Wave and Oscillation

What is simple harmonic motion? derive the differential equation of it.

Simple harmonic motion: Whenever a force acting on a particle and hence the acceleration of the particle is proportional to its displacement from its equilibrium position or any other fixed point in its path but is always directed in an opposite direction to ~~di~~ the direction of the displacement and if the maximum displacement of the particle is same on either side of the mean position.

Differential equation of SHM:

The general equation of displacement of a particle vibrating simple harmonically is

$$y = a\sin(\omega t+\alpha) \quad \text{—(i)}$$

Here,

y is the displacement and a is the amplitude and α is epoch of the vibrating particle.

Differentiating eq(i) with respect to time, we get,

$$\frac{dy}{dt} = \omega a\cos(\omega t+\alpha) \quad \text{—(ii)}$$

Here, $\frac{dy}{dt}$ represents the velocity of the vibrating patricle.

Differentiating equation (2) with respect to time, we get,

$$\frac{d^2y}{dt^2} = -\omega^2 a \sin(\omega t+\alpha)$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\omega^2 y \qquad [\text{Since, } y = a\sin(\omega t+\alpha)]$$

$$\Rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0$$

$$\therefore \frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{------ (3)}$$

Here,

$\frac{d^2y}{dt^2}$ represents the acceleration of the particle।

Equation (3) represents the differential equation of Simple harmonic motion.

প্র] Find the value of ω and T from the differential equation of SHM.

From the differential equation of SHM; we get,

Numerically, $$\frac{d^2y}{dt^2} = \omega^2 y$$

$$\Rightarrow \omega = \sqrt{\frac{(d^2y/dt^2)}{y}}$$

$$\therefore \omega = \sqrt{\frac{\text{Acceleration}}{\text{displacement}}}$$

Again, we know, $T = 2\pi/\omega$

$$= 2\pi\sqrt{\frac{\text{displacement}}{\text{Acceleration}}}$$

Calculate the average kinetic energy of a body executing simple harmonic motion.

Ans: The displacement of vibrating particle is given by

$$y = a\sin(\omega t + \alpha)$$

$$\therefore v = \frac{dy}{dt} = a\omega\cos(\omega t + \alpha)$$

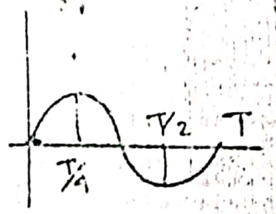
if ~~mis~~ m is the mass of the particle, the kinetic energy at any instant

$$K.E = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m a^2\omega^2\cos^2(\omega t + \alpha)$$

The average K.E of the particle in one complete vibration.

$$\text{Average, } K.E = \frac{1}{T}\int_0^T \frac{1}{2}ma^2\omega^2\cos^2(\omega t+\alpha)\,dt$$



$$= \frac{1}{T}\frac{ma^2\omega^2}{4}\int_0^T 2\cos^2(\omega t+\alpha)\,dt$$

$$= \frac{ma^2\omega^2}{4T}\int_0^T \{1+\cos 2(\omega t+\alpha)\}\,dt$$

$$= \frac{ma^2\omega^2}{4T}\left[\int_0^T dt + \int_0^T \cos 2(\omega t+\alpha)\,dt\right]$$

$$= \frac{ma^2\omega^2}{4T}\cdot T + 0 \qquad \left[\text{here, } \int_0^T \cos 2(\omega t+\alpha) = 0\right]$$

$$= \frac{ma^2 4\pi^2 n^2 T}{4T} \qquad [\omega = 2\pi n]$$

$$= m\pi^2 a^2 n^2$$

Where,

m is = mass

a = amplitude

n = frequency of particle.

$\therefore$ Average K.E $\propto$ a constant . a^2 [Here, $m\pi^2n^2$ = constant]

Average $\therefore$ KE $\propto a^2$ (showed)

[1] Calculate the total energy of a body executing simple harmonic motion.
or show that the total energy at any instant is constant.

Ans: The displacement of a particle executing simple harmonic motion is

$$y = a\sin(\omega t + \alpha)$$

$$\Rightarrow \sin(\omega t + \alpha) = \frac{y}{a}$$

Again, $\cos(\omega t+\alpha) = \sqrt{1-\sin^2(\omega t+\alpha)}$

$$= \sqrt{1-\frac{y^2}{a^2}}$$

$$= \sqrt{\frac{a^2-y^2}{a^2}}$$

$$= \frac{\sqrt{a^2-y^2}}{a}$$

Velocity of the particle, $v = \frac{dy}{dt}$

$$= \frac{d}{dt}\left(a \sin(\omega t + \alpha)\right)$$

$$= \omega a \cos(\omega t + \alpha)$$

$$= \omega \sqrt{a^2 - y^2}$$

The kinetic energy of instant displacement is

$$K.E = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \omega^2 (a^2 - y^2)$$

potential energy of the vibrating particle is the amount of workdone overcoming the force through a displacement y.

