frac{\partial^2}{\partial x^2}\left(x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z}\right)$$

$$= x \cdot \frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^3 u}{\partial x^2 \partial y} + z\frac{\partial^3 u}{\partial x^2 \partial z}$$

$$= \frac{\partial^2}{\partial y^2}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)\right\}$$

$$= \frac{\partial}{\partial y}\left\{x\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot\cancelto{0}{\frac{\partial x}{\partial y}} + y\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} + z\cdot\frac{\partial^2 u}{\partial y\partial z} + \frac{\partial u}{\partial z}\cdot\cancelto{0}{\frac{\partial z}{\partial y}}\right\}$$

$$= x\cdot\frac{\partial^3 u}{\partial y\partial x^2} + y\frac{\partial^3 u}{\partial y^3} + \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 u}{\partial y^2} + z\frac{\partial^3 u}{\partial y^2\partial z}$$

And,

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= x\frac{\partial^3 u}{\partial z^2\partial x} + y\frac{\partial^3 u}{\partial z^2\partial y} + z\frac{\partial^3 u}{\partial z^3} + 2\frac{\partial^2 u}{\partial z^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

$$= 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + x\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$+ y\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$= 0$$

If $\frac{\partial}{\partial x}\sin(x+y) = \cos(x+y)(1+0)$

$$\# \quad u = x\varphi(x+y) + y\psi(x+y)$$

$$\frac{\partial u}{\partial x} = x\varphi'(x+y)\cdot 1 + \varphi(x+y) + y\psi'(x+y)\cdot 1$$

$$u = f(x, y)$$

পরাধীন স্বাধীন

যে রাশিগুলো স্বাধীন চলক রয়েছে।

তাদের সাপেক্ষে differenciation করা যাবে।

$$\frac{\partial u}{\partial y} = x\varphi'(x+y) + y\psi'(x+y) + \psi(x+y)$$

Show that, $\frac{\partial^2 u}{\partial x^2} - 2\cdot\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial^2 u}{\partial x^2} = x\varphi''(x+y) + \varphi'(x+y) + \varphi'(x+y)\cdot 1 + y\psi''(x+y)\cdot 1$$

$$\frac{\partial^2 u}{\partial y^2} = x\varphi''(x+y) + y\psi''(x+y) + \varphi'(x+y) + \psi'(x+y)$$

$$\frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial y} = x\varphi''(x+y) + \varphi'(x+y) + y\psi''(x+y) + \psi'(x+y)\cdot 1$$

যোগ করে,

$$x\psi''(x+y) + 2\varphi'(x+y) + y\psi'(x+y) - 2x\varphi''(x+y) - 2\varphi'(x+y) - 2y\psi''(x+y)$$

$$-2\psi'(x+y) + x\varphi''(x+y) + y\psi''(x+y) + 2\psi'(x+y)$$

$$= 2x\varphi''(x+y) - 2x\varphi''(x+y) + 2y\psi''(x+y) - 2y\psi''(x+y) - 2\psi'(x+y) + 2\psi'(x+y)$$

$$= 0$$

Homogenous function:

A function $f(x,y)$ is said to be homogenous of degree n is the variable x and y if it can be expressed in the form $x^n \phi(y/x)$ or in the form $y^n \varphi(x/y)$.

$$f(x,y) = ax^2 + 2hxy + by^2$$

$$= x^2 \left[a + 2h\left(y/x\right) + b\left(y/x\right)^2\right]$$

$$= x^2 \left[a + 2hv + bv^2\right]$$

$$= x^2 \phi(v)$$

$$= x^2 \phi(y/x)$$

$$f(x,y) = \frac{\sqrt{x^2+y^2}}{x+y}$$

$$= \frac{x\sqrt{1+y^2/x^2}}{x\left(1+y/x\right)}$$

$$= x^0 \phi(y/x)$$

Euler's theorum on homogeneous function:

