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CE'18 (CR)

Integral Calculus

Defination: If $F(x)$ is the differential coefficient of $f(x)$ that is $\frac{d}{dx} f(x) = F(x)$ then to be the integral of $F(x)$, mathematically,

* $\int F(x)\,dx = f(x)$

$\frac{d}{dx}(x^2) = 2x$

$\int 2x\,dx = x^2 + c.$

Integration → Finding Area.
→ Anti derivative
→ Summation of a

Example:

1. $I = \int \frac{x^7 + 2x^2 + 3x + 4}{x}\,dx = \int \left(x^6 + 2x + 3 + \frac{4}{x}\right) dx$

$= \frac{1}{7}x^7 + 2\cdot\frac{x^2}{2} + 3x + 4\ln x + c.$

2. $I = \int \frac{4x^2 - \sqrt{x} + x^4}{x^3}\,dx = \int \left(\frac{4}{x} - x^{-5/2} + x\right) dx$

$= 4\ln x + \frac{2}{3}x^{-3/2} + \frac{x^2}{2} + c.$

3. $I = \int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx$

$= \int (\sec^2 x - \tan x \sec x)\,dx = \tan x - \sec x + c.$

4. $I = \int \frac{2-\sin 2x}{1-\cos 2x}\,dx = \int \frac{2-2\sin x\cos x}{2\sin^2 x}\,dx$

$= \int (\operatorname{cosec}^2 x - \cot x)\,dx$

$= -\cot x - \ln|\sin x| + c.$

Homework:

1. $I = \int \frac{dx}{\sqrt{x}+\sqrt{x+1}} = \int \frac{\sqrt{x}-\sqrt{x+1}}{x-x-1}\,dx = -\int(\sqrt{x}-\sqrt{x+1})\,dx$

$= -\frac{2}{3}x^{3/2} + \frac{2}{3}(x+1)^{2/3} + c.$

2. $I = \int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$

$= \int \sin x\,dx - \int \sin x\cos^2 x\,dx$

Let,
$\cos x = z$
$-\sin x\,dx = dz$

$= -\cos x + \int z^2\,dz$

$= -\cos x + \frac{z^3}{3} + c$

$= -\cos x + \frac{1}{3}\cos^3 x + c.$

Method of Substitution:

$I = \int (ax+b)^n\,dx$

Let,
$ax+b = z$
$\Rightarrow a\,dx = dz$
$\Rightarrow dx = \frac{1}{a}dz$

$= \int \frac{1}{a} z^n\,dz$

$= \frac{1}{a}\frac{z^{n+1}}{n+1} + c$

$= \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c.$

Ex:

1. $I = \int \frac{1}{\sqrt{x}} \cos\sqrt{x}\, dx$

Let, $\sqrt{x} = z$

$\Rightarrow \frac{1}{2\sqrt{x}} dx = dz$

$\Rightarrow \frac{dx}{\sqrt{x}} = 2\,dz$

$= 2\int \cos z\, dz$

$= 2\sin z + c$

$= 2\sin\sqrt{x} + c.$

2. $I = \int \frac{(x+1)e^x}{\cos^2(xe^x)} dx.$

Let, $xe^x = z$

$\Rightarrow (xe^x + e^x)\,dx = dz$

$\Rightarrow (x+1)e^x dx = dz$

$= \int \frac{dz}{\cos^2 z}$

$= \int \sec^2 z\, dz = \tan z + c = \tan(xe^x) + c.$

Home work.

(i) $I = \int \sin^2 x \cos x\, dx$

Let, $\sin x = z$

$\Rightarrow \cos x\, dx = dz$

$= \int \cancel{(1-\cos^2 x)\cos x\, dx}$

$= \int z^2\, dz = \frac{z^3}{3} + c = \frac{\sin^3 x}{3} + c.$

(ii) $I = \int \frac{\sqrt{x} + \ln x}{x} dx = \int \frac{1}{\sqrt{x}} + \frac{\ln x}{x}\, dx$

Let, $\ln x = z$

$\Rightarrow \frac{1}{x} dx = dz$

$= \int \frac{1}{\sqrt{x}} dx + \int \frac{\ln x}{x} dx$

$= 2\sqrt{x} + \int z\, dz = 2\sqrt{x} + \frac{z^2}{2} + c$

$= 2\sqrt{x} + \frac{(\ln x)^2}{2} + c$

(iii) $I = \int \tan^3 x \cancel{dx} \sqrt{\sec x}\, dx$

Let, $\sec x = z$

$\Rightarrow \sec x \tan x\, dx = dz$

$\Rightarrow z \tan x\, dx = dz$

$\Rightarrow \tan x\, dx = \frac{dz}{z}$

$= \int (\sec^2 x - 1) \tan x \sqrt{\sec x}\, dx$

$= \int (z^2 - 1)\sqrt{z}\, \frac{dz}{z}$

$= \int \frac{z^2 - 1}{\sqrt{z}}\, dz = \int \left(z^{3/2} - z^{-1/2}\right) dz$

$= \frac{2}{5} z^{5/2} - 2z^{1/2} + c = \frac{2}{5}(\sec x)^{5/2} - 2\sqrt{\sec x} + c.$

Integration by Parts:

$$\int uv\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \int v\, dx\right) dx$$

Rules of LIATE.

L = Logarithomic → $\log x$, $\ln x$.

I = Inverse Function → $\sin^{-1} x$, $\tan^{-1} 2x$.

A = Alzebric → $x, x^2, x^3, 1$.

T = Trigonometric → $\sin x, \cos x, \cot x$.

E = Exponential → e^x, a^x.

* If the both function are differentiable and integrable, and if one of them is of form x^n, then we take $u = x^n$.
* $\sin^{-1} x, \cos^{-1} x, \ldots\ldots\ldots, \ln x$ are taken as u.

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* If there is no function multiplied with $\sin^{-1}x, \cos^{-1}x$, ..., $\ln x$ etc. then we take $v=1$.

Example:

1. $I=\int \cos^{-1}x\,dx = \cos^{-1}x\int 1\,dx - \int\left\{\frac{d}{dx}(\cos^{-1}x)\int 1\,dx\right\}dx$

$= x\cos^{-1}x - \int\left(-\frac{1}{\sqrt{1-x^2}}\cdot x\right)dx = x\cos^{-1}x + \int\left(\frac{x}{\sqrt{1-x^2}}\right)dx$

$= x\cos^{-1}x - \frac{1}{2}\int\frac{-2x\,dx}{\sqrt{1-x^2}} = x\cos^{-1}x - \frac{1}{2}\,\frac{(1-x^2)^{1/2}}{1/2} + c$

$= x\cos^{-1}x - \sqrt{1-x^2} + c$.

2. $I = \int x\ln x\,dx = \ln x\int x\,dx - \int\left\{\frac{d}{dx}(\ln x)\int x\,dx\right\}dx$

$= \frac{1}{2}x^2\ln x - \int\left(\frac{1}{x}\cdot\frac{x^2}{2}\right)dx = \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + c$.

Home work:

1. $I = \int x^2\sin^2 x\,dx = \int x^2\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x^2dx - \frac{1}{2}\int x^2\cos 2x\,dx$

$= \frac{1}{6}x^3 - \frac{1}{2}\left\{x^2\,\frac{\sin 2x}{2} - \int\left(2x\,\frac{\sin 2x}{2}\right)dx\right\}$

$= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{2}\left\{x\,\frac{-\cos 2x}{2} - \int\left(1\cdot\frac{-\cos 2x}{2}\right)dx\right\}$

$= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x + c$.

2. $I = \int\frac{\ln(\ln x)}{x}\,dx = \int \ln z\,dz$ | Let, $\ln x = z$, $\frac{1}{x}dx = dz$

$= \ln z\cdot z - \int\left(\frac{1}{z}\cdot z\right)dz = z\ln z - z + c$

$= \ln x\cdot\ln(\ln x) - \ln x + c = \ln x\{\ln(\ln x) - 1\} + c$.

3. $I = \int \frac{\ln x}{(1+x)^3} dx$ *পরবর্তী পৃষ্ঠা করা আছে।

* $I = \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\tan x \sec^2 x \, dx}{\tan^4 x + 1}$

Let, $\tan^2 x = z$

$\Rightarrow 2 \tan x \sec^2 x \, dx = dz$

$\Rightarrow \tan x \sec^2 x \, dx = \frac{1}{2} dz$

$= \int \frac{1}{2} \frac{dz}{z^2+1} = \frac{1}{2} \cdot \frac{1}{1} \tan^{-1}\left(\frac{z}{1}\right) + c$

$= \frac{1}{2} \tan^{-1}(\tan^2 x) + c$

Standard Integral:

1. $I = \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

2. $I = \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$

3. $I = \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$

4. $I = \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$

5. $I = \int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x + \sqrt{a^2+x^2}\right) + c$

6. $I = \int \frac{dx}{\sqrt{x^2-a^2}} = \ln \left| x + \sqrt{x^2-a^2} \right| + c$

7. $I = \int \sqrt{a^2-x^2} \, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

8. $I=\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}|+c.$

9. $I=\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+c.$

☐ Prove that,

(i) $I=\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c.$

(ii) $I=\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c.$

Soln:

(i)

$$I=\int \frac{dx}{a^2\left\{1+\left(\frac{x}{a}\right)^2\right\}} = \frac{1}{a^2}\int \frac{a\left(\frac{1}{a}dx\right)}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\int \frac{d\left(\frac{x}{a}\right)}{1+\left(\frac{x}{a}\right)^2}$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a}+c. \quad \text{(Proved)}$$

(ii)

$$I=\int \frac{dx}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{1}{a}\int \frac{a\left(\frac{1}{a}dx\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= \sin^{-1}\frac{x}{a}+c. \quad \text{(Proved)}$$

Example:

1. $I=\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{(e^x)^2+1} = \int \frac{d(e^x)}{(e^x)^2+1} = \frac{1}{1}\tan^{-1}\left(\frac{e^x}{1}\right)+c$

$= \tan^{-1}(e^x)+c.$

2. $I=\int \frac{dx}{4\cos^2x+9\sin^2x} = \int \frac{\sec^2x\,dx}{4+9\tan^2x} = \int \frac{\sec^2x\,dx}{9\left(\frac{4}{9}+\tan^2x\right)}$

$= \frac{1}{9}\int \frac{d(\tan x)}{(\tan x)^2+\left(\frac{2}{3}\right)^2} = \frac{1}{9}\cdot\frac{1}{2/3}\tan^{-1}\left(\frac{\tan x}{2/3}\right)+c \cancel{= \frac{1}{6}\tan^{-1} 3\tan x}$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

[এই Math-এর ক্ষেত্রে $\tan x$ এ convert করতে হয়]

3. $$I = \int \frac{6x^5\,dx}{1+x^{12}} = \int \frac{d(x^6)}{1+(x^6)^2} = \frac{1}{1}\tan^{-1}\left(\frac{x^6}{1}\right) + c = \tan^{-1}(x^6) + c$$

Home Work.

1. $$I = \int \frac{dx}{4\cos^2 x - 9\sin^2 x} = \int \frac{\sec^2 x\,dx}{4 - 9\tan^2 x} = \int \frac{d(\tan x)}{9\left\{\left(\frac{2}{3}\right)^2 - (\tan x)^2\right\}}$$

$$= \frac{1}{9}\cdot\frac{1}{\cancel{2}\cdot\frac{2}{3}} \ln\left|\frac{\frac{2}{3}+\tan x}{\frac{2}{3}-\tan x}\right| + c = \frac{1}{3}\ln\left|\frac{2+3\tan x}{2-3\tan x}\right| + c.$$

Example:

4. $$I = \int \frac{dx}{x^4 - a^4} = \int \frac{dx}{(x^2+a^2)(x^2-a^2)} \cancel{= \int \frac{1}{x^2+a^2}}$$

$$= \frac{1}{2a^2}\int \left(\frac{1}{x^2-a^2} - \frac{1}{x^2+a^2}\right) dx$$

$$= \frac{1}{2a^2}\left\{\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{a}\tan^{-1}\frac{x}{a}\right\} + c.$$

$$= \frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + c.$$

5. $$I = \int \frac{dx}{(1+x^2)\sqrt{1-(\tan^{-1}x)^2}} = \int \frac{d(\tan^{-1}x)}{\sqrt{1-(\tan^{-1}x)^2}}$$

$$= \sin^{-1}(\tan^{-1}x) + c.$$

6. $$I = \int \sqrt{1+\sec x}\,dx = \int \sqrt{1+\frac{1}{\cos x}}\,dx = \int \sqrt{\frac{\cos x + 1}{\cos x}}\,dx$$

$$= \int \frac{\sqrt{2\cos^2 x/2}}{\sqrt{1-2\sin^2 x/2}}\,dx = \int \frac{\cos\frac{x}{2}}{\sqrt{(1/\sqrt{2})^2 - \sin^2 x/2}}\,dx$$

$$= 2\int \frac{\frac{1}{2}\cos\frac{x}{2}\,dx}{\sqrt{\left(\frac{1}{\sqrt2}\right)^2 - \sin^2 x/2}} = 2\int \frac{d(\sin\frac{x}{2})}{\sqrt{\left(\frac{1}{\sqrt2}\right)^2 - \sin^2 x/2}}$$

$$= 2\sin^{-1}\left(\frac{\sin x/2}{1/\sqrt2}\right) + c \quad = 2\sin^{-1}\left(\sqrt2 \sin\frac{x}{2}\right) + c.$$

H.W.

1. $$I = \int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{\sqrt{25}}\int \frac{dx}{\sqrt{x^2-\left(\frac{2}{5}\right)^2}} = \frac{1}{5}\ln\left|x+\sqrt{x^2-\left(\tfrac{2}{5}\right)^2}\right|+c$$

$$= \frac{1}{5}\ln\left|x+\sqrt{x^2-\frac{4}{25}}\right|+c.$$

Example:

1. $$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9-(9-x^2)}{\sqrt{9-x^2}}\,dx = 9\int \frac{1}{\sqrt{3^2-x^2}}\,dx - \int\sqrt{3^2-x^2}\,dx$$

$$\cancel{= 9\ln|x+\sqrt{\ }}$$

$$= 9\sin^{-1}\frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} - \frac{9}{2}\sin^{-1}\frac{x}{3} + c.$$

2. $$I = \int \frac{dx}{\sqrt{e^{2x}+1}} = \int \frac{dx}{e^x\sqrt{1+e^{-2x}}} = -\int \frac{-e^{-x}\,dx}{\sqrt{1+(e^{-x})^2}}$$

$$= -\int \frac{d(e^{-x})}{\sqrt{1+(e^{-x})^2}} = -\ln\left|e^{-x}+\sqrt{1+e^{-2x}}\right|+c.$$

H.W.

