

Hydrostatic Forces on Submerged Surfaces

When a static mass of fluid comes in contact with a surface, either plane or curved, a force is exerted by the fluid on the surface. This force is known as total pressure. It has been observed that:

- (i) The intensity of pressure p due to the weight of fluid at any point is directly proportional to its depth below the free liquid surface. This depth is known as pressure head.
- (ii) The fluid static pressure cause force which acts normally at every point of the surface of container or any submerged object.

Force on a Horizontal Submerged Plane Surface

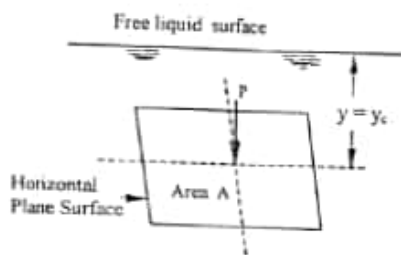


Fig. 2.14. Force on a Horizontal Submerged Plane Surface

Fig. 2.14 shows a plane surface submerged and held in a horizontal position at depth Y below the free surface of the liquid. Since every point on the surface is at the same depth, the pressure intensity is constant over the entire plane surface.

From hydrostatic equation $p = \rho y$, and if A is the total area of the surface then total pressure force on the horizontal surface is:

$$F = pA = \rho y A = A (\rho y)$$

For the given configuration, the depth $y = y_c$, depth of the centre of gravity (centroid) of the submerged surface below the free surface of the liquid:



Force on a Vertical Plane Submerged Surface

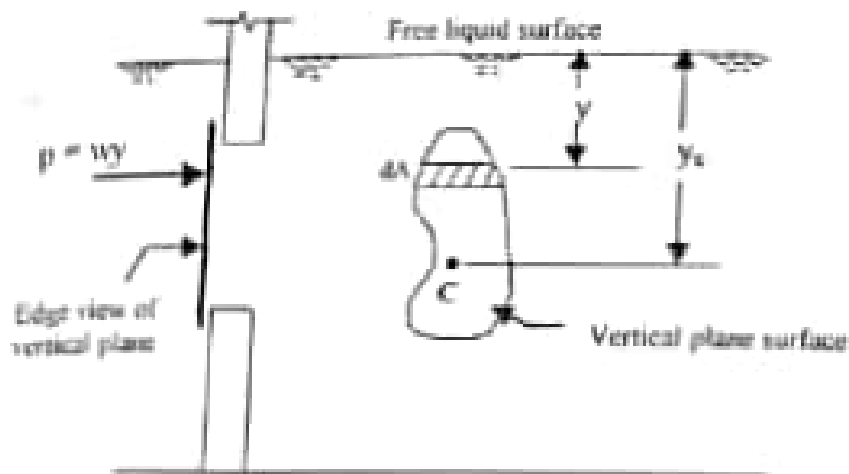


Fig. 2.15. Force on a Vertical Plane Submerged Surface

Consider a plane surface of arbitrary shape immersed vertically in a static mass of fluid (Fig. 2.15). Depth of the liquid varies from point to point; pressure intensity is thus not constant over the entire surface. Analysis for the total pressure force is then made by considering on the vertical plane surface an elementary horizontal strip lying at a depth y below the free surface of the liquid. The pressure intensity for this strip can be assumed to be constant and equal to

$$p = wy$$

Hence differential force on the strip is:

$$dF = p dA = wy dA$$

and the total force on the entire area is given by:

$$F = \int wy dA = w \int y dA$$

But, we know that $y_c = \frac{\int y dA}{A}$

$\Rightarrow y_c A = \int y dA$

The integral $\int y dA$ represents the sum of first moments of the areas of the strip about free liquid surface and equals Ay_c , where y_c is the depth to the centroid of the immersed surface. Thus

$$F = wAy_c = A(wy_c)$$

(2.12)

