CHAPTER-1

CHANGE OF AXES

Axes: Any two interesting st. lines in the plane may be regarded as axes.

Axes be two kind ==>>

(1)**Rectangular axes:** If the angle between the axes is $\pi/2$. Then this are called Rectangular axes.

(2)**Oblique axes:** If the angle between the axes is not equal to $\pi/2$. Then this axes are called Oblique axes.

1. Theorem: If origin o(0,0) is shie d at o(x,y) n d the effect in the co-ordinate of the point p(x,y).

Solution: Draw perpendicular from O' and P on OX. Suppose the perpendiculars intersect OX at M and M' respectively.

OM=x' and OM'=x',

Shift the origin from O to O'. Suppose O'X' and O'Y' are new axes. Suppose P(x,y) is the co-ordinate of P with respect to new axes.

So, O'N=X, and PN=Y.

Now, OM'=OM+MM'=OM+O'N=x_1+X.

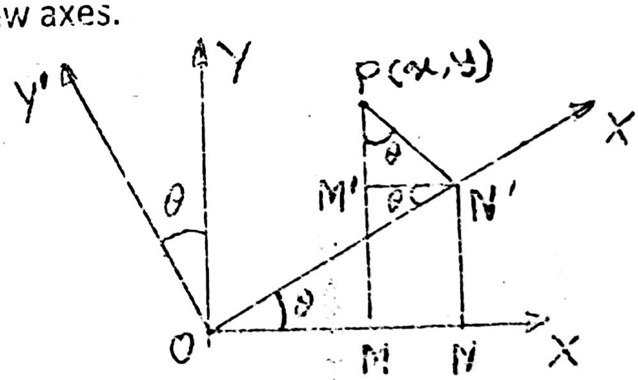
So, X=x-x_1.

Again, PM'=PN+NM'=PN+O'M=Y+y_1.

So, Y=y-y_1.

∴ P(X,Y)=(x-x_1, y-y_1).

Ans.



[Fig: Theorem.2]

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

3.Theorem: Remove the xy term from the expression $ax^{2+}2hxy+by^2$.

Or, Find the rotation angle θ axes.Show that second term of $ax^2+2hxy+by^2$ is removed.[EEE'09]

Or, If(x,y) be the co-ordinates of a point referred to rectangular axes.It is always possible to transform to rectangular axes with the same origin so tha $ax^2+2hxy+by^2$ becomes $a'x'^2+b'y'^2$in which the term in x'y' is wanting,(x',y') being the same point referred to the new axes.

Solution: Suppose, (x,y) be the co-ordinate of P w.r.to OX and OY and (x',y') is the co-ordinate of the same point P w.r.to OX' and OY'.

We know, $x=x'\cos\theta-y'\sin\theta$

And, $y=x'\sin\theta+y'\cos\theta$

Now, $ax^2+2hxy+by^2$

$= a(x'\cos\theta-y'\sin\theta)^2+2h(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)+b(x'\sin\theta+y'\cos\theta)^2$

Hence if the axes are rotated through an angle $\theta=\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$ then xy in the expression $ax^2+2hxy+by^2$ vanishes.

4. Theorem: By the orthogonal transformation without change of origin the expression $ax^2+2hxy+by^2$ may be written as $a_1x'^2+2h_1x'y'+b_1y'^2$ if and only if (i) $a+b=a_1+b_1$ (ii) $ab-h^2=a_1b-h_1^2$. **[Invariant Rule]**

[IPE'12,10ME'06,04,CSE'09,GCE'12]

Solution: If the axes are rotated through an angle θ about the origin, then,

$x=x'\cos\theta-y'\sin\theta$

$y=x'\sin\theta+y'\cos\theta$

putting these values in the expression $ax^2+2hxy+by^2$, we have

$a(x'\cos\theta-y'\sin\theta)^2+2h(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)+b(x'\sin\theta+y'\cos\theta)^2$

$=ax'^2\cos^2\theta-2ax'y'\sin\theta\cos\theta+ay'^2\sin^2\theta+2hx'^2\cos\theta\sin\theta+2hx'\cos^2\theta y'-2hx'y'\sin^2\theta-2hy'^2\sin\theta\cos\theta+bx'^2\sin^2\theta+2bx'y'\sin\theta\cos\theta+by'^2\cos^2\theta$

$=x'^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)+2x'y'(-a\sin\theta\cos\theta+h\cos^2\theta-h\sin^2\theta+b\sin\theta\cos\theta)+y'^2(a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta)$

$=a_1x'^2+2h_1x'y'+b_1y'^2$

Where, $a_1=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta$ .... (i)

$h_1=-a\sin\theta\cos\theta+h\cos^2\theta-h\sin^2\theta+b\sin\theta\cos\theta$ .... (ii)

$b_1=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta$ ....(iii)

from (i) and (ii),

$a_1+b_1=a(\cos^2\theta+\sin^2\theta)+b(\sin^2\theta+\cos^2\theta)$

$\therefore a+b=a_1+b_1$ (proved)

Now, $2a_1=2a\cos^2\theta+4h\cos\theta\sin\theta+2b\sin^2\theta$

$=a(1+\cos2\theta)+2h\sin2\theta+b(1-\cos2\theta)$

$\therefore 2a_1=(a+b)+2h\sin2\theta+(a-b)\cos2\theta$ .... (iv)

Similarly, $2b_1=(a+b)-2h\sin2\theta+(a-b)\cos2\theta$ .... (v)

From (ii),(iv) and (v) we have,

$2a_1.2b_1-(2h_1)^2=(a+b)^2-\{2h\sin2\theta+(a-b)\cos2\theta\}^2-\{2\sin\theta\cos\theta(b-a)+2h(\cos^2\theta-\sin^2\theta)\}^2$

$\Rightarrow 4(a_1b_1-h_1^2)=(a+b)^2-\{2h\sin2\theta+(a-b)\cos2\theta\}^2-\{2h\cos2\theta-(a-b)\sin2\theta\}^2$

$\Rightarrow 4(a_1b_1-h_1^2)=(a+b)^2-4h^2(\cos^22\theta+\sin^22\theta)-(a-b)^2 b(\sin^22\theta+\cos^22\theta)$

$=(a+b)^2-4h^2-(a-b)^2$

$=4ab-4h^2$

$\therefore ab-h^2=a_1b-{h_1}^2$ (proved)

Solved Problems

Q.1. Transfer the equao n $3x^2+2xy+3y^2-18x-22y+50=0$ in rectangular co-ordinates so as to remove the terms in x,y and xy. [ME'07,]

Or, how can you reduce the equao n $3x^2+2xy+3y^2-18x-22y+50=0$ to $4x^2+2y^2=1$.

Solution: Transforming to parallel axes through (h,k) we have, $x=x'+h$

$y=y'+k$

then the transformed equation becomes,

$3(x'+h)^2+2(x'+h)(y'+k)+3(y'+k)^2-18(x'+h)-22(y'+k)+50=0$

$\Rightarrow 3x'^2+6x'h+3h^2+2x'y'+2x'k+2y'h+2hk+3y'^2+6y'k+3k^2-18x'-18h-22y'-22k+50=0$

$\Rightarrow 3x'^2+2x'y'+3y'^2+x'(6h+2k-18)+y'(2h+6k-22)+(3h^2+2hk+3k^2-18h-22k+50)=0$ ……… (i)

Now, equating the co-efficients of x',y' to zero

$6h+2k-18=0$

} ……… (ii)

$2h+6k-22=0$

Solving the equations we have,

h= 2 and k=3

hence the equation (i) becomes,

$3x'^2+2x'y'+3y'^2-1=0$

Now, removing the suffixes, we get,

$3x^2+2xy+3y^2-1=0$ ……… ...(iii)

Now, rotating the axes through an angle θ

$$where, \theta=\frac{1}{2}\tan^{-1}\frac{2}{3-3}=\frac{\pi}{4} \quad [\tan 2\theta=\frac{2h}{a-b}, where, h=1, a=3, b=3]$$

We have,

$x=x'\sin\theta-y'\cos\theta=\frac{x'-y'}{\sqrt{2}}$

$y=x'\cos\theta+y'\sin\theta=\frac{x'+y'}{\sqrt{2}}$

putting these values in equation (iii), we have

$3(\frac{x'-y'}{\sqrt{2}})^2+2(\frac{x'-y'}{\sqrt{2}})(\frac{x'+y'}{\sqrt{2}})+3(\frac{x'+y'}{\sqrt{2}})^2-1=0$

$=(a+b)^2-4h^2-(a-b)^2$

$=4ab-4h^2$

$\therefore ab-h^2=a_1b-h_1^2$ (proved)

Solved Problems

Q.1. Transfer the equao n $3x^2+2xy+3y^2-18x-22y+50=0$ in rectangular co-ordinates so as to remove the terms in x,y and xy.[ME'07,]

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$\Rightarrow 3x'^2+6x'h+3h^2+2x'y'+2x'k+2y'h+2hk+3y'^2+6y'k+3k^2-18x'-18h-22y'-22k+50=0$

$\Rightarrow 3x'^2+2x'y'+3y'^2+x'(6h+2k-18)+y'(2h+6k-22)+(3h^2+2hk+3k^2-18h-22k+50)=0$... ... ... (i)

Now, equating the co-efficients of x', y' to zero

$6h+2k-18=0$

}... ... ...(ii)

$2h+6k-22=0$

Solving the equations we have,

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$3x'^2+2x'y'+3y'^2-1=0$

Now, removing the suffixes, we get,

$3x^2+2xy+3y^2-1=0$ ... ...(iii)

Now, rotating the axes through an angle θ

$$where, \theta=\frac{1}{2}\tan^{-1}\frac{2}{3-3}=\frac{\pi}{4} \quad [\tan 2\theta=\frac{2h}{a-b}, where, h=1, a=3, b=3]$$

We have,

$x= x'\sin\theta-y'\cos\theta=\frac{x'-y'}{\sqrt{2}}$

$y = x'\cos\theta+y'\sin\theta=\frac{x'+y'}{\sqrt{2}}$

putting these values in equation (iii), we have

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2+2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right)+3\left(\frac{x'+y'}{\sqrt{2}}\right)^2-1=0$$

$\Rightarrow 3x'^2-6x'y'+3y'^2+2x'^2-2y'^2+3x'^2+6x'y'+3y'^2-2=0$

$\Rightarrow 4x'^2+2y'^2-1=0$… … … … (iv)

Now, removing the suffixes we have,

$4x^2+2y^2-1=0$ **(Ans.)**

For second(or) question add this part: if the origin is shie d to (2,3)and rotate the axes through $\frac{\pi}{4}$,the given equao n becomes $4x^2+2y^2=1$

[Note: # To remove the first degree change the origin with parallel axes.

#To remove xy term or second degree rotate the axes through an angle θremaining the origin unchanged.]

Try yourself: Transfer the equao n $9x^2+24xy+2y^2-6x+20y+41=0$ in rectangular co-ordinates so as to remove the terms in x,y and xy.[ME'08,ETE'11]

Q.2.Determine the equation of the parabola $x^2-2xy+y^2+2x-4y+3=0$ ae r rotan g of axes through 45°.[EEE'06(Same)]

Solution:Ae r rotan g of axes through 45°,we have,

$x= x'\sin45°-y'\cos45°=\frac{x'-y'}{\sqrt{2}}$

$y = x'\cos45°+y'\sin45°=\frac{x'+y'}{\sqrt{2}}$

then the transferred equation is,

$(\frac{x'-y'}{\sqrt{2}})^2-2(\frac{x'-y'}{\sqrt{2}})(\frac{x'+y'}{\sqrt{2}})+(\frac{x'+y'}{\sqrt{2}})^2+2(\frac{x'-y'}{\sqrt{2}})-4(\frac{x'+y'}{\sqrt{2}})+3=0$

$\Rightarrow \frac{x'^2-2x'y'+y'^2}{2}-(x'^2-y'^2)+\frac{x'^2+2x'y'+y'^2}{2}+\sqrt{2}x'-\sqrt{2}y'-2\sqrt{2}x'-2\sqrt{2}y'+3=0$

$\Rightarrow x'^2-2x'y'+y'^2-2x'^2+2y'^2+x'^2+2x'y'+y'^2+2\sqrt{2}x'-2\sqrt{2}y'-4\sqrt{2}x'-4\sqrt{2}y'+6=0$

$$\Rightarrow 4y'^2-2\sqrt{2}x'-6\sqrt{2}y'+6=0$$

$$\Rightarrow 2y'^2-\sqrt{2}x'-3\sqrt{2}y'+3=0$$

Now removing the suffixes, we get

$2y^2-\sqrt{2}x-3\sqrt{2}y+3=0$ **(Ans.)**

Q.3.By transforming to parallel axes through a properly chosen point(h,k),prove that the equao n $12x^2-10xy+2y^2+11x-5y+2=0$ can be reduced

to one containing only the terms of 2^{nd} degree.[EEE'12,CSE'10(Same)IPE'10(Same)]

Solution: Transforming to parallel axes through (h,k),we have, $x=x'+h$

$y=y'+k$

then the transformed equation becomes,

$12(x'+h)^2-10(x'+h)(y'+k)+2(y'+k)^2+11(x'+h)-5(y'+k)+2=0$

$\Rightarrow 12x'^2+24x'h+12h^2-10x'y'-10x'k-10y'h-10hk+2y'^2+4y'k+2k^2+11x'+11h-5y'-5k+2=0$

$\Rightarrow 12x'^2-10x'y'+2y'^2+x'(24h-10k+11)+y'(4k-10h-5)+(12h^2-10hk+2k^2+11h-5k+2)=0$ ... ...(i)

Now, equating the co-efficient of x' and y' to zero,

$24h-10k+11=0$ ... ... (ii)

$4k-10h-5=0$

$\Rightarrow 10h-4k+5=0$ ... ... (iii)

Solving the equations (ii) and (iii) we have,

$h=-\frac{3}{2}$ & $k=-\frac{5}{2}$

Now putting the values of h & k in eq^n.(i) we get,

$12x'^2-10x'y'+2y'^2=0$

Removing the suffixes we get,

$12x^2-10xy+2y^2=0$ **(Ans.)**

Q.4.Prove that the value of g^2+f^2 in the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ remains unaffected by orthogonal transformation without change of origin.[ME'10;EEE'07]

Solution: After rotating the axes through an angle θ about the origin, we get

$x=x'cos\theta - y'sin\theta$

$y=x'sin\theta + y'cos\theta$

$\therefore\ a(x'cos\theta - y'sin\theta)^2+2h(x'cos\theta - y'sin\theta)(x'sin\theta + y'cos\theta)+b(x'sin\theta + y'cos\theta)^2$

$+2g(x'cos\theta - y'sin\theta)+2f(x'sin\theta + y'cos\theta)+c=0$

$\Rightarrow ax'^2cos^2\theta - 2ax'y'sin\theta cos\theta + ay'^2sin^2\theta + 2hx'^2cos\theta sin\theta + 2hx'cos^2\theta y' - 2hx'y'sin^2\theta - 2hy'^2sin\theta cos\theta + bx'^2sin^2\theta + 2bx'y'sin\theta cos\theta + by'^2cos^2\theta + 2gx'cos\theta - 2gy'sin\theta + 2fx'sin\theta + 2fy'cos\theta)+c=0$

$\Rightarrow x'^2(acos^2\theta + 2hcos\theta sin\theta + bsin^2\theta) + 2x'y'(-asin\theta cos\theta + hcos^2\theta - hsin^2\theta + bsin\theta cos\theta) + y'^2(asin^2\theta - 2hsin\theta cos\theta + bcos^2\theta) + 2x'(gcos\theta + fsin\theta) + 2y'(-gsin\theta + fcos\theta) + c = 0$

$= a_1x'^2 + 2h_1x'y' + b_1y'^2 + 2g_1x' + 2f_1y' + c = 0$

Where, $a_1 = acos^2\theta + 2hcos\theta sin\theta + bsin^2\theta$ ……… ……(i)

$h_1 = -asin\theta cos\theta + hcos^2\theta - hsin^2\theta + bsin\theta cos\theta$ ………… (ii)

$b_1 = asin^2\theta - 2hsin\theta cos\theta + bcos^2\theta$………..(iii)

$g_1 = gcos\theta + fsin\theta$ …… … … (iv)

$f_1 = -gsin\theta + fcos\theta$ ……… …… (v)

$\therefore g_1^2 + f_1^2 = g^2(cos^2\theta + sin^2\theta) + f^2(sin^2\theta + cos^2\theta)$

$\Rightarrow g_1^2 + f_1^2 = g^2 + f^2$

Therefore (g^2+f^2) remains unaffected by orthogonal transformation. **(Proved)**

Q.5.Transform the equao n $11x^2+24xy+4y^2-20x-40y-5=0$ to rectangular axes through the point (2,-1)and inclined at angle $\theta = \tan^{-1}(-\frac{4}{3})$.[IPE'12,EEE'05]

Solution: Given equao n is, $11x^2+24xy+4y^2-20x-40y-5=0$… … … …(i)

When the origin is shie d to (2,-1) then wehave,$x=x'+2$ and $y=y'-1$ … … … … …(ii)

We use (ii)in (i) then the equation implies that,$11(x'+2)^2+24(x'+2)(y'-1)+4(y'-1)^2-20(x'+2)-40(y'-1)-5=0$

$\Rightarrow 11x'^2+44x'+44+24x'y'-24x'+48y'-48+4y'^2-8y'+4-20x'-40-40y'+40-5=0$

$\Rightarrow 11x'^2+24x'y'+4y'^2-5=0$

Now dropping the suffixes we get, $11x^2+24xy+4y^2-5=0$… … … …. …. …. (iii)

Now we have,$\tan\theta = -\frac{4}{3}$

$$\therefore \sin\theta = -\frac{4}{5} \ and \ \cos\theta = \frac{3}{5}$$

When the axes are rotated through an angle $\theta = \tan^{-1}(-\frac{4}{3})$ then we get,

$x = x'cos\theta - y'sin\theta = \frac{3x'}{5} + \frac{4y'}{5}$

$y = x'sin\theta + y'cos\theta = \frac{3y'}{5} - \frac{4x'}{5}$

now we use these values in(iii),

$11(\frac{3x'}{5} + \frac{4y'}{5})^2 + 24(\frac{3x'}{5} + \frac{4y'}{5})(\frac{3y'}{5} - \frac{4x'}{5}) + 4(\frac{3y'}{5} - \frac{4x'}{5})^2 - 5 = 0$

$\Rightarrow 99x'^2+264x'y'+176y'^2+216x'y'-288x'^2+288y'^2-384x'y'+36y'^2-96x'y'+64x'^2-125=0$

$\Rightarrow -125x'^2+500y'^2-125=0$

$\Rightarrow x'^2-4y'^2+1=0$

Now dropping the suffixes the equation we get, $x^2-4y^2+1=0$ (Ans.)

<u>Try yourself:</u> Transform the equao n $14x^2-4xy+11y^2-36x+48y+41=0$ to rectangular axes through the point (1,-2)and inclined at angle $\theta = \tan^{-1}(-\frac{1}{2})$.[CSE'12]

Q.6.Determine the angle through which the axes must be rotated to remove xy term in the equao n $7x^2-6\sqrt{3}xy+13y^2=16$.[ME'11]

<u>Solution:</u> Let θ be the angle through which the axes are rotated then we have,

$x=x'\cos\theta - y'\sin\theta$

$y=x'\sin\theta + y'\cos\theta$

We use these values in the given equao n, $7(x'\cos\theta - y'\sin\theta)^2-6\sqrt{3}(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)+13(x'\sin\theta + y'\cos\theta)^2=16$

$\Rightarrow 7x'^2\cos^2\theta-14x'y'\sin\theta\cos\theta+7y'^2\sin^2\theta-6\sqrt{3}x'y'\cos^2\theta-6\sqrt{3}x'^2\cos\theta\sin\theta+6\sqrt{3}y'^2\sin\theta\cos\theta+6\sqrt{3}x'y'\sin^2\theta+13y'^2\cos^2\theta+26x'y'\sin\theta\cos\theta+13x'^2\sin^2\theta=16$

$\Rightarrow x'^2(7\cos^2\theta-6\sqrt{3}\cos\theta\sin\theta+13\sin^2\theta)+y'^2(7\sin^2\theta+6\sqrt{3}\sin\theta\cos\theta+13\cos^2\theta)+x'y'(-14\sin\theta\cos\theta - 6\sqrt{3}\cos^2\theta+6\sqrt{3}\sin^2\theta+26\sin\theta\cos\theta)=16$............(i)

In order to remove the xy term the co-efficient of x'y' from equation (i) is equal to zero,i.e,

$-14\sin\theta\cos\theta - 6\sqrt{3}\cos^2\theta+6\sqrt{3}\sin^2\theta+26\sin\theta\cos\theta=0$

$\Rightarrow 12\sin\theta\cos\theta-6\sqrt{3}\cos2\theta=0$

$\Rightarrow 6\sin2\theta=6\sqrt{3}\cos2\theta$

$\Rightarrow \tan2\theta=\sqrt{3}$

$\Rightarrow \tan2\theta=60°$

$\therefore\ \theta=30°$(Ans.)

