MATH

"Integration"

\# $\frac{d}{dx}(x^2) = 2x$ ⟶ 1. tangent
2. Instantaneous rate of change.

$\int 2x\,dx = x^2+c$ ⟶ Finding area
I. by anti- derivative
II. summation of series.

Definition of an integral:

If the differential co-efficient of a function $f(x)$ is $F(x)$, that is $\frac{d}{dx} f(x) = F(x)$ then $f(x)$ is said to be the integral of $F(x)$, mathmatically, ⟶

$$\int F(x)\,dx = f(x)$$

⟶ Integrating constant

$$\int_0^1 2x\,dx = [x^2+c]_0^1$$
$$= 1+c-0-c$$
$$= 1$$

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Ex: Integrate $\dfrac{x^3-x^2+1}{x}$ w.r to x.

or Evaluate $\displaystyle\int \frac{x^3-x^2+1}{x}\,dx$

$\Rightarrow$

$$I = \int \frac{x^3-x^2+1}{x}\,dx$$

$$= \int x^2 - x + \frac{1}{x}\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + \ln x + c$$

where c is the integrating constant.

Ex: $$I = \int \frac{4x^2-2\sqrt{x}}{x}\,dx$$

$$= \int 4x - \frac{2}{\sqrt{x}}\,dx$$

$$= 2x^2 - 4\sqrt{x} + c.$$

Ex: $$I = \int \frac{2-\sin 2x}{1-\cos 2x}$$

$$= \int \frac{2-2\sin x.\cos x}{2\sin^2 x}\,dx$$

$$= \int \operatorname{cosec}^2 x - \cot x\,dx$$

$$= -\cot x - \ln(\sin x) + c.$$

Ex: $$I = \int \frac{1}{1+\sin x}\,dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$= \int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x - \tan x.\sec x\,dx$$

$$= \tan x - \sec x + c.$$

① $I = \int \frac{dx}{\sqrt{x} + \sqrt{x+1}}$

② $I = \int \sin^3 x \, dx$

③ $I = \int \cos^4 x \, dx$

Method of substitution:

$$I = \int (ax+b)^n \, dx$$

$$= \frac{1}{a} \int z^n \, dz$$

$$= \frac{1}{a} \cdot \frac{z^{n+1}}{n+1} + c$$

$$= \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + c$$

let,

$z = ax+b$

বা, $\frac{dz}{dx} = a$

বা, $dx = \frac{1}{a} dz$

Ex: $I = \int \frac{1}{\sqrt{x}} \cos \sqrt{x} \, dx$

$$= \int 2 \cos z \, dz$$

$$= -2 \sin z + c$$

$$= -2 \sin \sqrt{x} + c$$

$z = \sqrt{x}$

$\Rightarrow dz = \frac{1}{2\sqrt{x}} dx$

$\Rightarrow \frac{1}{\sqrt{x}} dx = 2 dz$

Ex: $I = \int \frac{e^x (1+x)}{\cos^2 (x \cdot e^x)}$

$$= \int \sec^2 z \, dz$$

$$= \tan z + c$$

$$= \tan xe^x + c.$$

$z = xe^x$

$\Rightarrow dz = (e^x + xe^x) dx$

① $I = \int \frac{\sqrt{x} + \ln x}{x} \, dx$

② $I = \int \sin^2 x \cdot \cos x \, dx$

③ $I = \int \tan^3 x \cdot \sqrt{\sec x} \, dx$

Integration by parts:

$$\int uv \, dx = u \int v \, dx - \int \left\{ \frac{d}{dx}(u) \int v \, dx \right\} dx$$

If the integration and differentiation of the two functions are possible and one of them is of the form x^n, then we use $u = x^n$.

* The functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ ... are used as u function.
* When no function is multiplied with $\sin^{-1}x$, $\cos^{-1}x$ ... then we ~~tha~~ take $u = 1$.

Ex: $I = \int x \ln x \, dx$

$$= \ln x \int x \, dx - \int \left\{ \frac{d}{dx} \ln x \int x \, dx \right\} dx$$

$$= \frac{x^2}{2} \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + c$$

Ex: $\int \cos^{-1}x \, dx$

$$= \cos^{-1}x \int 1 \cdot dx - \int \left\{ \frac{d}{dx} \cos^{-1}x \int dx \right\} dx$$

$$= x\cos^{-1}x - \int - \frac{1}{\sqrt{1-x^2}} \, x \, dx$$

$$= x\cos^{-1}x + \int \frac{1}{2} \cdot \frac{2x}{\sqrt{1-x^2}} \, dx$$

$$= x\cos^{-1}x - \frac{1}{2} \int \frac{dz}{z^{1/2}}$$

$$= x\cos^{-1}x - \frac{1}{2} \cdot \frac{z^{1/2}}{1/2} + c.$$

$z = 1-x^2$

$\Rightarrow -dz = 2x \, dx$

① $I = \int x^2 \sin^2 x \, dx$

② $I = \int \frac{\ln(\ln x)}{x} \, dx$

③ $I = \int \frac{\ln x}{(1+x)^3} \, dx$

Standard Integrals:

1. $I = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

2. $I = \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$

3. $I = \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$

4. $I = \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

5. $I = \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + c$

6. $I = \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right) + c$

7. $I = \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

8. $I = \int \sqrt{x^2+a^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2+a^2}\right) + c$

9. $I = \int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2+a^2}\right) + c$

\# $I = \int \frac{6x^5}{1+x^{12}}\,dx$

$= \int \frac{6x^5}{1+(x^6)^2}\,dx$

let,

$z = x^6$

$\Rightarrow dz = 6x^5\,dx$

$= \int \frac{dz}{1+z^2}$

$= \frac{1}{1}\tan^{-1}\frac{z}{1} + c$

$= \tan^{-1}(x^6)$

\# $I = \int \frac{dx}{e^x + e^{-x}}$

let,

$z = e^x$

$\Rightarrow dz = e^x\,dx$

$= \int \frac{e^x}{(e^x)^2+1}\,dx$

$= \int \frac{dz}{z^2+1}$

$= \tan^{-1} z + c$

$= \tan^{-1} e^x + c$

\# $I = \int \frac{dx}{4\cos^2 x + 9\sin^2 x}$

$= \int \frac{\sec^2 x\,dx}{4 + 9\tan^2 x}$ [$\cos^2 x$ দিয়ে ভাগ]

* $\sin, \cos$ থাকলে $\tan$ করতে হবে।

$= \frac{1}{9}\int \frac{\sec^2 x\,dx}{\tan^2 x + 4/9}$

let,

$z = \tan x$

$= \frac{1}{9}\int \frac{dz}{z^2 + (2/3)^2}$

$= \frac{1}{9}\cdot\frac{1}{2/3}\tan^{-1}\frac{z}{2/3} + c$

$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\tan x\right) + c$

$$\# \; I = \int \frac{dx}{x^4-a^4}$$

$$= \int \frac{dx}{(x^2-a^2)(x^2+a^2)}$$

$$= \frac{1}{2a^2}\int \left(\frac{1}{x^2-a^2} - \frac{1}{x^2+a^2}\right) dx$$

$$= \frac{1}{2a^2}\int \frac{dx}{x^2-a^2} - \frac{1}{2a^2}\int \frac{dx}{x^2+a^2}$$

$=$

$$\# \; I = \int \frac{dx}{(1+x^2)\sqrt{1-(\tan^{-1}x)^2}}$$

$$= \int \frac{dz}{\sqrt{1-z^2}}$$

let,

$z = \tan^{-1}x$

$\Rightarrow dz = \frac{1}{1+x^2}dx$

$$= \sin^{-1}\frac{z}{1} + c$$

$$= \sin^{-1}(\tan^{-1}x) + c$$

$$\# \ I = \int \sqrt{1+\sec x}\, dx$$

$$= \int \sqrt{1+\frac{1}{\cos x}}\, dx$$

$$= \int \sqrt{\frac{\cos x+1}{\cos x}}\, dx$$

$$= \int \sqrt{\frac{2\cos^2 x/2}{1-2\sin^2 x/2}}\, dx$$

$$= \sqrt{2} \int \frac{\cos x/2 \ dx}{\sqrt{2}\sqrt{\frac{1}{\sqrt{2}} - \sin^2 x/2}}$$

let, $z = \sin x/2$

$\Rightarrow 2\, dz = \cos x/2 \, dx$

$$= \int \frac{2dz}{\left(\frac{1}{\sqrt{2}}\right)^2 - z^2}$$

$$= 2\sin^{-1}\left(\frac{z}{\frac{1}{\sqrt{2}}}\right) + c$$

$$= 2\sin^{-1}\left(\sqrt{2}\ \sin x/2\right) + c$$

$$\# \ I = \int \frac{dx}{\sqrt{e^{2x}+1}}$$

$$= \int \frac{dx}{e^x\sqrt{1+e^{-2x}}}$$

$$= \int \frac{e^{-x}}{\sqrt{1+(e^{-x})^2}}$$

let, $z = e^{-x}$

$\Rightarrow -dz = e^{-x} dx$

$$= \int - \frac{dz}{\sqrt{1+z^2}}$$

$$= -\ln\left(z+\sqrt{1+z^2}\right) + c$$

$$= -\ln\left(e^{-x} + \sqrt{1+e^{2x}}\right) + c$$

$I = \int \frac{dx}{\sqrt{25x^2-4}}$

$I = \int \frac{x^2 dx}{\sqrt{9-x^2}}$

$= \int \frac{9-(9-x^2)\,dx}{\sqrt{9-x^2}}$

$= \int \frac{9}{\sqrt{9-x^2}}\,dx - \int \sqrt{9-x^2}\,dx$

$= 9\sin^{-1}\frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} - \frac{9}{2}\sin^{-1}\frac{x}{3} + c$

$I = \int \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx$

$= \int \frac{\sin x \cdot \cos x}{\tan^4 x + 1}\,dx$

$= \int \frac{\tan x \cdot \sec^2 x}{\tan^4 x + 1}\,dx$

let,

$z = \tan^2 x$

$\Rightarrow dz = 2\tan x \cdot \sec^2 x\,dx.$

$= \int \frac{\frac{1}{2}\,dz}{z^2+1}$

$= \frac{1}{2}\tan^{-1}(\tan^2 x) + c$

$I = \int \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$

$I = \int \frac{x^3 dx}{\sqrt{a^6 - x^6}}$

$I = \int \frac{x\,dx}{a^4 - x^4} \longrightarrow z = x^2$

$I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\,dx$

$x^2 \pm a^2$ আকারে $\Rightarrow (x+\text{cons.})^2 \pm \text{cons}^2$

ax^2+bx+c

$= a\left(x^2+\frac{b}{a}+\frac{c}{a}\right)$

$= a\left(x^2+2\cdot\frac{1}{2}\cdot\frac{b}{a}\cdot x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)$

$= a\left\{\left(x+\frac{b}{2a}\right)^2 \pm (\text{cons})^2\right\}$

x^2+x+2

$= x^2+2\cdot\frac{1}{2}\cdot x+\frac{1}{4}-\frac{1}{4}+2$

$= \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2$

$I = \int \frac{\ln x}{(1+x)^3}\, dx$

$= \int (1+x)^{-3} \ln x \, dx$

$= \ln x \int (1+x)^{-3} dx - \int \left\{\frac{d}{dx} \ln x \int (1+x)^{-3} dx\right\} dx$

$= \ln x \cdot \frac{(1+x)^{-2}}{-2} - \int \frac{1}{x}\cdot\frac{(\ln x)^{-2}}{-2} dx$

$= -\frac{\ln x}{2(1+x)^2}+\frac{1}{2}\int \frac{1}{x(1+x)^2} dx$

By partial fraction method:

$$\frac{1}{x(1+x)^2} \equiv \frac{A}{x}+\frac{B}{1+x}+\frac{C}{(1+x)^2}$$

$\Rightarrow I = A(1+x)^2 + B.x.(1+x) + C.x$

Equaliting the co-efficents of x^2, x and constant terms:

$$\begin{aligned} 0 &= A+B \\ 0 &= 2A+B+C \\ 1 &= A \end{aligned} \qquad \Bigg| \qquad \begin{aligned} A &= 1 \\ B &= -1 \\ C &= -1 \end{aligned}$$

$$I = -\frac{\ln x}{2(1+x)^2} + \frac{1}{2}\int \left\{ \frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$$

$$= -\frac{\ln x}{2(1+x)^2} + \frac{1}{2}\left[\ln x - \ln(\ln x) + \frac{1}{1+x} \right] + c$$

From: ① $\int \frac{dx}{ax^2+bx+c}$

② $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

③ $\int \sqrt{ax^2+bx+c}$

$\rightarrow ax^2+bx+c = \left(x+\frac{b}{2a}\right)^2 + (con)^2$

① $\Rightarrow$ $I = \int \frac{dx}{\sqrt{x^2+x-2}}$

$$\text{Sl}^n = \int \frac{dx}{\sqrt{x^2+2\cdot\frac{1}{2}\cdot x+\frac{1}{4}-\frac{1}{4}-2}}$$

$$= \int \frac{dx}{\sqrt{(x+\frac{1}{2})^2-(\frac{3}{2})^2}}$$

$$= \ln\left\{\left(x+\frac{1}{2}\right)+\sqrt{(x+\tfrac{1}{2})^2-(\tfrac{3}{2})^2}\right\}+c$$

$I = \int \sqrt{4-3x-2x^2}\, dx$

$$= \int \sqrt{2}\cdot\sqrt{2-\frac{3}{2}x-x^2}\, dx$$

$$= \int \sqrt{2}\cdot\sqrt{2-\left(x^2+2\cdot\frac{1}{2}\cdot\frac{3}{2}x+\frac{9}{16}\right)+\frac{9}{16}}\, dx$$

$$= \sqrt{2}\sqrt{\frac{(\sqrt{41})^2}{4^2}-\left(x+\frac{3}{4}\right)^2}\, dx$$

$= \sqrt{2}\,..$

$I = \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

let,

$x-\alpha = z^2$

$\Rightarrow x = z^2+\alpha$

$\Rightarrow dx = 2z\,dz$

Using all these values the integrand be comes a standard integral.