Force on an Inclined Submerged Plane Surface

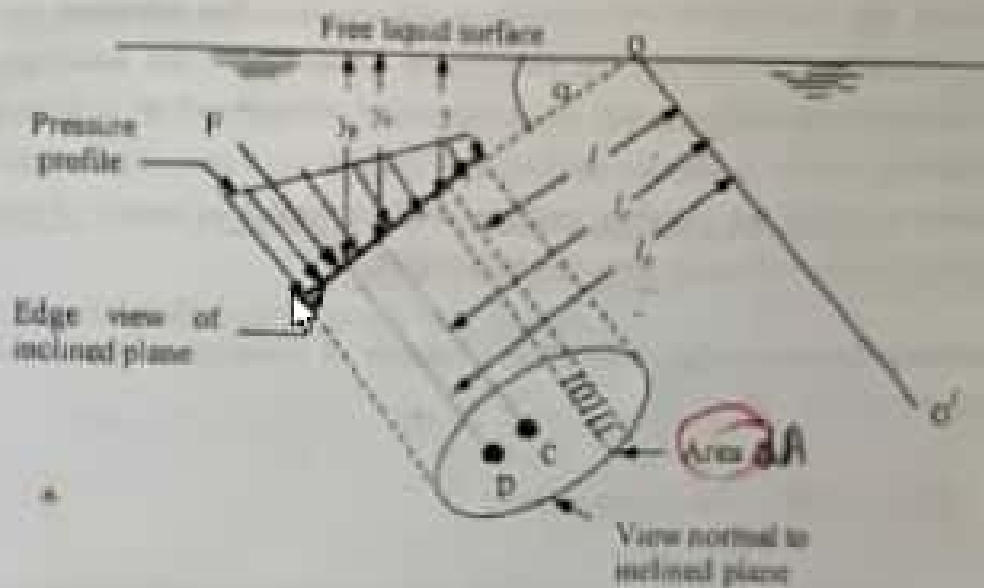


Fig. 2.16. Force on an Inclined Submerged Plane Surface

Let a plane surface of arbitrary shape be entirely submerged in a static mass of liquid. The plane of the surface intersects the horizontal liquid surface at axis $O - O'$ making an arbitrary angle α . The differential force dF on an elementary strip of area dA , where the pressure intensity is p , is given by:

$$dF = p dA = \rho g y dA = \rho g l \sin \alpha \times dA$$

Where y is the vertical depth of the elementary area from the free liquid surface, and l is its distance from the axis $O - O'$. Force on the entire immersed surface can be computed by integrating the differential force dF over the entire area A .

$$F = \int p dA = \rho g \sin \alpha \int l dA$$

Integral $\int l dA$ is the first moment of area A about $O - O'$ and is equal to $A l_c$

$$F = \rho g \sin \alpha A l_c$$

$$= \rho g A l_c \sin \alpha = \rho g A y_c$$

$$= A \rho g y_c$$

(2.13)

Evidently through equations (2.11), (2.12) and (2.13) we conclude that *whatever may be the inclination of the submerged plane surface to the free liquid surface, the magnitude of the resultant hydrostatic force equals the product of the area and the pressure at the centroid of the area.*

View normal to inclined plane

Fig. 2.16 Force on an Inclined Submerged Plane Surface

Let a plane surface of arbitrary shape be entirely submerged in a static mass of liquid. The plane of the surface intersects the horizontal liquid surface at axis $0 - 0'$ making an arbitrary angle α . The differential force dF on an elementary strip of area dA , where the pressure intensity is p , is given by:

$$dF = p dA = w y dA = w \times l \sin \alpha \times dA$$

Where y is the vertical depth of the elementary area from the free liquid surface, and l is its distance from the axis $0 - 0'$. Force on the entire immersed surface can be computed by integrating the differential force dF over the entire area A .

