Geometry

General equation of second Degree & Pair of Straight line

Two dimensional :

$$x^2 - y^2 = 1$$

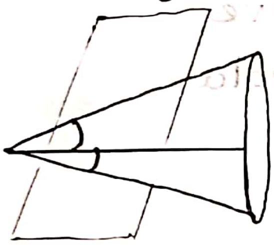
$$y^2 = x^2c + 2c$$

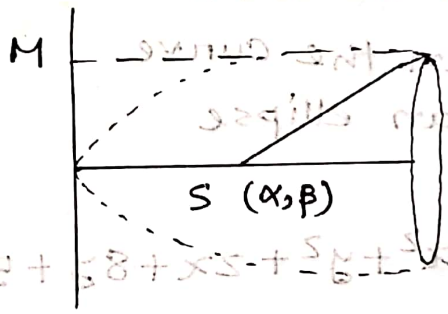
$$y = mx + c$$

General equation of Second Degree :

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

Cone :





angle দুইটাকে semi verticle angle বলে

উৎকেন্দ্রিকতা = eccentricity = e

$$\frac{SP}{PM} = e \Rightarrow SP^2 = e^2 \times PM^2 \Rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2 . \frac{(Ax+By+C)^2}{\left(\sqrt{A^2+B^2}\right)^2}$$

$$\Rightarrow (\quad)x^2 + (\quad)y^2 + (\quad)xy + (\quad)x + (\quad)y + (\quad) = 0$$

a   b   2h   2g   2f   c

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Case-I If $\Delta = 0$, then $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ is the equation of pair of straight line.

Case-II If $\Delta \neq 0$, then $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ is the equation of a curve.

* If $a = b$
  $h = 0$
  then the curve is a circle

* If $ab - h^2 = 0$
  then the curve is a parabola

* If $ab - h^2 > 0$
  then the curve is an ellipse

* If $ab - h^2 < 0$
  then the curve is a hyperbola

★ $x^2 + y^2 + 2x + 8y + 5 = 0$

$a = b$
$h = 0$

$\therefore$ circle

★ $x^2 + 9y^2 + 6xy + 4x + 12y - 5 = 0$

if $\Delta = 0$, then the eq^n will be the eq^n of pair of straight line.

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= -45 + 72 - 36 - 36 + 45 = 0$

$a = 1$	$2h = 6$
$b = 9$	$g = 2$
$c = -5$	$f = 6$

$\therefore \Delta = 0$

then pair of straight line.

$* \; 25x^2 + 2xy + 25y^2 - 130x - 130y + 169 = 0$

$a = 25 \quad h = 1$
$b = 25 \quad f = -65$
$g = -65 \quad c = 169$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 105625 + 8450 - 105625 - 105625 - 169$

$= -97344$

$\therefore \Delta \neq 0$
Pair of straight line ✗
Curve ✓

$ab - h^2 = 25 \times 25 - 1^2 = 624 > 0$

$\therefore$ The curve is ellipse.

$* \; 6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$

$a = 6 \quad h = \frac{5}{2}$
$b = -6 \quad g = -2$
$f = \frac{7}{2} \quad c = 11$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= -396 + (-35) - \frac{147}{2} - (-24) - \frac{55}{2}$

$= -508$

$\Delta \neq 0$
$\therefore$ curve ✓

$ab - h^2 = 6 \times (-6) - \frac{25}{4} = -36 - \frac{25}{4}$
$= -42.25 < 0$

$\therefore$ The curve is hyperbola

Conditions for being perpendicular/Parallel:

$a + b = 0 \Rightarrow$ those lines are perpendicular

$ab - h^2 = 0 \Rightarrow$ those lines are parallel

$$\# \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow ax^2 + (2hy + 2g)x + by^2 + 2fy + c = 0 \qquad \ldots\ldots(i)$$

শুধু x কে চলক ধরলে

By comparing equation (i) with $Ax^2 + Bx + C = 0$, we get.

