Fourier Transform

Mushfique Promel

"Engineering is an art
But no one wants to be the artist"

Fourier Transform

The function $F(s)$ is defined by

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

is called fourier transform of $f(x)$

[generally কোনো function দিবে $f(x)$. তার fourier transform হবে $F(s)$. আর formula apply করব $F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$]

Inverse fourier transform

এক্ষেত্রে $F(s)$ function টি জানা থাকবে. $f(x)$ বের করতে হবে

$$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

the function $f(x)$ is defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

is called inverse fourier transform of $F(s)$

~~A~~ এখন কিছু Problem দেখব

Ques. 1: i) Find the fourier transform of

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

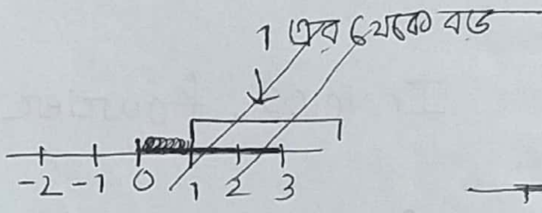
ii) Hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$

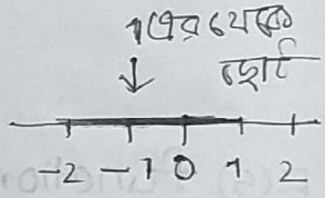
solve:

আমরা যখন ই Ques এ modulous দেখব তখন ই আগে modulous টা remove করে নিব, তারপর অন্য কাজ

$|x| < 1$

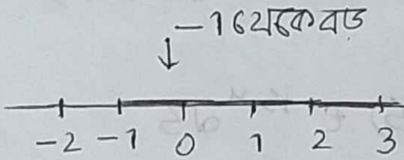
1. (+) নিলে $x < 1$



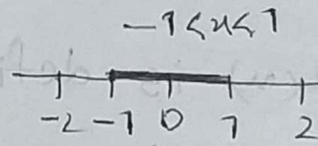


(−) নিলে $-x < 1$

$\therefore x > -1$



তাহলে পুরোটা দাঁড়াল: $-1 < x < 1$



এখন we know,

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

এখানে লিমিট $-1 < x < 1$.

Ques এ আবার $f(x) = 1$ কখন? যখন $|x| < 1$ মানে, $x < 1$

আবার $f(x) = 1$ কখন, যখন $x > -1$. তাহলে -1 থেকে 1 এর মধ্যে

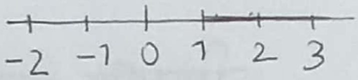
$f(x) = 1$.

আবার second modulous টা দেখি,

$|x|>1$

1 থেকে ∞

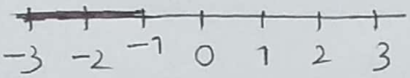
(+) নিয়ে, $x>1$



আবার, (-) নিয়ে,

$-x>1$ -1 থেকে ∞

$\therefore x<-1$



তাহলে, $f(x)=0 \quad |x|>1.$

$\therefore f(x)=0$ in $1<x<\infty$

and $-\infty<x<-1$

$|x|<1$ এর জন্য

এখন, $F(s)=\int_{-\infty}^{-1} f(x)\,e^{isx}dx+\int_{1}^{\infty} f(x)\,e^{isx}dx+\int_{-1}^{1} f(x)e^{isx}dx$

~~এখন~~ এখানে যে range এ $f(x)$ এর value তা বসিয়ে দিই

$$=\int_{-\infty}^{-1} 0\, e^{isx}dx+\int_{1}^{\infty} 0\cdot e^{isx}dx+\int_{-1}^{1} 1\, e^{isx}dx$$

$$=\int_{-1}^{1} e^{isx}dx$$

$$=\left[\frac{e^{isx}}{is}\right]_0^1$$

$$=\frac{e^{is}-e^{-is}}{is}$$

Now,

$$2 \times \frac{e^{is} - e^{-is}}{2is}$$

$$= 2 \times \frac{\sin s}{s}$$

$$\left[\sin x = \frac{e^{ix} - e^{ix}}{2i} \right]$$

$$= \frac{2 \sin s}{s}$$

$$\therefore F(s) = \frac{2 \sin s}{s}.$$

ii) যখনই কোনো কিছু evaluate করতে বলবে তখনই আমাদের inverse fourier transform করতে হবে

Now by inverse fourier transform,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} e^{-isx} ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-isx} ds$$

আমাদের evaluate করতে হবে $\int_{0}^{\infty} \frac{\sin x}{x} dx$

তাহলে উপরের লাইনে e^{-isx} term টা ~~zero~~ 1 করতে হবে।

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Putting $x=0$

$$f(x) = \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin s}{s}\, ds$$

আমরা x এর value zero বসিয়েছি।

$$\therefore f(0) = \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin s}{s}\, ds$$

[এখন, Ques এ দেখি,

$f(x) = 1 \quad |x|<1 \longrightarrow$ একে লেখা যায় $\underbrace{-1<x<1}_{\text{এই range এই zero আছে}}$.

$\therefore f(0) = 1$]

Now, $1 = \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin s}{s}\, ds$

$\therefore \pi = \int_{-\alpha}^{\alpha} \frac{\sin s}{s}\, ds$. → Pause

$[\, f(-s) = \frac{\sin(-s)}{(-s)} = \frac{\sin s}{s} = f(s)$

$\therefore \frac{\sin s}{s}$ হল even function]

$\therefore \pi = 2\times\int_{0}^{\alpha} \frac{\sin s}{s}\, ds$

wrong step.

$\therefore \int_{0}^{\alpha} \frac{\sin s}{s} = \frac{\pi}{2}$

এভাবেও answer চলে আসে
but this is wrong আমরা $\int_{-\alpha}^{\alpha}$ range থেকে দুই করে $\int_0^{\alpha}$ আনতে পারব না

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now, after pause line

$$= \lim_{a\to\infty} \int_{-a}^{a} \frac{\sin s}{s} ds$$

$$\Rightarrow \pi = \cancel{\lim_{a\to\infty}} \lim_{a\to\infty} 2\int_0^a \frac{\sin s\, ds}{s}$$

$$\Rightarrow \frac{\pi}{2} = \int_0^\infty \frac{\sin s\, ds}{s}$$

$$\therefore \int_0^\infty \frac{\sin s\, ds}{s} = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin x\, dx}{x} = \frac{\pi}{2}$$

(Ans)

Ques. 02.

Find the fourier transform of

$$f(x) = \begin{cases} 1-x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

আগের কাজ,

mod থেকে বের করি,

$|x| \leq 1$ $\pm$ (+) দিয়ে $x \leq 1$

(−) " $-x \leq 1$

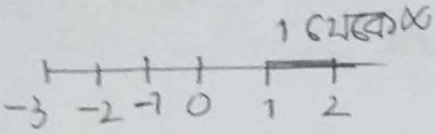
$\therefore x \geq -1$

$\therefore -1 \leq x \leq 1.$

$|x| > 1$

$\therefore$ (+) নিয়ে,

$x > 1$

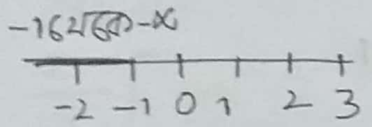


$\therefore \quad 1 < x < \infty$

(−) নিয়ে,

$-x > 1$

$\therefore x \not< -1$



$\therefore \quad -\infty < x < -1.$

$$\therefore \quad F(s) = \int_{-\infty}^{-1} f(x)\, e^{isx} dx + \int_{1}^{\infty} f(x)\, e^{isx} dx + \int_{-1}^{1} f(x)\, e^{isx} dx$$

$$= \int_{-\infty}^{-1} 0 \cdot e^{isx} dx + \int_{1}^{\infty} f(x)\, e^{isx} dx + \int_{-1}^{1} f(x)\, e^{isx} dx$$

$$= \int_{-1}^{1} f(x)\, e^{isx} dx$$

$$= \int_{-1}^{1} (1-x^2)\, e^{isx} dx.$$

$$= (1-x^2)\int_{-1}^{1} e^{isx}\,dx - \int_{-1}^{1}\left\{\frac{d}{dx}(1-x^2)\int e^{isx}dx\right\}dx$$

$$= \left[(1-x^2)\,\frac{e^{isx}}{is}\right]_{-1}^{1} - \int_{-1}^{1}(-2x)\cdot\frac{e^{isx}}{is}\,dx$$

$$= 0 + \frac{2}{is}\int_{-1}^{1} x\,e^{isx}\,dx$$

$$= \frac{2}{is}\left[x\int_{-1}^{1} e^{isx}\,dx - \int_{-1}^{1}\left\{\frac{dx}{dx}\int e^{isx}dx\right\}dx\right]$$

$$= \frac{2}{is}\left[\left[\frac{x\cdot e^{isx}}{is}\right]_{-1}^{1} - \left[\frac{e^{is}}{i^2s^2}\right]_{-1}^{1}\right]$$

$$= \frac{2}{is}\left[\frac{e^{is}}{is} + \frac{e^{is}}{is}\right] - \frac{2}{is}\left[\frac{e^{is}}{i^2s^2} - \frac{e^{-is}}{i^2s^2}\right]$$

$$= \frac{2}{is}\left[\frac{e^{is}+e^{-is}}{is}\right] - \frac{2}{is}\left[\frac{e^{is}-e^{-is}}{i^2s^2}\right]$$

$$= 2\times\left[\frac{e^{is}+e^{-is}}{i^2s^2}\right] - \frac{2}{is}\left[\frac{e^{is}-e^{-is}}{(-1)\,s^2}\right]$$

$$= -2\left[\frac{e^{is}+e^{-is}}{s^2}\right] + 2\left[\frac{e^{is}-e^{-is}}{is^3}\right]$$

$$= -4\left[\frac{e^{is}+e^{-is}}{2s^2}\right] + 4\left[\frac{e^{is}-e^{-is}}{2is^3}\right]$$

$$= -4\,\frac{\cos s}{s^2} + 4\,\frac{\sin s}{s^3} = \frac{-4}{s^3}\left[s\cos s - \sin s\right]$$