If $f(x,y)$ is a homogeneous function of degree n in x and y then,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$$

$$f(x,y) = x^n \varphi(y/x)$$

$$\frac{\partial f}{\partial x} = n x^{n-1}\varphi(y/x) + x^n \varphi'(y/x)(-y/x^2)$$

$$= n x^{n-1}\varphi\left(\frac{y}{x}\right) - y x^{n-2}\varphi'\left(\frac{y}{x}\right)$$

$$x\frac{\partial f}{\partial x} = n x^n \varphi\left(\frac{y}{x}\right) - y x^{n-1}\varphi'\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial y} = x^n \varphi'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)$$

$$y\frac{df}{\partial y} = y x^{n-1}\varphi'\left(\frac{y}{x}\right)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n x^n \varphi(y/x) - y x^{n-1}\varphi'\left(\frac{y}{x}\right) + y x^{n-1}\varphi'\left(\frac{y}{x}\right)$$

$$= n x^n \varphi\left(\frac{y}{x}\right)$$

$$= n f(x,y)$$

If $u = \cos^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$

then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

$\Rightarrow$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}$$

$$= x^{\frac{1}{2}}\,\varphi\left(\frac{y}{x}\right)$$

cos u is a homogeneous function.

According to Euler's theorum,

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

$$\Rightarrow x(-\sin u)\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

total differencial co-efficient:

$$v = f(x, y)$$

where,

$$x = \phi(t)$$

$$y = \varphi(t)$$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$v = f(x, y)$

$$x = \phi(t_1, t_2)$$

$$y = \varphi(t_1, t_2)$$

$$\frac{\partial v}{\partial t_1} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

$$\frac{\partial v}{\partial t_2} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial t_2}$$

Partial derivatives of a function of two functions.

$V = v(r, \theta)$

$x = r\cos\theta$

$y = r\sin\theta$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} y/x$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{-\frac{y}{x^2}}{\frac{x^2+y^2}{x^2}}$$

$$= \frac{-y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$= \left(\cos\theta \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta}\right) v$$

$$\therefore \frac{\partial}{\partial x} = \cos\theta \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$$

$$= \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{v}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+y^2/x^2} \times \frac{1}{x}$$

$$= \frac{1}{x \, \frac{x^2+y^2}{x^2}}$$

$$= \frac{x}{r^2}$$

$$= \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial v}{\partial y} = \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\cdot\frac{\cos\theta}{r}\right)v$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\frac{\cos\theta}{r}$$

2nd derivative

$$\frac{\partial^2}{\partial x^2} = \text{(Polar operation)}$$

$$\frac{\partial^2}{\partial y^2} = ?$$

If $u = F(y-z,\ z-x,\ x-y)$, then prove that,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ let,

$$x_1 = y-z$$
$$x_2 = z-x$$
$$x_3 = x-y$$

$$u = F(x_1, x_2, x_3)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial x}$$

$$= -\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}$$

$$\frac{\partial x_1}{\partial x} = 0$$

$$\frac{\partial x_2}{\partial x} = -1$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial y}$$

$$= \frac{\partial x}{\partial x_1} - \frac{\partial u}{\partial x_3}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial z}$$

$$= -\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} = 0$$

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If $u = f(x^2+2yz,\ y^2+2zx)$ then prove that,

$$(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ let,

$X_1 = x^2+2yz$

$X_2 = y^2+2zx$

$\therefore u = f(X_1, X_2)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial x} = 2x\cdot\frac{\partial u}{\partial x_1} + 2z\,\frac{\partial u}{\partial x_2}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y} = 2z\cdot\frac{\partial u}{\partial x_1} + 2y\cdot\frac{\partial u}{\partial x_2}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial z} = 2y\,\frac{\partial u}{\partial x_1} + 2x\cdot\frac{\partial u}{\partial x_2}$$

$$\Rightarrow \frac{\partial u}{\partial x_1}\left[2x(y^2-zx)+2z(x^2-yz)+2y(z^2-xy)\right] + \frac{\partial u}{\partial x_2}\left[2z(y^2-zx)+2y(x^2-yz) + 2x(z^2-xy)\right]$$

$\Rightarrow 0$

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If $x^2+y^2+z^2-2xyz=1$, then show that,

$$\frac{dx}{\sqrt{1-x^2}}-\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0$$

