Mathematics – 1201

⇒ Geometry

⇒ Matrix.

Matrix

Uses of Matrix:

→ To change a structure from one place to another place.

→ Solving system by using matrix.

Properties of matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} (3\times 3)$$

denoted by , [ ], ( ).

$m \times n$ — row, coloumn.

Square matrix: If the number of rows and columns of a matrix are equal then it is called square matrix. $m=n$.

Horizontal matrix: number of rows is ~~more~~ less than number of columns. $m<n$.

Vertical matrix: number of rows more than columns. $m>n$.

$\longrightarrow$ a matrix:

The summation of the diagonal elements of the square matrix. Is known as trace of the matrix.

As Trace of . $A = (a_{11} + a_{22} + a_{33})$.

Ex: If, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then show,

trace (A) + trace (B) = trace of (A+B)

$[A+B] = \begin{bmatrix} 1+a & 2+b \\ 3+c & 4+d \end{bmatrix}$

trac of $A = 5$

trace of $B = a+d$

trace of $A+B = 5+a+d$

trace of $(A+B) = 4+d+1+a$
$= 5+a+d$

(Showed)

Diagonal matrix: If $a_{ij} = 0$ for all $i \neq j$ then the square matrix is known as Diagonal matrix.

Ex: $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times3}$.

Scalar matrix: If the diagonal elements of a diagonal matrix is equal to a scalar say k. then the matrix is known as Scalar matrix.

Ex: $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$.

Identity matrix:

If the diagonal ~~matrix~~ elements of a diagonal matrix are equal to one then matrix is Identity matrix.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper and lower triangular matrix:

a square matrix $A = a_{ij}$. $i = j = 1, 2, 3 \dots n$ whose elements $a_{ij} = 0$ for $i > j$ is called upper triangular matrix. and for lower triangular matrix we have $a_{ij} = 0$ for $i < j$.

$j > i$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$i > j$

Commutative and anti-commutative matrix:

If A and B are square matrix of same order such that $AB = BA$ then A and B are called commutative. and if $AB = -BA$ then they are called anti-commutative matrix.

Periodic matrix:

A matrix for which $A^{k+1} = A$ where k is any scalar then A is periodic matrix.

Idempotent matrix:

A matrix A for which $A^2 = A$ is known as Idempotent matrix.

☐ Singular and Non-Singular matrix:

☐ Nilopotent matrix: A matrix A for which $A^p = O$ where p is any positive integer is known as is ~~know~~ a nilopotent matrix.

☐ Involutary matrix: A matrix A such that $A^2 = I$ It's known as Involutary matrix.

Theorem: A matrix is involutary iff $(I+A)(I-A) = O$

$\Rightarrow$

Singular and Non-Singular matrix: If the determinent of a square matrix is equal to zero, then that is singular matrix.
If it is not zero then it is non-singular matrix.

$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$|A| = 2-2=0$ Singular $|B| = 1-4=-3$ Non singular.

Transpose of a matrix: The matrix of order $m \times n$ is obtained by interchanging the row and column of ~~mxn~~ $n \times m$ matrix then A is called transpose of a matrix. Denoted by A'

$A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}_{2\times3}$ $A' = \begin{bmatrix} 1 & a \\ 2 & b \\ 3 & c \end{bmatrix}_{3\times2}$

Properties of transpose matrix:

→ $(A+B)' = A'+B'$

→ $(AB)' = \cancel{A}\, B'A'$ *

→ $(KA)' = KA'$

→ $(A')' = A$ * Prove করতে হবে।

~~# Ex~~

Symetric and skew symetric matrix:

If a square matrix A that $A = A'$ then it is called symetric matrix.

~~Aij~~ $A = a_{ij}$ is symetric if $a_{ij} = a_{ji}$ for all values of i and j

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = A'$$

Then it will be skew symetric if $a_{ij} = -a_{ji}$ for all values of i and j.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix} \quad A = -A' \quad \text{skew symetric.}$$

Diagonal Elements are $= a_{11}, a_{22}, a_{33}$

$a_{ii} \quad a_{jj}$

Theorem: The diagonal elements of a skew-matix is zero

Proff: If $A = a_{ij}$; $i = j = 1, 2, 3 \cdots n$ is a square ~~number~~ the matrix then the condition that matrix A, B, C, a skew-symetric is that $\boxed{a_{ij} = -a_{ji}}$ (1)

For diagonal elements the condition (1) becomes,

$$a_{ii} = -a_{ii}$$

$$\Rightarrow 2a_{ii} = 0$$

$$\Rightarrow a_{ii} = 0 \quad \text{(Prove)}$$

Theorem: Every square matrix can be expressed as a sum of a symmetric and skew-symmetric matrix.

Proof: Let 'A' be a square matrix then we write $A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$

So, we have to show that $\frac{1}{2}(A+A')$ ~~and~~ is a symmetric and $\frac{1}{2}(A-A')$ is a skew-symmetric matrix.