(Ans)

Fourier Sine transform

The function $F_s(s)$; defined by

$$F_s(s) = \int_0^\infty f(x) \sin sx \, dx$$

is called fourier sine transform of $f(x)$ in $0<x<\infty$

[মানে আমাদের কাছে কোনো function $f(x)$ থাকলে তার sine transform হবে $F_s(s) = \int_0^\infty f(x) \sin sx \, dx$]

Also the function $f(x)$, defined by

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds$$

is called inverse fourier sine transform of $F_s(s)$.

[$F_s(s)$ জানা থাকলে সেই ফাংশনটি ($f(x)$) নির্ণয় করা যায় এই formula apply করে, $f(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds$]

Problem 1:

Find fourier sine transform of $e^{-|x|}$. Hence show that

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2} \; ; m>0$$

solve: $f(x) = e^{-|x|}$

We know, $F_s(s) = \int_0^\infty f(x) \sin sx \, dx$

[0 থেকে ∞ interval এ $|x|$ অবশ্যই $+x$ ই হবে]

$$\therefore f(x) = e^{-|x|} = e^{-x}$$

$$\therefore F_s(s) = \int_0^{\infty} e^{-x} \sin sx \, dx$$

$$\left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right]$$

$$= \left| \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right|_0^{\infty}$$

$$= \left(0 - \frac{1}{1+s^2} (-0 - s) \right) \quad \left[\begin{array}{l} e^{\infty} \to \text{large} \\ e^{-\infty} = \frac{1}{e^{\infty}} \approx \frac{1}{\infty} = 0 \end{array} \right]$$

$$= \frac{s}{1+s^2}$$

[আমরা জানি, যখনই প্রশ্নে কোনো কিছু evaluate করতে বলবে বা show করতে বলবে তখন আমাদের inverse method এ আগানো লাগবে]

Now by inverse fourier sine transform,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{s}{1+s^2} \cdot \sin sx \, ds$$

changing s to m [কারণ question এ m আছে]

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2} \sin sm \, ds$$

$$\frac{\pi \cdot e^{-m}}{2} = \int_0^\infty \frac{s}{1+s^2} \cdot \sin sm \, ds$$

[প্রশ্নে আছে $\int_0^\infty \frac{x}{1+x^2} \sin mx \, dx = \frac{\pi e^{-m}}{2}$ দেখাও]

তাহলে s এর জায়গায় x বসালেই হচ্ছে]

$$\frac{\pi e^{-m}}{2} = \int_0^\infty \frac{x \sin mx}{1+x^2} \, dx$$

(showed)

#Example 2 Find fourier sine transform of $\frac{e^{-ax}}{x}$.

solve: Here, $f(x) = \frac{e^{-ax}}{x}$

$$F_s(s) = \int_0^\infty f(x) \cdot \sin sx \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin sx \, dx \quad \ldots \ldots \ldots (i)$$

[এই $\int_0^\infty e^{-ax} \sin sx \, dx$ এর formula আমরা জানি। কিন্তু এখানে extra x দ্বারা ভাগ আছে। এমন situation এ আমরা $F_s(s)$ কে differentiate করব s দ্বারা এর সাপেক্ষে]

$\therefore$ Differentiating eqn (i) w.r.to s

$$\frac{d}{ds}\left[F_s(s)\right] = \int_0^{\infty} \frac{e^{-ax}}{x} \cdot x \cdot \cos sx \, dx$$

$$= \int_0^{\infty} e^{-ax} \cdot \cos sx \, dx$$

এর formula আমাদের জানা

$$= \left| \frac{e^{-ax}}{a^2+s^2} \cdot \{-a\cos sx + s\sin sx\} \right|_0^{\infty}$$

$$= 0 - \left[\frac{1}{a^2+s^2}\{-a\}\right.$$

$$\frac{d}{ds}\left[F_s(s)\right] = \frac{a}{a^2+s^2}$$

Now, integrating with respect to s

$$F_s(s) = a \cdot \frac{1}{a} \tan^{-1}\frac{s}{a} \quad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \tan^{-1}\frac{s}{a}.$$

$$\therefore F_s(s) = \tan^{-1}\frac{s}{a} + c$$

(Ans)

Fourier Cosine Transform

The function $F_c(s)$ is defined by

$$F_c(s) = \int_0^\infty f(x) \cos sx \, dx$$

is called fourier cosine transform of $f(x)$ in $0 < x < \infty$

[কোনো একটি function $f(x)$ আমাদের জানা থাকলে তার fourier cosine transform হবে $F_c(s) = \int_0^\infty f(x) \cos sx \, dx$]

Also the function $f(x)$, defined by

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(s) \cos sx \, ds$$

is called inverse fourier cosine transform of $F_c(s)$

Problem 1: Find the fourier cosine transform of e^{-x^2}

$$\therefore F_c(s) = \int_0^\infty f(x) \cos sx \, dx$$

Let, $F_c(s) = I$

$$\therefore I = \int_0^\infty e^{-x^2} \cos sx \, dx \quad \ldots\ldots\ldots (1)$$

[খেয়াল করি, আমরা $\int e^{-ax} . \cos sx \, dx \, dx$ এর সূত্র জানি, কিন্তু $\int e^{-x^2} . \cos sx \, dx$ আছে এখানে। যখন আমাদের কোনো নির্দিষ্ট সূত্রের মধ্যে পড়বেনা তখন আমরা differentiate করে নিব আগে]

Differentiating eqn (i) w.r.to s

$$\frac{dI}{ds} = \int_0^\infty e^{-x^2}\cdot(-\sin sx)\cdot x\,dx$$

$$= -\int_0^\infty xe^{-x^2}\sin sx\,dx$$

$$= -\left[\sin sx\int_0^\infty xe^{-x^2}dx - \int_0^\infty\left\{\frac{d}{dx}\sin sx\int xe^{-x^2}dx\right\}\right]$$

Now

$$\int xe^{-x^2}dx$$

Let, $-x^2 = t$

$\therefore dt = -2x\,dx$

$$dx = \frac{dt}{-2x}$$

$$x\,dx = \frac{dt}{2}$$

$$= \int e^t\left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{2}e^t$$

$$= -\frac{1}{2}e^{-x^2}$$

$$= -\left[\sin sx\left(-\frac{1}{2}e^{-x^2}\right)\right]_0^\infty + \int_0^\infty s\cos sx\left(-\frac{1}{2}e^{-x^2}\right)dx$$

$$= \left[\sin sx\left(\frac{1}{2}e^{-x^2}\right)\right]_0^\infty - \int_0^\infty s\cos sx\cdot\frac{1}{2}e^{-x^2}dx$$

$$= 0 - \int_0^\infty s\cos sx\cdot\frac{1}{2}e^{-x^2}dx$$

$$= -\frac{s}{2}\int_0^\infty e^{-x^2}\cos sx\,dx$$

$$= -\frac{s}{2}I$$

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$$\therefore \frac{dI}{I} = -\frac{s}{2}\, ds$$

by integrating

$$\ln I = -\frac{s^2}{4} + c$$

$$\therefore I = e^{-\frac{s^2}{4} + c}$$

$$= e^{-s^2/4} \cdot e^c$$

$$= e^{-s^2/4} \cdot A$$

একটা definite value তে আনা লাগবে

Putting $s=0$ (কারণ আমাদের c টিকে সরাতে হবে)

$$I = A$$

যদি $s=0$ বসায় তাহলে আমাদের equation এও change আসবে

$$I = \int_0^\infty e^{-x^2} \cdot \cos 0 \, dx = \int_0^\infty e^{-x^2} dx$$

Let, $x^2 = t$

$\therefore 2x\,dx = dt$

$\therefore dx = \frac{dt}{2x}$

$= \frac{dt}{2\sqrt{t}}$

when $x = \infty$, $t = \infty$

$x = 0$, $t = 0$

$$\therefore I = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2} \int_0^\infty e^{-t} \cdot t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{2}\sqrt{\frac{1}{2}}$$

$$\left[\because \sqrt{n} = \int_0^\infty e^{-x}.x^{n-1}dx\right.$$

এখানে আছে, $\sqrt{\frac{1}{2}} = \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt\Big]$

$$= \frac{1}{2}.\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2} = A$$

$$\therefore I = e^{-s^2/4}.A$$

$$= e^{-s^2/4}.\frac{\sqrt{\pi}}{2} \quad \text{(Ans)}$$

\# Problem 2: Find the fourier cosine transform of

$$f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \\ 0 & x>2 \end{cases}$$

Solve: We know, $F_c(s) = \int_0^\infty f(x)\cos sx\,dx$

$$= \int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx + \int_2^\infty 0.\cos sx\,dx$$

$$= \left[x\int_0^1 \cos sx\,dx - \int_0^1 \frac{\sin sx}{s}dx\right] + \int_1^2 (2-x)\cos sx\,dx$$

$$= \left[\frac{x\sin sx}{s}\right]_0^1 + \left[\frac{\cos sx}{s^2}\right]_0^1 + \int_1^2 (2\cos sx\,dx - x\cos sx\,dx)$$

