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Partial Differential Equation:

An equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called PDE.

Ex: $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Linear partial Differential Equation: A PDE is said to be linear if it has 1st degree only in its derivatives.

Ex, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \cdot \partial y}$

☐ if $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be any two independent solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ then $\phi(u,v) = 0$ or $v = \phi(u)$ is a general solution of lagrange's linear equation $Pp + Qq = R$

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$F(z) \longrightarrow \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x}.$

Prove :

Let,

u, v any two function of x, y, z connected by the relation $\phi(u,v) = 0$ —— (1)

Partially differentiating (1) w.r.to x and y respectively,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x}\right) = 0$$

—— (ii)

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial y}\right) = 0$$

—— (iii)

Now elimating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (ii) and (iii),

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} \cdot \cancel{\frac{\partial z}{\partial x}} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

Here,

$\frac{\partial z}{\partial x} = p$

$\frac{\partial z}{\partial y} = q$

$$\Rightarrow \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \quad p\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}\right) + q\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} \quad \text{--- (iv)}$$

Let,

$$\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} = \lambda P$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} = \lambda Q$$

$$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} = \lambda R$$

Hence from (iv) we get,

$$\lambda Pp + \lambda Qq = \lambda R$$

$$\therefore \quad Pp + Qq = R \quad \cdots \text{(v)}$$

which is lagrange Linear PDE.

Again let,

$u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be any two sol^n of (v),

Differentiating these w.r.t. to x,y,z respectively,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \text{(vi)}$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad \text{(vii)}$$

Solving (vi) and (vii) $\Rightarrow$

$$\frac{dx}{-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\cdot\frac{\partial u}{\partial x}}$$

$$= \frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}}$$

Or, $\frac{dx}{\lambda P} = \frac{dy}{\lambda Q} = \frac{dz}{\lambda R}$

$\therefore \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ Ans.

"Math Problem"

Problem I : Solve : $\frac{y^2z}{x}p + x\cancel{y}zq = y^2$

Given,

$$\frac{y^2z}{x}p + x\cancel{y}zq = y^2$$

The lagrange's auxilary equations are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

* Simply solve করা যায় কারণ x,y,z গুণ আকারে আ

$$\therefore \text{Or, } \frac{dx}{\frac{y^2z}{x}} = \frac{dy}{x\cancel{y}z} = \frac{dz}{y^2}$$

Taking 1st and 2nd ratio,

$$\frac{x\,dx}{y^2z} = \frac{dy}{xz}$$

$$\Rightarrow \quad x^2dx = y^2dy$$

$$\Rightarrow \quad \frac{x^3}{3} - \frac{y^3}{3} = c \quad \text{[by integrating]}$$

$$\Rightarrow \quad x^3 - y^3 = 3c = c_1 \quad \text{--- (1)}$$

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Again, taking 1st and 3rd ratio,

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

$$\Rightarrow x\,dx = z\,dz$$

$$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = c'$$

$$\Rightarrow x^2 - z^2 = c_2 \quad \text{—— (ii)} \quad [\text{by integrating}]$$

Hence the general sol^n is $\phi\left(x^3 - y^3, x^2 - z^2\right) = 0$

where ϕ is an arbitary constant.

Ans

Problem 2: Solve: the PDE $(3x+y-z)p + (x+y-z)q = 2(z-y)$

Given,

$$(3x+y-z)p + (x+y-z)q = 2(z-y)$$

the lagrange's A.E are;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)}$$

Choosing $1, -3, -1$ as multiplier, then

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)} = \frac{dx-3dy-dz}{0}$$

i.e,

$$dx - 3dy - dz = 0$$

$$\Rightarrow x - 3y - z = c_1 \text{ (by integrating)}$$

Again choosing, $1, 1, -1$ and $1, -1, 1$

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)} = \frac{dx+dy-dz}{4(x+y-z)} = \frac{dx-dy+dz}{2(x-y+z)}$$

taking 4th and 5th ratio,

$$\frac{dx+dy-dz}{4(x+y-z)} = \frac{dx-dy+dz}{2(x-y+z)}$$

$$\Rightarrow \frac{1}{2}\log(x+y-z) = \log(x-y+z)$$

$$\Rightarrow \frac{\sqrt{x+y-z}}{x-y+z} = c_2 \text{ (by integrating)}$$

Hence, the general sol^n is $\phi\left(x-3y-z, \frac{\sqrt{x+y-z}}{x-y+z}\right)$

Problem 3 : (HW) Solve : Find the eqn of PDE

$$(y+zx)p - (x+yz)q = x^2-y^2$$

Soln

Given:

$$(y+zx)p - (x+yz)q = x^2-y^2$$

According to lagrange's auxilary equation.

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

Choosing $y, x, 1$ multiplier then

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} = \frac{ydx+xdy+dz}{0}$$

Now,

$$ydx+xdy+dz = 0$$

$$\Rightarrow d(xy) + dz = 0$$

$$\Rightarrow xy + z = c_1 \quad [\text{by integrating}]$$

Again choosing $x, y, -z$ as multiplier, then.

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} = \frac{x\,dx + y\,dy - z\,dz}{0}$$

Now,

$$x\,dx + y\,dy - z\,dz = 0$$

$\therefore$ $x^2+y^2-z^2 = c_2$ [by integrating]

Hence the general solution is $\phi(xy+z^2, x^2+y^2-z^2) = 0$

Here, ϕ is an arbitary constant.

HW Problem 4: Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Given, $(x^2-yz)p + (y^2-zx)q = z^2-xy$

According to lagrange's auxilary equation,

$$\frac{dx}{x^2-yz} = \frac{dy}{+(y^2-zx)} = \frac{dz}{z^2-xy}$$

$$\Rightarrow \frac{dx-dy}{x^2-y^2-yz+zx} = \frac{dy-dz}{y^2-z^2-zx+xy} = \frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

taking 1st and 2nd ratio,

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$$\Rightarrow \log(x-y) = \log(y-z) + \log c_1$$

$$\Rightarrow \frac{x-y}{y-z} = c_1$$

taking 2nd and 3rd ratio,

$$\frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

$$\Rightarrow \log(y-z) = \log(z-x) + \log c_2$$

$$\Rightarrow \frac{y-z}{z-x} = c_2$$

The general solution is $\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$

$\boxed{H.W}$ Problem 5: $(y-z)p+(x-y)q = z-x$

The lagrange's auxilary equation are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x}$$

choosing $(1,1,1)$ as multipliers,

$$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} = \frac{dx+dy+dz}{0}$$

i.e, $dx+dy+dz = 0$

$\Rightarrow x+y+z = c_1$ [by integrating] —(1)

Again choosing (x, z, y) as multiplier.

$$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} = \frac{xdx+zdy+ydz}{0}$$

i.e, $xdx+zdy+ydz = 0$

$\Rightarrow xdx + d(yz) = 0$

$\Rightarrow \frac{x^2}{2}+yz = c_2$ [by integrating]

Hence, the general solution is $\phi\left(x+y+z, \frac{x^2}{2}+yz\right) = 0$

~~H.W~~ **Pblm 6: Solve:** $x(2y^4 - z^4)p + y(z^4 - 2x^4)q = z(x^4 - y^4)$

The lagrange's auxilary equations are.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$$

choosing x^3, y^3 and z^3 as multiplier.

$$\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)} = \frac{x^3dx + y^3dy + z^3dz}{0}$$

i.e, $x^3dx + y^3dy + z^3dz = 0$

$\therefore x^4 + y^4 + z^4 = c_1$ [by integrating]

Again choosing $x, y, z/2$ as divider,

$$\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{2dz}{z}}{0}$$

i.e, $\frac{dx}{x} + \frac{dy}{y} + \frac{2dz}{z} = 0$

$\Rightarrow xyz^2 = c_2$ [by integrating]

Hence the general solution $\phi(x^4+y^4+z^4, xyz^2)=0$

Here, ϕ is an arbitary function.

Problem 6: Find the integral surface of the PDE $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$. which contain the line $x+y=0$, $z=1$

From,
The langrange's auxilary equation.

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

choosing $x, y, -1$ as multiplier,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx+y\,dy-dz}{0}$$

i.e.

$$x\,dx+y\,dy-dz=0$$

$$\Rightarrow x^2+y^2-2z=c_1 \text{ --- (i)}$$

Again choosing x, y, z as divider then.

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$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

i.e, $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

$\Rightarrow xyz = c_2$ (ii)

Since (i) and (ii) passes through $x+y=0, z=1$

so, $xy = c_2$

and, $x^2+y^2-z = c_1$

$\Rightarrow (x+y)^2 - 2xy - 2 = c_1$

Or, $0 - 2c_2 - 2 - c_1 = 0$

or, $c_1 + 2c_2 + 2 = 0$

or, $x^2+y^2-z + 2xyz + 2 = 0$

which is required integral surface.

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Problem 8: Find the integral surface of the PDE $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the $z=0$, $x^2+y^2 = 2x$.

Soln

From lagrange's auxilary equation,

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}$$

choosing $1, 2y, -2$ as multiplier,

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} = \frac{dx+2ydy-2dz}{0}$$

i.e, $dx+2ydy-2dz = 0$

$$\Rightarrow \cancel{xy^2}\; x+y^2-2z = c_1 \quad \text{—— (i)}$$

Taking 1st and 3rd ratio,

$$\frac{dx}{2(z-3)} = \frac{dz}{2x-3}$$

$$\Rightarrow (2x-3)\,dx = 2(z-3)\,dz$$

$$\Rightarrow x^2-3x-z^2+6z = c_2 \quad \text{—— (ii)}$$

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Adding (I) and (II)

$$x^2+y^2-z^2-2x+4z = c_1+c_2$$

$$\Rightarrow 2x-0-2x-0 = c_1+c_2$$

$$\Rightarrow c_1+c_2 = 0$$

$$\Rightarrow x^2+y^2-z^2-2x+4z = 0$$ Ans.