$I = \int \frac{dx}{\sqrt{x^2-7x+12}}$

$= \int \frac{dx}{\sqrt{(x-3)(x-4)}}$

let, $z^2 = x-3$

$\Rightarrow x = z^2+3$

$\Rightarrow dx = 2z\,dz$

$\therefore I = \int \frac{2z\,dz}{\sqrt{z^2(z^2+3-4)}}$

$= 2\int \frac{dz}{\sqrt{z^2-1^2}}$

$= 2\ln\left(z+\sqrt{z^2-1}\right)+c$

$= 2\ln\left\{\sqrt{x+3}+\sqrt{x}\right\}+c$

$I = \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

From : ① $I = \int \frac{Px+q}{ax^2+bx+c}\,dx$

ⓘⓘ $I = \int \frac{Px+q}{\sqrt{ax^2+bx+c}}\,dx$

ⓘⓘⓘ $I = \int (Px+q)\sqrt{ax^2+bx+c}\,dx$

Here we let:

$$\frac{d}{dx}(ax^2+bx+c) = 2ax+b$$

$2ax+b$

$5x \cdot 3x \cdot \frac{5}{3}(3x) = 5x$

$$Px+q = \frac{P}{2a}(2ax+b) - \frac{Pb}{2a} + q$$

Ex: $I = \int \frac{2x+5}{\sqrt{x^2-2x+2}}$

$$= \int \frac{(2x-2)+5+2}{\sqrt{x^2-2x+2}}\,dx$$

$$= \int \frac{2x-2}{\sqrt{x^2-2x+2}}\,dx + 7\int \frac{dx}{\sqrt{x^2-2x+2}}$$

$$= 2.\sqrt{x^2-2x+2} + 7\int \frac{dx}{(x-1)^2+(1)^2}$$

$$= 2\sqrt{x^2-2x+2} + 7\ln\sqrt{(x-1)+(x-1)^2+1^2} + c$$

Ex: $I = \int \frac{3x+2}{5x^2+2x+3}\,dx$

$$= \frac{1}{5}\int \frac{3x+2}{x^2+\frac{2}{5}x+\frac{3}{5}}\,dx$$

$$= \frac{1}{5}\int \frac{\frac{3}{2}(2x+2/5)-3/5+2}{x^2+\frac{2}{5}x+\frac{3}{5}}\,dx$$

From: $I = \int \frac{dx}{(ax+b)\sqrt{(cx+d)}}$

let, $z^2 = cx+d$

then, $x = \frac{z^2-d}{c}$

$\Rightarrow dx = \frac{2}{c} z\,dz$

$$\therefore I = \int \frac{\frac{2z}{c}\,dz}{\left(a\cdot\frac{z^2-d}{c}+b\right)z}$$

This is a standard integral...

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Ex: $I = \int \frac{x^2+x}{(x+2)\sqrt{(x+1)}}\,dx$

let,

$x-1 = z^2.$ them

$dx = 2z\,dz$

$$\therefore I = \int \frac{(z^2+1)^2 + z^2+1}{(z^2+1+2)\,z}\,2z\,dz$$

$$= \int 2\,\frac{z^4+2z^2\cdot 1+1+z^2+1}{z^2+3}\,dz$$

$$= \int \frac{z^4+3z^2+2}{z^2+3}\,dz$$

$$= 2\int \frac{z^2(z^2+3)+2}{z^2+3}\,dz$$

$$= 2\int z^2\,dz + 4\int \frac{dz}{z^2+(\sqrt{3})^2}$$

$$= 2\cdot\frac{z^3}{3} + 4\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right)+c$$

$$= \frac{2}{3}(x-1)^{3/2} + \frac{4}{\sqrt{3}}\tan^{-1}\left(\sqrt{\frac{x-1}{3}}\right)+c$$

Ex: $I = \int \frac{dx}{(x-3)\sqrt{x-2}} = ?$

Ex: $I = \int \frac{x^3}{\sqrt{x-1}}\,dx = ?$

From: $I = \int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$

let,

$$ax+b = \frac{1}{z}$$

$$\Rightarrow dx = -\frac{1}{a}\frac{dz}{z^2}$$

Using all these values the integrand I becomes a standard integral

Ex: $I = \int \frac{x\,dx}{(x+1)\sqrt{x^2+1}}$

$$= \int \frac{(x+1)-1}{(x+1)\sqrt{x^2+1}}\,dx$$

$$= \int \frac{dx}{\sqrt{x^2+1}} - \int \frac{dx}{(x+1)\sqrt{x^2+1}}$$

$$= \ln\left(x+\sqrt{x^2+1}\right) - I_1$$

$$= \ln\left(x+\sqrt{x^2+1}\right)$$

$-$

Where,

$$I_1 = \int \frac{dx}{(x+1)\sqrt{x^2+1}}$$

let,

$x+1 = \frac{1}{z}$, then $dx = -\frac{dz}{z^2}$

$$\therefore I_1 = \int -\frac{1}{z^2}\cdot\frac{dz}{\frac{1}{z}\left(\sqrt{\left(\frac{1}{z}-1\right)^2+1}\right)}$$

$$= -\int \frac{1}{z^2}\cdot\frac{z\,dz}{\sqrt{\frac{1}{z^2}-\frac{2}{z}+1+1}}$$

$$= -\int \frac{1}{z^2}\cdot\frac{z^2\,dz}{\sqrt{1-2z+2z^2}}$$

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$$= -\int \frac{dz}{\sqrt{2} \cdot \sqrt{z^2 - z + \frac{1}{2}}}$$

$$= -\int \frac{dz}{\sqrt{2} \cdot \sqrt{\left(z - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{1}{2}}}$$

$$= -\int \frac{dz}{\sqrt{2} \cdot \sqrt{\left(z - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{\sqrt{2}} \ln \left\{ \left(z - \frac{1}{2}\right) + \sqrt{\left(z - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \right\} + c$$

Ex: $I = \int \frac{dx}{(x+1)\sqrt{1+2x-x^2}} = ?$

Form: $I = \int \frac{x\,dx}{(ax^2+b)\sqrt{cx^2+d}} \Rightarrow$ let, $z^2 = cx^2+d$, $dx =$

$I = \int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}} \Rightarrow$ let, $x = \frac{1}{z}$, $dx =$

Ex: $I = \int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$

$\Rightarrow$ let,

$x = \frac{1}{z}$, $\tan dx = -\frac{dz}{z^2}$

$\therefore I = \int -\frac{1}{z^2} \cdot \frac{dz}{\left(\frac{1}{z^2}+1\right)\sqrt{\frac{1}{z^2}+4}}$

$= -\int \frac{1}{z^2} \cdot \frac{z^3\, dz}{(1+z^2)\sqrt{1+4z^2}}$

$= -\int \frac{z\, dz}{(z^2+1)\sqrt{4z^2+1}}$

$K = 4z^2+1 = z_1^2$

$\therefore 8z\, dz = 2z_1\, dz_1$

$\Rightarrow z\, dz = \frac{1}{4} z_1\, dz_1$

$\therefore I = -\int \frac{1}{4} \cdot \frac{z_1 dz}{\left(\frac{z_1^2+1}{4}+1\right) z_1}$

$= \int -\frac{dz_1}{z_1^2+(\sqrt{3})^2}$

$= -\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z_1}{\sqrt{3}}\right) + c_1$

$= -\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{4z^2+1}}{\sqrt{3}}\right) + c$

$= -\frac{1}{\sqrt{3}} \tan^{-1} \sqrt{\frac{4 \cdot \frac{1}{x^2}+1}{\sqrt{3}}} + c$ $= -\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{4+x^2}}{x\sqrt{3}}\right) + c$

Ex: (i) $I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

(ii) $I = \int \frac{(x+5)\,dx}{(x^2+2)\sqrt{x^2+3}}$

Form: $I = \int \frac{dx}{a+b\cos x}$

$I = \int \frac{dx}{a+b\sin x}$

Here converting cosx, sinx ... into tanx then the integrand becomes easier.

Ex: $I = \int \frac{dx}{5+3\cos x}$

$\Rightarrow I = \int \frac{dx}{5+3\cdot\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

$= \int \frac{\sec^2 x/2 \; dx}{5+5\tan^2 x/2 + 3 - 3\tan^2 x/2}$

$= \int \frac{\sec^2 x/2 \; dx}{2(\tan^2 x/2 + 4)}$

let, $z = \tan x/2$

$\Rightarrow 2dz = \sec^2 x/2 \; dx$

$$\therefore I = \int \frac{2dz}{2(z^2+2^2)}$$

$$= 2 \cdot \frac{1}{2} \tan^{-1}\left(\frac{z}{2}\right) + c$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{\tan x/2}{2}\right) + c$$

Ex: $I = \int \frac{dx}{3\sin x + 2\cos x + 5}$

Form: $I = \int \frac{P\cos x + 4\sin x + R}{a\cos x + b\sin x + c} dx = \frac{\text{Num}}{\text{Deno.}}$

Numerator = $L \times$ (Determinator) + $M \times \frac{d}{dx}$ (Deno.) + N

Now equality the ceffiecients of $\cos x$, $\sin x$ and constant term, we will get the value of L, M, N. Then the integrand becomes standard integral.

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Ex: $I = \int \frac{2+3\sin x - \cos x}{1+\cos x + \sin x}\, dx$

$\Rightarrow$ let,

$$2+3\sin x - \cos x = L(1+\cos x + \sin x) + M(0-\sin x + \cos x) + N$$

Equating the coefficent of $\sin x$, $\cos x$ and constant items,

$$3 = L - M$$
$$-1 = L + M$$
$$2 = L + N$$

Solving these equations.

$$L = 1, \quad M = -2, \quad N = 1$$

$$\therefore I = \int \frac{(1+\cos x + \sin x) - 2(\sin x + \cos x) + 1}{1+\cos x + \sin x}\, dx$$

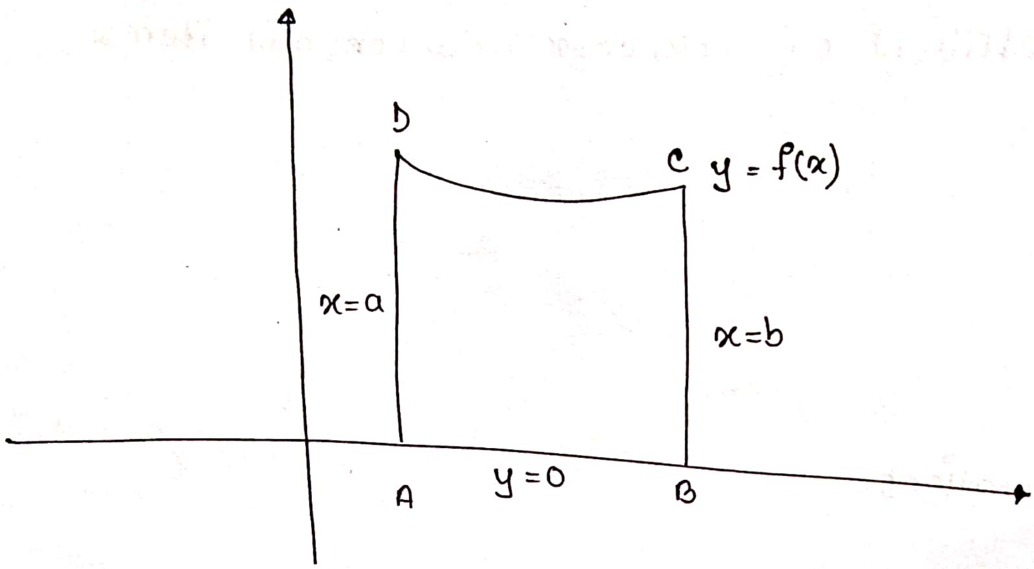
$$= \int dx - 2\int \frac{-\sin x + \cos x}{1+\cos x + \sin x}\, dx + \int \frac{dx}{1+\cos x + \sin x}$$

$$= x - 2\ln|1+\cos x + \sin x| + \int \frac{dx}{1 + \frac{1-\tan^2 x/2}{1+\tan^2 x/2} + \frac{2\tan x/2}{1+\tan^2 x/2}}$$

$$= x - 2\ln(1+\cos x + \sin x) + \ln(1+\tan x/2) + c$$

Definite Integral:

Geometrical interpretation of $\int_a^b f(x)\,dx$:



Geometrically $\int_a^b f(x)\,dx$ represents the area enclosed by the curve $y = f(x)$ the ordinates ~~by the curve~~ : $x = a$ and $x=b$ and the x-axis that $y=0$.