$A = a$
$B = 2(hy + g)$
$C = by^2 + 2fy + c$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2 + 2fy + c)}}{2a}$$

$$= \frac{-2(hy+g) \pm \sqrt{4(h^2y^2 + 2ghy + g^2) - (4aby^2 + 8afy + 4ac)}}{2a}$$

$$= \frac{-2(hy+g) \pm 2\sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}}{2a}$$

$$= \frac{-(hy+g) \pm \sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}}{a}$$

The roots of the eqn $A\ {}^2 + B\ + C = 0$

if $B^2 - 4AC = 0$

$A = h^2 - ab$
$B = 2(gh - af)$
$C = (g^2 - ac)$

$B^2 - 4AC = 0$

$$\Rightarrow 4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

Example :

$$(x+2y+3) = 0$$

$$\star \quad (2x+3y+1) = 0$$

Now, $(2x+3y+1)(x+2y+3) = 0$

$$\Rightarrow 2x^2 + 7xy + 6y^2 + 7x + 11y + 3 = 0$$

$a = 2 \qquad b = 6 \qquad f = \frac{11}{2}$

$2h = 7 \qquad g = \frac{7}{2} \qquad c = 3$

$$\star \quad 2x^2 + 7xy + 6y^2 + 7x + 11y + 3 = 0$$

$$\Rightarrow 2x^2 + (7y+7)x + 6y^2 + 11y + 3 = 0$$

$$\therefore x = \frac{-(7y+7) \pm \sqrt{49(y+1)^2 - 4 \cdot 2 \cdot (6y^2 + 11y + 3)}}{4}$$

$$\Rightarrow 4x = -7(y+1) \pm \sqrt{y^2 + 10y + 25}$$

$$\Rightarrow 4x = -7y - 7 \pm \sqrt{(y+5)^2}$$

$$\Rightarrow 4x = -7y - 7 \pm (y+5)$$

$(+) \Rightarrow \quad 4x = -7y - 7 + y + 5 \Rightarrow 4x + 6y + 2 = 0$

$$\Rightarrow 2x + 3y + 1 = 0$$

$(-) \Rightarrow \quad 4x = -7y - 7 - y - 5 \Rightarrow 4x + 8y + 12 = 0$

$$\Rightarrow x + 2y + 3 = 0$$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

If $\Delta = 0$, then this is the equation of pair of straight lines.

Let, Those straight lines are -

$$l_1 x + m_1 y + n_1 = 0 \quad \cdots\cdots (i)$$

$$l_2 x + m_2 y + n_2 = 0 \quad \cdots\cdots (ii)$$

Let, the angle between these straight lines is θ.

From (i) $\Rightarrow$ $y = -\frac{l_1}{m_1} x - \frac{n_1}{m_1}$

$$y = -\frac{l_2}{m_2} x - \frac{n_2}{m_2}$$

We know,

$$\tan\theta = \frac{M_1 - M_2}{1 + M_1 M_2}$$

$M_1 = -\frac{l_1}{m_1}$

$M_2 = -\frac{l_2}{m_2}$

$$\Rightarrow \tan\theta = \frac{-\frac{l_1}{m_1} + \frac{l_2}{m_2}}{1 + \frac{l_1}{m_1} \cdot \frac{l_2}{m_2}}$$

$$\Rightarrow \tan\theta = \frac{m_1 l_2 - m_2 l_1}{m_1 m_2 + l_1 l_2} \quad \cdots\cdots (iii)$$

Now, $(l_1 x + m_1 y + n_1)(l_2 x + m_2 y + n_2) = 0$

$\Rightarrow l_1 l_2 x^2 + l_1 m_2 xy + l_1 n_2 x + l_2 m_1 xy + m_1 m_2 y^2 + m_1 n_2 y +$
$l_2 n_1 x + m_2 n_1 y + n_1 n_2 = 0$

$$\Rightarrow l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y$$
$$+ m_1m_2y^2 + n_1n_2 = 0$$