(If it needs to determine the transferred equation for this question, write uptoend)

The transferred equation is,

$4x'^2+16y'^2=16$

$X'^2+4y'^2=4$

Dropping the suffixes we get,

$X^2+4y^2=4$ (Ans.)

CHAPTER-2

Pair of straight lines

Theorem.1. **Prove that the homogeneous equao n of 2nd degree $ax^2+2hxy+by^2=0$ will represent a pair of straight lines passing through the origin.**

Proof:

Given that,

$ax^2+2hxy+by^2=0$

$\Rightarrow a^2x^2+2ahxy+aby^2=0$

$\Rightarrow a^2x^2+2ahxy+h^2y^2+aby^2-h^2y^2=0$

$\Rightarrow (ax+hy)^2-y^2(h^2-ab)=0$

$\Rightarrow (ax+hy)^2-(y\sqrt{h^2-ab})^2=0$

$\Rightarrow (ax+hy-y\sqrt{h^2-ab})(ax+hy+y\sqrt{h^2-ab})=0$

$\therefore ax+(h-\sqrt{h^2-ab})y=0$ ______________ (1)

And $ax+(h+\sqrt{h^2-ab})y=0$ ______________ (2)

Equao n (1) & (2) represent the two straight lines passing through the origin.

Proved.

Theorem.2. **Determine the angle between the lines represented by the equation $ax^2+2hxy+by^2=0$.**

Sol^n: Let, $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by $ax^2+2hxy+by^2=0$.

Now, $\frac{a}{b}x^2+\frac{2h}{b}xy+y^2=(y-m_1x)(y-m_2x)$

$\Rightarrow \frac{a}{b}x^2+\frac{2h}{b}xy+y^2=m_1m_2x^2+(-m_1-m_2)xy+y^2$

$\therefore m_1+m_2=-\frac{2h}{b}$ and $m_1m_2=\frac{a}{b}$

Let, θ be the angle between the two lines.

So, $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}=\frac{\sqrt{(-\frac{2h}{b})^2-4\frac{a}{b}}}{1+\frac{a}{b}}=\frac{2\sqrt{\frac{h^2-ab}{b^2}}}{\frac{a+b}{b}}$

$\therefore \tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$

Where θ is the angle between the lines represented by the equation $ax^2+2hxy+by^2=0$.

Condition-01

#The lines will be coincident or parallel if $h^2 = ab$ [$\tan\theta = 0$]

Condition-02

#The lines will be perpendicular if a+b=0 [$\tan\theta = \infty$ or $\theta = 90^0$]

#The lines will be real if $h^2 > ab$

#The lines will be imaginary if $h^2 < ab$

Theorem.3. **Find the bisectors of the angles between the lines represented by $ax^2+2hxy+by^2=0$.**

Sol^n:

Let, $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by $ax^2+2hxy+by^2=0$.

So, $m_1+m_2=-\frac{2h}{b}$ and $m_1 m_2=\frac{a}{b}$

So, The equation of the bisectors are

$\Rightarrow (y-m_1x)^2(1+m_2^2) = (y-m_2x)^2(1+m_1^2)$

$\Rightarrow \frac{x^2-y^2}{a-b} = \frac{xy}{h}$

Theorem.4. **Find the condition that the general equao n of 2nd degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ may represent a pair of straight lines.[EEE'08,ME'05]**

Sol^n: If we transfer the origin to a point (α, β) the point of intersection of two straight lines without changing the direction of the axes, then we have

$a(x+\alpha)^2+2h(x+\alpha)(y+\beta)+b(y+\beta)^2+2g(x+\alpha)+2f(y+\beta)+c=0$

$\Rightarrow ax^2+2hxy+by^2+2(a\alpha+h\beta+g)x+2(h\alpha+b\beta+f)y+(a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c)=0$ ______________(1)

The equao n (1) may represent a pair of straight lines if it is reduced to a homogeneous equation in x and y.This is possible if the co-efficient of x and y and the constant terms are separately zero.

i.e

$a\alpha+h\beta+g=0$ ------(2) and $h\alpha+b\beta+f=0$ ------(3)

$a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0$ [from eq. (1) ∵ (2) & (3) = 0]

$\Rightarrow \alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+g\alpha+f\beta+c=0$

$\Rightarrow g\alpha+f\beta+c=0$__________(4) [∵ (2) = 0 , (3) = 0]

Now, solving (2)&(3) we have

$a\alpha + h\beta + g = 0$ ——— (2)

$h\alpha + b\beta + f = 0$ ——— (3)

$\frac{\alpha}{hf-bg} = \frac{\beta}{hg-af} = \frac{1}{ab-h^2}$

So, $\alpha = \frac{hf-bg}{ab-h^2}$ and $\beta = \frac{hg-af}{ab-h^2}$

Putting these values in equation (4) we have,

$g\left(\frac{hf-bg}{ab-h^2}\right) + f\left(\frac{hg-af}{ab-h^2}\right) + c = 0$

$\Rightarrow \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Which is the required condition.

Theorem 5. **Angle between the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$.**

Sol^n: Let, $lx+my+n=0$ and $l'x+m'y+n'=0$ be the lines represented by the given equation

So, $ax^2+2hxy+by^2+2gx+2fy+c = (lx+my+n)(l'x+m'y+n')$

Comparing the co-efficients we have,

$ll'=a$, $mm'=b$, $nn'=c$

$lm'+ml'=2h$; $ln'+nl'=2g$; $mn'+nm'=2f$;

Let, θ be the angle between the two lines.

So, $\tan\theta = \frac{m_1-m_2}{1+m_1m_2}$

Now, $lx+my+n=0$ and $l'x+m'y+n'=0$

So, $y = \frac{-l}{m}x - \frac{n}{m}$

and $y = \frac{-l'}{m'}x - \frac{n'}{m'}$

so,

$\tan\theta = \frac{\frac{-l}{m} + \frac{l'}{m'}}{1 + \frac{ll'}{mm'}}$

So, $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

Note [Formula in short].

 # $ax^2+2hxy+by^2+2gx+2fy+c=0$ will represent a pair of straight lines if **$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$**

 # Angle between the straight lines, $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

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The equation of the bisectors is, $\frac{(x-x_1)^2-(y-y_1)^2}{a-b}=\frac{(x-x_1)(y-y_1)}{h}$

Where x_1, y_1 is the point of intersection.

#Process of determining the point of intersection:

Let, $F(x, y) = ax^2+2hxy+by^2+2gx+2fy+c=0$.

$\frac{\partial F}{\partial x}$= - - - - - =0 ____________ (1)

And $\frac{\partial F}{\partial y}$ = - - - - - =0____________ (2)

Solving equations (1) & (2) we get,

$(x ,y) = (\alpha,\beta)$

Hence, the point of intersection (α, β).

Solved Problems

Q.1. Prove that the pair of the lines joining the origin to the point of intersection of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by the straight line $lx+my+n=0$ are coincident if $a^2l^2+b^2m^2= n^2$.

Sol^n: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ____________ (1)

$lx+my+n=0$

$\Rightarrow \left(\frac{lx+my}{-n}\right)^2=1$____________ (2)

Making homogeneous (1) with the help of (2), we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{lx+my}{-n}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(lx)^2}{n^2}+\frac{2lmnxy}{n^2}+\frac{(my)^2}{n^2}$$

$$\Rightarrow \left(\frac{l^2}{n^2}-\frac{1}{a^2}\right)x^2+\left(\frac{m^2}{n^2}-\frac{1}{b^2}\right)y^2+2\frac{lm}{n^2}xy=0$$

$\Rightarrow Ax^2+2Hxy+By^2=0$____________ (3)

Where $A=\left(\frac{l^2}{n^2}-\frac{1}{a^2}\right)$; $B=\left(\frac{m^2}{n^2}-\frac{1}{b^2}\right)$; $H=\frac{lm}{n^2}$

The lines represented by the eq^n(3)will be coincident if H^2=AB

So, $\frac{(lm)^2}{n^4}=\frac{(lm)^2}{n^4}-\frac{l^2}{n^2b^2}-\frac{m^2}{a^2n^2}+\frac{1}{a^2b^2}$

⇨ $\frac{l^2}{n^2b^2}+\frac{m^2}{a^2n^2}=\frac{1}{a^2b^2}$

⇨$a^2l^2+b^2m^2=n^2$ [Multiplying by $a^2b^2n^2$] **Proved**

Q.2. Prove that the equao n $2x^2+xy-y^2-x-7y-10=0$ represents a pair of straight lines.Find their point of intersection and the equation of the bisectors of the angle between them.

Sol^n: The general equation of second degree is

$ax^2+2hxy+by^2+2gx+2fy+c=0$_______ (1)

Comparing the co-efficients of the given equation with (1) we have,

A=2, $h=\frac{1}{2}$, b=-1, $g=-\frac{1}{2}$, $f=-\frac{7}{2}$, c=-10

Now, $\triangle=abc+2fgh-af^2-bg^2-ch^2=20+\frac{7}{4}-\frac{49}{2}+\frac{1}{4}+\frac{5}{2}=0$

Hence, the given equation represents a pair of straight lines. **Proved**

Point of intersection:

Let, $F(x, y) = 2x^2+xy-y^2-x-7y-10=0$__________ (2)

$\frac{\partial F}{\partial x}=4x+y-1=0$ __________ (3)

And $\frac{\partial F}{\partial y}=x-2y-7=0$ __________ (4)

Solving equations (3) & (4) we get,

X=1, y= -3

Hence, the point of intersection $(x_1, y_1) = (1,-3)$

The equation of the bisectors is $\frac{(x-x_1)^2-(y-y_1)^2}{a-b}=\frac{(x-x_1)(y-y_1)}{h}$

⇨ $\frac{(x-1)^2-(y+3)^2}{2-(-1)}=\frac{(x-1)(y+3)}{\frac{1}{2}}$

⇨$\frac{1}{2}(x^2-2x+1-y^2-6y-9)=3xy+9x-3y-9$

⇨ $x^2-y^2-2x-6y-8=6xy+18x-6y-18$

⇨ $x^2-6xy-y^2-20x+10=0$ Ans.

Q.3. Prove that the straight lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ will be equidistance from the origin if $f^4-g^4=c(bf^2-ag^2)$.[EEE'07,09,ME'07,IPE'10,GCE'12,]

Proof:

Let, lx+my+n=0 and l′x+m′y+n′=0 be the lines represented by the given equation

So, $ax^2+2hxy+by^2+2gx+2fy+c=(lx+my+n)(l'x+m'y+n')$

Comparing the co-efficients we have,

$ll'=a, mm'=b, nn'=c$

$lm'+ml'=2h$; $ln'+nl'=2g$; $mn'+nm'=2f$;

Since, the lines are equidistance from the origin.

So,

$\frac{n}{\sqrt{l^2+m^2}}=\frac{n'}{\sqrt{l'^2+m'^2}}$

$\Rightarrow n^2(l'^2+m'^2)=n'^2(l^2+m^2)$

$\Rightarrow n^2 l'^2-n'^2 l^2=m^2 n'^2-n^2 m'^2$

$\Rightarrow (nl'+ln')(nl'-ln')=(mn'+nm')(mn'-nm')$

$\Rightarrow (nl'+ln')\sqrt{(nl'+ln')^2-4nn'll'}=(mn'+nm')\sqrt{(mn'+nm')^2-4mm'nn'}$

$\Rightarrow 2g\sqrt{4g^2-4ac}=2f\sqrt{4f^2-4bc}$

$\Rightarrow 4g^2(4g^2-4ac)=4f^2(4f^2-4bc)$

$\Rightarrow c(f^2b-g^2a)=f^4-g^4$

$\Rightarrow c=\frac{f^4-g^4}{f^2b-g^2a}$ ———————— (1)

Since, the given equation represents a pair of straight lines, we have

$abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow ab\frac{f^4-g^4}{f^2b-g^2a}+2fgh-af^2-bg^2-\frac{f^4-g^4}{f^2b-g^2a}h^2=0$

$\Rightarrow abf^4-abg^4+2f^3bgh-2g^3fah-abf^4+a^2g^2f^2-b^2g^2f^2+abg^4-h^2f^4+h^2g^4=0$

$\Rightarrow (hg^2)^2-2hg^2.afg+(afg)^2=(hf^2)^2-2hf^2.bfg+(bfg)^2$

$\Rightarrow (hg^2-afg)^2=(hf^2-bfg)^2$

$\Rightarrow h(f^2-g^2)=fg(a-b)$ **Proved.**

Q.4. Show that the eq^n $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel straight lines if a:h=h:b =g:f Also show that the distance bet^n them is $\frac{2\sqrt{g^2-ac}}{\sqrt{a(a+b)}}$ [CSE'07,ME'05,08,11]

Sol^n: The given eq^n will represent two parallel straight lines if $\triangle=0$ and $h^2=ab$

So, $\triangle=abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow ch^2+2fgh-af^2-bg^2-ch^2=0$ $[h^2=ab]$

$\Rightarrow af^2-2fgh+bg^2=0$ $\Rightarrow$

$\Rightarrow (\sqrt{a}f-\sqrt{b}g)^2=0$

$\Rightarrow \sqrt{a}\, f - \sqrt{b}\, g=0$

$\Rightarrow \sqrt{a}\, f = \sqrt{b}\, g$

$\Rightarrow \frac{\sqrt{a}}{\sqrt{b}}=\frac{g}{f}$

$\Rightarrow \frac{\sqrt{a^2}}{\sqrt{ab}}=\frac{g}{f}$

$\Rightarrow \frac{a}{h}=\frac{g}{f}$ $[h^2 = ab]$

Again, $h^2 = ab$

$\Rightarrow \frac{h}{b}=\frac{a}{h}$

So, $\frac{a}{h}=\frac{h}{b} = \frac{g}{f}$

Which are the required conditions.

Let, the lines are $lx+my+n_1=0$ & $lx+my+n_2=0$

So, $ax^2+2hxy+by^2+2gx+2fy+c=(lx+my+n_1)(lx+my+n_2)$

Comparing the co-efficients we get,

$l^2=a$, $m^2=b$, $n_1n_2=c$, $l(n_1+n_2)=2g$

Let, the distance between the lines is P.

So $P=\frac{n_1-n_2}{\sqrt{l^2+m^2}}$

$=\frac{\sqrt{(n_1+n_2)^2-4n_1n_2}}{a+b}$

$=\frac{\sqrt{\frac{4g^2}{l^2}-4c}}{a+b}$

$=2\sqrt{\frac{\frac{g^2}{a}-c}{a+b}}$

$=2\sqrt{\frac{g^2-ac}{a(a+b)}}$ Showed.

Q.5. The axes being rectangular, find the equation to the pair of straight lines meeting at the origin which are perpendicular to the pair given by the equation $ax^2+2hxy+by^2=0$ [IPE'10]

Sol^n: Let, $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by $ax^2+2hxy+by^2=0$.

So, $m_1+m_2=-\frac{2h}{b}$ and $m_1 m_2=\frac{a}{b}$

Now, the equations of the lines perpendicular to the above lines and passing through the origin are $m_1y+x=0$ and $m_2y+x=0$

So, their joint equation is $(m_1y+x)(m_2y+x)=0$

$\Rightarrow m_1m_2y^2+(m_1+m_2)xy+x^2=0$

$\Rightarrow \frac{a}{b}y^2 - \frac{2h}{b}xy + x^2 = 0$

$\Rightarrow bx^2 - 2hxy + ay^2 = 0$

Ans.

Q.6. Find the area of the tringle formed by the lines represented by $ax^2+2hxy+by^2=0$ and $lx+my+n=0$.[CSE'10(SAME),ME'04,06,10,EEE'10,12,CSE'08,CSE'06]

Sol^n: Let, $y-m_1x=0$ ----- (1) and $y-m_2x=0$----- (2) be the lines represented by $ax^2+2hxy+by^2=0$.

So, $m_1+m_2=-\frac{2h}{b}$ and $m_1 m_2=\frac{a}{b}$

Given that, $lx+my+n=0$ ---------------------- (3)

Now, the two lines will meet the 3rd line at points

$\left(\frac{-n}{l+mm_1}, \frac{-nm_1}{l+mm_1}\right)$ [Solving (1) & (3)]

And $\left(\frac{-n}{l+mm_2}, \frac{-nm_2}{l+mm_2}\right)$ [Solving (2) & (3)]

Which will be two verc es of the triangle and the 3rd vertex is clearly (0, 0).Hence the area of the triangle, $\triangle=\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{-n}{l+mm_1} & \frac{-nm_1}{l+mm_1} & 1 \\ \frac{-n}{l+mm_2} & \frac{-nm_2}{l+mm_2} & 1 \end{vmatrix}$

$=\frac{1}{2}\left[\frac{n^2m_2}{(l+mm_1)(l+mm_2)} - \frac{n^2m_1}{(l+mm_1)(l+mm_2)}\right]$

$=\frac{1}{2}\left[\frac{n^2(m_2-m_1)}{(l+mm_1)(l+mm_2)}\right]$

$=\frac{1}{2}\left[\frac{n^2\sqrt{(m_1+m_2)^2-4m_1m_2}}{l^2+lmm_2+lmm_1+m^2m_1m_2}\right]$

$=\frac{1}{2}\left[\frac{n^2\sqrt{\frac{4h^2}{b^2}-4\frac{a}{b}}}{l^2+lm(m_1+m_2)+m^2m_1m_2}\right]$

$=\frac{1}{2}\left[\frac{\frac{2n^2}{b}\sqrt{h^2-ab}}{l^2+lm\left(\frac{-2h}{b}\right)+m^2\frac{a}{b}}\right]$

$=\frac{n^2\sqrt{h^2-ab}}{bl^2-2hlm+am^2}$ **Ans.**

Q.7. Show that one of the lines represented by $ax^2+2hxy+by^2=0$ will be coincident with one of the lines represented by $a_1x^2+2h_1xy+b_1y^2=0$, if $(ab_1-a_1b)^2=4(a_1h-ah_1)(bh_1-b_1h)$.[CSE`12]

Sol^n: $ax^2+2hxy+by^2=0$ ________ (1)

$a_1x^2+2h_1xy+b_1y^2=0$ ________ (2)

Let, $y=mx$ be the common line of (1) & (2)

From equao n (1),

ax^2+2hx(mx)+b$(mx)^2$ =0

⇨ bm^2+2hm+a=0________ (3) $[x^2 \neq 0]$

Similarly,

b_1m^2+$2h_1$m+a_1=0________ (4)

From (3) & (4) we get,

$$\frac{m^2}{2(a_1h-ah_1)}=\frac{m}{(a_1b-ab_1)}=\frac{1}{2(bh_1-hb_1)}$$

So, $m^2=\frac{a_1h-ah_1}{(bh_1-hb_1)}$ ________ (5)

And m=$\frac{(a_1b-ab_1}{2(bh_1-hb_1)}$ ________ (6)

From (5) & (6) we get,

$$[\frac{(a_1b-ab_1}{2(bh_1-hb_1)}]^2=\frac{a_1h-ah_1}{(bh_1-hb_1)}$$

⇒$(a_1b-ab_1)^2$=4(a_1h-ah_1) (bh_1-hb_1) **Proved.**

Q.8. If one of the lines ax²+2hxy+by²=0 be perpendicular to one of the lines $a'x^2$+2h'xy+$b'y^2$=0.Prove that $(aa'-bb')^2$+4$(a'h+bh')$ ($ah'+b'h$) =0.[CSE'11]

Soln: $ax^2+2hxy+by^2=0$ ________ (1)

$a'x^2$+2h'xy+$b'y^2$=0 ________ (2)

Let, y=mx & y=$\frac{-1}{m}$x are perpendicular from (1) & (2).

So from (1),

ax^2+2hx(mx)+b$(mx)^2$ =0

⇨ bm^2+2hm+a=0 ________ (3) $[x^2 \neq 0]$

From equao n (2),

$a'x^2$+2h'x$\frac{-x}{m}$+$b'(\frac{-x}{m})^2$=0

⇨$a'm^2$-2h'm+b'=0________ (4)

From (3) & (4) we get,

$$\frac{m^2}{2(b'h+ah')}=\frac{m}{(aa'-bb')}=\frac{1}{-2(bh'+a'h)}$$

So, $m^2=\frac{(b'h+ah')}{-(bh'+a'h)}$ ________ (5)

And =$\frac{(aa'-bb')}{-2(bh'+a'h)}$ ________ (6)

From (5) & (6) we get,

$$[\frac{(aa'-bb')}{-2(bh'+a'h)}]^2=\frac{(b'h+ah')}{-(bh'+a'h)}$$

$\Rightarrow (aa' - bb')^2 + 4(b'h + ah')(bh' + a'h) = 0$ **Proved.**

Q.9. Prove that the product of the perpendiculars from the point (x_1, y_1) on the lines $ax^2+2hxy+by^2=0$ is $\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}$.[EEE'10'CSE'09]

Sol^n: $ax^2+2hxy+by^2=0$ ________ (1)

Let, $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by $ax^2+2hxy+by^2=0$.