Therefore area ABCD $= \int_{x=a}^{x=b} f(x)\,dx$

Properties of Definite integrals:

1. $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b.$

✓ 4. $\int_0^4 f(x)\,dx = \int_0^4 f(a \mp x)\,dx$

5. $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \quad , \text{ if } f(a+x) = f(x).$

6. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad , \text{ if } f(2a-x) = f(x).$

$= 0 \quad , \text{ if } f(2a-x) = -f(x)$

✓ 7. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \;; \text{ if } f(x) \text{ is even.}$

$= 0, \text{ if } f(x) \text{ is odd.}$

that is $\int_{-a}^a f(x)\,dx = \int_0^a \{f(x) + f(-x)\}\,dx$

Ex: $I = \int_0^1 \sqrt{1-x^2}\, dx$

$$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x \right]_0^1$$

$$= 0 + \frac{1}{2} \cdot \pi/2 - 0 - 0$$

$$= \pi/4.$$

Ex: $I = \int_0^{\pi/3} \frac{\cos x}{3+4\sin x}\, dx$

$\Rightarrow$ let, $z = 3 + 4\sin x$

$\Rightarrow \frac{1}{4} dz = \cos x\, dx$

limit: when, $x = 0$, $z = 3$

$x = \pi/3$, $z = 3 + 2\sqrt{3}$

$$\therefore I = \int_3^{3+2\sqrt{3}} \frac{1}{4} \frac{dz}{dz}$$

$$= \frac{1}{4} \left[\ln z \right]_3^{3+2\sqrt{3}}$$

$$= \frac{1}{4} \left[\ln(3+2\sqrt{3}) - \ln(3) \right]$$

$$= \frac{1}{4} \ln\left(\frac{3+2\sqrt{3}}{3} \right)$$

Ex: ① $I = \int_{2}^{e} \left[\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right] dx$

② $I = \int_{1/2}^{1} \frac{dx}{x\sqrt{1-x^2}}$

③ $I = \int_{0}^{1} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

④ $I = \int_{0}^{1} x \sin^{-1} x \, dx$

Ex: $I = \int_{0}^{a} \sqrt{\frac{a+x}{a-x}}\, dx$

$\Rightarrow$ let $x = a\cos\theta$, then $dx = -a\sin\theta \cdot d\theta$

limit: when $x = 0$, $\theta = \pi/2$

$x = a$, $\theta = 0$

$\therefore I = -\int_{\pi/2}^{0} \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\, a\sin\theta \cdot d\theta$

$= a\int_{0}^{\pi/2} \frac{\cos\theta/2}{\sin\theta/2} \cdot 2\sin\theta/2 \cdot \cos\theta/2 \, d\theta$

$= 2a\int_{0}^{\pi/2} \cos^2 \theta/2 \, d\theta$

$= a\int_{0}^{\pi/2} (1+\cos\theta)\, d\theta \quad = a\left[\theta + \sin\theta\right]_{0}^{\pi/2}$

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Ex: $$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad \text{--- (1)}$$ [4 নং property]

By properties $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

We have from (1) →

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}\, dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \quad \text{--- (2)}$$

Adding eq. (1) and (2) →

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} + \int_0^{\pi/2} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}}$$

$$= \int_0^{\pi/2} dx$$

$$= [x]_0^{\pi/2}$$

$$= \pi/2$$

$$\therefore I = \pi/4$$

Ex: $I = \displaystyle\int_0^{\pi/2} \frac{dx}{1+\cot x} = ?$ $\longrightarrow$ $\cot x = \dfrac{\cos x}{\sin x}$

$$I = \int_0^{\pi} \frac{x\,dx}{1+\sin x\,d.} \quad \text{———} \quad (\mathrm{i})$$

$\Rightarrow$ By properties 4, we have,

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x} \quad \text{———} \quad (\mathrm{ii})$$

$(\mathrm{i}) + (\mathrm{ii}) \Rightarrow$

$$2I = \int_0^{\pi} \left(\frac{x}{1+\sin x} + \frac{\pi - x}{1+\sin x} \right) dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{dx}{1+\sin x}$$

$$= \pi \int_0^{\pi} \frac{1-\sin x}{\cancel{\cos^2 x}\ 1-\sin^2 x}$$

$$=$$

$$= \pi \int_0^{\pi}$$

$\therefore I = \pi$

$$\left\{ \begin{array}{l} \ln(ab) = \ln a + \ln b \\ \ln(a/b) = \ln a - \ln b \end{array} \right.$$

<u>Ex:</u> $I = \int_0^1 \ln\left(\frac{1}{x} - 1\right) dx$ —— (i)

$\Rightarrow I = \int_0^1 \ln\left(\frac{1-x}{x}\right) dx$ —— (ii)

we also have from 2 →

$$I = \int_0^1 \ln\left\{\frac{1-(1-x)}{1-x}\right\} dx$$

$\Rightarrow I = \int_0^1 \ln\left(\frac{x}{1-x}\right) dx$ —— (iii)

(ii) + (iii) ⇒

$$2I = \int_0^1 \left\{\ln\left(\frac{1-x}{x}\right) + \ln\left(\frac{x}{1-x}\right)\right\} dx$$

$$\Rightarrow 2I = \int_0^1 \ln 1 \, dx$$

$$= 0$$

$\therefore I = 0$

Ex: $I = \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$

$\Rightarrow$ Here $I = \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$

$= \int_{0}^{\pi/2} [\sin^7 x + \sin^7(-x)] \, dx$

$= \int_{0}^{\pi/2} (\sin^7 x - \sin^7 x) \, dx$

$= 0$

$$\left\{\begin{aligned} I_n &= x^n \\ &= \square\, x^{n-1}\, \square \\ &= \square\, x^{n-2}\, \square \end{aligned}\right.$$

Successive Reduction:

Ex: Find the reduction formula for $I_n = \int \sin^n x \, dx$ and hence evaluate $\int \sin^6 x \, dx$.

$\Rightarrow$ Given,

$$I_n = \int \sin^n x \, dx$$

$$\Rightarrow I_n = \int \underbrace{\sin x}_{v} . \underbrace{\sin^{n-1} x}_{u} \, dx$$

$$\Rightarrow I_n = \sin^{n-1} x \int \sin x \, dx - \int \left(\frac{d}{dx} \sin^{n-1} x \int \sin x \, dx \right) dx$$

$$= -\sin^{n-1} x . \cos x + \int (n-1) \sin^{n-2} x . \cos^2 x \, dx$$

$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$\Rightarrow I_n = -\sin^{n-1} x . \cos x + (n-1) . I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n (1+n-1) = -\sin^{n-1} x . \cos x + (n-1) I_{n-2}$$

$$\Rightarrow I_n = -\frac{\sin^{n-1} x . \cos x}{n} + \frac{n-1}{n} I_{n-2} \quad \text{——— (1)}$$

This is the reduction formula for $I_n = \int \sin^n x \, dx$

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From ①, we use $n = 6$, then

$$I_6 = -\frac{\sin^5 x . \cos x}{6} + \frac{5}{6}\int \sin^4 x \, dx$$

$$= -\frac{\sin^5 x . \cos x}{6} + \frac{5}{6}\left[-\frac{\sin^3 x . \cos x}{4} + \frac{3}{4}\int \sin^2 x \, dx\right]$$

$$= -\frac{\sin^5 x . \cos x}{6} - \frac{5}{24}\sin^3 x . \cos x + \frac{15}{24}\int \sin^2 x \, dx.$$

$$= \frac{-\sin^5 x . \cos x}{6} - \frac{5}{24}\sin^3 x . \cos x + \frac{15}{24}\left[-\frac{\sin x . \cos x}{2} + \frac{1}{2}\int \sin^0 x \, dx\right]$$

$$\Rightarrow I_6 = -\frac{\sin^5 x . \cos x}{6} - \frac{5}{24}\sin^3 x . \cos x - \frac{15}{48}\sin x . \cos x + \frac{15}{48}x$$

Ex: Find the reduction formula for

(i) $I_n = \int \cos^n x \, dx$

(ii) $I_n = \int \tan^n x \, dx$

Ex: If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$, then show that the reduction formula is $I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$, Hence evaluate I_5.

$\Rightarrow$ $I_n = \int x^n \sin x \, dx$

$I_{n-1} = \int x^{n-1} \sin x \, dx$

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\# $I_n = \int_0^{\pi/2} x^n \sin x \, dx$

$\Rightarrow x^n \int \sin x\, dx - \int \left(\frac{d}{dx} x^n \int \sin x\, dx \right) dx$

$= \left[-x^n \cos x \right]_0^{\pi/2} + \int_0^{\pi/2} n x^{n-1} \cos x\, dx$

$= -0 + 0 + n \int_0^{\pi/2} x^{n-1} \cos x\, dx$

$= n x^{n-1} \int_0^{\pi/2} \cos x\, dx - n \int_0^{\pi/2} \frac{d}{dx} x^{n-1} \int_0^{\pi/2} \cos x\, dx$

$= \left[n x^{n-1} \sin x \right]_0^{\pi/2} - n \int_0^{\pi/2} (n-1) x^{n-2} \sin x\, dx$

$\Rightarrow I_n = n \left(\pi/2 \right)^{n-1} - 0 - n(n-1) \int_0^{\pi/2} x^{n-2} \sin x\, dx$

$I_n = n \left(\pi/2 \right)^{n-1} - n(n-1)\, I_{n-2}$

$\Rightarrow I_n + n(n-1)\, I_{n-2} = n \left(\pi/2 \right)^{n-1}$ —— (i)

This is the reduction for I_n.

Hence Showed.

Now we put $n = 5$ in eqn (i), then –

$I_5 + (5 \times 4)\, I_3 = 5 \left(\pi/2 \right)^4$ —— (ii) we put again $n=3$ in eqn (I)

$$I_3 + (3\times 2) I_1 = 3\left(\frac{\pi}{2}\right)^2 \text{ ——— (III)}$$

Also for $n=1$, eqn (I) gives,

$$I_1 = 1$$

From eqn (III) →

$$\therefore I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6$$

and, From eqn (II) →

$$I_5 = 5\left(\frac{\pi}{2}\right)^4 - 60\left(\frac{\pi}{2}\right)^2 + 120.$$

Ex: If $U_n = \int_0^1 x^n \tan^{-1}x \, dx$, then show that its reduction formula is

$$(n+1)U_n + (n-1)U_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

⇒

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Ex: State and prove walli's formula.

Statement: $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)(n-5)\cdots 5.3.1}{n(n-2)(n-4)\cdots 6.4.2} \cdot \frac{\pi}{2}$

where n is even. (যুগ্ম)

$= \frac{(n-1)(n-3)(n-5)\cdots 6.4.2}{n(n-2)(n-4)\cdots \cancel{6.4.2}\ 5.3.1} \cdot 1$

when n is odd. (বিজোড়)

Ex: Evaluate (i) $\int_0^{\pi/2} \cos^6 x \, dx$

$= \frac{\cancel{6.4.2}\cdot 5.3.1}{6.4.2} \times \frac{\pi}{2}$

$= \frac{15}{48} \times \frac{\pi}{2}$

$= \frac{5}{32} \pi$

(ii) $\int_0^{\pi/2} \sin^7 x \, dx$

$= \frac{6.4.2}{7.5.3.1} \times 1$

$= \frac{16}{35}$

Gamma and Beta function:

Definition: The integral of the form $\int_0^{\infty} e^{-x} x^{n-1} \, dx$, $n>0$ is Known as Gamma, Gamma function or second eulerian function. it is denoted by Γn and defined as $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} \, dx$; $n>0$

The integral of the form $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx$; $m,n>0$ is known as Beta function or First Eulerian function, it is denoted by $\beta(m,n)$ and defined as $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$; $m,n>0$.

Ex: Show that (i) $\Gamma(n+1) = n \Gamma n$

(ii) $\Gamma(n+1) = \underline{|n}$

(iii) $\Gamma 1 = 1$

$\Rightarrow$ We know that the Gamma function is

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} \, dx \; ; \; n>0 \quad \text{---} (1)$$

(i) Now replacing n by $(n+1)$ in equation (1), then –

$$\Gamma_{n+1} = \int_0^\infty e^{-x} x^n \, dx$$

$$= x^n \int e^{-x} \, dx - \int \left(\frac{d}{dx} x^n \int e^{-x} \, dx \right) dx$$

$$= \left[-x^n e^{-x} \right]_0^\infty + \int_0^\infty n x^{n-1} e^{x} \, dx$$

$$= -0 + 0 + n \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$\Gamma_{n+1} = n \Gamma_n \qquad \text{——— (2)}$$

(ii) From equation (2), we replace n by $(n-1)$, then –

$$\Gamma_n = (n-1) \Gamma_{n-1} \qquad \text{——— (3)}$$

Again replacing n by $(n-2)$, $(n-3)$, ... successively, then –

$$\Gamma_{n-1} = (n-2) \Gamma_{n-2}$$

$$\Gamma_{n-2} = (n-3) \Gamma_{n-3}$$

$$\Gamma_{n-3} = (n-4) \Gamma_{n-4}$$ and so on

Substiting the above value in ② →

$$\Gamma{n+1} = n(n-1)\,\Gamma{n-1}$$

$$= n(n-1)(n-2)\,\Gamma{n-2}$$

$$= n(n-1)(n-2)(n-3)\,\Gamma{(n-3)}$$

$$= n(n-1)(n-2)(n-3)(n-4)\,\Gamma{n-4}$$

$$= n(n-1)(n-2)(n-3)\ldots\ 3.2.1\ \Gamma{1}$$

$$\Rightarrow \Gamma{n+1} = \underline{|n}$$

(iii) Using $n=1$, then equation ① becomes,

$$\Gamma{n} = \Gamma{1} = \int_0^\infty e^{-x} x^0\, dx$$

$$= \int_0^\infty e^{-x}\, dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= -e^{-\infty} + e^0$$

$$= 0 + 1$$

$$= 1$$

Ex: Derive the relation between the Gamma and Beta function or

Derive the relation, $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Ex: Show that $\displaystyle\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \; \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$

⇒ We know that the Beta function is –

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \; ; \quad m,n > 0 \quad \text{—— (1)}$$

We let,

$x = \sin^2\theta$, then $dx = 2\sin\theta \cdot \cos\theta \cdot d\theta$

limit, when $x = 0$, $\theta = 0$

$x = 1$, $\theta = \pi/2$

Therefore, (1) becomes

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} \, 2\sin\theta \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} 2\sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta \quad \text{—— (2)}$$

Again we let,

$2m-1 = p$ and $2n-1 = q$

$\Rightarrow m = \frac{p+1}{2}$ $\Rightarrow n = \frac{q+1}{2}$

From eq^n (2)

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+1}{2}} = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta \quad \left[\text{since } \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right.$$

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+1}{2}}$$

Ex: From the above relation evaluate —

(i) $\int_0^{\pi/2} \sin^5\theta \, d\theta$

(ii) $\int_0^{\pi/2} \cos^6\theta \, d\theta$

Ex: Show that $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx \; ; \; n>0$

Hence show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$\Rightarrow$ We know that, the Gamma function,

$$\Gamma n = \int_0^\infty e^{-x_1} x_1^{n-1}\, dx_1 \; ; \; n>0 \quad \text{——— (I)}$$

let,

$x_1 = x^2$, then $dx_1 = 2x\,dx$

limit: when $x_1 = 0$, $x = 0$

$x_1 = \infty$, $x = \infty$

$\therefore$ From eqn (I),

$$\Gamma n = \int_0^\infty e^{-x^2} (x^2)^{n-1}\, 2x\,dx$$

$$= \int_0^\infty 2e^{-x^2} x^{2n-2+1}\, dx$$

$$= \int_0^\infty 2\,e^{-x^2} x^{2n-1}\, dx$$

$$\therefore \Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx \quad \text{——— (II)}$$

Hence showed.

We use $n = \frac{1}{2}$ in (2), then,

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$$\Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-x^2}\,dx$$

$$\Rightarrow \int_0^\infty e^{-x^2}\,dx = \frac{\Gamma\tfrac{1}{2}}{2}$$

$$= \frac{\sqrt{\pi}}{2} \quad \left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

Hence showed.