$\therefore$ Acceleration $= -\omega^2 y$

Force $= -m\omega^2 y$

potential energy for a displacement dy is, $dE_p = -F\,dy$

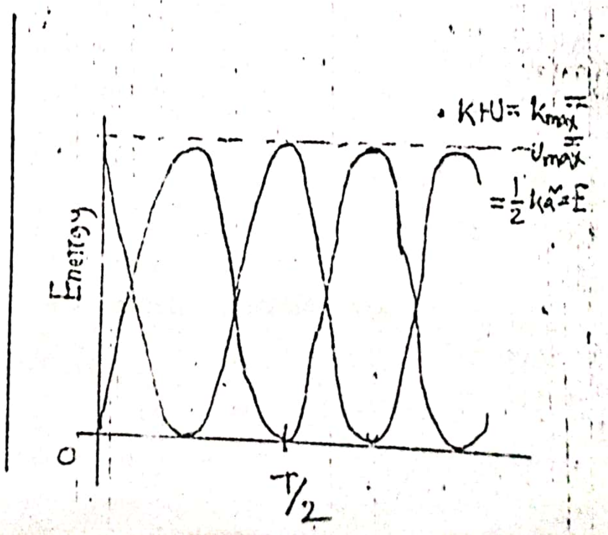
$\therefore$ Total amount of potential energy for a displacement y is

$$P.E = \int_0^y m\omega^2 y \, dy$$

$$= m\omega^2 \int_0^y y \, dy$$

$$= m\omega^2 \left[\frac{y^2}{2}\right]_0^y$$

$$= \frac{1}{2} m \omega^2 y^2$$



∴ Total Energy, $E_T = K.E + P.E$

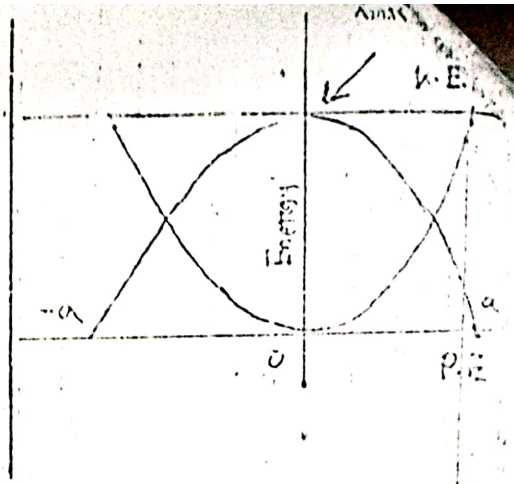
$$= \frac{1}{2} m\omega^2(a^2 - y^2) + \frac{1}{2} m\omega^2 y^2$$

$$= \frac{1}{2} m a^2 \omega^2$$

$$\therefore \boxed{E_T = \frac{1}{2} m a^2 \omega^2}$$

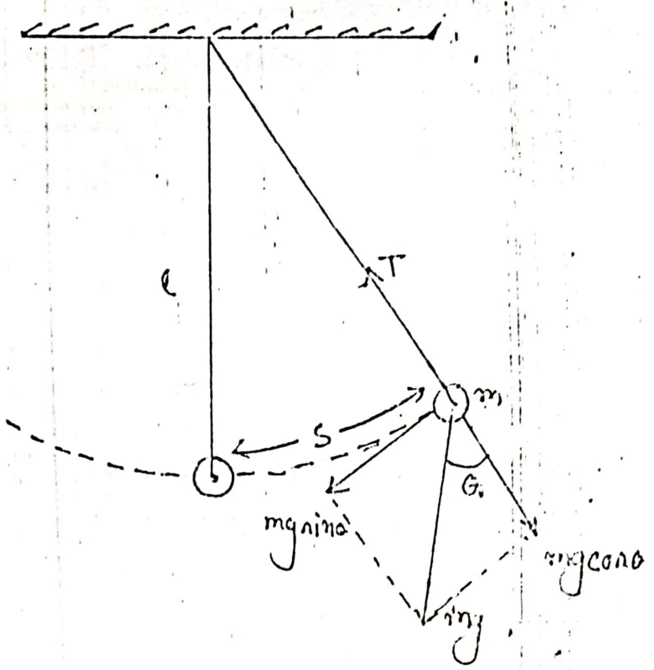
$$= \frac{1}{2} m a^2 \, 4\pi^2 n^2 \qquad [\because \omega = 2\pi n]$$

$$\boxed{\therefore E_T = 2\pi^2 a^2 n^2 m}$$



As the average kinetic energy of the vibrating particle is $\pi^2 m a^2 n^2$, and the average potential energy is $\pi^2 m a^2 n^2$, So the ~~th~~ total energy at any instant is a constant.

Show that the motion of a pendulam is SHM.



Let the mass of the bob be m and the length of the string be l. The path of the bob is not a straight line. The necessary condition for the motion to be simple harmonic is that the restoring force F shall be directly proportional to the co-ordinates and oppositely directed. i.e

$$F = -kx \qquad \text{where } k \text{ is the force constant.}$$

The weight (mg) is resolved into components along the radius and along the tangent to the circle. The restoring force F is

$$F = -mg\sin\theta .$$

However, if the angles θ is small, sinθ is very nearly equal to θ.

i.e. $\sin\theta \simeq \theta$.

$$F = -mg\theta .$$

Scanned by CamScanner

$\Rightarrow F = -mg \cdot \frac{s}{l}$ $\qquad [\because \theta = \frac{s}{l}]$

$\therefore F = -\frac{mg}{l} \cdot s \quad -(\mathrm{I})$

Again, we know, $F = ma$, $\qquad a =$ tangential acceleration.

$\Rightarrow F = m\frac{d^2s}{dt^2} \quad -(\mathrm{II}) \qquad = \frac{d^2s}{dt^2}$

From equation (I) and (II) we get,

$$m\frac{d^2s}{dt^2} = -\frac{mg}{l} \cdot s$$

$$\Rightarrow m\frac{d^2s}{dt^2} + \frac{mg}{l} \cdot s = 0$$

$$\Rightarrow \frac{d^2s}{dt^2} + \frac{g}{l} \cdot s = 0$$

Which is similar to the equation $\frac{d^2y}{dt^2} + \omega^2 y = 0$, so the motion of a pendulum is SHM.