1st .C- Day
20.08.19

: Matrix :

$$A = \begin{array}{c} \text{row} \curvearrowleft \curvearrowright \text{column} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \end{array}$$

m × n.
↓ row ↘ column

⊞ Definition :

1. <u>Square Matrix</u> : If the number of row and column of a matrix is equal then it will be called square matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3 \times 3}$$

⊛ Diagonal elements

$= a_{11}, a_{22}, a_{33}$.

2. <u>Trace of matrix</u> : The summation of the diagonal elements of a square matrix is known as trace of the matrix. Trace of $A = a_{11} + a_{22} + a_{33}$.

Ex: If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

then show that,

trace A + trace B = trace (A+B).

(*) when the order of the matrixs are same then they can be added or ~~ni~~ subtracted.

3. Diagonal matrix: If $a_{ij} = 0$ for all $i \neq j$, then the square matrix is known as diagonal matrix.

Ex: $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}_{3\times 3}$.

4. Scalar matrix: If the diagonal element of a diagonal matrix are equal to a scalar, say k, then the matrix is ~~sai~~ known as Scalar matrix. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

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5. Identity matrix: If the diagonal elements of a diagonal matrix are equal to one, then the matrix is called Identity matrix.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

6. Null matrix: Any matrix, in which all the elements are zeros, is called a zero matrix or null matrix.

2. Upper and lower triangular matrix:

A square matrix $A = (a_{ij})$, $i = j = 1, 2 \ldots n$, where elements $a_{ij} = 0$ for $i > j$; is called upper triangular matrix and for lower matrix $a_{ij} = 0$ for $i < j$.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

Upper triangular matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Lower triangular matrix.

8. Commutative and anti commutative matrix: If A and B are Square matrix of same order such that $AB = BA$, then A and B are called commutative and if $AB = -BA$. then they are anti-commutative matrix

9. Periodic matrix: A matrix 'A' for which $A^{K+1} = A$, where K is any scalar, then 'A' is periodic matrix of period K.

10. Idempotent matrix: A matrix 'A' for which $A^2 = A$ is known as idempotent matrix.

11. Nilpotent matrix: A matrix 'A' for which $A^P = O$, and $A^{P-1} \neq 0$, P is any positive integer is known as nilpotent matrix of index P.

order = P.

12. Involutory matrix: A matrix 'A' such that $A^2 = I$ is known an involutory matrix.

13. Theorem: A matrix A is involutory iff $(I + A)(I - A) = 0$. Proof Page - 68.

14. Singular and non-singular matrix:

If the determinant of a square matrix is equal to zero, then the matrix is known as singular matrix.

Again if the determinant of a square matrix is not equal to zero, then the matrix is known as non-singular matrix.

Ex: $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is singular.

$|A| = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$.

$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ non singular.

15. The transpose of matrix:

The matrix of order $n \times m$ is obtained by interchanging the rows and columns of a $m \times n$ matrix, A is called the transpose of a matrix. It is denoted by A'.

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}_{2\times3}$ then $A' = \begin{bmatrix} 1 & a \\ 2 & b \\ 3 & c \end{bmatrix}_{3\times2}$

Properties of the transpose matrix:

1. $(A+B)' = A' + B'$.

2. $(AB)' = B'A'$.

3. $(KA)' = KA'$, K is any scalar.

4. $(A')' = A$.

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1st E-day

16. Symmetric matrix and skew-symmetric matrix:

A square matrix 'A' such that $A = A'$ is called symmetric matrix. That is a square matrix $A = (a_{ij})$ is symmetric if $a_{ij} = a_{ji}$ for all the values of i and j.

Ex: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$, then $A = A'$, therefore A is symmetric.

A square matrix 'A' such that $A = -A'$ is called skew symmetric matrix. That is a square matrix $A = (a_{ij})$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all values of i and j.

Ex: $A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 5 \\ 2 & -5 & 0 \end{pmatrix}$, then $A = -A'$, therefore A is skew-symmetric matrix.

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Theorem: The diagonal elements of a skew-symmetric matrix are zero.

Proof: If $A = (a_{ij})$, $i = j = 1, 2, 3 \ldots n$ is a square matrix, then the condition that the matrix A be a skew-symmetric is that $a_{ij} = -a_{ji}$ —— ①

for diagonal elements the condition ① becomes

$a_{ii} = -a_{ii}$.

$\Rightarrow 2a_{ii} = 0$.

$\Rightarrow a_{ii} = 0$, which shows that the diagonal elements of a skew-symmetric matrix are zero.

(*) **Theorem:** Every suquare matrix can be expressed as a sum of a symmetric and a skew-symmetric matrix.

Proof: Let 'A' be a square matrix, then we can write, $A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$.

We have to show that $\frac{1}{2}(A+A')$ is symmetric and $\frac{1}{2}(A-A')$ is skew-symmetric matrix.

Now, $\left\{\frac{1}{2}(A+A')\right\}'$:-

$= \frac{1}{2}(A+A')'$ as $(KA)' = KA'$.

$= \frac{1}{2}\{A' + (A')'\}$ as $(A+B)' = A'+B'$.

$= \frac{1}{2}(A'+A)$ as $(A')' = A$.

$= \frac{1}{2}(A+A')$.

This shows that $\frac{1}{2}(A+A')$ is symmetric matrix

~~Now~~ we have to show that $\frac{1}{2}(A-A')$ is skew symmetric. ~~and $\frac{1}{2}(A-A')$ i~~

Now,

$$\left\{\frac{1}{2}(A-A')\right\}'$$

$$= \frac{1}{2}(A-A')' \quad \text{as} \quad (KA)' = KA'.$$

$$= \frac{1}{2}\left\{A' - (A')'\right\}$$

$$= \frac{1}{2}(A' - A) \quad \text{as} \quad (A')' = A.$$

$$= -\frac{1}{2}(A - A')$$

This show that $\frac{1}{2}(A-A')$ is skew symmetric matrix.

And hence, A is the sum of a symmetric and skew-symmetric matrix.

17. <u>Orthogonal matrix</u>: A square matrix 'A' is such that $A'A = I$, where I is the identity matrix is called orthogonal matrix.

<u>Ex</u>: Show that $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}_{2\times 2}$ is orthogonal matrix.

$\Rightarrow$ Here, $A' = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

~~We know~~

$\therefore A'A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha - \cos\alpha\sin\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$= I_2$. [Showed].

Ex: If, A and B are orthogonal matrix then show that AB and BA are also orthogonal.