So, $m_1+m_2=-\frac{2h}{b}$ and $m_1m_2=\frac{a}{b}$

Now the product of the perpendiculars from (x_1, y_1) on the lines is

$$\frac{y_1-m_1x_1}{\sqrt{1+m_1^2}} \times \frac{y_1-m_2x_1}{\sqrt{1+m_2^2}}$$

$$=\frac{y_1^2-m_2y_1x_2-m_1x_1y_1+m_1m_2x_1^2}{\sqrt{m_1^2+m_2^2+m_1^2m_2^2+1}}$$

$$=\frac{y_1^2-x_1y_1(m_1+m_2)+m_1m_2x_1^2}{\sqrt{(m_1+m_2)^2-2m_1m_2+m_1^2m_2^2+1}}$$

$$=\frac{y_1^2-x_1y_1\left(-\frac{2h}{b}\right)+\frac{a}{b}x_1^2}{\sqrt{(-\frac{2h}{b})^2-2\frac{a}{b}+(\frac{a}{b})^2+1}}$$

$$=\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{4h^2-2ab+a^2+b^2}}$$

$$=\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}$$ **Proved.**

Q.10. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, prove that the square of the distance of their point of intersection from the origin is $\frac{c(a+b)-f^2-g^2}{ab-h^2}$[CSE'11,EEE'12]

Sol^n: Let, $F(x, y) = ax^2+2hxy+by^2+2gx+2fy+c=0.$______(1)

$\frac{\partial F}{\partial x}= 2ax+2hy+2g=0$ __________ (2)

And $\frac{\partial F}{\partial y} = 2hx+2by+2f=0$__________ (3)

Let, the point of intersection be (α,β)

So we get,

$a\alpha+h\beta+g=0$ ------(4) and $h\alpha+b\beta+f=0$ ------(5)

Now, solving (4) and (5) we have

So, $\alpha=\frac{hf-bg}{ab-h^2}$ and $\beta=\frac{hg-af}{ab-h^2}$

Now,the square of the distance of $p(\alpha,\beta)$ from the origin,

$OP^2=\sqrt{(\alpha-0)^2+(\beta-0)^2} = (\frac{hf-bg}{ab-h^2})^2 + (\frac{hg-af}{ab-h^2})^2$

$$=\frac{h^2f^2-2fgbh+b^2g^2+g^2h^2-2afgh+a^2f^2}{(ab-h^2)^2}$$

$\therefore OP^2=\frac{h^2f^2+b^2g^2+g^2h^2+a^2f^2-2fgh(a+b)}{(ab-h^2)^2}$ __________(6)

Again, $\Delta=0$

$\therefore abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow 2fgh= af^2+bg^2+ch^2-abc$

Pun g the value in equao n (6) we have,

$\therefore OP^2=\frac{h^2f^2+b^2g^2+g^2h^2+a^2f^2-(af^2+bg^2+ch^2-abc)(a+b)}{(ab-h^2)^2}$

$=\frac{h^2f^2+b^2g^2+g^2h^2+a^2f^2-a^2f^2-abg^2+a^2bc-ach^2-abf^2-b^2g^2-bch^2+acb^2}{(ab-h^2)^2}$

$=\frac{h^2f^2+g^2h^2-abg^2+a^2bc-ach^2-abf^2-bch^2+acb^2}{(ab-h^2)^2}$

$=\frac{ab(-g^2+ac-f^2+bc)+h^2(f^2+g^2-ac-bc)}{(ab-h^2)^2}$

$=\frac{c(a+b)-f^2-g^2}{(ab-h^2)}$ **Proved.**

Q.12. If one of the straight lines given by the equao n $ax^2+2hxy+by^2=0$ coincides with one of those given lines by $a_1x^2+2h_1xy+b_1y^2=0$ and the othe lines represented by them be perpendicular, prove that $\frac{h\,a_1b_1}{b_1-a_1}$

$=\frac{h_1ab}{b-a}=\frac{1}{2}\sqrt{-aa_1bb_1}$.[EEE'08]

Sol^n: $ax^2+2hxy+by^2=0$ __________(1)

$a_1x^2+2h_1xy+b_1y^2=0$__________(2)

Let, $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by (1)

So, $m_1+m_2=-\frac{2h}{b}$ __________(3)

and $m_1m_2=\frac{a}{b}$ __________(4)

According to the question,

$y-m_1x=0$ and $y+\frac{x}{m_2}=0$ be the lines represented by the equao n (2)

$\therefore m_1-\frac{1}{m_2}=\frac{-2h_1}{b_1}$ __________(5)

And $-\frac{m_1}{m_2}=\frac{a_1}{b_1}$ __________(6)

Now, (4)×(6) we get,

$m_1m_2(-\frac{m_1}{m_2})=\frac{a}{b}.\frac{a_1}{b_1}$

$\Rightarrow m_1^2=-\frac{a}{b}.\frac{a_1}{b_1}=-\frac{aa_1bb_1}{(bb_1)^2}$

$\therefore m_1=\frac{\sqrt{-aa_1bb_1}}{bb_1}$ __________(7)

From (6),

$$-\frac{\frac{\sqrt{-aa_1bb_1}}{bb_1}}{m_2}=\frac{a_1}{b_1}$$

$$\Rightarrow m_2=\frac{\sqrt{-aa_1bb_1}}{ba_1} \quad \_\_\_\_\_\_\_\_ (8)$$

Now from (3),

$$m_1+m_2=-\frac{2h}{b}$$

$$\Rightarrow \frac{\sqrt{-aa_1bb_1}}{bb_1}+\frac{\sqrt{-aa_1bb_1}}{ba_1}=-\frac{2h}{b}$$

$$\Rightarrow \frac{\sqrt{-aa_1bb_1}\,(a_1-b_1)}{ba_1b_1}=-\frac{2h}{b}$$

$$\Rightarrow \frac{ha_1b_1}{(b_1-a_1}=\frac{1}{2}\sqrt{-aa_1bb_1} \quad \_\_\_\_\_\_\_\_\_\_\_\_ (9)$$

From equao n (5), $m_1-\frac{1}{m_2}=\frac{-2h_1}{b_1}$

$$\Rightarrow \frac{\sqrt{-aa_1bb_1}}{bb_1}-\frac{1}{\frac{\sqrt{-aa_1bb_1}}{ba_1}}=\frac{-2h_1}{b_1}$$

$$\Rightarrow \frac{-aa_1bb_1+a_1b_1b^2}{bb_1\sqrt{-aa_1bb_1}}=\frac{-2h_1}{b_1}$$

$$\Rightarrow \frac{a_1b_1(b-a)}{\sqrt{-aa_1bb_1}}=-2h_1$$

$$\Rightarrow \frac{\sqrt{-aa_1bb_1}\,a_1b_1}{2(-aa_1bb_1)}=\frac{-h_1}{b-a}$$

$$\Rightarrow \frac{1}{2}\sqrt{-aa_1bb_1}=\frac{h_1ab}{b-a} \quad \_\_\_\_\_\_\_\_\_\_ (10)$$

From equao n (9) & (10),

$$\frac{ha_1b_1}{a_1-b_1}=\frac{h_1ab}{b-a}=\frac{1}{2}\sqrt{-aa_1bb_1}$$ **Proved.**

CHAPTER-3

General Equation of Second Degree

[CONIC SECTION]

Definition: If a point P moves in a plane such a way that the ratio of its distance PS from a fixed point S in the plane to its perpendicular distance PM from a fixed straight line XM in it, is always a constant ,the locus of the point P is called a conic section or briefly a conic.

The constant ratio is called the eccentricity of the conic and generally represented by the letter 'e'. The fixed Point S is called the focus and the fixed straight line XM is called directrix of the conic.

Eccentricity, $e=\frac{SP}{PM}$ [e is a positive number].

e=o	Circle
e=1	Parabola
e<1	Ellipse
e>1	Hyperbola

⇒ **The general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ will represent**

1. A pair of straight lines if, $\Delta=0$
2. A pair of parallel straight lines if, $\Delta=o$ & $ab=h^2$
3. A pair of perpendicular straight lines if, $\Delta=0$ & $a+b=0$
4. A circle if, $h=0$ & $a=b$
5. A parabola if, $\Delta\neq0$ & $ab=h^2$
6. An ellipse if, $\Delta\neq0$ & $ab-h^2>0$
7. A hyperbola if, $\Delta\neq0$ & $ab-h^2<0$
8. A rectangular hyperbola if, $\Delta\neq0$, $ab-h^2<0$ & $a+b=0$

Note. $\Delta=abc+2fgh-af^2-bg^2-ch^2$

⇒ **Centre of a conic:**

The centre of a conic is the point on the plane of the conic such that every chord of the conic passes through it and bisects at it.

⇒ **Method of finding the centre:**

Let, $F(x,y)= ax^2+2hxy+by^2+2gx+2fy+c=0$ ----------------------(i)

Let, the centre of conic is (x_1,y_1)

We get, $F(x_1,y_1)=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0$

Now,

$\frac{\partial F}{\partial x} = 2ax_1+2hy_1+2g=0 \quad \Rightarrow ax_1+hy_1+g=0$ ---------------- (ii)

$\frac{\partial F}{\partial y} = 2hx_1+2by_1+2f=0 \quad \Rightarrow hx_1+by_1+f=0$ ----------------- (iii)

From (ii) & (iii) we get,

$$\frac{x_1}{hf-bg} = \frac{y_1}{gh-af} = \frac{1}{ab-h^2}$$

The centre of conic $(x_1,y_1)= \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$

⇒ If the origin is transferred to the centre (x_1,y_1) , the equation of the given conic becomes

$ax^2+2hxy+by^2+c_1=0$ --------------------------------(iv)

where c_1 is constant.

Now

$C_1= ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$

Or, $C_1=gx_1+fy_1+c$

Or, $C_1=\frac{\Delta}{ab-h^2}$ $[\Delta=abc+2fgh-af^2-bg^2-ch^2]$

⇒ Now to get the standard form of the equation of given conic, we have to remove the xy term from equation (iv) using invariant rule.

The standard form is **$a'x^2+b'y^2+c_1=0$ -------------------------(v)**

Note. To get the standard form of conic (which have centre like circle, ellipse & hyperbola) at first, we have to determine the centre and then, we have to remove x, y & xy terms. Parabola has no centre.

Q.1. **Reduce the equation.**

$f(x, y) = 8x^2 + 4xy + 5y^2 - 24x - 24y = 0$ ... [ME-06, 07 (same)] ... (1)

to the standard form and find its all properties.

In the Eq. (i) $a = 8, h = 2, b = 5, g = -12, f = -12, c = 0$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = -$ ve and $ab - h^2 = 36$.

Therefore, $\Delta \neq 0, ab - h^2 > 0$

(i) The equation (i) will represent an ellipse.

$\frac{\delta f(x, y)}{\delta x} \equiv 16x + 4y - 24 = 0$, and $\frac{\delta f(x, y)}{\delta y} \equiv 10y + 4x - 24 = 0$

or, $4x + y - 5 = 0$ and $2x + 5y - 12 = 0$, Centre of the conic is the intersection of these two lines. Solve $x_1 = 1, y_1 = 2$.

(ii) Centre (1, 2)

$C_1 = gx_1 + fy_1 + c = -12 \cdot 1 - 12 \cdot 2 + 0 = -36$

The eq. (1) referred to the centre (1, 2) is

$8x^2 + 4xy + 5y^2 - 36 = 0$, or, $\frac{2}{9}x^2 + \frac{1}{9}xy + \frac{5}{36}y^2 = 1$ ... ... (2)

Here $A = 2/9, H = 1/18, B = 5/36$ ... ... (3)

∴ The lengths of the semi-axes are given by

$\frac{1}{r^4} - (A + B)\frac{1}{r^2} + AB - H^2 = 0$ by 5, Art. 51.

Put the values of A, B and H from (3) then

$r^4 - 13r^2 + 36 = 0$ or, $r^2 = 9$ or, 4 i.e.; $r_1 = 3, r_2 = 2$ ... ... (4)

(iii) The lengths of the axes are 6 and 4.

The equations of the major and minor axes are

$(A - 1/r_1^2)x + Hy = 0$ and $(A - 1/r_2^2)x + Hy = 0$

or, $2x + y = 0$ and $x - 2y = 0$ by (3) and (4), referred to the centre (1, 2).

or, $(x - 1) + (y - 2) = 0$ and $(x - 1) - 2(y - 2) = 0$; referred to the old orgin.

or, $2x + y - 4 = 0$ and $x - 2y + 3 = 0$

(iv) Equation of the major axis is $2x + y - 4 = 0$ ... ... (5)

(v) Equation of the minor axis is $x - 2y + 3 = 0$ ... ... (6)

(Eccentricity. $e^2 = 1 - r_2^2/r_1^2 = 1 - 4/9$ ∴ $e = \sqrt{5}/3$ ... ... (7)

Now d for foci S and S' $= \pm ae = \pm r_1 e = \pm\sqrt{5}$ by (4) and (7)

(vi) Solpe of major axis, eq. (5)

$\tan\theta = -2$ and $\sin\theta = 2/\sqrt{5}, \cos\theta = -1/\sqrt{5}$ ... ... (8)

(h, k) *i.e.*: (1, 2) are the centre of the conic (1) by (II) of

(a) $d = \sqrt{(1^2 + 2^2)} = \sqrt{5}$

(vii) Focus S ; $(h + d\cos\theta, k + d\sin\theta) = (1 + \sqrt{5}. -1/\sqrt{5}, 2 + \sqrt{5}. 2/\sqrt{5}) = (0, 4)$.

Focus S', $(1 - \sqrt{5}. -1/\sqrt{5}, 2 - \sqrt{5}.2/\sqrt{5}) = (2, 0)$

d **for vertices A and A',** $= \pm a = \pm r_1 = \pm 3$

d **for feet of directrices Z and Z'** $= \pm a/e = \pm r_1/e = \pm 9/\sqrt{5}$.

(viii) Vertex A, $(1 + 3. -1/\sqrt{5}, 2 + 3.2/\sqrt{5}) = (1 - 3/\sqrt{5}, 2 + 6/\sqrt{5})$

Vertex A' $(1 - 3. -1/\sqrt{5}, 2 - 3.2/\sqrt{5}) = (1 + 3/\sqrt{5}, 2 - 6/\sqrt{5})$

(ix) Point Z $(1 + 9/\sqrt{5}. -1/\sqrt{5}, 2 + 9/\sqrt{5}. 2/\sqrt{5}) = (-4/5, 28/5)$

Point Z' $(1 - 9/\sqrt{5}. -1/\sqrt{5}, 2 - 9/\sqrt{5}. 2/\sqrt{5}) = (14/5, -8/5)$

(x) d for the points B and B' $= \pm b = \pm r^2 = \pm 2$

Slope of minor axis : $\tan\theta = \frac{1}{2}$, $\sin\theta = 1/\sqrt{5}$., $\cos = 2/\sqrt{5}$

(h, k) is the centre $(1, 2)$

Points B and B' $= (h \pm d\cos\theta, k \pm d\sin\theta)$

$= (1 \pm 2.2/\sqrt{5}, 2 \pm 2.1/1/\sqrt{5}) = (1 \pm 4 - \sqrt{5}, 2 \pm 2\sqrt{5})$

(xi) Length of the latus rectum $= \frac{2b^2}{a} = 2r_2^2/r_1 = 8/3$

(xii) Distance between S and L or, L' is given by

$d = \pm b^2/a = \pm r_2^2/r_1 = \pm 4/3$.

and (h, k) stands for S (0, 4) or, S' (2, 0) by (vii)

Solpe of latus rectum is the same as the solpe of minor axis

Points : L and L' $(0 \pm 4/3.2\sqrt{5}, 4 \pm 4/3.1/\sqrt{5}) = (+ 8/3\sqrt{5}, 4 + 4./3\sqrt{5})$

Similarly another pair of points with respect to S'(2, 0) can be determined.

(xiii) Equation of the latus rectum which is parallel to minor axis $x - 2y + 3 = 0$ is $x - 2y + k = 0$. Since it passes through the foci S (0, 4) or S' (2, 0), hence $k = 8, -2$. Hence equations of the later recta are $x - 2y + 8 = 0$ and $x - 2y - 2 = 0$

(xiv) Directrix through Z $(-4/5, 28/5)$ or, Z' $(14/5, -18/5)$ is also parallel to minor axis $x - 2y + 3 = 0$. Therefore the equation is $x - 2y + 3 = 0$, where $\lambda = 12, -10$.

Hence the equations of the directrices are $x - 2y + 12 = 0$ and $x - 2y - 10 = 0$

Q.2. Reduce the equation $6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$ to the standard form. Find also its, lengths, equations and directions of the axes. [CSE-06, EEE-10]

Let $f(x, y) \equiv 6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$ ... ... (1)

The centre is given by

$\frac{\delta f}{\delta x} \equiv 12x + 5y - 4 = 0, \frac{\delta f}{\delta x} \equiv 5x - 12y + 7 = 0$

Solving, centre is (1/13, 8/13)

$C = -C_1 = (gx_1 + fy_1 + c) = -\left(-2 \times \frac{1}{13} + \frac{7}{2} \times \frac{8}{13} + 11\right) = -13$

Transferring the origin to (1/13, 8/13) the equation (1) reduces to $6x^2 + 5xy - 5y^2 + 13 = 0$ by (5).

The squares of the semiaxes are given by

$(a - k/r_1^2)(b - k/r^2) = h^2$ see (9)

or, $(6 - {}^{-13}/r^2)(-6 - {}^{-13}/r^2) = (5/2)^2$

or, $169r^4 = 169 \times 4$ or, $r^2 = \pm 2$

i. e. $r_1^2 = 2$, and $r_2^2 = -2$

They are of opposite signs and equal, hence the conic (1) forms a rectangular hyperbola.

The lengths of the axes are $2\sqrt{2}$ and $2\sqrt{2}\,i$

The equations of axes are given with respect to the centre.

$(a - k/r_1^2)x + hy = 0$ and $(b - k/r_2^2)x + hy = 0$

or, $(6 + 13/2)x + 5/2.y = 0$ and $(6 - 13/2)x + 5y/2 = 0$

or, $y = -5x$ and $x - 5y = 0$

The equations with respect to the axes

are $(y - 8/13) = -5(x - 1/13) = 0$ and $x - 1/3 - 2(y - 8/13) = 0$

or, $y + 5x - 1 = 0$ and $x - 5y + 3 = 0$

Slopes are $\tan^{-1}(-5)$ and $\tan^{-1}(1/5)$ with the $x -$ axis.

Q.3. Reduce the equation — [EEE-09 (same)]

$16x^2 - 24xy + 9x^2 - 104x - 172y - 44 = 0$ ... ... (1)

to the standard form. Find all its properties.

As the second degree terms of (i) form a perfect square the eq.

(i) will represent a parabola since $\Delta \neq 0$ and $ab = h^2$

$\therefore (4x - 3y)^2 = 104x + 172y - 44$ ... ... (2)

The lines $4x - 3y = 0$ and $104x + 172y - 44 = 0$ are not at right angles, hence let us introduce a constant λ in (2)

$(4x - 3y + \lambda)^2 = (104x + 172y - 44) + 2\lambda(4x - 3y) + \lambda^2$

or, $(4x - 3y + \lambda)^2 = x(104 + 8\lambda) + y(172 - 6\lambda) + (\lambda^2 - 44)$

∴ The lines $4x - 3y + \lambda = 0$ ... ... (4)

and $x(104 + 8\lambda) + y(172 - 6\lambda) + (\lambda^2 - 44) = 0$ ... ... (5)

are at right angles if (apply $a_1a_2 + b_1b_2 = 0$

$4(104 + 8\lambda) + (-3)(172 - 6\lambda) = -1$ or, $\lambda = 2$ ... ... (6)

Now eq. (3) becomes by (6)

$(4x - 3y + 2)^2 = 40(3x + 4y - 1)$

which can be written as

$\left(\frac{4x - 3y + 2}{\sqrt{4^2 + 3^2}}\right)^2 . (25) = 40.5\left(\frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}}\right)$

or, $PN^2 = \frac{40.5}{25} PM$ or, $Y^2 = 8X$

Which is the stantard form of eq. (1)

or, $Y^2 = 4PX$. Where $P = \frac{1}{4} . 8 = 2$

For other properties of $Y^2 = 8x = 4pX$ ... ... (7)

Here $Y = \frac{4x - 3x + 2}{\sqrt{(4^2 + 3^2)}}$ and $X = \frac{3x + 4y - 1}{\sqrt{(3^2 + 4^2)}}$ ... ... (8)

(i) Axis of (7) is $Y = 0$ or, $4x - 3y + 2 = 0$, by (8) ... ... (9)

(ii) Tangent at vertex is $X = 0$, or, $3x + 4y - 1 = 0$ by (8) ... ... (10)

(iii) Vertex is the intersection of eq. (9) and (10) by (8)

i. e. $4x - 3y + 2 = 0$ and $3x + 4y - 1 = 0$, Solve.