Now, $\left\{\frac{1}{2}(A+A')\right\}' = \frac{1}{2}(A+A')'$ $\quad [(KA)' = KA']$

$= \frac{1}{2}\{A' + (A')'\}$ $\quad [(A+B)' = A'+B']$

$= \frac{1}{2}(A'+A)$

$= \frac{1}{2}(A+A')$

So, $\frac{1}{2}(A+A')$ is a symmetric matrix.

Again, $\left\{\frac{1}{2}(A-A')'\right\} = \frac{1}{2}\{A' - (A')'\}$

$= \frac{1}{2}\{A' - A\}$

$= -\frac{1}{2}(A-A')$

So, $\frac{1}{2}(A-A')$ is a skew symmetric matrix.

Orthogonal matrix: a square matrix 'A' such that $A'A = I$ where I is a identity matrix is called orthogonal matrix.

 # Ex: Show $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal matrix.

$A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$A'A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha - \cos\alpha\sin\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$

Ex: If a A and B are orthogonal matrix then show that AB and BA are also orthogonal matrix.

Soln: By defination we have,

$A'A = I \longrightarrow$ (i)

$B'B = I \longrightarrow$ (ii)

Now,

$(AB)'(AB) = (A'B')(AB)$

$= A\, B' (A'A) B$

$= (B' I) B$

$= B'B \quad [I \times A = A]$

$= I$

$\left[\begin{array}{l} C'C = I \\ \text{if } AB = C \\ (AB)'(AB) = I \end{array}\right.$

So, AB is orthogonal matrix if A and B is orthogonal matrix.

Conjugate of a matrix:

If $A = \begin{bmatrix} i & a+ib & 2 \\ 1-i & 0 & i \\ 3 & 3i & -i \end{bmatrix}$

then the conjugate of A is denoted by $\bar{A}$ and dfined as $\bar{A} = \begin{bmatrix} -i & a-ib & 2 \\ 1+i & 0 & -i \\ 3 & -3i & i \end{bmatrix}$

Properties of Conjugate matrix:

→ $\overline{A+B} = \bar{A} + \bar{B}$

→ $\overline{AB} = \bar{A}\,\bar{B}$

→ $\overline{KA} = K\bar{A}$

→ $\bar{\bar{A}} = A$

Hermitian and skew Hermitian matrix: A square matrix $A = a_{ij}$ such that $\bar{A}' = A$ is called Hermitian matrix. It's provided $\boxed{a_{ij} = \bar{a}_{ji}}$

If $\bar{A}' = -A$ is called skew-Hermitian matrix $\boxed{a_{ij} = -\bar{a}_{ji}}$

Theorem: The diagonal element of a Hermitian matrix are real and the diagonal elements of a skew Hermitian matrix are either zero or pure imaginary number.

Proof: $K+A = a_{ij}$ be a square matrix then the condition that 'A' be a hermitian matrix is

$a_{ij} = \bar{a}_{ji}$ for all i and j

For diagonal elements condition is,

$$a_{ii} = \bar{a}_{ii} \quad \text{(i)}$$

let,

$$a_{ii} = \alpha + i\beta \quad \text{(ii)}$$

$$\bar{a}_{ii} = \alpha - i\beta$$

Using (ii) we get from (i)

$$\alpha + i\beta = \alpha - i\beta$$

$$\Rightarrow 2i\beta = 0$$

$$\Rightarrow \beta = 0$$

This shows that the diagonal elements of a hermitian matrix is real.

Now, for ~~a~~ the diagonal elements the condition that a_{ij} be a skew hermitian is that

$$a_{ii} = -\bar{a}_{ii} \quad \text{(iii)}$$

Using (ii) we get from (iii)

$$\alpha + i\beta = -\alpha + i\beta$$

$$\Rightarrow 2\alpha = 0$$

$$\Rightarrow \alpha = 0$$

This shows that the diagonal elements ~~of~~ of a skew hermitian matrix are pure imaginary and can be zero ~~af~~ iff $\beta = 0$.

<u>Theorem:</u> Every Square matrix 'A' can be expressed as a Sum of a hermitian and a Skew hermitian matrix.

<u>Proof:</u> Let, $A = a_{ij}$ be a square matrix.

Than we can write,

$$A = \frac{1}{2}(A + \bar{A}') + \frac{1}{2}(A - \bar{A}')$$

We have to show that $\frac{1}{2}(A+\bar{A}')$ is hermitian and $\frac{1}{2}(A-\bar{A}')$ is a skew hermitian matrix.

Now, $\overline{\frac{1}{2}(A+\bar{A}')}' = \frac{1}{2}\overline{(A+\bar{A}')}'$ $\quad [\overline{KA} = K\bar{A})$

$= \frac{1}{2}(\bar{A} + \bar{\bar{A}}')'$ $\quad [\overline{A+B} = \bar{A} + \bar{B}]$

$= \frac{1}{2}[\bar{A}' + (A')')$

$= \frac{1}{2}(\bar{A}' + A)$

$= \frac{1}{2}(A + \bar{A}')$

which shows that, $\frac{1}{2}(A+\bar{A}')$ is a hermitian matrix.