Sol^n: By defination we have,

$A'A = I$. ——— (i)

$B'B = I$. ——— (ii)

Now, $(AB)'(AB) = (B'A')(AB)$.

$= \cancel{(AB)'(AB)}$

$= B'(A'A) \cdot B$.

$= B'IB$. as $A'A = I$.

$= (B'I)B$ as $B'I = B'$.

$= B'B$

$= I$ as $B'B = I$.

Therefore AB is orthogonal matrix.

again,

$$(BA)'(BA) = B(A'B')(BA) \quad \text{as } (AB)' = B'A'.$$

$$= A'(B'B).A$$

$$= A'.IA. \quad \text{as } B'B = I.$$

$$= (A'I).A$$

$$= A'A \quad \text{as } (A'I) = A'$$

$$= I \quad \text{as } A'A = I.$$

Hence, ~~we can~~

AB and BA are orthogonal.

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18. The conjugate of a matrix:

Ex: If ~~A = i a+ib 2~~

$$A = \begin{pmatrix} i & a+ib & 2 \\ 1-i & 0 & i \\ 3 & 3i & -i \end{pmatrix}$$

then the conjugate of A denoted by $\bar{A}$ and defined as.

$$\bar{A} = \begin{pmatrix} -i & a-ib & 2 \\ 1+i & 0 & -i \\ 3 & -3i & i \end{pmatrix}.$$

Properties of conjugate of a matrix:

i) $\overline{A+B} = \bar{A} + \bar{B}$.

ii) $\overline{AB} = \bar{A} \cdot \bar{B}$.

iii) $\overline{KA} = K\bar{A}$, K is any scalar

iv) $\overline{(\bar{A})} = A$.

D: Hermitian and skew-Hermitian matrix:

A square matrix $A = (a_{ij})$, such that $\bar{A}' = A$ is called Hermitian matrix. Hermitian matrix provided $a_{ij} = \overline{a_{ji}}$ for all values of i and j.

A square matrix $A = (a_{ij})$. Such that $\bar{A}' = -A$ is called skew-hermitian matrix. Skew-hermitian matrix provided $a_{ij} = -\overline{a_{ji}}$ for all the values of i and j.

(*) Theorem: The diagonal elements of a Hermitian matrix are real and the diagonal elements of a skew-Hermitian matrix are either zero or pure imaginary numbers.

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2m

$\boxed{a} + ib$

$\boxed{a} - ib$

real

$i^2 = -1.$

Proof: Let $A = (a_{ij})$ be a square matrix, Then the condition that A be a hermitian matrix is,

$$a_{ij} = \overline{a_{ji}} \text{ for all } i \text{ and } j.$$

For diagonal ~~elemnt~~ elements, the condition is

$$a_{ii} = \overline{a_{ii}} \quad \text{——— (i)}$$

Let, $a_{ii} = \alpha + i\beta$.
$\overline{a_{ii}} = \alpha - i\beta$. } ——— (ii)

Using eqn (ii), we get from (i),

$$\alpha + i\beta = \alpha - i\beta.$$

$$\Rightarrow 2i\beta = 0.$$

$$\therefore \beta = 0.$$

This shows that the diagonal elements of a hermitian matrix are real.

~~Now the condition~~

Now for the diagonal elements, the condition that $A = (a_{ij})$ be a skew-hermitian is that

$$a_{ii} = -\overline{a}_{ii} \longleftarrow \text{(III)}$$

using (R) we get from (III),

$a + i\beta = -(a \pm i\beta)$.

$\Rightarrow a + i\beta = -a + i\beta$.

$\Rightarrow 2a = 0, \Rightarrow a = 0$, this shows that the diagon elements of a skew-hermitian matrix are pure imaginary and can be zero iff $\beta = 0$.

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Theorem: Every square matrix A can be expressed as a sum of a hermitian matrix and a skew-hermitian matrix.

Proof: Let $A = (a_{ij})$ be a square matrix then we can write;

$$A = \frac{1}{2}(A + \bar{A}') + \frac{1}{2}(A - \bar{A}').$$

We have to show that $\frac{1}{2}(A + \bar{A}')$ is hermitian and $\frac{1}{2}(A - \bar{A}')$ is skew-hermitian matrix.

Now, $\left\{\overline{\frac{1}{2}(A + \bar{A}')}\right\}'$

$= \left\{\frac{1}{2}\left(\overline{A + \bar{A}'}\right)\right\}'$ as $\overline{KA} = K\bar{A}$.

$= \left[\frac{1}{2}\left\{\bar{A} + (\bar{\bar{A}}')\right\}\right]'$ as $\overline{A+B} = \bar{A} + \bar{B}$.

$= \left\{\frac{1}{2}(\bar{A} + A'\right\}'$ as $\bar{\bar{A}} = A$.

$= \frac{1}{2}(\bar{A} + A')'$ as $(KA)' = KA'$.

$= \frac{1}{2}\{\bar{A}' + (A')'\}$ as $(A+B)' = A' + B'$.

$= \frac{1}{2}(\bar{A}' + A)$ as $(A')' = A$.

$= \frac{1}{2}(A + \bar{A}')$

Which shows that $\frac{1}{2}(A + \bar{A}')$ is hermitian matrix

Now, $\left\{ \overline{\frac{1}{2}(A - \bar{A}')} \right\}'$

$= \left\{ \frac{1}{2}\left(\overline{A - \bar{A}'}\right) \right\}'$ as $(KA)' = KA'$.

$= \frac{1}{2}(\bar{A} - \bar{\bar{A}}')'$ as $(\overline{A+B}) = \bar{A} + \bar{B}$.

$= \frac{1}{2}(\bar{A} - A')'$ as $\bar{\bar{A}} = A$.

$= \frac{1}{2}\{\bar{A}' - (A')'\}$ as $(A-B)' = A' - B'$

$= \frac{1}{2}(\bar{A}' - A)$ as $(A')' = A$.

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$= \frac{1}{2}(A - \bar{A}')$.

This show that $\frac{1}{2}(A - \bar{A}')$ is a skew-hermitian matrix.

Hence A can be expressed as the sum of a hermitian and a skew-hermitian matrix.

20. Unitary matrix: A square matrix A is unitary matrix if $\bar{A}'A = I$, where I is the identity matrix.

Ex: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$, then express A as the sum of a symmetric and skew-symmetric matrix.

⇒ Sol^n: We know that the symmetric part is $\frac{1}{2}(A+A')$ and skew-symmetric part $\frac{1}{2}(A-A')$

Here, $A' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$.