Then,

$a = l_1l_2$ $\qquad 2g = l_1n_2 + l_2n_1$

$b = m_1m_2$ $\qquad 2f = m_1n_2 + m_2n_1$

$2h = l_1m_2 + l_2m_1$ $\qquad c = n_1n_2$

$$\tan\theta = \frac{m_1l_2 - m_2l_1}{m_1m_2 + l_1l_2}$$

$$\Rightarrow \tan^2\theta = \frac{(m_1l_2 - m_2l_1)^2}{(m_1m_2 + l_1l_2)^2}$$

$$\Rightarrow \tan^2\theta = \frac{(m_1l_2 + m_2l_1)^2 - 4m_1l_2m_2l_1}{(m_1m_2 + l_1l_2)^2}$$

$$\Rightarrow \tan\theta = \frac{\sqrt{(2h)^2 - 4ab}}{(a+b)}$$

$$\therefore \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

These straight lines will be parallel if $\theta = 0^\circ$

$$\tan 0 = \frac{2\sqrt{h^2 - ab}}{a+b} \Rightarrow h^2 - ab = 0$$

&

will be perpendicular if $\theta = 90^\circ$

$$\tan 90^\circ = \frac{2\sqrt{h^2 - ab}}{a+b} \Rightarrow a+b = 0$$

* Again we know, they will be parallel if

$$h^2 - ab = 0 \quad \ldots \ldots (i)$$

Now,

$$\Delta = 0$$

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow abc + 2fg\sqrt{ab} - af^2 - bg^2 - abc = 0 \qquad [h^2 = ab,\ h = \sqrt{ab}]$$

$$\Rightarrow 2fg\sqrt{ab} - af^2 - bg^2 = 0$$

$$\Rightarrow af^2 - 2fg\sqrt{ab} + bg^2 = 0$$

$$\Rightarrow (f\sqrt{a} - g\sqrt{b})^2 = 0$$

$$\Rightarrow f\sqrt{a} - g\sqrt{b} = 0$$

$$\Rightarrow f\sqrt{a} = g\sqrt{b}$$

$$\Rightarrow f\sqrt{ab} = gb \qquad [\text{by multiplying with } \sqrt{b}]$$

$$\Rightarrow fh = gb$$

$$\Rightarrow \frac{h}{b} = \frac{g}{f} \quad \ldots \ldots (ii)$$

From (i) $\Rightarrow$

$$h^2 = ab$$

$$\Rightarrow \frac{h}{a} = \frac{b}{h}$$

$$\Rightarrow \frac{a}{h} = \frac{h}{b} \quad \ldots \ldots (iii)$$

From (ii) & (iii) $\Rightarrow$

$$\frac{h}{b} = \frac{a}{h} = \frac{g}{f}$$

$$\therefore \frac{a}{h} = \frac{h}{b} = \frac{g}{f} \qquad [\text{If those lines are parallel}]$$

* If the general equation of 2nd degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines (when $\Delta=0$), then show that the equation $ax^2+2hxy+by^2=0$ represents a pair of straight lines parallel to them through the origin.

Soln:

$$l_1x+m_1y=0 \quad \ldots\ldots (i)$$
$$l_2x+m_2y=0 \quad \ldots\ldots (ii)$$

{ n_1 & $n_2=0$ because passes through the origin and parallel

By multiplying (i) and (ii),

$$(l_1x+m_1y)(l_2x+m_2y)=0$$
$$\Rightarrow l_1l_2x^2+(l_1m_2+l_2m_1)xy+m_1m_2y^2=0 \quad \ldots\ldots (iii)$$

Here, $l_1l_2=a$

$m_1m_2=b$

$l_1m_2+l_2m_1=2h$

From (iii),

$$ax^2+2hxy+by^2=0$$

General equation of 2nd degree:

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots (iv)$$

General eqn or eqn (iv) represents these pair of straight lines -

$$l_1x+m_1y+n_1=0$$
$$l_2x+m_2y+n_2=0$$

and

$l_1l_2=a$ $\quad m_1m_2=b$

$l_1m_2+l_2m_1=2h$

Area defining by pair of straight lines:

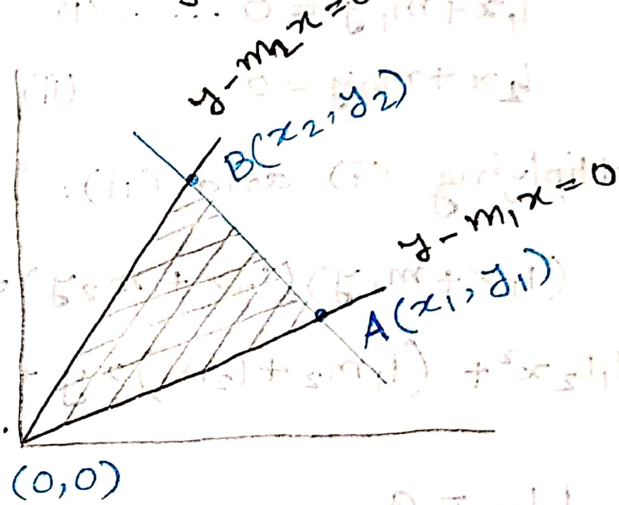
Find area of the triangle formed by the lines $ax^2+2hxy+by^2=0$ or $lx+my+n=0$

