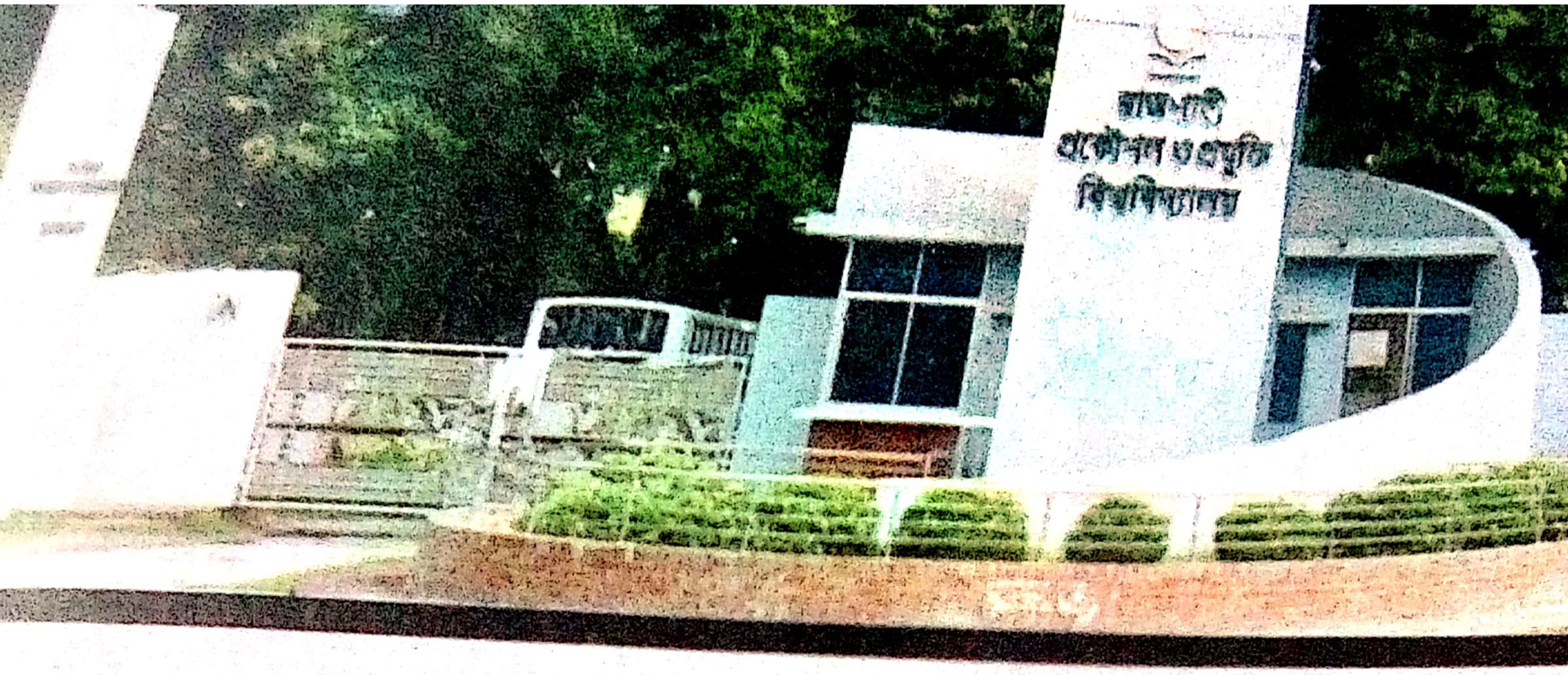

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Math - 2201

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# Fourier Series:

**Periodic function:** The function $f(x)$ is said to be periodic if there exists a non-zero number $T$ such that $f(T+x) = f(x)$

**Example:**

$f(x) = \sin x$ is a periodic function.

because $f(2\pi + x) = \sin(2\pi + x) = \sin x = f(x)$

**Even function:** A function $f(x)$ is said to be even if $f(-x) = f(x)$

**Example:** $f(x) = x^2$

**Odd function:** A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

**Example:** $f(x) = x^3$

**Fourier series:** Under certain condition the function $f(x)$ in a period $-\pi \leq x \leq \pi$ can be expressed in terms of a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$ which is known as

fourier series with fourier constants $a_0, a_n, b_n$.

# Determination of the Fourier constants / co-efficients in $(-\pi, \pi)$:

We know the fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —— ①

Now integrating ① w.r.t to $x$ in $(-\pi, \pi)$:

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum \left(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2} \times 2\pi + 0$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Multiplying ① by $\cos nx$ and integrating w.r.t to $x$ in $(-\pi, \pi)$:

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos nx\,dx + \sum \left(a_n \int_{-\pi}^{\pi} \cos^2 nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \cos nx\,dx\right)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + a_n \cdot \pi + 0$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

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Similarly multiplying ① by $\sin nx$ and integrating w.r.t. to $x$ in $(-\pi, \pi)$ we get,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Note: (a) If $f(x)$ is an even function then.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \neq 0, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \neq 0$$

and $b_n = 0$.