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$$= \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} + \int_1^2 2\cos sx\,dx - \int_1^2 x\cos sx\,dx$$

$$= \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} + \left[\frac{2\sin sx}{s}\right]_1^2 - \left[\frac{x\sin sx}{s}\right]_1^2 - \left[\frac{\cos sx}{s^2}\right]_1^2$$

$$= \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} + \frac{2\sin 2s}{s} - \frac{2\sin s}{s} - \frac{2\sin 2s}{s} + \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}$$

$$= -\frac{\cos 2s}{s^2} + \frac{2\cos s}{s^2} - \frac{1}{s^2}$$

$$= -\frac{1}{s^2}(\cos 2s - 2\cos s + 1)$$

* এখন দুইটা formula এর derivation শিখব আমরা

$$\text{i)}\quad F_s[x \cdot f(x)] = -\frac{d}{ds}\{F_c(s)\}$$

কোনো function এর আগে x গুন থাকলে

$$\text{ii)}\quad F_c[x \cdot f(x)] = \frac{d}{ds}\{F_s(s)\}$$

i) We have, $F_c(s) = \int_0^\infty f(x)\cos sx\,dx$

$$\frac{d}{ds}[F_c(s)] = \int_0^\infty f(x) \cdot x(-\sin sx)\,dx$$

$$= -\int_0^\infty f(x) \cdot x \cdot \sin sx\,dx$$

$$\therefore \frac{d}{ds} F_c(s) = -F_s[x \cdot f(x)]$$

$$\therefore F_s[x \cdot f(x)] = -\frac{d}{ds} F_c(s)$$

(Proved)

ii) $F_s(s) = \int_0^\infty f(x) \sin sx \, dx$

$$\therefore \frac{d}{ds} F_s(s) = \int_0^\infty f(x) \cdot x \cos sx \, dx$$

$$= F_c[x \cdot f(x)]$$

$$\therefore F_c(x f(x)) = \frac{d}{ds} F_s(s)$$

(Proved)

Problem 7: Find the fourier sine and cosine transform of $x \cdot e^{-ax}$.

i) $F_s(x \cdot f(x)) = -\frac{d}{ds} F_c(s)$

$f(x)$ functino টির fourier cosine transform হবে $F_c(s)$

তার মানে x গুন আকারে আছে এমন কোনো function $f(x)$ হলে আমরা তার fourier sine transform করার সময় উপরের formula টি apply করতে পারব

এখন,

আমাদের ques হল: $x \cdot e^{-ax}$

এখানে $x \cdot f(x)$ এর সাথে compare করলে পাই, $f(x) = e^{-ax}$

তাহলে, এখন, e^{-ax} এর আগে fourier cosine transform করতে হবে,

$$\therefore F_c(s) = \int_0^{\infty} e^{-ax} \cdot \cos sx \, dx$$

$$= \left| \frac{e^{-ax}}{a^2+s^2} (-a\cos sx + s\sin sx) \right|_0^{\infty}$$

$$= 0 - \frac{1}{a^2+s^2}(-a)$$

$$= \frac{a}{a^2+s^2}$$

এখন, $F_s\left[x \cdot e^{-ax}\right] = -\frac{d}{ds} F_c(s)$

$$\therefore \frac{d}{ds}\left\{\frac{a}{a^2+s^2}\right\}$$

$$= \frac{0 - 2s \cdot a}{(a^2+s^2)^2}$$

$$= \frac{-2as}{(a^2+s^2)^2}$$

$$\therefore -\frac{d}{ds} F_c(s) = \frac{2as}{(a^2+s^2)^2}$$

$$\therefore F_s(x \cdot e^{-ax}) = \frac{2as}{(a^2+s^2)^2}.$$

Now, $F_c\{x \cdot e^{-ax}\} = \frac{d}{ds}\{F_s\{e^{-ax}\}\}$

তাহলে, আগে $F_s(e^{-ax})$ বের করি

$$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \cdot \sin sx \, dx$$

$$= \left| \frac{e^{-ax}}{a^2+s^2}(-a \sin sx \mp s \cos sx) \right|_0^\infty$$

$$= \left[0 - \frac{(-s)}{a^2+s^2}\right]$$

$$= \frac{s}{a^2+s^2}$$

$$\therefore \frac{d}{ds} F_s(e^{-ax})$$

$$= \frac{d}{ds}\left\{\frac{s}{a^2+s^2}\right\}$$

$$= \frac{(a^2+s^2) - 2s^2}{(a^2+s^2)^2}$$

$$= \frac{a^2-s^2}{(a^2+s^2)^2}$$

$$\therefore F_c\{x \cdot e^{-ax}\} = \frac{a^2-s^2}{(a^2+s^2)^2}$$

= Problem: Find fourier sine and cosine transform of

$x^{n-1}, \underbrace{n>0}$

এইটুকু দ্বারাই বুঝা যাচ্ছে limit 0 থেকে ∞

[এই math টি কিছুটা exceptional. আমরা process মনে রাখবো]

Here, $f(x) = x^{n-1}$

$$\therefore F_s(x^{n-1}) = \int_0^\infty x^{n-1} \sin sx \, dx$$

Also, $$F_c(x^{n-1}) = \int_0^\infty x^{n-1} \cdot \cos sx dx$$

Now,

$$F_c(x^{n-1}) + iF_s(x^{n-1}) \dots\dots\dots (i)$$

$$= \int_0^\infty x^{n-1} \cos sx dx + i\int_0^\infty x^{n-1} \cdot \sin sx dx$$

$$= \int_0^\infty x^{n-1} (\cos sx + i\sin sx) dx$$

$$= \int_0^\infty x^{n-1} e^{isx} dx \quad \left[\because e^{ix} = \cos x + i\sin x. \quad \therefore e^{isx} = \cos sx + i\sin sx\right]$$

$\Rightarrow$ Let, $isx = -dt$

$$\therefore dx = -\frac{dt}{is}$$

$$= \int_0^\infty \left(\frac{-t}{is}\right)^{n-1} \cdot e^{-t} \left(-\frac{dt}{is}\right)$$

$$= \int_0^\infty \left(\frac{-1}{is}\right)^{n-1} \cdot t^{n-1} \cdot e^{-t} \cdot \left(-\frac{dt}{is}\right)$$

$$= \int_0^\alpha \frac{(-1)^n}{(is)^n} t^{n-1} e^{-t} dt$$

$$= \int_0^\alpha \left(\frac{-1}{is}\right)^n t^{n-1} e^{-t} dt$$

$$= \left(\frac{-1}{is}\right)^n \int_0^\alpha e^{-t} \cdot t^{n-1} dt$$

$$= \left(-\frac{1}{is}\right)^n \overline{|n}$$

$$= \left(\frac{i^2}{is}\right)^n \overline{|n}$$

$$= \left(\frac{i}{s}\right)^n \overline{|n}$$

$$= \frac{i^n}{s^n} \overline{|n}$$

$$= \frac{(0+i\cdot 1)^n}{s^n} \overline{|n}$$

$$= \frac{\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n}{s^n} \overline{|n}$$

$$= \frac{\cos^n\frac{\pi}{2} + i\sin^n\frac{\pi}{2}}{s^n} \overline{|n}$$

পরীক্ষায় আসলেও হুবহু এই math ই আসবে, তাই এই Process মনে রাখবে

Now, comparing real and imaginary part with eqn (1)

$$F_c(x^{n-1}) = \cos\frac{n\pi}{2}\frac{\Gamma n}{s^n}$$

$$F_s(x^{n-1}) = \sin\frac{n\pi}{2}\frac{\Gamma n}{s^n}$$

Finite Fourier Sine & Cosine transform

এতক্ষন আমরা limit দেখছিলাম 0 থেকে ∞ কিন্তু এখন আর infinity হবে না। কোনো এক নির্দিষ্ট range এর জন্য আমরা fourier transform করব।

The finite fourier sine transform of $f(x)$ in $0<x<c$ is defined by

কোনো নির্দিষ্ট value; ∞ না

$$F_s(n) = \int_0^c f(x)\cdot \sin\frac{n\pi x}{c}\,dx$$

integer

The function $f(x)$ [inverse finite fourier sine transform of $F_s(n)$] is given by

