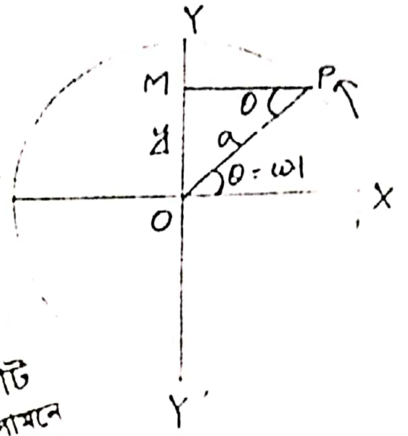
Simple harmonic motion:

Whenever a force acting on a particle and hence the acceleration of the particle is proportional to its displacement from its equilibrium position or any other fixed point in its path, but is always directed in a direction opposite to the direction of the displacement and if the maximum displacement of the particle is the same on either side of the mean position, the particle is said to execute a simple harmonic motion (SHM).

Derive the differential equation of a body executing simple harmonic motion?



Let P be a particle moving on the circumference of a circle of radius a with a uniform velocity V. Let ω be the uniform angular velocity of the particle. At any instant the distance of M from the centre O of the circle is called the displacement. If the particle moved from X to P in time t, then

$$\angle POX = \angle MPO = \theta = \omega t$$

From the ΔMPO

$$\sin\theta = \sin\omega t = \frac{OM}{a}$$

$$OM = y = a\sin\omega t \quad -(1)$$

OM is called the displacement of the vibrating particle. The maximum displacement of a vibrating particle is called its amplitude.

$$\text{Displacement} = y = a\sin(\omega t + \alpha)$$

Here, y is the displacement and a is the amplitude and α is epoch of the vibrating particle.

$$\frac{dy}{dt} = a\omega\cos(\omega t + \alpha)$$

$$\frac{d^2y}{dt^2} = -a\omega^2\sin(\omega t + \alpha) = -\omega^2 y$$

$$\boxed{\frac{d^2y}{dt^2} + \omega^2 y = 0}$$

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = a\omega\cos\omega t \quad -(2)$$

Here $\frac{dy}{dt}$ represents the velocity of the vibrating particle.

Differentiating equation (2) with respect to time,

$$\frac{d^2y}{dt^2} = -a\omega^2\sin\omega t = -\omega^2 . a\sin\omega t = -\omega^2 y$$

$$\boxed{\frac{d^2y}{dt^2} + \omega^2 y = 0}$$ represents the differential equation of a simple Harmonic Motion.

Here $\frac{d^2y}{dt^2}$ represents the acceleration of the particle

The time period of a vibrating particle can be calculated from equation

$$\omega = \sqrt{\frac{d^2y/dt^2}{y}}$$

$$\omega = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}} = 2\pi\sqrt{K}$$

K is the displacement per unit acceleration.