$x = -\frac{1}{3}, y = \frac{2}{3}$ Vertex $\left(-\frac{1}{3}, \frac{2}{3}\right)$

(iv) Latus rectum is $4p = 8$

(v) Focux $X = p, Y = 0$, by (8)

or, $\frac{3x + 4y - 1}{\sqrt{(3^2 + 4^2)}} = 2$ and $\frac{4x - 3y + 2}{\sqrt{(4^2 + 3^2)}} = 0$

or, $3x + 4y - 11 = 0$ and $4x - 3y + 2 = 0$, Solve

∴ $x = 1, y = 2$. The focus is $(1, 2)$

(vi) Foot of the directrix. $X = -p$ and $Y = 0$ by (8)

or, $\frac{3x + 4y - 1}{\sqrt{(3^2 + 4^2)}} = -2$ and $\frac{4x - 3y + 2}{\sqrt{4^2 + 3^2}}$

or, $3x + 4y + 9 = 0$ and $4x - 3x + 2 = 0$. Now solve. The foot of the directrix is $(-7/5 - 6/5)$

(vii) Equation of the directrix. $X = -p$ by (8)

or, $\frac{3x+4y-1}{\sqrt{(3^2+4^2)}} = -2$ ro, $3x + 4y + 9 = 0$

(VIII) Equation of the latus rectum $X = p$ by (8)

or, $\frac{3x+4y-1}{\sqrt{(3^2+4^2)}} =$ or, $3x + 4y - 11 = 0$

Q.4. Reduce the equation [CSE-11,12,09,]

$4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$ ... ... (1)

to the standard from. Find all its properties.

In the eq. (1), $a = 4$, $b = 1$, $h = -2$, $g = -4$, $f = -3$, $c = 5$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = -64$, $ab - h^2 = 4 - 4 = 0$

$\Delta \neq 0$ and $ab = h^2$. Hence eq. (1) represents a parabola. The eq. (1) may be written as

$(2x - y)^2 = 8x + 6y - 5$ ... ... (2)

or, $(2x - y + \lambda)^2 = 8x + 6y - 5 + \lambda^2 + 2\lambda(2x - y)$

or, $(2x - y + \lambda)^2 = x(4\lambda + 8) + y(-2\lambda + 6) + (\lambda^2 - 5)$ ... ... (3)

The lines $2x - y = \lambda = 0$ ... ... (4)

and $x(4\lambda + 1) + y(-2\lambda + 6) + (\lambda^2 - 5) = 0$ ... ... (5)

may be perpendicular if $a_1a_2 + b_1b_2 = 0$

i. e. $2(4\lambda + 1)(-1)(-2\lambda + 6) = 0$ or, $\lambda = -1$ ... ... (6)

The eq. (3) becomes

$(2x - y - 1)^2 = 4(x + 2y - 1)$ which can be written as

$\left(\frac{2x-y-1}{\sqrt{(2^2+1^2)}}\right)^2 5 = 4\sqrt{5}\left(\frac{x+2y-1}{\sqrt{(1^2+2^2)}}\right)$

It is of the form $PN^2 = 4(\sqrt{5})/5.\ PM.$

or, $Y^2 = 4/\sqrt{5}.\ X.$ which is the standard equation of a parabola.

(i) Latus rectum $= 4p = 4/\sqrt{5}$ or, $p = 1/\sqrt{5}$

$X = (x + 2y - 1)/\sqrt{5}$ and $Y = (2x - y - 1)/\sqrt{5}$ ... ... (7)

$\therefore\ Y^2 = 4pX.$

(ii) Axis is $Y = 0$ or, $2x - y - 1 = 0$ from (7) ... ... (8)

(iii) Tangent at the vertex $X = 0$ i. e. $x + 2y - 1 = 0$ from (7) ... ... (9)

(iv) Vertex A is the intersection of eqs. (8) and (9).

Solve $x = 3/5$, $y = 1/5$; A $(3/5, 1/5)$

(v) Focus is $S(p, 0)$ i.e. ; $X = p$ and $Y = 0$

or, $\frac{x+2y-1}{\sqrt{5}} = \frac{1}{\sqrt{5}}$ and $\frac{2x-y-1}{\sqrt{5}} = 0$ from (7)

or, $x + 2y - 2 = 0$ and $2x - y - 1 = 0$

Solve, $x = 4/5$, $y = 3/5$; $S(4/5, 3/5)$

(vi) Equation of latus rectum is $X = p$

or, $(x + 2y - 1)/\sqrt{5} = 1/\sqrt{5}$ or, $x + 2y - 2 = 0$

(vii) Foot of the directrix $Z(-p, 0)$ i. e. $X = -p$, $Y = 0$

or, $(x + 2y - 1)/\sqrt{5} = -1/\sqrt{5}$ and $(2x - y - 1)/\sqrt{5} = 0$

Solve, $x = 1/5$, $y = -1/5$ $\therefore$ $Z(4/5, -1/5)$.

(viii) Equation of the directrix is $X = -p$

or, $(x + 2y - 1)/\sqrt{5} = -1/\sqrt{5}$ or, $x + 2y = 0$

Q.5. Reduce the equation $x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$ to the standard form. Find also the equations of latus rectum, directrices and axes. [ME-08, EEE-07,06,]

Let $f(x, y) \equiv x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$ ... ... (1)

(i) its centre is at $\left(0, \frac{1}{2}\right)$

Now transfer the origin to the point $\left(0, \frac{1}{2}\right)$, the eq. (1) reduces to $x^2 + 12xy - 4y^2 + c_1 = 0$

where $c_1 = gx_1 + fy_1 + c = -3\cdot 0 + 2.\frac{1}{2} + 9 = 10$

(ii) The reduced equation is $x^2 + 12xy - 4y^2 + 10 = 0$

or, $-x^2/10 - 12xy/10 + 4y^2/10 = 1$

(iii) The lengths of the semi axes are given by

$(a - 1/r^2)(b - 1/r^2) = h^2$ or, $(-1/10 - 1/r^2)(4/10 - 1/r^2)$

$= (-6/10)^2$

or, $r^2 = 5/4$ or, -2, $\therefore$ $r_1^2 = 5/4$: $r_2^2 = -2$ ... ... (2)

The conic is a hyperbola.

(iv) Eq. of the transverse axis is

$(a - 1/r_1^2)x + hy = 0$ or, $(-1/10 - 4/5)x - 3/5y = 0$ or, $3x + 2y = 0$

referred to the centre $\left(0, \frac{1}{2}\right)$ or, $3(x - 0) + 2(y - 1/2) = 0$

referred to the origin, or, $3y + 2y - 1 = 0$ ... ... (3)

Slope of it, $\tan\theta = -3/2$. $\sin\theta = 3/\sqrt{13}$, $\cos\theta = -2/\sqrt{13}$ ... ... (4)

through $\left(0, \frac{1}{2}\right)$, then $k = 3/2$, The equation is $4x - 6y + 3 = 0$

(v) Eccentricity. $e^2 = 1 - r_2^2/r^2_1 = 1 + 2.4/5$

or, $e = \sqrt{(13/5)}$ ... ... (5)

(vi) d for foci S and S' $= \pm ae = \pm r_1 e \pm \sqrt{(13/2)}$

d for feet of the directriecs Z and Z' $= \pm a/e = \pm 5/2\sqrt{13}$

(vii) Here (h, k) are the centre $\left(0, \frac{1}{2}\right)$ of conic (1)

Points S and S' $= h + d\cos 0, k + d\sin\theta$

$= 0 \pm \frac{1}{2}\sqrt{\ } 13.2\sqrt{\frac{1}{13}}; \frac{1}{2} \pm \frac{1}{2}\frac{13}{}\sqrt{13}\, 3\sqrt{\frac{1}{13}} = (-1, 2)$ and $(1, -1)$

(viii) points Z and Z' $= 0 \pm \frac{5}{2}\sqrt{\frac{1}{13}}.-2\sqrt{\frac{1}{13}}, \frac{1}{2} \pm \frac{5}{2\sqrt{13}} - 3\sqrt{\frac{1}{13}}$

$= (-5/13, 14/13)$ and $(5/13, -1/13)$

(ix) Latus rectum and directrix are parallel to the conjugate axis $2x - 3y + 3/2 = 0$. Therefore their slopes are the same as conjugate axis.

(x) The equation of the latus rectum is $2x - 3y + k = 0$. Since it passes through S$(-1, 2)$ or, S'$(1, -1)$: hence $k = 8$ or, -4, The equations are $3x - 2y + 8 = 0$ and $3x - 2y - 4 = 0$

(xi) Similarly the equation of directrix is $3x - 2y + \lambda = 0$

Since it passes through Z$(-5/13, 14/13)$ or, $(5/13, -1/13)$

hence $\lambda = 43/13, -17/13$.

The equation of directrices $3x - 2y + 4 = 0$ and $3x - 2y - 1 = 0$

Q.6 **Find the lengths and equation of the axis of the conic** $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$. [ME-10]

Let $f(x, y) \equiv 36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$

The centre is given by

$\frac{\delta f}{\delta x} \equiv 72x + 24y - 72 = 0$ and $\frac{\delta f}{\delta x} \equiv 24x + 58y + 126 = 0$

Solving the centre is $(2, -3)$

$C_1 = gx_1 + fy_1 + c = -36 \cdot 2 + 63 . -3 + 81 = -180$

Therefore equation of the conic referred to centre $(2, -3)$ as origin is

$36x^2 + 24xy + 29x^2 - 180 = 0$ by (5) working Rule Art. 50 or, $x^2/5 + \frac{2}{15}xy + \frac{29}{180}y^2 = 1$

Now semi-axis of the conic are given by (See Eq. 4. Art. 51)

$(1/5 - 1/r^2)(29/180 - 1/r^2) = (1/15)^2$

or, $1/r^4 - \frac{13}{36}r + \frac{1}{36} = 0$ or, $r^4 - 13r^2 + 36 = 0$

or, $(r^2 - 9)(r^2 - 4) = 0$, $\therefore r = 3$, or, $+ 2$ i. e., $r_1 = 3$, $r_2 = 2$

The curve is an ellipse.

Major axis = 6, Minor axis = 4

Now equation to major axis referred to the centre $(2, -3)$ as origin is $(a - 1/r_1^2)x + hy = 0$

or, $(1/5 - 1/9)x + y/15 = 0$ by (6)

or, $4x + 3y = 0$

And referred to the old axis

$4(x - 2) + 3(y + 3) = 0$ or, $4x + 2y + 1 = 0$

Similarly the equation to minor axis referred to the centre

$(2, -3)$ is $(a - 1/r^2)x + hy = 0$ or, $(1/5 - 1/4)x + y/15 = 0$.

or, $3x = 4y$ and referred to the old axes is (see Art. 32)

$3(x - 2) = 4(y + 3)$ or, $3x - 4y - 18 = 0$

Lengths of the axes are 6 and 4.

And equations of major and minor axes respectively are $4x + 3y + 1 = 0$ and $3x - 4y - 18 = 0$.

CHAPTER-4

SYSTEM OF CIRCLES[2D]

Ex. 1. Prove that the circles $x^2 + y^2 - 3x + 8y - 2 = 0$ ... ... (i)

and $x^2 + y^2 + 4x - 5y - 24 = 0$ cut orthogonally ... ... (ii)

O_1 $(3/2, -4)$, $r_1 = [\sqrt{(9/4 + 16 + 2)}] = 9/2$, are the co-ordinates of centre and radius respectively of (i)

$O_2(-2, 5/2)$, $r_2 = \sqrt{[(4 + 25/4 + 24)]} = \sqrt{(137/4)}$, be the co-ordinates of the centre and radius respectively of ... ... (iii)

$O_1(3/2, -4) : r_1 = [\sqrt{(9/4 + 16 + 2)}] = 9/2$.

$O_1O_2^2 = (3/2 + 2)^2 + (-4 - 5/2)^2 = 218/4$.

$r_1^2 + r_2^2 = 81/41 + 37/4 = 218/4$.

$O_1O_2^2 = r_1^2 + r_2^2$ Hence they are orthogonal

Ex. 2. Find the equation of the circle which cuts the circle

$x^2 + y^2 - 6x + 8 = 0$ ... (i) and $x^2 + y^2 - 2x - 2y - 7 = 0$ ... ... (ii)

Orthogonally.

Let the equation to the circle passing through the origin be $x^2 + y^2 + 2gx + 2fy = 0$ ... ... (iii)

Since (iii) cuts (i) and (ii) orthogonally, then

$-6g + 2f.0 - 8 = 0$ and $-2g - 2f + 7 = 0$

or, $6g + 8 = 0$. or, $g = -4/3$ and $2g + 2f - 7 = 0$, or, $2f = 29/3$

The required equation is $x^2 + y^2 - 2(4/3)x + 29y/3 = 0$

or, $3x^2 + 3y^2 - 8x + 29y = 0$

Ex. 3. Find radical axis of the pairs of circles

$x^2 + y^2 = 144$ ... (i) $x^2 + y^2 - 15x + 11y = 0$ ... ... (ii)

The equation to the radical axis of (i) and (ii)

$(x^2 + y^2 - 144) - (x^2 + y^2 - 15x - 11y) = 0$ or, $15x - 11y - 144 = 0$

Ex. 4. Find the radical centre of the three circles

$x^2 + y^2 + x + 2y + 3 = 0$, $x^2 + y^2 + 2x + 4y + 5 = 0$

and $x^2 + y^2 - 7x - 8y - 9 = 0$

The radical axis of the circles is $x^2 + y^2 + x + 2y + 3 - (x^2 + y^2 + 2x + 4y + 5) = 0$

or, $x + 2y + 2 = 0$ ... ... (1)

$x^2 + y^2 + x + 2y + 3 = 0$ and $x^2 + y^2 - 7x - 8y - 9 = 0$

is $x^2 + y^2 - 7x - 8y - 9 - (x^2 + y^2 + x + 2y + 3) = 0$

or, $2x + 5y + 6 = 0$ ... ... (1)

Solve (1) and (2) for x and y.

$x = -2/3$ and $y = -2/3$. The centre $(-2/3, -2/3)$

Ex. 5. Find the equation of the circle which cuts orthogonally each of the three circles.

$S_1 \equiv x^2 + y^2 + x + 2y + 3 = 0$ ... ... (i)

$S_2 \equiv x^2 + y^2 + 2x + 4y + 5 = 0$ ... ... (ii)

$S_3 \equiv x^2 + y^2 - 7x - 8y + 9 = 0$ ... ... (iii)

The raidcal axes of the pairs of circles (i), (ii) and (iii) are respectively given by the equations

$S_2 - S_1 \equiv x + 2y + 2 = 0$ ... ... (iv)

$S_1 - S_3 = 8x + 10y - 6 = 0$ or, $4x + 5y - 3 = 0$ ... ... (v)

The intersecting of the lines (iv) and (v) is $(16/3, -11/3)$, which is the radical centre of

The square of the tangent from $(16/3, -11/3)$ to the circle (1) $= (16/3)^2 + (11/3)^2 + (16/3 - 22/3 + 3) = (16/3)^2 + (11/3)^2 + 1$

Hence the equation of the radical circles is

$(x - 16/3)^2 + (y + 11/3)^2 = (16/3)^2 + (11/3)^2 + 1$

or, $x^2 + y^2 - \frac{32}{3}x + \frac{22}{3}y = 1$ or, $3(x^2 + y^2 =) - 32x + 22y - 3 = 0$

Ex. 6. Find the co-ordinates of the limiting point of the co-axial system determined by circles $x^2 + y^2 + 4x + 2y + 5 = 0$ and $x^2 + y^2 + 2x + 4y + 7 = 0$ belong

The radical axis of two given circles is

$(x^2 + y^2 + 4x + 2y + 5) - (x^2 + y^2 + 2x + 4y + 7) = 0$

or, $2x - 2y - 2 = 0$ or, $x - y - 1 = 0$

Hence the equation of any circle co-axial with the given circles

is $x^2 + y^2 + 4x + 2y + 5 + \lambda(x - y - 1) = 0$

or, $x^2 + y^2 + x(4 + \lambda) + y(2 - \lambda) + (5 - \lambda) = 0$ ... (1) (By Art. 89)

Radius of this circle

$= \sqrt{[\{(4 + \lambda)/2\}^2 + \{(2 - 1)/2\}^2 - (5 - \lambda)\}]}$

Thus the radius is zero *i.e.*

$16 + \lambda^2 + 8\lambda + 4 - 4\lambda + \lambda^2 - 20 + 4\lambda = 0$ or, $2\lambda^2 + 8\lambda = 0$

$\therefore \lambda = 0, -4$

Co-ordinates of centre of (i) $-(4 + \lambda)/2, -(2 - \lambda)/2$

Putting the value λ the limiting points are $(-2, -1), (0, -3)$

Ex. 7. Find the equations of the circle through the intersection of the circles.

$x^2 + y^2 + 5x + 6y + 7 = 0$ ... ... (i)

$x^2 + y^2 + 4x + 3y + 2 = 0$ ... ... (ii)

whose centre lies on the line $x + 3y + 5 = 0$

Any circle through common points of the given circles is

$x^2 + y^2 + 5x + 5y + 7 - \lambda(x^2 + y^2 + 4x + 3y + 2) = 0$

or, $(1 - \lambda)x^2 + (1 - \lambda)y^2 + x(5 - 4\lambda) + y(6 - 3\lambda) + (7 - 2\lambda) = 0$ (iii) Centre of (ii) is

$\frac{5 - 3\lambda}{2(1 - \lambda)}, -\frac{6 - 3\lambda}{2}(1 - \lambda)$ or, $\left(\frac{4\lambda - 5}{2}(1 - \lambda)\right), \frac{3\lambda - 6}{2(1 - \lambda)}$

Since the centre lies on $x + 3y + 5 = 0$

or, $\frac{4\lambda -}{2(1 - \lambda)} + 3\frac{(3\lambda - 6)}{2(1 - \lambda)} + 5 = 0$ or, $\lambda = 13/3$

From (iii), putting the value of λ we have

$10(x^2 + y^2) - 37x + 21y + 5 = 0$

Ex. 8. Find the equation of the circle whose diameter is the common chord of the circle $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$ [GCE-12

The circle whose diameter is the common chord of the circle

$S_1 \equiv x^2 + y^2 + 2x + 3y + 1 = 0$ ... ... (i)

and $S_2 \equiv x^2 + y^2 + 4x + 3y + 2 = 0$ ... ... (ii)

passes through their points of intersection and has its centre on the radical axis of (i) and (ii). The radical axis of (i) and (ii) is $x^2 + y^2 + 2x + 3y + 1 - (x^2 + y^2 + 4x + 3y + 2) = 0$

or, $2x + 1 = 0$ ... ... (iii)

Any circle passing through the point of intersection of the two circles is $S_1 + \lambda S_2 = 0$

or, $x^2 + y^2 + \frac{2(1+2\lambda)}{1+\lambda}x + \frac{3(1+\lambda)}{1+\lambda}y + \frac{1+2\lambda}{1+\lambda} = 0$

Its centre is $\left(-\frac{1+2\lambda}{1-\lambda}, \frac{-3}{2}\right)$

As the centre lies on $2x + 1 = 0$

$\therefore \quad 2x\left(-\frac{1+2\lambda}{1-\lambda}\right) + 1 = 0$, or $\lambda = -\frac{1}{3}$

Hence the required circle is $S_1 - \frac{1}{3}S_2 = 0$

or, $2(x^2 + y^2) + 2x + 6y + 1 = 0$

Ex-9. Prove that the two circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

The radical axis of the two circles is

$x^2 + y^2 + 2ax + c^2 - (x^2 + y^2 + 2by + c^2) = 0$ or, $ax - by = 0$ (1)

The circles will touch each other if this line touches them i.e. if its distance from the centre of the circles is equal to the radius of that circle. Now the centre of the circle.

$x^2 + y^2 + 2ax + c^2 = 0$ is $(-a, o)$, and radius $= \sqrt{(a^2 - c^2)}$

$\therefore$ The required condition is

$\frac{a(-a) - b(0)}{\sqrt{(a^2 + b^2)}} = \sqrt{(a^2 - c^2)}$ or, $\frac{a^2}{a^2 + b^2} = a^2 - c^2$

or, $a^2 - a^2c^2 + b^2a^2 - b^2c^2 = a^2$ or, $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

Ex-10. The circle $x^2 + y^2 + 2x - 4y - 11 = 0$ and the line $x - y + 1 = 0$ intersect at A and B. Find the equation of the circle on AB as diameter and the equation of the circle through A, B orthogonal to the given circles.

The equation of the circle through the point of intersection

or $x^2 + y^2 + 2x - 4y - 11 = 0$ ... ... (1)

and $x - y + 1 = 0$ ... ... (2)

is $x^2 + y^2 + 2x - 4y - 11 + \lambda(x - y + 1) = 0$

Represents a circle through A and B. If its centre

$(-1 + \lambda/2, 2 - \lambda/2)$ lies on the line $x - y + 1 = 0$, $\lambda = -2$

Hence the equation of the circle on AB as diameter from (3) is

$x^2 + y^2 - 2y - 13 = 0$

The circle (3) is orthogonal to (1) if by cor.