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\cdot\sin\frac{n\pi x}{c}$$

The finite fourier cosine transform of $f(x)$ in $0<x<c$ is given by

$$F_c(n) = \int_0^c f(x) \cos\frac{n\pi x}{c}\, dx$$

The function $f(x)$ [inverse finite fourier cosine transform of $F_c(n)$] is given by

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos\frac{n\pi x}{c}.$$

Problem: If fourier sine transform of $f(x)$ is $\frac{1-\cos n\pi}{n^2\pi^2}$ $(0 \le x \le \pi)$; find inverse finite fourier sine transform.

solve: $0<x<c$ এর সাথে $0 \le x \le \pi$ compare করলে পাই, $c=\pi$

Here, $F_s(n) = \frac{1-\cos n\pi}{n^2\pi^2}$

$c=\pi$

$$\therefore f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \cdot \sin\frac{n\pi x}{c}$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^2\pi^2} \cdot \sin nx$$

(Ans)

Properties of fourier Transform

* Property এর proof আসবে exam এ

1) Linear Property

If $F(s)$ and $G(s)$ are fourier transform of $f(x)$ and $g(x)$ respectively,

$$\therefore F\{af(x) + b\,g(x)\} = aF(s) + b\cdot G(s)$$

$a, b \rightarrow$ constant

Proof:

We know,

$$F(s) = \int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx \quad \& \quad G(s) = \int_{-\infty}^{\infty} g(x)\, e^{isx}\,dx$$

Now, L.H.S. $= F\{af(x) + b\,g(x)\}$

$$= \int_{-\infty}^{\infty} \{af(x) + b\,g(x)\} e^{isx}\,dx$$

$$= a\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx + b\int_{-\infty}^{\infty} g(x)\cdot e^{isx}\,dx$$

$$= aF(s) + b\,G(s)$$

(Proved)

Shifting Property

If $F(s)$ is the complex fourier transform of $f(x)$,

then,

$$F\{f(x-a)\} = e^{isa}F(s)$$

Proof:

We know, $F\{f(x)\} = F(s) = \int_{-\alpha}^{\alpha} f(x)\, e^{isx}\, dx$

$\therefore F\{f(x-a)\} = \int_{-\alpha}^{\alpha} f(x-a)\, e^{isx} dx$

Let, $x-a=t$

$\therefore x = t+a$

$dx = dt$

$$= \int_{-\alpha}^{\alpha} f(t)\, e^{is(t+a)}\, dt$$

$$= \int_{-\alpha}^{\alpha} f(t) \cdot e^{ist} \cdot e^{isa}\, dt$$

$$= e^{isa} \int_{-\alpha}^{\alpha} f(t)\, e^{ist}\, dt$$

$$= e^{isa} \cdot F(s)$$

(Proved)

Change of scale property

If $F(s)$ is the complex fourier transform of $f(x)$

then, $F\{f(ax)\} = \frac{1}{a} F(\frac{s}{a}) \quad a \neq 0$

Proof: $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$\therefore F\{f(ax)\} = \int_{-\infty}^{\infty} f(ax) \cdot e^{isx}\, dx$

put, $ax = t$

$\therefore x = \frac{t}{a}$

$\therefore dx = \frac{dt}{a}$

$\therefore \int_{-\infty}^{\infty} f(t) \cdot e^{is \cdot \frac{t}{a}} \frac{dt}{a}$

$= \frac{1}{a} \int_{-\infty}^{\infty} f(t) \cdot e^{i(s/a)t}\, dt$

$= \frac{1}{a} F(s/a)$

(Proved)

$F_s\{f(ax)\} = \frac{1}{a} F_s(s/a)$

$F_c\{f(ax)\} = \frac{1}{a} F_c(s/a)$

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Modulation Theorem

If $F(s)$ is the complex fourier transform of $f(x)$

then,

$$F\{f(x)\cdot\cos ax\} = \int_{-\infty}^{\infty} f(x)\cdot\cos ax\cdot e^{isx}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\cdot\left(\frac{e^{iax}+e^{-iax}}{2}\right) e^{isx}\,dx$$

$$\left[\begin{aligned} e^{iax} &= \cos ax + i\sin ax \\ e^{-iax} &= \cos ax - i\sin ax \\ \therefore e^{iax} + e^{-iax} &= 2\cos ax \\ \therefore \cos ax &= \frac{e^{iax}+e^{-iax}}{2} \end{aligned}\right]$$

$$= \int_{-\infty}^{\infty} \frac{f(x)\cdot e^{i(s+a)x}\,dx}{2} + \int_{-\infty}^{\infty} \frac{f(x)\cdot e^{i(s-a)x}\,dx}{2}$$

$$= \frac{1}{2}\left[F(s+a)\right] + \frac{1}{2}F\left[(s-a)\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

$$* \; F_s\{f(x)\cdot\cos ax\} = \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$$

$$* \; F_{\cancel{s}c}\{f(x)\cdot\sin ax\} = \frac{1}{2}\left[F_s(s+a) - F_s(s-a)\right]$$

$$* \; F_s\{f(x)\cdot\sin ax\} = \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$$

Convolution theorem for Fourier Transform

The convolution theorem of two functions $f(x)$ and $g(x)$ over the interval $[-\infty, \infty]$ is given by $f(x) * g(x) =$

$$\int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du = h(x)$$

The fourier transform of convolution of $f(x)$ and $g(x)$ is the product of their fourier transforms i.e.

$$F[f(x) * g(x)] = F\{f(x)\} \cdot F\{g(x)\}$$

Proof: We have $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$\therefore F[f(x) * g(x)] = F\left[\int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du\right\} \cdot e^{isx}\, dx$$

Changing the order of integration

$$= \int_{-\infty}^{\infty} f(u) \underbrace{\left[\int_{-\infty}^{\infty} g(x-u) \cdot e^{isx}\, dx\right]}_{x \text{ related term গুলো এক করলাম}} du$$

Let $x - u = t$

$\therefore dx = dt$

$\therefore x = t + u$

$$= \int_{-\alpha}^{\alpha} f(u) \cdot \left[\int_{-\alpha}^{\alpha} g(t) \cdot e^{is(t+u)} dt \right] du$$

$$= \int_{-\alpha}^{\alpha} f(u) \cdot \left[\int_{-\alpha}^{\alpha} g(t) \cdot e^{ist} \cdot e^{isu} dt \right] du$$

$$= \int_{-\alpha}^{\alpha} f(u) \cdot \cancel{g}\, e^{isu} du \int_{-\alpha}^{\alpha} g(t) \cdot e^{ist} dt$$

[definite integral এর Property use করে]

$$= F\{f(x)\} \cdot F\{g(x)\}$$

(Proved)

Parseaval's identity for fourier transform

If fourier transform of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively

i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot \bar{G}(s)\, ds = \int_{-\infty}^{\infty} f(x)\cdot \bar{g}(x)\, dx$$

complex conjugate

ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} [f(x)]^2\, dx$$

Proof:

We know inverse fourier transform of $F(s)$ is given by.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

i) $$\text{R.H.S} = \int_{-\infty}^{\infty} f(x)\cdot \bar{g}(x)\, dx$$

e^{-isx} না হয়ে e^{isx} হল, কারণ এখানে complex conjugate নিচ্ছি

$$= \int_{-\infty}^{\infty} f(x)\cdot \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\, .e^{isx}\, ds\right] dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\left[\int_{-\infty}^{\infty} f(x)\, .e^{isx}\, dx\right] ds$$ [changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\cdot F(s)\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot \bar{G}(s)\, ds$$

$= \text{L.H.S}$ (Proved)

Parseval's identities for fourier cosine and sine transform

i) $\frac{2}{\pi}\int_0^\infty F_c(s)\cdot G_c(s)\,ds = \int_0^\infty f(x)\cdot g(x)\,dx$

ii) $\frac{2}{\pi}\int_0^\infty [F_c(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx$

iii) $\frac{2}{\pi}\int_0^\infty F_s(s)\cdot G_s(s)\,ds = \int_0^\infty f(x)\cdot g(x)\,dx$

iv) $\frac{2}{\pi}\int_0^\infty [F_s(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx$

কিছু গুরুত্বপূর্ণ formula
Parseval's identities
for fourier transform
solve করার

i) $f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$

$\therefore F_c(s) = \int_0^\infty f(x)\cdot \cos sx\,dx$

কি বুঝে
0 থেকে ∞ limit মানে fourier
transform এর মত $-\infty$ to ∞
কেন নিলাম না। কারণ: প্রশ্নে দেখি
১ম লিমিট 0 থেকে a আর ২য়
limit a থেকে ∞. তাই
$0 < x < \infty$ নিলাম limit.