Soln:

$$y-m_1x=0$$

and $y-m_2x=0$ as these are the pair of straight lines of $ax^2+2hxy+by^2=0$

$y-m_1x=0$ & $y-m_2x=0$ equations go through the point $A(x_1,y_1)$ and $B(x_2,y_2)$ respectively.



$\therefore\ y_1-m_1x_1=0 \quad \ldots\ldots (i)$

& $y_2-m_2x_2=0 \quad \ldots\ldots (ii)$

$lx+my+n=0$ also goes ~~tho~~ through those points.

$lx_1+my_1+n=0 \quad \ldots\ldots (iii)$

$lx_2+my_2+n=0 \quad \ldots\ldots (iv)$

For $A(x_1,y_1)$, let's consider eqn (i) and (iii),

$y_1-m_1x_1=0$

$\Rightarrow m_1x_1-y_1=0$

& $lx_1+my_1+n=0$

$$\frac{x_1}{-n}=\frac{y_1}{-m_1n}=\frac{1}{mm_1+l}$$

$\therefore x_1 = \dfrac{-n}{mm_1+l}$

$\therefore y_1 = \dfrac{-m_1 n}{mm_1+l}$

for $B(x_2, y_2)$, let's consider eqn (ii) & (iv)

$y_2 - m_2x_2 = 0 \Rightarrow m_2x_2 - y_2 = 0$

$lx_2 + my_2 + n = 0$

$$\frac{x_2}{-n} = \frac{y_2}{-m_2 n} = \frac{1}{mm_2 - l}$$

$\therefore x_2 = \dfrac{-n}{mm_2 - l}$

$\therefore y_2 = \dfrac{-m_2 n}{mm_2 - l}$

$$\therefore \text{Area of } \Delta OAB = \frac{1}{2}\begin{vmatrix} \frac{-n}{mm_1+l} & \frac{-m_1 n}{mm_1+l} & 1 \\ \frac{-n}{mm_2-l} & \frac{-m_2 n}{mm_2-l} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[0 - 0 + 1\left\{\left(\frac{-n}{mm_1+l}\right)\left(\frac{-m_2 n}{mm_2-l}\right)\right\} - \left(\frac{-m_1 n}{mm_1+l}\right)\left(\frac{-n}{mm_2-l}\right)\right\}\right]$$

$$= \frac{1}{2}\left[\frac{m_2 n^2}{(mm_1+l)(mm_2-l)} - \frac{m_1 n^2}{(mm_1+l)(mm_2-l)}\right]$$

$$= \frac{1}{2}\left[\frac{m_2 n^2 - m_1 n^2}{(mm_1+l)(mm_2-l)}\right]$$

Show that the straight lines joining the origin to the points of intersection of the two curves $ax^2+2hxy+by^2+2gx=0$ and $a'x^2+2h'xy+b'y^2+2g'x=0$ will be at 90° angles to one another if $g(a'+b') = g'(a+b)$

Sol^n:

Here, two curves have intersect each other.

So, $S+\lambda S'=0$

$(ax^2+2hxy+by^2+2gx) + \lambda(a'x^2+2h'xy+b'y^2+2g'x)=0$

If two lines intersect each other then

$L_1+\lambda L_2=0$

If two curves intersect each other then

$S_1+\lambda S_2=0$

$\Rightarrow (a+\lambda a')x^2 + 2(h+\lambda h')xy + (b+\lambda b')y^2 + 2(g+\lambda g')x=0$ ......(i)

It is given that the straight lines joining the origin to the points of intersection of the two curves, so the co-efficient of x should be zero.