∴ Symmetric part $= \frac{1}{2}(A+A')$.

$$= \frac{1}{2} \cdot \left[\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \right]$$

$$= \frac{1}{2} \cdot 2 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

$$= A.$$

And, skew-symmetric part,

$$\frac{1}{2}(A - A')$$

$$= 0.$$

$$\therefore A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$= A.$$

Ex: $A = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, then ~~show that A~~ express A as the part of symmetric and skew-symmetric matrix.

Ex: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$

then find the matrix C such that $A+B = 2C$.

Soln: Let, $C = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$.

Then by problem,

$$A+B = \begin{pmatrix} 3 & 5 & 7 \\ 3 & 5 & 7 \\ 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2k \end{pmatrix},$$

Equating corresponding elements.

$3 = 2a$

$a = 3/2$

$2b = 5$

$\Rightarrow b = 5/2$

$c = 7/2$

$d = 3/2$

$e = 5/2$

$f = 7/2$

$g = 3/2$

$h = 3/2$

$k = 2.$

$\therefore C = \begin{pmatrix} 3/2 & 5/2 & 7/2 \\ & & \\ & & \end{pmatrix}$

$1\times 3 \quad 3\times 3 \times 3\times 1$

Ex: If $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}_{2\times 2}$, then

evalute $A + 2B + 3I$.

Ex: solve the following equations for x and y.

$$2x - y = \begin{pmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{pmatrix}, \quad 2y + x = \begin{pmatrix} 4 & 7 & 5 \\ -1 & 4 & -4 \end{pmatrix}.$$

Ex: Find the value of a if

$$\begin{pmatrix} a & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ 4 \\ -1 \end{pmatrix} = O.$$

where O is the null matrix

~~2~~ 1×2. 3×3, ~~3~~ 3×1

Ex: If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then find the values of a and b so that $(aI + bA)^2 = A$.

2. Inverse matrix / Inverse of a matrix:

If A and B are two matrices such that $AB = BA = I$, then B is called the inverse of A and we write $B = A^{-1}$. and also A is called the inverse matrix of B and then $A = B^{-1}$.

Theorem: The inverse of a matrix is unique.

Proof: Let A be a square matrix, and if possible let B and C are two inverse of A. Then by definition.

$AB = BA = I$ ———— (I)

$AC = CA = I$ ———— (II)

From equation (I),

$BA = I$.

$\Rightarrow (BA)C = IC = C$ ——— (III)

and from equation (II)

$AC = I$.

$\Rightarrow B(AC) = IB = B$ ——— (IV)

According to the associative law of multiplicatio

$(BA)C = B(AC)$.

$\Rightarrow C = B$.

This shows that the inverse of a matrix is unique.

Theorem: The necessary condition for a square matrix A to ~~pre~~ possess an inverse is that A is non-singular matrix.

Proof: Let, B be the inverse of A, then by definition, $AB = BA = I$ ——— (1)

We take determinant on both sides of $AB = I$, then.

~~(AB) = |I|~~ $|AB| = |I|$.

$\Rightarrow |A||B| = 1$.

From this relation, $|A| \neq 0$ and also $|B| \neq 0$

Therefore the matrix A is non-singular matrix.

Theorem:
The inverse of the product of two matrices, having inverses is the product in reverse order of these matrix.

Proof: $(AB)^{-1} = B^{-1}A^{-1}$

A^{-1} and B^{-1} exist since A and B are non-singular

$\therefore (AB)(B^{-1}A^{-1}) = A(BB^{-1}).A^{-1}$, by associative law.

$= AIA^{-1}. = AA^{-1}.$

$= I.$

And $(B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B$, by associative law.

$= B^{-1}.I.B. \quad = B^{-1}.(IB).$

$= B^{-1}B$

$= I.$

$\therefore (B^{-1}A^{-1}).(AB) = (AB)(B^{-1}A^{-1}) = I$.

i.e, $B^{-1}.A^{-1}$ is the inverse of AB or $(AB)^{-1} = B^{-1}.A^{-1}$. and as such AB is ~~no~~ also non-singular.

22. Minor and Co-factor of a matrix:

The determinant of every square sub matrix of a matrix is called a minor of the matrix. The signed minor is called the cofactor of the matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}_{3\times 4}.$$

One sub matrix of order 3×3

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$ then $|A_1|$ = minor of order 3×3

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$ then $|A_2|$ = minor of order 2×2

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

minor of the element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$

$$a_{ij} = |M_{ij}|$$

minor of the element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}.$

The cofactor of $a_{11} = \alpha_{11} = (-1)^{1+1} \cdot |M_{11}|$

$$= (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$a_{12} = \alpha_{12} = (-1)^{1+2} |M_{12}| = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\text{Adj } A = \begin{pmatrix} \alpha_{11} & a_{12} & a_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}'$$

$$= \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}$$

Inverse from adjoint is given by.

$$A^{-1} = \frac{\text{adj } A}{|A|}.$$

$$A = \begin{array}{c|c} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 2 & 1 \end{pmatrix} & \begin{array}{l} H_1 \\ H_2 \\ H_3 \end{array} \\ \begin{array}{ccc} K_1 & K_2 & K_3 \end{array} & \end{array} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$H_2 = H_2 - 2H_1$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 3 & 2 & 1 \end{pmatrix}$$

$H_3 = H_3 - 3H_1$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & -4 & -8 \end{pmatrix}.$$

$$H_3 = H_3/(-4) \quad \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & 1 & 2 \end{pmatrix}.$$

$$H_3 = H_3 - H_2 \quad \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 7 \end{pmatrix}.$$

$$H_3 = H_3/7 \quad \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$H_2 = H_2 + 5H_3 \quad \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$H_1 = H_1 - 2H_2 \quad \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$H_1 = H_1 - 3H_3 \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$AB = I.$

$A = IA.$

$I = BA$

Ex: Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$ by elementary row transformation

Soln: We can write,

$$A = I_3 A.$$
$$I_3 = B A$$

Here, $$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

$H_2 = H_2 - 2H_1$

$H_3 = H_3 - 3H_1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} A.$$

Interchanging 2nd and 3rd row

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} A$$

$H_2 = H_2 \times (-1)$

$H_3 = H_3 \times (-1)$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} A.$$

$H_2 = H_2 - 3H_3$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix} A.$$

$H_1 = H_1 - 2H_2$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -6 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix} A$$

$H_1 = H_1 - 3H_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix} A$$

$I_3 = BA.$

$-6 + 3 \quad 2$

where $B = A^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$.