$$= \int_0^a 1 \cos sx\,dx + \int_a^\infty 0 \cdot \cos sx\,dx$$

$$= \left[\frac{\sin sx}{s}\right]_0^a$$

$$= \frac{\sin as}{s}$$

$$F_s(s) = \int_0^\infty f(x) \cdot \sin sx\,dx$$

$$= \int_0^a 1 \cdot \sin sx\,dx + \int_a^\infty 0 \cdot \sin sx\,dx$$

$$= \left[-\frac{\cos sx}{s}\right]_0^a$$

$$= -\left[\frac{\cos as}{s} - \frac{1}{s}\right]$$

$$= \frac{1}{s}\left[1 - \cos as\right]$$

ii) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$ এখানে $a = 1$

এখন, $F_c(s) = \int_0^\infty f(x) \cdot \cos sx$

$$= \int_0^1 1 \cos sx\,ds + \int_1^\infty 0 \cdot \cos sx\,dx$$

$$= \left[\frac{\sin sx}{s}\right]_0^1$$

$$= \left[\frac{\sin s}{s}\right]$$ একইভাবে, $F_s(s) = \frac{1 - \cos s}{s}$

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iii) $f(x) = e^{-ax}$

$$F_c(s) = \frac{a}{s^2+a^2}$$

$$F_s(s) = \frac{s}{s^2+a^2}$$

iv) $f(x) = \frac{1}{1+x^2}$

$$\therefore F_c(s) = \frac{\pi}{2} e^{-s}$$

$F_s(s) = \frac{\pi}{2} e^{-s} \longrightarrow$ যখন $f(x) = \frac{x}{1+x^2}$ তখন

[এই 4 টা formation মনে রাখব আমরা। math এ আমরা কিভাবে apply করব তা বুঝি সামনে]

Ques: Using Parseval's identity, Prove that

a) $$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

[আমরা question টা দেখি, এখানে এক side এ দেখি, $\frac{1}{(a^2+t^2)(b^2+t^2)}$ আর অন্য side এ $\frac{\pi}{2ab(a+b)}$. এখন প্রশ্ন হল উপরের 4 টার মধ্যে কোন formation টা use করা যায়? আমরা (iii) নং formation টা দেখি। $f(x) = e^{-ax}$ যখন তখন, $F_c(s) = \frac{a}{s^2+a^2}$, এখন আমরা যদি, $f(x) = e^{-ax}$ নিই ও $g(x) = e^{-bx}$ এই term টা নিই তাহলেই ques এর সাথে মিলে। হয়ে যাবে]

Let, $f(x) = e^{-ax}$ $\quad \therefore F_c(s) = \dfrac{a}{s^2+a^2}$

and $g(x) = e^{-bx}$ $\quad \therefore G_c(s) = \dfrac{b}{s^2+b^2}$

Now using parseval's identity for fourier cosine transform,

cosine transform ব্যবহার করে $F_c(s)$ ও $G_c(s)$ বের করেছি উপরের

$$\frac{2}{\pi}\int_0^{\infty} F_c(s)\cdot G_c(s)\,ds = \int_0^{\infty} f(x)\cdot g(x)\,dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\infty} \frac{a}{a^2+s^2}\cdot\frac{b}{b^2+s^2}\,ds = \int_0^{\infty} e^{-ax}\cdot e^{-bx}\,dx$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^{\infty} \frac{1}{(a^2+s^2)(b^2+s^2)}\,ds = \int_0^{\infty} e^{-(a+b)x}\,dx$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^{\infty} \frac{1}{(a^2+s^2)(b^2+s^2)}\,ds = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^{\infty}$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^{\infty} \frac{1}{(a^2+s^2)(s^2+b^2)}\,ds = +\left(\frac{1}{(a+b)}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{1}{(a^2+s^2)(b^2+s^2)}\,ds = \frac{\pi}{2ab(a+b)}$$

(Proved)

b) Prove $\int_0^\infty \frac{t^2}{(1+t^2)^2}\, dt = \frac{\pi}{4}$.

[$\frac{t^2}{(1+t^2)^2}$ এই টা ques এ আছে। আমরা 4টি formation এর (iv) নংটির 2nd part দেখি। $f(x) = \frac{x}{1+x^2}$. তাহলে $[f(x)]^2 = \frac{x^2}{1+x^2}$]

$\therefore$ Let, $f(x) = \frac{x}{1+x^2}$. $F_S(s) = \frac{\pi}{2} e^{-s}$

Now, using the parseval's identity for fourier sine transf

$$\frac{2}{\pi}\int_0^\infty [F_S(f(x))]^2\, ds = \int_0^\infty [f(x)]^2\, dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \left[\frac{\pi}{2}\cdot e^{-s}\right]^2 ds = \int_0^\infty \frac{x^2}{(1+x^2)^2}\, dx$$

$$\Rightarrow \frac{2}{\pi}\cdot\frac{\pi^2}{4}\int_0^\infty e^{-2s}\, ds = \int_0^\infty \left[\frac{x}{1+x^2}\right]^2 dx$$

$$\therefore \frac{2}{\pi}\cdot\frac{\pi^2}{4}\left[\frac{e^{-2s}}{-2}\right]_0^\infty = \int_0^\infty \left[\frac{x}{1+x^2}\right]^2 dx$$

$$\Rightarrow \frac{\pi}{2}\cdot\frac{1}{2} = \int_0^\infty \left[\frac{x}{(1+x^2)}\right]^2 dx$$

$$\therefore \int_0^\infty \frac{t^2}{(1+t^2)^2}\, dt = \frac{\pi}{4}$$

(Proved

Application of fourier transform to boundary value problems

প্রথমে আমরা mainly শিখব যে, কোনো একটি math দেখে কিভাবে approach হবে আমাদের, ques এ কোনো না কোনো hint থাকবে, ওটা দেখেই আমরা সামনে আগাব]

i) if $u(x,t)_{x=0}$ is then use infinite fourier sine transform

[$u(x,t)_{x=0}$ মানে হল $x=0$ যখন তখন u এর value দেওয়া থাকবে]

ii) if $u\left[\frac{\partial u}{\partial x}(x,t)\right]_{x=0}$ is then use infinite fourier cosine transform.

iii) if $-\infty < x < \infty$ is given, then use infinite fourier transform

iv) if $u(0,t)$ and $u(c,t)$ are given, then use finite fourier sine transform.

maximum limit

v) if $\left(\frac{\partial u}{\partial x}\right)_{x=0}$ and $\left(\frac{\partial u}{\partial x}\right)_{x=c}$ are given, then use finite fourier cosine transform.

Problem 1:

solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, $-\alpha < x < \alpha$; $t \geq 0$ with conditions

$u(x,0) = f(x)$

এই condition টা কই ছিল ?

$u(x,0)$
মানে t এর value zero হলে, $u(x,0) = f(x)$

আমাদের rules এর (iii) নং টি দেখি
$-\alpha < x < \alpha$ থাকলে infinite fourier transform apply করব

Solve: Here $u(x,t)$ and $-\alpha < x < \alpha$, so we apply fourier transform on both sides of $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$\therefore F\left(\frac{\partial u}{\partial t}\right) = \alpha^2 F\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\therefore \frac{d}{dt}\left[F(u)\right] = \alpha^2\left[-s^2 F(u)\right]$$

[formula: $F\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2 F(u)$]
মনে রাখব

Let, $F(u) = \bar{u}$ $\quad$ $\bar{u}(s,t)$ মানে $\bar{u}$ হল s ও t এর function একটা

$$\frac{d\bar{u}}{dt} = \alpha^2\left[-s^2 \bar{u}\right]$$

$$\Rightarrow \frac{d\bar{u}}{\bar{u}} = \alpha^2 - s^2 \cdot dt$$

Integrating both sides,

$$\log \bar{u} = -\alpha^2 s^2 t + \log c$$ ← $\log \bar{u}$ এর জন্য $\log c$ আসলো

$$\Rightarrow \log \bar{u} - \log c = -\alpha^2 s^2 t$$

$$\Rightarrow \log \frac{\bar{u}}{c} = \cancel{\alpha^2 t} - \alpha^2 s^2 t$$

$$\therefore \frac{\bar{u}}{c} = e^{-\alpha^2 s^2 t}$$

$$\therefore \bar{u} = c \cdot e^{-\alpha^2 s^2 t}$$

$$\therefore \bar{u}(s,t) = c \cdot e^{-\alpha^2 s^2 t} \quad \dots \quad \dots \dots (i)$$

Put, $t = 0$, $\bar{u}(s,0) = c \quad \dots \quad \dots \dots (ii)$

⟶ c এর value বের করার জন্য সামনে আগাই,

From fourier transform

$$F\{u(x,t)\} = F(s) = \int_{-\alpha}^{\alpha} f(x)\, e^{isx}\, dx$$

$$\therefore \bar{u} = \int_{-\alpha}^{\alpha} u(x,t)\, e^{isx}\, dx$$

$$\therefore F(u) = \bar{u}$$

Putting $t = 0$;

$$\bar{u}(s,0) = \int_{-\alpha}^{\alpha} u(x,0)\, e^{isx}\, dx$$

$$= \int_{-\alpha}^{\alpha} f(x)\, e^{isx}\, dx$$ [ques এ দেওয়া $u(x,0) = f(x)$]

$$\therefore \bar{u}(s,0) = F(s) \quad \dots \quad \dots (iii)$$ [fourier transform এর formula F(S) এর]

From (ii) and (iii)

$$c = F(s)$$

Now, from eqⁿ (i)

$$\bar{u}(s,t) = F(s) \cdot e^{-s^2\alpha^2 t}$$

From inversion theorem

$$u(x,t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} F(u) \cdot e^{-isx}\, dx$$