$(2 - \lambda) + 2(4 - \lambda) = -\lambda - 22$ or, $\lambda = 16$

Hence the equation of the orthogonal circle by putting the value of λ in (3) is $x^2 + y^2 - 14x - 12y - 27 = 0$

CHAPTER-5

RECTANGULAR CO-ORDINATES [3D]

Direction Cosines And Direction Ratios

- **Projection[IPE'12]:-** Let ,ST be a straight line and AB be a segment of ST line .
  Draw two perpendicular from A and B on ST . Suppose the perpendicular intersect ST at M and N Then MN is projection of AB on ST.
  MN=AB cos θ . Where θ is the angle between the line AB and ST.
  ## If P and Q be two points , then the projection of PQ on a line (whose direction cosines are *l , m , n) is*
  $$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

- **Direction Cosines[ME'11,'10,'08,'07'06,03,CSE'08,'06,ETE,'11,10,08'07,IPE'07]:-** If a given line makes angles α ,β,γ with the positive direction of X,Y & Z axes respectively, then cosα ,cosβ ,&cos γ are called the direction cosines of the line denoted by l, m & n respectively .

Provethat $cos^2\alpha+cos^2\beta+cos^2\gamma=1$[CSE'07] or $l^2+m^2+n^2=1$ [ME'10,'06,03,ETE'11,10,08,'07,'06,IPE,06] where *l,m,n* be the direction cosines of a line, or $sin^2\alpha+sin^2\beta+sin^2\gamma=2$[IPE'11,06]

Let, OP be the straight line whose direction cosines are cosα ,cosβ ,&cosγ .
Again ,suppose P(x, y, z) be any point on OP ,Now we take projection of OP on the axes

$x=r\cos\alpha$ ....(1)

$y=r\cos\beta$...............(2)

$z=r\cos\gamma$ ....(3)

now squaring the eqution (1), (2)&(3) and adding we get

$x^2+y^2+z^2=r^2(cos^2\alpha + cos^2\beta + cos^2\gamma)$

$r^2 = r^2(cos^2\alpha + cos^2\beta + cos^2\gamma)$ [where $x^2 + y^2 + z^2 = r^2$]

$\therefore cos^2\alpha+cos^2\beta+cos^2\gamma=1$

$\therefore l^2+m^2+n^2=1$

$\therefore 1 - sin^2\alpha+ 1 - sin^2\beta+1 - sin^2\gamma=1$ [$cos^2\alpha = 1 - sin^2\alpha$]

$\therefore \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

[*proved*]

R> **Direction- Ratios[ME'11,'10,CSE'08,'06,ETE'11,'10,'07]:-** Any three numbers a, b,c which are proportional to the direction cosines l, m, n respectively of the given line are called direction ratios of the given line.

Now from the definition of direction ratios we have

$$l:m:n=a:b:c$$

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}=\pm\frac{1}{\sqrt{a^2+b^2+c^2}} \qquad [l^2+m^2+n^2=1]$$

$$\therefore\ l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\quad m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}},\quad n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}};$$

Where the same sign either +ve or –ve is to be chosen through out .

- The relation from which we can determine the direction cosines from direction ratios are

$l=\frac{a}{\sqrt{a^2+b^2+c^2}}$, $m=\frac{b}{\sqrt{a^2+b^2+c^2}}$,$n=\frac{c}{\sqrt{a^2+b^2+c^2}}$; where l, m, n are direction cosines and a, b, c are direction ratios

- **Prove that $x_2-x_1, y_2-y_1, z_2-z_1$ *is* the direction ratios of a ST line which passing through P $(x_1,y_1,z_1)$and Q(x_2,y_2,z_2)[ME'11,10,IPE'07]**

__Solution:-__ Shift the origin from O(0,0,0) to P(x_1,y_1,z_1).Suppose $(X,Y,Z)$is the position

of the points Q with respect to new origin

There fore,

$$X=x_2-x_1$$

$$Y=y_2-y_1$$

$$Z=z_2-z_1$$

Now taking projection of PQ on X axis we get

$$X=rl \quad => \frac{l}{X}=\frac{1}{r}$$

Similarly Y=rm , Z=rn

$$\frac{m}{Y}=\frac{1}{r},\ \frac{n}{Z}=\frac{1}{r}$$

$$\therefore \frac{l}{X} = \frac{m}{Y} = \frac{n}{Z} = \frac{1}{r}$$

$$\therefore \frac{l}{x_2 - x_1} = \frac{m}{y_2 - y_1} = \frac{n}{z_2 - z_1} = \frac{1}{r}$$

Hence $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ *is* the direction ratios of a ST line.

(Proved)

Angle between two lines:

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Or, $$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note. 1. The lines will be perpendicular if, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

Or, $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

2. The lines will be parallel if, $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ or, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Solved Problems

Q.1. Find the direco n cosines of PQ, where P(1,2,3) & Q(2,5,6).[ETE'08(Same),'07]

Solution: Let, l,m,n are the d.c.s of PQ

$$\therefore \frac{l}{2-1} = \frac{m}{5-2} = \frac{n}{6-3} = \frac{l^2+m^2+n^2}{\sqrt{(2-1)^2+(5-2)^2+(6-3)^2}}$$

$\therefore l = \frac{1}{\sqrt{19}}$, $m = \frac{3}{\sqrt{19}}$, $n = \frac{3}{\sqrt{19}}$ Ans.

Q.2. The projeco n of a line on the axes are 2,3,6 ; what is the length of the line .[IPE'12,GCE'12]

Solution:- Let , the length of the line is r. and l,m,n be the direction cosines .Then

Projection on X-axis rl=2 ....(1)

Projection on Y-axis rm=3 ....(2)

Projection on Z-axis rn=6 ....(3)

Now squaring and adding equao n (1),(2)&(3) we get

$r^2\ (l^2+m^2+n^2)$= 4+9+36

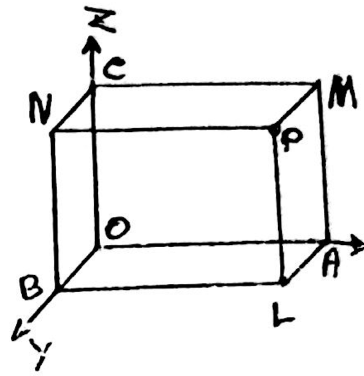
=>r^2=49

∴r=7, Hence the length of the line is 7.

Q.3. If the edges of a rectangular parallelepiped are a,b,c show that the angles between the four diagonals are given by $\cos^{-1}(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2})$.[CSE'12,IPE'11,10,ETE'10,GCE'12]

Solution: Let the eight vertices of the rectangular parallelepiped are O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c), L(a,b,0), N(0,b,c), M(a,0,c) & P(a,b,c)

Here, OP, AN, BM, & CL are the four diagonals.



Now, the d.c.s of OP are, $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$

Similarly, d.c.s of AN are, $\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$

d.c.s of BM are, $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$

d.c.s of CL are, $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}$

Angle between OP & AN is, $\cos\theta = \frac{-a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2}$

Or, $\cos\theta = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$ $[\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2]$

$\therefore \theta = \cos^{-1}(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2})$

Similarly, the angle between any one of the six pairs of diagonals can be found.

Hence the angle in general between the six pairs are given by

$\cos^{-1}(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2})$ **(Proved)**

Q.4. Find the direction cosine of a line that makes equal angles with the axes. Also find the equal angles.[ETE'12]

Solution:- Let, the line makes the angle α with the axes

∴ cos α, cos α , cos α are the direction cosine of a line

$\therefore \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Hence the direction cosine of a line are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$ (Ans)

$\therefore \alpha = \cos^{-1}\frac{1}{\sqrt{3}}$.

Equal angles are $\cos^{-1}\frac{1}{\sqrt{3}}$. (Ans)

Q.5. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosine of two ST line and α is the angle between them ,hence show that cos α $=l_1 l_2+m_1 m_2+n_1 n_2$.[ME'08,IPE'08]

Solution:- Through the origin O drawn two lines OP_1 of length r_1 and OP_2 of length r_2

Parallel respectively to the given line with direction cosine l_1, m_1, n_1 and l_2, m_2, n_2

$\therefore l_1=\frac{x_1}{r_1}$,or $x_1=l_1 r_1$

Similarly , $y_1=m_1 r_1,\ z_1=n_1 r_1$

Also $x_2=l_2 r_2$, $y_2=m_2 r_2$, $z_2=n_2 r_2$

Since α be the angle between the given line . Then α is also equal to the angle between OP_1 and OP_2

Now ,The projection of line OP_2 joining (0,0,0) and (x_2, y_2, z_2) on the lines OP_1 whose direction cosine are l_1, m_1, n_1 is

$(x_2-0)\, l_1+(y_2-0)\, m_1+(z_2-0)\, n_1=l_1 x_2+m_1 y_2+n_1 z_2$

But the projection of OP_1 is OP_2 cos α

$\therefore OP_2 \cos\alpha = l_1 x_2+m_1 y_2+n_1 z_2$

Or, $r_2 \cos\alpha = l_1 l_2 r_2 + m_1 m_2 r_2 + n_1 n_2 r_2$

Or, $r_2 \cos\alpha = r_2\,(l_1 l_2 + m_1 m_2 + n_1 n_2)$

$\therefore \cos\alpha = l_1 l_2+m_1 m_2+n_1 n_2$. **(Proved)**

(**Note** : if the two line are perpendicular then α=90

cos 90=0, then $l_1 l_2+m_1 m_2+n_1 n_2=0$)

Q.6. If $l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3$ are the direction cosines of three mutually perpendicular lines. The line whose direction cosines are perpendicular to $l_1+m_1+n_1$, $l_2+m_2+n_2$, $l_3+m_3+n_3$ makes equal angles with them. (prove it)

Or

Then find the direction cosine of a line whose direction cosines are $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$ and show that the line is equally inclined to given line.

Solution: Since The lines whose direction cosines are $l_1, m_1, n_1;$ $l_2, m_2, n_2;\ l_3, m_3, n_3$ are mutually perpendicular then

$$\cos 90^0 = l_1l_2+m_1m_2+n_1n_2 = 0 \text{ ....(1)}$$

$$\text{And } l_3l_1+m_3m_1+n_3n_1 = 0\text{......................................(2)}$$

$$\text{Also } l_2l_3+m_23+n_2n_3 = 0\text{......................................(3)}$$

Again $l_1{}^2+m_1{}^2+n_1{}^2 = l_2{}^2+m_2{}^2+n_2{}^2 = l_3{}^2+m_3{}^2+n_3{}^2 = 1$ ....(4)

Let the direction cosine of a line whose direction cosines are $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$ is l, m, n

$$\therefore \frac{l}{l_1+l_2+l_3} = \frac{m}{m_1+m_2+m_3} = \frac{n}{n+n_2+n_3} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{1}{\sqrt{\sum l_1{}^2+\sum m_1{}^2+\sum n_1{}^2}} = \frac{1}{\sqrt{3}}$$

$$l = \frac{l_1+l_2+l_3}{\sqrt{3}}, m = \frac{m_1+m_2+m_3}{\sqrt{3}}, n = \frac{n_1+n_2+n_3}{\sqrt{3}}$$

Let θ_1 be the angle between 1st and 4th line

Then we get

$$\cos\theta_1 = \frac{l_1(l_1+l_2+l_3)}{\sqrt{3}} + \frac{m_1(m_1+m_2+m_3)}{\sqrt{3}} + \frac{n_1(n_1+n_2+n_3)}{\sqrt{3}}$$

$$= \frac{l_1{}^2+m_1{}^2+n_1{}^2+l_1l_2+m_1m_2+n_1n_2+l_3l_1+m_3m_1+n_3n_1}{\sqrt{3}}$$

$$= \frac{1+0+0}{\sqrt{3}}$$

$$\therefore \theta_1 = \cos^{-1}\frac{1}{\sqrt{3}}$$

Again θ_2 be the angle between 2nd and 4th line

Then we have

$$\cos\theta_2 = \frac{l_2(l_1+l_2+l_3)}{\sqrt{3}} + \frac{m_2(m_1+m_2+m_3)}{\sqrt{3}} + \frac{n_2(n_1+n_2+n_3)}{\sqrt{3}}$$

$$= \frac{l_1{}^2+m_1{}^2+n_1{}^2+l_1l_2+m_1m_2+n_1n_2+l_2l_3+m_2m_3+n_2n_3}{\sqrt{3}}$$

$$= \frac{1+0+0}{\sqrt{3}}$$

$$\therefore \theta_2 = \cos^{-1}\frac{1}{\sqrt{3}}$$

Similarly the angle between 3rd and 4th line is $\cos^{-1}\frac{1}{\sqrt{3}}$

Hence the line is equally inclined to the given lines. (Proved)

Q.7. Find the angle between two diagonal of a cube.[ETE'12]

Solution: Let OABCDEFG be a cube. Suppose the length of each side of the cube is a.

Therefore, the co-ordinates are O(0,0,0), A(a,0,0), B(0,a,0), C(0,0,a), D(a,a,0), E(0,a,a), F(a,0,a), G(a,a,a).

In the figure OG and AE be the two diagonals.

Now, the direction cosines of OG is $\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{3a^2}}, \frac{a-0}{\sqrt{3a^2}}$ Or, $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Similarly, the direction cosines of AE is $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Let θ be the angle between OG and AE

$\therefore \cos\theta = (-\frac{1}{\sqrt{3}})\times\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}} = \frac{1}{3}$ [SEE FIG. Q.8]

$\therefore \theta = \cos^{-1}\frac{1}{3}$

Hence the angle between two diagonal of a cube is $\cos^{-1}\frac{1}{3}$

Q.8. A line makes angle $\alpha, \beta, \gamma, \delta$ with the four diagonal of a cube. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

[EEE-12;ME'10,'08,06;IPE'12,CSE'10,08,IPE'09,07,ETE,09]

Solution: Let OABCDEFG be a cube. Suppose the length of each side of the cube is a.

Therefore the co-ordinates are O(0,0,0,), A(a,0,0), B(0,a,0), C(0,0,a), D(a,a,0), E(a,0,a), F(0,a,a), G(a,a,a).

Here, OG,CD,AF,BE be the four diagonals.

The direction cosines of OG are $\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$

Or, $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Similarly, the direction cosines of CD are, $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

the direction cosines of AF are, $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

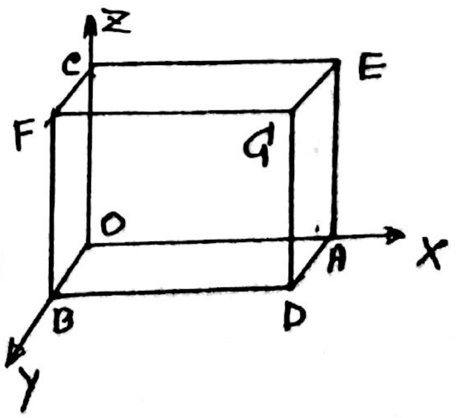
the direction cosines of BE are, $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Again let l, m, n be the direction cosines of a straight line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals

$\therefore \cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l+m+n}{\sqrt{3}}$ ....(1)

$\therefore \cos\beta = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} - \frac{n}{\sqrt{3}} = \frac{l+m-n}{\sqrt{3}}$ ....(2)

$\therefore \cos\gamma = \frac{-l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{-l+m+n}{\sqrt{3}}$ ....(3)



$\therefore \cos\delta = \frac{l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l-m+n}{\sqrt{3}}$ ....(4)

Now squaring (1),(2),(3),(4) and adding we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{(l+m+n)^2}{3} + \frac{(l+m-n)^2}{3} + \frac{(-l+m+n)^2}{3} + \frac{(l-m+n)^2}{3}$$

$$= \frac{(l+m+n)^2 + (l+m-n)^2 + (-l+m+n)^2 + (l-m+n)^2}{3}$$

$$= \frac{4(l^2+m^2+n^2)}{3}$$

$$= \frac{4}{3}$$

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$ (Proved)

Q.9. Find the angle between two lines whose d.c.s are given by the equations $l+m+n=0$ & $l^2+m^2+n^2=0$. [ME'09,ETE'08,'07]

Slution: Given that,

$l+m+n=0$ ------------------(1)

$l^2+m^2+n^2=0$ ----------(2)

From eq. (1) we have, $n=-(l+m)$ ------------(3)

$\therefore l^2+m^2+(l+m)^2=0$

Or, 2lm=0

When $l=0$ i.e. $1.l+0.m+0.n=0$ --------------(4)

From eq. (1) & (4) we have, $\frac{l}{0-0} = \frac{m}{1-0} = \frac{n}{0-1}$ or, $\frac{l}{0} = \frac{m}{1} = \frac{n}{-1}$

When m=0 i.e. $0.l+1.m+0.n=0$ --------------(5)

From eq. (1) & (5) we have, $\frac{l}{0-1} = \frac{m}{0-0} = \frac{n}{1-0}$ or, $\frac{l}{-1} = \frac{m}{0} = \frac{n}{1}$

Therefore, the d.c.s of the two lines are proporo nal to 0,1,-1 & -1,0,1

If θ be angle between them , then

$$\theta = \pm\frac{0.(-1)+1.0+(-1).1}{\sqrt{0^2+1^2+(-1)^2}\sqrt{(-1)^2+0^2+1^2}} = \frac{\pm 1}{2}$$

$\therefore \theta = \cos^{-1}\frac{\pm 1}{2}$

$\therefore \theta = \pi/3$ or, $\theta = 2\pi/3$ Ans.

Q.10. Find the distance of A(1,-2,3) from the line PQ through P(2,-3,5), which makes equal angle with the axes.

Solution: Since, PQ line makes equal angle with the axes, the **d.c.s** of the line are equal, i.e. $l=m=n$

We know, $l^2+m^2+n^2=1$ or, $3l^2=1$ or, $l=1/\sqrt{3}$

∴ The d.c.s of the PQ line are, $1/\sqrt{3}\ ,\ 1/\sqrt{3}\ ,\ 1/\sqrt{3}$

Now, $AP^2=(1-2)^2+(-2+3)^2+(3-5)^2=6$

Let, AC is the required distance.

∴ PC = Projection of AP on PQ

$$=1/\sqrt{3}(2-1)+1/\sqrt{3}(-3+2)+1/\sqrt{3}(5-3)$$

$$=1/\sqrt{3}+-1/\sqrt{3}+2/\sqrt{3}$$

∴ $PC=2/\sqrt{3}$

$\therefore PC^2=4/3$

$\therefore AC^2=AP^2-PC^2=6-\frac{4}{3}=\frac{14}{3}$ ∴ $AC=\frac{\sqrt{14}}{\sqrt{3}}$ ***Ans.***

Q.11. Prove that, two lines whose d.c.s are connected by two relations $al+bm+cn=0$ & $ul^2+vm^2+wn^2=0$ are perpendicular if $u(b^2+c^2)+v(c^2+a^2)+w(a^2+b^2)=0$ [ME'07] and parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$. [EEE-08,ME'09,CSE'06,IPE'10,ETE,'07]

Solution: Given, $al+bm+cn=0$ or, $n=\frac{-(al+bm)}{c}$ -------------(1)

$$ul^2+vm^2+wn^2=0 \quad \text{-----------------------------------(2)}$$

From (1) & (2) we have,

$$ul^2+vm^2+w\frac{(al+bm)^2}{c^2}=0$$

Or, $uc^2l^2+vm^2c^2+wa^2l^2+wb^2m^2+2wablm=0$

Or, $(uc^2+wa^2)l^2+2wablm+(vc^2+wb^2)m^2=0$

Or, $(uc^2+wa^2)(\frac{l}{m})^2+2wab(\frac{l}{m})+(vc^2+wb^2)=0$ -----------(3)

Let, l_1,m_1,n_1 and l_2,m_2,n_2 be the d.c.s of the given lines and also $\frac{l_1}{m_1}$, $\frac{l_2}{m_2}$ be the roots of eq. (3)

∴ Product of the roots of (3) is,

$\frac{l_1}{m_1}\frac{l_2}{m_2} = \frac{vc^2+wb^2}{uc^2+wa^2}$ or, $\frac{l_1 l_2}{vc^2+wb^2} = \frac{m_1 m_2}{uc^2+wa^2}$

Similarly, $\frac{m_1 m_2}{uc^2+wa^2} = \frac{n_1 n_2}{va^2+ub^2}$

$\therefore \frac{l_1 l_2}{vc^2+wb^2} = \frac{m_1 m_2}{uc^2+wa^2} = \frac{n_1 n_2}{va^2+ub^2}$

Now, the lines will be perpendicular if,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

Or, $(vc^2 + wb^2) + (uc^2 + wa^2) + (va^2 + ub^2) = 0$

Or, $u(b^2 + c^2) + v(c^2 + a^2) + w(a^2 + b^2) = 0$ ***[Proved]***

Again,

If the lines are parallel the d.c.s of the lines will be equal so that the roots of eq. (3) should be equal, I. e. $B^2 - 4AC = 0$

$\therefore (2abw)^2 - 4(uc^2 + wa^2)(vc^2 + wb^2) = 0$

Or, $4a^2b^2w^2 - (4uvc^4 + 4uwc^2b^2 + 4wva^2c^2 + 4w^2a^2b^2) = 0$

Or, $4uvc^4 + 4uwc^2b^2 + 4wva^2c^2 = 0$

Or, $\frac{c^2}{w} + \frac{b^2}{v} + \frac{a^2}{u} = 0$ [dividing by $4c^2uvw$] ***[Proved]***

Q.12. Prove that, the straight lines whose d.c.s are given by the relations *al+bm+cn=o* and *fml+gnl+hlm=0* are perpendicular if $f/a + g/b + h/c = 0$ and parallel if $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$.