1. $$I = \int \frac{x^3\,dx}{\sqrt{a^8-x^8}} = \frac{1}{4}\int \frac{4x^3\,dx}{\sqrt{(a^4)^2-(x^4)^2}} = \frac{1}{4}\int \frac{d(x^4)}{\sqrt{(a^4)^2-(x^4)^2}}$$

$$= \frac{1}{4}\sin^{-1}\left(\frac{x^4}{a^4}\right)+c.$$

$\boxed{*}$ 2. $I = \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$.

☹ → পারি নি ।
* পূর্ববর্তী পৃষ্ঠায় করা আছে ।

Ex:
1. $I = \int \frac{\ln x}{(1+x)^3} dx = \int (1+x)^{-3} \ln x \, dx$

$= \ln x \frac{(1+x)^{-2}}{-2} - \int \left\{ \frac{1}{x} \frac{(1+x)^{-2}}{-2} \right\} dx$

$= -\frac{\ln x}{2(1+x)^2} + \frac{1}{2} \int \frac{1}{x(1+x)^2} dx$

$= - \frac{\ln x}{2(1+x)^2} + \frac{1}{2} \int I_1$

Using partial fraction, we let,

$\frac{1}{x(1+x)^2} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$

$\Rightarrow 1 \equiv A(1+x)^2 + Bx(1+x) + Cx$

Equating the coefficient of x^2, x and constant term,

$0 = A+B$
$0 = 2A+B+C$
$1 = A$

solving this equation,
$A=1, \quad B=-1, \quad C=-1$

$\therefore I = -\frac{\ln x}{2(1+x)^2} + \frac{1}{2} \int \left\{ \frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$.

$= -\frac{\ln x}{2(1+x)^2} + \frac{1}{2} \left\{ \ln x - \ln(1+x) + \frac{1}{1+x} \right\} + c$.

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2. $I=\int \frac{dx}{a^2\cos^2x+b^2\sin^2x}=\int \frac{\sec^2x\,dx}{a^2+b^2\tan^2x}=\frac{1}{b^2}\int \frac{d(\tan x)}{\left(\frac{a}{b}\right)^2+\tan^2x}$

$=\frac{1}{b^2}\cdot\frac{1}{a/b}\tan^{-1}\left(\frac{\tan x}{a/b}\right)+c=\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)+c.$

□ Form:

(i) $I=\int \frac{dx}{ax^2+bx+c}$

(ii) $I=\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(iii) $I=\int \sqrt{ax^2+bx+c}\,dx$

⊛ এক্ষেত্রে ax^2+bx+c কে $(x\pm \text{constant})^2\pm(\text{constant})^2$ এই formate এ আনতে হবে।

Example:-

1. $I=\int \frac{dx}{\sqrt{x^2+x+2}}=\int \frac{dx}{\sqrt{x^2+2x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\frac{1}{4}+2}}$

$=\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}}=\int \frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}}$

$=\cancel{\ln\left|x+\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}\right|+c}$

$=\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}\right|+c.$

2. $I=\int \sqrt{4-3x-2x^2}\,dx$

$=\sqrt{2}\int \sqrt{2-\frac{3}{2}x-x^2}\,dx=\sqrt{2}\int \sqrt{2-\left\{x^2+2\cdot x\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2\right\}+\frac{9}{16}}\,dx$

$=\sqrt{2}\int \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}\,d\left(x+\frac{3}{4}\right)$

$=\sqrt{2}\,\frac{\left(x+\frac{3}{4}\right)\sqrt{\frac{41}{16}-\left(x+3/4\right)^2}}{2}+\frac{\frac{41}{16}}{2}\sin^{-1}\frac{x+\frac{3}{4}}{41/16}+c$

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Form: $I = \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

* এক্ষেত্রে $x-\alpha = z^2$ ধরতে হবে এবং এখান থেকে $x = z^2+\alpha$ এর মান পরের x এর জায়গায় বসাতে হবে।

Let, $x-\alpha = z^2 \Rightarrow x = z^2+\alpha$

$dx = 2z\,dz$.

$$\therefore I = \int \frac{2z\,dz}{\sqrt{z^2(z^2+\alpha-\beta)}} = \int \frac{2\cancel{z}\,dz}{\cancel{z}\sqrt{z^2+(\sqrt{\alpha-\beta})^2}}$$

$$= 2\frac{1}{\sqrt{\alpha-\beta}}\tan^{-1}\left(\frac{z}{\sqrt{\alpha-\beta}}\right)+c$$

$$= 2\frac{1}{\sqrt{\alpha-\beta}}\tan^{-1}\left(\frac{\sqrt{x-\alpha}}{\sqrt{\alpha-\beta}}\right)+c.$$

Example:-

1. $I = \int \frac{dx}{\sqrt{x^2-7x+12}} = \int \frac{dx}{\sqrt{(x-3)(x-4)}}$

Let, $x-3 = z^2 \Rightarrow dx = 2z\,dz$ and, $x = z^2+3$

$$\therefore I = \int \frac{2z\,dz}{\sqrt{z^2(z^2+3-4)}} = 2\int \frac{dz}{\sqrt{z^2-1}}$$

$$= 2\ln\left|z+\sqrt{z^2-1}\right|+c = 2\ln\left|\sqrt{x-3}+\sqrt{(\sqrt{x-3})^2-1}\right|+c$$

$$= 2\ln\left|\sqrt{x-3}+\sqrt{x-4}\right|+c.$$

Form:

(i) $I = \int \frac{px+q}{ax^2+bx+c}\,dx$

(ii) $I = \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

(iii) $I = \int (px+q)\sqrt{ax^2+bx+c}\,dx$

* এক্ষেত্রে, $ax^2+bx+c = z$ ধরতে হয়।

→ পরবর্তীতে ax^2+bx+c এর differential যে formate টি পাওয়া যাবে সেই formate টি লবে তৈরি করতে হবে।

Example:

1. $I = \int \frac{2x+5}{\sqrt{x^2-2x+2}}\,dx$

Let, $x^2-2x+2 = z$
$\Rightarrow (2x-2)\,dx = dz$

$$= \int \frac{(2x-2)+7}{\sqrt{x^2-2x+2}}\,dx$$

$$= \int \frac{(2x-2)\,dx}{\sqrt{x^2-2x+2}} + 7\int \frac{dx}{\sqrt{x^2-2x+2}}$$

$$= \int \frac{dz}{\sqrt{z}} + 7\int \frac{dx}{\sqrt{(x-1)^2+1^2}} = \frac{z^{1/2}}{1/2} + 7\ln\left|(x-1)+\sqrt{(x-1)^2+1^2}\right| + c$$

$$= 2\sqrt{x^2-2x+2} + 7\ln\left|(x-1)+\sqrt{(x-1)^2+1^2}\right| + c.$$

H.W. 1. $I = \int \frac{4+7x}{\sqrt{4-x^2}}\,dx = \int \frac{-\frac{7}{2}(-2x)+4}{\sqrt{4-x^2}}\,dx$

Let, $z = 4-x^2$
$dz = -2x\,dx$

$\int \frac{dz}{\sqrt{z}} = 2\sqrt{z}$

$$= 4\int \frac{dx}{\sqrt{4-x^2}} - \frac{7}{2}\int \frac{(-2x)\,dx}{\sqrt{4-x^2}} = 4\sin^{-1}\frac{x}{2} - \frac{7}{2}\int \frac{dz}{\sqrt{z}}$$

$$= 4\sin^{-1}\frac{x}{2} - \cancel{2}\times\frac{7}{\cancel{2}}\sqrt{z} + c = 4\sin^{-1}\frac{x}{2} - 7\sqrt{4-x^2} + c$$

Form:

(i) $I = \int \frac{dx}{(ax+b)\sqrt{px+q}}$

We. Let,
$px+q = z^2 \Rightarrow x = \square,\ dx = \square$

(ii) $I = \int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$

$ax+b = \frac{1}{z},\ x = \square,\ dx = \square.$

(iii) $I = \int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$

$px^2+q = z^2,\ x = \square,\ dx = \square$

(iv) $I = \int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$

$x = \frac{1}{z},\ dx = \square.$

(v) $I = \int \frac{x}{(ax^2+b)\sqrt{px^2+q}}dx$ $\quad px^2+q = z^2$, $dx = \boxed{}$

⇒ অর্থাৎ, $I = \int \frac{1}{g(x)\sqrt{\varphi(x)}}dx$

(i) $g(x)$ ও $\varphi(x)$ উভয় একঘাত হলে, $\varphi(x) = z^2$ ধরতে হয়।

(ii) $g(x)$ একঘাত ও $\varphi(x)$ দ্বিঘাত হলে, $g(x) = \frac{1}{z}$ ধরতে হয়।

(iii) $g(x)$ দ্বিঘাত ও $\varphi(x)$ একঘাত হলে, $\varphi(x) = z^2$ ধরতে হয়।

(iv) $g(x)$ ও $\varphi(x)$ উভয় দ্বিঘাত হলে, $x = \frac{1}{z}$ ধরতে হয়।

(v) $I = \int \frac{x}{g(x)\sqrt{\varphi(x)}}dx$ এর $g(x)$ ও $\varphi(x)$ উভয় দ্বিঘাত হলে, $\varphi(x) = z^2$ ধরতে হয়।

[→ Using all these values the integrand I becomes a standard integral.]

Example:

1. $I = \int \frac{dx}{(x-3)\sqrt{x-2}}$. $\quad$ Let, $x-2 = z^2$, $x = z^2+2$, $dx = 2z\,dz$

$= \int \frac{2z\,dz}{(z^2+2-3)z} = 2\int \frac{dz}{z^2-1} = 2.\frac{1}{2.1}\ln\left|\frac{z-1}{z+1}\right| + c$.

$= \ln\left|\frac{\sqrt{x-2}-1}{\sqrt{x-2}+1}\right| + c$.

2. $I = \int \frac{x^2+x}{(x+2)\sqrt{x-1}}dx$ $\quad$ Let, $z^2 = x-1$ $\Rightarrow 2z\,dz = dx$ and, $x = z^2+1$

$= \int \frac{(z^2+1)^2+(z^2+1)}{(z^2+1+2)z}2z\,dz$

$= 2\int \frac{}{z^2+3}$

3. $I=\int \frac{x^3}{\sqrt{x-1}}\,dx$ | Let, $x-1=z^2 \Rightarrow x=z^2+1 \Rightarrow dx=2z\,dz$

$$=\int \frac{(z^2+1)^3}{z}\,2z\,dz = 2\int (z^6+3z^4+3z^2+1)\,dz$$

$$=2\left(\frac{z^7}{7}+3\,\frac{z^5}{5}+3\cdot\frac{z^3}{3}+z\right)+c.$$

$$=\cancel{2}\,\frac{2}{7}(x-1)^7+\frac{6}{5}(x-1)^5+2(x-1)^3+(x-1)+c.$$

4. $I=\int\frac{x\,dx}{(x+1)\sqrt{x^2+1}}=\int\frac{(x+1)-1}{(x+1)\sqrt{x^2+1}}\,dx$ | Let, $x+1=\frac{1}{z} \Rightarrow x=\frac{1}{z}-1$
$\Rightarrow dx=-\frac{1}{z^2}\,dz$

$$=\int\frac{dx}{\sqrt{x^2+1}}-\int\frac{1}{(x+1)\sqrt{x^2+1}}\,dx$$

$$=\ln|x+\sqrt{x^2+1}|-I_1$$

$$I_1=\int\frac{-\frac{1}{z^2}\,dz}{\frac{1}{z}\sqrt{(\frac{1}{z}-1)^2+1}}=-\int\frac{dz}{z\sqrt{\frac{(1-z)^2+z^2}{z^2}}}=-\int\frac{dz}{\sqrt{2z^2-2z+1}}$$

$$=-\int\frac{dz}{\sqrt{2}\sqrt{z^2-z+\frac{1}{2}}}=-\frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{z^2-2.z.\frac{1}{2}+(\frac{1}{2})^2-1/4+\frac{1}{2}}}$$

$$=-\frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{(z-\frac{1}{2})^2+(\frac{1}{2})^2}}=-\frac{1}{\sqrt{2}}\ln\left|(z-\frac{1}{2})+\sqrt{(z-\frac{1}{2})^2+\frac{1}{4}}\right|+c$$

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$$= -\frac{1}{\sqrt{2}} \ln \left| \left(\frac{1}{x+1} - \frac{1}{2}\right) + \sqrt{\left(\frac{1}{x+1} - \frac{1}{2}\right)^2 + \frac{1}{4}} \right| + c.$$

$$\therefore I = \ln \left| x + \sqrt{x^2+1} \right| - \frac{1}{\sqrt{2}} \ln \left| \left(\frac{1}{x+1} - \frac{1}{2}\right) + \sqrt{\left(\frac{1}{x+1} - \frac{1}{2}\right)^2 + \frac{1}{4}} \right| + c.$$

5. $I = \int \frac{dx}{(x+1)\sqrt{1+2x-x^2}}$ | Let, $x+1 = \frac{1}{z}$ $\Rightarrow x = \frac{1}{z} - 1$, $dx = -\frac{1}{z^2} dz$

$$= -\int \frac{\frac{1}{z^2} dz}{\frac{1}{z}\sqrt{1+2\left(\frac{1}{z}-1\right) - \left(\frac{1}{z}-1\right)^2}}$$

$$= -\int \frac{dz}{z\sqrt{1+2\left(\frac{1-z}{z}\right) - \frac{(1-z)^2}{z^2}}}$$

$$= -\int \frac{dz}{z\sqrt{\frac{z^2+2z-2z^2-1+2z-z^2}{z^2}}}$$

$$= -\int \frac{dz}{\sqrt{-2z^2+4z-1}} = \int -\frac{dz}{\sqrt{2}\sqrt{-z^2+2z-\frac{1}{2}}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{1-(z^2-2z\cdot 1+1)-\frac{1}{2}}} = -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - (z-1)^2}}$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{z-1}{1/\sqrt{2}}\right) + c = -\frac{1}{\sqrt{2}} \sin^{-1}\left\{\frac{\sqrt{2}\left(\frac{1}{x+1}-1\right)}{1}\right\} + c$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1}\left\{\sqrt{2}\left(\frac{1}{x+1}-1\right)\right\}$$