এইটা এখানে $f(x)$

আর এইটা $F(s)$ মানে / $F\{f(x)\}$ $F\{u(x,t)\}$

$$\therefore u(x,t) = \frac{1}{2\pi}\int \bar{u}(s,t)\, e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int F(s) \cdot e^{-s^2\alpha^2 t} \cdot e^{-isx}\, ds$$

(Ans)

Problem 2: solve the eqn $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $(x>0, 7>0)$

subjected to conditions.

i) $u=0$, when $x=0$, $t>0$ iii) $u(x,t)$ is bounded

ii) $u = \begin{cases} 1, & 0<x<1 \\ 0 & x \geq 1 \text{ when } t=0 \end{cases}$

[এখানে দেওয়া আছে $u=0$ যখন, $x=0$.তো আমরা আমাদের formula 1 এ দেখেছিলাম, $u(x,t)_{x=0}$ মানে যখন $x=0$ হবে তখন $u(x,t)$ আমাদের জানা থাকবে। এই ques এও আমার একই জিনিস হচ্ছে। তাহলে আমরা formula 1 use করব। মানে fourier sine transform use করব]

Given eqn is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$\because u(x,t)$ and $u(0,t)=0$ is given so we apply fourier sine transform on both sides,

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\therefore \frac{d}{dt}\left[F_s(u)\right] = s(u)_{x=0} - s^2 F_s(u)$$

[formula : $F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s(u)_{x=0} - s^2 F_s(u)$]

Let, $F_s(u) = \bar{u}_s$

$$\therefore \frac{d}{dt}[\bar{u}_s] = -s^2 \bar{u}_s \quad [\because (u)_{x=0} = 0]$$

$$\frac{d\bar{u}_s}{\bar{u}_s} = -s^2 dt$$

by integrating,

$$\log \bar{u}_s = -s^2 t + \log c$$

$$\Rightarrow \log \frac{\bar{u}_s}{c} = -s^2 t$$

$$\therefore \frac{\bar{u}_s}{c} = e^{-s^2 t}$$

$$\therefore \bar{u}_s = c \cdot e^{-s^2 t}$$

$$\therefore \bar{u}_s(s,t) = c \cdot e^{-s^2 t} \quad \ldots \ldots \ldots (i)$$

Put ; $t=0$

$$\bar{u}_s(s,0) = c \quad \ldots \ldots \ldots (ii)$$

From fourier sine transform we have,

$$F_s(s) = \int_0^\infty f(x) \sin sx \, dx$$

$$\bar{u}_s(s,t) = \int_0^\infty u(x,t) \sin sx \, dx$$

Put $\cancel{\beta}$ $t=0$

$$\bar{u}_s(s,0) = \int_0^\infty u(x,0) \sin sx \, dx$$

$$= \int_0^1 1 \cdot \sin sx \, dx + \int_1^\infty 0 \cdot \sin sx \, dx$$

[কারন প্রশ্নেই দেওয়া, যখন, $t=0$, তখন, $u = \begin{cases} 1 & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$]

$$= \left| -\frac{\cos sx}{s} \right|_0^1 = -\frac{1}{s}(\cos s - \cos 0) = \frac{1-\cos s}{s} \quad \ldots\ldots\ldots (iii)$$

from (ii) and (iii)

$$u_s\, c = \frac{1-\cos s}{c}$$

from (i)

$$\bar{u}_s(s,t) = \left(\frac{1-\cos s}{s}\right) e^{-s^2 t}$$

From inversion formula, we know,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\, \sin sx\, ds$$

এখানে, তোমার আমাদের, $f(x) = u(x,t)$
এবং $F_s(s) = \bar{u}(s,t)$

$$\therefore\ u(x,t) = \frac{2}{\pi}\int_0^\infty \bar{u}(s,t)\, \sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos s}{s}\right)\cdot e^{-s^2 t}\, \sin sx\, ds.$$

(Ans)

Problem 3:

Solve the eqn $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if

i) $\frac{\partial u}{\partial x}(0,t) = 0$ for $t > 0$

ii) $u(x,0) = \begin{cases} x & 0 \le x < 1 \\ 0 & x > 1 \end{cases}$

iii) $u(x,t)$ is bounded for $x > 0,\ t > 0$

[এখানে (i) নং condition দেখি, $\frac{\partial u}{\partial x}(0,t) = 0$, মানে x এর value যখন 0 তখন $\frac{\partial u}{\partial x} = 0$. এইটা আমাদের u টা formula এর 2 নং টার সাথে মিলে। তাহলে আমরা fourier cosine transform করব]

Solve: Given, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$u(x,t)$ and here $\frac{\partial u}{\partial x}(0,t) = 0$ is given

we apply fourier cosine transform on both sides

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\left[\text{formula } F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2 F_c(u) - \left(\frac{\partial u}{\partial x}\right)_{x=0}\right]$$

$$\therefore \frac{d}{dt}\left[F_c(u)\right] = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 F_c(u)$$

Let, $F_c(u) = \bar{u}_c$

$\therefore \frac{d}{dt}\left[\bar{u}_c\right] = 0 - s^2 \bar{u}_c$ [প্রশ্নে আছে $\frac{\partial u}{\partial x}(0,t) = 0$ মানে $x=0$ যখন তখন $\frac{\partial u}{\partial x} = 0$]

$$\therefore \frac{d\bar{u}_c}{\bar{u}_c} = -s^2 dt$$

Integrating both sides,

$$\log \bar{u}_c = -s^2 t + \log c$$

$$\Rightarrow \log \frac{\bar{u}_c}{c} = -s^2 t$$

$$\therefore \frac{\bar{u}_c}{c} = e^{-s^2 t}$$

$$\therefore \bar{u}_c = c \cdot e^{-s^2 t} \quad \ldots \ldots \text{(i)}$$

$$\bar{u}_c(s,t) = c \cdot e^{-s^2 t}$$

Put, $t = 0$;

$$\therefore \bar{u}_c(s,0) = c \quad \ldots \ldots \text{(ii)}$$

From fourier cosine transform,

$$F_c(s) = \int_0^\infty f(x) \cdot \cos sx\,dx$$

$$\therefore \bar{u}_c(s,t) = \int_0^\infty u(x,t) \cos sx\,dx.$$

Put $t = 0$;

$$\bar{u}_c(s,0) = \int_0^\infty u(x,0) \cos sx\,dx$$

$$= \int_0^1 x \cos sx\,dx + \int_1^\infty 0 \cdot \cos sx\,dx$$ [ques এর (ii) নং condition]

$$= \left| x \cdot \frac{\sin sx}{s} - \left(-\frac{\cos sx}{s^2}\right) \right|_0^1$$

$$= \left| x \frac{\sin sx}{s} + \frac{\cos sx}{s^2} \right|_0^1$$

$$= \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \quad \ldots \ldots \text{(iii)}$$

From (ii) and (iii)

$$c = \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}$$

$\therefore$ From eqn (i)

$$\bar{u}_c(s,t) = \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) e^{-s^2 t}$$

From inversion theorem,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\, ds$$

$$\therefore u(x,t) = \frac{2}{\pi}\int_0^\infty \bar{u}_c(s,t)\cos sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) e^{-s^2 t}\cos sx\, ds$$

(Ans)

#Problem 4: Using finite fourier transform solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ given $u(0,t) = 0$, $u(4,t) = 0$ and $u(x,0) = 2x$ where $0 < x < 4$, $t > 0$

[এখানে প্রশ্নের শুরুতে বলে দিয়েছে finite fourier transform use করতে হবে। তবে এখানে নির্দিষ্ট করে বলেনাই যে fourier cosine/ fourier sine transform use করবো। এটা আমরা বুঝবো condition গুলো থেকে।]

অথবা আমাদের (iv) নং formula টি যদি, $u(0,t)$, and $u(c,t)$ are given, then use finite fourier sine transform.