Ex: Show that; $\Gamma n = k^n \int_0^\infty e^{-kx} x^{n-1}\,dx;\ n, k > 0$

$$\Gamma n = \int_0^\infty e^{-x_1} x_1^{n-1}\,dx_1\ ;\ n > 0$$

let, $x_1 = kx$

$dx_1 = k\,dx$

limit, $x_1 = 0$, $x = 0$

$x_1 = \infty$, $x = \infty$

Ex: Show that $\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Rightarrow$ We know that the beta function is $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \,;\; m,n>0$ —— (1)

We use $m = \frac{1}{2}$ and $n = \frac{1}{2}$ in eqn (1), that

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx$$

$$\Rightarrow \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

$$= \int_0^1 \frac{dx}{\sqrt{x-x^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}$$

$$= \left[\sin^{-1}\frac{x-\frac{1}{2}}{\frac{1}{2}}\right]_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(-1)$$

$$= \pi$$

$$\left\{ \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \right.$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}+\frac{1}{2}} = \pi$$

$$\Rightarrow \left(\Gamma\tfrac{1}{2}\right)^2 = \pi\,\Gamma 1 = \pi$$

$$\Rightarrow \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Ex: Show that; $2^n \Gamma\left(n+\frac{1}{2}\right) = 1.3.5 \cdots\cdots (2n-1)\sqrt{\pi}$

$\Rightarrow$ $\Gamma\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}-1\right)\Gamma\left(n+\frac{1}{2}-1\right)$

$$\left\{ \begin{array}{l} \Gamma(n+1) = n\Gamma(n) \\ \Gamma(n) = (n-1)\Gamma(n-1) \end{array} \right.$$

$= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$

$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$

$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)$

$\cdots\cdots$

$\cdots\cdots$

$\cdots\cdots$

$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\cdots\cdots \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\cdots\cdots \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

$\Rightarrow \Gamma\left(n+\frac{1}{2}\right) = \frac{1}{2^n}(2n-1)(2n-3)(2n-5)\cdots\cdots 5.3.1.\sqrt{\pi}$

$\Rightarrow 2^n\Gamma\left(n+\frac{1}{2}\right) = \cancel{(2}\ 1.3.5\cdots\cdots(2n-3)(2n-1)\sqrt{\pi}$

$\Rightarrow$

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Ex: Show that $2^{2m-1}\,\Gamma m\,\Gamma m+\frac{1}{2} = \Gamma 2m\,\sqrt{\pi}$, $m>0$

$\Rightarrow$ Given, that,

$$2^{2m-1}\,\Gamma m\,\Gamma m+\tfrac{1}{2} = 2^{2m-1}\,\Gamma m\,\Gamma \frac{2m+1}{2}$$

$$= 2^{2m-1}\,\Gamma m\left(\frac{2m+1}{2}-1\right)\Gamma \frac{2m+1}{2}$$

$$= 2^{2m-1}\,\Gamma m\left(\frac{2m-1}{2}\right)\Gamma \frac{2m-1}{2}$$

$$= 2^{2m-1}\,\Gamma m\left(\frac{2m-1}{2}\right)\left(\frac{2m-3}{2}\right)\Gamma \frac{2m-3}{2}$$

$$= 2^{2m-1}\,\Gamma m\left(\frac{2m-1}{2}\right)\left(\frac{2m-3}{2}\right)\left(\frac{2m-5}{2}\right)\Gamma \frac{2m-5}{2}$$

$$\cdots\cdots \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma \tfrac{1}{2}$$

$$= 2^{m-1}\,\Gamma m\,\frac{1}{2^m}(2m-1)(2m-3)(2m-5)\cdots 5.3.1\,\sqrt{\pi}$$

$$= 2^{m-1}\,\Gamma m\,\frac{(2m-1)(2m-2)(2m-3)(2m-4)(2m-5)\cdots 5.4.3.2.1}{(2m-2)(2m-4)\cdots 4.2.}\sqrt{\pi}$$

$$= 2^{m-1}\,\Gamma m\,\frac{\Gamma 2m\,\sqrt{\pi}}{2^{m-1}\,\Gamma m}$$

$$\Rightarrow 2^{2m-1}\,\Gamma m\,\Gamma m+\tfrac{1}{2} = \Gamma 2m\,\sqrt{\pi}$$

$$\left\{\begin{array}{l} \Gamma n+1 = \underline{|n} \\ \Gamma n+1 = n\,\Gamma n \\ \Gamma n = (n-1)\,\Gamma n-1 \end{array}\right.$$

Ex: Evaluate $\int_1^2 \int_y^{3y} (3x^2+y^2)\, dx\, dy$

$$= \int_1^2 \left[\int_y^{3y} (3x^2+y^2)\, dx\right] dy$$

$$= \int_1^2 \left[\frac{3x^3}{3} + y^2 x\right]_y^{3y} dy$$

Ex: Evaluate $\int_1^2 \int_0^1 \int_{-1}^1 (x^2+y^2+z^2)\, dx\, dy\, dz$

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Improper integral:

- If the range of integration is infinite and the integrand is bounded (that is discontinuity in the integral) then the integral is improper integral of the first kind.

- If the range of integration is finite and the integrand is unbouded, then the integral is improper of the second kind.

Ex: (i) $\int_0^{\infty} \frac{dx}{1+x^2}$ (ii) $\int_{-\infty}^{\infty} \frac{2x^2}{x^4+x^2+1} dx$ (iii) $\int_{-\infty}^{0} \frac{dx}{4+x^2}$

Ex: (i) $\int_0^{4} \frac{dx}{x-2}$ (ii) $\int_{-3}^{5} \frac{dx}{x+2}$ (iii) $\int_{-4}^{7} \frac{x^2+2}{\sqrt{x-5}} dx$

- If the integral both first and second kind, is called the improper integral of the 3rd kind.

Ex: (i) $\int_0^{\infty} \frac{e^x}{x-2} dx$ (ii) $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx$ (iii) $\int_{-\infty}^{\infty} \frac{dx}{x^2-7x+12}$

Ex: Show that $\int_1^{\infty} \frac{dx}{x^{3/2}} = 2$

$\Rightarrow$ We can write,

$$\int_1^{\infty} \frac{dx}{x^{3/2}} = \lim_{\varepsilon \to \infty} \int_1^{\varepsilon} \frac{dx}{x^{3/2}}$$

$$= \lim_{\varepsilon \to \infty} \left[\frac{x^{-1/2}}{-1/2} \right]_1^{\varepsilon}$$

$$= \lim_{\varepsilon \to \infty} -2 \left(\varepsilon^{-1/2} - 1^{-1/2} \right)$$

$$= \lim_{\varepsilon \to \infty} -2 \left(\varepsilon^{-1/2} - 1 \right)$$

$$= -2 \left(\infty^{-1/2} - 1 \right)$$

$$= -2 (0 - 1)$$

$$= 2$$

- $\infty^{-1/2} = 0$
- $e^{-\infty} = 0$
- $e^{\infty} = \infty$
- $\log 0 = \infty$
- $\log 1 = 0$
- $\log |-2| = \log 2$

Ex: Show that $\int_0^\infty \frac{e^x}{1+e^x} dx = \infty$

or show that $\int_0^\infty \frac{e^x}{1+e^x} dx$ is divergent

$\Rightarrow$ We can write,

$$= \int_0^\infty \frac{e^x}{1+e^x} dx$$

$$= \lim_{\varepsilon \to \infty} \int_0^\varepsilon \frac{e^x}{1+e^x} dx$$

$$= \lim_{\varepsilon \to \infty} \left[\log(1+e^x)\right]_0^\varepsilon$$

$$= \lim_{\varepsilon \to \infty} \left\{\log(1+e^\varepsilon) - \log(1+e^0)\right\}$$

$$= \log(1+e^\infty) - \log 2$$

$$= \log(1+\infty) - \log 2$$

$$= \log \infty - \log 2$$

$$= \infty - \log 2$$

$$= \infty$$

Ex: Evaluate $\int_0^{\infty} xe^{-x^2}\,dx$

$\Rightarrow$ We can write,

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{\epsilon\to\infty}\int_0^{\epsilon^2} xe^{-x^2}\,dx$$

$$= \lim_{\epsilon\to\infty}\int_0^{\epsilon^2}\frac{1}{2}e^{-z}\,dz$$

$$= \frac{1}{2}\lim_{\epsilon\to\infty}\left[-e^{-z}\right]_0^{\epsilon^2}$$

$$= -\frac{1}{2}\lim_{\epsilon\to\infty}\left(e^{-\epsilon^2}-e^{-0}\right)$$

$$= -\frac{1}{2}\left(e^{-\infty^2}-e^{-0}\right)$$

$$= -\frac{1}{2}\left(e^{-\infty}-1\right)$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

let,

$z = x^2$

$\Rightarrow dz = 2x\,dx$

$\Rightarrow x\,dx = \frac{1}{2}dz$

$x = 0,\ z = 0$

$x = \epsilon,\ z = \epsilon^2$

Ex: Evaluate $\int_{-\infty}^{0} \frac{e^x}{1+e^x}\, dx$

$\Rightarrow$ We can write,

$$\int_{-\infty}^{0} \frac{e^x\, dx}{1+e^x} = \lim_{\epsilon \to -\infty} \int_{\epsilon}^{0} \frac{e^x}{1+e^x}\, dx$$

$$= \lim_{\epsilon \to -\infty} \Big[\ln\left(1+e^x\right) \Big]_{\epsilon}^{0}$$

$$= \lim_{\varepsilon \to -\infty} \left\{ \ln\left(1+e^0\right) - \ln\left(1+e^{\varepsilon}\right) \right\}$$

$$= \ln 2 - \ln\left(1+e^{-\infty}\right)$$

$$= \ln 2 - \ln 1$$

$$= \ln 2.$$

Evaluate $\int_0^{\infty} \frac{x\,dx}{x^4+1}$

$$= \lim_{\epsilon \to \infty} \int_0^{\epsilon} \frac{x\,dx}{(x^2)^2+1}$$

let,

$x^2 = z$

$2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

$x = 0,\ z = 0$

$x = \epsilon,\ z = \epsilon^2$

$$= \lim_{\epsilon \to \infty} \frac{1}{2} \int_0^{\epsilon^2} \frac{dz}{z^2+1}$$

$$= \frac{1}{2} \lim_{\epsilon \to \infty} \left[\tan^{-1} z\right]_0^{\epsilon^2}$$

$$= \frac{1}{2} \lim_{\epsilon \to \infty} \left[\tan^{-1} \epsilon^2 - \tan^{-1} 0\right]$$

$$= \frac{1}{2}\left(\tan^{-1} \infty - \tan^{-1} 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

Ex: Evaluate $\int_{\infty}^{0} \frac{x\,dx}{x^4+1}$

$$= \lim_{\epsilon \to -\infty} \int_{\epsilon}^{0} \frac{x\,dx}{x^4+1}$$

$$= \frac{1}{2} \lim_{\epsilon \to -\infty} \int_{\epsilon^2}^{0} \frac{dz}{z^2+1}$$

$$= \frac{1}{2} \lim_{\epsilon \to -\infty} \left[\tan^{-1} z\right]_{\epsilon^2}^{0}$$

$$= \frac{1}{2} \lim_{\epsilon \to -\infty} \left(\tan^{-1} 0 - \tan^{-1} \epsilon^2\right)$$

$$= \frac{1}{2} \left(\tan^{-1} 0 - \tan^{-1} \infty\right)$$

$$= -\pi/4$$

let,

$x^2 = z$

$2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

$x = 0,\ z = 0$

$x = \epsilon,\ z = \epsilon^2$

Ex: Evaluate $\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1}$

$\Rightarrow$ We can write,

$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1} = \int_{-\infty}^{0} \frac{x\,dx}{x^4+1} + \int_{0}^{\infty} \frac{x\,dx}{x^4+1}$$

Ex: Evaluate $\int_1^3 \frac{dx}{1-x}$

$\Rightarrow$ We can write,

$$\int_1^3 \frac{dx}{1-x} = \lim_{\epsilon \to 0} \int_{1+\epsilon}^3 \frac{dx}{1-x}$$

$$= \lim_{\epsilon \to 0} -\left[\ln|1-x| \right]_{1+\epsilon}^3$$

$$= -\lim_{\epsilon \to 0} \left\{ \ln|2| - \ln|1-1-\epsilon| \right\}$$

$$= -\lim_{\epsilon \to 0} (\ln 2 - \ln \epsilon)$$

$$= -(\ln 2 - \ln 0)$$

$$= -[\ln 2 - \infty]$$

$$= \infty$$

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Ex: Show that $\int_1^3 \frac{x\,dx}{(x^2-1)^{3/2}} = 3$

Ex: Show that $\int_0^3 \frac{x\,dx}{\sqrt{9-x^2}} = \pi/2$

$$\Rightarrow \lim_{\epsilon \to 0} \int_0^{3-\epsilon} \frac{dx}{\sqrt{9-x^2}}$$

$$= \lim_{\epsilon \to 0} \left[\sin^{-1} \frac{x}{3}\right]_0^{3-\epsilon}$$

$$= \lim_{\epsilon \to 0} \left(\sin^{-1} \frac{3-\epsilon}{3} - \sin^{-1} 0\right)$$