$\overline{\frac{1}{2}(A+\bar{A}')}' = \frac{1}{2}\overline{(A-\bar{A}')}'$

$= \frac{1}{2}(\bar{A} - \bar{\bar{A}}')'$

$= \frac{1}{2}(\bar{A}' - (A')')$

$= \frac{1}{2}(\bar{A}' - A)$

$= -\frac{1}{2}(A - \bar{A}')$

So, $\frac{1}{2}(A-\bar{A}')$ is a skew-hermitian matrix.

Unitary Matrix: a Square matrix A is unitary matrix if $\bar{A}'A = I$ where I is the identity matrix.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ express a A as the sum of symmetric and skew symmetric matrix.

Soln: Symmetric part is $\frac{1}{2}(A+A')$

Skew- " part is $\frac{1}{2}(A-A')$

Now $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ So, $\frac{1}{2}(A+A') = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} = A$$

And, $\frac{1}{2}(A-A') = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right\}$

$$= \frac{1}{2} \times 0 = 0$$

So, $\frac{1}{2}(A+A') + \frac{1}{2}(A-A') = A$

Every square matrix can be written as the sum of symmetric matrix $\frac{1}{2}(A+A')$ and skew symmetric matrix $\frac{1}{2}(A-A')$.

Ex: $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ Prove, $\frac{1}{2}(A+A') + \frac{1}{2}(A-A') = A$

Soln: $A' = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 5 & 2 \\ 4 & 1 & 3 \end{bmatrix}$ Now, $\frac{1}{2}(A+A') = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 \\ 3 & 5 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 4 & 5 & 5 \\ 5 & 10 & 3 \\ 5 & 3 & 6 \end{bmatrix}$$

$$\frac{1}{2}(A-A') = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 1 \\ 3 & 5 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right\}$$

$$= \cancel{\frac{1}{2}\begin{bmatrix} 4 & 5 & 5 \\ 5 & 10 & \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A+A') + \frac{1}{2}(A-A') = \frac{1}{2}\begin{bmatrix} 4 & 5 & 5 \\ 5 & 10 & 3 \\ 5 & 3 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 6 & 8 \\ 4 & 10 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} = A \text{ (Proved)}$$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ Find the matrix C such that $\boxed{A+B = 2C}$

Soln: Let, $C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$

According to the question,

$$A+B = \begin{bmatrix} 3 & 5 & 7 \\ 3 & 5 & 7 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2k \end{bmatrix}$$

Ex

Now, equating the corresponding elements.

$3 = 2a \Rightarrow a = 3/2$ $\quad$ $3 = 2d \Rightarrow d = 3/2$ $\quad$ $3 = 2g \Rightarrow g = 3/2$

$5 = 2b \Rightarrow b = 5/2$ $\quad$ $5 = 2e \Rightarrow e = 5/2$ $\quad$ $3 = 2h \Rightarrow h = 3/2$

$7 = 2c \Rightarrow c = 7/2$ $\quad$ $7 = 2f \Rightarrow f = 7/2$ $\quad$ $4 = 2k = k = 2$.

Ex:

So,
$$C = \begin{bmatrix} 3/2 & 5/2 & 7/2 \\ 3/2 & 5/2 & 7/2 \\ 3/2 & 3/2 & 4/2 \end{bmatrix}$$ Ans

f

Ex: $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ \cancel{2} & \cancel{3} \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ \cancel{2} & \cancel{3} \end{bmatrix}$ then evaluate $A + 2B + 3I$

Soln: $2B = \begin{bmatrix} 2 & 4 \\ 10 & 12 \\ \cancel{4} & \cancel{6} \end{bmatrix}$ $\quad 3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$A + 2B + 3I = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 10 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 8 \\ 11 & 17 \end{bmatrix}$$

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Ex: Solve the following eqn for x and y.

$$2x - y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix} \qquad 2y + x = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

Ex: Find the value of 'a' if

$$(a, 4, 1)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ 4 \\ -1 \end{pmatrix} = O$$

O is a null matrix.

Ex: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Find the values of 'a' and 'b' so that

$$(aI + bA)^2 = A.$$

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the necessary condition for a

Inverse Matrix: If $AB = I = BA$ then 'A' is a Inverse matrix of B and B is the inverse matrix of A and $B = A^{-1}$ and $A = B^{-1}$.

$$\boxed{A^{-1} = \frac{\operatorname{Adj} A}{|A|}}$$

Theorem: The inverse matrix of a matrix is unique.

Proof: Let 'A' be a square matrix and if possible Let B and C are two inverses of A. Then by defination,

$$AB = BA = I \quad \text{——— (i)}$$

$$AC = CA = I \quad \text{——— (ii)}$$

from eqn. — (i)

$$BA = I$$

$$\Rightarrow (BA)C = IC = C \quad \text{——— (iii)}$$

from (ii)

$$AC = I$$

$$\Rightarrow B(AC) = BI = B \quad \text{——— (iv)}$$

According to the associative law of multiplication

$$(BA)C = B(AC)$$

$$\Rightarrow C = B$$

This shows that the inverse of a matrix is unique.

Theorem: The necessary condition for a square matrix 'A' to posses an inverse is that 'A' is non-singular matrix.