Ex: Find the adjoint of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$.

and hence evaluate A^{-1}.

Soln: cofactor of $1 = (-1)^2 \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = -1$.

$2 = -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = 3$.

$$\text{adj } A = \begin{pmatrix} \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{pmatrix}'$$

$$= \begin{pmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

Now $A^{-1} = \dfrac{\text{adj} A}{|A|}$,

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}.$$

$$= 1(24-25)+2(15-12)+3(10-12)$$

$$= -1 \neq 0.$$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \frac{\begin{pmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{pmatrix}}{-1}.$$

$$= \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}. \text{ Ans.}$$

Ex: Find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ with adjoint.

23. **Rank of a Matrix**: The rank of a matrix A is said to be r if every minor of order r+1 is zero and there is at least one minor of A of order r is diffierent from zero.

$$3\times3 = |0| = 0$$

$$2\times2 = |\ \ | \neq 0.$$

rank = 2.

Ex: Find the rank of $A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix}_{3\times4}$

Soln: The minor of order 3 is $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{vmatrix} = 0$

Similarly the other minor of order 3 are equal to zero.

Now the minor of order 2 $= \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1 \neq 0.$

Therefore the rank of the given matrix is 2.

24) The normal form or canonical form of a matrix : Every $m \times n$ matrix A of rank r can be reduced to any of the forms.

$$[I_r] ; \begin{bmatrix} I_r \\ 0 \end{bmatrix}, [I_r \; 0], \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

called normal forms by operation of elementary row and column transformations.

8.09.19
3rd F.day

Ex. Find the rank of $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 2 \end{pmatrix}$ reducing it to normal form.

Soln:

$$\underset{R_3 = R_3 - 2R_1}{\overset{R_2 = R_2 - 4R_1}{\sim}} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 6 & -5 & 6 \\ 0 & 4 & -4 & 4 \end{pmatrix}.$$

$$\overset{R_3 = R_3/4}{\sim} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 6 & -5 & 6 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

$$\overset{R_2 = R_2 - 6R_3}{\sim} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

$$\underset{\text{2nd and 3rd row}}{\overset{\text{Interchanging}}{\sim}} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\underset{\substack{K_3 = K_3 - K_1 \\ K_4 = K_4 + K_1}}{\overset{K_2 = K_2 + K_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\underset{K_4 = K_4 - K_2}{\overset{K_3 = K_3 + K_2}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\sim \begin{pmatrix} I_3 & 0 \end{pmatrix}.$$

This is the required normal form of the given matrix A and the rank of A is 3

Ex: Reduce the matrix, $A = \begin{pmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$

to normal form and hence find its rank.

Echelon form of a matrix:

A matrix A is said to be in echelon form if:

i) all the non zero rows, if any, precede the zero rows.

ii) the number of zero preceding the first non-zero element in a row is less than the number of such zero in the succeding row.

iii) the first non zero element in a row is unity.

The number of non zero rows in the reduce echelon form is the rank of the matrix.

<u>Ex</u>: Reduce the matrix,

$$A = \begin{pmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$ to echelon form and hence find its rank

Sol^n

Sol^n: Interchanging 1st and 2nd row
$$\sim \begin{pmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$H_2 = H_2 + 2H_1$
$$\sim \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$
$H_3 = H_3 - I_1$

$H_2 = H_2/3$
$$\sim \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix},$$
$H_3 = H_3/(-2)$

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$H_3 = H_3 - H_2$

$\sim$

$H_4 = H_4 - H_2$

$$\begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The above is the ~~the~~ echelon form of A and the no. of non-zero rows is 2 therefore the rank of A is 2.

Linear equations: consider a system of m line[ar] equations in the n unknowns $x_1, x_2, \ldots x_n$.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = h_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = h_2.$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = h_3.$$

$$- - - - - - - - - - - - - - -$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = h_m.$$

in which $a_{11}, a_{12}, \ldots a_{mn}$ are called coefficients and $h_1, h_2 \ldots h_m$ are constants : A system of linear equations is called consistant if it has at least one solution and inconsistante if it has no solution

The system of linear equations in matrix notation may be written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots & a_{3n} \\ - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_m \end{pmatrix}$$

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or more compactly

$$AX = H.$$

where, A is the coefficient matrix.

And the augmented matrix is.

$$[A \;\; H] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} & h_1 \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} & h_2 \\ a_{31} & a_{32} & a_{33} \cdots & a_{3n} & h_3 \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} & h_m \end{bmatrix}$$

14.09.19
4th e Day

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = h \rightarrow$ non homogeneous linear equation.

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \rightarrow$ homogeneous linear eqn.

System of linear equation, (non-homogeneous).

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 1 \\ a_{21}x_1 + a_{22}x_2 &= 2 \end{aligned}\right\}$$

System of homogeneous equation,

$$a_{11}x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + a_{22}x_2 = 1$$

A system of non-homogeneous linear equations will have,

i) no solution: if the rank of the augmented matrix, say, r_1, is different from the rank of the coefficient matrix, say r_2. That is $r_1 \neq r_2$.

ii) Unique solution: If $r_1 = r_2 =$ no. of unknowns, say n, then the system has unique solution.

iii) infinite no. of solution: If $r_1 = r_2 < n$, then the given system has infinite no of solution.

In this case, we assign $(n-r)$ arbitrary constants to $(n-r)$ unknowns and the other values of the remaining unknowns are determined interms of the $(n-r)$ arbitrary values.

If a system has atleast one solution then the system is known as consistent system.

$\boxed{r_1 = r_2}$

Ex: Show that the system.

$$x+2y-z=3$$

$$3x-y+2z=1$$

$$2x-2y+3z=2.$$

$$x-y+z=-1.$$

is consistant and solve them completely.

Soln: The augmented matrix is,

$$[A \quad H] = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

$H_2 = H_2 - 3H_1$

$H_3 = H_3 - 2H_1$

$H_4 = H_4 - H_1$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}.$$

$R_2 = -R_2 + R_3$ $\sim$
$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}$$

$R_3 = R_3 + 6R_2$ $\sim$

$R_4 = R_4 + 3R_2$
$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{pmatrix}$$

$R_3 = R_3/5$ $\sim$

$R_4 = R_4/2$
$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$R_4 = R_4 - R_3$ $\sim$
$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the above echelon form the rank of the augmented matrix is 3 and also the rank of the coefficient matrix is 3, therefore the given system is consistent. and the no. of unknowns is equal to the rank of the augmented matrix, thus the system has unique solution.