HW CT

Problem 9: Find the integral surface of the PDE $(x-y)y^2p+(y-x)x^2q = (x^2+y^2)z$ which passing through $xz=a^3$, $y=0$.

From lagrange's auxilary eqn.

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{z(x^2+y^2)} = \boxed{\cancel{\frac{x^2dx + y^2dy}{0}}}$$

choosing $x^2, y^2, 0$ and $1, -1, 0$ as multiplier

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{z(x^2+y^2)} = \frac{x^2dx+y^2dy}{0} = \frac{dx-dy}{(x-y)(x^2+y^2)}$$

Now, $x^2dx + y^2dy = 0$

$\Rightarrow x^3+y^3 = c_1$ [by integrating] —(i)

taking 3rd and 5th ratio,

$$\frac{dz}{z} = \frac{dx-dy}{x-y}$$

$\Rightarrow \log z = \log(x-y) + \log c_2$

$\Rightarrow \frac{z}{x-y} = c_2$ [by integrating] —(ii)

Since it passes through $xz = a^3$, $y=0$

$x^3 = c_1$

and from (ii) $\frac{z}{x} = c_2$

$$\Rightarrow \frac{a^3/x}{x} = c_2$$

$$\Rightarrow \frac{a^3}{x^2} = c_2$$

$$\Rightarrow \frac{a^9}{(x^3)^2} = c_2^3$$

$$\Rightarrow a^9 = c_1^2 c_2^3$$

$$\Rightarrow c_1^2 c_2^3 = a^9$$

$$\Rightarrow (x^3+y^3)^2 \frac{z^3}{(x-y)^3} = a^9$$

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Problem 10

#11 Find the integral surface of the PDE $4yzp+q+2y=0$ which passes through

$$z^2+y^2=1, \quad x+z=2$$

Sol^n

From the lagrange's auxilary equation.

$$\frac{dx}{4yz}=\frac{dy}{1}=\frac{dz}{-2y}$$

taking 1st and 3rd ratio.

$$\frac{dx}{2z}=\frac{dz}{-1}$$

$\Rightarrow \quad dx = -2z\,dz$

$\Rightarrow \quad x = -z^2+c_1$ [by integrating]

$\Rightarrow x+z^2=c_1$ —— (i)

taking 2nd and 3rd ratio.

$2y\,dy = -dz$

$\Rightarrow \quad y^2+z = c_2$ —— (ii) [by integrating]

(ii) − (i) ⇒

$y^2 - z - x + z^2 = c_2 - c_1$

$\Rightarrow \quad y^2 + z^2 - (z + x) = c_2 - c_1$

$\Rightarrow \quad 1 - 2 = c_2 - c_1$

$\Rightarrow \quad c_1 - c_2 = 1$

$\Rightarrow \quad x - z^2 - y^2 + z = 1$ Ans

−o−

(ii) + (i) ⇒

$y^2 + z^2 + x + z = c_1 + c_2$

$\Rightarrow \quad c_1 + c_2 = 3$

$\Rightarrow \quad y^2 + z^2 + x + z = 3$ Ans

Pblm 11

Obtain a PDE arising from $\varphi\left(\frac{z}{x^2}, x^2-y^2+z\right)=0$

Given,

$$\phi\left(\frac{z}{x^2}, x^2-y^2+z\right)=0 \quad \text{---} (i)$$

Let, $u=\frac{z}{x^2}$ and $v=x^2-y^2+z$

Differentiating (ii) w.r.t. to x and y respectivily,

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\left(\frac{-2z}{x^3}+p\frac{1}{x^2}\right)+\frac{\partial\phi}{\partial v}(2x+p)=0 \quad \text{---} (i)$$

And,

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\left(0+\frac{1}{x^2}\cancel{p}q\right)+\frac{\partial\phi}{\partial v}(-2y+q)=0$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from above equation,

$$\begin{vmatrix} -\frac{2z}{x^3}+\frac{p}{x^2} & 2x+p \\ \frac{q}{x^2} & -2y+q \end{vmatrix}=0$$

$$\Rightarrow \frac{4yz}{x^3} - \frac{2zq}{x^3} - \frac{2yp}{x^2} + \frac{pq}{x^2} - \frac{2q}{x} - \frac{pq}{x^2} = 0$$

$\Rightarrow xyp + q(x^2+2) = 2yz$ which is a required PDE.

$\boxed{\text{Non-Linear PDE}}$

↓

more than 1st degree.

Charpit's Method : If $F(x, y, z, p, q) = 0$ then the charpit's Auxilary Equation are.

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

Or,
$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

Find a complete sol^n of $p^2 - y^2 q = y^2 - x^2$ by Charpit's method.

$\underline{\underline{Sol^n}}$

$$F(x, y, z, p, q) = p^2 - y^2 q - y^2 + x^2 = 0 \quad \text{---(1)}$$

Charpit's auxilary equations are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{2x} = \frac{dq}{-2y} = \frac{dz}{-2p^2 + qy^2} = \frac{dx}{-2p} = \frac{dy}{y^2}$$

Here,
$F_x = 2x$
$F_z = 0$
$F_y = -2y$
$F_p = 2p$
$F_q = -y^2$

From 1st and 4th ratio,

$$\frac{dp}{2x} = \frac{dx}{-2p}$$

$$\Rightarrow -p\,dp = x\,dx$$

$$\Rightarrow p^2 + x^2 = a \quad \text{[by integrating]} \quad \text{---(ii)}$$

Solving (i) and (ii) $\Rightarrow$

$$p = \sqrt{a - x^2} \qquad q = \frac{a - y^2}{y^2} = \frac{a}{y^2} - 1$$

We know,

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

Or, $dz = p\,dx + q\,dy$

Or, $dz = \sqrt{a - x^2}\, dx + \left(\frac{a}{y^2} - 1\right) dy$

Or, $z = \int \sqrt{a - x^2}\, dx + \int \left(\frac{a}{y^2} - 1\right) dy$ [by integrati...]

$$\boxed{= \frac{x\sqrt{a - x^2}}{2} + \frac{a}{2} \sin^{-1} \frac{x}{\sqrt{a}} - \frac{a}{y} - y + K}$$

which is complete sol^n

Ans

Pblm 2: Find a complete sol^n of $z^2(p^2 z^2 + q^2) = 1$ by charpit's method.

Let,

$$F(x, y, z, p, q) = p^2 z^4 + q^2 z^2 - 1 = 0 \quad \text{——— (1)}$$

The charpit's eq^n are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

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$$\frac{dp}{4z^3p^3+2zpq^2} = \frac{dq}{4z^3p^2q+2zq^3} = \frac{dz}{-2p^2z^4-2q^2z^2}$$

$$= \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}$$

Here,
$F_x = 0$
$F_Y = 0$
$F_z = 4z^3p^2 + 2zq^2$
$F_p = 2pz^4$
$F_q = 2qz^2$

From 1st and 2nd ratio,

$$\frac{dp}{p(4z^3p^2+2zq^2)} = \frac{dq}{q(4z^3p^2+2zq^2)}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

$$\Rightarrow \ln p = \ln q + \ln c$$

$$\Rightarrow p = qc \quad \text{---} \ (ii)$$

Solving (i) and (ii) $\Rightarrow$

$$q^2c^2z^4 + q^2z^2 - 1 = 0$$

$$\Rightarrow q^2 = \frac{1}{c^2z^4+\cancel{z^2}}$$

$$\Rightarrow q = \frac{1}{z\sqrt{c^2z^2+1}}$$

$$\therefore p = \frac{c}{z\sqrt{c^2z^2+1}}$$

~~complete~~

:: Solution Process

(i) সূত্র লেখা

(ii) সূত্র অনুযায়ী কাজ

(iii) Ratio- এর সমীকরণ আর (i) এর সমীকরণ হতে p ও q এর value নির্ণয় করা

(iv) $dz = pdx + qdy$ এর Integration করা

then Singular.

(v) differential w.r.t a, b

(vi) a ও b এর value বের করা

(vii) a ও b এর value eqn এ বসানো

We know,

$$dz = p\,dx + q\,dy$$

$$\Rightarrow \quad dz = \frac{c}{2\sqrt{c^2z^2+1}}\,dx + \frac{1}{2\sqrt{c^2z^2+1}}\,dy$$

$$\Rightarrow \quad 2\sqrt{c^2z^2+1}\,dz = c\,dx + dy$$

$$\boxed{\Rightarrow \quad \frac{1}{3c^2}\left(c^2z^2+1\right)^{3/2} = cx + y + k}$$

Ans

Pblm 3 : Find a complete sol^n of

$pxy + pq + qy = yz$ by charpit's method.