(b) If $f(x)$ is an odd function then,

$$a_0 = 0, \quad a_n = 0 \text{ and } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \neq 0.$$

## (b) Fourier series with period $2l$ :

Consider a function $\phi(y)$ in $(-l, l)$ which is integrable and $y = \frac{lx}{\pi}$.

Then $\phi\left(\frac{lx}{\pi}\right)$ is the function of period $2\pi$ and $f(x) = \phi\left(\frac{lx}{\pi}\right) = \phi(y)$

In this case, the fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$ will be converted to a fourier series for $\phi(y)$ as

$$\phi(y) = \frac{a_0}{2} + \sum \left(a_n \cos \frac{n\pi y}{l} + b_n \sin \frac{n\pi y}{l}\right)$$

So the fourier constants are –

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} \phi(y)\,dy$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{l}\int_{-l}^{l} \phi(y)\cos\frac{n\pi y}{l}\,dy$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{l}\int_{-l}^{l} \phi(y)\sin\frac{n\pi y}{l}\,dy$$

↳ [illegible] range এর জন্য

## (ii) Parseval's Theorem:

Statement: If the fourier series of a function $f(x)$ is converges uniformly in the interval $(-L, L)$ then.

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \Sigma\left(a_n^2 + b_n^2\right),$$ here $a_0, a_n, b_n$ are fourier constants.

Proof: we know, the fourier series in $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \Sigma\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \text{--- (i)}$$

Multiplying (i) by $f(x)$ and integrating w.r.to $x$ in $(-L, L)$:

$$\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0}{2}\int_{-L}^{L} f(x)dx + \Sigma\left(a_n \int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx\right.$$

$$\left. + b_n \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\right) \quad \text{--- (ii)}$$

But we know,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

$$\Rightarrow \int_{-L}^{L} f(x)\, dx = L a_0$$

Similarly,

$$\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx = l a_n$$

and $$\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx = l b_n$$

Putting the values in (ii),

$$\int_{-l}^{l} \{f(x)\}^2 \, dx = \frac{a_0^2}{2} l + \sum (a_n^2 l + b_n^2 l)$$

$$\Rightarrow \frac{1}{l} \int_{-l}^{l} \{f(x)\}^2 \, dx = \frac{a_0^2}{2} + \sum (a_n^2 + b_n^2)$$

(Proved)

16/02/2020

## (iv) Half range cosine series:

The part which contains only the cosine term in Fourier series is called half range cosine series. Its range is $(0, \pi)$ which is half range of $(-\pi, \pi)$ of Fourier series. In even function cosine term remain.

Expansion:

The cosine series in the range: $0 \leq x \leq \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{a} a_n \cos nx \quad \text{--------} (1)$$

Now integrating with respect to $x$, in $(0, \pi)$,

$$\int_0^{\pi} f(x)\, dx = \frac{a_0}{2} \int_0^{\pi} dx + \sum a_n \int_0^{\pi} \cos nx\, dx$$

$$= \frac{a_0}{2} (\pi) + \sum a_n \left[ \frac{\sin nx}{n} \right]_0^{\pi}$$

$$= \frac{a_0}{2} \times \pi + \sum a_n \times 0$$

$$\therefore \quad a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

Multiplying (1) by $\cos nx$ and integrating in $(0, \pi)$:

$$\int_0^{\pi} f(x)\cos nx\, dx = \frac{a_0}{2}\int_0^{\pi} \cos nx\, dx + \sum a_n \int_0^{\pi} \cos^2 nx\, dx.$$

$$= 0 + \frac{a_n}{2}\int_0^{\pi} (1+\cos 2nx)\, dx$$

$$= \frac{a_n}{2}\left[x + \frac{\sin 2xn}{2n}\right]_0^{\pi}$$

$$= \frac{a_n \pi}{2}$$

$$\therefore \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

Hence, from (1) we get

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx + \sum \frac{2}{\pi}\int_0^{\pi} f(x)\cos^2 nx\, dx$$

# What do you mean by half range sine series and explain.

[11] Expand Fourier series for $f(x) = x^2$ within $-\pi \le x \le \pi$, Hence show that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$\Rightarrow$ Given that,

$$f(x) = x^2$$

$$\therefore f(x) = (-x)^2 = x^2 = f(x)$$

$\therefore f(x)$ is an even function and $b_n = 0$

In this case, the fourier series is $f(x) = \frac{a_0}{2} + \sum a_n \cos nx$ —— (1)

Where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} 2 \int_{0}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x^2\, dx$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi} \left[ x^2 \frac{\sin nx}{n} \right]_0^{\pi} - \frac{2}{\pi} \int_{0}^{\pi} 2x \frac{\sin nx}{n}\, dx$$

$$= \frac{2}{\pi} \times 0 - \frac{4}{n\pi} \left[ - \frac{x \cos nx}{n} \right]_0^{n} + \frac{4}{n\pi} \int_{0}^{\pi} - \frac{\cos nx}{n}\, dx$$

$$= \frac{4\pi \cos n\pi}{n^2 \pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Now, putting the values of $a_0, a_n, b_n$ in (ii) ⟶

$$f(x) = \frac{1}{2} \cdot \frac{2\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + 4 \left[ -\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots \right]$$

Putting $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4 \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

[Proved]