So, $2(g+\lambda g')=0$

$\Rightarrow \lambda g' = -g$

$\therefore \lambda = \frac{-g}{g'}$

Now, from equation (i),

$$\left(a-\frac{ga}{g'}\right)x^2+2\left(h-\frac{gh'}{g'}\right)xy+\left(b-\frac{gb'}{g'}\right)y^2=0$$

$$\Rightarrow (g'a-ga)x^2+2(g'h-gh')xy+(bg'-gb')y^2=0$$

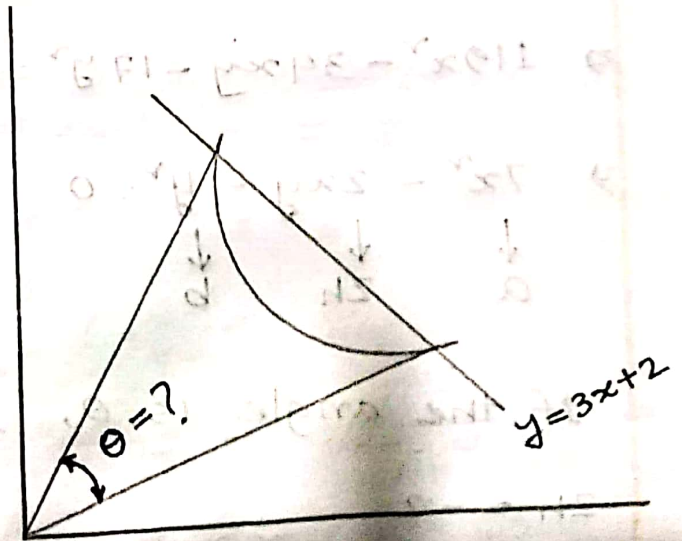
If these straight lines are perpendicular, then the sum of co-efficient of x^2 and y^2 would be zero.

$$\therefore (g'a-ga')+(bg'-gb')=0$$

$$\Rightarrow g'(a+b)-g(a'+b')=0$$

$$\therefore g'(a+b)=g(a'+b')$$

Find the angle between the lines joining the origin to the the point of intersection of the line $y=3x+2$ and the curve $x^2+2xy+3y^2+4x+8y-11=0$.



Given that,

$$y=3x+2$$

$$\Rightarrow y-3x=2$$

$$\therefore 1=\frac{y-3x}{2}$$

Now,

$$x^2+2xy+3y^2+4x+8y-11=0$$

$$\Rightarrow (x^2+2xy+3y^2)+(4x+8y)\times 1-11\times 1=0$$

$\Rightarrow x^2+2xy+3y^2+(4x+8y)\frac{(y-3x)}{2}-11\cdot\frac{(y-3x)}{2}=0$

$\Rightarrow x^2+2xy+3y^2+\frac{4xy-12x^2+8y^2-24xy}{2}-\frac{(11y-33x)}{2}=0$

$\Rightarrow 2x^2+4xy+6y^2+4xy-12x^2+8y^2-24xy-11y+33x=0$

$\Rightarrow -10x^2+14y^2-16xy+33x-11y=0$

$\Rightarrow 10x^2-14y^2+16xy-33x+11y=0$

$\Rightarrow 10x^2-14y^2+16xy-(33x-11y)\times 1=0$

$\Rightarrow 10x^2-14y^2+16xy-(33x-11y)\cdot\frac{(y-3x)}{2}=0$

$\Rightarrow 10x^2-14y^2+16xy-\frac{(33x-11y)(y-3x)}{2}=0$

$\Rightarrow 20x^2-28y^2+32xy-(33xy-99x^2-11y^2+33xy)=0$

$\Rightarrow 20x^2-28y^2+32xy-33xy+99x^2+11y^2-33xy=0$

$\Rightarrow 20x^2+99x^2-28y^2+11y^2+32xy-33xy-33xy=0$

$\Rightarrow 119x^2-17y^2-34xy=0$

$\Rightarrow 119x^2-34xy-17y^2=0$

$\Rightarrow \underset{a}{7x^2}-\underset{2h}{2xy}-\underset{b}{y^2}=0$ [Pair of straight lines through the origin]

If the angle is θ, then $\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$

$2h=-2$

$\therefore h=-1$

$a=7$

$b=-1$

$\Rightarrow \tan\theta=\frac{2\sqrt{1+7}}{6}$

$\theta=\tan^{-1}\frac{2\sqrt{2}}{3}$

Theory:

$ax^2 + 2hxy + by^2 = 0$ is the pair equation of pair of straight lines passes through the origin.