$\Rightarrow$

$$2x\,dx+2y\,dy+2z\,dz-2xy\,dz-2yz\,dx-2zx\,dy=0$$

$$\Rightarrow (x-yz)\,dx+(y-zx)\,dy+(z-xy)\,dz=0$$

1st

$$x^2+y^2+z^2-2xyz=1$$

$$\Rightarrow x^2-2xyz+y^2z^2=1-y^2-z^2+y^2z^2$$

$$\Rightarrow (x-yz)^2=1-y^2-z^2(1-y^2)$$

$$\Rightarrow (x-yz)^2=(1-y^2)(1-z^2)$$

$$\Rightarrow (x-yz)^2\cdot(1-x^2)=(1-x^2)(1-y^2)(1-z^2)$$

2nd

$$y^2-2xyz+x^2z^2=1-x^2-z^2+x^2z^2$$

$$\Rightarrow (y-zx)^2=(1-x^2)(1-z^2)$$

$$\Rightarrow (y-zx)^2(1-y^2)=(1-x^2)(1-y^2)(1-z^2)$$

Find the tangent at (a,b) to the curve $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2$.

$\Rightarrow \quad y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

$\Rightarrow \quad \frac{3x^2}{a^3} + \frac{3y^2}{b^3} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{3x^2}{a^3} \times \frac{b^3}{3y^2}$

at the point (a,b)

$$\left(\frac{dy}{dx}\right)_{(a,b)} = -\frac{3a^2}{a^3} \times \frac{b^3}{3b^2}$$

$$= -\frac{b}{a}$$

Equation of tangent at (a,b)

$$y - b = \left(-\frac{b}{a}\right)(x - a)$$

$\Rightarrow \quad \frac{y-b}{b} = -\frac{(x-a)}{a}$

$\Rightarrow \quad \frac{y}{b} - 1 + \frac{x}{a} - 1 = 0$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} - 2 = 0$

Find the tangent to the curve $xy^2 = 4(4-x)$ at the point where it is cut by the line $y = x$.

$\Rightarrow$

$$xy^2 = 16 - 4x \quad \text{—— (i)}$$

$$y = x \quad \text{—— (ii)}$$

$$\therefore y = 2$$

From (i)

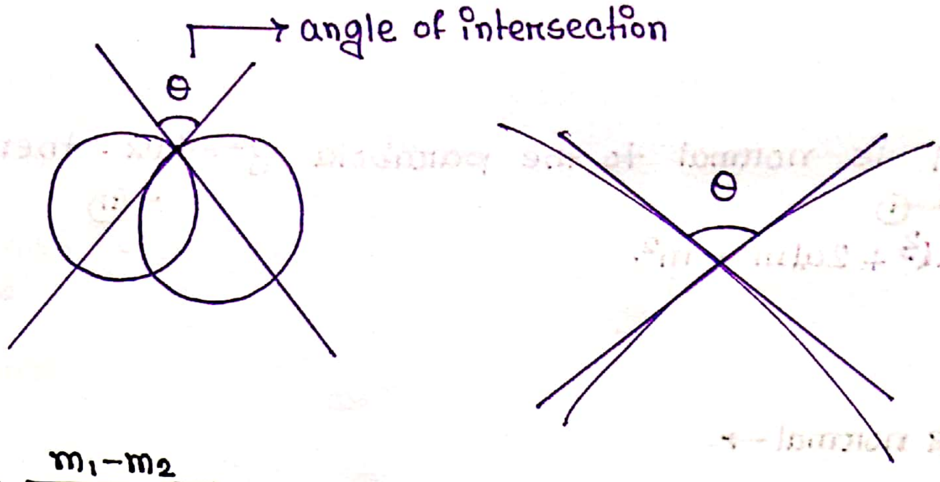
$$x \cdot x^2 = 4(4-x)$$

$$\Rightarrow x^3 - 16 + 4x = 0$$

$$\Rightarrow x^3 + 4x - 16 = 0$$

$$\Rightarrow x = 2$$

#



$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Show that the curves $x^3 - 3xy^2 = -2$ and $3x^2y - y^3 = 2$ cut orthogonaly.