Proof: Let, B the Inverse of A. then by defination $AB = BA = I$ ——— (1)

Take determinant on both sides of (1):

$|AB| = |I|$

$\Rightarrow |A|\,|B| = 1$ $\quad [|I| = 1]$

From this relation $|A| \neq 0$ and $|B| \neq 0$

Therefore the matries 'A' is non-singular.

যদি Non-singular হয় তবেই তার Inverse বের করা যাবে।

Theorem: The inverse of the product of two matrixes having inverse is the product in reverse order of these matrixes.

Minor and Co-factor of a matrix: The determinant of every square sub matrix of a matrix is called a minor of the matrix.

The signed minor is called the co-factor of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}$$

Sub-matrix of order 3×3, $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

And $|A_1|$ = minor of order 3×3.

$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ $\quad |A_2|$ = minor of order 2×2.

—x—

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

minor of the elemnt $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

co-factor of $a_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

—x—

—x—

$$\text{Adj } A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}' = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{\text{Adj } A}{|A|}}$$

$A = \begin{array}{c} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 2 & 1 \end{array}\right] \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ K_1 & K_2 & K_3 \end{array} \end{array}$

make it identical matrix by elementary row transformation.

Now, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & -4 & -8 \end{bmatrix}$ $\left[\begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array} \right]$

$= \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 1 & 2 \end{bmatrix}$ $\left[\begin{array}{l} R_1 = R_1 - 2R_2 \\ R_3 = R_3/-4 \end{array} \right]$

$= \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & 7 \end{bmatrix}$ $[R_3 = R_3 - R_2]$

$= \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$ $[R_3 = R_3/7]$

$= \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $[R_2 = R_2 + R_3 \times 5]$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $[R_1 = R_1 - 13 \times R_3]$

$= I_3$ Ans

$\boxed{\square}$ $AB = I$

$A = IA \longrightarrow$ multiple with A.

$\downarrow$

Make it Identical matrix

Ex: Find the Inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ by elementary row transformation

Soln: We can write.

$$\underset{\underset{I_3}{\downarrow}}{A} = \underset{\underset{B}{\downarrow}}{I_3}\ \underset{\underset{\boxed{A}}{\downarrow}}{A}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Interchanging the 2nd and 3rd row

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} A$$

$R_2 = R_2(-1)$

$R_3 = R_3(-1)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} A$$

$R_2 = R_2 - 3R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} A$$

$R_1 = R_1 - 2 \times R_2$

~~R_2~~ $R_1 = R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} A$$

$\sim I_3 = BA$

where, $B = A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Ex: Find the Adjoint of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and evaluate $\boxed{A^{-1}}$

Soln: $\text{Adj } A = \begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \end{bmatrix}'$

$= \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}' = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = -1$ So, $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

$= \dfrac{\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}}{-1}$

$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ Ans

Rank of matrix: The rank of a given matrix 'A' is said to be 'r' if every minor of order $(r+1)$ is 'zero' and there is at least one minor of A of order r is different from zero.

rank of

Ex: Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}_{3\times 4}$

Soln: The minor of order 3 is $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{vmatrix} = 0$

Similarly the other minor of order 3 are $= 0$

Now, the minor of order 2 is $= \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$

So, the rank of the matrix is '2'.

Normal or canonical form of a matrix:

Every 'm×n' matrix 'A' of rank 'r' can be reduced to any of the forms.

$$[I_r] \quad \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad \begin{bmatrix} I_r & 0 \end{bmatrix} \quad \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

I → Identical matrix. r → order

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is called $\boxed{\text{normal forms}}$ by operation of elementary row and column transformation.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank বের করা যাবে;
- → Normal or conical form
- → Minor
- → Echelon.

Find the rank of $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 2 \end{bmatrix}$ reducing it to normal form.

Solnº

$R_2 = R_2 - 4R_1$, $R_3 = R_3 - 2R_1$
$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 6 & -5 & 6 \\ 0 & 4 & -4 & 4 \end{bmatrix}$$

$R_3 = R_3/4$
$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 6 & -5 & 6 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$R_2 = R_2 - 6R_3$
$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Interchanging 2nd and 3rd row
$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$K_2 = K_2 + K_1$, $K_3 = K_3 + K_4$, $K_4 = K_4 + K_1$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$K_4 = K_4 - K_2$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim [I_3, 0]$$

This is the required normal form of given matrix A and the rank is '3'.

Echelon form:

A matrix 'A' is said to be in echelon form if,

(i) all the non zero rows if any predece the zero rows.

(ii) The number of zero preeciding the first non-zero element in a row is less than the number of such zero in the suceeding row.

(iii) The first non-zero element in a row is unity.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \rightarrow \text{non zero row} \\ \rightarrow \text{zero row.} \end{matrix}$$

(✱) Number of non-zero rows in the reduced echelon form is the rank of the matrix.