The corresponding linear equations are,

$$x + 2y - z = 3.$$
$$y = 4.$$
$$z = 4.$$

Therefore the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$.

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Ex: Solve the system, with matrix method.

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6.$$

$$x_1 + 3x_2 + x_3 - 2x_4 = 4.$$

$$2x_1 + 5x_2 - 2x_3 - 5x_4 = 10.$$

Soln: The augmented matrix is,

$$[A \; H] = \begin{pmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{pmatrix}$$

$$\begin{array}{l} H_2 = H_2 - H_1 \\ \sim \\ H_3 = H_3 - 2H_1 \end{array} \begin{pmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 1 & 3 & -2 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{pmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the above echelon from rank of the augmented matrix is equal to the rank of the coefficient matrix and the no. of unknowns is greater than the rank. Therefore the system is consistent and the system has infinite no of solutions.

We have to assign $4-3=1$ arbitrary value,

The corresponding linear equations are,

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6.$$

$$x_2 + 4x_3 + 2x_4 = -2.$$

$$x_4 = 0.$$

We let, $x_3 = K$, then,

$x_1 + 2x_2 - 3K - 4x_4 = 6.$

$x_2 + 4K + 2x_4 = -2.$

$x_4 = 0.$

$x_2 = -2-4K$

$x_1 + 0 - 4 - 8K - 3K = 6$

$\Rightarrow x_1 = 10 + 11K$

Thus the solution is,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10+11K \\ -2-4K \\ K \\ 0 \end{pmatrix}.$$

For the different values of K, we will get the infinite no. of solutions.

Assignment - 5 years question (previous).
non-homogeneous ৭২fয়,

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16.09.19

4th-E.

Ex: Find the value of λ so that, the equations,

$$ax + hy + g = 0.$$

$$hx + by + f = 0.$$

$$gx + fy + c = \lambda.$$

are ~~constan~~ consistent.

Sol^n: The augmented matrix is,

$$[A\ H] = \begin{pmatrix} a & h & -g \\ h & b & -f \\ g & f & \lambda - c \end{pmatrix}.$$

$$H_1 = H_1/a \qquad \begin{pmatrix} 1 & \frac{h}{a} & -g/a \\ 1 & b/h & -f/h \\ 1 & f/g & \frac{\lambda - c}{g} \end{pmatrix}.$$

$$H_2 = H_2/h$$

$$H_3 = H_3/g$$

$$\underset{A_3 = H_3 - H_1}{\overset{A_2 = H_2 - H_1}{\sim}} \begin{pmatrix} 1 & h/a & -g/a \\ 0 & \frac{b}{h} - \frac{h}{a} & -\frac{f}{h} + \frac{g}{a} \\ 0 & \frac{f}{g} - \frac{h}{a} & \frac{\lambda - c}{g} + g/a \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & h/a & -g/a \\ 0 & \frac{ab-h^2}{ah} & \frac{gh-af}{ah} \\ 0 & \frac{af-gh}{ag} & \frac{a(\lambda-c)+g^2}{ag} \end{pmatrix}.$$

$$\underset{H_3 = H_3 \times \frac{ag}{af-gh}}{\overset{H_2 = H_2 \times \frac{ah}{ab-h^2}}{\sim}} \begin{pmatrix} 1 & h/a & -g/a \\ 0 & 1 & \frac{gh-af}{ah} \times \frac{ah}{ab-h^2} \\ 0 & 1 & \frac{a(\lambda-c)+g^2}{ag} \times \frac{ag}{af-gh} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & h/a & -g/a \\ 0 & 1 & \frac{-af+gh}{ab-h^2} \\ 0 & 1 & \frac{a(\lambda-c)+g^2}{af-gh} \end{pmatrix}$$

Ex:

$$R_3 = R_3 - R_2 \sim \begin{pmatrix} 1 & h/a & -g/a \\ 0 & 1 & \dfrac{-af+gh}{ab-h^2} \\ 0 & 0 & \dfrac{a(\lambda-c)+g^2}{af-gh} - \dfrac{-af+gh}{ab-h^2} \end{pmatrix}.$$

In the above echelon form the rank of the augmented matrix is 3 and the rank of the coefficient matrix is 2. The given system will be consistent if.

$$\frac{a(\lambda-c)+g^2}{af-gh} - \frac{-af+gh}{ab-h^2} = 0.$$

$$\Rightarrow \lambda = \left[\left(\frac{-af+gh}{ab-h^2}\right)(af-gh) - g^2 + ac\right]\frac{1}{a}.$$

Ex: For what values of λ the system

$$x+y+z=1.$$

$$x+2y+4z=\lambda.$$

$$x+4y+10z=\lambda^2.$$

have a solution and hence solve completely in each case.

Soln: The augmented matrix is,

$$[A\ H] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{pmatrix}.$$

$H_2 = H_2 - H_1$

$H_3 = H_3 - H_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{pmatrix}$$

$$R_3 = R_3 - 3R_2 \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 1 - 3\lambda + 3 \end{pmatrix}.$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{pmatrix}$$

The given system has at least one solution if the rank of the augmented matrix and the rank of the coefficient matrix are same in the echelon form. Therefore we must have.

$$\lambda^2 - 3\lambda + 2 = 0.$$

$$\lambda^2 - 2\lambda - \lambda + 2 = 0.$$

$$\Rightarrow \lambda = 2, 1.$$

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For $\lambda = 2$, the echelon form,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding linear equations are,

$$x+y+z=1.$$

$$y+3z=1.$$

We have to assign $3-2=1$, arbitrary values,

let, $z=K$, then,

$$x+y+K=1 \qquad x+1-3K+K=1$$

$$x = 2K$$

$$y+3K=1.$$

Therefore the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2K \\ 1-3K \\ K \end{pmatrix}.$

For the different values of K we will get the infinite no. of solutions.

21.09.19
5th. C
JAY

The solution of homogeneous system of linear eqn:

(1) ~~Rank of the coefficient~~

A system of m homogeneous eqn s in n unknowns $x_1, x_2, \ldots, x_n$ has a nontrivial solution if the rank, say r, ~~of~~ the coefficient matrix is less than then ~~x~~ no of unknowns, say n.

If $r = n$, then the system has trivial solution, that is $x_1 = 0 = x_2 = \ldots = x_n$.