6. $I = \int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$

Let, $x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}dz$

$= -\int \frac{\frac{1}{z^2}dz}{\left(\frac{1}{z^2}+1\right)\sqrt{\left(\frac{1}{z}\right)^2+4}} = -\int \frac{\cancel{\frac{1}{z^2}}dz}{\cancel{\frac{1}{z^2}}(1+z^2)\cdot\frac{1}{z}\sqrt{1+4z^2}}$

$= -\int \frac{z.dz}{(1+z^2)\sqrt{4z^2+1}}$

$= -\frac{1}{2}\int \frac{z\ dz}{(1+z^2)\sqrt{z^2+\frac{1}{4}}}$

Again Let,

$z^2 + \frac{1}{4} = z_1^2$

$\Rightarrow \cancel{2}z\,dz = \cancel{2}z_1 dz_1$

$\Rightarrow z\,dz = z_1 dz_1$

and, $z^2 = z_1^2 - \frac{1}{4}$

$= -\frac{1}{2}\int \frac{\cancel{z_1}dz_1}{\left(1+z_1^2-\frac{1}{4}\right)\cancel{z_1}}$

$= -\frac{1}{2}\int \frac{dz_1}{z_1^2+\left(\frac{\sqrt{3}}{2}\right)^2}$ ~~$= \frac{1}{\sqrt{2}}\frac{1}{2\cdot\frac{\sqrt{3}}{2}}\ln\left|\frac{z_1-\frac{\sqrt{3}}{2}}{z_1+\frac{\sqrt{3}}{2}}\right|+c$~~

~~$= -\frac{1}{\sqrt{6}}\ln$~~ $= -\frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{z_1}{\frac{\sqrt{3}}{2}}\right)+c$

$= -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt{z^2+\frac{1}{4}}}{\sqrt{3}}\right)+c = -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\sqrt{\frac{1}{x^2}+\frac{1}{4}}\right)+c.$

H.W.

1. $I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Let, $x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}dz$

$= -\int \frac{\frac{1}{z^2}dz}{\left(1+\frac{1}{z^2}\right)\sqrt{1-\frac{1}{z^2}}}$

$= -\int \frac{\cancel{\frac{1}{z^2}}dz}{\cancel{\frac{1}{z^2}}(z^2+1)\frac{1}{z}\sqrt{z^2-1}} = -\int \frac{z\ dz}{(z^2+1)\sqrt{z^2-1}}$

Let, $z^2-1 = z_1^2$

$\Rightarrow \cancel{2}z\,dz = \cancel{2}z_1 dz_1$

$\Rightarrow z\,dz = z_1 dz_1$

and, $z^2 = z_1^2+1$

$$= -\int \frac{\cancel{z_1}\, dz_1}{(z_1^2+1+1)\cancel{z_1}} = -\int \frac{dz_1}{z_1^2+(\sqrt{2})^2} = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z_1}{\sqrt{2}}\right)+c$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{z^2-1}}{\sqrt{2}}\right)+c = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{x^2}-1}\right)+c.$$

Form:—

(i) $I = \int \frac{dx}{a+b\cos x}$, (ii) $\int \frac{dx}{a+b\sin x}$

We convert $\sin x$, $\cos x$ into $\tan x$ and then integrate.

Example:

1. $I = \int \frac{dx}{5+3\cos x} = \int \frac{dx}{5+3\,\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

$$= \int \frac{\sec^2 x/2 \, dx}{5+5\tan^2 x/2 + 3 - 3\tan^2 x/2} = \int \frac{\sec^2 x/2\, dx}{8+2\tan^2 x/2}$$

$$= \frac{1}{2}\int \frac{\sec^2 x/2\, dx}{(\tan^2 x/2 + 2^2)}$$

Let, $\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz$

$$= \frac{1}{2}\int \frac{2\, dz}{z^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right)+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\tan x/2}{2}\right)+c.$$

2. $I = \int \frac{dx}{3\sin x + 2\cos x + 5} = \int \frac{dx}{3\,\frac{2\tan x/2}{1+\tan^2 x/2} + 2\,\frac{1-\tan^2 x/2}{1+\tan^2 x/2} + 5}$

$$= \int \frac{\sec^2 x/2\, dx}{6\tan x/2 + 2 - 2\tan^2 x/2 + 5 + 5\tan^2 x/2}$$

$$= \int \frac{\sec^2 \frac{x}{2}\, dx}{3\tan^2 \frac{x}{2} + 6\tan\frac{x}{2} + 7} = \frac{1}{3}\int \frac{\sec^2 \frac{x}{2}\, dx}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2}\cdot 1 + 1 + 7 - 1}$$

$$= \frac{1}{3}\int \frac{\sec^2\frac{x}{2}\, dx}{(\tan\frac{x}{2}+1)^2 + 6}$$

Let,
$\tan\frac{x}{2} + 1 = z$
$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz$

$$= \frac{1}{3}\int \frac{2\, dz}{z^2 + (\sqrt{6})^2}$$

$$= \frac{2}{3}\cdot \frac{1}{\sqrt{6}} \tan^{-1}\left(\frac{z}{\sqrt{6}}\right) + c = \frac{2}{3\sqrt{6}} \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{6}}\right) + c.$$

Form: $I = \int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}\, dx$

* We let, Numerator $= L \times$ (Denominator) $+ M \times \frac{d}{dx}$(Denominator) $+ N$

∴ Equating the coefficients of $\sin x, \cos x$ and constant term, we shall get the value of L.M.N.

Ex:-

1. $I = \int \frac{2 + 3\sin x - \cos x}{1 + \cos x + \sin x}\, dx$

Let, $2 + 3\sin x - \cos x = L(1 + \cos x + \sin x) + M(-\sin x + \cos x) + N$

Equating the coefficient of $\sin x$, $\cos x$ and constant term, we get,

$L - M = 3$
$L + M = -1$
$L + N = 2$

solving these equations,
$L = 1$, $M = -2$, $N = 1$

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$$I=\int \frac{(1+\cos x+\sin x)-2(-\sin x+\cos x)+1}{1+\cos x+\sin x}\,dx$$

$$=\int 1\,dx-\int \frac{2(-\sin x+\cos x)\,dx}{1+\cos x+\sin x}+\int \frac{1}{1+\cos x+\sin x}\,dx$$

$$=x-2\ln|1+\cos x+\sin x|+\int \frac{1}{1+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}+\frac{2\tan x/2}{1+\tan^2 x/2}}\,dx$$

$$=x-2\ln|1+\cos x+\sin x|+\int \frac{\sec^2 x/2\,dx}{1+\cancel{\tan^2 x/2}+1-\cancel{\tan^2 x/2}+2\tan x/2}$$

$$=x-2\ln|1+\cos x+\sin x|+\int \frac{\sec^2 x/2\,dx}{2(1+\tan x/2)}$$

$$=x-2\ln|1+\cos x+\sin x|+\frac{1}{2}\times\frac{1}{2}\int \frac{\frac{1}{2}\sec^2 x/2\,dx}{(1+\tan x/2)}$$

$$=x-2\ln|1+\cos x+\sin x|+\frac{1}{4}\ln|1+\tan x/2|+c$$

Definite Integral:

$$* \quad I=\int_1^2 (x^2+2)\,dx=\left[\frac{x^3}{3}+2x\right]_1^2$$

$$=\left(\frac{8}{3}+2\times 2\right)-\left(\frac{1}{3}+2\times 1\right)$$

$$=\frac{20}{3}-\frac{7}{3}$$

$$=\frac{13}{3}$$

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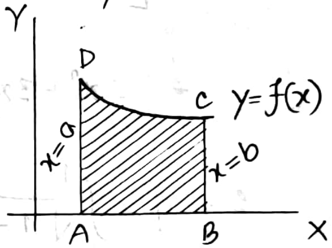
Golap Rahman (93) 'CE '18

* **Defination:** Geometrical interpretation of $\int_a^b f(x)\,dx$.

The definite integral $\int_a^b f(x)\,dx$ geometrically represents the area enclosed by the curve $y = f(x)$, the ordinates $x=a$ and $x=b$ and the x-axis that is $y=0$.

Thus the area of $ABCD = \int_{x=a}^{b} f(x)\,dx$



* Properties of Definite Integral:

1. $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx$; $a<c<b$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

5. $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(a+x) = f(x)$

6. $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even
$= 0$ if $f(x)$ is odd function

Standard Integral (1-9).

Example:

$$1.\ I=\int_{x=0}^{x=1}\left(\frac{3x^3+2x^2}{x^2}+x+1\right)dx=\int_0^1(3x+2+x+1)\,dx$$

$$=\cancel{\left[\frac{4x^2}{2}+3x\right]_0^1+2^2}=\int_0^1(4x+3)\,dx=\left[\frac{4x^2}{2}+3x\right]_0^1$$

$$=\left[2x^2+3x\right]_0^1=2\cdot1+3\cdot1=5.$$

$$2.\ I=\int_0^1\sqrt{1-x^2}\,dx=\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=0+\frac{1}{2}\cdot\frac{\pi}{2}-0-0=\frac{\pi}{4}$$

$$3.\ I=\int_0^{\pi/3}\frac{\cos x}{3+4\sin x}\,dx.$$

$$=\frac{1}{4}\int_0^{\pi/3}\frac{d(3+4\sin x)}{3+4\sin x}=\frac{1}{4}\Big[\ln|3+4\sin x|\Big]_0^{\pi/3}$$

$$=\frac{1}{4}\left\{\ln\left|3+4\times\frac{\sqrt{3}}{2}\right|-\ln|3|\right\}$$

$$=\frac{1}{4}\left\{\ln|3+2\sqrt{3}|-\ln 3\right\}$$

H.W.:

$$1.\ I=\int_2^e\left\{\frac{1}{\ln x}-\frac{1}{(\ln x)^2}\right\}dx$$

Let, $I_1=\int_2^e\frac{1}{\ln x}\,dx-\int\frac{1}{(\ln x)^2}\,dx$

$$= \frac{1}{\ln x}\cdot x - \int \left\{(-1)\frac{1}{(\ln x)^2}\cdot\frac{1}{x}\cdot x\right\}dx - \int \frac{1}{(\ln x)^2}\,dx$$

$$= \frac{x}{\ln x} + \int \frac{1}{(\ln x)^2}\,dx - \int \frac{1}{(\ln x)^2}\,dx = \frac{x}{\ln x} + c.$$

$$\therefore I = \left[\frac{x}{\ln x}\right]_2^e = \frac{e}{\ln e} - \frac{2}{\ln 2} = e - \frac{2}{\ln 2}.$$

2. $I = \int_{1/2}^{1} \frac{dx}{x\sqrt{1-x^2}}$

Let, $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

and Limit, when, $x = \frac{1}{2}$; $\theta = \frac{\pi}{6}$

$x = 1$; $\theta = \frac{\pi}{2}$

$$= \int_{\pi/6}^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta\sqrt{1-\sin^2\theta}}$$

$$= \int_{\pi/6}^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta\cos\theta} = \int_{\pi/6}^{\pi/2} \operatorname{cosec}\theta\, d\theta = \left[\ln\left(\tan\frac{\theta}{2}\right)\right]_{\pi/6}^{\pi/2}$$

$$= \ln\left(\tan\frac{\pi/2}{2}\right) - \ln\left\{\tan\frac{\pi/6}{2}\right\} = \ln\left(\tan\frac{\pi}{4}\right) - \ln\cancel{\left(\tan\frac{\pi}{12}\right)}\left(\tan\frac{\pi}{12}\right)$$

$$= \ln(1) - \ln(2-\sqrt{3}) = \ln\left(\frac{1}{2-\sqrt{3}}\right) = \ln\left(\frac{2+\sqrt{3}}{4-3}\right)$$

$$= \ln(2+\sqrt{3}).$$

Example:

1. $I = \int_0^a \sqrt{\frac{a+x}{a-x}}\,dx$

Let, $x = a\cos\theta$ $\quad dx = -a\sin\theta\, d\theta$

when, $x = 0$; $\theta = \frac{\pi}{2}$

$x = a$; $\theta = 0$

$$\therefore I = -\int_{\pi/2}^{0} \sqrt{\frac{a + a\cos\theta}{a - a\cos\theta}}\cdot a\sin\theta\, d\theta$$

$$= +\int_0^{\pi/2} \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\; a\sin\theta\, d\theta = a\int_0^{\pi/2} \sqrt{\frac{2\cos^2\theta/2}{2\sin^2\theta/2}}\;\sin\theta\, d\theta$$

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$$= a\int_0^{\pi/2} \frac{\cos\theta/2}{\cancel{\sin\theta/2}} \cdot 2\cancel{\sin\theta/2}\cos\theta/2 \, d\theta = a\int_0^{\pi/2} 2\cos^2\theta/2 \, d\theta$$

$$= a\int_0^{\pi/2}(1+\cos\theta)\,d\theta = a\left[\theta+\sin\theta\right]_0^{\pi/2} = a\left(\frac{\pi}{2}+1-0\right)$$

$$= a\left(1+\frac{\pi}{2}\right).$$

H.W.