Let,

$$F(x,y,z,p,q) = pxy + pq + qy - yz = 0 \quad \text{---(1)}$$

Charpit's eq^n are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{py + p(-y)} = \frac{dq}{px + q + q(-y) - z} = \frac{dz}{-p(xy+q) - q(p-2+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{p-2+y}$$

Here,

$F_x = py$

$F_y = px + q - z$

$F_z = -y$

$F_p = xy + q$

$F_q = p + y$

From 1st ratio,

$$dp = 0$$

$$\therefore p = c \quad \text{---(ii)}$$

From solving (i) and (ii) =)

$$q = \frac{yz - cxy}{c+y}, \quad p = c$$

we know,

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = c\,dx + \frac{y(z-cx)}{c+y}\,dy$$

$$\Rightarrow \frac{dz - c\,dx}{z - cx} = \frac{y}{c+y}\,dy = \left(1 - \frac{c}{c+y}\right)dy$$

$$\Rightarrow \log(z - cx) = y - c\log(c+y) + \log k$$

$$\Rightarrow (z - cx)(c+y)^c = ke^y$$

$$\Rightarrow z = cx + ke^y (c+y)^{-c}$$

Ans

Pblm 4: $16p^2z^2+9q^2z^2+4z^2-4=0$

Let, $f(x,y,z,p,q)=16p^2z^2+9q^2z^2+4z^2-4=0$ —(1)

The charpit's auxilary equation.

$$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{32p^3z+18pq^2z+8pz}=\frac{dq}{32p^2qz+18q^3z+8qz}=\frac{dz}{-32z^2p^2-18q^2z^2}=\frac{dx}{-32z^2p}=\frac{dy}{-18qz^2}=\frac{dx+4pdz+4zdp}{0}$$

Here,
$F_x=0$
$F_y=0$
$F_z=32p^2z+18q^2z+8z$
$F_p=32z^2p$
$F_q=18qz^2$

i.e,

$dx+4pdz+4zdp=0$

$\Rightarrow dx+4d(pz)=0$

$\Rightarrow x+4pz=a$ [contant] —(ii)

Solving (1) and (ii) $\Rightarrow$ we get,

$$p=\frac{a-x}{4z}$$

from (1) and (ii) $\Rightarrow$

$$16\frac{(a-x)^2}{16z^2}z^2+9q^2z^2+4z^2-4=0$$

$\Rightarrow (a-x)^2+9q^2z^2+4z^2-4=0$

$$\Rightarrow \cancel{9q^2z^2+4z^2=4}$$

$$\Rightarrow 9q^2z^2 = 4-4z^2-(a-x)^2$$

$$\Rightarrow q^2 = \frac{4}{9z^2} - \frac{4}{9} - \frac{(a-x)^2}{9z^2}$$

$$= \frac{4}{9}\left(\frac{1}{z^2} - 1 - \frac{(a-x)^2}{4z^2}\right)$$

$$= \frac{4}{9z^2}\left\{1 - z^2 - \frac{1}{4}(a-x)^2\right\}$$

$$\therefore q = \frac{2}{3z}\sqrt{1-z^2-\frac{1}{4}(a-x)^2}$$

We know,

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a-x}{4z}dx + \frac{2}{3z}\sqrt{1-z^2-\frac{1}{4}(a-x)^2}\,dy$$

$$z\,dz = \frac{a-x}{4}dx + \frac{2}{3}\sqrt{1-z^2-\frac{1}{4}(a-x)^2}\,dy$$

$$\Rightarrow dy = \frac{z\,dz - \frac{a-x}{4}dx}{\sqrt{1-z^2-\frac{1}{4}(a-x)^2}} \times \frac{3}{2}$$

$$\Rightarrow y + c = -\frac{3}{2}\sqrt{1-z^2-\frac{1}{4}(a-x)^2}$$

$$\Rightarrow \frac{9}{4}\left\{1-z^2-\frac{1}{4}(a-x)^2\right\} = (y+c)^2$$

$$\Rightarrow \frac{4}{9}(y+c)^2 + \frac{1}{4}(a-x)^2 + z^2 = 1$$ which is ellip.

Ans.

Pblm 5:

Solve: $p^2+q^2 = py - qx$ from Charpit's Auxilary Equation.

Let,

$$F(x,y,z,p,q) = p^2+q^2-py+qx = 0 \quad \text{---} (1)$$

Charpit's auxilary eqn are,

$$\frac{dp}{F_x+pF_z} = \frac{dq}{F_y+qF_z} = \frac{dz}{-pF_p-qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$F_p = 2p-y$

$F_q = 2q+x$

$F_z = 0$

$F_x = q$

$F_y = -p$

$$\Rightarrow \frac{dp}{q+0} = \frac{dq}{-p+0} = \frac{dz}{-p(2p-y)-q(2q+x)} = \frac{dx}{-(2p-y)} = \frac{dy}{-(2q+x)}$$

From 1st and 2nd ratio.

$$p\,dp = -q\,dq$$

$\Rightarrow p^2+q^2 = a$ — (ii) [constant (say)] [by integrating]

From (1) and (ii) $\Rightarrow$

$$a = py - qx$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$\Rightarrow a+qx = py$

$\Rightarrow py = a+qx$

$\Rightarrow \sqrt{a-q^2}\, y = a+qx$

$\Rightarrow ay^2 - q^2y^2 = a^2 + 2aqx + q^2x^2$

$\Rightarrow a^2 + 2aqx + q^2x^2 + q^2y^2 - ay^2 = 0$

$\Rightarrow q^2(x^2+y^2) + a^2 + 2aqx - ay^2 = 0$

$$\Rightarrow q = \frac{-2ax \pm \sqrt{4a^2x^2 - 4(x^2+y^2)(a^2-ay^2)}}{2(x^2+y^2)}$$

$$= \frac{-ax \pm \sqrt{a(x^2+y^2-a)y^2}}{(x^2+y^2)}$$

$$\therefore p = \left[a - \left\{\frac{-ax \pm \sqrt{a(x^2+y^2-a)y^2}}{x^2+y^2}\right\}^2\right]^{1/2}$$

We know,

$$dz = p\,dx + q\,dy$$

$$z = \int p\,dx + \int q\,dy + c \quad \underline{\text{Ans}}$$

p ও q এর মান বসাতে হবে।

H.W

Pblm 6: Solving: $2z+p^2+qy+2y^2=0$

Find out the complete solution.

let,

$$f(x,y,z,p,q) = 2z+p^2+qy+2y^2=0 \quad \text{---(1)}$$

The charpit's Auxilary eq^n are.

$$\frac{dp}{F_x+pF_z} = \frac{dq}{F_y+qF_z} = \frac{dz}{-pF_p-qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

Here,
$F_x = 0$
$F_y = q+4y$
$F_z = 2$
$F_p = 2p$
$F_q = y$

$$\Rightarrow \frac{dp}{2p} = \frac{dq}{q+4y+q2} = \frac{dz}{-p\times 2p-qy} = \frac{dx}{-2p}$$

$$= \frac{dy}{-y}$$

From 1st and 4th ratio,

$$\frac{dp}{2p} = \frac{dx}{-2p}$$

$\Rightarrow dp = -dx$

$\Rightarrow p+x = a$ [a constant]

$\Rightarrow p = a-x$.

From eq^n (1) =>

$$2z+(a-x)^2+ay+2y^2=0$$

$\Rightarrow qy = -2z-(a-x)^2-2y^2$

$$\Rightarrow q = \frac{-2z-(a-x)^2-2y^2}{y}$$

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Now,

$$dz = p\,dx + q\,dy$$

$$\int dz = (a-x)\,dx + \frac{-2z-(a-x)^2-2y^2}{y}\,dy$$

$$\int dz = \int (a-x)\,dx - \int \frac{2z}{y}\,dy - \int \frac{(a-x)^2}{y}\,dy - \int 2y\,dy.$$ solution কেমনে করলো?

An

$\boxed{\text{Complete and singular Solution.}}$ → উদ্দেশ্য

$p, q, \underline{a, b}$ এর value বের কর.
↳ constant

Pblm 7: Find the complete and Singular integral of $(p^2+q^2)y = qz$.

Soln

Let, $F(x,y,z,p,q) = (p^2+q^2)y - qz = 0$

$$\Rightarrow p^2y + q^2y - qz = 0 \quad \text{— (1)}$$

The charpit's auxilary eqn are,

$$\frac{dp}{F_x+pF_z} = \frac{dq}{F_y+qF_z} = \frac{dz}{-pF_p-qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

Here,
$F_x = 0$
$F_y = p^2+q^2$
$F_z = -q$
$F_p = 2py$
$F_q = 2qy - z$

$$\frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-2p^2y-2q^2y+qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

From 1st and 2nd ratio,

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$$\Rightarrow p^2+q^2 = a \quad \text{--- (ii)}$$

from eqn (i) and (ii) $\Rightarrow$

$$(a-q^2)y + q^2y - qz = 0$$

$$\Rightarrow ay - qz = 0 \quad \Rightarrow q = \frac{ay}{z}$$

eqn (ii) $\Rightarrow$

$$\therefore p^2 = a-q^2$$

$$= a - \frac{a^2y^2}{z^2}$$

$$= \frac{\sqrt{az^2-a^2y^2}}{z}$$

We know,

$$dz = pdx + qdy$$

Or, $$dz = \frac{\sqrt{az^2-a^2y^2}}{z}dx + \frac{ay}{z}dy$$

$$\Rightarrow dz - \frac{ay}{z}dy = \frac{\sqrt{az^2-a^2y^2}}{z}dx$$

$$\Rightarrow zdz - aydy = \sqrt{az^2-a^2y^2}\,dx$$

$$\Rightarrow \frac{az\,dz - a^2 y\,dy}{\sqrt{az^2 - a^2y^2}} = a\,dx$$

$\Rightarrow \sqrt{az^2 - a^2y^2} = ax + b$ which is complete Solution.