[ME'11,01,CSE'12,'11,09,08,CE'07,ETE'12,IPE,'06]

<u>Solution:</u> Given, $al+bm+cn=0$ or, $l=\frac{-(bm+cn)}{a}$ --------------(1)

And $fml+gnl+hlm=0$ ----------------------------(2)

From (1) & (2) we have,

$$fml+gn\frac{-(bm+cn)}{a} + hm\frac{-(bm+cn)}{a} = 0$$

or, $fmna - bmng - cn^2g - hm^2b - hmnc = 0$

or, $hbm^2 + (bg - fa + ch)mn + cgn^2 = 0$

or, $hb(\frac{m}{n})^2 + (bg - fa + ch)\frac{m}{n} + cg = 0$ ------------------(3)

Let, l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.s of the given lines and also $\frac{m_1}{n_1}$, $\frac{m_2}{n_2}$ be the roots of eq. (3)

∴ Product of the roots of (3) is,

$$\frac{m_1m_2}{n_1n_2} = \frac{cg}{hb} = \frac{g/_b}{h/_c} \qquad \therefore \frac{m_1m_2}{g/_b} = \frac{n_1n_2}{h/_c} \quad \text{-------------(4)}$$

Similarly, $$\frac{m_1m_2}{g/_b} = \frac{l_1l_2}{f/_a} \quad \text{--------------------------(5)}$$

Now, the lines will be perpendicular if,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Or, $$f/_a + g/_b + h/_c = 0 \qquad \textbf{[Proved]}$$

Again,

If the lines are parallel the d.c.s of the lines will be equal so that the roots of eq. (3) should be equal, I. e. $B^2 - 4AC = 0$

$\therefore \quad (bg - fa + ch)^2 - 4hbcg = 0$

Or, $bg - fa + ch = \pm 2\sqrt{hbcg}$

Or, $(\sqrt{bg})^2 \pm 2\sqrt{bg}\sqrt{ch} + (\sqrt{ch})^2 = fa$

Or, $(\sqrt{bg} \pm \sqrt{ch})^2 = (\sqrt{fa})^2$

Or, $\pm[\sqrt{bg} \pm \sqrt{ch}] = \sqrt{fa}$

Or, $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$ ***[Proved]***

Q.13. Show that, the lines whose d.c.s are connected by *l+m+n=0* and *2mn+3nl-5lm=0* are perpendicular to each other.[CSE'09]

<u>**Solution:**</u> Given, $l+m+n=0$ or, $n=-(m+l)$ -----------------(1)

And $2mn+3nl-5lm=0$ ---------------------------(2)

From (1) & (2) we have,

$$-2ml - 2m^2 - 3l^2 - 3ml - 5ml = 0$$

Or, $2m^2 + 10ml + 3l^2 = 0$

Or, $2(\frac{m}{l})^2 + 10\frac{m}{l} + 3 = 0$ ------------------------------(3)

Let, l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.s of the given lines and also $\frac{m_1}{l_1}$, $\frac{m_2}{l_2}$ be the roots of eq. (3)

$\therefore$ Product of the roots of (3) is,

$$\frac{m_1m_2}{l_1l_2} = \frac{3}{2} \qquad or, \frac{m_1m_2}{3} = \frac{l_1l_2}{2}$$

Similarly, $$\frac{m_1m_2}{3} = \frac{n_1n_2}{-5}$$

$$\therefore \frac{l_1l_2}{2} = \frac{m_1m_2}{3} = \frac{n_1n_2}{-5}$$

The lines will be perpendicular if, $l_1l_2 + m_1m_2+n_1n_2 = 0$

Or, 2+3-5=0 **[proved]**

Q.14. Prove that, the straight lines whose d.c.s are given by the relations *2l+2m-n=0 & mn+nl+lm=0 are right angles.*[ETE'09]

Solution: Given, *2l+2m-n=0* or, *n=2l+2m* ---------------(1)

And *mn+nl+lm=0* -------------------------(2)

From (1) & (2) we have,

m(2l+2m)+(2l+2m)l+lm=0

Or, $2m^2 + 2l^2 + 5ml = 0$

Or, (m+2l) (2m+l)=0

∴ 2l+m=0 --------------(3) and l+2m=0 -----------------(4)

From (1) & (3) we have,

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{-2}$$

Therefore 1,-2 ,-2 are the direco n rao s of r st line.

Again, from (1) & (4) we get, $\frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$

Therefore 2, 1 ,2 are the direco n rao s of second line.

Now, the two lines will be perpendicular or at right angle if,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

∴ *1.2+(-2).(-1)+(-2).2 = 2+2-4 = 0 which is true.* **[Proved]**

THE PLANE[3D]

The Plane: A plane is a surface such that if any two points are taken on it,the straight line joining them lies wholly and the surface i.e. every point on the line joining the two points will be on the plane.

Theorem: Prove that the general equao n of r st degree in x ,y,z i.e. Ax+By+Cz+D=0 represents a plane.

Proof: Let (x_1,y_1,z_1) and (x_2,y_2,z_2) be the coordinates any two points lying on the locus whose equation is Ax+By+Cz+D=0 … … … … … … …(i)

As the two points lying the line therefore these two points satisfies (i).Then we get,

$$Ax_1+By_1+Cz_1+D=0 \text{ … … … … … …(ii)}$$

$$Ax_2+By_2+Cz_2+D=0 \text{ … … … … … …(iii)}$$

Multiplying (iii) by γ and adding with (ii),we get

$$Ax_1+B\,y_1+C\,z_1+D+\gamma(Ax_2+By_2+C\,z_2+D)=0$$

or, $A(x_1+\gamma\,x_2)+B(y_1+\gamma\,y_2)+C(z_1+\gamma\,z_2)+D(1+\gamma)=0$

Dividing by $(1+\gamma)$,we get

$$A\,.\frac{(x1+\gamma\,x2)}{(1+\gamma)}+B\,.\frac{(y1+\gamma\,y2)}{(1+\gamma)}+C\,.\frac{(z1+\gamma\,z2)}{(1+\gamma)}+D=0 \text{ … … … … … … … (iv)}$$

Result (iv) shows that equation (i) is satisfied by the point whose coordinates are

$$\left(\frac{x1+\gamma\,x2}{1+\gamma},\frac{y1+\gamma\,y2}{1+\gamma},\frac{z1+\gamma\,z2}{1+\gamma}\right)$$

It is evidently that the equation (i) is a plane.

So, we can say that the general equation of first degree in x,y,z represents a plane.

General equation of a plane:-Ax+By+Cz+D=0

General equation of a plane passing through a point (x_1,y_1,z_1) : - $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

General equation of a plane passing through the orgin:- Ax+By+Cz+D=0

Where A,B,C are direction ratios of the normal to the plane.

Normal: A line is perpendicular to the plane is called normal.

Theorem: Find the equation to the plane in terms of the intercepts a,b and c which it makes on the axis.

Proof: Let the equation of the plane be Ax+By+Cz+D=0.. … … … … … (i)

Since it makes gnterceptsa,b and c on the axes of x,y and z respectively,then the points(a,0,0),(0,b,0),(o,0,c) will sas fy (i)

So, Aa+0+0+D=0

Or,Aa+D=0

Or, $A=-\frac{D}{a}$

Similarly, $B=-\frac{D}{b}$, $C=-\frac{D}{c}$

Putting these values in (i),we get

$$-\frac{D}{a}x-\frac{D}{b}y-\frac{D}{c}z+D=0$$

Or, $\frac{D}{a}x+\frac{D}{b}y+\frac{D}{c}z=D$

$\therefore \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is the required equao n of the plane.

Theorem: To find the condition that the general homogeneous equation of second degree in x,y and z may represent a pair of planes and to find the angle between them and the condition that these two planes may be perpendicular to each other.

Solution: The general homogeneous equation of second degree in x,y and z is

$Ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ ... . (i)

Let $Ax^2+by^2+cz^2+2fyz+2gzx+2hxy=(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)$ ... ... (ii)

Where, $l_1x+m_1y+n_1z=0$ and $l_2x+m_2y+n_2z=0$ are the equao n of two planes represented by equation (i).

Comparing the coefficients of like terms both sides of (ii), we get,

$$l_1l_2=a;\quad m_1m_2=b;\quad n_1n_2=c;$$

$$m_1n_2+m_2n_1=2f;\quad n_1l_2+n_2l_1=2g;\quad l_1m_2+l_2m_1=2h$$

Now, consider the product of two determinants, we have

$$\begin{bmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{bmatrix} \times \begin{bmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{bmatrix} = 0$$

Or, $\begin{bmatrix} l_1l_2+l_1l_2 & m_1l_2+m_2l_1 & n_1l_2+n_2l_1 \\ l_1m_2+l_2m_1 & m_1m_2+m_1m_2 & m_1n_2+m_2n_1 \\ n_1l_2+n_2l_1 & m_1n_2+m_2n_1 & n_1n_2+n_1n_2 \end{bmatrix}$

Or, $\begin{bmatrix} 2l_1l_2 & 2h & 2g \\ 2h & 2m_1m_2 & 2f \\ 2g & 2f & 2n_1n_2 \end{bmatrix}=0$

Or, $\begin{bmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{bmatrix}=0$

Or, $8\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}=0$

Or, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}=0$

Or, $abc+2fgh-af^2-bg^2-ch^2=0$ which is the required condition.

Let, θ be the angle between them.

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

$$=\frac{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_1l_2)^2+(l_1m_2-l_2m_1)^2}}{l_1l_2+m_1m_2+n_1n_2}$$

Now, $(m_1n_2-m_2n_1)^2=(m_1n_2+m_2n_1)^2-4m_1n_2m_2n_1$

$=4f^2-4bc$

Similarly, $(n_1l_2-n_1l_2)^2=4g^2-4ca$

$(l_1m_2-l_2m_1)^2=4h^2-4ab$

$$\therefore \tan\theta=\frac{\sqrt{4(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c}$$

Or, $\theta=\tan^{-1}\frac{\sqrt{4(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c}$ (Ans.)

If the two planes are perpendicular to each other, then $\theta=90°$

$$\therefore a+b+c=0$$

Every equation of the first degree in x,y,z represents a plane.General equation of a given plane ax+by+cz+d=0

General equation of a given plane that passes through a given point: Let the point (x_1,y_1,z_1)

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

Angle between two planesax+by+cz+d=0 and $ax_1+by_1+cz_1+d=0$

$$;\cos\theta=\frac{aa_1+bb_1+cc_1}{\sqrt{a^2+b^2+c^2}\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}}$$

Ex-1: A plane meets the co-ordinate axes in A,B ,C such that the centroid of the triangle ABC is the point (p,q,r).Show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{y}{r}=3$ **[CSE'06,ETE'10]**

Solution: Let the equation of the plane be

$\frac{x}{a}+\frac{y}{b}+\frac{y}{c}=1$ … … … … … (i)

If it meets the coordinate axes in A,B,C then the coordinates of A,B,C are A(a,0,0),B(0,b,0),C(o,0,c) respecv ely.

Now, centroid of the triangle ABC is $(\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3})=(\frac{a}{3},\frac{b}{3},\frac{c}{3})$

But we are given that the centroid is the point (p,q,r) then we can write,

$P=\frac{a}{3}$; a=3p

q=b/3; b=3p

r=c/3; c=3r

Putting this values in (i),we get

x/3p+y/3q+z/3r=1

or, x/p+y/q+z/r=3

(proved)

Ex-2: A variable plane passes through a fixed point (a,b,c) and meets the axes in A,B,C. Show that the locus of the point of intersection of the planes through A,B,C parallel to the coordinate planes is a/x+b/y+c/z=1 or $ax^{-1}+by^{-1}+cz^{-1}=1$ or ayz+bzx+cxy=xyz.

Solution: Let the plane be

$x/\alpha+y/\beta+z/\gamma=1$ … … … … … … (i)

It passes through (a,b,c)

$\therefore a/\alpha+b/\beta+c/\gamma=1$ … … … … … (ii)

Also the plane (i) meets the axes in A,B,C.

Coordinates of A,B,C are $(\alpha,0,0),(0,\beta,0),(0,0,\gamma)$

The planes through A,B ,C which is parallel to the coordinate planes $x=\alpha, y=\beta$ and $z=\gamma$ respectively.

Putting the values of α, β and γ in equation (ii),we get

$$a/x+b/y+c/z=1$$

$$\text{or, } ax^{-1}+by^{-1}+cz^{-1}=1$$

$$\text{or } ayz+bzx+cxy=xyz. \text{ (proved)}$$

Ex-3: A variable plane is a constant distance 'p' from the origin and meets the axes in A,B and C. Show that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$ **(EEE-06.CE'07,'05,CSE'03,IPE'12'10,'07,ETE'09,'07)**

Solution: Let the equation of the plane be $x/a+y/b+z/c=1$ … … … … … (i)

It meets the axes in A,B,C. So the coordinates of A,B,C are A(a,0,0),B(0,b,0),C(o,0,c).

The distance of the plane from the origin is $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

According to the question, $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p$

$$\text{Or, } \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \text{ … … …. …. …. …. …. (ii)}$$

Let (x_1,y_1,z_1) be the centroid of the tetrahedron OABC, then

$$x_1=a/4 \;\therefore a=4x_1$$

$$y_1=b/4 \;\therefore b=4y_1$$

$$z_1=c/4 \;\therefore c=4z_1$$

Putting the values of a,b,c in (ii) ,we get

$$\frac{1}{(4x1)^2}+\frac{1}{(4y1)^2}+\frac{1}{(4z1)^2}=\frac{1}{p^2}$$

$$\text{Or, } \frac{1}{x1^2}+\frac{1}{y1^2}+\frac{1}{z1^2}=\frac{16}{p^2}$$

$$\text{Or, } x^{-2}+y^{-2}+z^{-2}=16p^{-2}$$

Hence the locus is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$

Ex-4: Prove that the equation $2x^2+6y^2-12z^2+6yz+2zx+7xy=0$ represents a pair of planes and find the angle between them.

Solution: The given equao n is $2x^2+6y^2-12z^2+6yz+2zx+7xy=0$

Or, $2x^2+2zx+7xy+6y^2-12z^2+6yz=0$

Or, $2x^2+2zx+7xy+6\{(y+2z)-z(y+2z)\}=0$

Or, $2x^2+2zx+7xy+6(y-z)(y+2z)=0$

Or, $2x^2+3(y+2z)x+4(y-z)x+6(y-z)(y+2z)=0$

Or, $x(2x+3y+6z)+(2y-2z)(2x+3y+6z)=0$

Or, $(2x+3y+6z)(x+2y-2z)=0$

i.e, $(2x+3y+6z)=0$

$(x+2y-2z)=0$

Hence the given equations represents a pair of planes.

Let, θ be the angle between the two planes

$$\cos\theta = \frac{2.1+3.2+6.(-2)}{\sqrt{4+9+36}\sqrt{1+4+4}} = \frac{-4}{21}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-4}{21}\right) (Ans.)$$

An alternative solution of this type

Ex-5: Prove that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents a pair of plane. Find the angle between them.

Solution: Given that, $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ … … … … … … (i)

Here a=2; b= - 6; c= - 12; f=9; g=1; h=1/2

The equation(i) represents a pair of planes if

$abc+2fgh-af^2-bg^2-ch^2=0$

L.H.S. $=abc+2fgh-af^2-bg^2-ch^2$

$=2.(-6).(-12)+2.9.1.1/2- 2.81+6.1+12.1/4$

$=144+9-162+6+3$

$=162-162=0$

Hence the given equation represents a pair of planes.

Let, θ be the angle between them

$$\tan\theta = \frac{\sqrt{4(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c}$$

$$= -\frac{\sqrt{185}}{16}$$

$$\therefore \theta = \tan^{-1}\left(-\frac{\sqrt{185}}{16}\right) \text{(Ans.)}$$

Ex-6: A variable plane is a constant distance 'p' from the origin and meets the axes in A,B and C. Through A,B,C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$. **(EEE-07,CE'08,GCE'12,ME'11,'05ETE'12,'08)**

Solution: Let the equation of the plane be $x/a+y/b+z/c=1$ … … … … … (i)

It meets the axes in A,B,C. So the coordinates of A,B,C are A(a,0,0),B(0,b,0),C(o,0,c).

The distance of the plane from the origin is $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

According to the question, $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p$

Or, $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$ … … …. …. …. …. (ii)

Now, equation of the planes through A,B,C and parallel to the coordinate planes are x=a,y=b,z=c

Putting these values in (ii),we get $\frac{1}{x1^2}+\frac{1}{y1^2}+\frac{1}{z1^2}=\frac{1}{p^2}$

$\therefore x^{-2}+y^{-2}+z^{-2}=p^{-2}$ (proved)

Ex-7: Find the equation of the plane passing through the lines of interseco n of the planes 2x – y=0 and 3z – y=0 and perpendicular to the plane 4x+5y-3z=8.[CSE'11]

Solution: The given planes are $2x - y=0$ … … … … … (i)

$3z - y=0$ … … … … … … (ii)

and $4x+5y-3z=8$… … … … … … (iii)

Now, the equation of any plane through the line of intersection of (i) and (ii), is

$2x - y + k(3z - y)=0$

$\Rightarrow 2x-(1+k)y+3kz=0$ … … … … … … … … (iv)

Since the plane (iv) is perpendicular to the plane (iii) we get

$4.2-5(1+k)-3.3k=0$

$\Rightarrow 8-5-5k-9k=0$

$\Rightarrow k=3/14$

Putting the value of k in (iv) we get

$2x-(1+3/14)y+3.3/14.z=0$

$\Rightarrow 28x-17y+9z=0$ (Answer)

Ex-8: Find the equation of the plane that passes through (2,-3,1) and is normal to the line joining the points (3,4,-1) and (2,0,1).

Solution: Any plane passes through the points (2,-3,1) is

$A(x-2)+B(x+3)+c(z-1)=0$ … … … … … … … … …(i)

Direco n rao s of the normal to the plane (2-3,0-4,1+1)

= - 1,-4,2

Or,(3-2,4-0,-1-1)

=(1,4,-2)

Since the normal is perpendicular to the plane, then we get A/1=b/4=c/-2=k(let)

A=k; B=4k ; c=-2k

Putting the value of A,B and C in (i), we get

$K(x-2)+4k(y+3)+(-2k)(z-1)=0$

$\Rightarrow (x-2)+4(y+3)- 2(z-1)=0$ which is required equation of plane.

$\Rightarrow x+4y-2y+3+2-2=0$

$\Rightarrow x+4y-2y+3=0$ which is the equation of plane.

Ex-9: Find the equation of the plane through the line of intersection of the planes p=ax+by+cz+d=0 and Q= a′x+b′y+c′z+d′=0 under the following condio ns

(i) Perpendicular to xy plane i.e, z=0

(ii) Parallel to x- axis i.e, perpendicular to yz plane

(iii) Perpendicular to plane lx+my+nz+p=0

Solution: Any plane through the line of intersection of the given planes

$ax+by+cz+d+k(a'x+b'y+c'z+d')=0$

$\Rightarrow (a+a'k)x+(b+b'k)y+(c+c'k)z+(d+d'k)=0$ … … … … … … … (i)

(i)If the plane (i) is perpendicular to xy plane i.e, parallel to z axis i.e, the coefficient of z is equal to zero then

$c+c'k=0$

$k=-c/c'$

putting the value of k in (i) we get, $(a-a'c/c')x+(b-b'c/c')y+(c-c'c/c')z+(d-d'c/c')=0$

$\Rightarrow(ac'-a'c)x+(bc'-b'c)y+0+(dc'-d'c)=0$ which is equao n of plane perpendicular to xy plane.

(ii) If the plane (i) is parallel to x axis

i.e, perpendicular to yz plane.

i.e, the coefficient of x is equal to zero then $a+a'k=0$

$\Rightarrow k=-a/a'$

Putting the value of k in (i),we get

$\Rightarrow 0+y(b-ab'/a')+z(c-ac'/a')+(d-ad'/a')=0$

$\Rightarrow(a'b-ab')y+(a'c-ac')z+(a'd-ad')=0$ which is the equao n of plane parallel to x- axis.

(iii)The equation of plane (i) is perpendicular to plane $lx+my+nz=p$

From the condition of perpendicularity of the two planes,we get

$l(a+a'k)+m(b+kb')+n(c+c'k)=0$

$(al+bm+cn)+k(a'l+b'm+c'n)=0$

$\therefore k=-\frac{(al+bm+cn)}{(a'l+b'm+c'n)}$

Putting the value in (i),we get

$(ax+by+cz+d)-\frac{(al+bm+cn)}{(a'l+b'm+c'n)}(a'x+b'y+c'z+d')=0$

$\Rightarrow(ax+by+cz+d)(a'l+b'm+c'n)-(al+bm+cn)(a'x+b'y+c'z+d')=0$ which is the required equation of plane.