$$F = \int p dA = w \sin \alpha \int l dA$$

Integral $\int l dA$ is the first moment of area A about $0 - 0'$ and is equal to $A \bar{c}$

$$\bar{c} = \frac{\int l dA}{A} \Rightarrow \bar{c} A = \int l dA$$

$$F = w \sin \alpha A \bar{c}$$

$$= w A (l_c \sin \alpha) = w A y_c$$

$$= A (w y_c)$$

(2.13)

Evidently through equations (2.11), (2.12) and (2.13) we conclude that *whatever may be the inclination of the submerged plane surface to the free liquid surface, the magnitude of the resultant hydrostatic force equals the product of the area and the pressure at the centroid of the area.*

Centre of Pressure

We know that the intensity of pressure is not uniform, but increases with depth of a point on the immersed surface. As the pressure is greater over the lower portion of the surface, therefore the resultant pressure, on an immersed surface will act at some point, below the centre of gravity of the immersed surface and towards the lower edge of the figure. The point of the area at which the resultant pressure acts, is known as centre of pressure and is always expressed in terms of depth from the liquid surface.

Considering the above figure location of centre of pressure is determined by taking moments about the axis $O-O'$.

$$F l_p = \int (dF) l = \int w \times (\sin \alpha \times dA \times l)$$

$$= w \sin \alpha \int l^2 dA$$

$$\therefore l_p = \frac{w \sin \alpha \int l^2 dA}{F}$$

Substituting the value of $F = w \sin \alpha \int l dA$, we obtain

$$l_p = \frac{w \sin \alpha \int l^2 dA}{w \sin \alpha \int l dA}$$

$$= \frac{\int l^2 dA}{\int l dA}$$

Recognizing that $\int l^2 dA = I_o =$ second moment of area, or the moment of inertia of area conserved about axis $O-O'$, the above equation can be rewritten as

$$l_p = \frac{I_o}{A l_c}$$

Moment of inertia is generally quoted about an axis through the centroid; shifting the axis on the surface to parallel axis passing through the centroid gives:

$$I_o = I_c + A l_c^2 \text{ (parallel axis theorem)}$$

where I_c is the moment of inertia of the area about an axis through the centroid

$$\text{or } I_o = A k_c^2 + A l_c^2$$

$$l_p = \frac{I_c + A l_c^2}{A l_c}$$

$$= \frac{I_c}{A l_c} + l_c \quad \text{--- (2.14)}$$

$$\text{or, } l_p - l_c = \frac{I_c}{A l_c} = \frac{k_c^2}{l_c} \quad \text{--- (2.14a)}$$

In terms of vertical distance from the free surface,

$$y_p - y_c = \frac{I_c \sin^2 \alpha}{A y_c} = \frac{k_c^2 \sin^2 \alpha}{y_c} \quad \text{--- (2.15)}$$

The following facts can be gleaned from the above equation:

1. Centre of pressure lies below the centroid, because for any plane surface the factor $(k_c^2 \sin^2 \alpha / y_c)$ is always positive.
2. Deeper the surface is lowered into the liquid i.e., greater is the value of y_c , closer comes the centre of pressure to the centroid of the area.
3. Depth of centre of pressure is independent of the specific weight of the liquid and is consequently same for all liquids.
4. For a horizontal surface $\alpha = 0$ and so $y_p = y_c$, i.e., the centre of pressure coincides with the centroid.
5. For a vertical surface $\alpha = 90^\circ$ and so

$$y_p - y_c = \frac{I_c}{A y_c}$$

6. If the vertical rectangular surface has breadth b and depth d , then

$$y_p - y_c = \frac{d}{2} + \frac{bd^3 \cdot 12}{(bd)d \cdot 2} = \frac{2}{3}d$$

i.e., the centre of pressure is at a depth equal to $\frac{2}{3}$ rd of the submerged height of the surface below the liquid level.