Ex:

$$x + y + z + w = 0.$$

$$x + 3y + 2z + 4w = 0.$$

$$2x + z - w = 0.$$

Sol^n: The coefficient matrix,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{pmatrix}.$$

$$\underset{R_3 = R_3 - 2R_1}{\underset{\sim}{R_2 = R_2 - R_1}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & -2 & -1 & -3 \end{pmatrix}.$$

$$\underset{\sim}{R_3 = R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank of the given system is 2.

and the no. of unknowns is 4.

Therefore the given system has nontrivial solution.

The corresponding linear equations, are,

$$x + y + z + w = 0.$$

$$2y + z + 3w = 0.$$

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$$2a - \frac{(2a+b)}{3} - (a+2b) \quad 3a+3b-\frac{2a-b}{3}$$

$$x = -a + 2b + b - \frac{(2a+b)}{3}$$

We ~~x~~ have to assign $4-2=2$ arbitrary values.

let, ~~$x=a$~~ $y=a$

~~$y=b$~~, $z=b$

Then the solution is, $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -(a+2b) \\ a \\ b \\ -\frac{(2a+b)}{3} \end{pmatrix}$.

Ex: solve ;

$2x_1 - x_2 + x_3 = 0.$

$3x_1 + 2x_2 + x_3 = 0.$

$x_1 - 3x_2 + 5x_3 = 0.$

Soln: The coefficient matrix.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{pmatrix}.$$

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$H_1 = \frac{H_1}{2}$

$$\begin{pmatrix} 1 & -1/2 & 1/2 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{pmatrix}.$$

Interchanging H_1 and H_3

$$\begin{pmatrix} 1 & -3 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$-1+6$

$1-10$

$2 \quad +3\times 3$

$1-15$

$H_2 = H_2 - 3H_1$

$H_3 = H_3 - 2H_1$

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 11 & -14 \\ 0 & 5 & -9 \end{pmatrix}.$$

16

$\frac{14+9}{23}$

$\frac{11-5}{6}$ -14

$H_2 = H_2 - H_3$

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 6 & -6 \\ 0 & 5 & -9 \end{pmatrix}.$$

$5 - 3$

Interchanging H_2 and H_3

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 5 & -9 \\ 0 & 6 & -6 \end{pmatrix}$$

$H_3 = \frac{H_3}{6}$

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 5 & -9 \\ 0 & 1 & -1 \end{pmatrix}.$$

$-7+5$

~~$H_1 = H_1 + 3H_3$~~

$H_2 = H_2 - 5H_3$

$$\begin{pmatrix} 1 & 0 & 2 \\ \cancel{0} & \cancel{0} & \cancel{4} \\ 0 & 1 & -1 \end{pmatrix}.$$

$H_3 = H_3 - 5H_2$

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 5 & -9 \\ 0 & 0 & 44 \end{pmatrix}.$$

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$R_3 = \frac{R_3}{44}$

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 5 & -9 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_3 = R_3 +$

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$2x_1 - x_2 + x_3 = 0$

$3x_1 + 2x_1 + x_3 = 0.$

$x_1 - 3x_2 + 5x_3 = 0.$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{pmatrix}$$

$C_1 = C_1 - 2C_3$

$C_2 = C_2 + C_3$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ -9 & 2 & 5 \end{pmatrix}.$$

$C_2 = C_2 - 3C_1$

$C_3 = C_3 - C_1$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 9 & 29 & 14 \end{pmatrix}$$

$R_3 = R_3 - 9R_2$

$$\sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 29 & 14 \end{pmatrix}.$$

$R_3 = R_3 - 14R_1$

$C_2 = \frac{C_2}{29}$

$$\sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

interchanging R_1 and R_2

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

interchanging R_2 and R_3

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the above echelon form the rank of the matrix is 3. And the unknown is 3.

So the system has trival solution.

$\therefore x_1 = x_2 = x_3 = 0$.

23.09.19
5th E

* $AX = H$.

$X = A^{-1} H$

Ex: Solve by matrix method:

$x + y + z = 6$.

$x + 2y + 3z = 14$

$x + 4y + 9z = 36$.

Soln: In matrix notation, we can write.

$AX = H$.

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 36 \end{pmatrix}.$$

$H_2 = H_2 - H_1$

$H_3 = H_3 - H_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 30 \end{pmatrix}.$$

$R_3 = R_3 - 3R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 6 \end{pmatrix}.$$

$R_3 = \frac{R_3}{2}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix}.$$

The corresponding linear equations are

$x+y+z=6$

$y+2z=8$

$z=3$

Therefore the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Ex: Using matrix method solve :

$$2x - y + 3z = 9.$$

$$x + y + z = 6.$$

$$x - y + z = 2.$$

Ex: Solve by :

$$x + 2y + 3z = 6$$

$$2x + 4y + z = 7$$

$$3x + 2y + 9z = 14.$$

using determinants.

Soln: By Cramer's Rule:

$$\begin{array}{l} \;\; x \\ \begin{vmatrix} 2 & 3 & 6 \\ 4 & 1 & 7 \\ 2 & 9 & 14 \end{vmatrix} \end{array} = \begin{array}{l} \;\; y \\ \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} \end{array} = \begin{array}{r} z \;\; \\ \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} \end{array}$$

$$= \frac{1}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}}.$$

$$= \frac{x}{\cancel{636}\ -20} = \frac{y}{-20} = \frac{z}{-20} = \frac{1}{-20}.$$

$$\therefore x = \frac{-20}{-20},\ y = \frac{-20}{-20},\ z = \frac{-20}{-20} = 1.$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Ex: Using determinants, solve:

$$x+y+z=9.$$

$$2x+5y+7z=52.$$

$$2x+y-z=0.$$

$\lceil 9(-5-7)-1(-52-7)+1($

$\Rightarrow$ Sol^n:

$$\frac{x}{\begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}}.$$

$$\Rightarrow \frac{x}{\quad}$$

28.02.19
6th C

Characteristic equation of a matrix:

Any vector x which by the transformation is carried into ~~x~~ λX, that is any vector ~~x~~ X for which $AX = \lambda X$ is called an eigen vector under the transformations.

Now, $\lambda x - AX = 0$.

$$(\lambda I - A)X = 0.$$

$$\Rightarrow \begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

The system of homogeneous equations has non-trivial solution if ~~any~~ and only if

$$|\lambda I - A| = 0.$$

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$$= \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ - & - & - & - \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = 0.$$

The equation $|\lambda I - A| = 0$ is called the characteristic equation and the values $\lambda_1, \lambda_2 \cdots \lambda_n$ are characteristic values or eigen values.