$\Rightarrow az^2 - a^2y^2 = (ax+b)^2 \quad \text{---} (iii)$

eqn (iii) P. differentiating w.r.t a and b respectively,

$$z^2 - 2ay^2 = 2x(ax+b) \quad \text{---} (iv)$$

and $0 = 2(ax+b)$

$\Rightarrow ax + b = 0 \quad \text{---} (v)$

from eqn (iv) and (v) $\Rightarrow$

$$a = \frac{z^2}{2y^2} \quad \text{and} \quad b = \frac{-xz^2}{2y^2}$$

Then putting the value of a and b in eqn (iii) $\Rightarrow$

$$az^2 - a^2y^2 = (ax+b)^2$$

$$\Rightarrow \frac{z^2 \cdot z^2}{2y^2} - \frac{z^4}{4y^4}y^2 = \left(\frac{z^2x}{2y^2} - \frac{xz^2}{2y^2}\right)^2$$

$$\Rightarrow \frac{z^4}{2y^2} - \frac{z^4}{4y^2} = 0$$ which is singular soln.

Pblm 8 : Find the complete and singular integral of $2xz - px^2 - 2qxy + pq = 0$

Soln

let, $F(x,y,z,p,q) = 2xz - px^2 - 2qxy + pq = 0$ —(i)

Then the charpit's Auxilary eqn ;

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{2z - 2px - 2qy + 2px} = \frac{dq}{-2qx + q2x} = \frac{dz}{\ } = \frac{dx}{\ } = \frac{dy}{\ }$$

Here,
$F_x = 2z - 2px - 2qy$
$F_y = -2qx$
$F_z = 2x$
$F_p = -x^2 + q$
$F_q = -2xy + p$

from 2nd ratio ,

$dq = 0$

$\Rightarrow q = a$ [by integrating] —(ii)

eqn (1) and (ii) $\Rightarrow$

$$2xz - px^2 - 2axy + ap = 0$$

$$\Rightarrow p(ax^2 - a) = 2xz - 2axy$$

$$\Rightarrow p = \frac{2xz - 2axy}{x^2 - a} \quad \text{—(iii)}$$

we know , $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{2xz - 2axy}{x^2 - a}\,dx + a\,dy$$

$$\Rightarrow dz - a\,dy = \frac{2x(z - ay)}{x^2 - a}\,dx.$$

$$\Rightarrow \frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}$$

$$\Rightarrow \log(z - ay) = \log(x^2 - a) + c$$

$\Rightarrow \log(z - ay) - \log(x^2 - a) = \log b$

$\Rightarrow \log \frac{z - ay}{x^2 - a} = b$ ✗

$$\boxed{\Rightarrow (z - ay) = b(x^2 - a) \quad [b \text{ is a constant}]} \quad \text{—— (III)}$$

P.d. eqn (III) with r. to a and b respectivily,

$0 = y - b \quad \Rightarrow b = y$

$0 = 0 + x^2 - a \quad \Rightarrow a = x^2$.

Then putting the value in eqn (III) $\Rightarrow$

$z - x^2 y = y(x^2 - a)$ which is the singular sol^n.

An

Non-Linear PDE

First order non linear PDE

Type-1

$$F(p,q)=0$$

A complete soln is $z = ax + h(a)y + c$

where, $F(a, h(a)) = 0$ and a, c are the arbitrary constant.

IF, we put,

$$c = \phi(a)$$

then $z = ax + h(a)y + \phi(a)$

Example:

Solve: $p^2 - q^2 = 1$

Soln: Given,

$$p^2 - q^2 = 1 \quad -(1)$$

Let, $F(p,q) = p^2 - q^2 - 1 = 0$

$$F(a, h(a)) = a^2 - [h(a)]^2 - 1 = 0$$

$$\Rightarrow h(a) = (a^2 - 1)^{1/2}.$$

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$\Rightarrow h(a) = (\sec^2\alpha - 1)^{1/2} = \tan\alpha$

Put, $a = \sec\alpha$,

A complete solⁿ of ① is,

$$z = ax + h(a)y + c$$
$$= ax + (a^2-1)^{1/2}y + c$$
$$= x\sec\alpha + y\tan\alpha + c$$
$$= x\sec\alpha + y\tan\alpha$$

Also, let, $c = \phi(\alpha) = 0$

p. Differentiate w.r.to α.

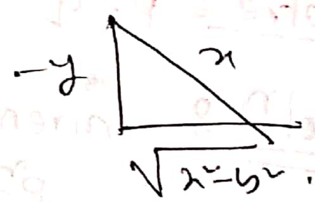
$$0 = x\sec\alpha\tan\alpha + y\sec^2\alpha$$

$$\Rightarrow x\tan\alpha + y\sec\alpha = 0$$

$$\Rightarrow x\frac{\sin\alpha}{\cos\alpha} + y\frac{1}{\cos\alpha} = 0$$

$$\Rightarrow x\sin\alpha + y = 0$$

$$\Rightarrow \sin\alpha = -y/x$$



$$\therefore \sec\alpha = \frac{x}{\sqrt{x^2-y^2}}$$

$$\tan\alpha = \frac{-y}{\sqrt{x^2-y^2}}.$$

Hence the singular sol^n is

$$z = \frac{x^2}{\sqrt{x^2-y^2}} - \frac{y^2}{\sqrt{x^2-y^2}}$$

$$\Rightarrow z^2 = x^2 - y^2 .$$
Ans

$\boxed{\text{Type - 2}}$

If $z = px + qy + F(p,q)$

A complete sol^n is

$$z = ax + by + F(a,b)$$

Solve:

$$z = px + qy + 3p^{1/3} q^{1/3}$$

Solution: Given,

$$z = px + qy + 3p^{1/3} q^{1/3}$$

~~Diff. w.r.to a and b respectivily~~,

A complete sol^n is

$$z = ax + by + 3a^{1/3} b^{1/3}$$

Diff. w.r. to a and b respetivily

$$0 = x + a^{-2/3} b^{1/3}$$

$$\Rightarrow x = -a^{-2/3} b^{1/3}$$

and
$$y + a^{1/3} b^{-2/3} = 0$$
$$\Rightarrow y = -a^{1/3} b^{-2/3}$$

then,
$$ax + by = -a^{1/3} b^{1/3} - a^{1/3} b^{1/3}$$
$$= -2a^{1/3} b^{1/3}$$

Also, $xy = a^{-1/3} b^{-1/3}$
$$\Rightarrow \frac{1}{xy} = a^{1/3} b^{1/3}$$

Hence,
$$z = ax + by + 3a^{1/3} b^{1/3}$$
$$= -2a^{1/3} b^{1/3} + 3a^{1/3} b^{1/3}$$

Or, $z = a^{1/3} b^{1/3}$
$$\Rightarrow z = \frac{1}{xy}$$
$$\Rightarrow xyz = 1$$ Ans

Type III

if $p(z, p, q) = 0$

A complete soln is

$$z = F(x+ay) \qquad u = ay + x$$
$$= F(u)$$

$$p = \frac{\partial z}{\partial x}$$
$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y}$$
$$= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} = a \frac{dz}{du}$$

$$\because F(z, p, q) = 0$$

$$\Rightarrow F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$$

whose soln is the complete soln.

Solve: $z = p^2 + q^2$

Soln Given $z = p^2 + q^2$ $\qquad u = x + ay$

let $z = F(x+ay)$

$= F(u)$

then, $p = \frac{dz}{du}$ $q = a\frac{dz}{du}$.

then, eqn (1) becomes.

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$= (1+a^2)\left(\frac{dz}{du}\right)^2 \cancel{a}$$

$$\Rightarrow \frac{dz}{du} = \sqrt{\frac{z}{1+a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}} du$$

$$= 2\sqrt{z} = \frac{1}{\sqrt{1+a^2}} u + K$$ where, $b = \sqrt{1+a^2}\, K$.

$$\Rightarrow 2\sqrt{z} = \frac{1}{\sqrt{1+a^2}} u + \frac{b}{\sqrt{1+a^2}}$$

$$\Rightarrow 2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(u+b)$$

$\Rightarrow 4z(1+a^2) = (x+ay+b)^2$, which is complete soln

Type - IV

$$F_1(x, p) = F_2(y, q)$$

let, $F_1(x,p) = F_2(y,q) = a$

then, $p = F_1(x,a)$

$q = F_2(y,a)$

we know, $dz = p\,dx + q\,dy$

or, $dz = F_1(x,a)\,dx + F_2(y,a)\,dy$

or, $z = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + b.$

Example:

Solve: $p - q = x^2 + y^2$

$\Rightarrow p - x^2 = q + y^2$

let, $p - x^2 = y^2 + q = a$

then,

$p = a + x^2$

$q = a - y^2$

we know, $dz = p\,dx + q\,dy$

$= (a + x^2)\,dx + (a - y^2)\,dy$

$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + k$ which is a comple Solution

Homogenous and homogenous Linear PDE with constant cofficient

Case I

$D = D_x = \frac{\partial}{\partial x}$

$D' = D_y = \frac{\partial}{\partial y}$.

Distinct Factor.

if $F(D, D')z = 0$

and, $z = \phi_1(y + m_1 x)$

$z = \phi_2(y + m_2 x)$ . . . $z_n = \phi_n(y + m_n x)$

then the general solution is -

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x).$$

Example.

Solve: $(D^3 - 3D^2D' + 2DD'^2)z = 0$ $\left[D = \frac{\partial}{\partial x}\; D' = \partial/\partial y\right]$

Soln Given,

$$(D^3 - 3D^2D' + 2DD'^2)z = 0 \quad -(1)$$

let, $z = \phi(y + mx)$ be a trail soln of (1)

the putting $D = m$ and $D' = 1$.