1. $$I = \int_0^{\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{\pi} \frac{d\theta}{2+\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}}$$

$$= \int_0^{\pi} \frac{\sec^2\theta/2 \, d\theta}{2+2\tan^2\theta/2+1-\tan^2\theta/2} = \int_0^{\pi} \frac{\sec^2\theta/2 \, d\theta}{3+\tan^2\theta/2}$$

Let, $\tan\frac{\theta}{2} = z \Rightarrow \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = dz \Rightarrow \sec^2\theta/2 \, d\theta = 2\,dz$

When, $\theta = 0$; $z = 0$ and $\theta = \pi/2$; $z = 1$

$$\therefore I = 2\int_0^1 \frac{dz}{(\sqrt{3})^2+z^2} = 2\,\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{z}{\sqrt{3}}\right)\right]_0^1$$

$$= 2\cdot\frac{1}{\sqrt{3}}\left\{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}(0)\right\} = \frac{2}{\sqrt{3}}\times\frac{\pi}{6} = \frac{\sqrt{3}\,\pi}{\sqrt{3}.\sqrt{3}.3}$$

$$= \frac{\sqrt{3}\,\pi}{9}$$

2. $$I = \int_2^4 \frac{x-3}{\sqrt{(x-1)(5-x)}}\,dx = \int_2^4 \frac{x-3}{\sqrt{5x-5-x^2+x}}\,dx$$

$$= \int_2^4 \frac{x-3}{\sqrt{-x^2+6x-5}}\,dx = \int_2^4 \frac{x-3}{\sqrt{4-(x^2-6x+9)}}\,dx$$

$$= \int_2^4 \frac{x-3}{\sqrt{4-(x-3)^2}}\,dx.$$

Let, $(x-3)^2 = z \Rightarrow 2(x-3)\,dx = dz \Rightarrow (x-3)dx = \frac{1}{2}\,dz$

When limit, $x=2$; $z=1$ and $x=4$; $z=1$

$$\therefore I = \frac{1}{2}\int_1^1 \frac{dz}{\sqrt{4-z}} = \frac{1}{2}\left[2\sqrt{4-z}\right]_1^1 = \left[\sqrt{4-z}\right]_1^1 = \sqrt{3}-\sqrt{3}$$

$= 0$

3. $I = \int_0^1 x \sin^{-1}x \, dx$, Let, $I_1 = \sin^{-1}x \int x\,dx - \int\left(\frac{d}{dx}\sin^{-1}x \int x\,dx\right)dx$

$$= \sin^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}}\,\frac{x^2}{2}\,dx = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\int \sqrt{1-x^2}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x - \sin^{-1}x\right] + c$$

$$\therefore I = \left[\frac{x^2}{2}\sin^{-1}x + \frac{1}{4}\cdot x\sqrt{1-x^2} - \frac{1}{4}\sin^{-1}x\right]_0^1$$

$$= \frac{1}{2}\frac{\pi}{2} + 0 - \frac{1}{4}\times\frac{\pi}{2} + 0 = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$$

4. $$I = \int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

Let, $x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}\,dz$

when, $x = 0$; $z = \infty$, $x = 1$; $z = 1$

$$= \int_\infty^1 \frac{-\frac{1}{z^2}\,dz}{\left(1+\frac{1}{z^2}\right)\sqrt{1-\frac{1}{z^2}}}$$

$$= -\int_\infty^1 \frac{\cancel{\frac{1}{z^2}}\,dz}{\cancel{\frac{1}{z^2}}(z^2+1)\frac{1}{z}\sqrt{z^2-1}} = -\int_\infty^1 \frac{z\,dz}{(z^2+1)\sqrt{z^2-1}}$$

Again Let, $z^2-1 = z_1^2 \Rightarrow \cancel{2}z\,dz = \cancel{2}z_1\,dz_1$

When, $z=\infty$; $z_1 = \infty$

$z = 1$, $z_1 = 0$

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$$\therefore I = -\int_{\infty}^{0} \frac{z_1\,dz_1}{(z_1^2+1+1)\sqrt{z_1^2}} = -\int_{\infty}^{0} \frac{\cancel{z_1}\,dz_1}{(z_1^2+2)\cancel{z_1}} = -\int_{\infty}^{0} \frac{dz_1}{\sqrt{z_1^2+(\sqrt{2})^2}}$$

$$= -\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{z_1}{\sqrt{2}}\right]_{\infty}^{0} = -\left(0-\frac{1}{\sqrt{2}}\times\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}.$$

Properties-4

$$I = \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

Example:

1. $$I = \int_0^{\pi/2} \frac{dx}{1+\cot x} = \int_0^{\pi/2} \frac{dx}{1+\frac{\cos x}{\sin x}} = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx \quad \text{—— (i)}$$

Using the properties $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)}\,dx = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x} \quad \text{—— (ii)}$$

Adding the equation (i) and (ii) $\Rightarrow$

$$2I = \int_0^{\pi/2}\left(\frac{\sin x}{\sin x+\cos x} + \frac{\cos x}{\cos x+\sin x}\right)dx$$

$$= \int_0^{\pi/2}\left(\frac{\sin x+\cos x}{\sin x+\cos x}\right)dx = \int_0^{\pi/2} 1.\,dx$$

$$= [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$2I = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}.$$

2. $I = \int_0^{\pi} \frac{x\,dx}{1+\sin x}$ —— (i)

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we have from (i)

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx = \int_0^{\pi} \frac{\pi - x}{1+\sin x} \text{ —— (ii)}$$

Adding the equation (i) and (ii)

$$2I = \int_0^{\pi} \frac{\cancel{x} + \pi - \cancel{x}}{1+\sin x}\,dx = \pi \int_0^{\pi} \frac{dx}{1+\sin x} = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x)\,dx = \pi\,[\tan x - \sec x]_0^{\pi}$$

$$= \pi(0+1-0+1) = 2\pi$$

$\therefore I = \pi$

H.W.

1. $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ —— (i)

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)}+\sqrt{\cos(\pi/2 - x)}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \text{ —— (ii)}$$

Adding the equation (i) and (ii) ⇒

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} 1\,dx = [x]_0^{\pi/2}$$

$$= \pi/2$$

$\therefore I = \frac{\pi}{4}$.

2. $I=\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$ —— (i)

Using the properties $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

$I=\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$ —— (ii)

(i) + (ii) ⇒

$2I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$

Let, $\cos x = z$
$\Rightarrow -\sin x\,dx = dz$
$\Rightarrow \sin x\,dx = -dz$
when, $x=0$; $z=1$
$x=\pi$; $z=-1$

$= -\pi\int_1^{-1} \frac{dz}{1+z^2}$

$= -\pi\left[\frac{1}{1}\tan^{-1}\frac{z}{1}\right]_1^{-1}$

$= -\pi\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) = -\pi\left(-\frac{2\pi}{4}\right) = \frac{\pi^2}{2}$

$\therefore I = \frac{\pi^2}{4}$

Ex: 1. $I=\int_0^{1} \ln\left(\frac{1}{x}-1\right)dx = \int_0^1 \ln\left(\frac{1-x}{x}\right)dx$ —— (i)

Using $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$,

$I=\int_0^1 \ln\left(\frac{1-(1-x)}{1-x}\right)dx = \int_0^1 \ln\frac{x}{1-x}\,dx$. —— (ii)

(i) + (ii) ⇒

$2I = \int_0^1 \left\{\ln\frac{1-x}{x} + \ln\left(\frac{x}{1-x}\right)\right\}dx = \int_0^1 (\ln 1)\,dx$

$\therefore I = 0$

→ Wallis's Formula

** Proof that,

$$\int_{-a}^{+a} f(x)\,dx = \int_0^a \{f(x)+f(-x)\}\,dx.$$

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***Proof: $\int_{-a}^{+a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{+a} f(x)\,dx$

Now putting $x = -z$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-z)\,dz = \int_{0}^{a} f(-z)\,dz = \int_{0}^{a} f(-x)\,dx$$

$$\therefore \int_{-a}^{+a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a}\big(f(x)+f(-x)\big)dx$$

So, If $f(x)$ is odd function of x $\therefore f(-x) = -f(x)$

$$\therefore \int_{-a}^{a} f(x)\,dx = 0$$

and $f(x)$ is even function of x $\therefore f(-x) = f(x)$

$$\therefore \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Ex:

1. $I = \int_{0}^{\pi/2} \sin^7 x\,dx = \frac{6\times 4\times 2}{7\times 5\times 3\times 1} = \frac{16}{35}$

2. $I = \int_{0}^{\pi/2} \sin^4 x\,dx = \frac{3\times 1}{4\times 2}\times\frac{\pi}{2} = \frac{3\pi}{16}$

3. $I = \int_{0}^{\pi/2} \cos^6 x\,dx = \frac{5\times 3\times 1}{6\times 4\times 2}\times\frac{\pi}{2} = \frac{5\pi}{32}$

4. $I = \int_{0}^{\pi/2} \sin^4 x\cos^5 x\,dx = \frac{\cancel{3}\times 1\times 4\times 2}{9\times 7\times 5\times \cancel{3}\times 1} = \frac{8}{315}$

5. $I = \int_{0}^{\pi/2} \sin^4 x\cos^8 x\,dx = \frac{\cancel{3}\times 1\times 7\times \cancel{5}\times \cancel{3}\times 1}{\underset{4}{\cancel{12}}\times \underset{2}{\cancel{10}}\times 8\times \underset{2}{\cancel{6}}\times \cancel{4}\times 2}\times\frac{\pi}{2} = \frac{7\pi}{2048}$.

*Statement: $\int_{0}^{\pi/2} \sin^n x\,dx = \int_{0}^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)\ldots 3.1}{n(n-2)(n-4)\ldots 4.2}\times\frac{\pi}{2}$ if n is even number or $= \frac{(n-1)(n-3)\ldots 4.2}{n(n-2)(n-4)\ldots 3.1}$ if n is odd number.

→ Walli's Formula

** Proof that,

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \text{if } f(2a-x) = f(x)$$

And $$\int_0^{2a} f(x)\,dx = 0 \quad \text{if } f(2a-x) = -f(x)$$

→ **Proof:** $$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

Since, $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$.

$$\therefore \int_0^{\pi} \sin x\,dx = 2\int_0^{\pi/2} \sin x\,dx \text{ and } \int_0^{\pi} \cos x\,dx = 0$$

And generally $\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx$

and $\int_0^{\pi} f(\cos x)\,dx = 0$ if $f(\cos x)$ is an odd function of $\cos x$.

* Proof that. $$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

→ **Proof:** $$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

Put, $x = 2a - z$ in the 2nd integral; then $dx = -dz$

also when, $x = a$, $z = a$ and when $x = 2a$, $z = 0$.

the second integral on the right side,

$$\int_a^{2a} f(x)\,dx = \int_0^{a} f(2a-z)\,dz = \int_0^{a} f(2a-z)\,dz$$

$$= \int_0^{a} f(2a-x)\,dx.$$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx.$$

Successive Reduction Formula:

The formula in which a certain integral involving some parameters is connected with some integrals of lower order is called a Reduction Formula. In most of the cases the reduction formula is obtained by the process of integra-tion by parts.

Example:

1. Find the reduction formula for $I_n = \int \sin^n x\, dx$ and hence evaluate I_5.

Sol^n: Given that,

$$I_n = \int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$$

$\int uv\,dx$ এর rules

$$= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\, dx \qquad [\because \sin x \neq x] \quad \cos x \cdot \cos x = \cos^2 x$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\, dx.$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$\Rightarrow I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n (1+n-1) = -\sin^{n-1} x \cos x + (n-1) I_{n-2}.$$

$$\therefore I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

This is the reduction formula for $I_n = \int \sin^n x\, dx$

Now we put $n=5$,

$$\therefore I_5 = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} I_3$$

$$= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\left(-\frac{\sin^2 x \cos x}{3} + \frac{2}{3} I_1\right)$$

$$= -\frac{\sin^4 x \cos x}{5} - \frac{4}{15}\sin^2 x \cos x + \frac{8}{15}(-\cos x + 0) + c$$

$$= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + c.$$

H.W. * Find out the reduction formula for
(i) $I_n = \int \cos^n x \, dx$ (ii) $I_n = \int \tan^n x \, dx$.

Sol^n: (i)

$$I_n = \int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

$$= \cos^{n-1} x \sin x - \int (n-1)\cos^{n-2} x (-\sin^2 x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx$$

$$\Rightarrow I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n (1 + n - 1) = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$\therefore I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$$

So this is the reduction formula for $I_n = \int \cos^n x \, dx$.

Sol^n: (ii) $I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx$

$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

This is the reduction formula for $I_n = \int \tan^n x \, dx$

Example:

1. If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$, then show that the reduction formula is $I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ and hence evaluate I_5.

⇒ Soln:

$$I_n = \int_0^{\pi/2} x^n \sin x \, dx = \left[-x^n \cos x\right]_0^{\pi/2} - \int_0^{\pi/2} (-n x^{n-1} \cos x) \, dx$$

$$= 0 + n\left[x^{n-1} \sin x\right]_0^{\pi/2} - n \int_0^{\pi/2} (n-1) x^{n-2} \sin x \, dx$$

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$$

$$\therefore I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

We put 5, then equation (i) gives,

$$I_5 + (5\times4) I_3 = 5\left(\frac{\pi}{2}\right)^4$$

$$\Rightarrow I_5 = 5\left(\frac{\pi}{2}\right)^4 - 20 I_3$$

$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6I_1$

$I_1 = 1$

$$= 5\left(\frac{\pi}{2}\right)^4 - 20\left\{3\left(\frac{\pi}{2}\right)^2 - 6I_1\right\}$$

$$= 5\left(\frac{\pi}{2}\right)^4 - 60\left(\frac{\pi}{2}\right)^2 + 120$$

Ans.

H.W.