এখানে ques এ $c=4$ ধরলে আমাদের (iv) নং formular আগে ছিলো]

solve: Given eqn is: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$\because u(0,t)=0$ and $u(4,t)=0$ are given, so we apply finite fourier transform on both sides

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

[formula: finite fourier sine transform এর জন্য:

$$F_s\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{n^2\pi^2}{c^2} F_s(u) + \frac{n\pi}{c}\left[u(0,t) - (-1)^n u(c,t)\right]$$

$$\therefore \frac{d}{dt} F_s(u) = -\frac{n^2\pi^2}{4^2} F_s(u) + \frac{n\pi}{4}\left[u(0,t) - (-1)^n u(4,t)\right]$$

$\because c=4$

let, $F_s(u) = \bar{u}_s$

$$\therefore \frac{d}{dt}\bar{u}_s = -\frac{n^2\pi^2}{4^2}\bar{u}_s + 0 \cdot \left[\because \begin{array}{l} u(0,t)=0 \\ u(4,t)=0 \end{array}\right]$$

$$\therefore \frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2\pi^2}{4^2} dt$$

$$\therefore \log \bar{u}_s = -\frac{n^2\pi^2}{16} t + \log c$$

$$\Rightarrow \log \frac{\bar{u}_s}{c} = -\frac{n^2\pi^2 t}{16}$$

$$\therefore \bar{u}_s(n,t) = c \cdot e^{-\frac{n^2\pi^2 t}{16}} \quad \ldots \ldots \ldots (i)$$

finite তাই s এর জায়গায় n

Put, $t=0$

$$\bar{u}_s(n,0) = c \quad \ldots \ldots \ldots (ii)$$

From finite fourier sine transform we know,

$$F_s(n) = \int_0^c f(x) \cdot \sin\frac{n\pi x}{c}\, dx \quad [n \to \text{integer}]$$

$$\therefore \bar{u}_s(n,t) = \int_0^c u(x,t) \cdot \sin\frac{n\pi x}{c}\, dx$$

Put $t=0$

$$\bar{u}_s(n,0) = \int_0^4 u(x,0) \sin\frac{n\pi x}{4}\, dx$$

$$= \int_0^4 2x \cdot \sin\frac{n\pi x}{4}\, dx \quad [\because u(x,0) = 2x]$$

$$= \left| 2x\left(\frac{-\cos n\pi x/4}{n\pi/4}\right) - 2\left(-\frac{\sin n\pi x/4}{n^2\pi^2/16}\right)\right|_0^4$$

$$= \left| \frac{-8x}{n\pi} \cos\frac{n\pi x}{4} + \frac{32}{n^2\pi^2} \sin\frac{n\pi x}{4}\right|_0^4$$

$$= \left(\frac{-8\cdot 4}{n\pi} \cos n\pi + 0 + 0 - 0 \right)$$

$$= -\frac{32}{n\pi} (-1)^n$$

$$\therefore \bar{u}_s(n,0) = -\frac{32}{n\pi}(-1)^n \quad \ldots \ldots \ldots (iii)$$

From (ii) and (iii)

$$c = -\frac{32}{n\pi}(-1)^n$$

$\therefore$ From (i),

$$\bar{u}_s(n,t) = -\frac{32}{n\pi}(-1)^n \cdot e^{-\frac{n^2\pi^2}{16}t}$$

Now, taking inverse finite fourier sine transform

$$f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \cdot \sin \frac{n\pi x}{c}$$

$$\therefore u(x,t) = \frac{2}{4} \sum_{n=1}^{\infty} \frac{-32}{n\pi} (-1)^n e^{-\frac{n^2\pi^2 t}{16}} \cdot \sin \frac{n\pi x}{4}$$

$$= \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(-1)\cdot(-1)^n}{n} \cdot e^{-\frac{n^2\pi^2 t}{16}} \cdot \sin \frac{n\pi x}{4}$$

$$= \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n^2\pi^2 t}{16}} \cdot \sin \frac{n\pi x}{4}$$

(Ans)

Problem 5: solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0<x<6$, $t>0$; given

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(6,t) = 0, \quad u(x,0) = 2x$$

[এই Problem টি solve করব formula v) অনুযায়ী। কারণ $\frac{\partial u}{\partial x}(0,t) = 0$ ও $\frac{\partial u}{\partial x}(6,t) = 0$ দেওয়া. এখানে $c=6$.]

Solve: Given eqn is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ are given! so we apply finite fourier cosine transform on both sides.

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(\frac{\partial^2 u}{\partial x^2}\right)$$

[formula: $F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{n^2\pi^2}{c^2} F_c(u) + \frac{\partial u}{\partial x}(c,t)\cos n\pi - \frac{\partial u}{\partial x}(0,t)$]

$$\therefore \frac{d}{dt} F_c(u) = -\frac{n^2\pi^2}{6^2} F_c(u) + \frac{\partial u}{\partial x}(6,t)\cos 6\pi - \frac{\partial u}{\partial x}(0,t)$$

$\Rightarrow$ Let, $F_c(u) = \bar{u}_c$

$$\therefore \frac{d}{dt}\bar{u}_c = -\frac{n^2\pi^2}{36}\bar{u}_c \quad \left[\because \frac{\partial u}{\partial x}(6,t) = 0 \text{ and } \frac{\partial u}{\partial x}(0,t) = 0\right]$$

$$\therefore \frac{d\bar{u}_c}{\bar{u}_c} = -\frac{n^2\pi^2}{36}dt$$

on integrating

$$\log \bar{u}_c = -\frac{n^2\pi^2}{36}t + \log c$$

$$\Rightarrow \log \frac{\bar{u}_c}{c} = -\frac{n^2\pi^2}{36} t$$

$$\Rightarrow \bar{u}_c = c \cdot e^{-\frac{n^2\pi^2}{36} t} \quad \ldots \ldots \ldots (ii)$$

Put $t=0$,

$$\bar{u}_c(n,0) = c \ldots \ldots \ldots (ii)$$

Now from finite fourier cosine transform,

$$F_c(n) = \int_0^c f(x) \cos\frac{n\pi x}{c}\, dx$$

$$\therefore \bar{u}_c(n,t) = \int_0^6 u(x,t) \cdot \cos\frac{n\pi x}{6}\, dx$$

Put, $t=0$

$$\therefore \bar{u}_c(n,0) = \int_0^6 u(x,0) \cdot \cos\frac{n\pi x}{6}\, dx$$

$$= \int_0^6 2x \cdot \cos\frac{n\pi x}{6}\, dx$$

$$= \left| 2x \cdot \frac{\sin n\pi x/6}{n\pi/6} - (2)\left(\frac{-\cos n\pi x/6}{n^2\pi^2/36}\right) \right|_0^6$$

$$= \left| \frac{12x}{n\pi} \cdot \sin\frac{n\pi x}{6} + \frac{72}{n^2\pi^2} \cdot \cos\frac{n\pi x}{6} \right|_0^6$$

$$= 0 - 0 + \frac{72}{n^2\pi^2} \cdot \cos n\pi - \frac{72}{n^2\pi^2} \cos 0$$

$$= \frac{72}{n^2\pi^2}\left[(-1)^n - 1\right] \ldots \ldots \ldots (iii)$$

From (ii) and (iii),

$$c = \frac{72}{n^2\pi^2}\left[(-1)^n - 1\right]$$

From (i),

$$\bar{u}_c(n,t) = \frac{72}{n^2\pi^2}\left[(-1)^n - 1\right] \cdot e^{-\frac{n^2\pi^2}{36}t}$$

Now by inversion theorem

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n) \cdot \cos\frac{n\pi x}{c}$$

$$\therefore u(x,t) = \frac{1}{6} F_c(0) + \frac{2}{6}\sum_{n=1}^{\infty} \bar{u}_c \cos\frac{n\pi x}{6} \quad \ldots\ldots (iv)$$

এর value বের করতে হবে.

$$F_c(n) = \int_0^c f(x) \cdot \cos\frac{n\pi x}{c}\, dx$$

$$\therefore F_c(0) = \int_0^c f(x)\, dx$$

$$= \int_0^6 u(x,t)\, dx$$

Put, $t=0$

$$F_c(0) = \int_0^6 u(x,0)\, dx$$

$$= \int_0^6 2x\, dx \quad [\text{ques এ আছে } u(x,0) = 2x]$$

$$= [x^2]_0^6$$

$$= 36.$$

from (iv)

$$u(x,t) = \frac{36}{6} + \frac{1}{3}\sum_{n=1}^{\infty} \frac{72}{n^2\pi^2}\left[(-1)^n - 1\right] e^{-\frac{n^2\pi^2}{36}t} \cos\frac{n\pi x}{6}$$

$$\therefore u(x,t) = 6 + \frac{1}{3}\sum_{n=1}^{\infty} \frac{72}{n^2\pi^2}\left[(-1)^n - 1\right] e^{-\frac{n^2\pi^2}{36}t} \cos\frac{n\pi x}{6}.$$

(Ans)

Fourier Integral

মনে করি, কোনো একটি function $f(x)$ দেওয়া

$\therefore$ The fourier integral of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \underbrace{\int_{-\infty}^{\infty} f(t)\cdot\cos\lambda(t-x)\, dt}\, d\lambda$$

আমাদের এইটুকু part solve করা লাগবে। তাহলেই ans চলে আসবে।

The fourier cosine integral of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda t\cdot\cos\lambda x\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\cos\lambda x\left[\int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt\right] d\lambda$$

এইটুকু solve করলেই ans

$$= \frac{2}{\pi}\int_0^{\infty}\cos\lambda x\int_0^{\infty} f(t)\cos\lambda t\, dt\, d\lambda$$

0 থেকে ∞ আনার জন্য সামনে 2 আনলাম

The fourier sine integral of $f(x)$ is given by.

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cdot \sin\lambda t \, \sin\lambda x \, dt \, d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \sin\lambda x \int_{-\infty}^{\infty} f(t) \cdot \sin\lambda t \, dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin\lambda x \int_0^{\infty} f(t) \cdot \sin\lambda t \, dt$$

[প্রশ্নে যদি উল্লেখ না করে দেয় fourier sine integral / fourier cosine integral করব তখন আমাদের check করতে হবে $f(x)$ odd না even?

যদি $f(x)$ odd হয়: মানে $f(-x) = -f(x)$.