Graphical Representation of SHM

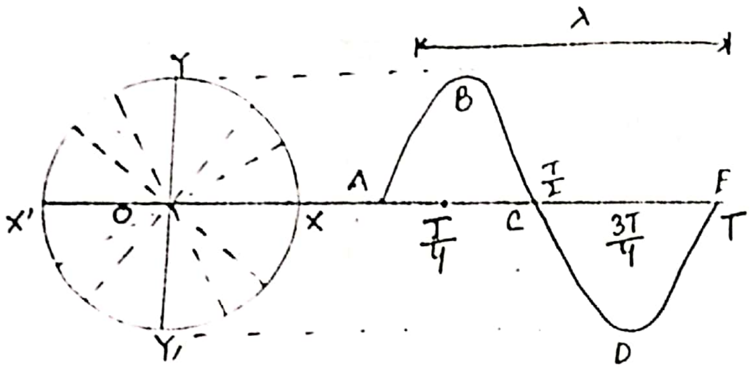


Fig: Displacement time curve

The displacement curve is a sine curve represented by ABCDF

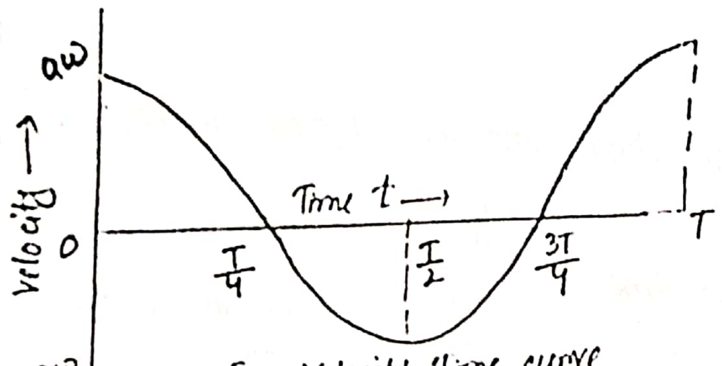


Fig: Velocity time curve

$V = \frac{dy}{dt} = a\omega\cos\omega t$

The velocity-time curve is a cosine curve

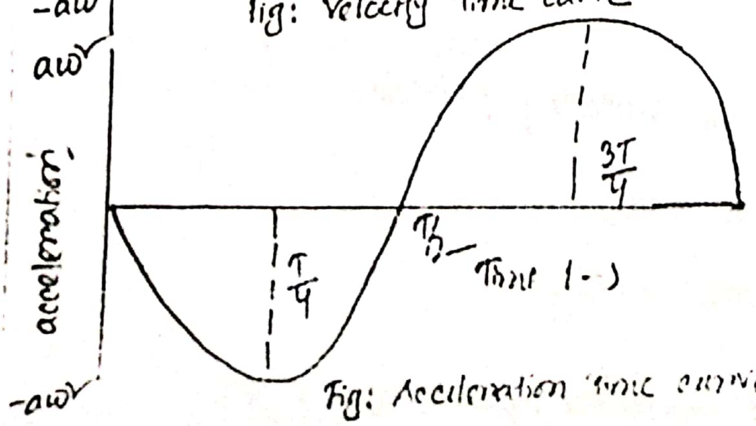


Fig: Acceleration time curve.

$a = \frac{d^2y}{dt^2} = -a\omega^2\sin\omega t$

The acceleration time graph is a negetive sine graph.

culate the average kinetic energy and the total energy of a body executing simple Harmonic motion?

The displacement of a vibrating particle is given by

$$y = a\sin(\omega t + \alpha)$$

$$v = \frac{dy}{dt} = a\omega\cos(\omega t + \alpha)$$

If m is the mass of the vibrating particle, the kinetic energy at any instant $= \frac{1}{2}mv^2 = \frac{1}{2}m \cdot a^2\omega^2\cos^2(\omega t + \alpha)$

The $\boxed{\text{average}}$ kinetic energy of the particle in $\boxed{\text{one}}$ complete vibration

$$= \frac{1}{T}\int_0^T \frac{1}{2}ma^2\omega^2\cos^2(\omega t + \alpha)\,dt$$

$$= \frac{1}{T} \cdot \frac{ma^2\omega^2}{4}\int_0^T 2\cos^2(\omega t + \alpha)\,dt$$

$$= \frac{ma^2\omega^2}{4T}\int_0^T \left[1 + \cos 2(\omega t + \alpha)\right]_0^T dt$$

$$= \frac{ma^2\omega^2}{4T}\left[\int_0^T dt + \int_0^T \cos 2(\omega t + \alpha)\,dt\right]$$

But, $\int_0^T \cos 2(\omega t + \alpha)\,dt = 0$

$$\boxed{\text{Average K.E}} = \frac{ma^2\omega^2}{4T} \cdot T + 0$$

$$= \frac{ma^2\omega^2}{4}$$

$$= \frac{ma^2(2\pi n)^2}{4}$$

$$= \frac{ma^2 \times 4\pi^2 n^2}{4}$$

$$= \boxed{\pi^2 m a^2 n^2}$$

where m is the mass of the vibrating particle, a is the amplitude of vibration and n is the frequency of vibration.