1. If $U_n = \int_0^1 x^n \tan^{-1} x \, dx$, then show that its reduction formula is $(n+1) U_n + (n-1) U_{n-2} = \frac{\pi}{2} - \frac{1}{n}$

⇒ Soln: $U_n = \int_0^1 x^n \tan^{-1} x \, dx$

$$= \left[\tan^{-1}x \cdot \frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \left(\frac{1}{1+x^2} \cdot \frac{x^{n+1}}{n+1}\right) dx$$

$$= \left(\frac{\pi}{4} \cdot \frac{1}{n+1} \cancel{- 0}\right) -$$

Let, $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$

when. limit, $x = 0$; $\theta = 0$ and $x = 1$; $\theta = \frac{\pi}{4}$

$$\therefore U_n = \int_0^{\pi/4} \theta \tan^n\theta \sec^2\theta \, d\theta$$

$$= \left[\theta \tan^n\theta \tan\theta\right]_0^{\pi/4} - \int_0^{\pi/4} \left\{\left(\tan^n\theta + n\theta \tan^{n-1}\theta \sec^2\theta\right)\tan\theta\right\} d$$

$$= \frac{\pi}{4} - \int_0^{\pi/4} \left(\tan^{n+1}\theta\right) d\theta - n \int_0^{\pi/4} \theta \tan^n\theta \sec^2\theta \, d\theta$$

$$\therefore U_n = \frac{\pi}{4} - \int_0^{\pi/4} \tan^{n+1}\theta \, d\theta - nU_n$$

$$\Rightarrow (n+1)U_n = \frac{\pi}{4} - \int_0^{\pi/4} \tan^{n+1}\theta \, d\theta \quad \text{—— (i)}$$

Put ~~when~~, $n = n-2$

$$(n-1)U_{n-2} = \frac{\pi}{4} - \int_0^{\pi/4} \tan^{n-1}\theta \, d\theta \quad \text{—— (ii)}$$

adding (i) and (ii) we get,

$$(n+1)U_n + (n-1)U_{n-2} = \frac{\pi}{4} + \frac{\pi}{4} - \int_0^{\pi/4} \left(\tan^{n+1}\theta + \tan^{n-1}\theta\right)$$

$$= \frac{\pi}{2} - \int_0^{\pi/4} \tan^{n-1}\theta \left(\tan^2\theta + 1\right)$$

$$= \frac{\pi}{2} - \int_0^{\pi/4} \tan^{n-1}\theta \sec^2\theta \, d\theta$$

$$= \frac{\pi}{2} - \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4} = \frac{\pi}{2} - \frac{1}{n}$$

$$\therefore (n+1)U_n + (n-1)U_{n-2} = \frac{\pi}{2} - \frac{1}{n}$$ [Showed]

2. Find the reduction formula for $I_n = \int x^n e^{ax}\,dx$.

$\Rightarrow$ **Sol^n:**

$$I_n = \int x^n e^{ax}\,dx = x^n \frac{e^{ax}}{a} - \int \left(n x^{n-1} \frac{e^{ax}}{a}\right) dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} \cdot e^{ax}\,dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

3. Find the reduction formula for $I_n = \int e^{ax}\sin^n x\,dx$.

$\Rightarrow$ **Sol^n:**

$$I_n = \int e^{ax}\sin^n x\,dx = \sin^n x\,\frac{e^{ax}}{a} - \int n\sin^{n-1}x\cos x\,\frac{e^{ax}}{a}\,dx$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\cdot\sin^{n-1}x\cos x\,\frac{e^{ax}}{a} + \frac{n}{a}\int\left\{\sin^{n-1}x(-\sin x) + \cos^2 x\,(n-1)\sin^{n-2}x\right\}\frac{e^{ax}}{a}\,dx$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{ne^{ax}}{a^2}\sin^{n-1}x\cos x + \frac{n\cancel{(n-1)}}{a^2}\int e^{ax}\left\{-\sin^n x + (n-1)\sin^{n-2}x(1-\sin^2 x)\right\}dx$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{ne^{ax}}{a^2}\sin^{n-1}x\cos x + \frac{n}{a^2}\int e^{ax}\left\{\sin^n x + (n-1)(\sin^{n-2}x - \sin^n x)\right\}dx$$

$$= \frac{e^{ax}(a\sin^n x - n\sin^{n-1}x\cos x)}{a^2} + \frac{n}{a^2}\int e^{ax}\left\{\sin^n x(-1-n+1) + (n-1)\sin^{n-2}x\right\}dx$$

$$\Rightarrow I_n = \frac{e^{ax}(a\sin^n x - n\sin^{n-1}x\cos x)}{a^2} - \frac{n^2}{a^2} I_n + \frac{n(n-1)}{a^2} I_{n-2}$$

$$\Rightarrow \left(1+\frac{n^2}{a^2}\right) I_n = \frac{e^{ax}(a\sin^n x - n\sin^{n-1}x\cos x)}{a^2} + \frac{n(n-1)}{a^2} I_{n-2}$$

$$\Rightarrow \left(\frac{a^2+n^2}{a^2}\right) I_n = \frac{e^{ax}\sin^{n-1}x(a\sin x - n\cos x) + n(n-1) I_{n-2}}{a^2}$$

$$\therefore I_n = \frac{e^{ax}\sin^{n-1}x(a\sin x - n\cos x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2} I_{n-2}$$

(Ans:)

Beta and Gamma function:

▣ $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ denoted by $\beta(m,n)$. $(m,n>0)$ is called the First Eulerian integral or Beta function.

▣ $\int_0^\infty e^{-x}x^{n-1}\,dx$ denoted by $\Gamma(n)$ $[n>0]$ is called the second Eulerian integral or Gamma function.

$$\therefore \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+1}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$\therefore \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$$

* Properties of β & Γ Function:

1. $\beta(m,n) = \beta(n,m)$
2. $\Gamma(1) = 1$.
3. $\Gamma(n+1) = n!$
4. $\int_0^\infty e^{-kx} x^{n-1}\, dx = \frac{\Gamma(n)}{k^n}, \; k,n>0$
5. $\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$
6. $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi} \; [0<m<1]$

* Put $m = \frac{1}{2}$, $\Gamma(1/2)\,\Gamma(1/2) = \frac{\pi}{\sin \frac{1}{2}\pi}$

$$\Rightarrow \Gamma(1/2)^2 = \pi$$

$$\therefore \Gamma(1/2) = \sqrt{\pi}$$

State and Prove the relation between Beta and Gamma function :-

⇒ Statement.

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

→ **Proof:**

We know,

$$\int_0^\infty e^{-zx} x^{n-1}\, dx = \frac{\Gamma(n)}{z^n}$$

$$\Rightarrow \Gamma(n) = \int_0^\infty e^{-zx} x^{n-1} z^n\, dx \quad \text{(i)}$$

Multiplying on both sides of (i) by $z^{m-1} e^{-z}$.

We get,

$$\Gamma(n)\, z^{m-1} e^{-z} = \int_0^\infty z^{m+n-1} e^{-z(x+1)} x^{n-1}\, dx$$

Integrating both side we respect to z with in limits 0 to ∞, we get,

$$\Rightarrow \Gamma(n)\int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty \left[\int_0^\infty z^{(m+n)-1} e^{-z(x+1)} dz\right] x^{n-1}\, dx.$$

$$\Rightarrow \Gamma(n)\int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{n-1}\, dx.$$

$$\Rightarrow \Gamma(m)\Gamma(n) = \Gamma(m+n)\ \beta(m,n).$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$ (Proved)

* Show that,

(i) $\Gamma(n+1) = n! = \lfloor n$

(ii) $\Gamma(n+1) = n\Gamma(n)$

(iii) $\Gamma(1) = 1$

$\Rightarrow$ Solⁿ: We know that,

Gamma function is $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \;;\; n>0$ — (i)

Replacing n by $(n+1)$ in the equation (i)

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx \quad \text{— (ii)}$$

$$\Rightarrow \Gamma(n+1) = \left[-x^n \cdot e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\, dx$$

$$= 0 + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= n\,\Gamma(n).$$

$$\therefore \Gamma(n+1) = n\,\Gamma(n) \quad \text{--- (iii)} \quad \text{(Proved)}$$

Replacing n by $(n-1)$ in the equation (iii)

$$\Gamma(n) = (n-1)\,\Gamma(n-1) \quad \text{--- (iv)}$$

Again replacing n by $(n-2), (n-3), (n-4) \ldots$ successively

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$

$$\Gamma(n-2) = (n-3)\,\Gamma(n-3)$$

$$\Gamma(n-3) = (n-4)\,\Gamma(n-4)$$

and so on.

Using all these values in equation (iii) becomes,

$$\Gamma(n+1) = n(n-1)\,\Gamma(n-1)$$

$$= n(n-1)(n-2)\,\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3) \ldots \ldots 3.2.1.$$

$$= \underline{|n} = n!$$

$$\therefore \Gamma(n+1) = \underline{|n} = n! \quad \text{(Proved)}$$

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For $n=1$, equation (1) becomes,

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty$$

$$= -e^{-\infty} + e^0 = 0+1 = 1.$$

$\therefore \Gamma(1) = 1$ (Proved)

Example:

1. Show that,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$$

Hence evaluate $\int_0^{\pi/2} \sin^5\theta \, d\theta$.

⇒ Sol^n: We know that, Beta function is

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{--- (i)}$$

We let, $x=\sin^2\theta$ then, $dx = 2\sin\theta\cos\theta\, d\theta$

and limit when $x=0$; $\theta=0$ and $x=1$; $\theta=\pi/2$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \; 2\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta. \quad \text{--- (ii)}$$

Let, $2m-1=p \quad \therefore m=\frac{p+1}{2}$

and $2n-1=q \quad \therefore n=\frac{q+1}{2}$

Then from equation (ii)

$$\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=\int_0^{\pi/2} 2\sin^p\theta\cos^q\theta\, d\theta$$

$$\Rightarrow \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}=\int_0^{\pi/2} 2\sin^p\theta\cos^q\theta\, d\theta .$$

$$\therefore \int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \quad \text{(Proved)}$$

Here, $\int_0^{\pi/2}\sin^5\theta\, d\theta=\int_0^{\pi/2}\sin^5\theta\cos^0\theta\, d\theta$

$$=\frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{5+0+2}{2}\right)}=\frac{\Gamma(3)\Gamma(1/2)}{2\Gamma(7/2)}=\frac{\Gamma(2+1)\Gamma(1/2)}{2\Gamma\left(\frac{5}{2}+1\right)}$$

$$=\frac{\lfloor 2\,\Gamma\left(\frac{1}{2}\right)}{2\,\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}\right)}=\frac{\cancel{2}}{\cancel{2}\cdot\frac{15}{4}}=\frac{4}{15} . \text{ Ans:}$$

H.W.

1. $\int_0^{\pi/2}\cos^5\theta\, d\theta$

2. $\int_0^{\pi/2}\sin^5\theta\cos^5\theta\, d\theta$.

Sol^n:

1. $$\int_0^{\pi/2} \sin^0\theta \cos^5\theta \, d\theta = \frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{0+5+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2+1)}{2\,\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{1}{2}\right)} = \frac{\lfloor 2}{\cancel{2}\,\frac{15}{2}} = \frac{4}{15}$$

2. $$\int_0^{\pi/2} \sin^5\theta \cos^5\theta \, d\theta = \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\,\Gamma\left(\frac{5+5+2}{2}\right)} = \frac{\Gamma(3)\,\Gamma(3)}{2\,\Gamma(6)}$$

$$= \frac{\cancel{2}\times\cancel{2}}{\cancel{2}\times 5\times 4\times 3\times\cancel{2}\times 1} = \frac{1}{60}.$$

* Show that, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$\Rightarrow$ Sol^n: We know,

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) \quad \text{—— (i)}$$

Replacing m, n by $\frac{1}{2}$ in the equation (i), we get,

$$\Rightarrow \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \beta\left(\frac{1}{2},\frac{1}{2}\right)$$

$$\Rightarrow \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}{1} = \int_0^{\pi/2} x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$\Rightarrow \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \int_0^{\pi/2} x^{-\frac{1}{2}}(1-x)^{-1/2}\,dx$$

On putting $x = \sin^2\theta$.

$$\Rightarrow \left\{\Gamma\left(\tfrac{1}{2}\right)\right\}^2 = 2\int_0^{\pi/2} \cancel{\sin^{-1}\theta}\ \cancel{(1-\sin^2\theta)}^{\cancel{-1}} d\theta$$

$$= \cancel{\int_0^{\pi/2}} \cancel{\sin^{-1}\theta \cos^{-1}\theta\, d\theta} \Rightarrow F^{-1}$$

$$= 2\left[\theta\right]_0^{\pi/2} = 2\times\frac{\pi}{2} = \pi$$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}. \qquad \text{(Proved)}$$

Or, $$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) \text{ ——— (i)}$$

Again, $$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta \text{ ——— (ii)}$$

(i) & (ii) we get,

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$$

Put, $m = \frac{1}{2}$ $\Rightarrow 2m-1 = 0$

$n = \frac{1}{2}$ $\Rightarrow 2n-1 = 0$

$$\therefore \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = 2\int_0^{\pi/2} 1.1.\,d\theta$$

$$\Rightarrow \left\{\Gamma\left(\tfrac{1}{2}\right)\right\}^2 = 2\left[\theta\right]_0^{\pi/2} = 2\times\frac{\pi}{2} = \pi$$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}. \qquad \text{(Proved)}$$

☒ Show that, $\Gamma(n) = k^n \int_0^{\infty} e^{-kx} x^{n-1}\, dx$; $n, x > 0$

We know that, the Gamma function is :

$$\Gamma(n) = \int_0^{\infty} e^{-x_1} x_1^{n-1}\, dx_1 \ ; \ n>0 \text{ ——— (i)}$$

Let,
$x_1 = kx$ then $dx_1 = k\,dx$.

Therefore,

$$\Gamma(n) = \int_0^\infty e^{-kx}(kx)^{n-1} k\,dx$$

$$\therefore \Gamma(n) = k^n \int_0^\infty e^{-kx} x^{n-1} dx \quad \text{(Showed)}$$

Again,

$$\int_0^\infty e^{-kx} \cdot x^{n-1} dx = \frac{\Gamma(n)}{k^n}. \quad \text{(Showed)}$$

* Show that,

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx \;;\; n>0$$

Hence evaluate $\int_0^\infty e^{-x^2} dx$.

→ Sol^n: We know, The Gamma function is

$$\Gamma(n) = \int_0^\infty e^{-x_1} \cdot x_1^{n-1} dx_1 \;;\; n>0 \quad \text{—(i)}$$

Let, $x_1 = x^2$ then, $dx_1 = 2x\,dx$

Therefore,

$$\Gamma(n) = \int_0^\infty e^{-x^2} (x^2)^{n-1} 2x\,dx$$

$$= 2\int_0^\infty e^{-x^2} x^{2n-1} dx \quad \text{—(ii)}$$

(Showed)

Now using $n = \frac{1}{2}$, then from equation (ii).

