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Wave Motion

[redacted]

17 **Wave motion:** Wave motion, is a form of disturbance which travels through the medium due to the repeated periodic motion of the particles of the medium about their mean position, the disturbance being handed over from one particle to the next.

Characteristics of wave motion:

1. Wave motion is a disturbance produced in the medium by the repeated periodic motion of the particles of the medium.
2. Only the wave travels forward, whereas the particles of the medium vibrate about their mean positions.
3. There is a regular phase change between the various particles of the medium.
4. The velocity of the wave is different from the velocity with which the particles of the medium that are vibrating about their mean positions.

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Q. Derive the equation of wavemotion in the form of $y = a\sin\frac{2\pi}{\lambda}(vt - x)$

Consider a particle O in the medium. Let the displacement at any instant of time be given by,

$$y = a\sin\omega t \quad \text{---(1)}$$

Consider another particle A at a distance x from the particle O. The displacement of A is given by

$$y = a\sin(\omega t - \alpha) \quad \text{---(2)}$$

Where α is the phase difference between the particles O and A.

For a phase difference of 2π, the path difference is λ. suppose for a phase difference of α, the path difference is x.

$$\frac{\alpha}{x} = \frac{2\pi}{\lambda}$$

$$\alpha = \frac{2\pi x}{\lambda}$$

Also, $\omega = \frac{2\pi}{T} = \frac{2\pi v}{\lambda}$

Substituting the values of α and ω in equation (2)

$$y = a\sin\left(\frac{2\pi vt}{\lambda} - \frac{2\pi x}{\lambda}\right)$$

$$\boxed{y = a\sin\frac{2\pi}{\lambda}(vt - x)}$$

which represents the equation of wavemotion.

Q. **Differential Equation of wavemotion**

(2017) The general equation of a simple harmonic motion

$$\boxed{y = a\sin\frac{2\pi}{\lambda}(vt-x) \quad -(1)}$$

Differentiating eqn -(1) with respect to time,

$$\frac{dy}{dt} = \frac{2\pi a v}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x) \quad -(2)$$

Differentiating eqn-(2) with respect to time,

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2 a v^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt-x) \quad -(3)$$

Differentiating eqn-① with respect to x to find the value of compression.

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt-x) \quad —④$$

Differentiating eqn-④ with respect to x

$$\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt-x) \quad —⑤$$

From eqn ② and ④

$$\frac{dy}{dt} = -v\frac{dy}{dx} \quad —⑥$$

$$\boxed{\frac{dy}{dt} = -v\frac{dy}{dx}}$$

Particle velocity = (– wave velocity) × (slope of the compression curve)

Show that: $\boxed{U = -v\frac{dy}{dx}}$

where U is the particle velocity, v is the wave velocity.

equation ③ and ⑤

$$\boxed{\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}} \quad —⑦$$

Equation ⑦ represents the differential equation of wave motion. the general differential equation of wave motion can be written as,

$$\frac{d^2y}{dt^2} = K.\frac{d^2y}{dx^2}$$

$v = \sqrt{K}$

$\therefore v^2 = K$

$\therefore \boxed{v = \sqrt{K}}$

Distribution of velocity and pressure in a plane progressive wave

For a plane progressive wave

$$y = a\sin\frac{2\pi}{\lambda}(vt-x) \quad \text{--- (1)}$$

The particle velocity,

$$U = \frac{dy}{dt} = \frac{2\pi a v}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)$$

The strain in the medium is $\frac{dy}{dx}$. When $\frac{dy}{dx}$ is $\boxed{\text{positive}}$, it represents a region of $\boxed{\text{rarefaction.}}$ If $\frac{dy}{dx}$ is $\boxed{\text{negative}}$, it represents a region of $\boxed{\text{compression}}$. The bulk modulus of elasticity of the medium,

$$K = \frac{\text{change in pressure}}{\text{volume strain}}$$

$$K = \frac{-dP}{dy/dx}$$

$$dP = -K\left(\frac{dy}{dx}\right)$$

$$\boxed{dP = K\left(-\frac{dy}{dx}\right)}$$

Differentiating eqn (1) with respect to x,

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)$$

$$dP = K\left(-\frac{dy}{dx}\right)$$

$$\boxed{dP = \frac{2\pi K a}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)}$$

Show that the energy of a plane $\boxed{\text{progressive}}$ wave is given by $\boxed{2\pi^2 \rho n^2 a^2}$

The equation of simple harmonic wave is

$$y = a\sin\frac{2\pi}{\lambda}(vt-x) \quad \text{--- (1)}$$

The particle velocity

$$U = \frac{dy}{dt} = \frac{2\pi a v}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x) \quad \text{--- (2)}$$

The acceleration of the particle,

$$f = \frac{du}{dt} = \frac{d^2y}{dt^2} = -\frac{4\pi^2 a v^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt-x) \quad \text{--- (3)}$$

The -ve sign shows that the acceleration is directed towards the mean position.

$\boxed{\text{Potential Energy}}$

To move the particle from its mean position to a distance y, work done for a displacement $dy = f dy$

Work done per unit volume for a displacement dy

$$= \rho\left(\frac{4\pi^2 a v^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt-x)\right)dy$$

$\boxed{F = \frac{mf}{V} = \rho f}$ → Per unit volume

where, ρ be the density of the medium.