Ex: Find the characteristic equation of

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix},$$ hence find the characteristic values and vectors.

Soln: The characteristic equation for given A is

$$|\lambda I - A| = 0.$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \right| = 0.$$

$$\Rightarrow \begin{vmatrix} \lambda-2 & -2 & -1 \\ -1 & \lambda-3 & -1 \\ -1 & -2 & \lambda-2 \end{vmatrix} = 0.$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\Rightarrow \lambda^3 \lambda^2 - 6\lambda^2 + 6\lambda + 5\lambda - 5 = 0.$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0.$$

$$\Rightarrow \lambda = 1, 5, 1.$$

These are the characteristic or eigen value.

For $\lambda = 1$, the equation $(\lambda I - A) X = 0$, becomes.

$$\begin{pmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The corresponding system of linear equation:

$$-x_1 - 2x_2 - x_3 = 0.$$

$$-x_1 - 2x_2 - x_3 = 0.$$

$$-x_1 - 2x_2 - x_3 = 0.$$

The linearly independent solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

similarly, the linearly independent solution for $\lambda = 1$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

for $\lambda = 5$, the equation $(\lambda I - A)X = 0$ becomes.

$$\begin{pmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$3-2-1=0$

$1, 1$

The corresponding system of linear equations,

$$3x_1 - 2x_2 - x_3 = 0.$$

$$-x_1 + 2x_2 - x_3 = 0.$$

$$-x_1 - 2x_2 + 3x_3 = 0.$$

the linearly independent solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The eigen values are 1, 5, 1 and the corresponding eign vectors are;

$$\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Caylay- Hamilton Theorem:

Statement: Every square matrix satisfies its characteristic equation

Ex

$$\lambda^2-7\lambda^2+11\lambda-5=0$$

$$A^3-7A^2+11A-5I=0$$

Proof: Home work.

Ex: Find the characteristic equation for A

$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, Hence find eigen value and the corresponding eign vectors. Show that the given matrix A satisfies the characteristic equation. Hence find A^{-1}.

⇒ characteristic equation for A. is

$$\lambda^2 - 4\lambda - 5 = 0 \text{ ——— } (1)$$

characteristic values, are,

$$\lambda = -1, \ \lambda = 5.$$

$$(\lambda I - A)X = 0.$$

$$\Rightarrow \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

for $\lambda = 1$

$$\cancel{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda=5$, the eigen vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

to variefy cayley-Hamilton theorem we have to show that,

$$A^2 - 4A - 5I = 0 \quad \text{------------} \quad (I)$$

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

$$= \begin{pmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}.$$

Now $A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the matrix A satisfies the Cayley Hamilton theorem.

From eq^n (11), we can write,

$$A^2 - 4A = 5I.$$

Multiplying this equation by A^{-1}.

$$A^{-1}A^2 - 4AA^{-1}I = 5A^{-1}.$$

$$\Rightarrow A - 4I = 5A^{-1}.$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 5A^{-1}.$$

$$\Rightarrow 5A^{-1} = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}.$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}.$$

Ex: If $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$ satisfies cayley-Hamilton theorem $\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0$, then find its inverse.

Soln: since A satisfies cayley-Hamilton then,

$5I = A^3 - 5A^2 + 6A$.

$5A^{-1} = A^2 - 5A + 6I$.

$$\Rightarrow \quad = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{pmatrix} - 5\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{pmatrix}.$$

~~Asse~~ System of linear equation

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Cayley - Hamilton Theorem

Statement: Every square matrix satisfies its characteristic equation.

$$\left[\begin{array}{l} \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \\ A^3 - 7A + 11A - 5I = 0 \end{array}\right]$$

Proof: Home work.

Ex: Find the characteristic equation for $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ hence find the eigen values and the corresponding eigen ~~values~~ vector. Show that the given matrix A satisfies the characteristic equation. Hence A^{-1}

Soln: The characteristic for given A is

$\Rightarrow \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \right| = 0$

$\Rightarrow \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = 0$

$\Rightarrow (\lambda - 1)(\lambda - 3) - 8 = 0$

$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\therefore \lambda = 5, -1,$

There are the characteristics or eigen value

For $\lambda = 5$, the equation $(\lambda I - A)X = 0$ becomes

$\begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

The equations

$4x_1 - 2x_2 = 0$

$-4x_1 + 2x_2 = 0$

The linearly independent solution is, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

For same

$\lambda = -1$ the equation $(\lambda I - A)X = 0$ becomes

$$\begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

The corresponding system of linear equations

$$-2x_1 - 2x_2 = 0$$
$$-4x_1 - 4x_2 = 0$$

The linearly independent solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The eigen values are 5, −1 and the corresponding eigen values are, $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The characteristic equation is

$$A^2 - 4A - 5I \quad -(2)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3-4-5 & 2+6-8-6 \\ 4+12-16 & 8+9-12-5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

The given matrix satisfied the characteries equation from equation (2)

$A^2 - 4A = 5I$

Multiplying this equation by A^{-1}

$A^{-1}A^2 - 4A^{-1}A = 5A^{-1}I$

$\Rightarrow A - 4I = 5A^{-1}$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 5A^{-1}$

$\Rightarrow \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix} = 5A^{-1}$

$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$

Ex: If $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$ satis Cayley - Hamilton theorem $\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0$

Soln: Since A satisfies Cyley - Hamilton thm,

$\therefore A^3 - 5A^2 + 6A - 5I = 0$

$\Rightarrow 5I = A^3 - 5A^2 + 6A$

$\Rightarrow 5A^{-1} = A^2 - 5A + 6I$

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$$= \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{pmatrix} - 5 \begin{pmatrix} 4 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} + 6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{pmatrix}$$

\# Show that the set of vectors $\{\underset{v_1}{(2, 1, 2)}, \underset{v_2}{(0, 1, -1)}, \underset{v_3}{(4, 3, 3)}\}$ in linearly dependent.