(✱) Reduce the matrix to echelon form and find the rank:

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[\text{1st and 2nd row}]{\text{Interchanging}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[H_3 = H_3 - H_1]{H_2 = H_2 + 2H_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[H_3 = H_3/2,\ H_2 = H_2/3]{H_3 = H_3 + 2H_4} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{H_3 = H_3 + H_2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

this is the echelon form of A. and number of non-zero rows is 2.

So, the rank is '2'.

$$[A\ H] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r_1 = 3$

$r_2 = 2$

$r_1 \neq r_2$

Echelon form এ 1st row এর 1st element '1' হবে।

Linear Equations:

$$\left.\begin{array}{l} y = mx \\ y = mx + c \\ y^2 = 4ax \end{array}\right\} \text{linear equations.}$$

Consider a system of 'm' linear equations in the 'n' unknowns $x_1, x_2 - - - x_n$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + - - - - + a_{1n}x_n = h_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + - - - - + a_{2n}x_n = h_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + - - - - + a_{3n}x_n = h_3$$
$$\cdots \quad - \quad - \quad - \quad - \quad - \quad -$$
$$- \quad - \quad - \quad - \quad - \quad - \quad -$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + a_{m4}x_4 + - - - - + a_{mn}x_n = h_m$$

in which $a_{11}, a_{12}, a_{13} - - - a_{mn}$ are called co-efficients. and $h_1, h_2 .. h_m$ are consiting a a system of linear equations is called consistant if it has at least one soln and called ~~incont~~ inconsistant if it has no soln.

The system of linear eqn. in matrix may be written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_m \end{pmatrix}$$

or, more compuctly

$$AX = H$$ where A is the co-efficient matrix.

and the argumented matrix is,

$$[A \quad H] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & h_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & h_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & h_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & h_m \end{array}\right]$$

* $a_1x_1 + a_2x_2 + a_3x_3 + \cdots a_nx_n = h_1$ → non-homogenous linear eqn,

$a_1x_1 + a_2x_2 + a_3x_3 + \cdots a_nx_n = 0$ → homogenous linear eqn.

* A system of non-homogenous linear eqn. will have,,

(i) no soln: If the rank of the argumented matrix say r_1 is different from the rank of the co-efficient matrix - say r_2 that $r_1 \neq r_2$

(ii) unique soln: If $r_1 = r_2$, no. of unknowns say 'n' then the system has unique soln.

(iii) Infinite soln: If $r_1 = r_2 < n$ then the system has infinte number of soln. It this case we assign $(n-r)$ orbitrary constants to $(n-r)$ unknowns and the other values of the remaining unknowns

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are determined in terms of the (n-r) orbitory values.

If a system at least one soln then it is called as consistant system.

[4]
$$x+2y-z=3$$
$$3x-y+2z=1$$
$$2x-2y+3z=2$$
$$x-y+z=-1$$

show that the system is consistant and solve them completely.

Soln The argumented matrix is.

$$[A\ H] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$\underline{H_2 = H_2 - 3H_1}$, $H_3 = H_3 - 2H_1$, $H_4 = H_4 - 1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$\underline{H_2 = H_2 - H_3}$
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$\underline{H_3 = H_3 + -6H_2}$, $H_4 = H_4 - 3H_2$
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$H_3 = \frac{H_3}{5}$, $H_4 = \frac{H_4}{2}$
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\underline{H_4 = H_4 - H_3}$
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In the above achelon form; the rank of argumented matrix is '3' and also the rank of co-efficient matrix is '3'. Therefore the given system is consistant and the number of unknowns is equal to the rank of the argumented matrix. Thus the system has unique solution.

Therefore, the corresponding linear eqn are

$$x + 2y + z = 3$$
$$y = 4$$
$$z = 4$$

Therefore soln is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$

$\boxed{7}$ Solve the system with matrix method:

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6$$
$$x_1 + 3x_2 + x_3 - 2x_4 = 4$$
$$2x_1 + 5x_2 - 2x_3 - 5x_4 = 10$$

Soln: The argumented matrix is,

$$[A\,|\,H] = \left[\begin{array}{ccccc} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{array}\right]$$

$H_2 = H_2 - H_1$

$H_3 = H_3 - 2H_1$

$$\left[\begin{array}{ccccc} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{array}\right]$$

$H_3 = H_3 - H_2$

$$\left[\begin{array}{ccccc} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Here, In the echelon form, rank of argumented matrix is equal to rank of co-efficient matrix. The number of unknowns is greater than the rank. So, the system is consistant and has infinite number of Soln.