So,

Auxilary eqn is

$$m^3 - 2m^2 + 2m = 0$$

$$\Rightarrow m(m-1)(m-2) = 0$$

$$\Rightarrow m = 0, 1, 2$$

Hence, the general solⁿ is $z = \phi_1(y) + \phi_2(y+x) + \phi_3(y+2x)$

Ans

$\boxed{\text{Case II}}$

Repeated factors:

$$m = m_1, m_2$$

General Solution

$$z = \phi_1(y+m_1x) + x\phi_2(y+m_2x) + x^2\phi_3(y+m_3x) + \cdots$$

Example:

Solve: $(25D^2 - 40DD' + 16D'^2)z = 0$

Solⁿ Given

$$(25D^2 - 40DD' + 16D'^2)z = 0 \quad \text{--- (1)}$$

Let, $z = \phi(y+mx)$ be a trail solⁿ of (1)

the putting $D = m$ and $D' = 1$

so.

Auxilary eqn is,

$$25m^2 - 40m + 16 = 0$$

$$\Rightarrow (5m-4)(5m-4) = 0$$

$$\therefore m = 4/5, 4/5.$$

Hence, the general soln is

$$z = \phi_1(y + \tfrac{4}{5}x) + x\phi_2(y + \tfrac{4}{5}x)$$

$\boxed{\text{H.W}}$

Solve: $(D^4 - 2D^3D' + 2DD' - D'^4)z = 0$

Given.

Auxilary eqn is,

$$\cancel{D^4 - 2D^3 + 2D - 1 = 0}$$

$$\Rightarrow m^4 - 2m^3 + 2m - 1 = 0$$

$$\Rightarrow (m-1)^3(m+1) = 0$$

$$\therefore m = 1, 1, 1, -1$$

General Soln is,

$$z = \phi_1(y-x) + \phi_2(y+x) + x\phi_3(y+x) + x^2\phi_4(y+x)$$

Case – III

Complex factors.

Let, $m_1 = a + ib$

and $m_2 = a - ib$

then the general solution

$$z = \phi_1\{y + (a+ib)x\} + \phi_1\{y + (a-ib)x\}$$
$$i\left[\phi_2\{y + (a+ib)x\} - \phi_2\{y + (a-ib)x\}\right]$$

Example.

Solve: $(D^2 - 2DD' + 5D'^2)z = 0$

Or, $r - 2s + 5t = 0$

$D_x = D = \frac{\partial}{\partial x}$

$D_y = D' = \frac{\partial}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Soln

Given

Auxilary eqn is,

$$m^2 - 2m + 5 = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

Hence, the general soln is

$$y = \phi_1\{y + (1+2i)x\} + \phi_1\{y \pm (1 \pm 2i)x\} + i\left[\phi_2\{y + (1+2i)x\} - \phi_2\{y + (1-2i)x\}\right]$$

Non-homogenous PDE

CASE - I Algebraic Function.

Example: Solve: $(D_x^3 + D_x^2 D_y - 6 D_x D_y^2) z = x^2 + y^2$

Sol^n Given, ______ (1)

let $z = \phi(y+mx)$ be a trail sol^n of corresponding homogenous part of (1)^n

the putting $D = m$ and $D' = 1$

Auxilary. Eq^n is,

$$m^3 + m^2 - 6m = 0$$

$$\therefore m = 0, 2, -3$$

Hence, the complementary sol^n is

$$z = \phi_1(y + 0.x) + \phi_2(y+2x) + \phi_3(y-3x)$$

$$= \phi_1(y) + \phi_2(y+2x) + \phi_3(y-3x)$$

Again the particular integral be

$$z_p = \frac{1}{D_x^3 + D_x^2 D_y - 6 D_x D_y} (x^2+y^2)$$

$$= \frac{1}{D_x (D_x - 2D_y)(D_x + 3D_y)} x^2+y^2$$

$$= \frac{1}{D_x^3} \left(1 - \frac{2D_y}{D_x}\right)^{-1} \left(1 + \frac{3D_y}{D_x}\right)^{-1} (x^2+y^2)$$

$$= \frac{1}{D_x^3} \left(1 + \frac{2D_y}{D_x} + \frac{4D_y^2}{D_x^2} + \cdots\right)\left(1 - \frac{3D_y}{D_x} + \frac{9D_y^2}{D_x^2} \cdots\right)(x^2+y^2)$$

$$= \frac{1}{D_x^3} \left(1 - \frac{3D_y}{D_x} + \frac{9D_y^2}{D_x^2} + \frac{2D_y}{D_x} - \frac{6D_y^2}{D_x^2} + \frac{4D_y^2}{D_x^2} + \cdots\right)(x^2+y^2)$$

$$= \frac{1}{D_x^3} \left(1 - \frac{D_y}{D_x} + \frac{7D_y^2}{D_x^2} - \cdots\right)(x^2+y^2)$$

$$= \left(\frac{1}{D_x^3} - \frac{D_y}{D_x^4} + \frac{7D_y^2}{D_x^5} - \cdots\right)(x^2+y^2)$$

$$\cancel{= \frac{1}{D_x^3}(x^2+y^2) - \frac{D_y}{D_x^4}(x^2+y^2) + \frac{7D_y^2}{D_x^5}(x^2+y^2) + \cdots}$$

$$= \cancel{\frac{x^5}{60}} + \Rightarrow \frac{1}{D_x^3}(x^2+y^2) - \frac{D_y}{D_x^4}(x^2+y^2) + \frac{7D_y^2}{D_x^5}(x^2+y^2)$$

$$\Rightarrow \frac{x^{2+1+1+1}}{2 \times 3 \times 4 \times 5} + \frac{x^3}{6} \times y^2 - \frac{1}{D_x^4}(2y) + \frac{7}{D_x^5}(2) - 0$$

$$\Rightarrow \frac{x^5}{60} + \frac{x^3y^2}{6} - \frac{x^4}{12}y + \frac{7x^5}{60}.$$

$\therefore z = z_c + z_p.$

Case 11

Solve: $(D_x - 2D_y)^2 (D_x + 3D_y) z = e^{2x+y}$

Soln: Given,

— — — — — — — — — — — — — — —

— — — — — —

then the auxilary eqn is

$$(m-2)^2 (m+3) = 0$$

$$\therefore m = 2, 2, -3$$

$$\therefore z_c = \phi_1(y+2x) + \phi_2 x(y+2x) + \phi_3(y-3x)$$

Now, the particular integral is

$$z_p = \frac{1}{(D_x - 2D_y)^2 (D_x + 3D_y)} e^{2x+y}$$

$$= \frac{1}{(D_x - 2D_y)^2} \frac{1}{5} e^{2x+y}$$

$$= \frac{x}{5.2(D_x - 2D_y)} e^{2x+y}$$

$$= \frac{x^2}{5.2.1} e^{2x+y}$$

$$= \frac{x^2}{10} \cdot e^{2x+y}$$

$\therefore$ The general Soln is $= z_c + z_p$

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Case 2

Exponential Function:

(a) $\frac{1}{F(D_x, D_y)} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$.

Solve: $(D_x^2 + 2D_x D_y + D_y^2)z = e^{2x+3y}$

Sol^n, Given

— — — — — — — —

A.E

$m^2 + 2m + 1 = 0$

$\therefore m = -1, -1$

Hence, the — — — —

$z_c = \phi_1(y-x) + x\phi_2(y-x)$

$z_p = \frac{1}{D_x^2 + 2D_x D_y + D_y^2} e^{2x+3y}$

$= \frac{1}{2^2 + 2.2.3 + 3^2} e^{2x+3y}$

$= \frac{1}{25} e^{2x+3y}$

Hence:

$\therefore$ — — — $z = z_c + z_p$.

$=$ — — — — — —

H.W

Solve : $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$

Soln Given,

$$D^3 - 2D^2D' - DD'^2 + 2D'^3 = 0$$

$$\Rightarrow D_x^3 - 2D_x^2 D_y - D_x D_y + 2D_y^3 = 0 \quad \text{--- (1)}$$

— — — — — — — — —

— — — — — — — — —

— — — — homogenouns part of (1)

$$m^3 - 2m^2 - m + 2 = 0$$

$\therefore m = -1, 2, 1$

$$\therefore z_c = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+2x)$$

$$z_p = \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3} e^{x+y}$$

$$= \cancel{\frac{1}{1-2-1+1} e^{x+y}}$$

$$= \cancel{- e^{x+y}}$$

$$= \frac{1}{3D^2 - 4DD' - D'^2} e^{x+y}$$

$$= -\frac{x}{2} e^{x+y}$$

$$z_p = \frac{1}{(D_x+D_y)(D_x-D_y)(D_x-2D_y)} e^{x+y} = \frac{x}{-2} e^{x+y}$$

$$\boxed{z = z_p + z_c}$$

Case - 3

Sine and cosine Function. [দ্বিঘাত হতে হবে]