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \pi/2$$

Ex: Evaluate $\int_0^3 \frac{dx}{(3-x)^2}$

$$= \lim_{\epsilon \to 0} \int_0^{3-\epsilon} (3-x)^{-2}\,dx$$

$$= \lim_{\epsilon \to 0} \left[\frac{-(3-x)^{-2+1}}{-2+1}\right]_0^{3-\epsilon}$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{3-x}\right]_0^{3-\epsilon}$$

$$= \lim_{\epsilon \to 0} \left(\frac{1}{3-3+\epsilon} - \frac{1}{3-0}\right)$$

$$= \frac{1}{0} - \frac{1}{3}$$

Ex: Show that $\int_0^2 \frac{dx}{1-x^2} = \infty$

$\Rightarrow$ We can write,

$$\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$$

$$= \int_0^1 \frac{dx}{1-x^2} - \int_1^2 \frac{dx}{x^2-1}$$

$$= \lim_{\epsilon_1 \to 0} \int_0^{1-\epsilon_1} \frac{dx}{1-x^2} - \lim_{\epsilon_2 \to 0} \int_{1+\epsilon_2}^{2} \frac{dx}{x^2-1}$$

$$= \lim_{\epsilon_1 \to 0} \left[\frac{1}{2} \cdot \ln \left|\frac{1+x}{1-x}\right|\right]_0^{1-\epsilon_1} - \lim_{\epsilon_2 \to 0} \left[\frac{1}{2} \ln \left|\frac{x-1}{x+1}\right|\right]_{1+\epsilon_2}^{2}$$

$$= \frac{1}{2} \lim_{\epsilon_1 \to 0} \left\{\ln \frac{1+1-\epsilon_1}{1-1+\epsilon_1} - \ln 1\right\} - \frac{1}{2} \lim_{\epsilon_2 \to 0} \left\{\ln \frac{2-1}{2+1} - \ln \cdot \frac{1+\epsilon_2-1}{1+\epsilon_2+1}\right\}$$

$$= \frac{1}{2} \lim_{\epsilon_1 \to 0} \left\{\ln \frac{2-\epsilon_1}{\epsilon_1} - 0\right\} - \frac{1}{2} \lim_{\epsilon_2 \to 0} \left\{\ln \left|\frac{1}{3}\right| - \ln \left|\frac{\epsilon_2}{2+\epsilon_2}\right|\right\}$$

$$= \frac{1}{2} \left\{\ln \infty - 0 - \ln \frac{1}{3} - \ln 0\right\}$$

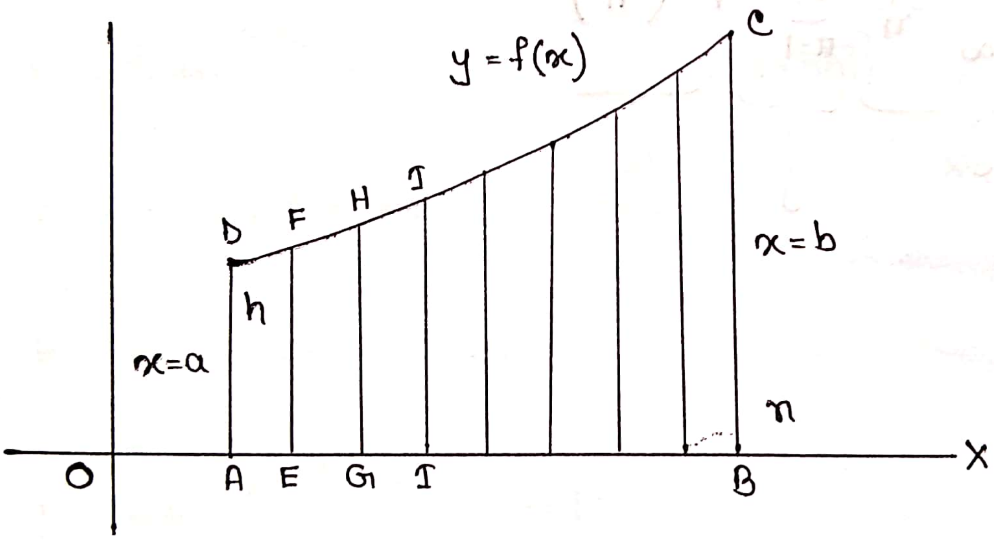
$$= \infty$$

Ex: Show that $\int_{-1}^{1} \frac{dx}{x}$ is divergent.

Definite integral as the limit of a sum:

let, $y = f(x)$ is bounded on $[a, b]$ and continuous on (a, b) are finite and $b > a$. let us divide the integral into n equal parts with length h, then —

$$a + nh = b$$
$$\Rightarrow nh = b - a$$



Area of AEFD $= hf(a+h)$

EGHF $= hf(a+2h)$

Now the area of the region enclosed by $x = a$, $x = b$, x axis and the curve $y = f(x)$.

$$\int_a^b f(x)\,dx = \lim_{h \to 0} \left[hf(a+h) + hf(a+2h) + \dots + hf(a+nh)\right]$$

$$\therefore \int_a^b f(x)\,dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh) \quad \text{——— (1)}$$

If $a=0$, $b=1$, then $nh=1$ and $h=0$ implies $n \to \infty$

eqn (1) becomes,

$$\int_0^1 f(x)\,dx = \underbrace{\lim_{n \to \infty} \frac{1}{n}}_{dx} \underbrace{\sum_{r=1}^{n}}_{\int} \underbrace{f\left(\frac{r}{n}\right)}_{x}$$

Ex: $\int_a^b x\,dx$, evaluate the integration as a process of summation

⇒ we know that,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h \int_{r=1}^{n} (a+rh) \quad , \quad nh = b-a \qquad (1)$$

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h \sum_{r=1}^{n} (a+rh)$$

$$= \lim_{h \to 0} \left[h(a+h) + h(a+2h) + h(a+3h) + \cdots + h(a+nh)\right]$$

$$= \lim_{h \to 0} h \left\{(a+h+a+2h+a+3h+ \cdots + a+nh)\right\}$$

$$= \lim_{h \to 0} h \left\{na + h(1+2+3+\cdots+n)\right\}$$

$$= \lim_{h \to 0} h \left[na + h\frac{n(n+1)}{2}\right]$$

$$= \lim_{h \to 0} \left[anh + \frac{nh(nh+h)}{2}\right]$$

$$= \lim_{h \to 0} \left[a(b-a) + \frac{(b-a)(b-a+h)}{2}\right]$$

$$= a(b-a) + \frac{(b-a)^2}{2}$$

$$= ab - a^2 + \frac{b^2 - 2ab + a^2}{2}$$

$$= ab - a^2 + \frac{a^2}{2} + \frac{b^2}{2} - ab$$

$$= \frac{b^2 - a^2}{2}$$

Ex: Evaluate $\int_0^1 x^3 \, dx$ as a process of summation.

$\int_{x=a}^{b} f(x)\, dx = \lim_{h\to 0} \sum_{r=1}^{n} f(a+rh)$; $nh = b-a$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$

$\frac{r}{n} \downarrow x$

Ex: Evaluate : $\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right]$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+\frac{r}{n}}$

$= \int_0^1 \frac{1}{1+x}\, dx$

$\left\{\begin{array}{l} nh = b - a \\ \quad\;\, \parallel \quad \parallel \\ \quad\;\, 1 \quad\; 0 \end{array}\right.$

$= \ln(1+x) \Big|_{x=0}^{1}$

$= \ln 2 - \ln 1$

$= \ln 2$

Ex: Evaluate: $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

$$\Rightarrow \lim_{n\to\infty}\sum_{n=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

$$=\int_{x=0}^{1}\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=\left[\sin^{-1}x\right]_0^1$$

$$=\pi/2$$

Ex: Evaluate $\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\right]$

$\Rightarrow$ $1.2n$ $\quad 2.2n \quad \ldots\ldots\ldots \quad \sqrt{n.2n-n^2}=\sqrt{n^2}=n$

$$=\lim_{n\to\infty}\sum_{r=1}^{r=n}\frac{1}{\sqrt{r.\,2n-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2\cdot\frac{r}{n}-\left(\frac{r}{n}\right)^2}}$$

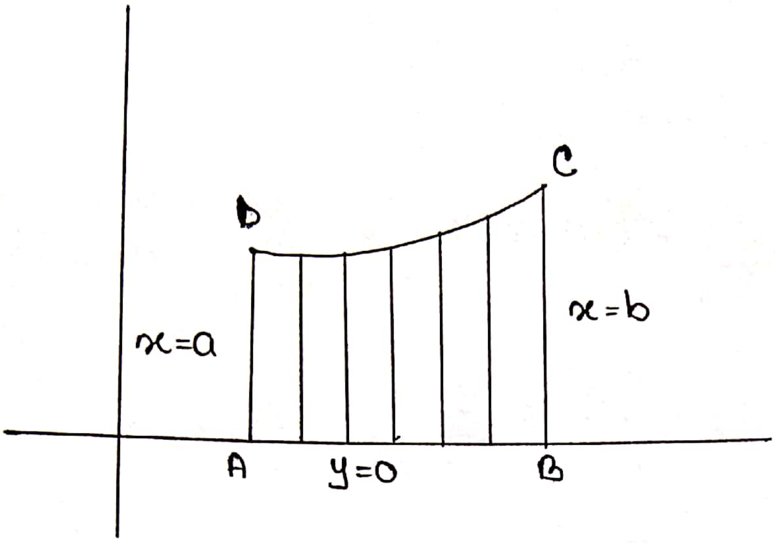
$$=\int_{x=0}^{1}\frac{1}{\sqrt{2x-x^2}}\,dx$$

$$=\int_0^1\frac{dx}{\sqrt{1-(1-2x+x^2)}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-(x-1)^2}}$$

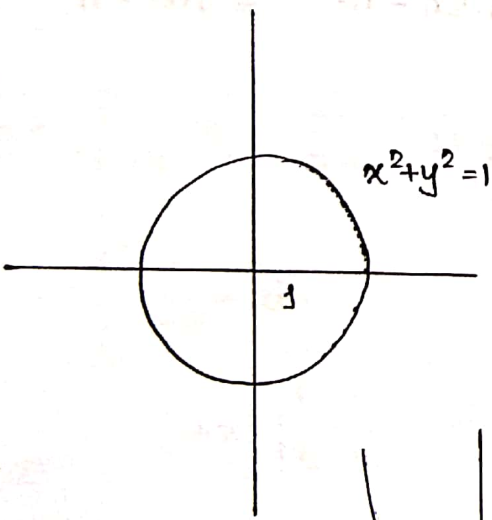
$$=\left[\sin^{-1}(x-1)\right]_0^1=\left\{\sin^{-1}(1-1)-\sin^{-1}(0-1)\right\}=\pi/2$$

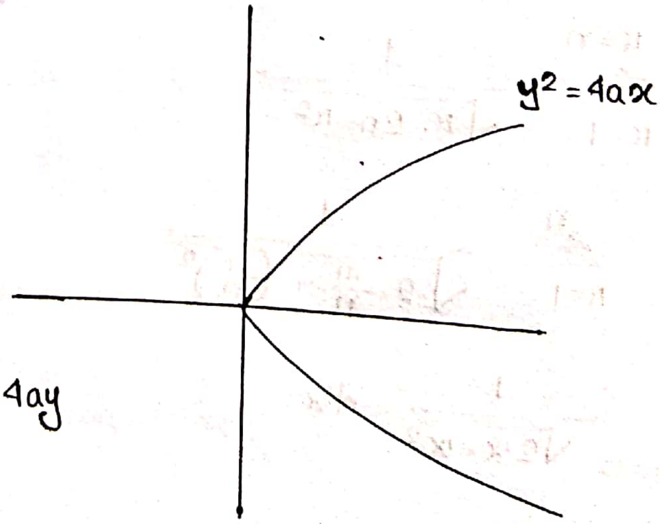
Finding Area formed by different curves:

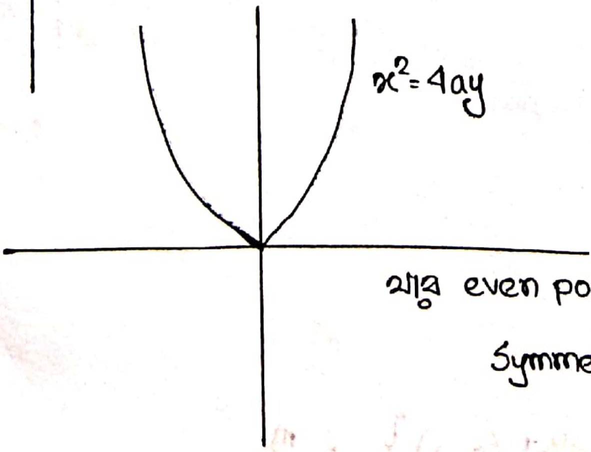


$$\text{Area ABCD} = \int_{x=a}^{x=b} f(x)\,dx$$

$$= \int_{x=a}^{b} y\,dx$$







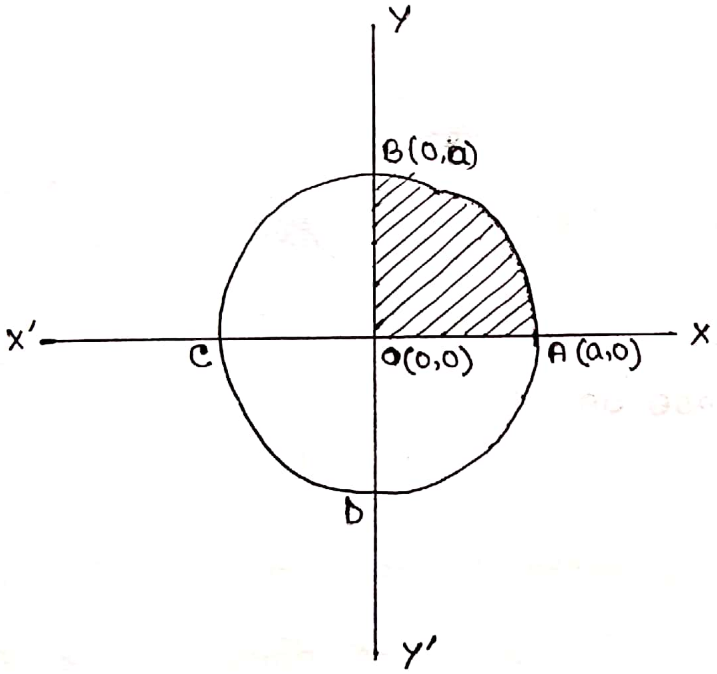
যার even power আছে, বিপরীত অক্ষের সাপেক্ষে Symmetric.

⇒ If any equation contain even power of x, then the eqn is symmetric about y axis.

⇒ " " " " " " " y. " " " " " x axis

⇒ " " " " " " x and y " " " about x, y axis.

Ex: Find the area of $x^2+y^2=a^2$.



The given circle is symmetric about both x and y axis. on x-axis, $y=0$ then the equation becomes $x^2=a^2$

$$\Rightarrow x = \pm a$$

$$\int_a^b y\,dx.$$

Also on y-axis $x=0$, then we get $y^2=a^2$

$$\Rightarrow y = \pm a$$

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Thus the intersection point with x and y axis are $A(a,0)$ and $B(0,a)$.