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Total energy of a vibrating particle:

$$\boxed{\cos\theta = \sqrt{1-\sin^2\theta}}$$

$$y = a\sin(\omega t+\alpha)$$

$$\sin(\omega t+\alpha) = \frac{y}{a}$$

$$\cos(\omega t+\alpha) = \sqrt{1-\frac{y^2}{a^2}} = \sqrt{\frac{a^2-y^2}{a^2}} = \frac{\sqrt{a^2-y^2}}{a}$$

Velocity $v = a\omega\cos\omega t = a\omega \cdot \frac{\sqrt{a^2-y^2}}{a} = \omega\sqrt{a^2-y^2}$

The kinetic energy of the particle at the instant the displacement is y

$$= \frac{1}{2}mv^2 = \boxed{\frac{1}{2}m\omega^2(a^2-y^2)}$$

Potential energy of the vibrating particle is the amount of work in overcoming the force through a distance y.

$$\text{Acceleration} = -\omega^2 y$$

$$\text{force} = -m\omega^2 y$$

$$P.E = \int_0^y m\omega^2 y\,dy = m\omega^2 \cdot \frac{y^2}{2} = \boxed{\frac{1}{2}m\omega^2 y^2}$$

Total energy of the particle at the instant the displacement is y

$$= K.E + P.E$$

$$= \frac{1}{2}m\omega^2(a^2-y^2) + \frac{1}{2}m\omega^2 y^2$$

$$= \frac{1}{2}m\omega^2 a^2$$

$$= \frac{1}{2}m(2\pi n)^2 a^2$$

$$= \boxed{2\pi^2 m a^2 n^2}$$

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Calculate the simple harmonic motion of a loaded spring.

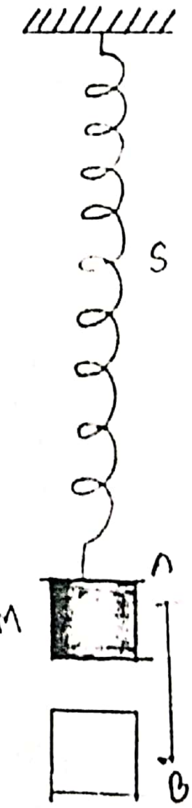
Simple harmonically motion of a loaded spring.

Consider a spring S whose upper end is fixed to a rigid support and the lower end is attached to a mass A. In the equilibrium position, the mass is at A. When the mass is displaced, it oscillates simple harmonically in the vertical direction.

Suppose, at any instant, the mass is at B. The distance $AB = y$. Let the tension per unit displacement of the spring be K.

Force exerted by the spring $= Ky$

According to Newton's second law,



$$Force = M.\frac{d^2y}{dt^2} = -Ky$$

$$M\frac{d^2y}{dt^2} + Ky = 0$$

$$\frac{d^2y}{dt^2} + \left(\frac{K}{M}\right)y = 0 \quad \text{--(I)}$$

The equation is similar to the equation of simple harmonic motion.

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{--(II)}$$

Comparing (I) and (II)

$$\omega^2 = \frac{K}{M}$$

$$\omega = \sqrt{\frac{K}{M}}$$

Time period, $T = \frac{2\pi}{\omega}$

$$T = 2\pi\sqrt{\frac{M}{K}}$$

$$\boxed{= 2\pi\sqrt{\frac{Mx}{mg}}} \qquad \left[K = \frac{mg}{x}\right]$$

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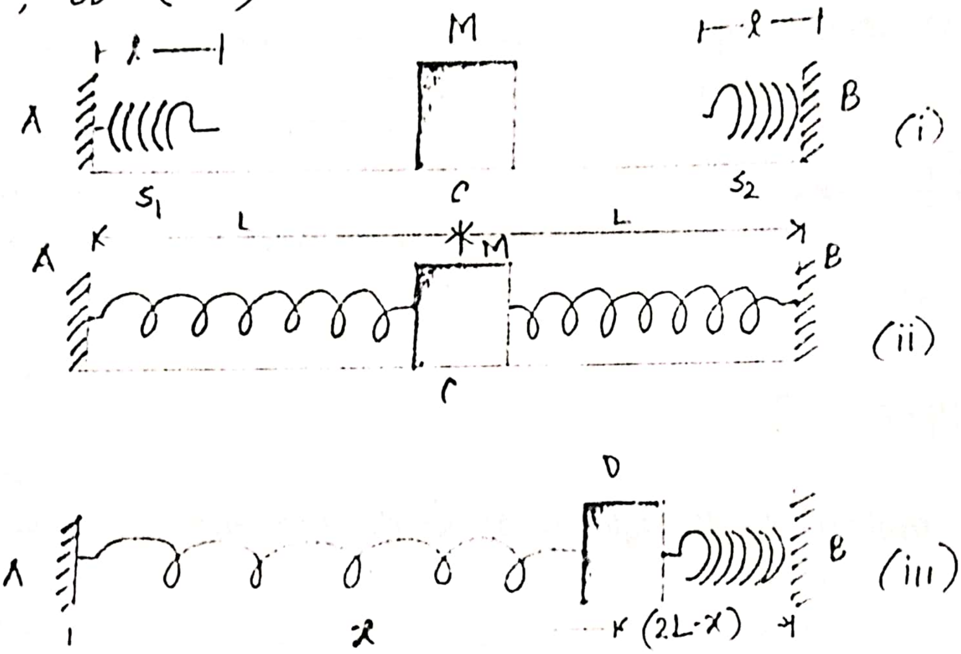
Calculate simple harmonic motion of a mass between two springs.