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} \cdot 1 \cdot dx$$

$$\Rightarrow \sqrt{\pi} = 2\int_0^\infty e^{-x^2} dx$$

$$\therefore \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad \text{(Ans.)}$$

Multiple Integrals:

* $\int_{x=a}^{b} f(x)\,dx$. * $\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dx\,dy$.

* $\int_{a}^{b}\int_{c}^{d} (ax+by)\,dx\,dy$.

Example:

1. Show that, → Multiple + β & Γ function

$$\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)} \cdot x^{2m-1} \cdot y^{2n-1}\,dx\,dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}$$

L.S. ⇒ $\int_0^\infty e^{-ax^2} x^{2m-1}\,dx \int_0^\infty e^{-by^2} \cdot y^{2n-1}\,dy$

Let, $x^2 = x_1 \Rightarrow 2x\,dx = dx_1$ & $x=0$; $x_1 = 0$ and $x=\infty$; $x_1=\infty$

$y^2 = y_1 \Rightarrow 2y\,dy = dy_1$ & $y=0$; $y_1=0$ and $y=\infty$; $y_1=\infty$

$$\therefore L.S. = \frac{1}{2}\int_0^\infty e^{-ax_1}(x_1)^{m-1}\,dx_1 \cdot \frac{1}{2}\int_0^\infty e^{-by_1} y_1^{n-1}\,dy_1$$

$$= \frac{1}{2}\frac{\Gamma(m)}{a^m} \cdot \frac{1}{2}\frac{\Gamma(n)}{b^n} = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n} = R.S.$$

(Showed)

2. Evaluate: $\int_1^2\int_y^{3y} (3x^2+y^2)\,dx\,dy$.

→ Sol^n: $\int_1^2 \left[3\frac{x^3}{3} + y^2 x\right]_y^{3y} dy = \int_1^2 \{(3y)^3 + y^2 \cdot 3y - y^3 - y^3\}\,dy$

$$= \int_1^2 (27y^3 + 3y^3 - 2y^3)\,dy = \int_1^2 28y^3\,dy = 28\left[\frac{y^4}{4}\right]_1^2$$

$$= 28\left(\frac{16}{4} - \frac{1}{4}\right) = 28 \times \frac{15}{14} = 105.$$

3. $$\int_{-1}^{2}\int_{0}^{1}\int_{-1}^{1} (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_{-1}^{2}\int_{0}^{1} \left[\frac{x^3}{3} + y^2x + z^2x\right]_{-1}^{1} dy\,dz$$

$$= \int_{-1}^{2}\int_{0}^{1} \left(\frac{1}{3} + y^2 + z^2 + \frac{1}{3} + y^2 + z^2\right) dy\,dz$$

$$= \int_{-1}^{2} \left[\frac{2}{3}y + 2\frac{y^3}{3} + 2z^2y\right]_{0}^{1} dz$$

$$= \int_{-1}^{2} \left(\frac{2}{3} + \frac{2}{3} + 2z^2\right) dz = \left[\frac{4}{3}z + \frac{2}{3}z^3\right]_{-1}^{2}$$

$$= \frac{8}{3} + \frac{16}{3} + \frac{4}{3} + \frac{2}{3} = \frac{30}{3} = 10.$$

Proper Integrals:

An integral is said to be proper integral when it is bounded and the range of the integration in finite $\int_{1}^{2} (x^2+1)\,dx$.

Improper Integrals:

If the range of integration is infinite and the integrand is bounded (that is there is no discontinuity in the interval), then the integral is called improper integral of first kind.

Ex: (i) $\int_0^{\infty} \frac{dx}{1+x^2}$, (ii) $\int_{-\infty}^{0} \frac{dx}{4+x^2}$, (iii) $\int_{-\infty}^{\infty} \frac{2x^2}{x^4+x^2+1}\,dx$

If the range of integration is finite and the integral is unbounded, then the integral is unbounded, then the integral is improper integral of the second kind.

Ex: (i) $\int_0^{4} \frac{dx}{x-2}$, (ii) $\int_{-4}^{1} \frac{x^2+2}{\sqrt{x-5}}\,dx$, (iii) $\int_{-3}^{5} \frac{dx}{x+2}$

If the integral both first and second kind, then it is Known as improper integral of third kind.

Ex: (i) $\int_0^{\infty} \frac{e^x}{x-2}\,dx$, (ii) $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx$, (iii) $\int_{-\infty}^{\infty} \frac{dx}{x^2-7x+12}$

** Show that, $\int_1^{\infty} \frac{dx}{x^{3/2}} = 2$

→ Sol^n: We can write,

$$\int_1^{\infty} \frac{dx}{x^{3/2}} = \lim_{\varepsilon\to\infty} \int_1^{\varepsilon} \frac{dx}{x^{3/2}} = \lim_{\varepsilon\to\infty} \left[\frac{x^{-1/2}}{-1/2}\right]_1^{\varepsilon}$$

$$= \lim_{\varepsilon\to\infty} (-2\varepsilon^{-1/2}+2) = -2(\infty^{-1/2}-1) = -2(0-1) = 2.$$

** Show that, $\int_0^{\infty} \frac{e^x}{1+e^x}\,dx = \infty$

→ Sol^n: We can write,

$$\int_0^{\infty} \frac{e^x}{1+e^x}\,dx = \lim_{\varepsilon\to\infty} \int_0^{\varepsilon} \frac{e^x}{1+e^x}\,dx$$

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$$= \lim_{\varepsilon\to\infty} \left[\ln(1+e^{x})\right]_0^{\varepsilon} = \lim_{\varepsilon\to\infty} \left\{\ln(1+e^{\varepsilon}) - \ln 2\right\}$$

$$= \ln(1+e^{\infty}) - \ln 2 = \ln(1+\infty) - \ln 2 = \infty - \ln 2$$

$= \infty$ (Showed)

** Show that, $\int_0^{\infty} xe^{-x^2}\,dx = \frac{1}{2}$

→ Solⁿ: We can write,

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{\varepsilon\to\infty}\int_0^{\varepsilon} xe^{-x^2}\,dx.$$

Let, $x^2 = z \Rightarrow 2x\,dx = dz$

when, $x=0$; $z=0$ and $x=\varepsilon$; $z=\varepsilon^2$

$$\therefore \int_0^{\infty} xe^{-x^2}\,dx = \lim_{\varepsilon\to\infty}\int_0^{\varepsilon^2} \frac{1}{2}e^{-z}\,dz$$

$$= \frac{1}{2}\lim_{\varepsilon\to\infty}\left[-e^{-z}\right]_0^{\varepsilon^2} = \frac{1}{2}\lim_{\varepsilon\to\infty}\left(-e^{-\varepsilon^2}+1\right) = \frac{1}{2}\left(-e^{-\infty}+\right.$$

$$= \frac{1}{2}(0+1) = \frac{1}{2} \text{ (Showed)} \qquad \left[\because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{1/0}\right.$$

**Ex:

1. $\int_0^{\infty} \frac{x\,dx}{x^4+1}$

→ Solⁿ: $\int_0^{\infty} \frac{x\,dx}{x^4+1} = \lim_{\varepsilon\to\infty}\int_0^{\varepsilon} \frac{x\,dx}{(x^2)^2+1}$

Let, $x^2 = z \Rightarrow 2x\,dx = dz$.

When limit $x=0$; $z=0$ and $x=\varepsilon$; $z=\varepsilon^2$.

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$$\therefore \lim_{\varepsilon\to\infty}\int_0^{\varepsilon^2}\frac{1}{2}\frac{dz}{z^2+1} = \lim_{\varepsilon\to\infty}\frac{1}{2}\left[\tan^{-1}z\right]_0^{\varepsilon^2}$$

$$=\frac{1}{2}\lim_{\varepsilon\to\infty}\left(\tan^{-1}\varepsilon^2-0\right) = \frac{1}{2}\left(\tan^{-1}\infty\right) = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}.$$

H.W. ① $\displaystyle\int_{-\infty}^{0}\frac{x\,dx}{x^4+1}$

→ Solⁿ: We can write,

$$\int_{-\infty}^{0}\frac{x\,dx}{x^4+1} = \lim_{\varepsilon\to-\infty}\int_{\varepsilon}^{0}\frac{x\,dx}{x^4+1} = \lim_{\varepsilon\to-\infty}\int_{\varepsilon}^{0}\frac{x\,dx}{(x^2)^2+1}$$

Let, $x^2 = z \Rightarrow 2x\,dx = dz$.

When limit, $x=\varepsilon$; $z=\varepsilon^2$ and $x=0$; $z=0$.

$$\therefore \lim_{\varepsilon\to-\infty}\int_{\varepsilon^2}^{0}\frac{1}{2}\frac{dz}{z^2+1} = \lim_{\varepsilon\to-\infty}\frac{1}{2}\left[\tan^{-1}z\right]_{\varepsilon^2}^{0} = \frac{1}{2}\lim_{\varepsilon\to-\infty}\left(-\tan^{-1}\varepsilon^2\right)$$

$$=\frac{1}{2}\left\{-\tan^{-1}(-\infty)\right\} = -\frac{1}{2}\times\frac{\pi}{2} = -\frac{\pi}{4}.$$

→ $\log 1 = 0$, $\log 0 = \infty$, $e^{-\infty}=0$, $e^{\infty}=\infty$.

→ $\log(1\cancel{-\infty}+e^{-\infty}) = \log 1 = 0$

** Evaluate : $\displaystyle\int_{-\infty}^{\infty}\frac{x\,dx}{x^4+1}$.

→ Solⁿ: We can write,

$$\int_{-\infty}^{\infty}\frac{x\,dx}{x^4+1} = \int_{-\infty}^{0}\frac{x\,dx}{x^4+1} + \int_{0}^{\infty}\frac{x\,dx}{x^4+1} = -\frac{\pi}{4}+\frac{\pi}{4} = 0.$$

Ex:

1. Evaluate: $\int_1^3 \frac{dx}{1-x}$

We can write,

$$\int_1^3 \frac{dx}{1-x} = \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^{3} \frac{dx}{1-x} = \lim_{\varepsilon \to 0} \left[-\ln(1-x)\right]_{1+\varepsilon}^{3}$$

$$= \lim_{\varepsilon \to 0} \left\{-\ln(1-3)+\ln(-\varepsilon)\right\} = -\ln(2) + \ln(0)$$

$$=$$

H.W.

1. Show that, $\int_0^3 \frac{dx}{(3-x)^2}$ is divergent

→ Sol^n:

2. Evaluate: $\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$

$$= \lim_{\varepsilon_1 \to 0} \int_0^{1-\varepsilon_1} \frac{dx}{1-x^2} + \lim_{\varepsilon_2 \to 0} \int_{1+\varepsilon_2}^{2} \frac{dx}{1-x^2}$$

$$= \lim_{\varepsilon_1 \to 0} \frac{1}{2}\left[\ln\left|\frac{1+x}{1-x}\right|\right]_0^{1-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} \frac{1}{2}\left[\ln\left|\frac{1+x}{1-x}\right|\right]_{1+\varepsilon_2}^{2}$$

$$= \lim_{\varepsilon_1 \to 0} \frac{1}{2}\left[\ln\left|\frac{1+1-\varepsilon_1}{1-1+\varepsilon_1}\right| - \ln 1\right] + \lim_{\varepsilon_2 \to 0} \frac{1}{2}\left[\ln\left|\frac{1+2}{1-2}\right| - \ln\left|\frac{1+1+\varepsilon_2}{1-1-\varepsilon_2}\right|\right]$$

$$= \lim_{\varepsilon_1 \to 0} \frac{1}{2} \ln \frac{2}{\varepsilon_1}$$

Example:

1. Show that, $\int_{-1}^{1} \frac{dx}{x}$ is divergent.

Sol/n:

$$\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} = \lim_{\varepsilon_1 \to 0} \int_{-1}^{0-\varepsilon_1} \frac{dx}{x} + \lim_{\varepsilon_2 \to 0} \int_{0+\varepsilon_2}^{1} \frac{dx}{x}$$

$$= \lim_{\varepsilon \to 0} [\ln x]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} [\ln x]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon \to 0} (\ln \varepsilon_1 - \ln 1) + \lim_{\varepsilon_2 \to 0} (\ln 1 - \ln \varepsilon_2)$$

$$= \ln 0 - \ln 1 + \ln 1 - \ln 0 = 0.$$

H.W.

1. $\int_{1}^{3} \frac{x\,dx}{(x^2-1)^{3/2}}$

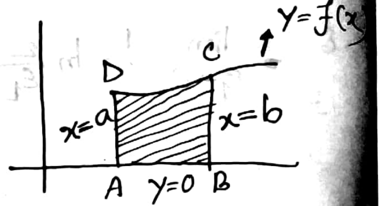
Golap (93)

Geometrical interpretation of definite integral:

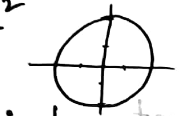
Area in Cartesian form:

* Area of ABCDA $= \int_{x=a}^{b} f(x)\,dx$

$$= \int_{x=a}^{b} y\,dx .$$

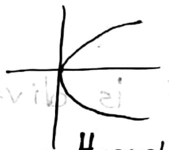


* $x^2+y^2=2^2$



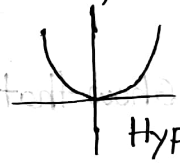
circle

* $y^2=4ax$



Hyperbola

* $x^2=4ay$



Hyperbola.

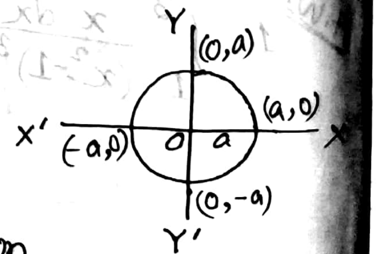
→ If any equation of a curve contains the even powered x variable, then the curve is symmetric about y-axis.

→ If any equation of curve contains the even powered y variable, then the curve is symmetric about x-axis.

→ And if the equation has both even powered x and y variable then the curve is symmetric about both axis.