Centre of Pressure of a Composite Section

The centre of pressure of a composite section (i.e., a section with cut out hole or other composite section) is obtained as discussed below:

1. Split up the composite section into convenient sections (i.e., rectangles, triangles or circles).
2. Calculate the pressures, P_1, P_2, \dots on all the sections.
3. Then calculate the total pressure P on the whole section by the algebraic sum of the pressures.
4. Then calculate the depths of centres of pressure h_1, h_2, \dots for all the sections from the water surface.
5. Then equate $Ph = P_1h_1 + P_2h_2 + \dots$ where h = Depth of centre of pressure of the section from the water level.

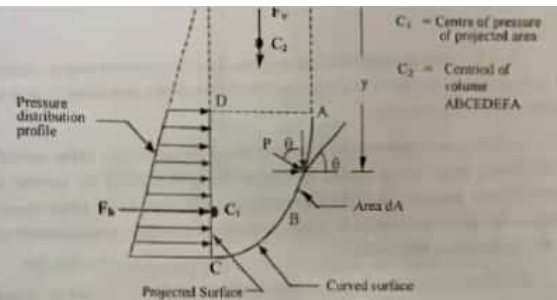


Fig. 2.17 Force on Curved Submerged Surface

Let the Fig. 2.17 represent the trace of curved surface submerged wholly in a static mass of liquid. Consider on the curved surface an elementary area dA lying at a vertical depth y below the free surface of the liquid. If p is the normal pressure intensity at the elementary area, then the differential force acting in direction normal to the surface is

$$dF = p dA = wy dA$$

and the total force on the entire curved surface is

$$F = \int wy dA.$$

Since for a curved surface direction of the pressure force varies from point to point, straightforward integration procedure cannot longer be applied. Computation of total pressure on a curved surface is then made possible by assessing the pressure forces acting on projected horizontal and vertical planes. For the elementary area:

$$dF_h = dF \sin \theta = p dA \sin \theta$$

$$dF_v = dF \cos \theta = p dA \cos \theta$$

Where θ is the inclination of the elementary area dA with the horizontal. Substituting $p = wy$ in the above expressions and subsequent integration yields:

$$F_h = \int dF_h = w \int y dA \sin \theta$$

$$F_v = \int dF_v = w \int y dA \cos \theta$$

In these expressions, $dA \sin \theta$ and $dA \cos \theta$ represent respectively the vertical and horizontal projections of the elementary area dA .

Consequently,

$w \int y dA \sin \theta$ represents the total pressure force on projected area of curved surface on the vertical plane. The point of application of the horizontal component F_h is at the centre of pressure of the projected area.

$w \int y dA \cos \theta$ represents the total pressure on projected area of the curved surface on the horizontal plane, and it equals the weight of liquid lying in the portion ABCDEFA; weight of liquid extending from the curved surface to the free surface of the liquid. The point of application of the component F_v , acting vertically downward is at the centroid of the liquid volume above the curved surface.

The resultant pressure force F is then equal to $\sqrt{F_h^2 + F_v^2}$; acting at angle $\tan^{-1} \left(\frac{F_v}{F_h} \right)$ with the horizontal.

In some engineering applications, the liquid acts from below the curved surface. In that case, the vertical component of pressure force acts upwards, and equals the weight of an imaginary column of liquid above the curved surface up to the free surface.

Pressure Diagrams

A pressure diagram may be defined as a graphical representation of the variation in the intensity of pressure over a surface. Such diagrams are very useful for finding out the vertical surface (i.e., wall or dam). A vertical surface may be subjected to the following types of pressure.

1. Pressure due to one kind of liquid on one side,
2. Pressure due to one kind of liquid, over another, on one side, and
3. Pressure due to liquids on both sides.