Simple harmonic motion of a mass between two springs:

Consider two springs S_1 and S_2 each having a length l in the relaxed position. Mass M is placed midway between the two springs on a frictionless surface. $AC = BC = L$

When the mass M is displaced, it executes simple harmonic motion. Let any instant, D be the displaced position of mass M.

Here $AD = x$, $BD = (2L-x)$



Let the tension per unit displacement in the spring be K. The displacement of the spring S_1 is $(x-l)$ and it exerts a force $= K[x-l]$ in the direction DA. The displacement of the spring S_2 is $(2L-x-l)$ and it exerts a force $= K(2L-x-l)$ in the direction DB.

The resultant force on the mass M $= K[2L-x-l] - K[x-l]$ in the direction DB

$= -2K[x-L]$ in the direction DB

According to Newton's second law of motion,

$$F = M\frac{d^2x}{dt^2} = -2K[x-L]$$

$$M\frac{d^2x}{dt^2} + 2K[x-L] = 0 \quad \text{--- (1)}$$

Taking the displacement from the mean position

$$x - L = y$$

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$$

From equation (I) we have,

$$M \cdot \frac{d^2y}{dt^2} + 2K(x-L) = 0$$

$$M \frac{d^2y}{dt^2} + 2Ky = 0$$

$$\boxed{\frac{d^2y}{dt^2} + \left(\frac{2K}{M}\right) y = 0 \quad \text{— (II)}}$$

we know that the equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{— (III)}$$

from equation (II) and (III) we can write,

$$\omega^2 = \frac{2K}{M}$$

$$\omega = \sqrt{\frac{2K}{M}}$$

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{M}{2K}}}$$

Lissajous figures

What is Lissajous figures? Write it uses?

When a particle is influenced simultaneously by two simple harmonic motions at right angle to each other, the resultant motion of the particle traces a curve. These curves are called a Lissajous figures.

Lissajous figures depends on the time period, phase difference and the amplitude of the two constituent vibrations.

Uses:

(I) To determine the ratio of the time period.

(II) Determination of the frequency of a tuning fork.

Calculate the composition of two simple harmonic motions of equal time p

Let us consider two simple harmonic motions of the same time period and are of different amplitudes and different phase angles, acting simultaneously on a particle in perpendicular directions. If α is the phase difference between the two vibrations, then their equations can be written as,

$$x = a\sin(\omega t + \alpha) \quad \text{(I)}$$

$$y = b\sin\omega t \quad \text{— (II)}$$

From equation (I).

$$\frac{x}{a} = \sin(\omega t + \alpha) = \sin\omega t \cdot \cos\alpha + \cos\omega t \cdot \sin\alpha$$

$$= \sin\omega t \cdot \cos\alpha + \sqrt{1-\sin^2\omega t} \cdot \sin\alpha \quad \text{— (III)}$$

From equation (II).

$$\frac{y}{b} = \sin\omega t$$

$$\therefore \cos\omega t = \sqrt{1-\frac{y^2}{b^2}}$$

Putting the value of $\sin\omega t$ and $\cos\omega t$ in equation (III)

$$\frac{x}{a} = \left[\frac{y}{b}\cos\alpha + \sqrt{1-\sin^2\omega t} \cdot \sin\alpha\right]$$

$$\frac{x}{a} = \left[\frac{y}{b}\cos\alpha + \sqrt{1-\frac{y^2}{b^2}}\sin\alpha\right] \quad \left(\frac{x}{a} - \frac{y}{b}\cos\alpha\right) = \sqrt{1-\frac{y^2}{b^2}}\sin\alpha$$

Squaring both sides, we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\alpha - \frac{2xy}{ab}\cos\alpha = \left(1-\frac{y^2}{b^2}\right)\sin^2\alpha$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\alpha - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha - \frac{y^2}{b^2}\sin^2\alpha$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}(\sin^2\alpha + \cos^2\alpha) - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha} \quad \text{— (IV)}$$

This represents the general equation of an $\boxed{\text{ellipse}}$.