তাহলে fourier sine integral করব

আর যদি $f(x)$ even হয়: মানে $f(-x) = f(x)$

তাহলে fourier cosine integral করব

আর যদি ~~fourier~~ even/odd কোনটাই না হয়:

তাহলে fourier integral এর normal formula apply করব]

#Problem 1: Find the fourier integral representation of function, $f(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases}$

এবার আগে check করব function odd না even:

put $x = -x$

$\therefore f(x) = f(x)$ কারণ function এ কোথাও x নাই তাই $-x$, x এর জায়গায় বসালেও কোনো

$\therefore$ even function change আসবে না

so, the fourier cosine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\lambda x \int_0^{\infty} f(t) \cos\lambda t \, dt \, d\lambda$$

আমরা আগে limit চিন্তা করি:

$|x|<1$, x হতে পারে (+)/(−)

(+) নিলে,

$x<1$

(−) নিলে, $-x<1$

$\therefore x>-1$

$\therefore -1<x<1$

-1 0 1 2

এখন $|x|>1$ এর জন্য

(+) নিয়ে $x>1$

(−) " $-x>1 \therefore x<-1$

$x<-1$ $x>1$

-2 -1 0 1 2

তাহলে,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\lambda x \left[\int_0^1 1 \cdot \cos\lambda t \, dt\right] d\lambda$$

আমাদের formula তে ছিল 0 থেকে ∞. তাই আমরা -1 থেকে 0 তে লিখব না। শুধু 0 থেকে 1 এ f(x) কে নিব

আমরা 1 থেকে ∞ এর Part টা লিখলাম না। কারণ তখন $f(t) = 0$. মানে $f(x) = 0$

$$= \frac{2}{\pi}\int_0^{\infty} \cos\lambda x \left[\frac{\sin\lambda t}{\lambda}\right]_0^1 d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos\lambda x \left[\frac{\sin\lambda}{\lambda}\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\cos\lambda x \cdot \sin\lambda}{\lambda} d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\lambda \cdot \cos\lambda x}{\lambda} d\lambda$$

$$\therefore \int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

[just $\frac{\pi}{2}$ দিয়ে f(x) কে গুণ দিলাম]

#Problem 2: Find the fourier integral representation of function

$$\therefore f(x) = \begin{cases} -e^{ax} & ; x<0 \\ e^{-ax} & ; x>0 \end{cases}$$

এই দুইটা এখন compare করব

solve: Put, $x = -x$

$$f(-x) = \begin{cases} -e^{-ax} & -x<0 \\ e^{ax} & -x>0 \end{cases} = \begin{cases} -e^{-ax} & ; x>0 \\ e^{ax} & ; x<0 \end{cases}$$

[যখন, $x<0$ তখন, আগে ছিল $f(x) = -e^{ax}$

আর পরে হল, $f(-x) = e^{ax}$

$\therefore f(-x) = -f(x)$ বলা যায়

আবার যখন, $x>0$ ছিল, তখন, আগে $f(x) = e^{-ax}$

পরে, $f(-x) = -e^{-ax}$

$\therefore f(-x) = -f(x)$

তাহলে, $f(x)$ হল odd function.]

$\therefore$ the function $f(x)$ is an odd function

$\therefore$ The fourier sine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \int_0^{\infty} f(t)\sin\lambda t \; dt \; d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \int_0^{\infty} e^{-at}.\sin\lambda t \; dt\, d\lambda$$

↳ $x>0$ শুধু সেটাই আসবে। কারন 0 থেকে ∞. আর

$f(x) = e^{-ax}$

$\therefore f(t) = e^{-at}$ তাই e^{-ax} এর বদলে e^{-at} নিয়েছি

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left| \frac{e^{-at}}{a^2+\lambda^2}(-a\sin\lambda t - \lambda\cos\lambda t)\right|_0^\infty d\lambda$$

$$\left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[0 - \frac{1}{a^2+\lambda^2}(-\lambda)\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{a^2+\lambda^2}\right]d\lambda$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda\cdot\sin\lambda x}{a^2+\lambda^2}\,d\lambda$$

$$\therefore \int_0^\infty \frac{\lambda\sin\lambda x}{a^2+\lambda^2}\,d\lambda = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2}(-e^{ax}) & ; x<0 \\ \frac{\pi}{2}(e^{-ax}) & ; x>0 \end{cases}$$

(Ans)

Problem 3: Find the fourier sine integral for $f(x) = e^{-\beta x}$. Hence show that,

$$\frac{\pi}{2} e^{-\beta x} = \int_0^{\infty} \frac{\lambda \sin \lambda x}{\beta^2 + \lambda^2} d\lambda$$

[এখানে ques এ direct বলা আছে fourier sine integral করতে]

solve: The fourier sine integral of $f(x)$ is,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} e^{-\beta t} \sin \lambda t \, dt \, d\lambda$$

$$\left[\int e^{at} \sin bt \, dt = \frac{e^{at}}{a^2+b^2} (a \sin bt - b \cos bt) \right]$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[\frac{e^{-\beta t}}{\beta^2 + \lambda^2} (-\beta \sin \lambda t - \lambda \cos \lambda t) \Big|_0^{\infty} d\lambda \right.$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[0 - \frac{1 \, (-\lambda)}{\beta^2 + \lambda^2} \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \cdot \frac{\lambda}{\beta^2 + \lambda^2} d\lambda$$

$$\therefore \int_0^{\infty} \frac{\sin \lambda x \cdot \lambda}{\beta^2 + \lambda^2} d\lambda = \frac{\pi}{2} \times f(x)$$

$$= \frac{\pi}{2} \cdot e^{-\beta x}$$

(showed)

Problem 4: Express the function $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

as a fourier sine integral and show that

$$\int_0^\infty \frac{\sin\lambda x \cdot \sin\lambda\pi}{1-\lambda^2}\, d\lambda = \frac{\pi}{2}\sin x, \quad 0 \le x \le \pi$$

solve: The fourier sine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \cdot \int_0^\pi \sin t \sin\lambda t\, dt\, d\lambda$$

[0 থেকে π এর টা ~~lim~~ লিখলাম
π থেকে ∞ তে 0. তাই আর লিখলাম না]

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \, \frac{1}{2}\int_0^\pi 2\sin\lambda t \sin t\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \sin\lambda x \int_0^\pi [\cos(\lambda-1)t - \cos(\lambda+1)t]\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \sin\lambda x \left[\frac{\sin(\lambda-1)t}{(\lambda-1)} - \frac{\sin(\lambda+1)t}{(\lambda+1)}\right]_0^\pi d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \sin\lambda x \left[\frac{\sin(\lambda\pi-\pi)}{(\lambda-1)} - \frac{\sin(\lambda\pi+\pi)}{\lambda+1}\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \sin\lambda x \left[\frac{-\sin(\pi-\lambda\pi)}{(\lambda-1)} - \frac{\sin(\pi+\lambda\pi)}{\lambda+1}\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \sin\lambda x \left[\frac{-\sin\lambda\pi}{\lambda-1} + \frac{\sin\lambda\pi}{\lambda+1}\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^{\alpha} \sin\lambda x \sin\lambda\pi \left(\frac{1}{\lambda+1} - \frac{1}{\lambda-1}\right) d\lambda$$

$$\therefore f(x) = \frac{1}{\pi}\int_0^{\alpha} \sin\lambda x \cdot \sin\lambda\pi \frac{(-2)}{\lambda^2-1} d\lambda$$

$$\Rightarrow \int_0^{\alpha} \sin\lambda x \cdot \sin\lambda\pi \cdot \frac{(-2)}{\lambda^2-1} d\lambda = \pi \cdot f(x)$$

$$\Rightarrow \int_0^{\alpha} \sin\lambda x \cdot \sin\lambda\pi \cdot \frac{2}{1-\lambda^2} d\lambda = \pi \cdot f(x)$$

$$\therefore \int_0^{\alpha} \frac{\sin\lambda x \cdot \sin\lambda\pi}{1-\lambda^2} d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2} \cdot \sin x & 0 \le x \le \pi \\ 0 & ; x > \pi \end{cases}$$

(Proved)

Problem 5:

Express $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$ as a Fourier sine integral and hence evaluate $\int_0^\infty \left(\frac{1-\cos\pi\lambda}{\lambda}\right) \sin x\lambda \, d\lambda$

Solve: The Fourier sine integral of $f(x)$ is given by:

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \int_0^\infty f(t) \cdot \sin(\lambda t) \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \int_0^\pi 1 \cdot \sin\lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \int_0^\pi \sin\lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[-\frac{\cos\lambda t}{\lambda} \right]_0^\pi d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\frac{-\cos\lambda\pi + 1}{\lambda} \right] d\lambda$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\lambda x \, (1-\cos\lambda\pi)}{\lambda} \, d\lambda$$

$$\therefore \int_0^\infty \left(\frac{1-\cos\pi\lambda}{\lambda}\right) \cdot \sin\lambda x \, d\lambda = \frac{f(x) \cdot \pi}{2}$$

$$= \frac{\pi}{2} \cdot f(x)$$

$$= \begin{cases} \frac{\pi}{2} \cdot 1, & 0 \le x \le \pi \\ 0 ; & x > \pi \end{cases}$$

(Ans)