(i) (ii) $[\theta = 90°]$

$\Rightarrow \quad 3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy_1}{dx} =$

$6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy_2}{dx} =$

$$\frac{dy_1}{dx} \times \frac{dy_2}{dx} = -1$$

If $lx+my=1$ (i) is normal to the parabola $y^2=4ax$ (ii) then show that $al^3+2alm^2=m^2$.

(h,K) (ii) এর normal →

$$y-K=-\frac{1}{\left(\frac{dy}{dx}\right)_{(h,K)}}(x-h)$$

eqn of normal at (h,K).

$$y^2=4ax$$

$$\Rightarrow 2y\cdot\frac{dy}{dx}=4a$$

$$\Rightarrow \frac{dy}{dx}=\frac{2a}{y}$$

$\Rightarrow$

at (h,K) →

$$\left(\frac{dy}{dx}\right)_{(h,K)}=\frac{2a}{K}$$

eqn of normal →

$$y-K=\frac{-K}{2a}(x-h)$$

$$\Rightarrow 2ay-2aK=-Kx+Kh$$

$$\Rightarrow Kx+2ay-Kh+2aK=0$$

$$\Rightarrow \frac{K}{Kh+2aK}x+\frac{2a}{Kh+2aK}y=\cancel{0}\,1$$

the point (h,k) is on the parabola.

$$\therefore k^2 = 4ah \quad \text{——— (ii)}$$

given the normal

$$lx + my = 1 \quad \text{——— (iii)}$$

Eqn (i) and (iii) are identical,

$$l = \frac{k}{kh + 2ak} \quad \text{——— (iv)}$$

and $$m = \frac{2a}{kh + 2ak} \quad \text{——— (v)}$$

Solve করে (ii) এ বসাব
$h = ?,\ k = ?$

If $lx + my = 1$ touches the curve $(ax)^n + (by)^n = 1$

then show that,

$$\left(\frac{l}{a}\right)^{\frac{n}{n-1}} + \left(\frac{m}{b}\right)^{\frac{n}{n-1}} = 1$$

⇒ let,
(h,k) be any point on the curve.

$$(ax)^n + (by)^n = 1 \quad \text{——— (i)}$$

eqn of tangent at (h,k)

$$(y-k) = \left(\frac{dy}{dx}\right)_{(h,k)} (x-h) \quad \text{——— (1)}$$

From (i)

$$a^n . nx^{n-1} + b^n ny^{n-1} . \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = - \frac{a^n x^{n-1}}{b^n y^{n-1}}$$

at the point (h,k)

$$\left(\frac{dy}{dx}\right)_{(h,k)} = - \frac{a^n h^{n-1}}{b^n k^{n-1}}$$

(h,k) is a point on (i)

$$\therefore (ah)^n + (b\cancel{y}k)^n = 1 \text{ ——— (iii)}$$

Now from (ii)

$$y-k = - \frac{a^n h^{n-1}}{b^n k^{n-1}} (x-h)$$

$$\Rightarrow b^n k^{n-1} y - b^n k^n = -a^n h^{n-1} + a^n h^n$$

$$\Rightarrow a^n h^{n-1} x + b^n k^{n-1} y = a^n h^n + b^n k^n = 1 \text{ ——— (iv)}$$

(using (iii))

given the tangent,

$$lx + my = 1 \text{ ——— (v)}$$

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eq^n (iv) and (v) are identical.

$$l = a^n h^{n-1}$$

$$m = b^n k^{n-1}$$

$$\text{L.H.S.} = \left(\frac{l}{a}\right)^{n/n-1} + \left(\frac{m}{b}\right)^{n/n-1}$$

$$= \left(\frac{a^n h^{n-1}}{a}\right)^{n/n-1} + \left(\frac{b^n k^{n-1}}{b}\right)^{n/n-1}$$

$$= \left(a^{n-1} h^{n-1}\right)^{n/n-1} + \left(b^{n-1} k^{n-1}\right)^{n/n-1}$$

$$= (ah)^n + (bk)^n = 1$$