Let,

$$y - m_1 x = 0$$
$$y - m_2 x = 0$$

Now, $(y - m_1 x)(y - m_2 x) = 0$

$$\Rightarrow y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0$$

$$\Rightarrow m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0 \quad \ldots \ldots (i)$$

We know,

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots \ldots (ii)$$

From (i) and (ii),

$$\frac{m_1 m_2}{a} = \frac{-(m_1 + m_2)}{2h} = \frac{1}{b}$$

Now,

$$m_1 m_2 = \frac{a}{b}$$
$$m_1 + m_2 = -\frac{2h}{b}$$

We know,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \tan^2\theta = \frac{(m_1 - m_2)^2}{(1 + m_1 m_2)^2}$$

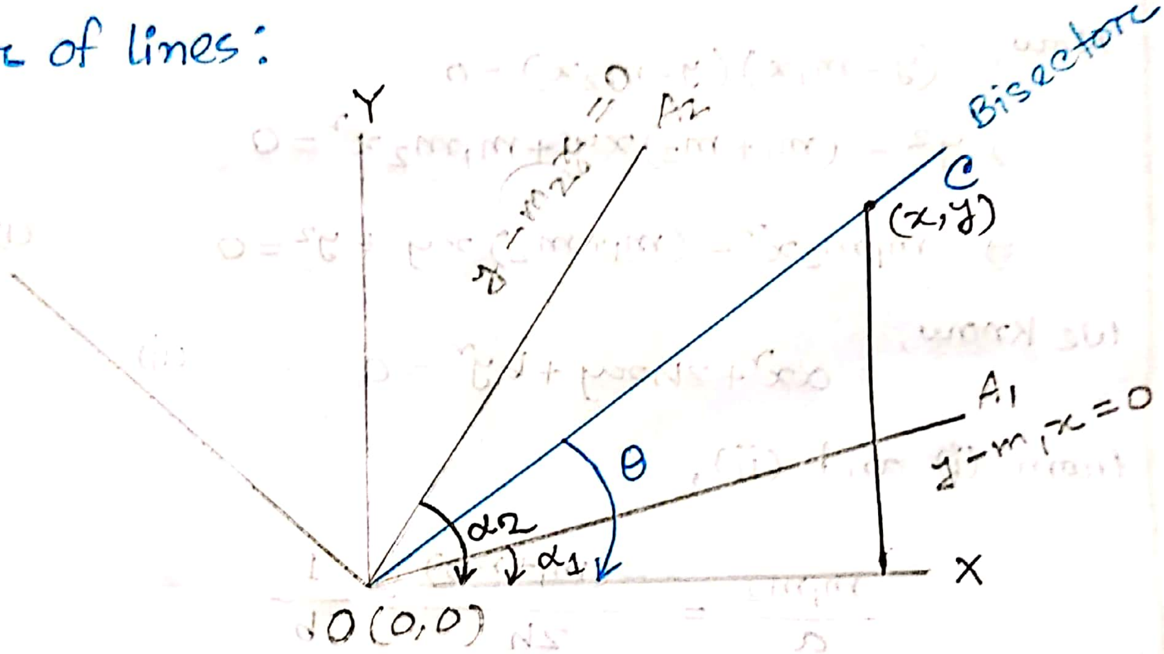
$$\Rightarrow \tan^2\theta = \frac{(m_1 + m_2)^2 - 4m_1 m_2}{(1 + m_1 m_2)^2}$$

$$\Rightarrow \tan^2\theta = \frac{\frac{4h^2}{b^2} - \frac{4a}{b}}{} = \frac{\frac{4h^2 - 4ab}{b^2}}{}$$

$$\Rightarrow \tan\theta = \frac{\frac{2\sqrt{h^2-ab}}{b}}{\frac{a+b}{b}}$$

$$\therefore \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

Bisector of lines:



Here, $\angle A_1OC = \angle A_2OC$

and $\tan\alpha_1 = m_1$

$\tan\alpha_2 = m_2$

Now, $\angle XOC = \angle XOA_1 + \angle A_1OC$

$$\Rightarrow \theta = \angle XOA_1 + \frac{1}{2}\angle A_1OA_2$$

$$\Rightarrow \theta = \angle XOA_1 + \frac{1}{2}(\angle XOA_2 - \angle XOA_1)$$

$$\Rightarrow \theta = \alpha_1 + \frac{1}{2}(\alpha_2 - \alpha_1)$$

$$\Rightarrow \theta = \frac{1}{2}(\alpha_1 + \alpha_2)$$

$$\Rightarrow 2\theta = \alpha_1 + \alpha_2$$

$$\Rightarrow \tan 2\theta = \tan(\alpha_1 + \alpha_2)$$

$$\Rightarrow \tan 2\theta = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \tan\alpha_2}$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{m_1+m_2}{1-m_1m_2}$$

$$\Rightarrow \frac{2\cdot\frac{y}{x}}{1-\frac{y^2}{x^2}} = \frac{-\frac{2h}{b}}{1-\frac{a}{b}}$$

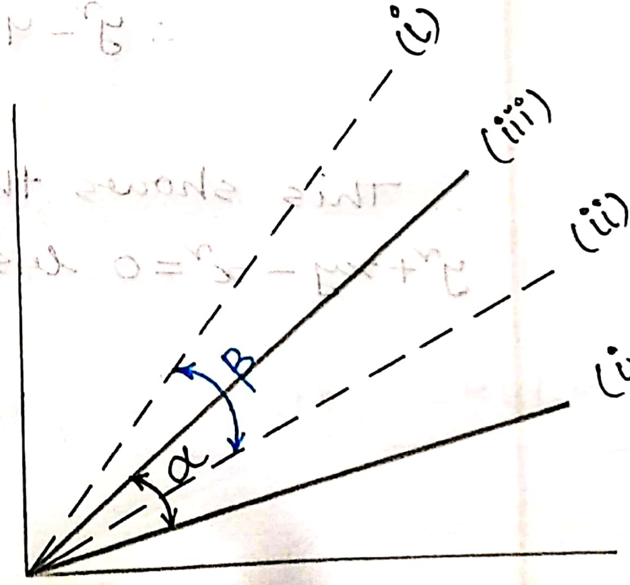
$$\Rightarrow \frac{\frac{2y}{x}}{\frac{x^2-y^2}{x^2}} = \frac{-\frac{2h}{b}}{\frac{b-a}{b}}$$

$$\Rightarrow \frac{2xy}{x^2-y^2} = \frac{2h}{a-b}$$

$$\Rightarrow \frac{xy}{x^2-{}^2} = \frac{h}{a-b}$$

$$\therefore \frac{x^2-y^2}{a-b} = \frac{xy}{h}$$ bisected equation

* Prove that the lines $y^2-4xy-x^2=0$ and $y^2+xy-x^2=0$ bisect the angle between one another



Here the angles between one another is bisected by each other.

(i) & (ii) are the lines of equation $y^2-4xy-x^2=0$.

By comparing $y^2-4xy-x^2=0$ with $ax^2+2hxy+by^2=0$, we get,

$a=-1$ $\quad h=-2$

$b=1$

bisector's equation of $y^2-4xy-x^2=0$ is -

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

$$\Rightarrow \frac{x^2-y^2}{-1-1}=\frac{xy}{-2}$$

$$\Rightarrow x^2-xy-y^2=0$$

$$\therefore y^2+xy-x^2=0$$

Again bisector's equation of $y^2+xy-x^2=0$ is -

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h} \qquad \left|\; \begin{array}{l} a=-1 \\ b=1 \\ h=\frac{1}{2} \end{array}\right.$$