Thus the area,

$$OABO = \int_{x=a}^{0} y\,dx$$

we let,

$$x = a\sin\theta$$

then $\quad dx = a\cos\theta$

limit: $\quad x = a, \quad \theta = \pi/2$

$\qquad\quad x = 0, \quad \theta = 0$

$$\therefore \text{Area, } OABO = \int_{\theta=\pi/2}^{0} \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= \int_{\pi/2}^{0} a^2\cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_{\pi/2}^{0} 2\cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_{\pi/2}^{0} (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{0}$$

$$= \frac{a^2}{2}\left(0+0-\pi/2-0\right)$$

$$= \frac{a^2}{2}\times\frac{\pi}{2} = \frac{\pi a^2}{4} \text{ sq. units.}$$

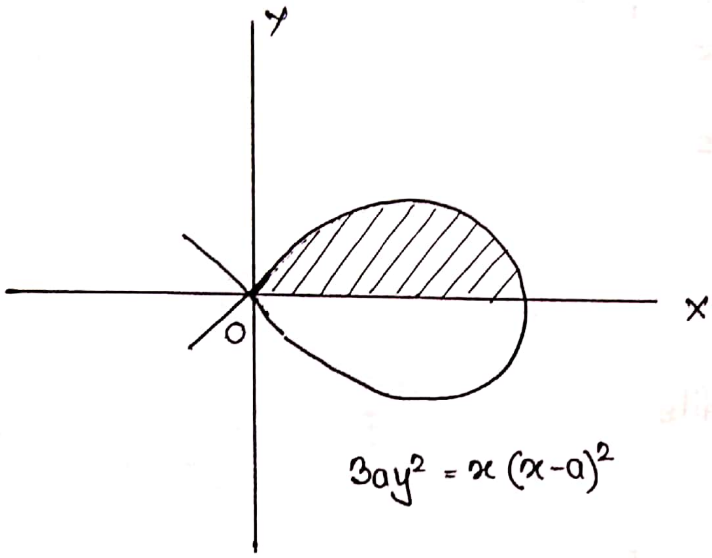
Thus the total area $= 4 \times \frac{\pi a^2}{4}$ sq. units.

$= \pi a^2$ sq. units.

Symmetric Curve:

Ex: Find the area enclosed by $3ay^2 = x(x-a)^2$

→



$3ay^2 = x(x-a)^2$

The equation has ~~been~~ even powered y variable, therefore the given curve is symmetric about x axis.

On x axis $y=0$, then from the given equation, $x(x-a)^2 = 0$

$\Rightarrow x = 0, a$

For $x=0$, $y=0$

$x=a$, $y=0$

Thus the intersection points are $O(0,0)$ and $A(a,0)$.

Now the area OABO $= \int_{x=0}^{a} y\,dx$

$$= \int_{x=0}^{a} \frac{1}{\sqrt{3a}} x^{\frac{1}{2}} (x-a)\, dx$$

$$= \frac{1}{\sqrt{3a}} \int_{x=0}^{a} \left(x^{3/2} - a x^{1/2}\right) dx$$

$$= \frac{1}{\sqrt{3a}} \cdot \frac{2}{5} \left[x^{5/2}\right]_0^a - \frac{a}{\sqrt{3a}} \cdot \frac{2}{3} \times \left[x^{3/2}\right]_0^a$$

$$= \frac{2}{5\sqrt{3a}} a^{5/2} - \frac{2\sqrt{a}}{3\sqrt{3}} a^{3/2}$$

$$= \frac{1}{\sqrt{3a}} \left(\frac{2}{5} - \frac{2}{3}\right) a^{5/2}$$

$$= \frac{1}{\sqrt{3a}} \left(-\frac{4}{15}\right) a^{5/2}$$

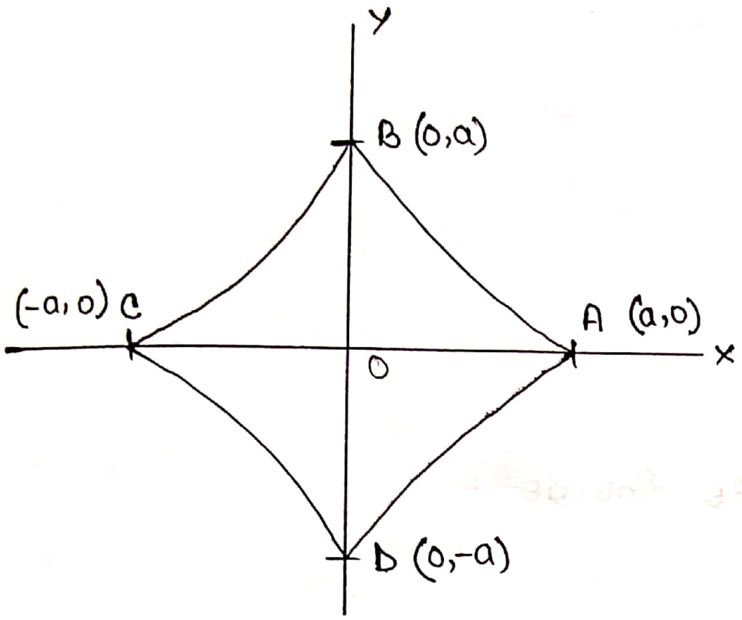
$$= \frac{-4}{15\sqrt{3}} a^2 \text{ sq units}$$

Thus the total area $= 2 \times \dfrac{4}{15\sqrt{3}} a^2$

$$= \frac{8}{15\sqrt{3}} a^2 \text{ units.}$$

<u>Ex:</u> Find the area enclosed by astroid $x^{2/3}+y^{2/3}=a^{2/3}$

$\Rightarrow$ The given astroid is symmetric about both x and y axis.



2 দ্বারা ভাগযোগ্য

↑

even power

তাই দুই অক্ষের সাপেক্ষে প্রতিসম;

$$\left(x^{\frac{1}{3}}\right)^2+\left(y^{\frac{1}{3}}\right)^2=a^{\frac{2}{3}}$$

On x axis, $y=0$ then $x^{2/3}=a^{2/3}$

$\therefore x=\pm a$

y axis, $x=0$, then $y^{2/3}=a^{2/3}$

$\therefore y=\pm a$

Thus the intersection point are $A(a,0)$, $B(0,a)$, $C(-a,0)$, $D(0,-a)$

Now the area $OABO=\int_{x=0}^{a} y\,dx$ ——— (1)

$$\left\{\begin{array}{l} x^2+y^2=a^2 \\ x=a\cos\theta \\ y=a\sin\theta \end{array}\right.$$

The parametric equation for astroid is

$x = a\cos^3\theta \Rightarrow dx = -3\cos^2\theta.\sin\theta.d\theta$

and $y = a\sin^3\theta$

limit:

when $x=0$, $\theta = \pi/2$

$x=a$, $\theta=0$

$$\text{Area } OABO = -\int_{\theta=\pi/2}^{0} a\sin^3\theta.\ 3a\cos^2\theta.\ \sin\theta.\ d\theta$$

$$= 3a^2 \int_{\theta=0}^{\pi/2} \sin^4\theta.\ \cos^2\theta.d\theta$$

$$= 3a^2 \frac{\Gamma 5/2\ \Gamma 3/2}{2\ \Gamma 8/2}$$

$$= 3a^2 \frac{\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}.\frac{1}{2}\Gamma\frac{1}{2}}{2\times3\times2} \quad \left[\Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

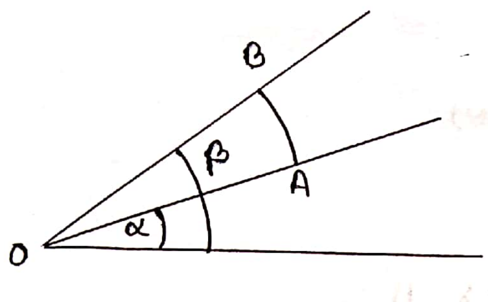
$$= \frac{3a^2\pi}{32} \text{ sq units.}$$

Thus the total area $= 4\times\frac{3a^2\pi}{32}$

$$= \frac{3a^2\pi}{8} \text{ sq units.}$$

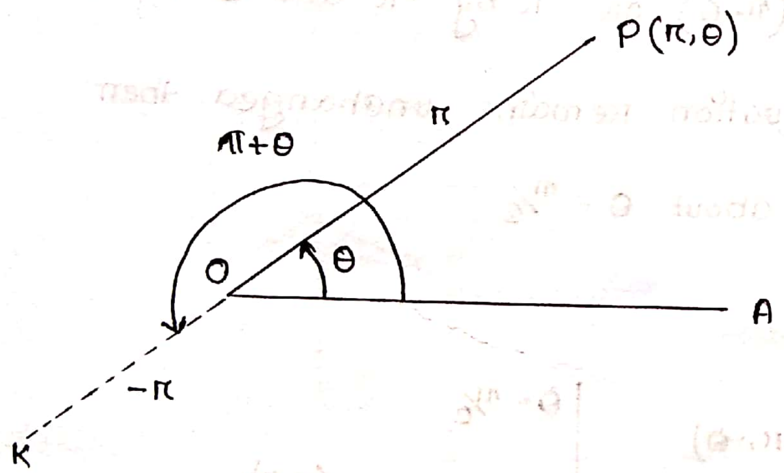
Area in polar Co-ordinates:

If $r = f(\theta)$ is a single valued continuous function of θ in $[\alpha, \beta]$, then the area enclosed by $r = f(\theta)$, $\theta = \alpha$



$$\text{Area } ABO = \frac{1}{2} \int_{\theta=\alpha}^{\beta} r^2 d\theta$$

In polar Co-ordinate:



Where,

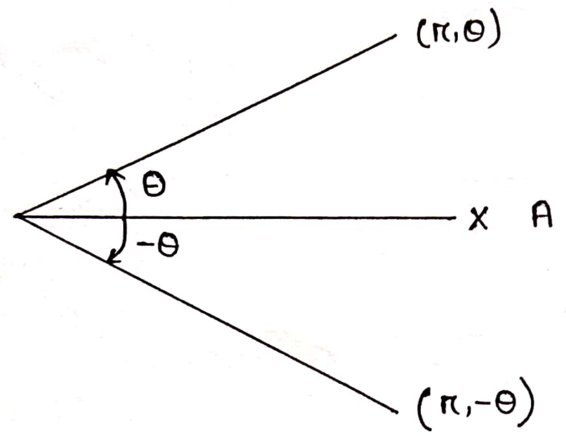
θ = vectorial angle

r = radius vector.

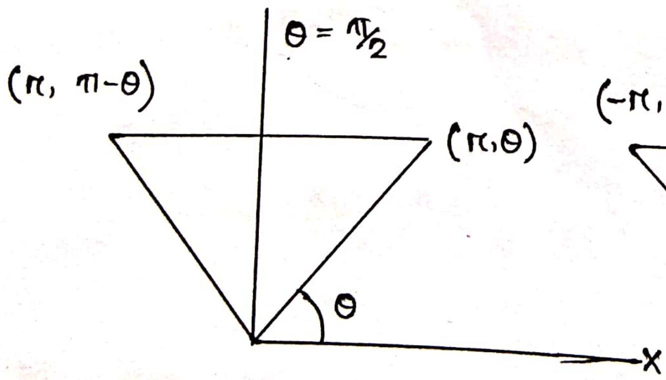
O = pole

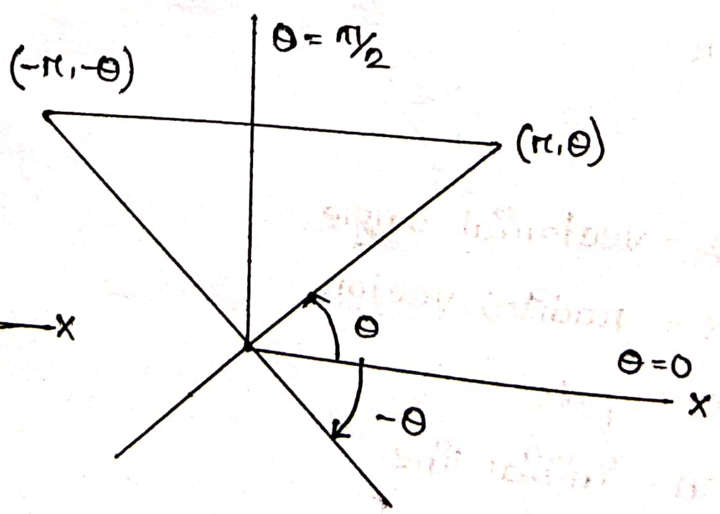
OA = initial line.

(i) If θ is replaced by $-\theta$ in $r = f(\theta)$ and the eqn remains unchanged, then the curve is symmetric about the initial line.

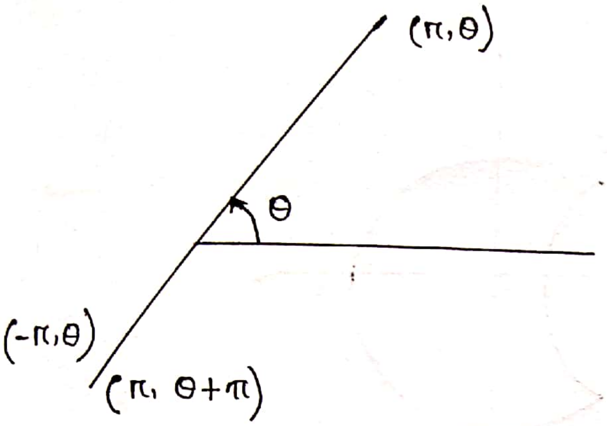


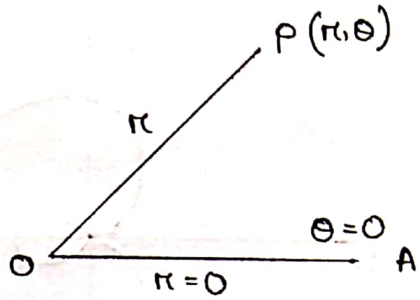
(ii) Either θ is replaced by $(\pi-\theta)$ or r by $-r$ and θ by $-\theta$ simultaneously and the equation remains unchanged, then the curve is symmetric about $\theta = \pi/2$.





(iii) Either θ is replaced by $\theta+\pi$ or r by $-r$ and the eq^n remains unchanged, then the curve is symmetric about pole.