Example:

1. Find the area of $x^2+y^2=a^2$.



→ Sol^n: The given circle is symmetric about both x and y-axis. On x-axis $y=0$, then from the equation, we have $x=\pm a$.
and on y-axis $x=0$ then, $y=\pm a$.

Now the area $ABOA = \int_0^a y\,dx = \int_0^a \sqrt{a^2-x^2}\,dx$

Let, $x = a\sin\theta$ and limit, when $x=0$; $\theta=0$

$dx = a\cos\theta\,d\theta$ $\quad x=a$; $\theta = \pi/2$

$\therefore$ Area of $ABOA = \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$$= \int_0^{\pi/2} a^2\cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{a^2}{2}(1+\cos 2\theta)\,d\theta$$

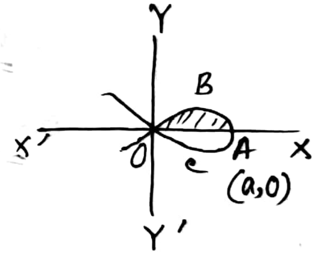
$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{a^2}{2}\left(0+\frac{\pi}{2}-0-0\right)$$

$$= \frac{a^2\pi}{4}$$

Thus the total area $= 4\times\frac{a^2\pi}{4} = a^2\pi$ square units.

2. Find the area enclosed by $3ay^2 = x(x-a)^2$

→ Solⁿ: Now, on x-axis $y=0$, then the given curve, $x = a, a$



$\therefore$ Area of $ABOA = \int_{x=0}^{a} y\,dx$

$$= \int_0^a \sqrt{\frac{x}{3a}}\,(x-a)\,dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a x^{1/2}(x-a)\,dx = \frac{1}{\sqrt{3a}}\int_0^a \left(x^{3/2} - ax^{1/2}\right)dx$$

$$= \frac{1}{\sqrt{3a}}\left[\frac{2}{5}x^{5/2} - \frac{2}{3}ax^{3/2}\right]_0^a = \frac{1}{\sqrt{3a}}\left(\frac{2}{5}a^{5/2} - \frac{2}{3}a^{5/2}\right)$$

$$= \frac{1}{\sqrt{3a}}\cdot\frac{4}{15}a^{5/2} = -\frac{4}{15\sqrt{3}}a^{5/2-1/2} = -\frac{4a^2}{15\sqrt{3}} \text{ sq. units.}$$

Thus the total area $= 2\times\frac{4a^2}{15\sqrt{3}} = \frac{8a^2}{15\sqrt{3}}$ square units.

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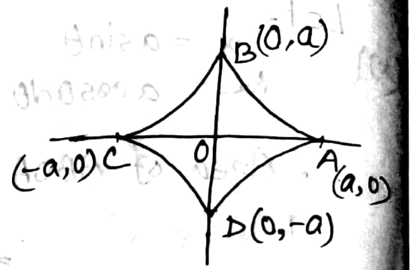
3. Find the are of astroid whose equation is

$$x^{2/3}+y^{2/3}=a^{2/3} \quad \text{---(i)}$$



→ **Sol^n:** On x-axis $y=0$, then from the equation (i) → $x=\pm a$

On y-axis $x=0$, then we have $y=\pm a$.

Now the area of ABDA $=\int_{x=0}^{a} y\,dx$ ---(ii)

The parametric equation for astroid is $x=a\cos^3\theta$, $y=a\sin^3\theta$

Then from the equation (ii).

$$\text{Area of ABOA} =\int_{\theta=\pi/2}^{0} - a\sin^3\theta \; 3a\cos^2\theta \sin\theta \, d\theta.$$

$$=\int_0^{\pi/2} 3a^2 \sin^4\theta \cos^2\theta \, d\theta$$

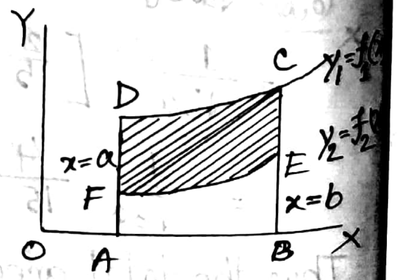
$$= 3a^2 \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)} = 3a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$$

$$= 3a^2 \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot3\cdot2\cdot1} = \frac{3a^2\pi}{32} \text{ sq. units.}$$

Thus the total area $=4\times\frac{3a^2\pi}{32}=\frac{3a^2\pi}{8}$ square units.

Area of ABCDA $=\int_{x=a}^{b} y_1\,dx$

Area of ABEFA $=\int_{x=a}^{b} y_2\,dx$



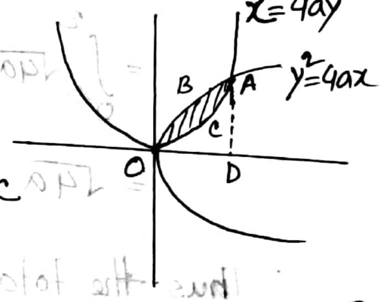
$\therefore$ Area of ECDFE = Area of ABCDA − Area of ABEFA

$$= \int_{x=a}^{b} Y_1 dx - \int_{x=a}^{b} Y_2 dx = \int_{x=a}^{b} (Y_1 - Y_2)\, dx.$$

Example:

1. Find the area enclosed by $y^2 = 4ax$ and $x^2 = 4ay$.

→ Solⁿ:



The parabola $y^2 = 4ax$ is symmetric about x-axis and $x^2 = 4ay$ is symmetric about y-axis.

Now area of OCABO = area ODABO − area ODACO. —— (i)

Now the intersection points are

$$x = \frac{y^2}{4a} \text{ and } x^2 = 4ay \Rightarrow \frac{y^4}{16a^2} = 4ay$$

$$\Rightarrow y^4 - (4a)^3 y = 0 \Rightarrow y\{y^3 - (4a)^3\} = 0$$

$O(0,0)$ $A(4a, 4a)$

From Equation (i) ⇒

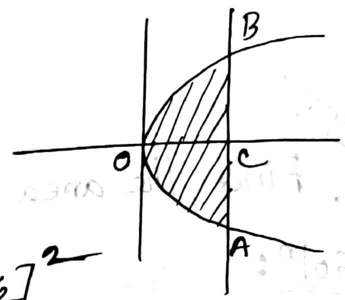
$$\text{Area of OCABO} = \int_{x=0}^{4a} \sqrt{4a}\sqrt{x}\, dx - \int_{x=0}^{4a} \frac{1}{4a} x^2 dx$$

$$= \sqrt{4a}\left[\frac{2}{3}x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a} = \sqrt{4a}\,\frac{2}{3}(4a)^{3/2} - \frac{1}{4a}\cdot\frac{1}{3}(4a)^3$$

$$= \frac{2}{3}(4a)^{\frac{1}{2}+\frac{3}{2}} - \frac{1}{3}(4a)^2 = \frac{2}{3}(4a)^2 - \frac{1}{3}(4a)^2$$

$$= \frac{1}{3}(4a)^2 = \frac{16a^2}{3} \text{ square units.}$$

2. Find the area enclosed by $y^2=4ax$ and $x=2$



→ **Sol^n:**

On x-axis $y=0$, then $x=0$.
and $x=2$.

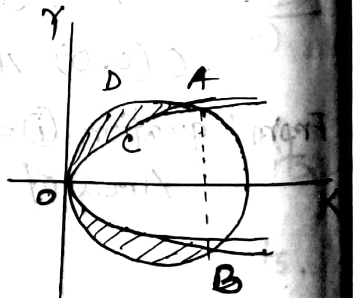
∴ Area of OCBO $= \int_{x=0}^{2} y\,dx$

$$= \int_0^2 \sqrt{4a}\sqrt{x}\,dx = \sqrt{4a}\left[\frac{2}{3}x^{3/2}\right]_0^2$$

$$= \sqrt{4a}\,\frac{2}{3}(2)^{3/2} = \sqrt{4a}\times\frac{2}{3}.2\sqrt{2} = \frac{8\sqrt{2}\sqrt{a}}{3}$$

∴ Thus the total area $= 2\times\frac{8\sqrt{2}}{3}\sqrt{a} = \frac{16\sqrt{2}\sqrt{a}}{3}$ sq. units.

3. Find the area interior $y^2=2ax-x^2$ and exterior to $y^2=ax$ lying in the first quadrant. Hence find the corresponding total area.



→ **Sol^n:**

First curve $y^2=2ax-x^2$ is a circle with centre at $(a,0)$ and radius a and the second curve $y^2=ax$ is parabola with vertex at $(0,0)$ and focus at $(\frac{a}{4},0)$. Both curves are symmetric about x-axis. The two curves intersect each other O, A(a,a) and B$(a,-a)$.

Therefore the area included between the curves in the first quadrant =

$ax = 2ax - x^2$
$\Rightarrow x^2 - ax = 0$
$\Rightarrow x(x-a) = 0$
$\therefore x = 0, a.$
$y = 0, \; y = \pm a.$

$$\int_{x=0}^{a} \sqrt{2ax - x^2}\, dx - \int_{x=0}^{a} \sqrt{ax}\, dx.$$

$$= \int_{x=0}^{a} \left(\sqrt{2ax - x^2} - \sqrt{ax}\right) dx$$

$$= \int_{x=0}^{a} \sqrt{a^2 - (a-x)^2}\, dx - \int_{x=0}^{a} \sqrt{a}\, x^{1/2}\, dx$$

Let, $a - x = a\sin\theta \Rightarrow dx = -a\cos\theta\, d\theta$

when limit, $x = 0$; $\theta = \pi/2$, $x = a$; $\theta = 0$

$$= \int_{\pi/2}^{0} -\sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta - \sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^a$$

$$= \int_0^{\pi/2} a^2\cos^2\theta\, d\theta - \frac{2}{3}a^2 = \int_0^{\pi/2} \frac{a^2}{2}(1+\cos 2\theta)\, d\theta - \frac{2}{3}a^2$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} - \frac{2}{3}a^2 = \frac{a^2}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right) - \frac{2}{3}a^2$$

$$= \left(\frac{a^2\pi}{4} - \frac{2a^2}{3}\right) \text{ square units.}$$

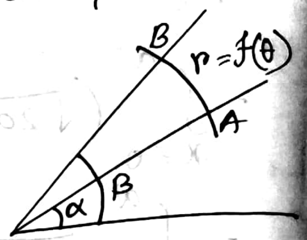
Thus the corresponding total area.

$$= 2 \times \left(\frac{a^2\pi}{4} - \frac{2a^2}{3}\right) = \left(\frac{\pi a^2}{2} - \frac{4a^2}{3}\right) \text{ sq. units.}$$

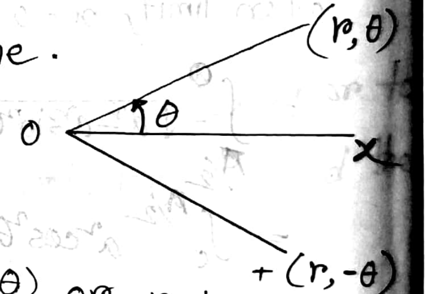
Area in Polar Form:

If $r=f(\theta)$ is a single valued continuous function of θ in $[\alpha, \beta]$, then the area enclosed by $r=f(\theta)$, $\theta=\alpha$, $\theta=\beta$ is

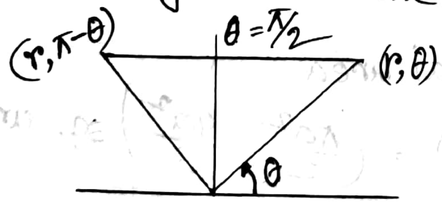
$$\text{Area } OABO = \frac{1}{2}\int_{\theta=\alpha}^{\beta} r^2\, d\theta$$

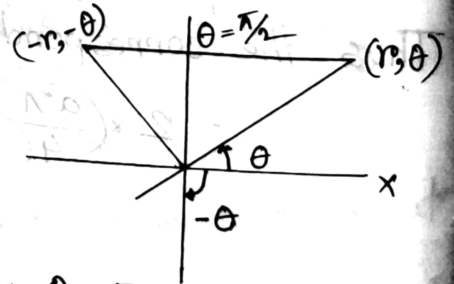


(i) If θ is replaced by $-\theta$ in $r=f(\theta)$ and if the equation remains unchanged, then the curve is symmetric about the initial line.

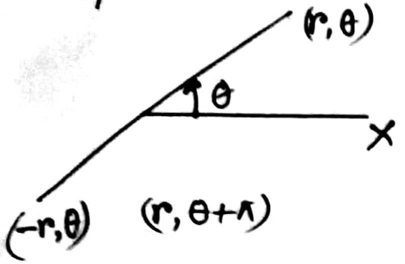


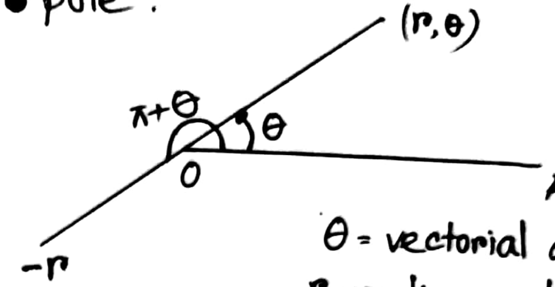
(ii) Either θ is replaced by $(\pi-\theta)$ or r by $-r$ and θ by $-\theta$ simultaneously and the equation remains unchanged then the curve is symmetric about $\theta=\frac{\pi}{2}$





(iii) Either θ is replaced by $\theta+\pi$ or r by $-r$ and the curve remains unchanged then the curve i symmetric about the pole.





θ = vectorial angle
r = radius vector
O = pole
OA = Initial line.