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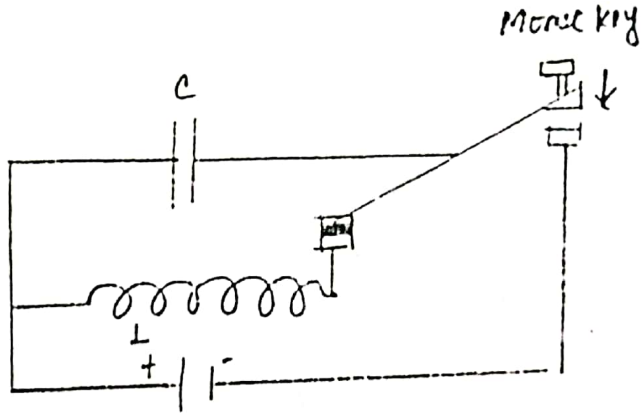
Discuss the LC circuit and calculate the expression for the frequency of oscillations.

A condenser is charged by pressing the Morse key and when the Morse key is released, the condenser gets discharged through the Inductance L. Let the charge on the condenser at any instant of time be Q.

$$\boxed{\frac{Q}{C} + L\frac{dI}{dt} = 0}$$



$$L\frac{dI}{dt} = -\frac{Q}{C}$$

$$L.\frac{dI}{dt} = -\frac{1}{C}.Q$$

$$L.\frac{d^2I}{dt^2} = -\frac{1}{C}\frac{dQ}{dt} \quad \text{(I)}$$

But, $\frac{dQ}{dt} = I$

from (I)

$$L.\frac{d^2I}{dt^2} = -\frac{1}{C}I$$

$$\frac{d^2I}{dt^2} = -\frac{1}{LC} \times I$$

$$\frac{d^2I}{dt^2} + \left(\frac{1}{LC}\right)I = 0 \quad \text{(II)}$$

This equation is similar to the equation of simple harmonic motion.

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{(III)}$$

From equation (II) and (III)

$$\omega^2 = \frac{1}{LC}$$

$$\omega = \sqrt{\frac{1}{LC}}$$

$$\omega = \frac{2\pi}{T}$$

Time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$

Frequency, $n = \frac{1}{T} = \boxed{\frac{1}{2\pi\sqrt{LC}}}$

Special cases:

① If $\alpha = 0$ or 2π

$\cos\alpha = 1$, $\sin\alpha = 0$

The equation (iv) becomes,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

$$\frac{x}{a} = \frac{y}{b}$$

$$\boxed{y = \frac{b}{a}x}$$

This represents the equation of the straight line.

(II) If $\alpha = \pi$, $\sin\alpha = 0$, $\cos\alpha = -1$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

$$\frac{x}{a} = -\frac{y}{b}$$

$$\boxed{y = -\frac{b}{a}x}$$

This represents the equation of straight line.

(III) If $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\sin\alpha = 1$, $\cos\alpha = 0$

$$\therefore \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

This represents equation of the ellipse.

(IV) If $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$a = b$

$$\therefore \boxed{x^2 + y^2 = a^2}$$

This represent the equation of a circle.

(V) $\alpha = \frac{\pi}{4}$ $\sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{\sqrt{2}xy}{ab} = \frac{1}{2}$$

This represent the equation of $\boxed{\text{oblique ellipse}}$.

(VI) $\alpha = \frac{3\pi}{4}$, $\sin\alpha = \frac{1}{\sqrt{2}}$

$\cos\alpha = -\frac{1}{\sqrt{2}}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\sqrt{2}xy}{ab} = \frac{1}{2}$$

$\boxed{\text{oblique ellipse}}$