[যে কয়টি vector কে কয়টি scalar দিয়ে গুণ করে যোগ করলে u : linear combination]

$\Rightarrow$ Let x, y, z three scalars, then the linear combination of the given vectors is

$$x(2, 1, 2) + y(0, 1, -1) + z(4, 3, 3) = (0, 0, 0)$$

$\Rightarrow$ এখন $\hat{i}, \hat{j}, \hat{k}$ এর coefficient সমান করে পাই

$$\Rightarrow (2x + 0.y + 4z,\ x + y + 3z,\ 2x - y + 3z) = (0, 0, 0)$$

The corresponding system of linear equations

$$2x + 0.y + 4z = 0$$
$$x + y + 3z = 0$$
$$2x - y + 3z = 0$$

Linear combination:

The vectors $v_1, v_2, v_3 \ldots v_n$ are said to be linear combination of a vector u if there exists scalars $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ such that

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_n v_n$$

The vectors v_1, v_2, v_n are said to be linearly dependent if there exists a non-trivial combination of them equal to zero vector that is,

$\alpha_1 v_1 + \alpha_2 v_2 \ldots + \alpha_n v_n = 0$, Where $\alpha_i \neq 0$ for at least one i

[That is non zero solution] → linearly independent non-trivial infinite solution

And the vectors v_1, v_2, v_m are

Here coefficient matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & -1 & 3 \end{pmatrix}$

Interchanging 1st and 2nd row $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 4 \\ 2 & -1 & 3 \end{pmatrix}$

$H_2 = H_2 - 2H_1$, $H_3 = H_3 - 2H_1$ $\begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -3 \end{pmatrix}$

$H_2 = H_2/(-2)$, $H_3 = H_3/(-3)$ $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$H_3 = H_3 - H_2$ $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

The rank of A is 2 and the no. of unknowns is 3, therefore the solution of the system is non-trivial or infinite num of solution.

Therefore the given set of vectors are linearly dependent

said to be linearly independent if only linear combination of them equal to zero is the trivial one that is

$\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_m V_m = 0$ iff

$\alpha_1 = 0 = \alpha_2 = \alpha_3 = \dots = \alpha_m$

[That is zero solution.

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Ex: Show that the vectors,

$\{(2,-1,4), (3,6,2), (2,10,-4)\}$ are linearly independent vectors.

Let the x, y, z are the scalar,

the linear combination

$$x(2,-1,4) + y(3,6,2) + z(2,10,-4) = (0,0,0)$$

$$\Rightarrow (2x+3y+2z,\ -x+6y+10z,\ 4x+2y-4z) = (0,0,0)$$

The corresponding equation of linear equation,

$$2x+3y+2z=0$$
$$-x+6y+10z=0$$
$$4x+2y-4z=0$$

So the coefficients are,

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{vmatrix}$$

$R_1 \rightleftarrows$ $\Rightarrow \begin{vmatrix} -1 & 6 & 10 \\ 2 & 3 & 2 \\ 4 & 2 & -4 \end{vmatrix}$

$R_2 = R_2 + 2R_1$
$R_3 = R_3 + 4R_1$
$$\begin{vmatrix} -1 & 6 & 10 \\ 0 & 15 & 22 \\ 0 & 26 & 36 \end{vmatrix}$$

$\Rightarrow R_1 = R_2 + R_1$ $\begin{vmatrix} 1 & 9 & 12 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{vmatrix}$

$\Rightarrow \underbrace{R_2 = R_2 + R_1}$
$R_3 = R_3 - 4R_1$
$$\begin{vmatrix} 1 & 9 & 12 \\ 0 & 15 & 22 \\ 0 & -34 & -52 \end{vmatrix}$$

$\underbrace{R_2 = R_2 \div 15}$
$R_3 = R_3 \div (-34)$
$$\begin{vmatrix} 1 & 9 & 12 \\ 0 & 1 & 22/15 \\ 0 & 1 & \frac{52}{34} \end{vmatrix}$$

$\underbrace{R_3 = R_3 - R_2}$
$$\begin{vmatrix} 1 & 9 & 12 \\ 0 & 1 & 22/15 \\ 0 & 1 & 52/34 - \frac{22}{15} \end{vmatrix}$$

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$$\sim \begin{pmatrix} 1 & 9 & 12 \\ 0 & 1 & 22/15 \\ 0 & 0 & 16/255 \end{pmatrix}$$

$$r_3/(16/255) \begin{pmatrix} 1 & 9 & 12 \\ 0 & 1 & 22/15 \\ 0 & 0 & 1 \end{pmatrix}$$

Here echolon - form rank is 3, and it is trivial so the vector is lieanly independent.

Ex: A man buys 8 dozens of mangoes, 10 dozens of apple, 4 dozens of bananas. Mangoes cost 180 per dozen, apple tk 90 per dozen and bananas tk 60 per dozens. Represent the quantities bought by a row matrix and the prices by a colomn matrix and hence obtain the total cost.

Soln:

$$A = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix}_{1\times 3} \quad B = \begin{bmatrix} 180 \\ 90 \\ 60 \end{bmatrix}_{3\times 1}$$

Cost = element (1×1) হবে এবং matrix গুণের ভাবে নিলে 1×1 হয়।

The total cost, $A\times B = \begin{pmatrix} 8 & 10 & 4 \end{pmatrix} \times \begin{pmatrix} 180 \\ 90 \\ 60 \end{pmatrix}$

$= \begin{bmatrix} 1440 + 900 + 240 \end{bmatrix}$

$= 2580$ tk

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Ex : A store has in stock 30 dozens shirts, 15 dozens trousers and 25 dozens pair of socks. If the selling prices are tk 500 per shirt tk. 900 per trousers and tk 120 per pair socks, then find the total amount the store owner will get after selling all the item in the stock

$$A = \begin{bmatrix} 360 & 180 & 300 \end{bmatrix} \times \begin{bmatrix} 500 \\ 900 \\ 120 \end{bmatrix}.$$

Total cost $= 3,78,000$ tk.

A manufacture produces three products A, B, C which he sells in the market. Annual sale volumes are indicated as follows:

Market :	A	B	C
I	8000	10,000	15000
II	10000	2000	20,000

If unit sell prices of A, B, C are tk 225, tk, 150 tk and 125 respectively then find the total revenue in each market with the help of matrices.

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selling price in row matrix Let A, A = (225 150 125)

Num of products say B =

$$\begin{array}{cc} I & II \\ \left[\begin{array}{cc} 8000 & 10,000 \\ 10,000 & 2,000 \\ 15,000 & 20,000 \end{array}\right] \end{array}$$

$$A \times B = \begin{bmatrix} 225 & 150 & 125 \end{bmatrix} \begin{bmatrix} 8000 & 10,000 \\ 10,000 & 2000 \\ 15,000 & 20,000 \end{bmatrix}$$

$$= \begin{bmatrix} 5175000, & 50,50,000 \end{bmatrix}$$

that for market I revenue is 51,75000, for market II the revenue is 50,50000 tk