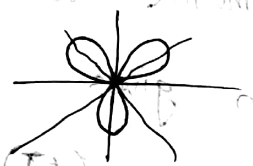
Math-1101

Date: 17-06-19

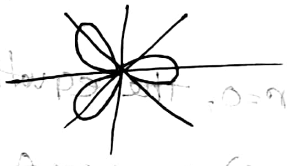
Some well Known curves:

1. Rose Petals:

(i) $r = a\sin 3\theta$

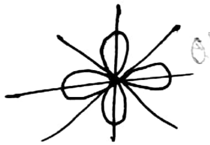


(ii) $r = a\cos 3\theta$



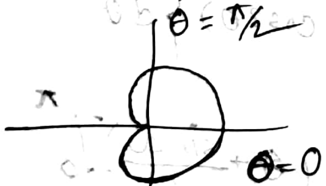
(iii) $r = a\sin 2\theta$

(iv) $r = a\cos 2\theta$

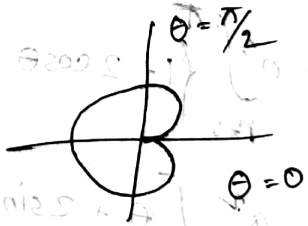


2. Cardioide:

(i) $r = a(1+\cos\theta)$

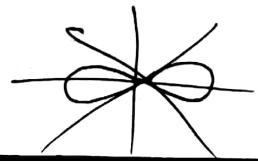


(ii) $r = a(1-\cos\theta)$

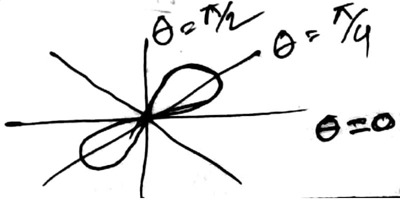


3. Lemniscate:

(i) $r^2 = a^2\cos 2\theta$



(ii) $r^2 = a^2\sin 2\theta$



$r = f(\theta)$

$$\text{Area} = \frac{1}{2}\int_{\theta=\alpha}^{\beta} r^2\, d\theta$$

<u>Example:</u> Find the area of the cardioide $r = a(1+\cos\theta)$

Sol^n: For $\theta = -\theta$, the equation remains unchanged, therefore, the cardioide is symmetric about the initial line.

Now for $r=0$, the equation gives

$$a(1+\cos\theta) = 0 \Rightarrow \cos\theta = -1 = \cos(\pm\pi)$$

$$\therefore \theta = \pm\pi$$

Thus the area $= 2\times\frac{1}{2}\int_{r=0}^{\pi} r^2\, d\theta$

$$= \int_{r=0}^{\pi} a^2(1+\cos\theta)^2\, d\theta$$

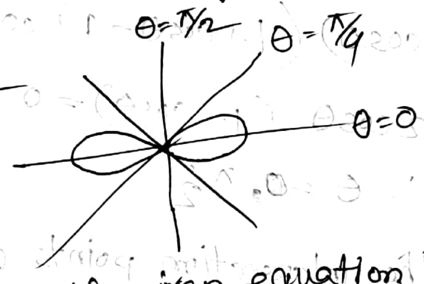
$$= a^2\int_{r=0}^{\pi}\left\{1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right\} d\theta$$

$$= a^2\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

$$= \frac{3\pi a^2}{2} \text{ sq. units.}$$

Ex: Find the total area of $r^2 = a^2\cos 2\theta$

Soln: For, $r = -r$ and $\theta = -\theta$, the equation remains unchanged, therefore the given curve is symmetric about $\theta = \pi/2$



Now for $r=0$ the given equation gives $a^2\cos 2\theta$

$\therefore \theta = \pm \pi/4$

$$\text{Area on one loop} = 2 \times \frac{1}{2}\int_{\theta=0}^{\pi/4} r^2\, d\theta$$

$$= \int_0^{\pi/4} a^2 \cos 2\theta \, d\theta = a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= a^2 \cdot \frac{1}{2} \times 1 = \frac{a^2}{2} \text{ sq. units.}$$

$$\text{Total area} = 2 \times \frac{a^2}{2} = a^2 \text{ sq. units.}$$

Ex: Determine the area inside the circle $r = \sin\theta$ and outside the cardioide $r = 1 - \cos\theta$.

Soln: Given that,

$r = \sin\theta$ and $r = 1 - \cos\theta$, then we can write,

$\sin\theta = 1 - \cos\theta$

$\Rightarrow \sin^2\theta = (1-\cos\theta)^2$

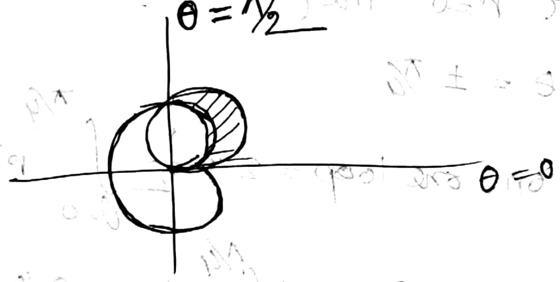
$\Rightarrow (1-\cos\theta)(1+\cos\theta) - (1-\cos\theta)^2 = 0$

$\Rightarrow (1-\cos\theta)(1+\cos\theta - 1 + \cos\theta) = 0$

$\Rightarrow 2\cos\theta\,(1-\cos\theta) = 0$

$\therefore \theta = 0, \pi/2$

Thus the intersection points are $(0,0)$ and $(1, \pi/2)$



Now, the required area

$$= \int_{\theta=0}^{\pi/2} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta, \qquad \text{where } r_1 = \sin\theta,\ r_2 = 1-\cos\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(-1 + 2\cos\theta - \cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left[-\theta + 2\sin\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

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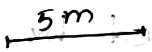
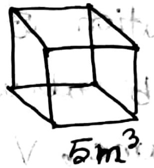
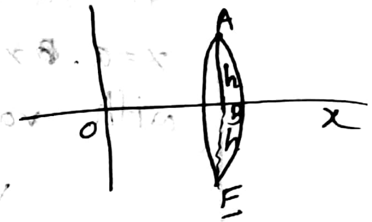
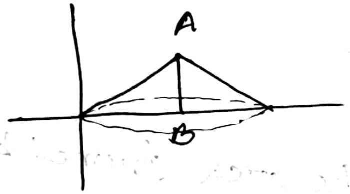
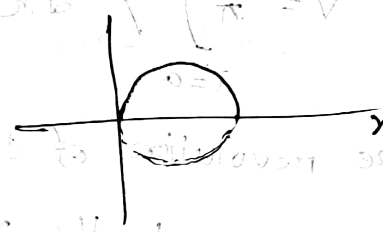
Date: 24-06-19.

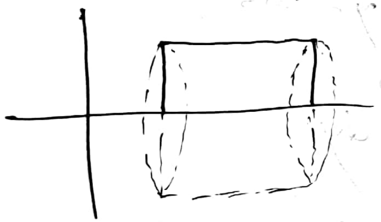


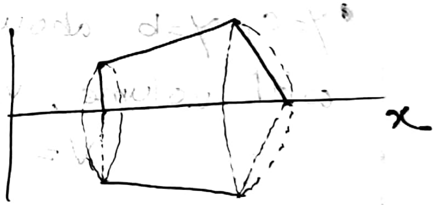


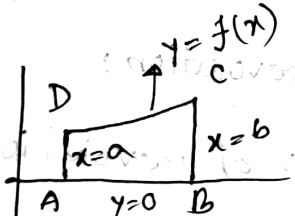












$$\therefore \text{Area } ABCD = \int_{x=a}^{b} f(x)\,dx = \int_{x=a}^{b} y\,\delta x$$

Determination of volume:

The revolution of the area formed by $y=f(x)$, $x=a$, $x=b$ about the x-axis produce solid with volume, V, then

$$V = \pi \int_{x=a}^{b} y^2 \, dx$$

Or, The revolution of the area formed by $x=f(y)$, $y=a$, $y=b$ about the y-axis produce solid with volume, V, then

$$V = \pi \int_{y=a}^{b} x^2 \, dy$$

Surface of ~~th~~ solids of revolution:

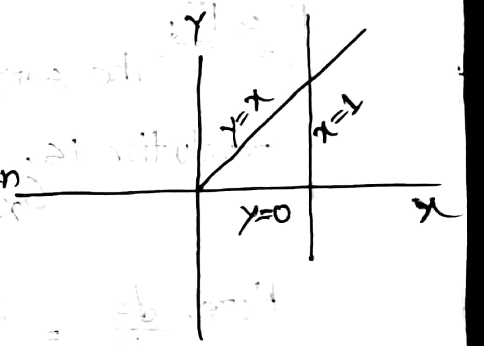
Surface area of solids of revolution about x-axis is

$$S_x = \int_{x=a}^{b} 2\pi f(x) \, ds = 2\pi \int_{x=a}^{b} y \frac{ds}{dx} dx$$

$$= 2\pi \int_{x=a}^{b} y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

where ds is the arc. length defined $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Ex: Find the volume of the solid formed by revolution of the area enclosed by $y=x, y=0, x=1$ about x-axis.



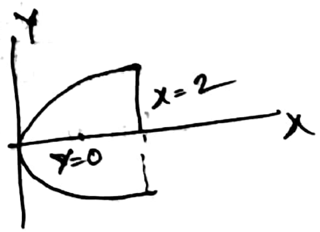
Soln:

Let the formed volume, V, then

$$V = \pi\int_{x=0}^{1} y^2 dx = \pi\int_{0}^{1} x^2 dx$$

$$= \pi\left[\frac{x^3}{3}\right]_0^1 = \frac{\pi}{3} \text{ cubic units.}$$

Ex: Find the volume of the solid formed by the revolution of the enclosed area $y^2 = x, y=0, x=2$ about x-axis.



Soln:

The formed volume,

$$V = \pi\int_{x=0}^{2} y^2 dx = \pi\int_{0}^{2} x\, dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^2 = 2\pi \text{ cubic units.}$$

Ex: Find the surface area of the solid formed by revolution of the circle $x^2+y^2=a^2$ about x-axis

Soln:

The surface area of the solid formed by revolution is,

$$S_x = 2\pi\int_{x=-a}^{a} y \frac{ds}{dx} \cdot dx \quad \text{--- (1)}$$

$y = 0$ for x-axis

$\Rightarrow x^2 = a^2$

$\therefore x = \pm a$

Here; $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ --- (2) and given that $x^2+y^2=a^2$ --- (3)

Differentiat (3) w.r.to x.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

using this value eqn. (2) becomes.

$$\frac{ds}{dx} = \sqrt{1+\frac{x^2}{y^2}} = \sqrt{\frac{x^2+y^2}{y^2}} = \frac{a}{y}$$

From equation (1),

$$S_x = 2\pi\int_{-a}^{a} y \cdot \frac{a}{y}\, dx = 2\pi a\int_{-a}^{a} dx$$

$$= 2\pi a\left[x\right]_{-a}^{a} = 2\pi a(a+a)$$

$$= 4\pi a^2. \quad \text{square units.}$$

Definite integral as the limit of a sum:

Let, $y=f(x)$ is bounded on $[a,b]$ and continuous on $[a,b]$ where a,b are finite and $b>a$. Let us divide the interval into n equal parts with length h, then $a+nh=b \Rightarrow nh=b-a$

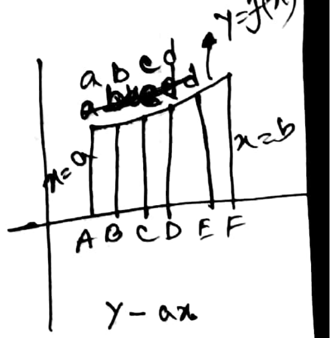
So the area of the region enclosed by $x=a, x=b$, x-axis and the curve $y=f(x)$ is $\int_{x=a}^{b} f(x)\,dx =$

$$\int_{x=a}^{b} f(x)\,dx = \lim_{h\to 0}\left[hf(a+h)+hf(a+2h)+\ldots\ldots+hf(a+nh)\right]$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) \quad ; \; nh=b-a$$

If $a=0$ and $b=1$, then $nh=1$ and $h\to 0$ implies that $\frac{1}{n}\to\infty$

$$\int_{x=0}^{1} f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$



We put $\lim\limits_{n\to\infty}\frac{1}{n}=dx$ $\Sigma=\int$ and $\frac{r}{n}=x$,

The lower limit is equal to zero

AB = h
Bb = f(a+h)
∴ area ABba = hf(a+h)

and the upper limit is the coefficient of n about the summation.

Ex: Evalute $\int_a^b x\,dx$ as a process of summation.

Soln. We know that,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh), \quad nh = b-a$$

$$\therefore \int_a^b x\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (a+rh)$$

$$= \lim_{h\to 0} h\left[(a+h)+(a+2h)+(a+3h)+\dots+(a+nh)\right]$$

$$= \lim_{h\to 0} h\left[na + h(1+2+3+\dots\dots+n)\right]$$

$$= \lim_{h\to 0} h\left(na + h\frac{n(n+1)}{2}\right)$$

$$= \lim_{h\to 0}\left\{hna + h^2\frac{n(n+1)}{2}\right\}$$

$$= \lim_{h\to 0}\left[(b-a)a + \frac{nh(nh+h)}{2}\right]$$

$$= \lim_{h\to 0} \left\{ ba - a^2 + \frac{(b-a)(b-a+h)}{2} \right\}$$

$$= ba - a^2 + \frac{a^2+b^2-2ab}{2}$$

$$= \frac{2ab - 2a^2 + a^2 + b^2 - 2ab}{2}$$

$$= \frac{b^2-a^2}{2}$$

Ex: Evaluate: $\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}\right]$

Solution: Given that,

$$\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \frac{1}{1+r/n} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+r/n}$$

$$= \int_{x=0}^{1} \frac{1}{1+x}\, dx = \left[\ln(1+x)\right]_0^1$$

$$= \ln 2 - \ln 1 = \ln 2.$$

Ex: **Evalute:** $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

Soln:

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}\frac{1}{\sqrt{1-(r/n)^2}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{1-(r/n)^2}}$$

$$= \int_{x=0}^{1}\frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1}x\right]_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(0) = \pi/2.$$

Ex: **Evalute:** $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\right]$

Soln:

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{2rn-r^2}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2r/n-(r/n)^2}}$$

$$= \int_{x=0}^{1} \frac{dx}{\sqrt{2x-x^2}} = \int_{x=0}^{1} \frac{dx}{\sqrt{1-(1-2x+x^2)}}.$$

$$= \int_0^1 \frac{dx}{\sqrt{1-(1-x)^2}} = \left[-\sin^{-1}(1-x)\right]_0^1$$

$$= -\sin^{-1} 0 + \sin^{-1}(1) = \pi/2.$$