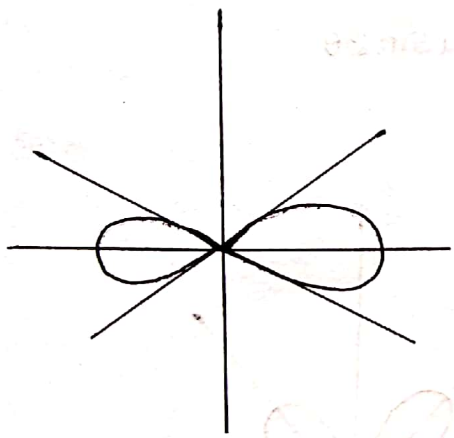




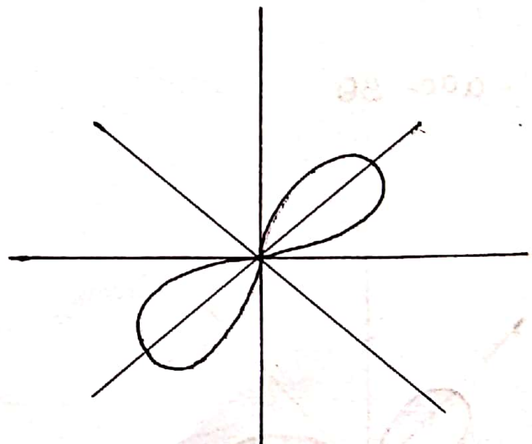
Some unknown curves:

1. Lemniscate:

(i) $r^2 = a^2 \cos 2\theta$

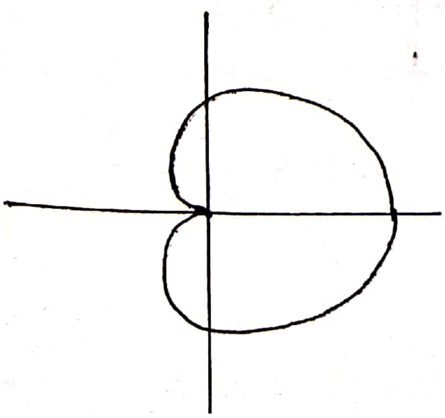


(ii) $r^2 = a^2 \sin 2\theta$

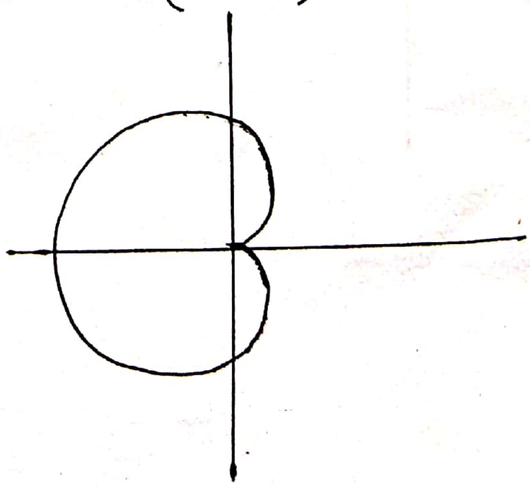


2. Cardioide:

(i) $r = a(1+\cos\theta)$

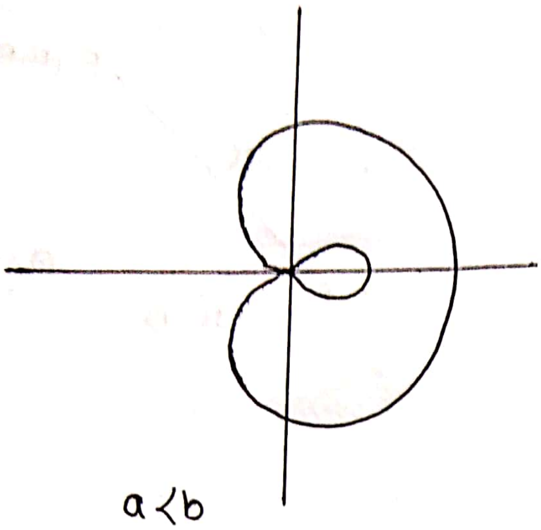


(ii) $r = a(1-\cos\theta)$

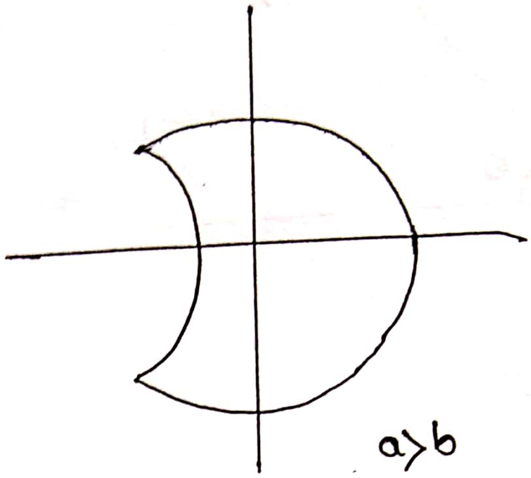


3. Limacon:

$r = a + b\cos\theta$



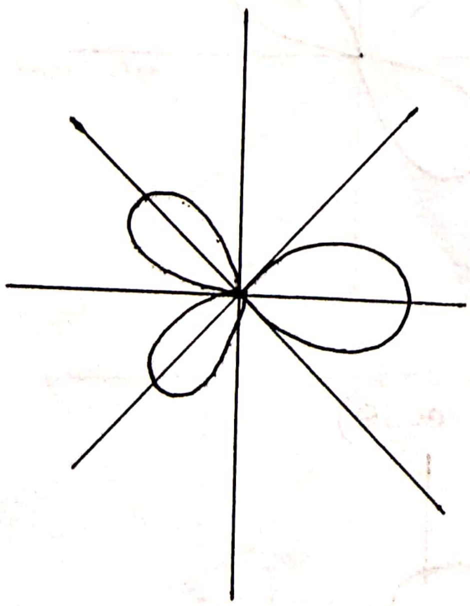
$a < b$



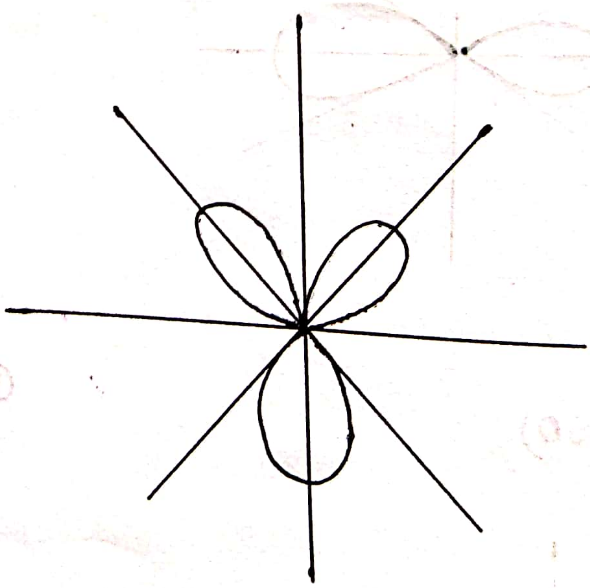
$a > b$

4. Rose Petals:

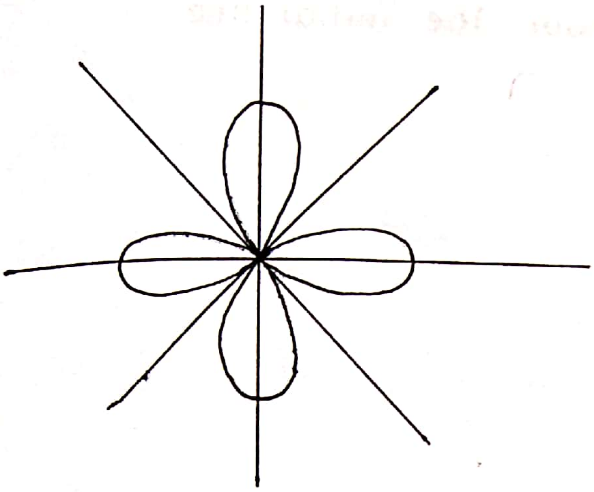
(i) $r = a\cos 3\theta$



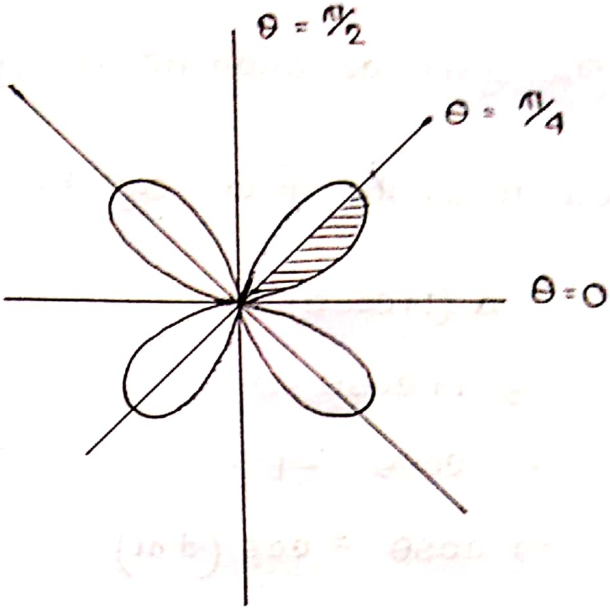
(ii) $r = a\sin 3\theta$



(iii) $r = a\cos 2\theta$



(iv) $r = a\sin 2\theta$



θ সংখ্যা odd হলে petal ততটাই হবে,

" " even " " দ্বিগুন "

$$\frac{1}{2}\int_{\theta=\alpha}^{\beta} r^2\, d\theta = \text{area}$$

Ex: Find the area of $r = a(1+\cos\theta)$

$\Rightarrow$ The given co-ordinate is symmetric about the initial line

For $r=0$, the given eqn becomes,

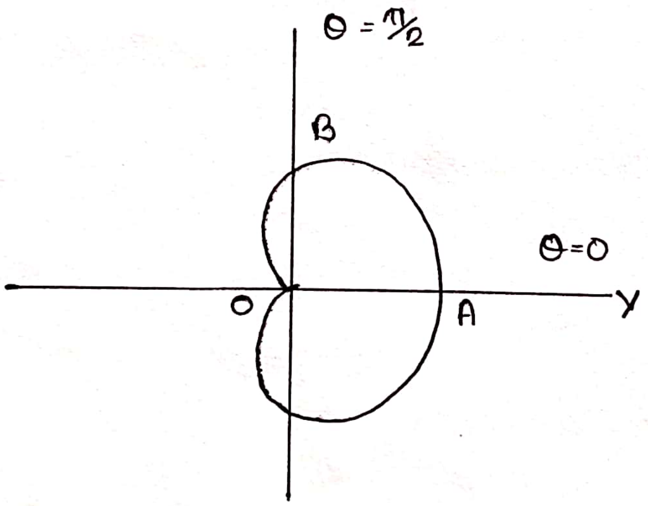
$a(1+\cos\theta) = 0$

$\Rightarrow 1+\cos\theta = 0$

$\Rightarrow \cos\theta = -1$

$\Rightarrow \cos\theta = \cos(\pm\pi)$

$\Rightarrow \theta = \pm\pi$



Now the area ABOA $= \frac{1}{2}\int_{\theta=0}^{\pi} r^2 d\theta$

$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2 (1+2\cos\theta+\cos^2\theta)\, d\theta$

$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2 \left\{ (1+2\cos\theta + \frac{1}{2}(\cos 2\theta+1)) \right\} d\theta$

$$= \frac{a^2}{2}\left[\left(\theta + 2\sin\theta + \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right)\right]_0^{\pi}$$

$$= \frac{a^2}{2}\left[\pi + 0 + \frac{1}{2}(0+\pi) - 0 - 0 - \frac{1}{2}(0)\right]$$

$$= \frac{1}{2}a^2\left(\pi + \frac{\pi}{2}\right)$$

$$= \frac{3\pi a^2}{4} \text{ sq. units.}$$

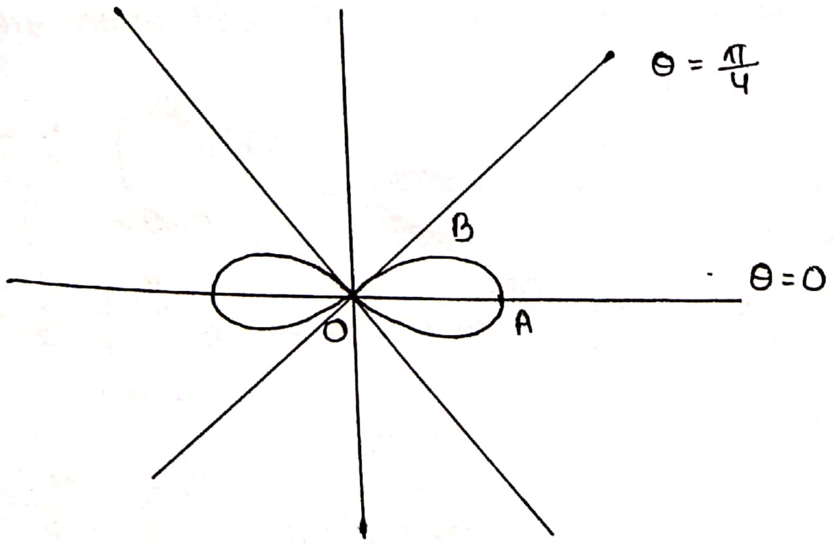
Thus the total area $= 2 \times \frac{3\pi a^2}{4}$ sq. units.

Ex: Find the area of $r^2 = a^2\cos 2\theta$

$\Rightarrow$ Since the curve is unchanged for $r = -r$ and $\theta = -\theta$, therefore, the given curve is symmetric about with the initial line and $\theta = \pi/2$.