Pressure Due to One Kind of Liquid on One Side

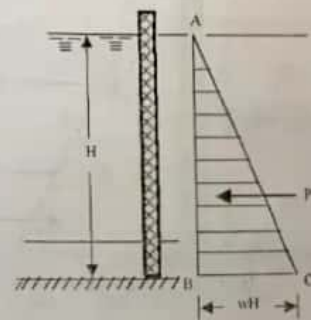


Fig. 2.18 Pressure Due to One Kind of Liquid on One Side

Consider a vertical wall subjected to pressure due to one kind of liquid, on one of its sides as shown in Fig. 2.18

Let H = Height of liquid

w = Specific weight of the liquid

P = Total pressure on the wall, per unit length.

We know that the pressure on the wall is zero at the liquid surface, and will increase by a straight line law to wH at the bottom. Therefore the pressure diagram will be a triangle ABC as shown in the figure. The total pressure on the wall per unit length

$$P = \text{Area of triangle ABC} = \frac{1}{2} \times H \times wH = \frac{wH^2}{2}$$

This pressure will be act at the centre of gravity of the triangle, i.e., at a distance of $\frac{1}{3}H$ from the bottom of the liquid.

Pressure Due to One Kind of Liquid over Another on One Side

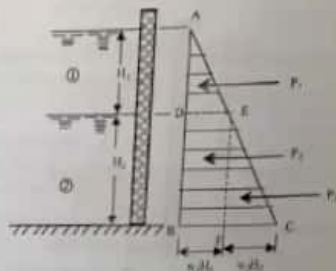


Fig. 2.19 Pressure Due to One Kind of Liquid over Another on One Side

Consider a vertical wall, subjected to pressure due to one kind of liquid, over another, on one side as shown in the figure. This will happen, when one liquid is insoluble into the other.

- Let
- H_1 = Height of liquid 1,
 - w_1 = Specific weight of liquid 1,
 - H_2 = Height of liquid 2
 - w_2 = Specific weight of liquid 2
 - P = Total pressure on the wall per unit length

We know that the pressure in such a case is zero at the liquid surface, and will increase by a straight line law to $w_1 H_1$ up to a depth of H_1 . It will further increase, by a straight line law, to $w_1 H_1 + w_2 H_2$ as shown in the figure.

The pressure P_1 on the surface AD, due to liquid 1, may be found out, as usual, from the area of triangle ADE (i.e., $P_1 = \frac{w_1 H_1^2}{2}$). The pressure on the surface DB will consist of pressure P_2 due to superimposed liquid 1, as well as pressure P_3 due to liquid 2. This pressure will be given by the area of the trapezium BCED (i.e., area of rectangle DBFE due to superimposed liquid i.e., $P_2 = w_1 H_1 \times H_2$ and the area of triangle FCE due to liquid 2 (i.e., $P_3 = \frac{w_2 H_2^2}{2}$).

The total pressure P will be sum of these pressures (i.e., $P = P_1 + P_2 + P_3$). The line of action of the total pressure may be found out by equating the moments of P , P_1 , P_2 and P_3 about A.

Pressure Due to Liquids on Both Sides

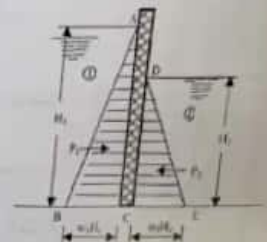


Fig. 2.20 Pressure Due to Liquids on Both Sides

Consider a vertical wall subjected to pressures due to liquids on both sides as shown in Fig. 2.20.

- Let
- H_1 = Height of liquid 1,
 - w_1 = Specific weight of liquid 1,
 - H_2 = Height of liquid 2
 - w_2 = Specific weight of liquid 2
 - P = Resultant pressure on the wall per unit length.

We know that the pressure of liquid 1 is zero at the liquid surface and will increase by a straight line law, to $w_1 H_1$ at the bottom as shown in the above figure.