P.I. Particular integral

$$z_p = \frac{1}{F(D_x^2, D_x D_y, D_y^2)}$$

$$= \frac{1}{F(-a^2, -ab, -b^2)}$$

$\sin(ax+by)$

Or,

$\cos(ax+by)$

if $F(-a^2, -ab, -b^2) \neq 0$

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Solve

$$(D_x^2 - 5D_xD_y + 6D_y^2)\, z = \sin(4x+y)$$

Given,

— — - — —

— — — —

— — — —

$m^2 - 5m + 6 = 0$

$\therefore m = 3, 2.$

$\therefore z_c = \phi_1(y+2x) + \phi_2(y+3x)$

$$\text{PI of } z_p = \frac{1}{D_x^2 - 5D_xD_y + 6D_y^2} \sin(4x+y)$$

$$= \frac{1}{-4^2 - 5\times(-4\times 1) + 6\times -1^2} \sin(4x+y)$$

$$= \frac{\sin(4x+y)}{-16+20-6} = -\frac{1}{2}\sin(4x+y) \text{ Ans.}$$

— — — — — —

$\therefore z = z_c + z_p$

$= - - - - - -$

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Solve:

$$(D_x^3 - 7D_x D_y^2 - 6D_y^3)z = \cos(x+2y)$$

Given,

A.E

$$m^3 - 7m - 6 = 0$$

$\therefore m = 3, -1, -2.$

$$\therefore z_c = \phi_1(y+3x) + \phi_2(y-x) + \phi_3(y-2x)$$

$$z_p = \frac{1}{D_x^3 - 7D_x D_y^2 - 6D_y^3}\cos(x+2y)$$

$$= \frac{1}{\cancel{-1^3 - 7\times + 1\times}}$$

How and Ans

$m \rightarrow D_x$. $\quad m = -1$

$\ominus \rightarrow D_y$. $\quad \Rightarrow (m+1) = 0$

$\Rightarrow (D_x + D_y) = 0$

$$= \frac{1}{(D_x + D_y)(D_x^2 - D_x D_y - 6D_y^2)}\cos(x+2y)$$

$$= \frac{1}{-1^2 - 1\times -1\times 2 - 6\times -2^2} \cdot \frac{1}{D_x + D_y}\cos(x+2y)$$

$$= \frac{1}{25} \cdot \frac{D_x - D_y}{D_x^2 - D_y^2}\cos(x+2y)$$

$$= \frac{1}{25.(-1^2-1\times-2^2)} (D_x - D_y)\cos(x+2y)$$

$$= \frac{1}{25.3}\{-\sin(x+2y) + 2\sin(x+2y)\}$$

$$= \frac{1}{75}\sin(x+2y)$$

$$\therefore z = z_c + z_p.$$

$D' =$ constant $\phi(x+2y) = (x)$ } $\&$ $\sim$

Exponential

Solve: $(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$

Given,

A.E,

$$4m^2 - 4m + 1 = 0$$

$$\therefore m = \frac{1}{2}, \frac{1}{2}$$

$$\therefore z_c = \phi_1\left(y+\frac{x}{2}\right) + \phi_2 x\left(y + x/2\right)$$

$$z_p = \frac{1}{4D^2 - 4DD' + D'^2} 16\log(x+2y)$$

$$= \frac{1}{(D - \frac{1}{2}D')^2}$$

$$\boxed{= \frac{1}{4\times1^2 - 4\times1\times2 + 2^2} 16\log(x+2y)}$$

$$= \frac{x}{2(D - \frac{1}{2}D')} 16\log(x+2y)$$

Only D, D' ...
... Derivative,
$D/D' = 1$

$$\Rightarrow \frac{x^2}{8D - 4D'} \; 1 \; 6\log(x+2y)$$

$$\Rightarrow \frac{x^2}{8} \; 16\log(x+2y)$$

$$= 2x^2 * \log(x+2y)$$

$z = z_c + z_p.$

case – IV

A general method.

Considere $(D - mD')z = f(x,y)$

$$p - mq = f(x,y)$$

$$y + mx = c$$

$$PI = \int f'(x)$$

Solve $r + s - 2t = y\cos x$

Or, $(D^2 + DD' - 6D'^2)z = y\cos x$ —(1)

$m^2 + m - 6 = 0$

$\Rightarrow m = -3, 2$

Hence, the complementary equation

$$z_c = \phi_1(y+2x) + \phi_2(y-3x)$$

Now, the particular soln is

$$z_p = \frac{1}{(D+3D')(D-2D')} y\cos x$$

$y + 2x = c_2$

$y - 3x = c_1$

$$= \frac{1}{D+3D'} \int (c_2 - 2x)\cos x \, dx$$

$$= \frac{1}{D+3D'} \left\{ c_2 \sin x - 2(x\sin x + \cos x) \right\}$$

$$= \frac{1}{D+3D'} \left\{ (c_2 - 2x)\sin x - 2\cos x \right\}$$

$$= \frac{1}{D+3D'} (y\sin x - 2\cos x)$$

$$= \int \left\{ (c_1 + 3x)\sin x - 2\cos x \right\}$$

$$= -c_1\cos x + 3\sin x - 3x\cos x - 2\sin x$$

$$= -(c_1 + 3x)\cos x + \sin x$$

$$= -y\cos x + \sin x$$

Hence, the general solution is $z = z_c + z_p$.

$\boxed{\text{Solve}}$ $(D_x^3 + D_x^2 D_y - D_x D_y^2 - D_y^3) z = e^x \cos 2y$

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$$m^3 + m^2 - m - 1 = 0$$

$$\therefore m = 1, -1, -1$$

$$z = \phi_1 (y-x) + x\phi_2 (y-x) + \phi_3 (y+x)$$

$$z_p = \frac{1}{D_x^3 + D_x^2 D_y - D_x D_y^2 - D_y^3} e^x \cos 2y$$

$$= e^x \frac{1}{(D_x+1)^3 + (D_x+1)^2 D_y - (D_x+1) D_y^2 - D_y^3} \cos 2y$$

$\cos(0 \cdot x + 2y)$

$D_x^2 = 0$

D_x

$\therefore D_x = 0$

$$= e^x \frac{1}{D_x^3 + 3D_x^2 + 3D_x + 1 + D_x^2 \cdot D_y + 2D_x \cdot D_y + D_y - D_x D_y^2 - D_y - [illegible]} \cos 2y$$

$$= e^x \frac{1}{0+0+0+1+0+0+D_y - 0 + 4 + 4D_y} \cos 2y$$

$$= \frac{e^{x}}{5} \frac{1}{D_y+1} \cos 2y$$

$$= \frac{e^{x}}{5} \frac{D_y-1}{D_y^2-1} \cos 2y$$

$$= \frac{e^{x}}{-25} (D_y-1) \cos 2y$$

$$= \frac{e^{x}}{-25} (-2\sin 2y - \cos 2y)$$

$$= \frac{e^{x}}{25} (2\sin 2y + \cos 2y)$$

$\therefore z = z_c + z_p$.

Solve: $z_{xx} + z_{yy} = 30(2x+y)$

$$z_{xx} = \left(\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x}\right) z = \frac{\partial^2}{\partial x^2} z = D_x^2 z$$

Ch, $(D_x^2 + D_y^2) z = 30(2x+y)$.

Sol^n

$m^2+1=0$

$\therefore m = \pm i$

$$z_c = \phi_1(y+ix) + \phi_1(y-ix) + i\{\phi_2(y+ix) - \phi_2(y-ix)\}$$

$\boxed{\text{(Ans.)}}$

$$\therefore \quad z_p = \frac{1}{f(D_x, D_y)} 30(2x+y)$$

$$= 30 \frac{1}{D_x^2 + D_y^2}(2x+y)$$

$$= 30 \frac{1}{D_x^2}\left(1 + \frac{D_y^2}{D_x^2}\right)^{-1}(2x+y)$$

$$= 30 \frac{1}{D_x^2}\left(1 \cancel{+} \frac{D_y^2}{D_x^2} + \frac{D_y^4}{D_x^4} + \cdots\right)(2x+y)$$

$$\Rightarrow 30\left(\frac{1}{D_x^2} \cancel{+} \frac{D_y^2}{D_x^4} + \frac{D_y^4}{D_x^6} + \cdots\right)(2x+y)$$

$$\Rightarrow 30\left\{\frac{1}{D_x^2}(2x+y) + \frac{D_y^2}{D_x^4}(2x+y) + \frac{D_y^4}{D_x^6}(2x+y) + \cdots\right\}$$

~~Solve: $(D^2 - 2DD' + D'^2)z = 2\cos y - x\sin y$~~

$$\Rightarrow 30 \frac{1}{D_x^2}(2x) + 30\frac{1}{D_x^2}(y) + 0$$

$$\Rightarrow 30 \times \frac{2x^3}{2\times 3} + 30y \times \frac{x^2}{2}$$

$$z_p \Rightarrow 10x^3 + 15yx^2.$$

$$\therefore z = z_c + z_p$$

** Solve: $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$.

$m^2 - 2m + 1 = 0$

$\Rightarrow m = -1, -1$ $\qquad \cos(ax+by)$

$z_c = \phi_1(y - x) + \phi_2 x(y - x)$

$$z_p = \frac{1}{D^2 - 2DD' + D'^2}(2\cos y - x\sin y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2}(2\cos y) - \frac{1}{D^2 - 2DD' + D'^2}(x\sin y)$$

(m-1)

$$= \frac{2}{-1^2}\cos y - \frac{1}{(D+D')(D+D')}x\sin y$$

$$= \frac{2}{-1}\cos y - \frac{1}{(D+D')(D+D')}(y-c_1)\sin y$$

$$= -2\cos y - \frac{1}{(D+D')}\int (y-c_1)\sin y\, dy$$

$$= -2\cos y - \frac{1}{(D+D')}\left\{-y\cos y + \sin y + c_1\cos y\right\}$$

$$= -2\cos y - \frac{1}{D+D'}\left\{\sin y - \cos y(y-c_1)\right\}$$

$= 2\cos y + \frac{1}{(D+1)}\{x\cos y - \sin y\}$

$= 2\cos y + \int\{(y - c_1)\cos y - \sin y\}dy$

$= 2\cos y - c_1 \sin y + \cos y + \cos y + y \sin y$

$= (y - c_1)\sin y$

$= x \sin y.$

$z = 2c + 2\phi$.