We have to assign $(4-3) = 1$ arbitrary value.

$$\underset{3}{r_1} = \underset{3}{r_2} < \underset{4}{n}$$

So, the corresponding linear eqn,

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6$$
$$x_2 + 4x_3 + 2x_4 = -2$$
$$x_4 = 0$$

let, $x_3 = K$.

then,
$$x_1 + 2x_2 - 3K - 4x_4 = 6$$
$$x_2 + 4K + 2x_4 = -2$$
$$x_4 = 0$$

Soln is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 + 11K \\ -2 - 4K \\ K \\ 0 \end{pmatrix}$$

Find the value of λ so that the eqns,

$$ax+hy+g=0$$
$$hx+by+f=0$$
$$gx+fy+e=\lambda$$

are consistant,

$$ax+hy=-g$$
$$hx+by=-f$$
$$gx+fy=\lambda-e.$$

Soln: The argumented matrix is,

$$[A\cdot H] = \begin{pmatrix} a & h & -g \\ h & b & -f \\ g & f & \lambda-e \end{pmatrix}$$

$$\xrightarrow[\substack{H_2 = H_2/h \\ H_3 = H_3/g}]{H_1 = H_1/a} \begin{bmatrix} 1 & h/a & -g/a \\ 1 & b/h & -f/h \\ 1 & f/g & \frac{\lambda-e}{g} \end{bmatrix}$$

$$\xrightarrow[H_3 = H_3 - H_1]{H_2 = H_2 - H_1} \begin{pmatrix} 1 & h/a & -g/a \\ 0 & b/h - h/a & g/a - f/h \\ 0 & f/g - b/h & \frac{\lambda-e}{g} - g/a \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & h/a & -g/a \\ 0 & \frac{ab-h^2}{ah} & \frac{-af+gh}{ah} \\ 0 & \frac{af-gh}{ag} & \frac{a\lambda-ae+g^2}{ag} \end{pmatrix}$$

$$\xrightarrow[H_3 = H_3 \times \frac{ag}{af-gh}]{H_2 = H_2 \times \frac{ab}{ab-h2}} \begin{pmatrix} 1 & h/a & -g/a \\ 0 & 1 & \frac{-af+gh}{ah} \times \frac{ab}{ab-h2} \\ 0 & 1 & \frac{a(\lambda-e)+g^2}{ag} \times \frac{ag}{af-gh} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & h/a & -g/a \\ 0 & 1 & \frac{-af+gh}{ah-h2} \\ 0 & 1 & \frac{a(\lambda-e)+g^2}{af-gh} \end{pmatrix}$$

$$\xrightarrow{H_3 = H_3 - H_2} \begin{pmatrix} 1 & h/a & -g/a \\ 0 & 1 & \frac{-af+gh}{ah-h2} \\ 0 & 0 & \frac{a(\lambda-e)+g^2}{af-gh} - \frac{-af+gh}{ah-h^2} \end{pmatrix}$$

In the echelon form, rank of argumented matrix is '3' and rank of co-efficient matrix is '2'

So, given system will be consistant if

$$\frac{a(\lambda-c)+g^2}{af-gh} - \frac{-af+gh}{ab-h^2} = 0$$

$$\Rightarrow \lambda = \left[\frac{1}{a}\left(\frac{-af+gh\,(af-gh)}{ab-h^2} - g^2 + ac\right)\right]$$

□ For what values of 'λ' the system have a soln and hence solve completely in each case.

$$x+y+z=1$$
$$x+2y+4z=\lambda$$
$$x+4y+10z=\lambda^2$$

Soln: The argumented matrix is,

$$[A\ H] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix} \quad \begin{matrix} H_2 = H_2 - H_1 \\ H_3 = H_3 - H_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{bmatrix}$$

$$H_3 = H_3 - 3H_2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-1-3\lambda-3 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right]$$

The given system have at least one value if the rank of the argumented matrix and the rank of the co-efficient matrix are same in the echelon form, therefore we must have, $\lambda^2-3\lambda+2=0$

$$\Rightarrow \lambda^2-2\lambda-\lambda+2=0$$

$$\Rightarrow \lambda = 2, 1$$

That is, for $\lambda=2$ and $\lambda=1$ the system will be consistant.

For $\lambda=2$ the echelon form is,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding linear eqns. are $x+y+z=1$, $y+3z=1$

we have to assign $3-2=1$ arbitrary values,

let, $z=K$ then, $x+y+K=1$
$y+3K=1$

So, the soln is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2K \\ 1-3K \\ K \end{pmatrix}$

Now, For. $\lambda=1$ the echelon form is, $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The coresponding linear eqn are,

$x+y+z=1$
$y+3z=0$

let. $z=K$ then, $x+y+K=1$
$y+3K=0$

So, the soln is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2K \\ -3K \\ K \end{pmatrix}$

The system of l...

The soln of homogenous system of linears:

A system of 'm' homogenous eqn's in 'n' unkowns, $x_1, x_2, x_3 \cdots x_n$ has a non-trivial soln if the rank of the co-efficient matrix is less than number of unknowns says 'n'. If 'r=n' then the system has trivial value. that is $x_1 = 0 = x_2 = \cdots = x_n$.

Ex:

$$x + y + z + w = 0$$
$$x + 3y + 2z + 4w = 0$$
$$2x + z - w = 0$$

Soln: The co-efficient matrix is,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & -2 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is non-trivial and has ifinite number of soln. The rank of the given system is '2' and the number of unknowns is '4' Therefore the given system has non-trival soln.