The total pressure on the wall per unit length due to liquid 1

$$P_1 = \frac{1}{2} \times H_1 \times w_1 H_1 = \frac{w_1 H_1^2}{2}$$

Similarly, total pressure on the wall per unit length due to liquid 2,

$$P_2 = \frac{1}{2} \times H_2 \times w_2 H_2 = \frac{w_2 H_2^2}{2}$$

As the two pressures are acting in the opposite directions, therefore the resultant pressure will be given by the difference of the two pressures (i.e., $P = P_1 - P_2$). The line of action of the resultant pressure may be found out by equating the moments of P , P_1 and P_2 about the bottom of the wall.

Hoop Tension in a Pressure Pipe

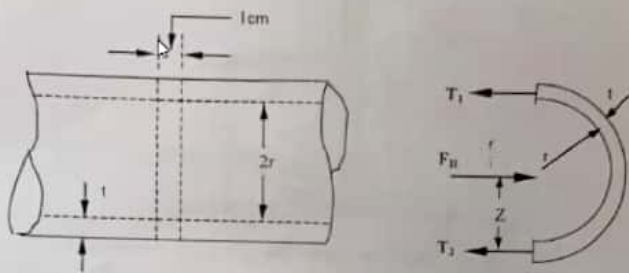


Fig. 2.21 Hoop Tension in a Pressure Pipe

A circular pipe under the action of an internal pressure is in tension around its circumference. This hoop tension causes stretching of the pipe wall and the pipe material is subjected to tensile stresses. Consider the above figure showing longitudinal section and one-half cross-section of a circular pipe. Assuming that no longitudinal stress occurs, the walls are in tension as shown. Considering 1 cm length of pipe and one-half of its circular section as a free body, let the tensile force per cm of pipe length at top and bottom be T_1 and T_2 , respectively as shown in Fig.2.21

The horizontal hydrostatic force F_H acts through the centre of pressure of the projected area and has a magnitude $2rp$, in which p is the pressure at the pipe centre in kg/cm^2 and r is the pipe radius in cm. For penstock pipes which are subjected to high pressures, the centre of pressure may be taken at the pipe centre, and then,

$$T_1 + T_2 = F_H$$

$$= 2pr$$

or $2T = 2pr$

Therefore the hoop tension force

$$T = p.r \tag{2.21}$$

For the pipe wall thickness of t cm, the tensile stress σ in the pipe-wall

If the pressures at the top and bottom of pipe differ appreciably, the centre of pressure is computed and Z determined. To determine T_1 and T_2 , the following equation may be written.

$$T_1 + T_2 = 2pr$$

taking moments of forces about the pipe bottom

$$T_1 \cdot 2r = F_H \cdot Z$$

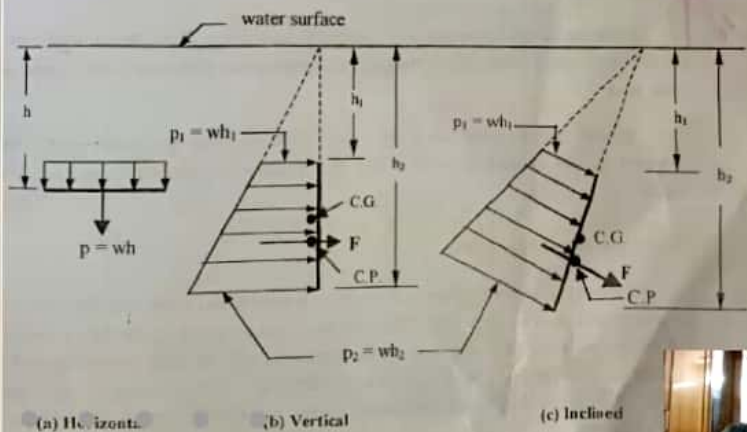
$$= 2pr \times Z$$

Solving the above two equation

$$T_1 = pZ \tag{2.22}$$

$$T_2 = p(2r - Z) \tag{2.23}$$

Pressure Diagrams for Horizontal, Vertical and Inclined surfaces



(a) Horizontal (b) Vertical (c) Inclined

Fig. 2.22 Pressure Diagrams for Horizontal, Vertical and Inclined surfaces