sec → sec + tan

csc − csc → cot

$y(\cos y)$ 2

$c_1 \cos y$

$\sin y$

$\cos y$

Canonical Form

Comparing the given equation with

$$Rr + Ss + Tt + F(x, y, z, p, q) = 0$$

The quadratic eqn can be form

$$R\alpha^2 + S\alpha + T = 0$$

HPE

(i) hyperbolic if $S^2 - 4RT > 0$

(ii) Parabolic if $S^2 - 4RT = 0$

(iii) Elliptic if $S^2 - 4RT < 0$

Case I

If $S^2 - 4RT > 0$ then the required Canonical form is $\frac{\partial^2 z}{\partial u \partial z} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$

Now, $\frac{dy}{dx} + \alpha_1 = 0$ and $\frac{dy}{dx} + \alpha_2 = 0$

if roots are distinct

Integrating we get, $u = F_1(x, y)$ and $v = F_2(x, y)$

Case II

If $S^2 - 4RT = 0$

then the required canonical form is

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Now,

$$\frac{dy}{dx} + \alpha_1 \; (\alpha_1 = \alpha_2) = 0$$

if roots are equal

Integrating, $u = F_1(x, y)$ and Let, $v = F_2(x, y)$

Case III.

If $S^2 - 4RT < 0$, then the required canonical form is

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \varphi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$$

Now, $\frac{dy}{dx} + \alpha_1 = 0$ and $\frac{dy}{dx} + \alpha_2 = 0$

integrating when roots are complex conjugates.

$u = f_1(x,y)$ and $v = f_2(x,y)$ ——— (ii)

To get, real Connonical form (i) transform (ii) into.

$$\alpha = \frac{1}{2}(u+v) \quad \text{and} \quad \beta = \frac{1}{2} i (v-u).$$

Formulas

(i) $$p = \frac{\partial z}{\partial x}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}.$$

(ii) $$q = \frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

(iii) $$r = \frac{\partial p}{\partial x}$$

$$= \frac{\partial p}{\partial x}\left(\frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$+ \frac{\partial z}{\partial v} \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}$$

$$+ \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

$$+ \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial v \partial u}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}$$

$$+ \frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$+ \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}.$$

$$\therefore r = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$+ \cancel{\frac{\partial^2 z}{\partial y}}\ \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

(iv) $t = \frac{dq}{dy}$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2$$

$$+ \frac{\partial z}{\partial u}\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

$$s = \frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial x \cdot \partial y}$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\right) + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x \cdot \partial y} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}.$$

Problem 1 : Reduce to cannonical form of the equation $(y-1)r-(y^2-1)s+y(y-1)t+p-q=2ye^{2x}(1-y)^3$ and hence solve it.

Given,

$$(y-1)r-(y^2-1)s+y(y-1)t+2ye^{2x}(1-y)^3=0 \quad \text{--- (1)}$$

Comparing the given the equation with

$$Rr+Ss+Tt+F(x,y,z,p,q)=0$$

$$R=(y-1),\quad S=-(y^2-1),\quad T=y(y-1)$$

$$S^2-4RT=(y^2-1)^2-4y(y-1)^2$$

$$=(y-1)^2\{(y+1)^2-4y\}$$

$$=(y-1)^2(y-1)^2$$

$$=(y-1)^4>0$$

$\therefore$ So the PDE is hyperbolic.

The α quadratic eqn will be,

$$R\alpha^2+S\alpha+T=0$$

$$\Rightarrow (y-1)\alpha^2-(y^2-1)\alpha+y(y-1)=0$$

~~$(y-1)=(y+1)$~~

$$\Rightarrow \alpha^2 - (y+1)\alpha + y = 0$$

$$\Rightarrow \alpha^2 - y\alpha - \alpha + y = 0$$

$$\Rightarrow \alpha(\alpha - y) - 1(\alpha - y) = 0$$

$$\Rightarrow (\alpha - y)(\alpha - 1) = 0$$

$$\therefore \alpha = y, 1$$

Let, $\alpha_1 = y$ and $\alpha_2 = 1$

Then the eq^n are,

$$\frac{dy}{dx} + \alpha_1 = 0 \quad \text{and,} \quad \frac{dy}{dx} + \alpha_2 = 0$$

$$\Rightarrow \frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{y} + dx = 0$$

$$\Rightarrow \log y + x = \log(\text{const.})$$

$$\Rightarrow \log y + \log e^x = \log(\text{const})$$

$$\Rightarrow ye^x = \text{constant.}$$

$$\Rightarrow \frac{dy}{dx} + 1 = 0$$

$$\Rightarrow dy + dx = 0$$

$$\Rightarrow y + x = \text{constant}$$

The transformation of indopent variables from x, y to u, v

so, made by $u = x+y$

and $v = ye^{x}$.

Now, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & ye^{x} \\ 1 & e^{x} \end{vmatrix} = e^{x}(1-y) \neq 0$

Hence transformation is valid.

Now,

$$p = \frac{\partial z}{\partial x}$$

$$= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}.$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \cdot ye^{x}$$

$$q = \frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \cdot e^{x}$$

Here

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial v}{\partial x} = ye^{x}$

$\frac{\partial u}{\partial y} = 1$

$\frac{\partial v}{\partial y} = e^{x}$

$\frac{\partial^2 u}{\partial x^2} = 0$

$\frac{\partial^2 v}{\partial x^2} = ye^{x}$

$\frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 v}{\partial y^2} = 0$

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$$r = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} ye^x + \frac{\partial^2 z}{\partial v^2}(ye^x)^2 + \frac{\partial z}{\partial v} ye^x .$$

Now, ~~Putting~~ $t = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}\right)$

$$+ \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} e^x + \frac{\partial^2 z}{\partial v^2} e^{2x}$$

$$s = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y}\right) + \cancel{2}\frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}.$$

$$= \frac{\partial^2 z}{\partial u^2}(1) + \frac{\partial^2 z}{\partial u \partial v}(e^x + ye^x) + \frac{\partial^2 z}{\partial v^2} ye^{2x} + e^x \frac{\partial z}{\partial v}$$

Now, Putting the values of p, q, r, t, s in given eqn.

$$(y-1)\left\{\frac{\partial^2 z}{\partial u^2}+2ye^x\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}y^2e^{2x}+ye^x\frac{\partial z}{\partial v}\right)$$

$$-(y^2-1)\left\{\frac{\partial^2 z}{\partial u^2}+e^x\frac{\partial^2 z}{\partial u\partial v}+ye^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial^2 z}{\partial u\partial v}+e^x\frac{\partial z}{\partial v}\right)$$

$$+y(y-1)\left\{\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}\right\}$$

$$-p-q=2ye^{2x}(1-y)^3$$

$$\Rightarrow e^x(1-y)^3\frac{\partial^2 z}{\partial u\partial v}=2ye^{2x}(1-y)^3$$

$$=\frac{\partial^2 z}{\partial u\partial v}=2ye^x$$

$$\boxed{\Rightarrow \frac{\partial^2 z}{\partial u\partial v}=2v \quad —(i)}$$

This is the required Cannonical form.

Now integrating ① w.r. to v.

$$\frac{\partial z}{\partial u}=v^2+\phi_1(u) \quad —(ii)$$ where $\phi_1(u)$ is integrating constant.

Again, integrating (11) w.r. to u,

$$z = v^2u + \int \phi_1(u)\,du + \phi_2(v)$$ when $\phi_2(u)$ is integrating

$$\boxed{\therefore\ z = v^2u + \phi_1(u) + \phi_2(v)}$$

$$\therefore\ z = y^2e^{2x}(x+y) + \phi_1(x+y) + \phi_2(ye^{x})$$ which is completed sol^n.

Pblm 2: Reduce to Cannonical form of the equation $r+2s+t=0$

Given,

$$r+2s+t = 0 \quad \text{---(1)}$$

Comparing eq^n (1) with $Rr+Ss+Tt+F(x,y,z,p,q$

then, $R=1,\ S=2,\ T=1$

Now, $S^2-4RT = 4-4=0$ which is parabolic.

The α quadratic eq^n

$$R\alpha^2+S\alpha+T=0 \qquad \therefore\ \alpha^2+2\alpha+1=0$$

$$\Rightarrow (\alpha+1)^2 = 0$$

$$\therefore\ \alpha = -1,-1$$

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Now, the eq^n are, $\frac{dy}{dx} + 1 = 0$

$$\frac{dy}{dx} - 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = 1$$

$$\Rightarrow \quad dx - dy = 0$$

$$\Rightarrow \quad x - y = c$$

To change the independent variables we take.

$$u = x - y$$

$$\text{and } v = x + y$$

Here,

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial v}{\partial x} = +1$$

$$\frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial y} = +1$$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial^2 y} = 0$$

$$\frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial y^2} = 0$$

$$p = \frac{\partial z}{\partial x}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}.$$

$$r = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$+ \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + 0 + 0$$

$$t = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \cdot \partial v}\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right) + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2$$

$$+ \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \cdot \partial v} + \frac{\partial^2 z}{\partial v^2} + 0 + 0$$

$$s = \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} * \frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x \cdot \partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y \partial x}$$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}(1-1) + \frac{\partial^2 z}{\partial v^2} + 0 + 0$$

$$= \frac{\partial^2 z}{\partial v^2} - \frac{\partial^2 z}{\partial u^2}$$

Putting the value of r, s, t in (1)

$$\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial v^2} - 2\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u^2}$$

$$- 2\frac{\partial^2 z}{\partial u . \partial v} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$\therefore \quad \frac{\partial^2 z}{\partial v^2} = 0$$

which is cannonical form.

integrating w.r. to v

$$\frac{\partial z}{\partial v} = \phi_1(u)$$

Again integrating w.r. to v

$$z = v\phi_1(u) + \phi_2(u)$$

$$= (x+y)\ \phi_1(x-y) + \phi_2(x-y).$$

which is completed solution.