The corresponding linear eqn are,

$$x + y + z + w = 0$$
$$2y + z + 3w = 0$$

we have to assign $4 - 2 = 2$ arbitory values,

let, $y = a$
$z = b$

Then, the Soln is, $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{2a+b}{3} - a - b \\ a \\ b \\ \frac{-(2a+b)}{4} \end{pmatrix}$

Ex: Solve by matrix method.

$$x+y+z=6$$
$$x+2y+3z=14$$
$$x+4y+9z=36$$

Soln: In matrix notation we can write,

$$AX = H$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 36 \end{pmatrix}$$

(*) lower triangle matrix এ রূপান্তর করলে সুবিধা

(*) $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ কলামের কোন পরিবর্তন হবেনা।

$R_2 = R_2 - R_1$
$R_3 = R_3 - R_1$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 30 \end{pmatrix}$

$R_3 = R_3 - 3R_2$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 6 \end{pmatrix}$

$R_3 = R_3/2$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix}$

the corresponding linear eqns are,

$$x+y+z=6$$
$$y+2z=8$$
$$z=3$$

Therefore the soln is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Ex: Solve
$$x+2y+3z=6$$
$$2x+4y+z=7$$
$$3x+2y+9z=14$$
using determinents.

Sol: By cramers rule:

$$\frac{x}{\begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{6(36-2)-2(63-14)+3(14-56)} = \frac{y}{1(63-14)-6(18-3)+3(28-21)} = \frac{z}{1(56-14)-2(28-21)+6(4-12)}$$
$$= \frac{1}{1(36-2)-2(18-3)+3(4-12)}$$

$$\Rightarrow \frac{x}{-20} = \frac{y}{-20} = \frac{z}{-20} = \frac{1}{-20}$$

So, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ Ans

Characteristic eqn of matrix:

Any vector 'X' which by the transformation is carried into λX, that is any vector 'X' for which $AX = \lambda X$ is called an eigon vector under the transformation.

Now, $\lambda X - AX = 0$

$$\Rightarrow (\lambda I - A)X = 0$$

$$\Rightarrow \begin{bmatrix} \lambda-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda-a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda-a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

The system of homogeneous eqns has non-trivial soln. if and only if $(\lambda I - A) = 0$

$$\Rightarrow \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{bmatrix} = 0$$

The eqn. $|\lambda I - A| = 0$ is called the characteristics eqn. and the values $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are characteristic values of eigen values.

Ex: Find the characteristic eqn. of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$; hence find the characteristic values and vectors.

Soln: The characteristic eqn for given 'A' is

$$|\lambda I - A| = 0$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 2 & -2 & -1 \\ -1 & \lambda - 3 & -1 \\ -1 & -2 & \lambda - 2 \end{vmatrix} = 0$$

$$\Rightarrow \lambda - 2(\lambda^2 - 3\lambda - 2\lambda + 6 - 2) - (-2)(-\lambda + 2 - 1) + (-1)(2 + \lambda - 3) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 - 5\lambda + 4) - 2\lambda + 2 + 1 - \lambda = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 4\lambda - 2\lambda^2 + 10\lambda - 8 - 2\lambda + 3 - \lambda = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 6\lambda^2 + 6\lambda + 5\lambda - 5 = 0$$

$$\Rightarrow \lambda = 1, 5, 1$$

These are the characteristic or eigen values.

For $\lambda = 1$, the eqn, $(\lambda I - A)X = 0$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The corresponding system of linear eqns,

$$-x_1 - 2x_2 - x_3 = 0$$
$$-x_1 - 2x_2 - x_3 = 0$$
$$-x_1 - 2x_2 - x_3 = 0$$

linearly independent eqn

Cause, Actually '1' eqn not '3'

So, linearly independent soln is $-x_1 - 2x_2 - x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \quad \text{again} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

এরকম অনেক হবে।

for $\lambda = 5$ the eqn $(\lambda I - A)X = 0$ gives

$$\begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The corresponding system of linear eqns,

$$3x_1 - 2x_2 - x_3 = 0$$
$$-x_1 + 2x_2 - x_3 = 0$$
$$-x_1 - 2x_2 + 3x_2 = 0$$

The linearly independent soln is

$$\begin{pmatrix} x_1 \\ \cancel{x_2} \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So, The eigen values are $1, 5, 1$ and the corresponding eigen vectors are,

$$\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Cayley - Hamilton Theorem:

statement: Every square matrix satisfies its characteristic equation.

Proof:

Ex: Find the characteristic equation for $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ hence find eigen values and the corresponding eigen vectors. show that the given matrix A satisfies the characteristic equation. Hence find A^{-1}

14-10-19
7th : E Day

Ex: Show that the set of vectors $\{(2,1,2), (0,1,-1), (4,3,3)\}$ is linearly dependent.

Linear combination: The vectors $v_1, v_2, v_3 \dots v_n$ are said to be linear combination of a vector u if there exists scalars $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ such that
$$u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_n v_n$$

Soln:

The vectors $v_1, v_2, v_3 \dots v_n$ are said to be linearly dependent if there exists a non-trivial combination of them equal to zero vector that is
$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ where $\alpha_i \neq 0$ for at least one i.