$$\Rightarrow \frac{x^2-y^2}{-1-1}=\frac{xy}{\frac{1}{2}}$$

$$\Rightarrow x^2-y^2=-4xy$$

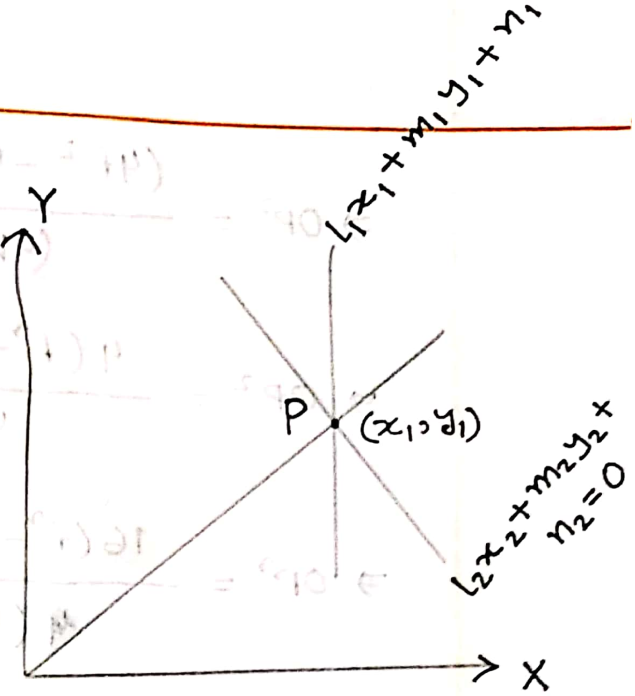
$$\therefore y^2-4xy-x^2=0$$

$\therefore$ This shows that the lines $y^2-4xy-x^2=0$ and $y^2+xy-x^2=0$ bisect the angles between one another.

~~Let~~ Let

$l_1x + m_1y + n_1 = 0 \quad \ldots\ldots (i)$

$l_2x + m_2y + n_2 = 0 \quad \ldots\ldots (ii)$



Equation (i) & (ii) goes through the point $P(x_1, y_1)$.

$\therefore \; l_1x_1 + m_1y_1 + n_1 = 0 \quad \ldots\ldots (iii)$

$l_2x_1 + m_2y_1 + n_2 = 0 \quad \ldots\ldots (iv)$

$O(0,0)$

Show that, $OP^2 = \dfrac{c(a+b) - f^2 - g^2}{ab - h^2}$

from (iii) & (iv),

$$\frac{x_1}{m_1n_2 - m_2n_1} = \frac{y_1}{l_2n_1 - l_1n_2} = \frac{1}{l_1m_2 - l_2m_1}$$

$$\therefore x_1 = \frac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1} \quad \text{and} \quad y_1 = \frac{l_2n_1 - l_1n_2}{l_1m_2 - l_2m_1}$$

$$OP^2 = (x_1 - 0)^2 + (y_1 - 0)^2 = x_1^2 + y_1^2$$

$$\Rightarrow OP^2 = \left(\frac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1}\right)^2 + \left(\frac{l_2n_1 - l_1n_2}{l_1m_2 - l_2m_1}\right)^2$$

$$\Rightarrow OP^2 = \frac{(m_1n_2 - m_2n_1)^2 + (l_2n_1 - l_1n_2)^2}{(l_1m_2 - l_2m_1)^2}$$

$$\Rightarrow OP^2 = \frac{\{(m_1n_2 + m_2n_1)^2 - 4m_1m_2n_1n_2\} + \{(l_2n_1 + l_1n_2)^2 - 4l_1l_2n_1n_2\}}{\{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2\}}$$

$$\Rightarrow OP^2 = \frac{\{(2f)^2 - 4bc\} + \{(2g)^2 - 4ac\}}{\{(2h)^2 - 4ab\}}$$

$l_1l_2 = a$
$m_1m_2 = b$
$n_1n_2 = c$
$m_1n_2 + m_2n_1 = 2f$
$n_1l_2 + l_2m_1 = 2g$
$l_1m_2 + l_2m_1 = 2h$

$$\Rightarrow OP^2 = \frac{(4f^2 - 4bc) + (4g^2 - 4ac)}{(4h^2 - 4ab)}$$

$$\Rightarrow OP^2 = \frac{4(f^2 - bc) + 4(g^2 - ac)}{4(h^2 - ab)}$$

$$\Rightarrow OP^2 = \frac{(f^2 - bc) + (g^2 - ac)}{\cancel{4}(h^2 - ab)}$$

$$\Rightarrow OP^2 = \frac{(f^2 - bc) + (g^2 - ac)}{(h^2 - ab)}$$

$$\Rightarrow OP^2 = \frac{f^2 + g^2 - c(a+b)}{h^2 - ab}$$