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Pblm 3: Reduce the canonical form and solve the PDE $x^2r - 2xys + y^2t - xp + 3yq = 8\frac{y}{x}$ —(1)

Comparing eqn (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$$R = x^2, \quad S = -2xy, \quad T = y^2$$

Now, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$ which is para

The α quadratic eqn is

$$R\alpha^2 + S\alpha + T = 0$$

$$\Rightarrow x^2\alpha^2 + (-2xy)\alpha + y^2 = 0$$

$$\Rightarrow x^2\alpha^2 - xy\alpha - xy\alpha + y^2 = 0$$

$$\Rightarrow x\alpha(x\alpha - y) - y(x\alpha - y) = 0$$

$$\Rightarrow (x\alpha - y)^2 = 0$$

$$\therefore \alpha = \frac{y}{x}, \frac{y}{x}$$

Then the equation,

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\Rightarrow xy = \text{const} \quad \text{(by integrating)}$$

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Therefore we suppose,

$$u = xy$$

and choose v to be any function of x, y which is independet of u, hence we can be may choice.

let, $v = \frac{y}{x}$

Here,

$\frac{\partial u}{\partial x} = y$

$\frac{\partial v}{\partial x} = -y/x^2$

$\frac{\partial u}{\partial y} = x$

$\frac{\partial v}{\partial y} = \frac{1}{x}$

$\frac{\partial^2 u}{\partial x^2} = 0$

$\frac{\partial^2 v}{\partial x^2} = \frac{2y}{x^3}$

$\frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 v}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial x \partial y} = 1$

$\frac{\partial^2 v}{\partial y \partial x} = -\frac{1}{x^2}$

$$p = \frac{\partial z}{\partial x}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} y + \frac{\partial z}{\partial v} \cdot \frac{-y}{x^2}$$

$$q = \frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} x + \frac{1}{x} \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial u^2} \left(\frac{\partial u}{\partial x}\right)^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2} \left(\frac{\partial v}{\partial x}\right)^2$$

$$+ \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

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$$= y^2\frac{\partial^2 z}{\partial u^2} - 2\frac{y^2}{x^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2} + 0 + \frac{2y}{x^3}\frac{\partial z}{\partial v}$$

$$t = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2$$

$$+ \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial^2 y}$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2} + 0 + 0$$

$$s = \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial^2 z}{\partial^2 v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x\partial y}$$

$$+ \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y.\partial x}$$

$$= \frac{\partial^2 z}{\partial u^2}xy + \frac{\partial^2 z}{\partial u\partial v}\left(y/x - \frac{y}{x}\right) + \frac{\partial^2 z}{\partial^2 v}\left(-\frac{y}{x^3}\right)$$

$$+ \frac{\partial z}{\partial u} - \frac{1}{x^2}\frac{\partial z}{\partial v}.$$

$$= xy\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial^2 v}\left(-y/x^3\right) - \frac{1}{x^2}\frac{\partial z}{\partial v} + \frac{\partial z}{\partial u}$$

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Putting the value of p, q, r, s, t in (i) we get,

** বমা ৩ ***

$$\boxed{\Rightarrow \frac{y}{x}\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v} = 2}$$

$$\Rightarrow v\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v} = 2$$ which is the required cannonical form

Let, $z_1 = \frac{\partial z}{\partial v}$

$$\frac{\partial z_1}{\partial v} = \frac{\partial^2 z}{\partial v^2}$$

$$v\frac{\partial z_1}{\partial v} + 2z_1 = 2$$

$$\Rightarrow \frac{\partial z_1}{\partial v} + \frac{2}{v} z_1 = \frac{2}{v}$$

This is 1st order linear PDE in z_1

Integrating Factor $= e^{\int \frac{2}{v} dv} = v^2$, multiplying (11) by v^2

$$v^2 \frac{\partial z_1}{\partial v} + 2v z_1 = 2v$$

$$\Rightarrow v^2 z_1 = v^2 + \phi_1(u) \quad \text{[by integrating]}$$

$$\Rightarrow z_1 = 1 + \frac{1}{v^2}\phi_1(u)$$

$$\Rightarrow \frac{\partial z}{\partial v} = 1 + \frac{1}{v^2}\phi_1(u)$$

$$\Rightarrow z = v + \int \frac{1}{v^2}\phi_1(u)\,dv \quad \text{[by integrating]}$$

$$= v - \frac{1}{v}\phi_1(u) + \phi_2(u)$$

$$\therefore z = \frac{y}{x} - \frac{x}{y}\phi_1(xy) + \phi_2(xy)$$

which is completed solⁿ of (1).

Problem 4: Reduce to Canonical Form and solve the PDE $y(x+y)(r-s) - xp - yq = z \quad -(1)$

Comparing eqn (1) with $Rr + Ss + Tt + F(x,y,z,p,q) = 0$

$$R = y(x+y), \quad S = -y(x+y), \quad T = 0$$

Now,

$S^2 - 4RT = y^2(x+y)^2 > 0$ which is hyperbolic.

Then the α quadratic eqn,

$$R\alpha^2 + S\alpha + T = 0$$

$$\Rightarrow \quad y(x+y)\alpha^2 - y(x+y)\alpha = 0$$

$$\Rightarrow \quad \alpha(xy\alpha + y^2\alpha - xy - y^2) = 0$$

Then transformation of independent variable x, y to u, v

$u = x+y, \quad v = y$

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 1$

$\frac{\partial v}{\partial y} = 1$

$\frac{\partial^2 u}{\partial x^2} = 0$

$\frac{\partial^2 v}{\partial x^2} = 0$

$\frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 v}{\partial y^2} = 0$

$$p = \frac{\partial z}{\partial x}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}$$

$$q = \frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 0 + 0 + 0 + 0$$

$$= \frac{\partial^2 z}{\partial u^2}$$

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$$s = \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x \cdot \partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y \partial x}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + 0 + 0 + 0$$

Putting p, q, r, s in eqn ① =>

$$y(x+y)\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) - x\frac{\partial z}{\partial u} - y\frac{\partial z}{\partial u} - y\frac{\partial z}{\partial v} = z$$

$$\Rightarrow uv\left(-\frac{\partial^2 z}{\partial u \partial v}\right) - (x+y)\frac{\partial z}{\partial u} - y\frac{\partial z}{\partial v} - z = 0$$

$$\Rightarrow uv\left(-\frac{\partial^2 z}{\partial u \partial v}\right) - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} - z = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} + \frac{z}{uv} = 0$$

$$\Rightarrow \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial u} + \frac{1}{u}\right)\cdot\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0$$

let,

$$\frac{\partial z}{\partial v} + \frac{z}{v} = w$$

$$\left(\frac{\partial}{\partial u}+\frac{1}{u}\right)w=0$$

$$\Rightarrow \frac{\partial w}{\partial u}+\frac{w}{u}=0$$

$$\Rightarrow \frac{\partial w}{w}+\frac{\partial u}{u}=0$$

$\Rightarrow$ ~~we~~ ~~we~~ $wu=c$ [integrating,

$$\Rightarrow wu=\phi_1(v)$$

$$\Rightarrow u\left(\frac{\partial z}{\partial v}+\frac{z}{v}\right)=\phi_1(v)$$

$\Rightarrow \dfrac{\partial z}{\partial v}+\dfrac{z}{v}=\dfrac{1}{u}\phi_1(v)$ — which is linear.

$$\Rightarrow IF=e^{\int\frac{1}{v}dv}=v$$

multiply v.

$$v\frac{\partial z}{\partial v}+z=\frac{v}{u}\phi_1(v)$$

$$\Rightarrow d(vz)=\frac{v}{u}\psi_1(v)$$

$$\Rightarrow vz=\frac{1}{u}\int v\,\phi_1(v)\,dv+\phi_2(u)$$

$$\boxed{\Rightarrow z=\frac{1}{vu}\phi_1(v)+\frac{1}{v}\phi_2(u)}$$

4) Reduce $xyr - (x^2-y^2)s - xyt + py - qx = 2(x^2-y^2)$ into canonical form and hence solve it.

1st order 1st degree (Only $\frac{dy}{dx}$ or p),

* Order and degree চিহ্নিত করা
* Linear / Bernoil's eqn কিনা দেখা
* Separable করা যায় কিনা দেখা
* Homogenous কিনা দেখা
* Exact কিনা দেখা

1st order higher degree. (p^2)

* lagrange's কিনা দেখা
* Claiurault's কিনা দেখা
* xy এক সাথে কিনা দেখা
* x একটা solvable for x
* Y একটা থাকলে solvable for Y.

higher Order ~~higher~~ 1st degree.

* e^x, Sin ax / cos ax, x^m, $x^m v$, e^{ax}, v. → constant co-efficient
* Euler. → Variable Co-efficient.

$\downarrow$

$x = e^z$

$$* \; x^n \frac{d^n y}{dx^n} = \{D(D-1) \cdots (D-n+1)\} y$$

higher order higher degree