Ex-10: Find the equation of the plane through the points (2,2,1) and (9,3,6) and perpendicular to the plane $2x+6y+6z=9$ **[CE'04,CSE'10,IPE'06,ME'08,'06,'04]**

Solution: Any plane passing through (2,2,1) is

$A(x-2)+B(y-2)+C(z-1)=0$… … … … … … … (i)

Since it passes through (9,3,6)then (i) becomes

$A(9-2)+B(3-2)+C(6-1)=0$

$\Rightarrow 7A+B+5C=0$ … … … … … … … … … …(ii)

The plane (i) perpendicular to the given plane. So we can write

$2A+6B+6C=0$ … … … … …. … …. … …(iii)

Now from (ii) & (iii) by cross multiplication we get

$A/-24=B/-32=C/40$

$\Rightarrow A/-3=B/-4=C/5=k$(let)

$\therefore A=-3k$; $B=-4k$; $C=5k$;

Putting the value of A,B & C in (i)we get,

$-3k(x-2)-4k(y-2)+5k(z-1)=0$

$\Rightarrow-3(x-2)-4(y-2)+5(z-1)=0$ [Dividing by k]

$\Rightarrow-3x+6-4y+8+5z-5=0$

$\Rightarrow 3x+4y-5z=9$ (Answer)

Solve same problem by yourself⊗Find the equao n of the plane through the points (1,0,-1) and (2,1,3) and perpendicular to the plane $2x+y+z=1$⊗ Ans. $3x-7y+z=2$(EEE-09)

☹Find the equation of the plane through the points (1,-2,2) and (-3,1,-2) and perpendicular to the plane 2x+y+z+6=0 Ans.x-12y-10z=5.

Ex-11: Show that equation of the plane through the point (-1,3,2) and perpendicular to the planes x+2y+2z=5. and 3x+3y+2z=8 is 2x-4y+3z+8=0.[CE'03,'01,ETE'11,ME'07]

Solution: Any plane passing through (-1,3,2) is

A(x+1)+B(y-3)+C(z-2)=0... ... ... (i)

Since (i) is perpendicular to the planes x+2y+2z=5... .... (ii)

and 3x+3y+2z=8 ... . (iii)

So, A+2B+2C=0 ... ... (iv)

And 3A+3B+2C=0... .... (v)

Now from (iv) & (v) by cross multiplication we get

A/-2=B/4=C/-3=k(let)

∴A=-2k ; B=4k ; C=-3k ;

Putting the value of A,B & C in (i)we get,

-2k(x+1)+4k(y-3)-3k(z-2)=0

⇒-2(x+1)+4(y-3)-3(z-2)=0 [Dividing by k]

⇒2x-4y+3z+8=0 (Proved)

Ex-12: Find the equation of a plane passing through the point(1,-2,1)and perpendicular to the line with direction ratios 2,3,5.

Solution: Any plane passing through (1,-2,1) is

A(x-1)+B(y+2)+C(z-1)=0... ... ... (i)

The plane is perpendicular to the line with direction ratios 2,3,5.

So, A/2=B/3=C/5=k(let)

⇒A=2k; B=3k;C=5k

Putting the value of A,B & C in (i)we get,2k(x-1)+3k(y+2)+5k(z-1)=0

∴ 2x+3y+5z=1 (Answer)

Ex-13: Find the equation of the plane through the point (4,0,1) and parallel to the plane 4x+3y-12z+6=0.(EEE-09)

Solution: We know,(From H.Sc)The equations of the parallel planes differ by a constant only.

So the equation of the plane parallel to the plane 4x+3y-12z+6=0 is 4x+3y-12z+k=0.

Since it passes through the point (4,0,1)

16+0-12+k=0

⇒k=-4

So the required plane is 4x+3y-12z-4=0. (Answer)☹☹☹

Ex-14: A point moves on the plane x/a+y/b+z/c=1 which is fixed.The plane through p perpendicular to op meets the axes in A,B and C. The planes through A,B and C parallel to yz,zx,xy planes intersect in Q. Prove that if the axes be rectangular the locus of Q is $1/x^2+1/y^2+1/z^2=1/ax+1/by+1/cz$(EEE-05,CE'02,ME'10.'09)

Solution: Given plane,x/a+y/b+z/c=1 ... ... (i)

Let (α,β,γ) be the coordinates of P, which moves on the given plane. Then

$\alpha/a+\beta/b+\gamma/c=1$ ... ... (ii)

Here OP be the normal to the required plane, then the direction ratios(**d.r's**) of OP are α,β,γ

Hence the equation of the plane is $\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$ ... ...(iii)

It meets the coordinate axes in A,B,C. So the coordinates of A,B and C are

$A(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha},0,0), B(0,\frac{\alpha^2+\beta^2+\gamma^2}{\beta},0), C(0,0,\frac{\alpha^2+\beta^2+\gamma^2}{\gamma})$

Now, the equation of the planes passing through A,B and C and parallel to the coordinate axes are $x=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}$; $y=\frac{\alpha^2+\beta^2+\gamma^2}{\beta}$; $z=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$

$$\therefore 1/x^2+1/y^2+1/z^2=\frac{\alpha^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\beta^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$$

$$=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$$

$$=\frac{1}{\alpha^2+\beta^2+\gamma^2} \text{ ... ... (iv)}$$

$$\text{Also, } 1/ax+1/by+1/cz=\frac{\frac{\alpha}{a}}{\alpha^2+\beta^2+\gamma^2}+\frac{\frac{\beta}{b}}{\alpha^2+\beta^2+\gamma^2}+\frac{\frac{\gamma}{c}}{\alpha^2+\beta^2+\gamma^2}$$

$$=\frac{1}{\alpha^2+\beta^2+\gamma^2}[\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1] \text{ ... ...(iv)}$$

$\therefore 1/x^2+1/y^2+1/z^2=1/ax+1/by+1/cz$ (Proved)

15. ...d the equation of the plane through the line $\frac{x-2}{3}=\frac{y-3}{5}=\frac{z}{7}$ and ...ssing through the point (1,-2,3)

<u>Solution</u>: Given that, $\frac{x-2}{3}=\frac{y-3}{5}=\frac{z}{7}$ ... ...(i)

$\therefore 5(x-2)-3(y-3)=0$ and $7(x-2)-3z=0$

$\therefore$ Any plane through the line (i) is

$\{5(x-2)-3(y-3)\}+k\{7(x-2)-3z\}=0$ ... ...(ii)

The plane passes through (1,-2,3)

$\therefore 5(1-2)-3(-2-3)+k\{7(1-2)-3\cdot3\}=0$

$\Rightarrow -5+15-7k-9k=0$

$\Rightarrow k=\frac{10}{16}=\frac{5}{8}$

$\therefore$ From (ii) we have,

$(5x-3y-1)+\frac{5}{8}(7x-3z-14)=0$

$\Rightarrow 40x-24y-8+35x-15z-70=0$

$\therefore 25x-8y-5z-26=0$ (Ans)

CHAPTER-7

THE STRAIGHT LINE[3D]

(With shortest distance)

⇒Eeqution of a straight line [General form]:

Any two planes (Equations of first degree in x,y & z) taken together represent a general equation of straight line. i.e.

$a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0$

Or, $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$

Represents a general equation of straight line.

NOTE. This equation also represent the line of intersection of the two given planes.

⇒Eeqution of a straight line [Symmetrical form]:

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

Which also represent the eq. of straight line through (x_1, y_1, z_1), whose direction cosines are l,m,n.

In terms of direction ratios,

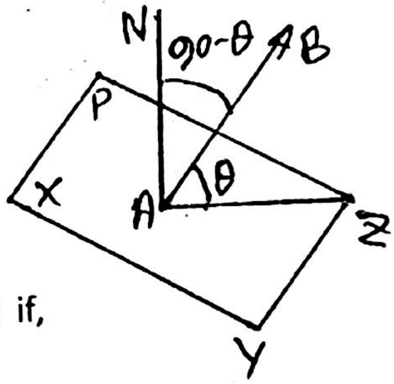
$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

⇒Angle between a line and a plane:

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ [eq. of a st. line]

$ax+by+cz=0$ [eq. of a plane][a,b,c are the direction ratios of the normal to the plane]

$$\sin\theta = \cos(90-\theta) = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+c^2}}$$



1. The line and the plane will be perpendicular if, $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$
2. The line and the plane will be parallel if, $al+bm+cn=0$

⇒Condition for a line lie on a plane:

The line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ will lie on the plane $ax+by+cz+d=0$ if,

$al+bm+cn=0$ &

$ax_1+by_1+cz_1+d=0$

⇒Eeqution of a straight line through two points:

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2}$$

⇒Distance of a point from a line whose d.c.s are l,m,n :

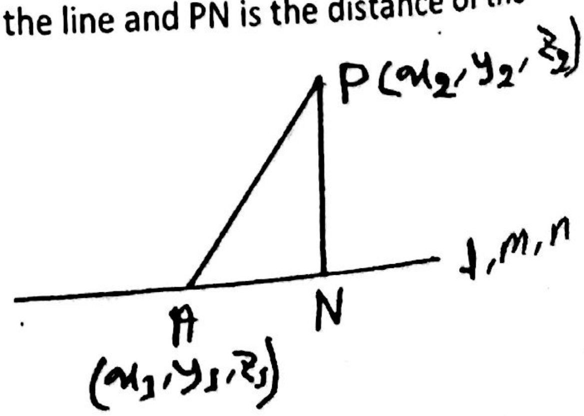
If $P(x_2, y_2, z_2)$ any given point and $A(x_1, y_1, z_1)$ is any point on the line and PN is the distance of the point from the line then,

$$PN = \sqrt{AP^2 - AN^2}$$

Where, AN=projection of AP on the line

$$= (x_2-x_1)l + (y_2-y_1)m + (z_2-z_1)n$$

$$AP = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$



Solved problems

Q.1. Find the distance of the point (-1,-5,-10) from the point of interseco n of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane x-y+z=5.[ME'07]

Solution: Let, $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=r$ ----------------------------(1)

$\therefore x=3r+2,\quad y=4r-1,\quad z=12r+2$

The line (1) intersect the plane x-y+z=5. So we get,

$3r+2-4r+1+12r+2=5 \qquad \therefore r=0$

∴ The point of interseco n is (2,-1,2). Hence the required distance is

$\sqrt{9+16+144}=13$ **Ans.**

Q.2. Find the eq. of the line perpendicular to both the line $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{3}, \frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2}$ and passing through their intersection.[CE'04,CSE'11(SAME}ETE,10]

Solution: Let, $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{3}=r_1 \qquad \therefore x=r_1+1,\quad y=2r_1+1,\quad z=3r_1-2$

And $\frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2}=r_2 \qquad \therefore x=2r_2-2,\quad y=5-r_2,\quad z=2r_2-3$

If the two lines intersect, we have

$x=r_1+1=2r_2-2$ ------(1) $\quad y=2r_1+1=5-r_2$ -----(2) $\quad z=3r_1-2=2r_2-3$ -----(3)

Solving any two eq. we have, $r_1=1$ & $r_2=2$

Hence the point of intersection is (2,3,1)

Let, l,m n be the d.c.s of the required line. Since it is perpendicular to the given lines, we have

l+2m+3n=0-----------(4) and 2l-m+2n=0 ---------------(5)

solving (4) &(5) we get $\frac{l}{7}=\frac{m}{4}=\frac{n}{-5}=\frac{1}{\sqrt{90}}$

hence the eq^n of the required line is, $\frac{x-2}{7}=\frac{y-3}{4}=\frac{z-1}{-5}$ **Ans.**

Q.3. Find the eq. of the line of interseco n of two plane 2x-y+2z+7=0 & x+2y-3z+6=0.[CSE'11]

Solution: Let, l,m,n be the d.c.s of the line of intersection of the two planes. So we have

2l-m-2n=0 -------------(1) $\quad$ l+2m-3n=0 ---------(2) $\quad$ [al+bm+cn=0]

Solving (1) & (2) we have, $\frac{l}{3-4}=\frac{m}{2+6}=\frac{n}{4+1}$ or, $\frac{l}{1}=\frac{m}{-8}=\frac{n}{-5}=\frac{1}{\sqrt{90}}$

Let, (x',y',0) be a point on the line of intersection. Then we have

$\frac{x-x'}{1}=\frac{y-y'}{-8}=\frac{z-0}{-5}$ ----------------------(3)

Again (x',y',0) lie on the both planes. So we have

2x'-y'+7=0 -----------(4) and x'+2y'+6=0 -------------(5)

Solving (4) & (5) we have, x'= -4 & y'=-1

Hence the required line is, $\frac{x+4}{1}=\frac{y+1}{-8}=\frac{z}{-5}$ **Ans.**

NOTE: The required line is parallel to both the planes and perpendicular to the normal of the planes. So angle between the required line and each plane is zero. i.e. $\sin\theta=0$ i.e. al+bm+cn=0.

Q.4. Find the the symmetrical form of the eq. of a line x+y+z+1=0=4x+y-2z+2 and n d its d.c.s.[ME11(SAME),ETE'12]

[Note: x+y+z+1=0 and 4x+y-2z+2=0 represent two planes and x+y+z+1=0=4x+y-2z+2 represents the line of intersection of the two planes which is the general form of st. line. **Q.3 & Q.4 are same]**

Solution: Let, l,m,n be the d.c.s of the given line. So we have

L+m+n=0 -----------(1) & 4l+m-2n=0 -------------(2)

Solving (1) & (2) we have, $\frac{l}{-3}=\frac{m}{6}=\frac{n}{-3}$ or, $\frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}=\frac{1}{\sqrt{6}}$

$\therefore$ direction cosines are $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$ Ans

Let, $(x',y',0)$ be a point on the given line , then its eq. is, $\frac{x-x'}{-1} = \frac{y-y'}{2} = \frac{z-0}{-1}$ ---------------(3)

Again $(x',y',0)$ lie on the both planes. So we have,

$x'+y'+1=0$ ----------(4) & $4x'+y'+2=0$ ------(5)

Solving (4) & (5) we have, $x'=\frac{-1}{3}$ & $y'=\frac{-2}{3}$

Hence the required eq. or the symmetrical form of the given eq. is, $\frac{x+1/3}{-1} = \frac{y+2/3}{2} = \frac{z}{-1}$ Ans

Q.5 Find the co-ordinate of the point where the joining the points (2,-3,1),(3,-4,-5) cuts the plane 3x+4y+5z=5.

Solution: Let l,m,n be the d.c.s of the joining the points (2,-3,1),(3,-4,-5).

So, we have, $\frac{l}{3-2} = \frac{m}{-4+3} = \frac{n}{-5-1}$ or $\frac{l}{1} = \frac{m}{-1} = \frac{n}{-6} = \frac{1}{\sqrt{38}}$

Hance, the equation of the line is $\frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6} = R$ (say)

$\therefore$ $x=R+2$, $y=-R-3$, $z=1-6R$

If $(R+2, -R-3, 1-6R)$ lie on the plane $3x+4y+5z=5$

We have, $3R+6-4R-12+5-30R=5$ or, $R=\frac{-6}{31}$

$\therefore$Co ordinates of the required point is $(\frac{56}{31}, -\frac{87}{31}, \frac{67}{31})$ *(Ans).*

SHORTEST DISTANCE

- ***Shortest distance:*** When two lines do not interest and not parallel as well , i.e they do not lie in the same plane , then these lines are said to be non-intersecting lines.The straight line which is perpendicular to each of these two non-intersecting lines is called the line of shortest distance and the length of this line intercepted between the given lines is called the length of the shortest distance. [EEE'10]

- ***Theorem:*** **To find the shortest distance between two given lines and also to obtain the equations of the shortest distance.[ME08(SAME),ETE'07]**

- ***Solution:*** Let the equations of the given lines be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n}$

Let LM be the line which perpendicular to both the given lines and let it's direction ratio be l,m,n

$$\therefore ll_1 + mm_1 + nn_1 = 0$$
$$ll_2 + mm_2 + nn_2 = 0$$

By cross multiplication rule,

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Hence the actual direction cosines L,M,N are

$$\frac{m_1n_2 - m_2n_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \frac{n_1l_2 - n_2l_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \frac{l_1m_2 - l_2m_1}{\sqrt{\Sigma(m_1m_2 - m_2n_1)^2}}$$

The S.D i.e LM is clearly the projection of AC on LM whose d,c's are L,M,N

$$\therefore S.D = L(x_2 - x_1) + M(y_2 - y_1) + N(z_2 - z_1)$$

$$= \frac{(m_1n_2 - m_2n_1)(x_2 - x_1) + (n_1l_2 - n_2l_1)(y_2 - y_1) + (l_1m_2 - l_2m_1)(z_2 - z_1)}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}}$$

Equations of S.D

The equation of the plane containing the first of the given lines and the line of shortest distance

whose direction ratio are l, m, n is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$

Similarly, the equation of the plane containing the second of the given lines and the line of shortest distance is

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

These two plane taken together represent the equation of the line of the shortest distance

Incase the shortest distance LM=0 , then the two given lines intersect , i.e they are coplanar.

S.D=0 if $\sum(x_2 - x_1)(m_1n_2 - m_2n_1 = 0$

This is also the condition for the given lines to be coplanar.

OR

Let the equation of the given lines be

$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$ .... (1)

$\frac{x - \alpha'}{l'} = \frac{y - \beta'}{m'} = \frac{z - \gamma'}{n'}$ .... (2)

It is clear that the two points A and A' on (1) and (2) (α, β, γ) and $(\alpha', \beta', \gamma')$

The shortest distance between (1) and (2) is at right angles to (1) and (2) and is therefore equal to the projection of AA' on a perpendicular line PP' .

Let λ, μ, σ be the d,c's of the PP'

$\lambda l + m\mu + n\sigma = 0$

$\lambda l' + m'\mu + n'\sigma = 0$

$\therefore \frac{\lambda}{mn' - m'n} = \frac{\mu}{nl' - n'l} = \frac{\sigma}{lm' - l'm}$

Shortest distance, S.D= The projection of the line joining A and A' TO the line joining point P'

$= \lambda(\alpha - \alpha') + \mu(\beta - \beta') + \sigma(\gamma - \gamma')$

$$= \begin{vmatrix} \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\sum(mn' - m'n)}$$

Equation of the shortest line (PP') then we get

$$\begin{vmatrix} x - \alpha & y - \beta & z - \gamma \\ l & m & n \\ \lambda & \mu & \sigma \end{vmatrix} = 0 , \quad \begin{vmatrix} x - \alpha' & y - \beta' & z - \gamma' \\ l' & m' & n' \\ \lambda & \mu & \sigma \end{vmatrix} = 0$$

EX-1: Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$, $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$

State whether the lines are coplanar. [EEE'05,07CSE'09(SAME),12(SAME),ME'04,05,09(SAME)]

Solution: $S.D = \begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \div \sqrt{(8-9)^2+(15-16)^2+(10-12)^2}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \div \sqrt{6}$

$= \{1(15-16) - 1(10-12) + 1(8-9)\} \div \sqrt{6}$

$= (-1+2-1) \div \sqrt{6}$

$= 0$

Since shortest distance=0 So, The lines are coplanar (Proved)

EX-2: Show that the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is $\frac{1}{\sqrt{6}}$ [ETE'12,GCE,12(Same),IPE,10] and that is equations are $11x+2y-7z+6=0=7x+y-5z+7$ [EEE'09(SAME)]

Solution: Shortest distance,

$S.D = \begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \div \sqrt{(8-9)^2+(15-16)^2+(10-12)^2}$

$= \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \div \sqrt{6} \quad = (-1+4-2) \div \sqrt{6} \quad = \frac{1}{\sqrt{6}}$

$\therefore S.D = \frac{1}{\sqrt{6}}$ (Proved)

Now, let l,m,n be the direction cosines of the $S.D$ then

$\therefore 2l + 3m + 4n = 0 \cdots\cdots\cdots (1)$

$3l + 4m + 5n = 0 \cdots\cdots\cdots (2)$

From (1) and (2), By cross mulp licao n rule,

$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9}$

$\therefore \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

The equao n of plane containing the 1st line and show the line S.D or

The equation of $S.D$ when the plane through 1st line and $S.D$ as

$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0$

$\Rightarrow (x-1)(-3-8) - (y-2)(-2+4) + (z-3)(4+3) = 0$

$\Rightarrow -11(x-1) - 2(y-2) + 7(z-3) = 0$

$\Rightarrow -11x + 11 - 2y + 4 + 7z - 21 = 0$

$\Rightarrow 11x + 2y - 7z + 6 = 0$ (Proved)

Again, the eq. of plane containing the 2nd line and the line of shortest distance is

$\begin{vmatrix} x-2 & y-4 & z-5 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0$

$\Rightarrow (x-2)(-4-10) - (y-4)(-3+5) + (z-5)(6+4) = 0$

$\Rightarrow -14x + 28 - 2y + 8 + 10z - 50 = 0$

$\Rightarrow 7x + y - 5z + 7 = 0$

Hence the eq. of SD is

$11x + 2y - 7z + 6 = 0 = 7x + y - 5z + 7$ (Proved)

EX-3: Find the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$, $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.[EEE'06,10,CE'03]

Solution: Try yourself ; Ans: $3\sqrt{30}$

EX-4: Find the length of the S.D between the lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$, $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$.[ETE'08,'06IPE'08]Find also the equations and the points where it intersects the lines.[CE'08,CE'06(SAME)CSE'11(SAME),IPE'11,09,'07]

Solution: Let $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = r \cdots\cdots (1)$

$\Rightarrow x = r + 3$, $y = -2r + 5$, $z = r + 7$ $\therefore (x, y, z) = (r + 3, -2r + 5, r + 7)$ and

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = r_1 \cdots\cdots\cdots (2)$

$\Rightarrow x = 7r_1 - 1$, $y = -6r_1 - 1$, $z = r_1 - 1$ $\therefore (x, y, z) = (7r_1 - 1, -6r_1 - 1, r_1 - 1)$

Let the S.D meet the given lines in L and M

$L = (r + 3, -2r + 5, r + 7) \cdots\cdots\cdots\cdots (3)$

and $M = (7r_1 - 1, -6r_1 - 1, r_1 - 1) \cdots\cdots\cdots (4)$

D.R'S of LM are $r - 7r_1 + 4, -2r + 6r_1 + 6, r - r_1 + 8$

LM is the perpendicular to both the given line

$\therefore 1(r - 7r_1 + 4) - 2(-2r + 6r_1 + 6) + 1(r - r_1 + 8) = 0$

$\Rightarrow 6r - 2r_1 = 0 \cdots\cdots\cdots (5)$

and $7(r - 7r_1 + 4) - 6(-2r + 6r_1 + 6) + 1(r - r_1 + 8) = 0$

$\Rightarrow 20r - 86r_1 = 0 \cdots\cdots\cdots (6)$

Solving (5) and (6) $r = 0$, $r_1 = 0$

Putting these value in (3) & (4), we get

$L = (3, 5, 7)$, $M = (-1, -1, -1)$ (Ans)

D.R'S of LM are 4, 6, 8 =2, 3, 4

$S.D = LM = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = 2\sqrt{29}$ (Ans)

Equation of S.D i.e LM is $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$ and $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4}$ (Ans)

EX-5: Find the S.D between the z- axis and the lines $x + y + 2z + 3 = 0$ and $2x + 3y + 4z + 4 = 0$.