For $r = 0$, the given eqn gives $a^2\cos 2\theta = 0$

$$\Rightarrow \theta = \frac{\pi}{4}$$



Therefore half of the one loop lies between $\theta = 0$ to $\theta = \frac{\pi}{4}$

Now the area OABO $= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{4}} r^2 d\theta$

$$= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta$$

$$= \frac{a^2}{2} \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{a^2}{4}$$

Area of one loop $= 2 \times \frac{a^2}{4}$

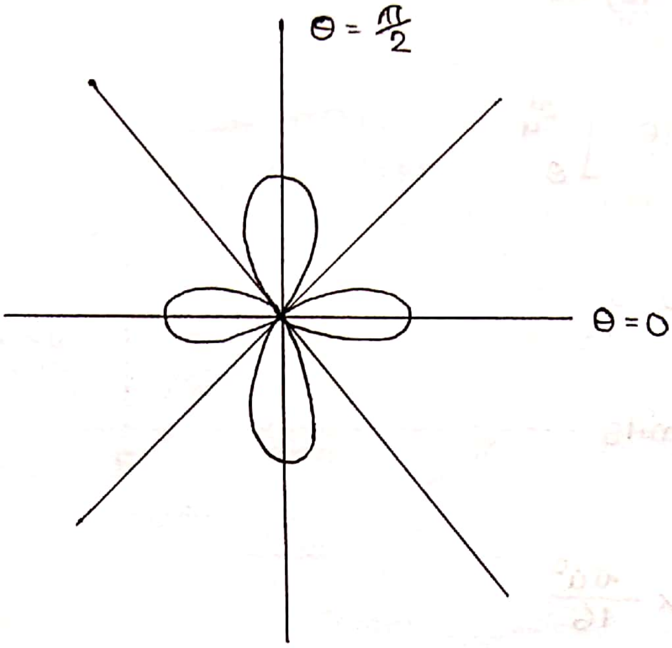
$= \frac{a^2}{2}$ sq units

And area of total loop $= 2 \times \frac{a^2}{2}$ sq units.

$= a^2$ sq units.

Ex: Find the area of one loop of $r = a\cos 2\theta$ and hence find its total area.

→



Since the given curve is unchanged for $\theta = 0$ and $(\pi - \theta)$, then the given curve is symmetric about the initial line and the line $\theta = \frac{\pi}{2}$.

The given curve contains four loops.

For $r = 0$, the given eqn gives $\theta = \pm \frac{\pi}{4}$

That is one loop lies between $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$.

Now the area lies in the first quadrant,

$$= \frac{1}{2} \int_{\theta=0}^{\pi/4} r^2 d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{4}} a^2 \cos^2 2\theta \, d\theta$$

$$= \frac{a^2}{4} \int_{\theta=0}^{\frac{\pi}{4}} 2\cos^2 2\theta \, d\theta$$

$$= \frac{a^2}{4} \int_0^{\frac{\pi}{4}} (1+\cos 4\theta)\, d\theta$$

$$= \frac{a^2}{4} \left[\theta + \frac{\sin 4\theta}{4}\right]_\theta^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4} \left(\frac{\pi}{4} + 0 - 0 - 0\right)$$

$$= \frac{\pi a^2}{16} \text{ sq units}$$

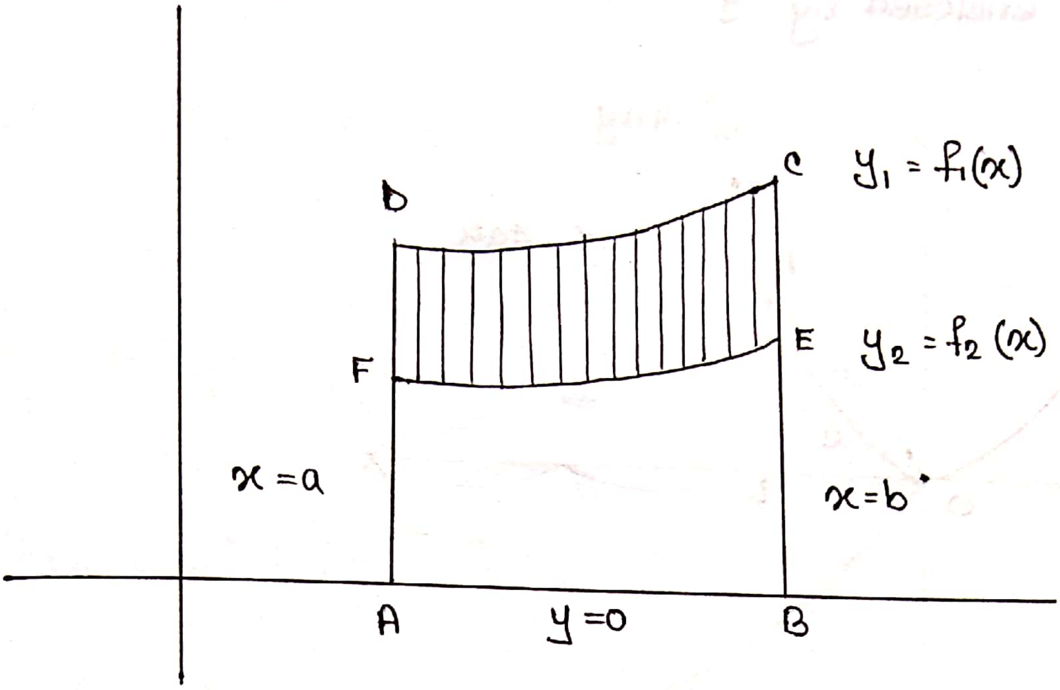
Area of one loop $= 2 \times \frac{\pi a^2}{16}$

$$= \frac{\pi a^2}{8} \text{ sq units}$$

$\therefore$ Total area $= 4 \times \frac{\pi a^2}{8}$

$$= \frac{1}{2} \pi a^2 \text{ sq units.}$$

#



$$\text{Area } ABCD = \int_{x=a}^{b} y_1\,dx$$

$$\text{Area } ABEF = \int_{x=a}^{b} y_2\,dx$$

$$\text{Area } ECDF = \text{Area } ABCD - \text{Area } ABEF$$

$$= \int_{x=a}^{b} y_1\,dx - \int_{x=a}^{b} y_2\,dx$$

$$= \int_{x=a}^{x=b} (y_1 - y_2)\,dx$$

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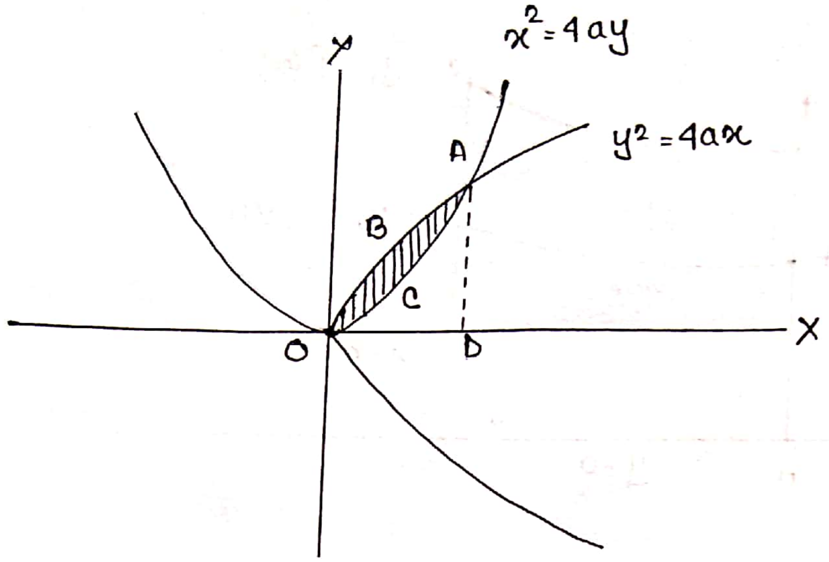
Ex: Find the area enclosed by $y^2=4ax$ and $x^2=4ay$.

$\Rightarrow$



The given two parabolas are symmetric about x axis and y axis respectively.

Now the intersection points of the two parabolas are O(0,0), A (4a, 4a)

$\therefore$ Area OCABO = Area ODABO − Area ODACO

$= A_1 - A_2$

$x^2 = 4ay$

and $y^2 = 4ax$

$\Rightarrow \left(\frac{y^2}{4a}\right)^2 = 4ay$

$\Rightarrow y^4 = (4a)^3 y$

$\Rightarrow y\,(y^3 - (4a)^3) = 0$

$\Rightarrow$ $y=0$ $\quad x=0$

$y = 4a$ $\quad x = 4a$

where,

$$A_1 = \int_{x=0}^{4a} \sqrt{4a}\; x^{\frac{1}{2}}\, dx$$

$$= \sqrt{4a}\left[\frac{x^{3/2}}{3/2}\right]_{x=0}^{4a}$$

$$= \frac{2}{3}\sqrt{4a}\left[(4a)^{3/2}\right]$$

$$= \frac{2}{3}(4a)^2$$

$$= \frac{2}{3}\,16a^2$$

$$= \frac{32a^2}{3} \text{ sq units}$$

$$A_2 = \int_0^{4a} \frac{1}{4a} x^2\, dx$$

$$= \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{1}{12a}(4a)^3 \text{ sq units}$$

Area, $OCABO = \frac{2}{3}(4a)^2 - \frac{1}{12a}(4a)^3$

$$= \frac{16a^2}{3} \text{ sq units}$$

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Ex: Find the area interior to $y^2 = 2ax - x^2$ and exterior to $y^2 = ax$ lying in the first quadrant. Hence find the total area.

→

$$x^2+y^2+2gx+2fy+c=0$$

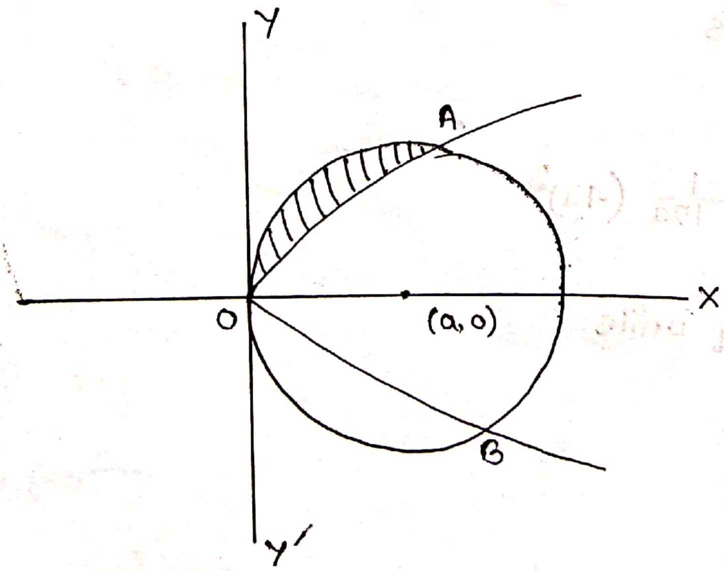
$(-g,-f)$, $r=\sqrt{g^2+f^2-c}$

Here,

$$x^2+y^2-2ax=0$$

$(a,0)$, $r=a$

The first equation $y^2 = 2ax - x^2$ is a circle with the centre at $(a,0)$ and whose radious is a and the 2nd curve is parabola with verten at $(0,0)$ and focus at $(a/4, 0)$. Both curves are symmetric about the x-axis. The two curves intersect each other at $(0,0)$ and $(a, \pm a)$



$$y^2 = 2ax - x^2$$

$$\Rightarrow ax = 2ax - x^2 \quad [y^2 = ax]$$

$$\Rightarrow x^2 - ax = 0$$

$$\Rightarrow x(x-a) = 0$$

$$\Rightarrow x = 0, a$$

$$y = 0, \pm a$$

Therefore the area included between the curves in the first quardant is.

$$= \int_0^a \{ f_1(x) - f_2(x) \}$$

where,

$$f_1(x) = \sqrt{2ax - x^2}$$

$$f_2(x) = \sqrt{ax}$$

$$= \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{ax} \right) dx$$

$$= \int_0^a \sqrt{2ax - x^2} \, dx - \int_0^a \sqrt{ax} \, dx$$

$$= I_1 - I_2$$

Where, $I_1 = \int_0^a \sqrt{2ax - x^2} \, dx$

$$= \int_0^a \sqrt{a^2 - (a-x)^2} \, dx$$

$$= -a^2 \int_{\frac{\pi}{2}}^0 \cos^2\theta \, d\theta$$

let,

$a - x = a\sin\theta$

$-dx = a\cos\theta \, d\theta$

limit $\rightarrow$

$x = 0, \quad \theta = \pi/2$

$x = a, \quad \theta = 0$

$$= a^2 \int_0^{\pi/2} \frac{1}{2} (\cos 2\theta + 1)$$

$$= \frac{a^2}{2} \left[\frac{\sin 2\theta}{2} + \theta \right]_0^{\pi/2}$$

$$= \frac{\pi a^2}{4} \text{ sq. unit.}$$

and,

$$I_2 = \int_0^a \sqrt{ax}\, dx$$

$$= a^{1/2} \left[\frac{x^{3/2}}{3/2} \right]_0^a$$

$$= \frac{2}{3} a^2 \text{ sq. unit.}$$

Thus the area $= \left(\frac{\pi a^2}{4} - \frac{2}{3} a^2 \right)$ sq units

$$= a^2 \left(\frac{\pi}{4} - \frac{2}{3} \right) \text{ sq. units}$$

Hence the total area $= 2a^2 \left(\frac{\pi}{4} - \frac{2}{3} \right)$ sq. units.

Ex: Determine the area inside the circle $r = \sin\theta$ and outside the cardioide $r = 1-\cos\theta$.

$\Rightarrow$ Given that,

$$r = \sin\theta = 1-\cos\theta$$

$$\Rightarrow \sin^2\theta = (1-\cos\theta)^2$$

$$\Rightarrow (1-\cos\theta)(1+\cos\theta) = (1-\cos\theta)^2$$

$$\Rightarrow (1-\cos\theta)(1+\cos\theta-1+\cos\theta) = 0$$

$$\Rightarrow 2\cos\theta(1-\cos\theta) = 0$$

$$\Rightarrow \theta = \pi/2, 0$$

Therefore the intersection points are $(0,0)$ and $(1, \pi/2)$

Thus the required area $= \int_0^{\pi/2} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \left\{\sin^2\theta - (1-\cos\theta)^2\right\} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (-1 + 2\cos\theta - \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[-\theta + 2\sin\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[-\frac{\pi}{2} + 2\right]$$

$$= \left(1 - \frac{\pi}{4}\right) \text{ sq units.}$$