Total work done for a displacement y

$$= \int_0^y \rho \left(\frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt-x) \right) dy$$

But, $y = a \sin \frac{2\pi}{\lambda}(vt-x)$

Potential energy per unit volume

$$= \left(\frac{4\pi^2 \rho v^2}{\lambda^2} \right) \int_0^y y \, dy$$

$$= \frac{4\pi^2 \rho v^2 y^2}{2\lambda^2}$$

$$P.E = \frac{2\pi^2 \rho v^2 y^2}{\lambda^2}$$

$$\boxed{P.E = \frac{2\pi^2 \rho v^2}{\lambda^2} a^2 \sin^2 \left[\frac{2\pi}{\lambda}(vt-x) \right]}$$

Kinetic energy per unit volume,

$$K.E = \frac{1}{2} \frac{m}{V} \left(\frac{dy}{dt} \right)^2$$

$$= \frac{1}{2} \rho \left(\frac{dy}{dt} \right)^2$$

$$= \frac{1}{2} \rho \left[\frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda}(vt-x) \right]^2$$

$$K.E = \frac{1}{2} \rho . \frac{4\pi^2 a^2 v^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda}(vt-x)$$

$$\boxed{K.E = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda}(vt-x)}$$

Total energy per unit volume,

$$\boxed{E = PE + KE}$$

$$= \frac{2\pi^2 \rho v^2 a^2}{\lambda^2}\left[\sin^2 \frac{2\pi}{\lambda}(vt-x) + \cos^2 \frac{2\pi}{\lambda}(vt-x)\right]$$

$$E = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2}$$

But, $\boxed{v = n\lambda}$

$$E = \frac{2\pi^2 \rho n^2 \lambda^2 a^2}{\lambda^2}$$

$$\boxed{E = 2\pi^2 \rho n^2 a^2}$$

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Q. Show that in the case of a $\boxed{\text{stationary}}$ wave $\boxed{\text{no}}$ energy is transferred across any section of the medium.

$$y_1 = a\sin \frac{2\pi}{\lambda}(vt-x) \quad \text{—(i)}$$

$$y_2 = a\sin \frac{2\pi}{\lambda}(vt+x) \quad \text{—(ii)}$$

For stationary wave,

$$y = y_1 + y_2$$

$$= 2a\left[\sin \frac{2\pi}{\lambda} vt + \cos \frac{2\pi}{\lambda} x\right]$$

$$y = 2a\cos \frac{2\pi}{\lambda} x \, \sin \frac{2\pi}{\lambda} vt \quad \text{—(iii)}$$

Now, $$\frac{dy}{dt} = \frac{4\pi a v}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda} \quad \text{—(iv)}$$ [differentiating eqn (iii)

Again, $\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi v t}{\lambda}$ — (V)

Now,

$$E = \frac{-P}{\frac{dy}{dx}}$$

$$\boxed{P = -E\frac{dy}{dx}} \quad \text{— (VI)}$$

Again,

$$v = \sqrt{\frac{E}{\rho}}$$

$$v^2 = \frac{E}{\rho}$$

$$\boxed{v^2\rho = E} \quad \text{— (VII)}$$

$$P = -v^2\rho \cdot \frac{dy}{dx} \quad \text{— (VIII)}$$

From (V) taking value of $\frac{dy}{dx}$

⇒(VIII)

$$P = -v^2\rho\left[-\frac{4\pi a}{\lambda} \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi v t}{\lambda}\right]$$

$$\boxed{P = v^2\rho\,\frac{4\pi a}{\lambda} \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi v t}{\lambda}}$$

Maximum pressure P_0,

$$\therefore P_0 = \frac{v^2\rho \cdot 4\pi a}{\lambda}$$

$$\therefore \boxed{P = P_0 \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi v t}{\lambda}}$$

let, $P_x = P_0 \sin \frac{2\pi x}{\lambda}$

$$\boxed{P = P_x \sin \frac{2\pi v t}{\lambda}}$$

Particle velocity-

$$\frac{dy'}{dx} = U = \frac{dy}{dt} = \boxed{\frac{4\pi a v}{\lambda} \cos \frac{2\pi x}{\lambda}} \cos \frac{2\pi v t}{\lambda}$$

$$\boxed{U = U_x \cos \frac{2\pi v t}{\lambda}}$$

Work done,

$$W = \int_0^T P.U \, dt$$

$$= \int_0^T P_x U_x \sin \frac{2\pi v t}{\lambda} \cos \frac{2\pi v t}{\lambda} \, dt$$

Rate of energy transfer of stationary wave

$$= \frac{1}{T} \int_0^T P_x U_x \sin \frac{2\pi v t}{\lambda} \cos \frac{2\pi v t}{\lambda} \, dt$$

$$= \frac{1}{2T} \int_0^T P_x U_x \, 2\sin \frac{2\pi v t}{\lambda} \cos \frac{2\pi v t}{\lambda} \, dt$$

$$= \frac{1}{2T} \int_0^T P_x U_x \sin \frac{4\pi v t}{\lambda} \, dt$$

$$= \frac{P_x U_x}{2T} \int_0^T \sin \frac{4\pi v t}{\lambda} \, dt$$

$$= \boxed{0}$$ (showed)

Transverse wave motion

In this type of wave motion, the particle of the medium vibrate at right angles to the direction of propagation of the wave.

Longitudinal wave motion

In this type of wave motion, the particle of the medium vibrate along the direction of the propagation of the wave.