Solution: Equation of the plane containing the lines is

$$x + y + 2z + 3 + \lambda(2x + 3y + 4z + 4) = 0$$

$\Rightarrow (1 + 2\lambda)x + (1 + 3\lambda)y + (2 + 4\lambda)z = 0 \cdots\cdots\cdots (1)$

If it is parallel to z-axis then the coefficient of z is zero, $2 + 4\lambda = 0$ $\Rightarrow \lambda = -\frac{1}{2}$

Putting the value of λ in (1), we get, $-{}^1/_2\lambda + 1 = 0$

$\Rightarrow \lambda = 2$

The S.D between the given lines is the distance of the point (0,0,0) on z-axis from the plane (2) is

$$= \frac{0+0+0+2}{\sqrt{1^2}} = 2$$

S.D=2 (Ans)

EX-6: **Prove that the shortest distance between the diagonals of a rectangular parallelepiped and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$ where a, b, c are the length of edges.**

Solution: Let us coterminous edges OA, OB and OC be chosen along the axis of coordinates. Let us find the S.D between the diagonal AL and the Edges OB which does not meet it.

A is (a,0,0) and L is (0,b,c)

$\therefore AL\ is\ \frac{x-a}{-a}=\frac{y-0}{b}=\frac{z-0}{c}$ ………………… (1)

$\therefore OB\ is\ \frac{x}{0}=\frac{y}{b}=\frac{z}{0}$ ………………… (2)

If l, m, n be the d,c's of S.D then, $-al+bm+cn=0\ and\ 0+bm+0=0$

$\therefore m=0\ and\ \frac{l}{c}=\frac{n}{a}$ $\qquad \therefore \frac{l}{c}=\frac{m}{0}=\frac{n}{a}=\frac{1}{\sqrt{c^2+a^2}}$

$l=\frac{c}{\sqrt{c^2+a^2}},\quad m=0,\quad n=\frac{a}{\sqrt{c^2+a^2}}$ ………………………… (3)

S.D is the projeco n of join of (a, 0, 0) and (0, 0, 0) the points of (1) and (2) respecv ely on a line whose d,c's are given by (3) is $\quad l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$

$=l(a-0)+m.0+n.0$

$=al \quad =\frac{ca}{\sqrt{c^2+a^2}}$

Similarly we can find the other S.D. is $\frac{bc}{\sqrt{b^2+c^2}}$ and $\frac{ab}{\sqrt{a^2+b^2}}$

EX-7: **Find the shortest distance between the z-axis and the line $ax+by+cz+d=0$; $a'x+b'y+c'z+d'=0$**

Solution: Equao n of the plane containing the 2nd line is

$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$

$\Rightarrow (a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+d+\lambda d'=0$ ………………… (1)

If it is parallel to z-axis, whose d,c's are 0, 0, 1; then

$(a+\lambda a').0+(b+\lambda b').0+(c+\lambda c').1=0$

$\therefore (c+\lambda c')=0 \qquad \Rightarrow \lambda=-({}^{c}/_{c'})$

Putting the value of λ in (1), we get

$(ac'-a'c)x+(bc'-b'c)y+(dc'-d'c)=0$ ………………………… (2)

The S.D between the givens lines is the distance of the point (0, 0, 0) on z-axis from plane (2) is

$$\frac{dc'-d'c}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}\quad (ans)$$

EX-8: Show that, the equations of the plane containing the line $\frac{y}{b}+\frac{z}{c}=1,\ x=0$ and parallel to the line $\frac{x}{a}-\frac{z}{c}=1, y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if 2d is the shortest distance, show that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$;(CSE-09,10,CE'07,EEE'08)

Solution: Given that $\frac{y}{b}+\frac{z}{c}=1, \quad x=0$ ……………………… (1)

Any plane containing the line is $\frac{y}{b}+\frac{z}{c}-1+\lambda x=0$

$$\Rightarrow \lambda x+\frac{y}{b}+\frac{z}{c}-1=0 \quad \text{………………… (2)}$$

Also, again $\frac{x}{a}-\frac{z}{c}=1, \quad y=0$

$\Rightarrow \frac{x}{a}=\frac{z}{c}+1, \quad y=0 \qquad \Rightarrow \frac{x}{a}=\frac{z+c}{c}, \quad y=0 \qquad \therefore \frac{x}{a}=\frac{y}{0}=\frac{z+c}{c}$ --------(3)

Plane (2) is parallel to the line (3)

$\therefore \lambda a+\frac{1}{b}.0+\frac{1}{c}.c=0 \quad \Rightarrow \lambda a+1=0 \quad \therefore \lambda=-1/a$ [al+bm+cn=0]

Putting the value of λ in (2), we get

$-\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0 \qquad \Rightarrow \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ ***(proved)***

From (1), we get $\frac{y}{b}+\frac{z}{c}=1, \quad x=0$

$\Rightarrow \frac{y}{b}=1-\frac{z}{c}, \quad x=0 \qquad \Rightarrow \frac{y}{b}=\frac{c-z}{c}, \quad x=0$

$\therefore \frac{x}{0}=\frac{y}{b}=\frac{c-z}{c} \qquad \Rightarrow \frac{x}{0}=\frac{y}{b}=\frac{z-c}{-c}$ ……………………………… (4)

Now, shortest distance (3) and (4) is

$$2d=\begin{vmatrix} 0-0 & 0-0 & c+c \\ 0 & b & -c \\ a & 0 & c \end{vmatrix} \div \sqrt{(0-ab)^2+(bc-0)^2+(0+ac)^2}$$

$$\Rightarrow 2d=\begin{vmatrix} 0 & 0 & 2c \\ 0 & b & -c \\ a & 0 & c \end{vmatrix} \div \sqrt{a^2b^2+b^2c^2+a^2c^2}$$

$$\Rightarrow 2d=2c(0-ab) \div \sqrt{a^2b^2+b^2c^2+a^2c^2}$$

$$\Rightarrow 2d=\frac{-2abc}{\sqrt{a^2b^2+b^2c^2+a^2c^2}}$$

$$\Rightarrow d^2=\frac{a^2b^2c^2}{a^2b^2+b^2c^2+a^2c^2}$$

$$\Rightarrow \frac{1}{d^2}=\frac{a^2b^2+b^2c^2+a^2c^2}{a^2b^2c^2}$$

$$\Rightarrow \frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$ ***(Proved)***

The Sphere[3D]

Definition: A sphere is the locus of a point which moves so that is always at a constant distance from a fixed point. The fixed point is called the centre of the sphere and the distance is called the radius of the sphere.

#If the centre of the sphere is (a,b,c) and radius r,then the equation of the sphere becomes $(x-a)^2+(y-b)^2+(z-c)^2=r^2$.

#General equation of a sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ where the centre is (-u,-v,-w) and radius is $\sqrt{u^2+v^2+w^2-d}$.

#Equation of tangent plane to a sphere at any point (x_1,y_1,z_1) is $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

#The plane lx+my+nz=p will touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ if $(lu+mv+nw+p)^2=(u^2+v^2+w^2-d)(l^2+m^2+n^2)$

Or $\frac{-(lu+mv+nw-p)}{\sqrt{l^2+m^2+n^2}}=\pm(\sqrt{u^2+v^2+w^2-d})$

#Equation of a sphere which passes through two circles $x^2+y^2+z^2=m$ and $ax+by+cz=n$ is $x^2+y^2+z^2-m+k(ax+by+cz-n)=0$ From this the centre will be determined.

#Two spheres $s_1=x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

And $s_2=x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ be orthogonal if $2u_1u_2+2v_1v_2+2w_1w_2=(d_1+d_2)$

Hence $c_1^2+c_2^2=r_1^2+r_2^2$ where c_1 and c_2 are the centres of the spheres respectively.

1. A sphere of radius k passes through the origin and meets the axes in A,B,C. Prove that the centroid of the ΔABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.(CE'03,'06 '08,;)

Solution: Let the equation of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Since it passes through the origin (0,0,0) and meet the axes at A(-2u,0,0),B(0,-2v,0),C(0,0,-2w). The Centre at (-u,-v,-w).

$\therefore d=0$ and $k^2=u^2+v^2+w^2$ ... ... (i)

From the ΔABC centroid is at$\left(-\frac{2u}{3},-\frac{2v}{3},-\frac{2w}{3}\right)$

$\therefore x^2+y^2+z^2=\frac{4}{9}(u^2+v^2+w^2)$

Now from equation (i)we get,

$9(x^2+y^2+z^2)=4k^2$(Proved)

2.If any tangent plane to the spheres $x^2+y^2+z^2=r^2$ makes intercepts a,b and c on the co-ordinate axes,prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$.(CE-'07)

__Solution:__ Let, the equation of tangent plane to a sphere at a point (x_1,y_1,z_1) is $xx_1+yy_1+zz_1=r^2$... ... ...(i)

It meets the axes at (a,0,0),(0,b,0),(0,0,c).

$\therefore x_1=\frac{r^2}{a},\ y_1=\frac{r^2}{b}\quad ,z_1=\frac{r^2}{c}.$

$\therefore x_1^2+y_1^2+z_1^2=r^4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$

From the given sphere satisfying by the point (x_1,y_1,z_1) we get,$x_1^2+y_1^2+z_1^2=r^2$

$\therefore r^2=r^4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$

$\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$(Proved)

3. Find the equation to the sphere which passes through the circle $x^2+y^2+z^2=5,x+2y+3z=3$ and touches the plane $4x+3y=15$.(CE'08)

Solution:Letthe equation of the sphere is $x^2+y^2+z^2-5+k(x+2y+3z-3)=0$

Comparing with the standard equation, the Centre is(-k/2,-k,-3k/2) and

$r=\sqrt{\frac{k^2}{4}+k^2+\frac{9k^2}{4}+5+3k}$

$=\sqrt{\frac{14k^2}{4}+5+3k}$

Since this touches the plane 4x+3y=15 then,$\frac{\frac{-4k}{2}-3k-15}{\sqrt{16+9}}=\pm$ r

$\therefore\ \frac{-2k-3k-15}{5}=\pm\left(\sqrt{\frac{14k^2}{4}+5+3k}\right)$

$\Rightarrow\frac{-5k-15}{5}=\pm\left(\sqrt{\frac{7k^2}{2}+5+3k}\right)$

$\Rightarrow -k-3=\pm\left(\sqrt{\frac{7k^2+10+6k}{2}}\right)$

Squaring both sides we get,

$2k^2+12k+18=7k^2+10+6k$

$\therefore 5k^2-6k-8=0\qquad \therefore k=-4/5,2$

Here k=2 is valid and the equao n of the sphere be $x^2+y^2+z^2+2x+4y+6z-11=0$(Ans.)

4. A sphere of constant radius r passes through the orgin O and cuts the axes A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC. [IPE-07] [CE-06,04]

Let the equation of the phere through the origin, be

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \quad \ldots \quad (1)$$

Its radius is r and $r^2=u^2+v^2+w^2 \quad \ldots \quad (2)$

Its cuts the axes at the point A $(-2u, 0, 0)$ B$(0, -2v, 0)$ and C $(0, 0, -2w)$.

The equation of the plane ABC is $\frac{x}{2u}+\frac{y}{2v}+\frac{z}{2w}+1=0 \quad \ldots \quad (3)$

The line through (0, 0, 0) and perpendicular to (3) is

$$\frac{x-0}{0-1/2u}=\frac{y-0}{0-1/2v}=\frac{z-0}{0-1/2w}$$

$$\text{or, } \frac{x}{1/u}=\frac{y}{1/v}=\frac{z}{1/w}=k \text{ (say)} \quad \ldots \quad (4)$$

The equation (4) cuts the eq. (3)

where $\frac{k}{2u^2}+\frac{k}{2v^2}+\frac{k}{2w^2}+1=0$

$$\text{or, } k=\frac{-2}{1/u^2+1/v^2+1/w^2}=\frac{-2}{\lambda} \text{ (say)} \quad \ldots \quad (5)$$

The co-ordinates of the foot of the perpendicular are

$$x=\frac{-2}{u\lambda},\ y=\frac{-2}{v\lambda},\ z=\frac{-2}{w\lambda} \quad \ldots \quad (6)$$

Squaring and adding

$$\therefore x^2+y^2+z^2=\frac{4}{\lambda^2}\left(\frac{1}{u^2}+\frac{1}{v^2}+\frac{1}{w^2}\right)=\frac{4}{\lambda} \quad \ldots \quad (7)$$

Again from (6) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{(u^2+v^2+w^2)\lambda^2}{4}=\frac{r^2}{4}\frac{16}{(x^2+y^2+z^2)^2}=\frac{4r^2}{(x^2+y^2+z^2)^2}$

or, $(x^2+y^2+z^2)(1/x^2+1/y^2+1/z^2)=4r^2$

5. **Find the two tangent planes to the sphere**

$x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ ... (1)

which are parallel to the plane

$2x + 2y = z$ ... (2)

The centre of the sphere (1) is $(2, -1, 3)$ and radius or 3

The general equation of a plane parallel to the plane

$2x + 2y - z = 0$ is $2x + 2y - z + k = 0$ ... (3)

this will be a tangent plane, if its distance from the centre $(2, -1, 3)$ of the sphere is equal to the radius 3 therefore, we have

$$\pm \frac{2.2 + 2(-1) - 3 + k}{\sqrt{(2^2 + 2^2 + 1)}} = 3 \text{ or, } \frac{-1 + k}{3} \pm 3 \quad k = 10, \text{ or } -8$$

Hence the required tangent planes are

$2x + 2y - z + 10 = 0$ and $2x + 2y - z - 8 = 0$

6. **A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of centre of the sphere OABC is $ax + by + cz = 2$**

Let the equation of the sphere through the origin be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ where u, v, w are parameters

It cuts the axes at $A(-2u, 0, 0)$, $B(0, -2v, 0)$ $C(0, 0, -2w)$.

The equation of the plane ABC is

$$\frac{x}{-2u} + \frac{y}{-2v} + \frac{z}{-2w} = 1 \text{ or, } \frac{x}{-u} + \frac{y}{-v} + \frac{z}{-w} = 2$$

As it passes through (a, b, c) therefore $\frac{a}{-u} + \frac{b}{-v} + \frac{c}{-w} = 2$

Hence the locus of the centre $(-u, -v, -w)$ is $ax + by + cz = 2$

7. **Find the equations of the two spheres which pass through the circle $x^2 + y^2 + z^2 - 4x - y + 3z + 12 = 0$, $2x + 3y - 7z - 10 = 0$ and touch the plane $x - 2y + 2z = 1$** [IPE-11,08]

Let the equations of the sphere be

$x^2 + y^2 + z^2 - 4x - y + 3z + 12 + \lambda(2x + 3y - 7z - 10) = 0$

or, $x^2 + y^2 + z^2 - 2x(2 - \lambda) - y(1 - 2\lambda) - z(7\lambda - 3) + 12 - 10\lambda = 0$ ... (1)

The centre of the sphere is at the point $2 - \lambda.\ (1 - 3\lambda)/2.\ (7\lambda - 3)/2$

Radius of the sphere

$r^2 = (2 - \lambda)^2 + (1 - 3\lambda)^2/4 + (7\lambda - 3)^2/4 - 12 + 10\lambda = (62\lambda^2 - 24\lambda - 22)/4$

The length of the perpendicular from the centre of the sphere to the plane $x - 2y + 2z = 1$ ... (2)

is equal to the radius of the sphere

$$\therefore \frac{(2 - \lambda - 1 + 3\lambda + 7\lambda - 3 - 1)^2}{1^2 + 2^2 + 2^2} = \frac{(62\lambda^2 - 24\lambda - 22)/4}{4}$$

or, $36\lambda^2 - 24\lambda + 4 = 62\lambda^2 - 24\lambda - 22$ or, $\lambda^2 = 1$ or, $\lambda = \pm 1$

Put the values of λ in (1) The equations of the required spheres are $x^2 + y^2 + z^2 - 2x + 2y - 4z + 2 = 0$, $x^2 + y^2 + z^2 - 6x - 4y + 10z + 22 = 0$

Obtain the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ which pass through the line.

$$\frac{16 - x}{2} = \frac{z}{2} = \frac{y + 15}{3}$$

The equation of the line can be put as

$P \equiv x + z - 16 = 0$ and $Q \equiv 2y - 3z + 30 = 0$

Any plane through the intersection of P and Q is

$x + z - 16 + \lambda(2y - 3z + 30) = 0$

or, $x + 2\lambda y + (\lambda - 2\lambda) z - 16 + 30\lambda = 0$ ... ... (1)

The centre of the sphere is $(-3, 0, 1)$ and radius is 3

If it is tangent plane to the sphere (given) then perpendicular through $(-3, 0, 1)$ to the plane is 3

$$\pm \frac{-3 + 0 + (1 - 3\lambda) - 16 + 30\lambda}{\sqrt{[1 + 4\lambda^2 + (1 - 3\lambda^2)]}}$$

$$\pm \frac{18 + 27\lambda}{\sqrt{[13\lambda^2 - 6\lambda + 21]}} = 3 \text{ or, } 36 - 108\lambda^2 + 81\lambda = 13\lambda^2 - 6\lambda^2 + 2$$

or, $2\lambda^2 - 3\lambda + 1 = 0$, or, $\lambda = 1, 12$

Putting the values of λ in (1) we have

$x + 2y - z - 2 = 0$, $x + 2y - 2z + 14 = 0$

Two spheres of radii r_1 and r_2 cut orthogonally.

Prove that the radius of the common circle be

$\frac{r_1 r_2}{\sqrt{(r_1^2 + r_2^2)}}$ [C.U. - 05]

Let the equation of the common circle be

$x^2 + y^2 = a^2, z = 3$ ... ... (1)

The equations of the given spheres through the circle are

$x^2 + y^2 + z^2 + 2k_1 z - a^2 = 0$ ... ... (2)

and $x^2 + y^2 + z^2 + 2k_2 z - a^2 = 0$ ... ... (3)

where $r_1^2 = k_1^2 + a^2$, $r_2^2 = k_2^2 + a^2$ ... ... (4)

Again (2) and (3) cut orthogonally if

$2k_1 k_2 = a^2 + a^2$ or, $k_1 k_2 = a^2$ or, $k_1^2 k_2^2 = a^4$

or, $(r_1^2 - a^2)(r_2^2 - a^2) = a^4$ $\therefore$ from (4)

or, $r_1^2 r_2^2 = (r_1^2 + r_2^2) a^2$

$$a = \frac{r_1 r_2}{\sqrt{(r_1^2 + r_2^2)}}$$