[That is non-zero solution]

And the vectors $v_1, v_2, \ldots v_n$ are said to be linearly independent if only linear combination of them equal to zero is the trivial one that is

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0 \quad \text{iff} \quad \alpha_1 = 0 = \alpha_2 = \alpha_3 = \cdots = \alpha_m$$

[That is zero solution]

Soln:

Let x, y, z be three scalars, then the linear combination of the given vectors is

$$x(2,1,2) + y(0,1,-1) + z(4,3,3) = (0,0,0)$$

$$\Rightarrow (2x + 0.y + 4z,\ x + y + 3z,\ 2x - y + 3z) = (0,0,0)$$

Now,

the corresponding system of linear equations

$$2x + 0.y + 4z = 0$$

$$x + y + 3z = 0$$

$$2x - y + 3z = 0$$

Here,

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & -1 & 3 \end{pmatrix}$$

Interchanging 1st and 2nd row

$$\approx \begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 4 \\ 2 & -1 & 3 \end{pmatrix}$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 2R_1$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -3 \end{pmatrix}$$

$R_2 = R_2/(-2)$

$R_3 = R_3/(-3)$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of A is 2 and the number of unknowns is 3, therefore the solution of the system is non-trivial or infinite no of solution.

Therefore the given set of vectors are linearly dependent.

Ex: show that the vectors $\{(2,-1,4), (3,6,2), (2,10,-4)\}$ are linearly independent vectors.

Soln: Let, x, y, z be three scalars, then linear combination of three vectors are

$$x(2,-1,4) + y(3,6,2) + z(2,10,-4) = (0,0,0)$$

$$(2x+3y+2z, \; -x+6y+10z, \; 4x+2y-4z) = (0,0,0)$$

Now, the corresponding linear equations are

$$2x+3y+2z = 0$$
$$-x+6y+10z = 0$$
$$4x+2y-4z = 0$$

Here,

$$A = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 6 & 10 \\ 2 & 3 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 6 & 10 \\ 0 & 15 & 22 \\ 0 & 26 & 40 \end{pmatrix}$$

$H_1 = H_1 + H_2$

$$\begin{pmatrix} 1 & 9 & 12 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{pmatrix}$$

$H_2 = H_1 + H_2$

$H_3 = H_3 - 4H_1$

$$\begin{pmatrix} 1 & 9 & 12 \\ 0 & 15 & 22 \\ 0 & -37 & -52 \end{pmatrix}$$

$H_2 = H_2/15$

$H_3 = H_3/-37$

$$\begin{pmatrix} 1 & 9 & 12 \\ 0 & 1 & 22/15 \\ 0 & 1 & 52/37 \end{pmatrix}$$

$H_3 = H_3 - H_2$

$$\begin{pmatrix} 1 & 9 & 12 \\ 0 & 1 & 22/15 \\ 0 & 0 & 16/25 \end{pmatrix}$$

$H_3 = H_3/(16/25)$

$$\begin{pmatrix} 1 & 9 & 12 \\ 0 & 1 & 22/15 \\ 0 & 0 & 1 \end{pmatrix}$$

02.11.19
7th E-day

Ex: A man buys 8 dozens of mangoes, 10 dozens of apples, 4 dozens of bananas. Mangoes cost tk 180. per dozens, apples tk 90. per dozens and bananas tk 60 per dozen. Represent the quantities bought buy a row matrix and the prices by a column matrix and hence obtain the total cost.

Sol^n: Let,

$$A = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix}_{1\times3}$$

And $$B = \begin{bmatrix} 180 \\ 90 \\ 60 \end{bmatrix}_{3\times1}$$

Then total cost $= A \times B = \begin{pmatrix} 8 & 10 & 4 \end{pmatrix} \begin{pmatrix} 180 \\ 90 \\ 60 \end{pmatrix}$

$$= \begin{pmatrix} 2580 \end{pmatrix}_{1\times1}$$

$\therefore$ That is 2580 tk.

Ex: A store has in stock 30 dozens shirts, 15 dozens trousers and 25 dozens pair of socks. If the selling prices are tk 500 per shirts, tk. 900 per trousers and tk 120 per socks, then find the total amount the store owner will get after selling all the items in the stock.

Soln:

Ex: A manufacturer produces three products A, B, C which he sells in the market. Annual sale volumes are indicated as follows:

Markets	A	B	C
I	8.000	10,000	15,000
II	10,000	2,000	20,000

If unit sell prices of A, B, C are tk 225, tk 150 and tk 125 respectively. Then find the total revenue in each market with the help of matrices.

Soln: Selling price in row matrix let, $A = \begin{pmatrix} 225 & 150 & 125 \end{pmatrix}_{1\times3}$

No of products in matrix say B, B =

$$\begin{array}{cc} \text{I} & \text{II} \\ \end{array}$$
$$\begin{pmatrix} 8,000 & 10,000 \\ 10,000 & 2,000 \\ 15,000 & 20,000 \end{pmatrix}_{3\times2}$$

$$\therefore \text{Total revenue} = AXB = \begin{array}{cc} I & II \\ (517500 & 50,5000) \end{array}$$

that for market I revenue is 5,17,5000 tk

and for market II the revenue is 5,05,0000 tk